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**VECTOR AND MATRIX DIFFERENTIATION**

# 1 Derivatives of $Ax$

Let $a \in R^n$, $x \in R^n$ (all vectors are column vectors). Then

$$
\begin{aligned}
\frac{\partial\left(a'x\right)}{\partial x} &= \begin{pmatrix} \frac{\partial\left(a'x\right)}{\partial x_1} \\ \vdots \\ \frac{\partial\left(a'x\right)}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial\left(a_1x_1+\ldots+a_nx_n\right)}{\partial x_1} \\ \vdots \\ \frac{\partial\left(a_1x_1+\ldots+a_nx_n\right)}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\
&= a.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\left(a'x\right)}{\partial x'} &= \begin{pmatrix} \frac{\partial\left(a'x\right)}{\partial x_1} & \ldots & \frac{\partial\left(a'x\right)}{\partial x_n} \end{pmatrix} \\
&= \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \\
&= a'.
\end{aligned}
$$

Let $A$ be a $m \times n$ matrix,

$$
A = \begin{pmatrix} a_1' \\ \vdots \\ a_m' \end{pmatrix},
$$

where $a_j \in R^n$ for $j = 1, \ldots, m$.

$$
\begin{aligned}
\frac{\partial\left(Ax\right)}{\partial x'} &= \begin{pmatrix} \frac{\partial\left(a_1'x\right)}{\partial x'} \\ \vdots \\ \frac{\partial\left(a_m'x\right)}{\partial x'} \end{pmatrix} \\
&= \begin{pmatrix} a_1' \\ \vdots \\ a_m' \end{pmatrix} \\
&= A.
\end{aligned}
$$

Similarly,

$$\begin{aligned}
\frac{\partial\left(x'A'\right)}{\partial x} &= \begin{pmatrix} \frac{\partial\left(x'a_1\right)}{\partial x} & \ldots & \frac{\partial\left(x'a_m\right)}{\partial x} \end{pmatrix} \\
&= \begin{pmatrix} a_1 & \ldots & a_m \end{pmatrix} \\
&= A'.
\end{aligned}$$

Let $\alpha \in R^r$, and $x = x\left(\alpha\right)$. Then

$$\frac{\partial x}{\partial \alpha'} = \begin{pmatrix} \frac{\partial x_1}{\partial \alpha'} \\ \vdots \\ \frac{\partial x_n}{\partial \alpha'} \end{pmatrix},$$

an $n \times r$ matrix.

Let $x = x\left(\alpha\right)$. Then

$$\begin{aligned}
\frac{\partial\left(a'x\right)}{\partial \alpha} &= \begin{pmatrix} \frac{\partial(a_1x_1+\ldots+a_nx_n)}{\partial \alpha_1} \\ \vdots \\ \frac{\partial(a_1x_1+\ldots+a_nx_n)}{\partial \alpha_r} \end{pmatrix} \\
&= \begin{pmatrix} a_1\frac{\partial x_1}{\partial \alpha_1} + \ldots + a_n\frac{\partial x_n}{\partial \alpha_1} \\ \vdots \\ a_1\frac{\partial x_1}{\partial \alpha_r} + \ldots + a_n\frac{\partial x_n}{\partial \alpha_r} \end{pmatrix} \\
&= \frac{\partial x'}{\partial \alpha}a,
\end{aligned}$$

an $r$-vector, and

$$\frac{\partial\left(a'x\right)}{\partial \alpha'} = a'\frac{\partial x}{\partial \alpha'}.$$

Next,

$$\begin{aligned}
\frac{\partial\left(Ax\right)}{\partial \alpha'} &= \begin{pmatrix} \frac{\partial\left(a_1'x\right)}{\partial \alpha'} \\ \vdots \\ \frac{\partial\left(a_m'x\right)}{\partial \alpha'} \end{pmatrix} \\
&= \begin{pmatrix} a_1'\frac{\partial x}{\partial \alpha'} \\ \vdots \\ a_m'\frac{\partial x}{\partial \alpha'} \end{pmatrix} \\
&= A\frac{\partial x}{\partial \alpha'}.
\end{aligned}$$

Note that

$$\frac{\partial x}{\partial x'} = I_n.$$

## 2 Derivatives of $z'Ax$

Let $x \in R^n$, $z \in R^m$, and $A$ is $m \times n$. Define $c = A'z$, a $n$-vector. Then,

$$\begin{aligned} \frac{\partial (z'Ax)}{\partial x} &= \frac{\partial (c'x)}{\partial x} \\ &= A'z. \end{aligned}$$

Next,

$$\frac{\partial (z'Ax)}{\partial z} = Ax.$$

Let $x = x(\alpha)$, and $z = z(\alpha)$. Then

$$\begin{aligned} \frac{\partial (z'Ax)}{\partial \alpha} &= \frac{\partial z'}{\partial \alpha}\frac{\partial (z'Ax)}{\partial z} + \frac{\partial x'}{\partial \alpha}\frac{\partial (z'Ax)}{\partial x} \\ &= \frac{\partial z'}{\partial \alpha}Ax + \frac{\partial x'}{\partial \alpha}A'z. \end{aligned}$$

Next, if $m = n$,

$$\begin{aligned} \frac{\partial (x'Ax)}{\partial x} &= Ax + A'x \\ &= (A + A')\,x. \end{aligned}$$

If $A$ is symmetric,

$$\frac{\partial (x'Ax)}{\partial x} = 2Ax.$$

If $x = x(\alpha)$, $A$ is symmetric and independent of $\alpha$,

$$\frac{\partial (x'Ax)}{\partial \alpha} = 2\frac{\partial x'}{\partial \alpha}Ax.$$

The derivative of $z'Ax$ with respect to $A$ is given by

$$\begin{aligned} \frac{\partial (z'Ax)}{\partial A} &= \begin{pmatrix} \frac{\partial (z'Ax)}{\partial A_{11}} & \cdots & \frac{\partial (z'Ax)}{\partial A_{1n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial (z'Ax)}{\partial A_{m1}} & \cdots & \frac{\partial (z'Ax)}{\partial A_{mn}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial\left(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j\right)}{\partial A_{11}} & \cdots & \frac{\partial\left(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j\right)}{\partial A_{1n}} \\ \cdots & \cdots & \cdots \\ \frac{\partial\left(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j\right)}{\partial A_{m1}} & \cdots & \frac{\partial\left(\sum_{i=1}^m \sum_{j=1}^n z_i A_{ij} x_j\right)}{\partial A_{mn}} \end{pmatrix} \\ &= \begin{pmatrix} z_1 x_1 & \cdots & z_1 x_n \\ \cdots & \cdots & \cdots \\ z_m x_1 & \cdots & z_m x_n \end{pmatrix} \\ &= zx'. \end{aligned}$$

## 3 GMM first-order conditions

Let $z \in R^l$, $x \in R^k$, $b \in R^k$, $y \in R$, and $W$ is $l \times l$ and symmetric. Then

$$\begin{aligned}\frac{\partial}{\partial b}\left[(z(y-x'b))' W(z(y-x'b))\right] &= 2\frac{\partial (z(y-x'b))'}{\partial b} W(z(y-x'b)) \\ &= -2\frac{\partial (zx'b)'}{\partial b} W(z(y-x'b)) \\ &= -2\frac{\partial (b'xz')}{\partial b} W(z(y-x'b)) \\ &= -2xz'W(zy-zx'b).\end{aligned}$$

Similarly, suppose that $Z$ is $n \times l$, $X$ is $n \times k$, $b$ is $k \times 1$, $Y$ is $n \times 1$, and $W$ is $l \times l$. Then

$$\begin{aligned}\frac{\partial}{\partial b}\left[(Z'(Y-Xb))' W(Z'(Y-Xb))\right] &= 2\frac{\partial (Z'(Y-Xb))'}{\partial b} W(Z'(Y-Xb)) \\ &= -2\frac{\partial (Z'Xb)'}{\partial b} W(Z'(Y-Xb)) \\ &= -2\frac{\partial (b'X'Z)}{\partial b} W(Z'(Y-Xb)) \\ &= -2X'ZW(Z'Y-Z'Xb).\end{aligned}$$

## 4 ML first-order conditions

Suppose $x$ is a $k$-vector, and $\Sigma$ is $k \times k$, symmetric and positive definite. Define

$$Q(\Sigma) = -\log|\Sigma| - x'\Sigma^{-1}x.$$

We use the fact that

$$\frac{\partial \log|\Sigma|}{\partial \Sigma} = \Sigma^{-1},$$

and, hence,

$$\begin{aligned}-\frac{\partial \log|\Sigma|}{\partial \Sigma^{-1}} &= \frac{\partial \log\left|\Sigma^{-1}\right|}{\partial \Sigma^{-1}} \\ &= \Sigma.\end{aligned}$$

Next,

$$\frac{\partial\left(x'\Sigma^{-1}x\right)}{\partial \Sigma^{-1}} = xx'.$$

Therefore, the first-order conditions are given by

$$\Sigma = xx'.$$