Theoretical Astrophysics

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Contents

Units, Definitions and Data

We will use CGS units, throughout, as they make electromagnetism look simple.

Units of various quantities in the cgs system:

[length] = cm	[mass] = g
[time] = s	[force] = dyne = g cm s^{-2}
[pressure] = dyne cm^{-2}	[energy] = erg = g cm^2 s^{-2}
[temperature] = Kelvin	

Useful definitions:

1 inch = 2.54 cm	1 G = 10^{-4} T
1 eV = $1.602176487 \times 10^{-12}$ erg	$0°C = 273.15$ K
1 erg = 10^{-7} J	1 dyne = 10^{-5} N
1 C = 2.99792458×10^9 esu	1 atm = 1.01325 barr
1 bar = 10^6 dyne cm^{-2}	1 Jansky = 10^{-23} erg s^{-1} cm^{-2} Hz^{-1}
1 AU = $1.49597870700 \times 10^{13}$ cm	1 parsec = 3.085677×10^{18} cm
1 ly = 0.3066 pc	

Some useful physical constants:

$c = 2.99792458 \times 10^{10}$ cm s^{-1}	$G = 6.67 \times 10^{-8}$ cm^3 g^{-1} s^{-2}
$h = 6.62606896 \times 10^{-27}$ erg s	$\hbar = h/2\pi = 1.054571628 \times 10^{-27}$ erg s
$e = 4.8032042710^{-10}$ esu	$k_b = 1.38 \times 10^{-16}$ erg K^{-1}
$m_e = 9.10938214 \times 10^{-28}$ g	$m_p = 1.672621637 \times 10^{-24}$ g
$\alpha = e^2/\hbar c = 1/137.036$	$r_e = e^2/m_e c^2 = 2.8179402894 \times 10^{-13}$cm
$a_\infty = \hbar^2/m_e e^2 = 0.52917720859 \times 10^{-8}$cm	$m_e e^4/2\hbar^2 = 13.60569193$ eV
$\sigma_T = 8\pi r_e^2/3 = 0.6652458558 \times 10^{-24}$ cm^2	$\sigma_{\mathrm{sb}} = 5.670400 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4}

Useful data:

$\pi = 3.141592653589793238$
$e = 2.718281828459045235$

$\gamma = 0.577215664901532861$

sidereal year $= 3.15581498 \times 10^7$ s $\simeq \pi \times 10^7$

number of seconds in a day $= 86400$

Wien displacement law $\lambda = 0.2898$ cm K$/T$

electron cyclotron frequency $= 1.759 \times 10^7 B$ rad s^{-1} G^{-1}.

proton cyclotron frequency $= 9.579 \times 10^3 B$ rad s^{-1} G^{-1}.

density of air at STP $\rho \simeq 10^{-3}$ g cm^{-3}

density of water at STP $\rho \simeq 1$ g cm^{-3}

room temperature $T = 273$ K

$M_{\rm Earth} = 6.0 \times 10^{27}$ g

$R_{\rm Earth} = 6.4 \times 10^8$ cm

$GM_{\rm Earth}/R^2_{\rm Earth} = 980$ cm s^{-2}

$M_\odot = 1.9884 \times 10^{33}$ g

$R_\odot = 6.9551 \times 10^{10}$ cm

$L_\odot = 3.8427 \times 10^{33}$ erg s^{-1}

distance to the galactic center $= 8.4$ kpc

circular velocity at solar position $= 240$ km s^{-1}

local galactic disk density $= 3 - 12 \times 10^{-24}$ g cm^{-3}

Chapter 1

the Fluid Equations

Fluid mechanics can describe a vast number of phenomena you see everyday, from the flow of air around you, the mixing of a drop of milk in coffee, motion of clouds, and large scale weather systems. Since most of the universe is gaseous, knowledge of fluid flow is required for many astrophysical systems. Even the motions of individual stars in the galaxy can be approximately described by fluid concepts and equations.

The equations of fluid dynamics are relatively simple, and are just statements of conservation of mass, momentum ($f = ma$) and energy. However, the types of possible fluid systems and solutions are vast, and our aim here is only to introduce some basic concepts and toy problems applicable to astronomy.

First we will start with the simplest situation of a gas at rest. The gravitational and pressure gradient forces per unit volume will be discussed. Solutions will be found for static systems in "hydrostatic balance", where pressure balances gravity. Many objects in the Universe are roughly in hydrostatic balance, including some we experience in everyday life. In other words, they are not exploding due to pressure overwhelming gravity, nor collapsing due to gravity overwhelming pressure; there is a balance. Hence astronomers use the concept of hydrostatic balance often, and must understand it well. As a simplest problem, consider the run of pressure with depth in a swimming pool. Hydrostatic balance leads to the idea that "the more stuff is over your head, the higher the pressure you feel." In a swimming pool, you can feel the higher pressure in your ears when you go down about 10m.

In hydrostatic balance, the sum of all forces is zero at every point. Next we turn to the more general problem of moving fluids, where unbalanced forces can cause acceleration. Hence we need to understand the "ma" part of "f=ma" for a fluid. In many situations, pressure and gravity are *mostly* in balance, but there are slight deviations which cause fluid to be accelerated and move. For example, pressure at sea level is $P \simeq 1$ bar, while "weather" is accompanied by slight pressure changes $\delta P \sim 10^{-3}$ bar. In other cases, the imbalance between pressure and gravity is extreme and either explosion (e.g. supernovae, classical novae, ...) or collapse (star formation, core collapse, ...) occurs. Small perturbations about an equilibrium state can be described by a *linearized* form of the fluid equations. We'll study stellar oscillations in stars using the linear equations. Violent behavior in fluids, involving discontinuous changes such as "shocks", are highly nonlinear, and instead we use a method similar to the "Gaussian pillbox" from electromagnetism to describe changes in the fluid across the discontinuity.

1.1 gravity and pressure forces in hydrostatic balance

In this section we will derive the gravity and pressure forces (per unit volume) and look at the simplest solution of the hydrostatic balance equation. Our first (important) application will be to hydrostatic balance of a layer of water, like a swimming pool or ocean.

First introduce some definitions. See figure 1.1 for the geometry. Let the z direction increase upward, and gravity $g \simeq 10^3$ cm s^{-2} point downward. That is, the vector gravity is $\boldsymbol{g} = -g\boldsymbol{e}_z$, where $\boldsymbol{e}_z$ is the upward unit vector. The water has a mass density $\rho \simeq 1$ g cm^{-3}. Let P be the pressure. Pressure can be defined as the force per unit area on a surface, or equivalently the momentum per unit time through some surface area. In CGS units, the unit of pressure is $[P] =$ dyne cm^{-2}. At sea level, atmospheric pressure, due to the weight of the atmosphere above us, is 1bar $= 10^6$ dyne cm^{-2}.

The force per unit volume on the water is $\rho\boldsymbol{g} = -\rho g\boldsymbol{e}_z$. What is the origin of the pressure force? From figure 1.1, we see the pressure is not the same at each height, it has a run given by $P(z)$. Consider water in a column of horizontal area A in between the two levels z and $z + dz$, where dz is an infinitesimal length. Then the force on this column of water due to the

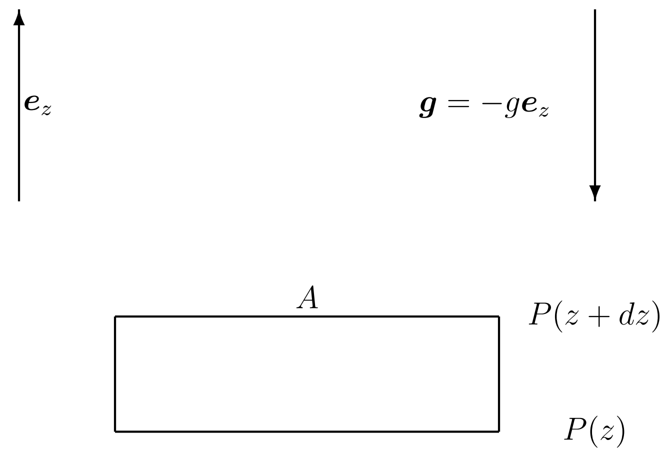

Figure 1.1: Plane parallel geometry for hydrostatic balance. The fluid element under consideration is in the box with area A on the top and bottom faces, between altitudes z and $z + dz$. Here $P(z)$ is the pressure at altitude z.

pressure force is

$$\begin{aligned} & \text{pressure force on column in the } + \text{ z direction} \\ = & \; AP(z) - AP(z+dz) \\ \simeq & \; -A\frac{dP}{dz}dz, \end{aligned} \tag{1.1}$$

where in the second expression we Taylor expanded the pressure. Dividing by the volume of the column, Adz, the pressure force per unit volume is

$$\text{pressure force per unit volume} \;=\; -\boldsymbol{\nabla} P = -\frac{dP}{dz}\boldsymbol{e}_z. \tag{1.2}$$

The equation of hydrostatic balance applies to static gas, and says that the pressure force (per unit volume), $-\boldsymbol{\nabla} P$, is balanced by the gravitational force per unit volume, $\rho \boldsymbol{g}$. In a general vector form this equation is

$$0 = -\boldsymbol{\nabla} P + \rho \boldsymbol{g}. \tag{1.3}$$

and in our case where all variables only depend on z

$$\frac{dP}{dz} = -\rho g. \tag{1.4}$$

Let's solve this equation for the swimming pool, with surface at $z = 0$, and depth $-z > 0$ when below the surface. Since ρ and g are constant, we can directly integrate. Below the water level, for $z < 0$,

$$P(z) = 1\text{bar} - \rho g z = 1 \text{ bar}\left(1 + \frac{|z|}{10 \text{ m}}\right), \tag{1.5}$$

where in the last step we plugged in numbers.

Q: In a depth of just $|z| = 10$ m, the water gives the same hydrostatic pressure as the atmosphere! How can this be?

A: It's because the atmosphere has a much smaller density. Notice that it's the quantity $\rho \times z$ that matters. The integral $\int_z^\infty dz \rho(z)$ is called the "mass column" and is the mass per unit horizontal area above height z. Since the density of air is $\rho \simeq 10^{-3}$ g cm^{-3} at sea level, and (as we'll see) the "height" of the atmosphere is about $z = 10$ km, the small density is offset by the larger size of the atmosphere.

We started with the ρ and g constant case is it is easiest. However, in a gaseous atmosphere, density will decrease upward. What is needed is a relation between density ρ and pressure P which allows us to solve the hydrostatic balance equation. This relation is called the "equation of state." We will start with the familiar case of a non-degenerate, ideal gas.

1.2 the "equation of state" of a non-degenerate ideal gas

Briefly, non-degenerate just means the temperature is high and the density low, and ideal means that the potential energy between particles is much smaller than their kinetic energy. These concepts will be revisited later when we study statistical mechanics.

We are perhaps most used to the ideal gas law for a box of volume V containing N gas particles at temperature T:

$$P = \frac{Nk_bT}{V} = \left(\frac{N}{V}\right) k_bT \equiv nk_bT. \tag{1.6}$$

In astrophysics, we don't have any boxes, hence in the third step we've rewritten this expression in terms of the number density of particles $n = N/V$. Rather, we idealize small volumes of fluid in which the number density is roughly constant, and so we always need to think in per unit volume or per unit mass terms.

Often, instead of n, it's more convenient to use mass density ρ, and so *in the non-degenerate case* we define a "mean molecular weight" μm_p, where μ is a dimensionless number and m_p is the proton mass (or more precisely, the atomic mass unit). The number density is then rewritten as

$$n = \frac{\rho}{\mu m_p} \tag{1.7}$$

and the pressure becomes

$$P = \frac{\rho k_b T}{\mu m_p}. \tag{1.8}$$

What is μ for the Earth's atmosphere? It is composed, by number, of about 78% N_2 (mass $28m_p$) and 22% O_2 (mass $32m_p$). In other words, if n is the *total* number density then $n_{N_2}/n = 0.78$ and $n_{O_2}/n = 0.22$. To derive μ we then compute

$$\begin{aligned} \rho &\equiv \mu m_p = 28 m_p n_{N_2} + 32 m_p n_{O_2} \\ &= m_p n \left(0.78 \times 28 + 0.22 \times 32\right) \simeq 29 m_p n, \end{aligned} \tag{1.9}$$

so that $\mu \simeq 29$ for the Earth's atmosphere. In planetary physics, fractions of often by number while in astronomy fractions are by mass. We'll do that later.

Now let's compute the run of pressure in a simple hydrostatic model of a thin atmosphere which has constant temperature T and mean molecular weight μ.

1.3 plane parallel isothermal atmosphere

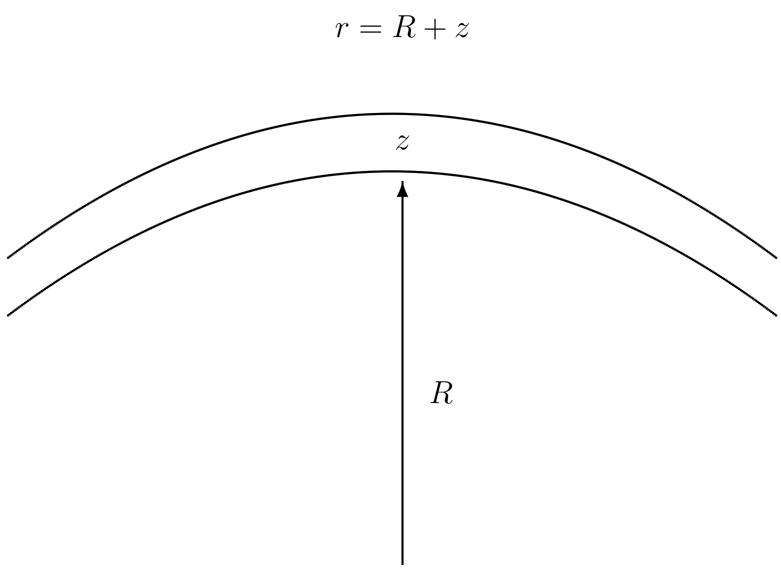

Figure 1.2: Spherical geometry looks plane parallel locally when the region of the atmosphere under consideration is thin in comparison to the size of the sphere.

The density in the Earth's atmosphere decreases so fast that we're often only interested in regions near the surface, for which the altitude z, measured from ground, satisfies $z \ll R$, where R is the radius of the Earth (see figure 1.2). So plane parallel with constant gravity is good in this region.

Plugging the pressure from eq. 1.8 into eq.1.4, and treating μ and T as constant, we find

$$\frac{dP}{dz} = \frac{k_b T}{\mu m_p} \frac{d\rho}{dz} = -\rho g. \tag{1.10}$$

This can be written in the more intuitive form

$$\frac{d\ln\rho}{dz} = -\frac{1}{H}, \tag{1.11}$$

where the "scale height"

$$H \equiv \frac{k_b T}{\mu m_p g} \simeq 10 \text{ km} \tag{1.12}$$

and the numerical value is for Earth at sea level. As we'll see, *the "thickness" of the atmosphere as basically the scale height H*, hence it is an important parameter in any atmosphere. The scale height is larger for larger temperature, and smaller for larger mean molecular weight or gravity. Since H is treated as constant, eq.1.11 can be immediately integrated to find

$$\rho(z) = \rho(0)e^{-z/H}, \tag{1.13}$$

i.e. the density efolds over each scale height. Note that 10 km $\sim 30,000$ ft, the cruising altitude for airplanes. So the density is significantly lower there. As we'll see, the temperature decreases by about 10 K km^{-1}, so it's about ~ 100 K colder than ground as well.

Q: what is the scale height in the non-degenerate atmosphere of a $M = M_\odot$, $R = 10$ km neutron star composed of ionized hydrogen and with temperature $T = 10^6$ K?

A: the surface gravity is $g = GM_\odot/(10\text{km})^2 \simeq 10^{14}$ cm s^{-2}. For ionized hydrogen, there are two particles supplying the pressure, proton and electron, with equal density due to charge neutrality, so $\mu = 1/2$. The scale height is then

$$H = \frac{2k_b T}{m_p g} \simeq 1 \text{ cm}. \tag{1.14}$$

So the surface gravity is so large that the atmospheric thickness is only a centimeter!

What is the thickness of an atmosphere? When you first think about this, it doesn't make sense, as there is gas no matter how high you go up.

But the density becomes very small as you go up. For instance, the mass of the atmosphere above a height z is

$$\begin{aligned} M(>z) &= 4\pi R^2 \int_z^\infty dz \rho(z) = 4\pi R^2 H \rho(0) e^{-z/H} \\ &\simeq 10^{-6} M_{\rm Earth} e^{-z/H}. \end{aligned} \tag{1.15}$$

So, since the base density is small, there's not much mass in the atmosphere as compared to the interior, and the mass a few scale heights up is exponentially small. As far as mass is concerned, the thickness of the atmosphere is a few scale heights. Since the opaqueness of gas is proportional to the column along the line of sight, radiation also seems to emerge from "photospheres" of order a few scale heights thickness. Due to larger base density and temperature, Venus has a much thicker atmosphere than Earth does. The giant planets (Jupiter, Saturn, Uranus and Neptune) have solid surface very, very deep in the planet, if at all. The Sun is entirely gaseous.

Q: Consider a satellite orbiting Earth subject to drag forces due to the Earth's atmosphere. The size of the drag force is proportional to the gas density. Given that the density at the Earth's surface is basically fixed, how will the density at the altitude of the satellite change if the atmospheric temperature increases? How sensitive is density to temperature?

A: As temperature increases, scale height increases and density increases. If you want to design safe orbits for satellites, keep in mind that temperature appears in an exponential! If the Earth's atmosphere heats up a bit, the density increases exponentially with the temperature increase. Temperature increases high up in the Earth's atmosphere can be much larger than at sea level.

This increase in temperature and hence density is what led to large atmospheric drag forces on the *Skylab* satellite that caused it to crash.

Lastly, hydrostatic balance takes on a very simple form in a plane-parallel, constant gravity atmosphere. Integrate $dP/dz = -\rho g$ from z to ∞, with $P(\infty) = 0$. We find

$$P(z) = g \int_z^\infty dz' \, \rho(z') \equiv g\Sigma(z), \tag{1.16}$$

where we have defined the "mass column density"

$$\Sigma(z) = \int_z^\infty dz' \, \rho(z'). \tag{1.17}$$

Σ has the interpretation as the amount of mass (per unit area) above your head, and we see that $P \propto \Sigma$. This is the origin of the statement that pressure is just due to the mass over your head. As we've already seen, in an isothermal atmosphere, Σ takes on the simple form $\Sigma(z) = H\rho(z)$, so to convert from mass per unit volume to mass per unit area, you multipy by the scale height.

When we get to radiative transfer, we'll see the "optical depth" of a material involves the "number column" of each species, i.e. number of particles per unit area. When multiplied by a cross-sectional area, this gives a dimensionless number called "optical depth."

Before deriving the fluid equations, we will first discuss the physical conditions under which a collection of particles is called a fluid.

1.4 what is a fluid?

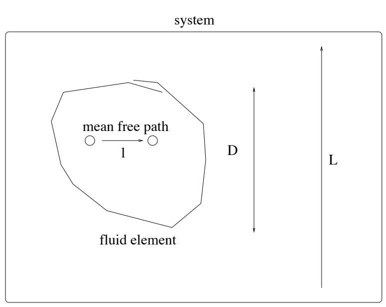

Figure 1.3: The three size scales of interest when defining a fluid element: the mean free path ℓ, the "fluid element" D, and the system size L.

When can you treat a collection of particles as a continuum, so that one particle knows what the other ones are doing (the fluid limit)? When does each particle move ballistically, independent of the other particles (the collisionless limit)? [1]

[1] Here we focus on neutral atoms or molecules. In an ionized plasma, the particles can also be coupled through both mean and fluctuating electric and magnetic fields.

Let the size of the system be L, and the mean free path traveled between collisions be ℓ. Particles travel ballistically between collisions. Clearly when $\ell \ll L$, a particle can't travel very far before it collides, transferring energy and momentum with the other particle. This interaction lets particles know what the neighboring particles are doing, and tends to organize the particles so that they have similar average directions and random thermal motions. By contrast, when $\ell \gg L$, particles travel ballistically through the system, and don't know what the other particles are doing. They hit boundaries without any prior knowledge the boundary is there, whereas in the collisional case particles reflecting off the boundary hit particles in the incoming flow, so that they realize the boundary is there without hitting it.

The fluid limit is much simpler, as the *statistical distribution* of particles can be characterized a few parameters, such as average density, temperature and mean velocity (i.e. the Boltzmann distribution for classical gases). By contrast, in the collisionless case, one needs to understand the full distribution function for the particle positions and velocities.

Let's say we are in the fluid limit so that $\ell \ll L$. "Fluid elements" of size D are defined such that they contain many particles, and are many mean free paths in size, $D/\ell \gg 1$, so that we can define macroscopic averages over the particles. They must also satisfy $D \ll L$, so that there are many fluid elements in the system, and that the macroscopic properties such as density and temperature vary only slightly over a fluid element, so that calculus can be used to understand their properties.

1.4.1 the small but finite mean free path limit – diffusion and "transport"

When the mean free path is small ($\ell \ll L$), but nonzero, particles undergo a random walk in space. In between each collision, they "hop" a distance ℓ, in a random direction (relative to the bulk flow). As a result, particles carry with them the properties of one fluid element and deposit them in another fluid element. This causes the property to mix between the two fluid elements.

Consider heat for example. If one fluid element is hotter than its neighbor, then when the particle hops from one fluid element to the other, it carries with it a larger thermal motion, and deposits it in that fluid element, increasing its temperature. This gives rise to the process of diffusion of heat.

As a second example, consider diffusion of momentum, i.e. viscosity,

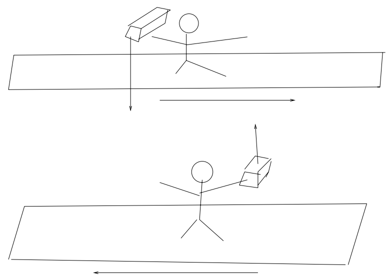

Figure 1.4: Toy problem to understand diffusion of momentum. Two people on train cars moving in opposite directions throw bricks from their train to the the other. When the other person catches the brick, the net momentum of the train decreases. This causes the two trains to try to have zero relative motion with respect to each other.

between neighboring fluid elements in relative motion. See figure 1.4 for a diagram. When a particle hops from its fluid element to the neighbor, it deposits its momentum, decreasing the relative motion. This viscous force is what causes velocity gradients in air to decrease over time, and for air to come to rest near solid surfaces. [2]

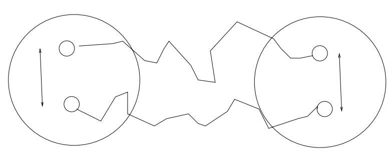

Figure 1.5: Motion of two particles in a fluid element under the action of both advection, the smooth bulk flow to the right, and diffusion, the small random motions. It makes sense to talk about fluid elements when the relative motion between particles is sufficiently small.

[2] In many fluid flows, this "molecular viscosity" due to the finite mean free path is a small effect, but the velocity gradients can give rise to "turbulence" in which swirling eddies of fluid mix momentum in a manner similar to microscopic motion. This effect is called "turbulent viscosity."

When can we think of a fluid element as containing the same particles as it moves through space? When is diffusion so large that fluid elements have particles rapidly moving in and out? There are two considerations. First, particles tend to diffuse due to their finite mean free path. Second, particles are "advected" with the mean flow. See figure 1.5 for a diagram. The true particle motion is a combination of both types of motion, like cloud of gnats in a wind. When the distance moved by the random walk is small compared to the bulk, advective motion, all particles stay near their neighbors.

Let's estimate the sizes of these two effects, and their relative importance. Diffusion is characterized by a "diffusion coefficient" D, with units cm^2 s^{-1}. In a time t a particle random walks a distance $x_{\rm diff} = \sqrt{Dt}$. The square root represents the random walk nature of the particles undergoing diffusion, namely that the average displacement is zero, while the root mean square deviation is nonzero.

By contrast, the distance traveled due to "advection" of particles by the bulk flow is $x_{\rm advect} = vt = L$, where v is the bulk flow velocity and L is the distance traveled. Over the time t, advection moves particles further than diffusion, $x_{\rm advect} > x_{\rm diff}$, if

$$vt \quad > \quad \sqrt{Dt}, \tag{1.18}$$

or

$$t \quad > \quad \frac{D}{v^2}. \tag{1.19}$$

So advection wins at late times, and diffusion wins at early times.

Put another way, this can be stated in terms of distance traveled as $L > \sqrt{DL/v}$ or

$$L \quad > \quad \frac{D}{v}. \tag{1.20}$$

Advection wins over large distances, diffusion wins over short distances.

In fluid dynamics, there are many dimensionless numbers which quantify advection versus diffusion for different fluid properties (heat, momentum, etc). For instance, for diffusion of momentum, the diffusion coefficient coefficient $D \rightarrow \nu$, where ν is the "kinematic viscosity." The ratio of advection to diffusion for bulk flow of characteristic size v and distance scales L is called the "Reynold's number":

$$\mathrm{Re} \quad = \quad \frac{vL}{\nu}. \tag{1.21}$$

It often takes some thought to figure out the right v and L for the problem at hand. For Re $\gg 1$, viscous effects can be ignored and one has a "perfect fluid". This is most of your experience in daily life. For Re $\ll 1$, diffusion of momentum is dominant as in, e.g. the fall of a marble in honey, or the fall of (small) chalkdust in air.

Diffusion coefficients differ slightly depending on the quantity being diffused. For air and water, the values are $D = 10^{-1}, 10^{-2}$ cm^2 s^{-1}, respectively.

Q: How long does it take heat to diffuse across this classroom?

A: The time is $t \sim L^2/D \sim (10 \text{ m})^2/(0.1 \text{ cm}^2 \text{ s}^{-1}) \sim 10^7 \text{ s} \sim (1/3)$year! Clearly when you turn on a heater and feel it get warmer on the other side of the room, this is due to advection (bulk motion of air) rather than diffusion.

This is generally true in daily life, than diffusion only matters on very short length and time scales.

A thanksgiving Q: Consider a spherical turkey of radius $L = 10$ cm, with heat diffusion coefficient $D = 10^{-2}$ cm^2 s^{-1}. How long will it take this turkey to cook?

A: $t = L^2/D = 10^4$ s $\sim$ a few hours.

How do we estimate the mean free path for different types of gases? Let's run through some examples.

1.4.2 neutral atoms and molecules in this room

The size of atoms and molecules is $\simeq 10^{-8}$ cm, giving cross sectional areas $\sigma \sim 10^{-16}$cm^2 (**Q:** what is the force law between atoms and molecules that causes this scattering?). The number density of particles is $n = P/(k_b T) \simeq 10^{19}$ cm^{-3}. The mean free path is defined as the distance over which a particle must travel before it hits another particle. See figure 1.6. After moving a distance ℓ, the atom has swept out a volume $\sigma\ell$. the number of target particles contained in the swept out volume is $n\sigma\ell$. If we require that one target particle was hit over the distance ℓ, then the mean free path is

$$\ell = \frac{1}{\sigma n} \simeq 10^{-4} \text{ cm}, \tag{1.22}$$

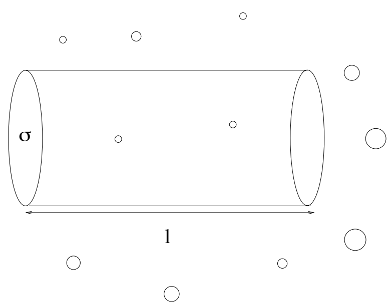

Figure 1.6: Geometry for mean free path calculation. The test particle has cross sectional area σ. The number density of target particles is n, and the mean free path ℓ is defined to be that length for which the volume swept out by the test particle contains one target, on average.

where we plugged in numbers for this room. Since the size of the room is $L \sim 10$ m $= 10^3$ cm, the mean free path is tiny compared to the room, and the fluid approximation is excellent.

1.4.3 photons in the Sun

Later we'll derive that free electrons scatter radiation, and the cross section for this process ("Thompson scattering") has cross section $\sigma \simeq 10^{-24}$ cm^2. An average electron density in the sun is about $n \simeq (1 \text{ g cm}^{-3})/\mathrm{m_p} \simeq 10^{24}$ cm^{-3}. The mean free path for photons is then

$$\ell \simeq 1 \text{ cm}. \tag{1.23}$$

The radius of the Sun is $R_\odot = 7 \times 10^{10}$ cm, so the fluid approximation is again excellent. So the photons can act as a fluid.

1.4.4 charged particles in the Sun

We can estimate the "Coulomb" cross section for the collision of electrons and ions by computing the impact parameter for which the potential and

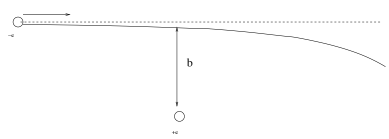

Figure 1.7: Deflection of an electron from straight line motion due to a neighboring ion. Here b is the impact parameter and the characteristic thermal speed of the electron is $v \sim (k_b T/m_e)^{1/2}$.

kinetic energies are comparable:

$$\frac{e^2}{b} \simeq k_b T. \tag{1.24}$$

This gives cross section

$$\sigma \sim b^2 \sim \frac{e^4}{(k_b T)^2} \simeq 10^{-20}\ \mathrm{cm}^2, \tag{1.25}$$

where we plugged in a central temperature $T \simeq 10^7$ K. Again using charged particle density $n \simeq 10^{24}$ cm^{-3}, the mean free path for charged particles is

$$\ell = \frac{1}{n\sigma} \simeq 10^{-4}\mathrm{cm}. \tag{1.26}$$

This is even smaller than the mean free path for photons. Again, the fluid approximation for particles is excellent.

The short mean free path for charged particles is one reason why heat transport in the Sun is dominated by photons, as they move further between collisions. The second reason is that photons travel much faster, and move further between collisions. In some degenerate stars, the electron mean free path becomes quite long and heat is transported by particles in the core (this is called conduction, rather than radiative diffusion). Also, above the photosphere in stars, the radiation doesn't interact with matter and cannot transport heat from one place to another, and conduction is again important.

Having discussed when the fluid approximation is valid, what are the equations of motion for a fluid? First we'll discuss conservation of mass.

1.5 mass conservation

On the timescale of one class, graduate students are neither created nor destroyed [3]. The only way they get in or out of the room is through the door. Similarly, mass is neither created nor destroyed [4], and the only way to change the mass in a box is to have it enter or leave through the boundaries.

Consider a box of volume V with mass density ρ. The mass in the box is then $M = \rho V$. The rate of change of mass in the box is then

$$\frac{dM}{dt} = -\sum_{\text{faces}} \boldsymbol{v} \cdot \boldsymbol{n} \rho A \tag{1.27}$$

where $\boldsymbol{v}$ is the fluid velocity, $\boldsymbol{n}$ is the *outward* unit vector for a given face, and A is the surface area of the face. The left hand side can be written

$$\frac{d}{dt} \int d^3x \rho = \int d^3x \frac{\partial \rho}{\partial t}, \tag{1.28}$$

for fixed boundaries. Similarly, turning the area integral into a volume integral, the right hand side becomes

$$-\int d^3x \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) . \tag{1.29}$$

Since mass conservation must hold even for infinitesimal, arbitrarily shaped volumes, it must hold locally at each point. The end result is the "Eulerian" form of mass conservation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0. \tag{1.30}$$

This is also called "the continuity equation". The quantity $\rho \boldsymbol{v}$ is called the mass flux.

Eq.1.30 is a partial differential equation involving the four variables $\rho(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$. Clearly we need more equations.

So far we have discussed the "Eulerian" form of the continuity equation, in which the observer sits at a fixed position and watches the fluid flow past. Another form of the continuity equation is the "Lagrangian" version,

[3] The timescale of the qualifier exam is perhaps more relevant. Ha ha. Just kidding.

[4] In relativistic physics, you can trade rest mass for energy of motion, but the total mass-energy is conserved.

in which the observer is riding along with the fluid element. Let the fluid element follow a path $\boldsymbol{x}(t)$ in time, with velocity $\boldsymbol{v}(t) = d\boldsymbol{x}(t)/dt$. Then the density of this fluid element is $\rho(\boldsymbol{x}(t), t)$. The time rate of change *following the fluid element* is then found using the chain rule:

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial\boldsymbol{x}} \cdot \frac{d\boldsymbol{x}}{dt} = \frac{\partial\rho}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}\rho. \tag{1.31}$$

Here the "comoving derivative" D/Dt is defined as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla}. \tag{1.32}$$

In the frame riding with the fluid, eq.1.30 can be rewritten in the Lagrangian form

$$\frac{D\rho}{Dt} = -\rho\boldsymbol{\nabla} \cdot \boldsymbol{v}. \tag{1.33}$$

In other words, the density increases as the fluid compresses ($\boldsymbol{\nabla} \cdot \boldsymbol{v} < 0$) and vice versa.

Why bother with this second form? *When you are applying thermodynamics to fluid elements, you need to use the Lagrangian form.*

1.6 momentum conservation

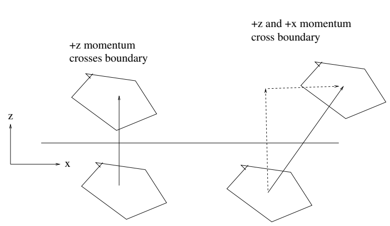

Figure 1.8: Change in momentum of a box as fluid moves through a boundary. Both momentum perpendicular (left) and parallel (right) to the boundary can leave the box.

The derivation is similar to that of mass conservation, except now we have to deal with a vector quantity. See the diagram in figure 1.8. We'll first consider a box containing fluid with no forces like pressure or gravity acting on it. The momentum per unit volume in an infinitesimal box is $\rho \boldsymbol{v}$, so the total momentum in the box is $\int d^3 x \rho \boldsymbol{v} = M\boldsymbol{v}$. Momentum in *any* direction can be carried out of each face. The equation of momentum conservation is then

$$\frac{d(M\boldsymbol{v})}{dt} = -\sum_{\text{faces}} A\boldsymbol{v}\cdot\boldsymbol{n}\rho\boldsymbol{v}, \tag{1.34}$$

where again $\boldsymbol{v}\cdot\boldsymbol{n}\rho\boldsymbol{v}$ is the flux (momentum per unit area per unit time) through the face, A is the area of the face, and $\boldsymbol{n}$ is the outward normal vector. Turning the area integral into a volume integral, the momentum equation *in the absence of any forces on the fluid* for the *i′th* component is

$$\frac{\partial}{\partial t}(\rho v_i) + \sum_j \frac{\partial}{\partial x_j}(\rho v_i v_j) = 0. \tag{1.35}$$

Here $i = x, y, z$ in cartesian coordinates. The index i represents the component of momentum we are interested in. The index j represents the face the fluid is entering or leaving. It's conventional to move this term to the left hand side of the equation, leaving the right hand side for forces due to pressure, gravity, etc. Let's take a moment to introduce some notation to make dealing with this equation simpler and more intuitive.

1.6.1 Vectors and tensors (math) and stress (physics)

A *vector* is defined as a quantity with certain transformation properties under rotation, translation, etc. In a certain coordinate system with basis vectors $\boldsymbol{e}_i$ (think cartesian $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$), we can write down the components of the vector as $\boldsymbol{V} = V_i\boldsymbol{e}_i$. In a different basis with different basis vectors $\boldsymbol{e}_{i'}$ there will be different components $V_{i'}$.

Vectors have one index. *Tensors* are quantities with one or more indices associated with them, e.g. $\boldsymbol{T} = T_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j$. Here T_{ij} are the components of the tensor, and we've introduced the *tensor product* notation $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ for the basis vectors. This is a new sort of quantity with two "slots", one for each basis vector. Vectors are then a subset of tensors, with only one slot. Tensors can have arbitrarily many indices. Often we'll get tired of writing the tensor product symbol, and just omit it, i.e. $\boldsymbol{e}_i \otimes \boldsymbol{e}_j \to \boldsymbol{e}_i\boldsymbol{e}_j$

What can you do with a tensor? First, you can dot it with a vector *on either side*, to produce a vector:

$$\boldsymbol{e}_i \cdot (\boldsymbol{e}_j \otimes \boldsymbol{e}_k) \equiv (\boldsymbol{e}_i \cdot \boldsymbol{e}_j)\boldsymbol{e}_k = \delta_{ij}\boldsymbol{e}_k \tag{1.36}$$

$$(\boldsymbol{e}_j \otimes \boldsymbol{e}_k) \cdot \boldsymbol{e}_i = \boldsymbol{e}_j(\boldsymbol{e}_i \cdot \boldsymbol{e}_i) = \boldsymbol{e}_j\delta_{ki}, \tag{1.37}$$

where δ_{ij} is the Kronecker delta (1 or 0).

In eq.1.35 we've seen an example of a tensor, $\rho v_i v_j$, which is called the *Reynolds stress tensor*:

$$\rho \boldsymbol{v} \otimes \boldsymbol{v}. \tag{1.38}$$

Eq.1.35 can then be rewritten as a vector equation

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) = 0. \tag{1.39}$$

The meaning of eq.1.39 is spelled out in eq.1.35, namely, it shows how to take the divergence of a tensor, which produces a vector (the force per unit volume). To compute the momentum in direction $\boldsymbol{e}_i$ moving through face with normal vector $\boldsymbol{e}_j$, we take the dot product

$$\boldsymbol{e}_i \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v}) \cdot \boldsymbol{e}_j = \rho v_i v_j. \tag{1.40}$$

Momentum per unit area per unit time is called "stress". So pressure P has units of stress. So does the Reynolds stress ρv^2. As seen from eq.1.39, it is the divergence of the stress tensor that enters the momentum equation as force per unit volume.

Q: For fluid flows along one direction, sometimes ρv^2 is called the "ram pressure". You can think of it as a force per unit area that tries to blow you over in a wind, or that slows down a meteorite as it enters the atmosphere, or that the solar wind is exerting on the Earth's magnetosphere. At sea level, what velocity is required for the ram pressure to equal atmospheric pressure?

A: Equating $\rho v^2 = P \simeq 1$ bar, we find

$$v = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{10^6 \text{ dyne cm}^{-2}}{10^{-3} \text{ g cm}^{-3}}\right)^{1/2} \simeq 300 \text{ m s}^{-1}. \tag{1.41}$$

This is roughly the sound speed in air! It is very difficult for confined, messy flows to exceed the sound speed, due to strong interactions with the neighboring fluid elements trying to slow them down.

1.6.2 including pressure and gravity forces

We've already written down these forces in Section 1.1. Now including them in the momentum equation we find

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}\boldsymbol{v}) = -\boldsymbol{\nabla} P + \rho \boldsymbol{g}, \tag{1.42}$$

where $\boldsymbol{g} = -\boldsymbol{\nabla}\Phi$ is the gravitational acceleration. Clearly when the fluid is not moving, we recover the hydrostatic balance equation.

Sometimes, like when solving the fluid equations on a computer, it's convenient to write pressure and gravity as the divergence of stresses as well. For pressure this can be done using the Kronecker delta:

$$-\frac{\partial P}{\partial x_i} = -\sum_j \frac{\partial}{\partial x_j}(\delta_{ij} P) \tag{1.43}$$

or

$$-\boldsymbol{\nabla} P = -\boldsymbol{\nabla}\cdot(P\boldsymbol{\delta}). \tag{1.44}$$

For gravity, we can do a similar trick using the Poisson equation,

$$\nabla^2\Phi = 4\pi G\rho. \tag{1.45}$$

That is

$$\begin{aligned}\rho g_i &= -\rho\frac{\partial \Phi}{\partial x_i} = -\frac{1}{4\pi G}\left(\sum_j \frac{\partial^2\Phi}{\partial x_j^2}\right)\frac{\partial\Phi}{\partial x_i} = -\frac{1}{4\pi G}\sum_j\left(\frac{\partial}{\partial x_j}\left(\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\right) - \frac{\partial\Phi}{\partial x_j}\frac{\partial^2\Phi}{\partial x_i x_j}\right)\\ &= -\frac{1}{4\pi G}\sum_j\left(\frac{\partial}{\partial x_j}\left(\frac{\partial\Phi}{\partial x_i}\frac{\partial\Phi}{\partial x_j}\right) - \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{\partial\Phi}{\partial x_j}\right)^2\right).\end{aligned} \tag{1.46}$$

In tensor form this can be written

$$\rho\boldsymbol{g} = -\frac{1}{4\pi G}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\Phi\otimes\boldsymbol{\nabla}\Phi - \frac{1}{2}\boldsymbol{\delta}|\boldsymbol{\nabla}\Phi|^2\right) = -\frac{1}{4\pi G}\boldsymbol{\nabla}\cdot\left(\boldsymbol{g}\otimes\boldsymbol{g} - \frac{1}{2}\boldsymbol{\delta}|\boldsymbol{g}|^2\right) \tag{1.47}$$

The final form for the momentum equation, in "conservative form", can then be written

$$\frac{\partial}{\partial t}(\rho\boldsymbol{v}) + \boldsymbol{\nabla}\cdot\left(\rho\boldsymbol{v}\otimes\boldsymbol{v} + P\boldsymbol{\delta} + \frac{\boldsymbol{g}\otimes\boldsymbol{g} - \frac{1}{2}\boldsymbol{\delta}|\boldsymbol{g}|^2}{4\pi G}\right) = 0. \tag{1.48}$$

This form of the momentum is useful for solving problems numerically on the computer as you can't lose momentum! The only way momentum gets into or out of a box is by moving through the boundaries.

1.6.3 the Euler form of the momentum equation

While eq.1.48 is useful for numerical work, it turns out to be inconvenient for analytics. Rather than the equation of momentum conservation, it's useful to solve the "Euler" form of the equation. The left hand side of eq.1.42 can be simplified using the continuity equation (eq.1.30) as follows:

$$\begin{aligned}\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) &= v_i\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j)\right] + \rho\left[\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right] \\ &= \rho \frac{D v_i}{D t}, \end{aligned} \tag{1.49}$$

where the first term is zero from the continuity equation, and we have noticed the second term is just the comoving derivative of the velocity. The "Euler equation", including pressure and gravity forces, can then be written in the intuitive form

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v}\right) = -\boldsymbol{\nabla} P + \rho \boldsymbol{g}. \tag{1.50}$$

A vast number of phenomena can be described with such an innocuous-looking equation.

Combining the Euler and continuity equation, we have 4 equations and 5 variables in the equations (ρ, $3 v_i$'s and P). We still need an equation of state to relate P back to the other fluid variables. To do this right involves, at a minimum, learning some thermodynamics (adiabatic changes in a gas). But in order to get started, let's do the simplest case – assume the gas is isothermal.

1.7 an isothermal stellar wind

So far we've considered static gas and ignored the ma part of $f = ma$. Now let's consider a steady, isothermal wind coming off a star. Our goal is to compute the run of density and velocity in the wind, as well as the mass loss rate.

For an isothermal gas,

$$P = P(\rho) = a^2 \rho \tag{1.51}$$

where the "isothermal sound speed"

$$a = \left(\frac{k_b T}{\mu m_p}\right)^{1/2} \tag{1.52}$$

is treated as given constant. This assumption let's us find P given ρ (the "equation of state"). For this to be valid means there are heating and cooling processes which happen to balance near this temperature, in spite of the fluid flow, which may try to raise or lower the temperature adiabatically. The isothermal approximation can be good, but also this lets us get maximum intuition with minimum effort.

We assume spherical symmetry so that all fluid quantities depend only on spherical radius r. We also assume the wind properties are independent of time, so that $\partial/\partial t \to 0$. The gravity felt by the gas is due to the mass M of the star, i.e. $g(r) = GM/r^2$ for $r > r_b$, where r_b is the radius of the base of the wind. The velocity is radially outward, with

$$\boldsymbol{v} = v(r)\boldsymbol{e}_r, \tag{1.53}$$

where the radial basis vector is

$$\boldsymbol{e}_r = \boldsymbol{e}_x \sin\theta\cos\phi + \boldsymbol{e}_y \sin\theta\sin\phi + \boldsymbol{e}_z \cos\theta. \tag{1.54}$$

The convective derivative becomes

$$\frac{D\boldsymbol{v}}{Dt} = \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = v(r)\frac{\partial}{\partial r}\left(v(r)\boldsymbol{e}_r\right) = v\frac{dv}{dr}\boldsymbol{e}_r. \tag{1.55}$$

Evaluating the divergence in spherical coordinates, the continuity equation becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho v\right) = 0. \tag{1.56}$$

This implies the quantity $r^2\rho v$ is constant in r. As this is just the mass loss per unit solid angle, integrating over 4π steradians we find the total mass loss rate

$$\dot{M} = \text{constant in t and r } = 4\pi r^2\rho(r)v(r). \tag{1.57}$$

This equation relates v and ρ.

Next, the radial component of the momentum equation gives

$$\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2} = -a^2\frac{d\rho}{dr} - \frac{GM\rho}{r^2}. \tag{1.58}$$

Dividing by ρ gives

$$v\frac{dv}{dr} = -a^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2}. \tag{1.59}$$

Our goal is to solve the differential eq.s 1.57 and 1.59, subject to appropriate boundary conditions.

First, we can eliminate ρ in favor of v in eq.1.59 using eq.1.57:

$$\frac{d\ln\rho}{dr} = \frac{d\ln(\dot{M}/(4\pi r^2 v))}{dr} = -\frac{2}{r} - \frac{d\ln v}{dr}. \tag{1.60}$$

Plugging into eq.1.59 and rearranging gives

$$(v^2 - a^2)\frac{d\ln v}{dr} = \frac{2a^2}{r} - \frac{GM}{r^2} \tag{1.61}$$

or

$$\frac{d\ln v}{dr} = \frac{\frac{2a^2}{r} - \frac{GM}{r^2}}{v^2 - a^2}. \tag{1.62}$$

Here's the crucial point about eq.1.62. We don't expect the gradients in velocity, accelerations in other words, to become infinite, as this is unphysical. But at first sight it seems this is exactly what will happen on the right hand side when $v \to a$ in the denominator. On further thought, we can avoid infinite acceleration by making the numerator of the right hand side to zero at this same point, so that the velocity gradient stays finite. This is called the regularity condition:

$$\frac{2a^2}{r_s} = \frac{GM}{r_s^2} \qquad \text{at the radius } \mathrm{r} = \mathrm{r_s} \text{ where } v = a. \tag{1.63}$$

This point in the flow is called the "sonic point". In the isothermal approximation, we know exactly where the sonic point is:

$$r_s = \frac{GM}{2a^2}. \tag{1.64}$$

The ratio of the sonic point radius to the base radius, r_b, is

$$\frac{r_s}{r_b} = \frac{GM}{2r_b a^2}. \tag{1.65}$$

For a non-degenerate star, we found that $k_b T_{\rm central}/\mu m_p \sim GM/r_b$, and so this ratio is approximately $r_s/r_b \sim T_{\rm central}/T_{\rm wind}$ for a nondegenerate star. Since the center of the Sun is at $T_{\rm central} \sim 10^7$ K, and the solar wind is at $T_{\rm wind} \sim 10^6$ K, the sonic point is at $r_s/r_b \sim 10$.

Next, let's integrate equation 1.61 to find an algebraic equation to solve for $v(r)$. Each integral is simple, and the final result is

$$\frac{1}{2}v^2 - a^2\ln(v) = 2a^2\ln(r) + \frac{GM}{r} + \text{constant}. \tag{1.66}$$

We can set the constant by requiring $v = a$ at $r = r_s$, with the result

$$\frac{1}{2}v^2 + a^2\ln\left(\frac{r_s^2 a}{r^2 v}\right) - \frac{GM}{r} = \frac{1}{2}a^2 - \frac{GM}{r_s} = \frac{1}{2}a^2 - 2a^2 = -\frac{3}{2}a^2 \tag{1.67}$$

or, simplifying,

$$\frac{1}{2}v^2 + a^2\ln\left(\frac{r_s^2 a}{r^2 v}\right) - \frac{GM}{r} = -\frac{3}{2}a^2 \tag{1.68}$$

This looks like an energy conservation statement. We'll see that the weird logarithmic term arises from a thermodynamic energy called the "enthalpy." Eq.1.68 is a transcendental equation to solve for $v(r)$. One could use, e.g. Newton's method to solve it numerically. Let's look at some analytic limits.

Near the star, $r \ll r_s$, the velocity becomes very small due to the rapid increase of density ($v \propto 1/r^2\rho$). The $v^2/2$ term can be ignored in this limit. Solving the remaining equation for v gives

$$\frac{v(r)}{a} \simeq \frac{r_s^2}{r^2}e^{3/2 - GM/ra^2} = \left(\frac{r_s}{r}\right)^2 e^{3/2}e^{-2r_s/r}. \tag{1.69}$$

This equation says the velocity goes down exponentially fast moving into the star. The exponential factor just comes from hydrostatic balance in a $1/r^2$ gravity field. Using this continuity equation, we can relate the density ρ_b at the base radius r_b to the density ρ_s at the sonic point:

$$\rho_s = \rho_b \frac{r_b^2 v_b}{r_s^2 a} = \rho_b e^{3/2 - 2r_s/r_b}. \tag{1.70}$$

We can now compute the mass loss rate in terms of ρ_b at the base of the wind:

$$\dot{M} = 4\pi r_s^2 \rho_s a = 4\pi r_s^2 a \rho_b e^{3/2 - 2r_s/r_b}. \tag{1.71}$$

Given base properties, we can compute $\dot{M}$.

Q: The base of the wind, where the flow velocities become large, is in the solar corona. Appropriate parameters for the gas are $T = 1 \times 10^6$ K, $\mu \simeq 0.63$ (fully ionized), $n_{\rm protons} \sim 10^{10}$ cm^{-3} $\rightarrow \rho_{\rm b} \simeq 10^{-14}$ g cm^{-3}, $r_b \simeq R_\odot$. Compute the gas sound speed, a, sonic point radius, r_s, and mass loss rate, $\dot{M}$.

A:

$$a = \left(\frac{k_b T}{\mu m_p}\right)^{1/2} = 110 \text{ km s}^{-1}, \tag{1.72}$$

$$r_s = GM/(2a^2) = 3.6 \times 10^{11} \text{ cm} = 5.2 \text{ R}_\odot \tag{1.73}$$

$$\rho_s = \rho_b e^{3/2 - 2r_s/r_b} = 10^{-14} \text{ g cm}^{-3} \times \text{e}^{3/2-7} \simeq 1.4 \times 10^{-18} \text{ g cm}^{-3} \tag{1.74}$$

$$\dot{M} = 4\pi r_s^2 \rho_s a = 3 \times 10^{13} \text{ g s}^{-1} = 4 \times 10^{-13} \text{ M}_\odot \text{ yr}^{-1}. \tag{1.75}$$

This is too high by about a factor of 10. Perhaps our base density is too high....

We found that the velocity is very small near the star, as it has to be for the star to live a long time, otherwise the mass flux is huge. We also found the velocity goes from below to above the sound speed at the sonic point. What does the velocity do outside the sonic point?

For $r \gg r_s$, we can drop the GM/r term in eq.1.68. Assuming $v \sim a$ inside the logarithm and ignoring constant terms we find the rough approximation

$$v \simeq 2a\sqrt{\ln(r/r_s)}. \tag{1.76}$$

So the velocity continues to increase outside the sonic point, albeit very slowly. This is an artifact of the isothermal approximation, in which energy is continually added to the gas to keep the temperature constant. In reality, as the gas expands it will tend to cool, and (for adiabatic gas flow) the velocity will asymptote to a constant.

The isothermal wind has many of the qualitative concepts from more general cases, though: combining the continuity and momentum equations; sonic point; energy integral, etc.

1.8 mini-review of thermodynamics as needed for fluid dynamics

As we saw in section 1.6.3, the pressure must be related to the other variables to have a self-contained system of equations. One assumption to make analytic progress was to set T/μ to a constant (section 1.7). A first step toward doing the problem right is to consider *adiabatic flow* in which there is no microscopic dissipation or heat exchange between fluid elements. The extra equation we then add in just says that the entropy of each fluid element stays constant (if no discontinuities). Equivalently, we can rewrite this as an energy conservation equation for the fluid.

Here we will consider the simplest of equations of state, an ideal gas with n degrees of freedom. Some examples are

- n=3: ionized gas with only translational motion
- n=5: diatomic molecule with 3 translation, 2 rotation (why not 3?)
- n=6: molecule, 3 translation, 2 rotation, 1 vibration
- n=6: solid, 3 translational, 3 vibrational

The pressure is given by our standard formula

$$P = \frac{\rho k_b T}{\mu m_p}. \tag{1.77}$$

Following the same derivation as for pressure, with each degree of freedom having energy $k_bT/2$, and each species have number density n_i, the internal energy per unit mass is

$$\varepsilon = n \times \frac{k_b T}{2} \times \frac{\sum_i n_i}{\rho} = \frac{n}{2}\frac{k_b T}{\mu m_p}. \tag{1.78}$$

In thermodynamics, the equation of energy conservation (the 1st law) can be written as

$$dQ = T ds = d\varepsilon + P d(1/\rho) = \text{heat added to the system}. \tag{1.79}$$

Here we have defined

- s = entropy per unit mass
- $1/\rho$ = volume per unit mass
- ε = internal energy per unit mass.

Recall that physicists use the convention that $-PdV$ is the work done *by* the fluid.

We now derive an explicit expression for the entropy. Plugging eq.1.77 and 1.78 into eq.1.79, we find

$$T ds = \frac{n}{2}\frac{k_b}{\mu m_p} dT - \frac{P}{\rho^2} d\rho \tag{1.80}$$

or, dividing by T and collecting terms,

$$ds = \frac{k_b}{\mu m_p}\left(\frac{n}{2}\frac{dT}{T} - \frac{d\rho}{\rho}\right). \tag{1.81}$$

This can be immediately integrated to find

$$s(\rho, T) = \frac{k_b}{\mu m_p} \ln\left(\frac{T^{n/2}}{\rho}\right) + \text{constant}. \tag{1.82}$$

The constant is not of interest here. We change variables from T and ρ to P and ρ by plugging in $T \propto P/\rho$, finding

$$s(\rho, P) = \frac{k_b}{\mu m_p} \ln\left(\frac{P^{n/2}}{\rho^{n/2+1}}\right) + \text{constant}. \tag{1.83}$$

Similarly, we can eliminate ρ in favor of P and T to find

$$s(T, P) = \frac{k_b}{\mu m_p} \ln\left(\frac{T^{n/2+1}}{P}\right) + \text{constant}. \tag{1.84}$$

Next we use these expressions to understand a very important concept – the “adiabat”.

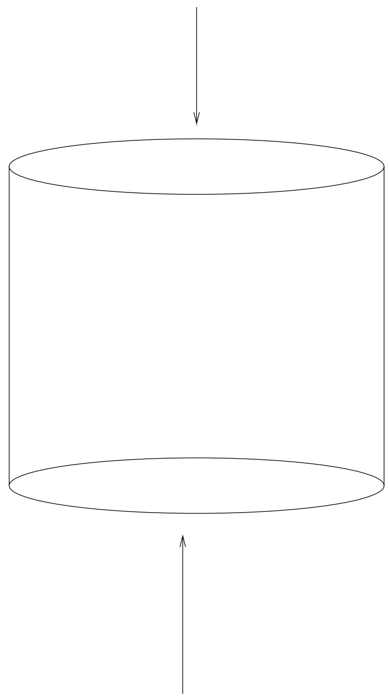

Figure 1.9: Squeeze a cylinder of gas, not letting heat get in or out of walls.

1.8.1 the adiabat

Consider a cylinder of gas with insulated walls, so that no heat is exchanged with the surroundings. Further assume that no dissipative processes occur (e.g. viscous dissipation), so that no heat is generated internally. The end result is then that when we compress the cylinder, the gas motion is adiabatic:

$$dQ = Tds = 0. \tag{1.85}$$

From eq.1.82, 1.83 and 1.84 this implies the "adiabatic relations" between the fluid variables:

$$T \propto \rho^{2/n} \tag{1.86}$$
$$P \propto \rho^{1+2/n} \tag{1.87}$$
$$T \propto P^{2/(2+n)}. \tag{1.88}$$

So as you compress the cylinder, how hot it gets depends on how many degrees of freedom there are to put the heat into.

For a more general equation of state, rather that these functions of n, people usually define general thermodynamic functions such as "the first adiabatic index"

$$\Gamma_1 \equiv \frac{\partial \ln P}{\partial \ln \rho}\rfloor_s \to 1 + \frac{2}{n} \tag{1.89}$$

and the "adiabatic temperature gradient."

$$\nabla_{\rm ad} \equiv \frac{\partial \ln T}{\partial \ln P}\rfloor_s \to \frac{2}{n+2}. \tag{1.90}$$

As we'll see, Γ_1 determines the adiabatic speed of sound, and $\nabla_{\rm ad}$ sets the temperature profile of convective regions of stars, or the Earth's atmosphere near ground.

Perhaps the most often used values are for an ionized gas with $n = 3$: $\Gamma_1 = 5/3$ and $\nabla_{\rm ad} = 2/5$.

In the absence of heating (nuclear burning, irradiation) and transport effects (viscosity, heat diffusion, etc), and shocks, fluid motion takes place with each fluid element conserving its entropy, i.e. the fluid motion is adiabatic. Hence this is the most important case to understand. If a particular fluid element conserves it's entropy, then the *Lagrangian* change in entropy, comoving with the fluid, is zero:

$$\frac{Ds}{dt} = \frac{\partial s}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} s = 0. \tag{1.91}$$

Often, instead of eq.1.91, an energy equation for the fluid is used. The equations are equivalent, as we now see.

1.9 the energy equation

The derivation takes a couple pages, so first let's state the answer. The energy conservation equation for an adiabatic fluid is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \boldsymbol{\nabla} \cdot \left[\rho \boldsymbol{v}\left(\frac{1}{2}v^2 + w\right)\right] = \rho \boldsymbol{v} \cdot \boldsymbol{g}. \tag{1.92}$$

This has the standard form of a conservation law, with energy density (bulk flow plus internal energy)

$$e = \frac{1}{2}\rho v^2 + \rho\varepsilon \tag{1.93}$$

and energy flux

$$\boldsymbol{v}\left(\frac{1}{2}\rho v^2 + \rho w\right). \tag{1.94}$$

All quantities are defined except the "enthalpy per unit mass", w, which has its standard thermodynamic definition using the "Legendre transformation". That is, starting from the first law,

$$d\varepsilon \quad = \quad Tds - Pd(1/\rho), \tag{1.95}$$

add $d(P/\rho)$ to each side,

$$d(\varepsilon + P/\rho) \quad \equiv \quad dw = Tds + \frac{1}{\rho}dP \tag{1.96}$$

where we have defined enthalpy as

$$w \quad = \quad \varepsilon + P/\rho. \tag{1.97}$$

For our simple "n degree of freedom" gas, the enthalpy is

$$w \quad = \quad \varepsilon + \frac{P}{\rho} = \left(\frac{n}{2} + 1\right)\frac{k_b T}{\mu m_p}. \tag{1.98}$$

What does enthalpy mean? The ε term is the internal energy of the system, and the P/ρ term can be interpreted as the energy required to "make room" for the system given its external environment. Whew!

Let's get back to the derivation. Taking the derivative of the bulk kinetic energy (per unit volume), and using the continuity and momentum equations, and eq.1.96,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\boldsymbol{v}\cdot\frac{\partial \boldsymbol{v}}{\partial t} \\
&= -\frac{1}{2}v^2\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) + \boldsymbol{v}\cdot(-\rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} - \boldsymbol{\nabla}p + \rho\boldsymbol{g}) \\
&= -\frac{1}{2}v^2\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) - \frac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{\nabla}v^2 - \boldsymbol{v}\cdot\boldsymbol{\nabla}p + \rho\boldsymbol{v}\cdot\boldsymbol{g} \\
&= -\boldsymbol{\nabla}\cdot\left(\boldsymbol{v}\frac{1}{2}\rho v^2\right) - \boldsymbol{v}\cdot(\rho\boldsymbol{\nabla}w - \rho T\boldsymbol{\nabla}s) + \rho\boldsymbol{v}\cdot\boldsymbol{g}.
\end{aligned} \tag{1.99}$$

Next, the derivative of the internal energy. Again using the continuity equation and the first law, we find

$$\begin{aligned}
\frac{\partial(\rho\varepsilon)}{\partial t} &= \rho\frac{\partial\varepsilon}{\partial t} + \frac{\partial\rho}{\partial t}\varepsilon \\
&= \rho\left(T\frac{\partial s}{\partial t} + \frac{P}{\rho^2}\frac{\partial\rho}{\partial t}\right) + \frac{\partial\rho}{\partial t}\varepsilon = w\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} \\
&= -w\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) + \rho T\frac{\partial s}{\partial t}. \qquad (1.100)
\end{aligned}$$

Adding eq.1.99 and 1.100, and collecting together terms, the final result including non-adiabatic effects is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{v}\left(\frac{1}{2}v^2 + w\right)\right] = \rho\boldsymbol{v}\cdot\boldsymbol{g} + \rho T\frac{Ds}{Dt}. \qquad (1.101)$$

Setting the entropy derivative to zero recovers eq.1.92. Alternatively, we have have a model for entropy generation by, say, viscosity, heat diffusion, particle diffusion, etc we can include that on the right hand side.

Now that we're have thermodynamics and the energy equation at our disposal, let's look at a one of the most important fluid phenomena, sound waves.

1.10 sound waves in a uniform fluid

First we will *linearize* the continuity and momentum equations, then linearize the adiabat, and combine the three equations into one p.d.e., the wave equation.

Consider a uniform fluid with density ρ_0, pressure P_0 and temperature T_0, all constant. This background state is not moving ($\boldsymbol{v}_0 = 0$), and is time-independent. We are ignoring gravity in this problem.

Now introduce perturbations to each quantity: $\delta\rho(\boldsymbol{x},t)$, $\delta P(\boldsymbol{x},t)$, and $\delta\boldsymbol{v}(\boldsymbol{x},t)$ so that the total quantities of the combined system are

$$\begin{aligned}
\rho(\boldsymbol{x},t) &= \rho_0 + \delta\rho(\boldsymbol{x},t) \qquad (1.102)\\
P(\boldsymbol{x},t) &= P_0 + \delta P(\boldsymbol{x},t) \qquad (1.103)\\
\boldsymbol{v}(\boldsymbol{x},t) &= \delta\boldsymbol{v}(\boldsymbol{x},t). \qquad (1.104)
\end{aligned}$$

We further assume that the perturbations are small, so that quadratic terms such as $\delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\delta\boldsymbol{v}$ can be ignored. Plugging eq.1.102, 1.103 and 1.104 into eq.1.30 and 1.50, the derivatives of the background quantities drop out, as do the nonlinear terms (by assumption), leaving the following equations:

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \boldsymbol{\nabla}\cdot\delta\boldsymbol{v} = 0 \tag{1.105}$$

$$\rho_0 \frac{\partial \delta\boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\delta p. \tag{1.106}$$

Again, we need a relation between pressure and density. From the definition of the first adiabatic index in eq.1.89, for $s =$ constant this gives

$$P = P(\rho, s) \tag{1.107}$$

$$\frac{dP}{P} = \frac{\partial \ln P}{\partial \ln \rho}\Big\rfloor_s \frac{d\rho}{\rho}, \tag{1.108}$$

which should be applied to *fluid elements*, i.e. to Lagrangian changes in the fluid. Since the background fluid is homogeneous, Lagrangian changes (following a fluid element) are the same as Eulerian changes (at fixed position). So we find

$$\delta P = \frac{\Gamma_1 P_0}{\rho_0}\delta\rho \equiv c_s^2 \delta\rho, \tag{1.109}$$

where

$$c_s = \left(\frac{\Gamma_1 P}{\rho}\right)^{1/2}. \tag{1.110}$$

Plugging eq.1.109 into eq.1.105, we are left with two equations and two unknowns. Taking the time derivative of eq.1.105 and plugging in eq.1.106 we find

$$\frac{\partial^2 \delta P}{\partial t^2} = -c_s^2 \boldsymbol{\nabla}\cdot\left(\rho_0 \frac{\partial \delta\boldsymbol{v}}{\partial t}\right) = -c_s^2 \boldsymbol{\nabla}\cdot(-\boldsymbol{\nabla}\delta P) = c_s^2 \nabla^2 \delta P. \tag{1.111}$$

Eq.1.111 is the wave equation.

To solve eq.1.111, we use the fact that the background is uniform, and hence that the coefficient c_s^2 is constant, in order to expand the spatial dependence in a plane wave (i.e. a Fourier series in space and time):

$$\delta P(\boldsymbol{x}, t) = \delta P e^{i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t} \tag{1.112}$$

where δP is now a constant. Plugging this form of the space and time-dependence into the wave equation, we find

$$\partial/\partial t \rightarrow -i\omega \tag{1.113}$$

$$\boldsymbol{\nabla} \rightarrow i\boldsymbol{k}, \tag{1.114}$$

so that the wave equation becomes

$$-\omega^2 \delta P e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t} = -c_s^2 k^2 \delta P e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}. \tag{1.115}$$

Canceling the amplitude and space-time dependence, we end up with the **dispersion relation** for sound waves:

$$\omega^2 = c_s^2 k^2. \tag{1.116}$$

Defining the wave frequency, in cycles per second, $\nu = \omega/2\pi$, and wavelength $\lambda = 2\pi/|\boldsymbol{k}|$, the dispersion relation takes on the familiar form

$$\lambda\nu = c_s, \tag{1.117}$$

just as for electromagnetic waves.

Q: Compute the sound speed in air for a diatomic gas with $T = 273$ K and $\mu = 29$. Compute the wavelength of a sound wave with frequency characteristic of human speech, $\nu = 100$ Hz.

A: Plugging in $\Gamma_1 = 7/5$ for a diatomic gas, and $T = 273$ K with $\mu = 29$, we find $c_s = 330$ m s^{-1}. For a sound wave with $\nu = 1000$ Hz, the wavelength is $\lambda = 3.3$m. This is much bigger than the region where sound is produced! How is this possible?

If we follow a pressure maximum as it moves through the air, this means we are setting $\boldsymbol{k}\cdot\boldsymbol{x} - \omega t =$ constant. Differentiating this expression with respect to time we find the velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$ is

$$\boldsymbol{k}\cdot\boldsymbol{v} = \omega, \tag{1.118}$$

and the speed of sound in air has magnitude $v = \omega/k = c_s$. We call this the "adiabatic speed of sound." Since Γ_1 is involved, the sound speed directly involves the number of microscopic degrees of freedom of the gas. Since

$$\Gamma_1 = \frac{n+2}{n}, \tag{1.119}$$

sound travels faster in gases that have less degrees of freedom. The limit of $n \to \infty$ recovers the isothermal sound speed $a = (P/\rho)^{1/2}$.

When the speed of sound was first derived (ref?), people didn't understand thermodynamics yet, and got the sound speed slightly wrong. For a diatomic gas with $n = 5$, $\Gamma_1 = 7/5$ and the error was about 20% using using the isothermal formula, when the adiabatic formula should have been used.

But why is the isothermal formula wrong? If would be correct, if heat could diffuse one wavelength in one wave period. For a diffusion coefficient for heat of $D = 0.1$ cm^2 s^{-1}, the distance diffused in a time ν^{-1} is $x = (D/\nu)^{1/2}$. Setting $x = \lambda = c_s/\nu$ and squaring, we can solve for the frequency at which the sound wave becomes isothermal due to heat diffusion:

$$\nu_{\rm crit} = \frac{c_s^2}{D}. \tag{1.120}$$

But we derive the diffusion coefficient to be of size $D \simeq c_s\ell$, where ℓ is the mean free path. So

$$\nu_{\rm crit} = \frac{c_s}{\ell} \simeq 3 \times 10^7 \text{ Hz}, \tag{1.121}$$

that is, a sound wave becomes isothermal when the wavelength becomes of order the mean free path.

Let's finish the discussion of sound waves with the "eigenvector" that goes along with the "eigenfrequency" found in the dispersion relation. The linearized momentum equation is

$$-i\omega\rho_0\delta\boldsymbol{v} = -i\boldsymbol{k}\delta P, \tag{1.122}$$

or, using the dispersion relation to simplify,

$$\delta\boldsymbol{v} = \frac{\boldsymbol{k}}{\omega}\frac{\delta P}{\rho_0} = \frac{\hat{\boldsymbol{k}}}{c_s}\frac{\delta P}{\rho_0}. \tag{1.123}$$

This equation says the fluid motion in the wave, given by $\boldsymbol{v}$, is in the same direction as the wavevector $\boldsymbol{k}$, and is proportional to the pressure perturbation. We call waves that propagate in the direction of the fluid motion "longitudinal." This situation can be contrasted to the "transverse" wave case, where the fluid motion is perpendicular to the direction the wave is moving.

Next, we turn out attention to discontinuous fluid motion in "shocks."

1.11 shocks

Discontinuous solutions are allowed by the fluid equations. In nature, they arise when supersonic (motion faster than the speed of sound) fluid smacks into something – the fluid is moving so fast that no sound waves can travel "upstream" to warn the fluid it's about to hit something. They can also arise naturally in sound waves in a uniform gas, as nonlinear fluid effects tend to make a sinusoidal wave evolve to have sharp features after a time (the "N-wave"). For the moment, we'll ignore the question of precisely how shocks arise and just discuss how the up- and down-stream fluids are matched through the discontinuity. We'll also ignore the structure *inside* shocks which gives rise to dissipation and increase of entropy.

Our goal is to determine downstream quantities (ρ, P, v, T, etc) from upstream quantities. Briefly, fluid moving supersonically is decelerated, and bulk kinetic energy goes into thermal energy.

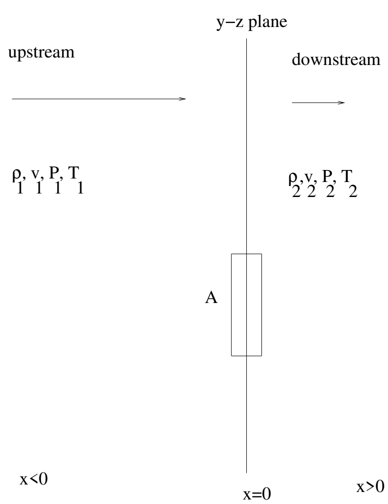

Figure 1.10: Geometry for a shock in the y-z plane. Gas comes from the upstream side, is decelerated in the shock, and moves slower on the downstream side.

We consider fluid motion in the x direction, with $\partial/\partial y = \partial/\partial z = 0$. Upstream quantities ($x < 0$) have subscript 1, and downstream have subscript 2. We will also work in a reference frame in which $v_y = v_z = 0$, and denote $v_x \equiv v$. See figure 1.10 for the geometry.

First consider the continuity equation, which given our assumptions becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) \quad = \quad 0. \tag{1.124}$$

We integrate this equation over a small "Gaussian pillbox" surrounding the discontinuity, of perpendicular area A, in the y-z plane, and length ϵ in the x-direction on each side. Integrating over this volume we find

$$A \int_{-\epsilon}^{\epsilon} dx \frac{\partial \rho}{\partial t} \quad = \quad -A \int_{-\epsilon}^{\epsilon} dx \frac{\partial}{\partial x}(\rho v) = -A\left(\rho_2 v_2 - \rho_1 v_1\right). \tag{1.125}$$

As long as $\partial\rho/\partial t$ is not a delta function, i.e. it is smooth in x, then as $\epsilon \to 0$ the time derivative term goes to zero, and we obtain the "shock jump condition" for mass:

$$\rho_1 v_1 \quad = \quad \rho_2 v_2. \tag{1.126}$$

Eq.1.126 says that the mass *flux* must be the same on either side. This makes sense, as if it wasn't true if means either mass would be piling up or being taken away from the infinitesimal region of the shock, which is unphysical. Note that eq.1.126 applies even if the flow is time-dependent.

Ignoring gravity (which wouldn't change the result), the momentum equation in the x-direction is

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial}{\partial x}\left(\rho v^2 + P\right) \quad = \quad 0. \tag{1.127}$$

Again integrating over the pillbox we get

$$\rho_1 v_1^2 + P_1 \quad = \quad \rho_2 v_2^2 + P_2. \tag{1.128}$$

Lastly, the energy equation in this case becomes

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\varepsilon\right) + \frac{\partial}{\partial x}\left[\rho v\left(\frac{1}{2}v^2 + w\right)\right] \quad = \quad 0, \tag{1.129}$$

which gives

$$\rho_1 v_1\left(\frac{1}{2}v_1^2 + w_1\right) \quad = \quad \rho_2 v_2\left(\frac{1}{2}v_2^2 + w_2\right), \tag{1.130}$$

where $w = \varepsilon + P/\rho$ is the enthalpy.

The mass, momentum and energy equations are 3 equations for the 4 variables ρ, v, P, ε. The last equations is the EOS.

We now need to decide on an equation of state for the fluid to relate P and ε to the other fluid variables. If there is little heat flow outside the shock, then the adiabatic equation of state

$$P = K\rho^\gamma \tag{1.131}$$
$$\varepsilon = \frac{P/\rho}{\gamma - 1} \tag{1.132}$$
$$w = \varepsilon + \frac{P}{\rho} = \left(\frac{\gamma}{\gamma-1}\right)\left(\frac{P}{\rho}\right) = \frac{c^2}{\gamma - 1}, \tag{1.133}$$

where K is a constant, γ is the first adiabatic index ($\equiv \Gamma_1$), and $c^2 = \gamma p/\rho$ is the adiabatic sound speed.

There is a fair amount of algebra to get the solution for downstream quantities in terms of upstream for the adiabatic case. Let's discuss the solution first. Define the *upstream mach number*

$$M_1 = \frac{v_1}{c_1}. \tag{1.134}$$

The the results are given in *Astrophysical Flows* pg. 40. We'll give the general result, then take the high mach number limit, and lastly set $\gamma = 5/3$, the ionized case:

$$\frac{v_1}{v_2} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2 + 2} \to \frac{\gamma+1}{\gamma-1} \to 4 \tag{1.135}$$
$$\frac{P_2}{P_1} = 1 + \left(\frac{2\gamma}{\gamma+1}\right)\left(M_1^2 - 1\right) \to \left(\frac{2\gamma}{\gamma+1}\right) M_1^2 \to \frac{5}{4}M_1^2 \tag{1.136}$$
$$\begin{aligned}\frac{T_2}{T_1} &= \frac{P_2\rho_1}{P_1\rho_2} = \frac{[\gamma + 1 + 2\gamma(M_1^2 - 1)]\,[(\gamma-1)M_1^2 + 2]}{(\gamma+1)^2 M_1^2} \\ &= \frac{2\gamma(\gamma-1)}{(\gamma+1)^2}M_1^2 \to \frac{5}{16}M_1^2.\end{aligned} \tag{1.137}$$

Note that while the density ratio goes to a constant at high mach number, the other quantities increase with M_1. The faster the upstream gas, the hotter the downstream gas. Also note that as $M_1 \to 1^+$, the upstream and downstream quantities are equal.

Next, consider the case where cooling is efficient and the gas stays nearly isothermal. This can happen for adiabatic shocks as well if there is subsequent cooling that makes the up and downstream temperatures equal. The isothermal limit is where there are an infinite number of degrees of freedom n in the gas, so that $\gamma = 1 + 2/n \to 1$. Taking $\gamma \to 1$ at fixed M_1 gives a density ratio

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma-1)M_1^2+2} \to M_1^2. \tag{1.138}$$

So in the isothermal case, the downstream density can also increase rapidly with mach number. This can lead to dense shells of downstream gas which produce lots of radiation.

Fast outflows like supernovae ($v \sim 10^3$ km s^{-1}) can heat gas up to very high temperatures such that the radiation is then observable.

Q: A supernova explosion causes fast moving gas with $v = 1000$ km s^{-1} to plow into the interstellar medium. Take $\mu = 0.63$ for fully ionized gas, and assume $M_1 \gg 1$. What is the downstream temperature? What wavelength band does this radiate in?

A: We don't need the upstream temperature, as it cancels:

$$\frac{T_2}{T_1} \simeq \frac{5}{16}\frac{v_1^2}{c_1^2} = \frac{5}{16}\frac{3}{5}\frac{\mu m_p v_1^2}{k_b T_1} \tag{1.139}$$

$$T_2 = \frac{3}{16}k_b^{-1}\mu m_p v_1^2 \simeq 10^7 \text{ K}. \tag{1.140}$$

Hence keV particle energies are produced, which radiate in X-rays. The central parts of supernovae contain this keV gas which can be seen in X-rays.

Now that we've done an example of violent fluid motion, let's go back to hydrostatic gas, and decide if it will just sit there, if it is unstable and will move away from its initial static state to give a "convection zone."

1.12 The buoyancy force

In water, wood floats while metal sinks. Hot air balloons go up. Basements are cold and attics hot. These are all manifestations of the buoyancy force:

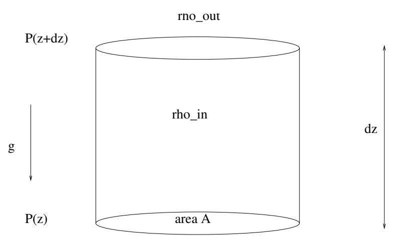

Figure 1.11: Cylinder of gas with density $\rho_{\rm in}$ inside and $\rho_{\rm out}$ outside. Gravity points in the $-z$ direction, and the background pressure decreases upward.

light fluid, relative to the surroundings, experiences an upward force, while heavy fluid is accelerated downward. Let's derive this.

Consider a cylinder of cross-sectional area A containing fluid of density $\rho_{\rm in}$ surrounded by a fluid of density $\rho_{\rm out}$. The fluid outside is in hydrostatic balance with $dP_{\rm out}/dz = -\rho_{\rm out} g$. The mass inside the cylinder is $M = A\ dz\ \rho_{\rm in}$. Considering the gravitational force, and the pressure force on the upper and lower surfaces, Newton's second law gives the acceleration (in the z direction):

$$\begin{aligned} ma &= -mg + P(z)A - P(z+dz)A \simeq -mg - \frac{dP_{\rm out}}{dz}Adz \\ &= -mg + \rho_{\rm out} gAdz = -mg + \frac{\rho_{\rm out}}{\rho_{\rm in}}mg. \end{aligned} \tag{1.141}$$

Dividing by m, the final result for the *buoyancy acceleration* is

$$\boldsymbol{a} = -g\left(\frac{\rho_{\rm in}-\rho_{\rm out}}{\rho_{\rm in}}\right)\boldsymbol{e}_z = \left(\frac{\rho_{\rm in}-\rho_{\rm out}}{\rho_{\rm in}}\right)\boldsymbol{g}. \tag{1.142}$$

So when the density inside is greater than outside, the force is down, and vice versa.

In astrophysics, what causes density variations that give buoyancy?

1) temperature variation (hot air rises)

2) mean molecular weight variation (helium baloons rise).

What is the role of the buoyancy force for objects in hydrostatic balance? It turns out this question has profound effects on stellar and planetary atmospheres.

1.13 atmospheric stability

Perturbation theory: give something a push and see if it oscillates about the initial equilibrium or moves away from it. Similarly, a perturbation to a stable atmospheres rings in the *acoustic* and *gravity* waves, while an unstable atmosphere will contain a *convection zone* in which the fluid is basically boiling.

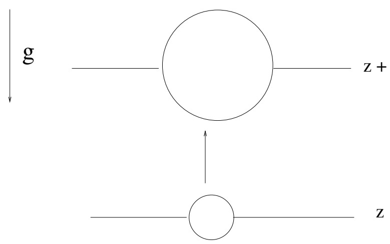

Figure 1.12: A hydrostatic atmosphere, with gravity pointing down and the z direction pointing up.

Consider a stratified atmospheres with $\rho(z)$, $P(z)$ and $T(z)$. Hydrostatic balance implies $dP/dz = -\rho g$. Now made a small perturbation in the fluid position $z \to z + \xi$, i.e. the fluid is perturbed upward by an amount ξ. Picture you are moving a little bubble upward. We do this carefully, under the two conditions:

> a) slowly, so that the bubble is in pressure equilibrium with the surroundings. In practice, this means move the bubble at much less than the sound speed.
>
> b) fast enough that there is no heat flow into or out of the bubble. This implies the fluid motion is adiabatic with $P \propto \rho^{\Gamma_1}$.

When the bubble gets to its new position, what is its density relative to the background? If heavier it will sink and be stable. If lighter, it will continue to rise unstably.

The density of the *background* at $z + \xi$ is

$$\rho(z + \xi) \simeq \rho(z) + \xi \frac{d\rho}{dz}. \tag{1.143}$$

The density of the *bubble* at $z+\xi$ is

$$\rho(z)\left(\frac{P(z+\xi)}{P(z)}\right)^{1/\Gamma_1} \simeq \rho(z)\left(1+\xi\frac{d\ln P}{dz}\right)^{1/\Gamma_1}$$
$$\simeq \rho(z)+\xi\rho(z)\frac{d\ln P}{dz}\Gamma_1^{-1}. \qquad (1.144)$$

The density of the bubble minus the density of the background, or the *over-density* $\delta\rho$, is

$$\delta\rho = \left(\rho(z)+\xi\rho(z)\frac{d\ln P}{dz}\Gamma_1^{-1}\right)-\left(\rho(z)+\xi\frac{d\rho}{dz}\right)$$
$$= -\xi\rho(z)\left(\frac{d\ln\rho}{dz}-\frac{1}{\Gamma_1}\frac{d\ln P}{dz}\right). \qquad (1.145)$$

The buoyancy acceleration is then

$$a = -g\frac{\delta\rho}{\rho} = +\xi\; g\left(\frac{d\ln\rho}{dz}-\frac{1}{\Gamma_1}\frac{d\ln P}{dz}\right) \equiv -N^2\xi \qquad (1.146)$$

where we have defined the "Brunt-Vaisalla" frequency

$$N^2 = -g\left(\frac{d\ln\rho}{dz}-\frac{1}{\Gamma_1}\frac{d\ln P}{dz}\right). \qquad (1.147)$$

Eq.1.146 looks like a harmonic oscillator: force is proportional to displacement.

The key point. The Brunt is a property of the atmosphere, as it depends on gravity, density and pressure profiles, and the adiabatic index. But from eq.1.146, if $N^2 > 0$, the atmosphere is stable, as an upwardly perturbed fluid element feels a downward restoring force, causing a stable oscillation. By contrast, if $N^2 < 0$, an upwardly perturbed fluid element feels a stronger upward force as it moves up, and accelerates away from its initial position.

$N^2 > 0$ **is required for an atmosphere to be stable. Low frequency perturbations result in oscillatory "gravity waves". We see these waves in the Earth's atmosphere, and in pulsating stars.**

$N^2 < 0$ **means an atmosphere is unstable. Perturbations grow, and the fluid is set in motion. This is called "convection", and**

regions with $N^2 < 0$ are called "convection zones." Picture the fluid is like a pot of boiling water.

Why do convection zones occur in nature?

- The Sun heats the ground, and air near the ground is hotter than the air above it. This causes the lower ~ 10 km of the Earth's atmosphere – the "troposhere" – to convect. This moving fluid is the origin of density perturbations which cause light rays to be bent, causing "seeing", which causes stars to twinkle (but not planets!), and vexes observers.

- Nuclear reactions in the cores of hot stars are quite temperature sensitive, and hence generate most of the luminosity in a tiny region. The temperature gradient is too steep, and you get *a convective core*.

- Near the surfaces of stars, hydrogen and helium go from being ionized (deep) to neutral (shallow). Neutral atoms cause the gas to be much more opaque than the ionized gas, and the temperature gradient becomes very steep, causing convection.

The criterion $N^2 < 0$ for convection is concise, but not intuitive. Let's reexpress this criterion in terms of entropy and temperature.

First, for uniform composition, any thermodynamic function can be expressed in terms of two others. Let $\rho = \rho(P, S)$. Then

$$\begin{aligned} -\frac{N^2}{g} &= \frac{\partial \ln \rho}{\partial \ln P}\rfloor_S \frac{d \ln P}{dz} + \frac{\partial \ln \rho}{\partial S}\rfloor_P \frac{dS}{dz} - \frac{1}{\Gamma_1}\frac{d\ln P}{dz} \\ &= \frac{\partial \ln \rho}{\partial S}\rfloor_P \frac{dS}{dz} \end{aligned} \tag{1.148}$$

where we used $\partial \ln P/\partial \ln \rho\rfloor_S = \Gamma_1$. So the Brunt is related the entropy gradient in the atmosphere. We can shed light on the thermodynamic derivative as follows. The specific heat (per unit mass) at constant pressure is defined as

$$C_P = T\frac{\partial S}{\partial T}\rfloor_P \tag{1.149}$$

and the adiabatic temperature gradient is

$$\nabla_{\rm ad} = \frac{\partial \ln T}{\partial \ln P}\rfloor_S. \tag{1.150}$$

Now expressing $S = S(T, P)$ and using these definitions gives

$$\frac{dS}{C_p} = \frac{dT}{T} - \nabla_{\rm ad}\frac{dP}{P}. \tag{1.151}$$

Hence, for an ideal gas with $P = \rho k_b T/\mu m_p$,

$$\frac{\partial \ln \rho}{\partial S}\rfloor_P = \frac{1}{C_p}\frac{\partial \ln \rho}{\partial \ln T}\rfloor_P = -\frac{1}{C_p}. \tag{1.152}$$

The end results for Brunt in terms of entropy is then

$$N^2 = +\frac{g}{C_p}\frac{dS}{dz}. \tag{1.153}$$

Since g and C_p are both positive, the end result is that

Atmospheres are stable if entropy increases outward, and vice versa.

Now the temperature profile. Eq.1.151 implies

$$\frac{N^2}{g} = \frac{1}{C_p}\frac{dS}{dz} = \frac{d\ln T}{dz} - \nabla_{\rm ad}\frac{d\ln P}{dz} = \frac{d\ln T}{d\ln P}\frac{d\ln P}{dz} - \nabla_{\rm ad}\frac{d\ln P}{dz} \tag{1.154}$$

Define the temperature gradient in the actual atmosphere, $\nabla = d\ln T/d\ln P$ (not to be confused with the thermodynamic quantity $\nabla_{\rm ad}$). Then we can rewrite Brunt as

$$N^2 = +\frac{g}{H}\left(\nabla_{\rm ad} - \nabla\right), \tag{1.155}$$

where $H = -1/(d\ln P/dz)$ is the pressure scale height.

Atmospheres are stable if the temperature decreases outward slower than the adiabatic temperature gradient, and vice versa.

If the temperature decreases outward too fast convection occurs. A neutrally stable atmosphere with $N^2 = 0$ has $\nabla = \nabla_{\rm ad}$. Let's derive this temperature gradient:

$$\frac{dT}{dz} = \frac{dT}{dP}\frac{dP}{dz} = \frac{T}{P}\nabla_{\rm ad}(-\rho g) = -\nabla_{\rm ad}\frac{\mu m_p g}{k_b}, \tag{1.156}$$

where in the last step we used the ideal gas law. For Earth, $\mu = 29$, $g = 10^3$ cm s^{-2} and the atmosphere is mainly composed of diatomic gases with $n = 5$, so that $\nabla_{\rm ad} = 2/(2+n) = 2/7$. Then

$$\frac{dT}{dz} = -\frac{2}{7}\frac{\mu m_p g}{k_b} \simeq -10 \text{ K km}^{-1}. \qquad (1.157)$$

This is called the "dry adiabatic lapse rate", since atmospheric scientists do not have the knack for clarity. It's more or less the temperature gradient near the surface of the Earth. If it's 273 K at ground an an airplane flies at 10 km, the temperature outside is only $273 - 100 = 173$ K. This is actually a bit too cold. Condensation of water wafted up from lower down releases a bit of latent heat which keeps the temperature gradient more shallow than 10 K km^{-1}; that profile is called the "wet adiabat", an oxymoron.

What is the actual temperature gradient in a convective atmosphere?

1.14 the mixing length theory of convection

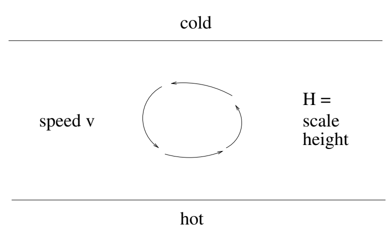

Figure 1.13: Schematic of a convective eddy carrying heat from the hot base to the cold top.

Up to factors of order unity, this is how people relate heat flux and temperature gradients in convective atmospheres of planets and stars. The idea is that unstable buoyancy ($N^2 < 0$) causes fluid motions to start which carry heat upward. Figure 1.13 shows an eddy carrying heat from the hot base to the cold top. Eddies have a maximum size of order a pressure scale height H, can the fluid moves at velocity v, which we will solve for, given the heat flux F and density ρ.

What is v? In a time t, the buoyancy acceleration $a = -N^2\xi \simeq |N^2|H$ causes the fluid to move a distance

$$\frac{1}{2}at^2 \sim |N^2|Ht^2 \sim H, \qquad (1.158)$$

so that $t \sim 1/\sqrt{|N^2|} \equiv 1/|N|$, were we define $|N| \equiv \sqrt{-N^2}$. So the absolute value of the Brunt gives the eddy turnover time. The velocity of the eddy is then

$$v \sim at \sim |N|^2 H/|N| \sim |N|H. \tag{1.159}$$

If the fluid parcel rose adiabatically, it would not be carrying any *excess* heat from the base to the top. We would like to know how much the temperature changed relative to the adiabatic change to derive the heat flux. First, define the "excess" heat carried up as

$$\begin{aligned} C_p \Delta T &= C_p \left(\frac{dT}{dz} - \frac{dT}{dz}\Big\rfloor_{\rm ad} \right) H = C_p T \left(\nabla - \nabla_{\rm ad} \right) \\ &= C_p T \times \frac{(-N^2)H}{g} = C_p T \times \frac{v^2}{gH} = v^2 \times \frac{C_p T}{gH} \sim v^2, \end{aligned} \tag{1.160}$$

since $C_p T \sim k_b T/\mu m_p \sim gH \sim P/\rho \sim k_b T/\mu m_p$ up to a constant of order unity. So the excess heat per unit mass carried up is comparable to the kinetic energy, per unit mass, of the eddies. The heat flux is then

$$F = \rho \times C_p \Delta T \times v \sim \rho v^3. \tag{1.161}$$

Usually this result is turned around to give $v = (F/\rho)^{1/3}$ once you know F and ρ.

Q: What are the eddy velocity and turnover time for the Earth's atmosphere?

A: The flux from the Sun is $F = \sigma T_\odot^4 (R_\odot / 1 \text{ AU})^2 \simeq 10^6$ erg cm^{-2} s^{-1}. The density is $\rho \simeq 10^{-3}$ g cm^{-3}. This gives a velocity

$$v \simeq \left(\frac{F}{\rho} \right)^{1/3} \simeq 10 \text{ m s}^{-1}. \tag{1.162}$$

The eddy turnover time is how long it takes to travel a scale height at this speed. We find

$$t_{\rm eddy} = \frac{H}{v} \sim 10^3 \text{ s} \sim \text{hrs}. \tag{1.163}$$

These eddies from convection are one source of atmospheric turbulence that causes "seeing" that makes stars twinkle.

1.15 waves in a stratified atmosphere

We already discussed linear waves in a constant density and pressure background, in which case there is only the acoustic wave. Let's now discuss waves in a stratified atmosphere in which the background quantities depend on altitude z. This will lead to to the existence of an additional type of wave called "internal gravity waves", which are restored by the buoyancy force ("buoyancy waves" would have been a better name). Gravity waves only exist in stable regions of an atmosphere where $N^2 > 0$. Regions where $N^2 < 0$ have unstable buoyancy and are convective.

Our goal is to find the dispersion relation $\omega = \omega(\boldsymbol{k})$ and to use that to understand the different types of waves possible (acoustic and gravity) and where the waves propagate or are evanescent.

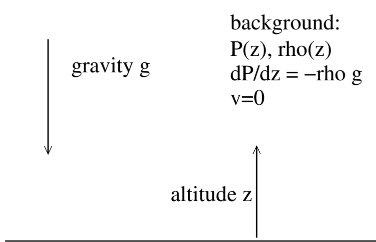

Figure 1.14: Plane parallel geometry.

The setup is shown in figure 1.14. Let there be a background profile of pressure $P(z)$ and density $\rho(z)$ with altitude z, in hydrostatic balance with $dP/z = -\rho g$. Let the background be at rest, with $\boldsymbol{v} = 0$ and let the gravitational acceleration be $\boldsymbol{g} = -g\boldsymbol{e}_z$, with constant g. We will work in Cartesian coordinates (x, y, z). We want to look for solutions to the linearized fluid equations again, but this time allow the background to vary with alitude [5].

[5] This discussion can easily be generalized to a sphere with dependence only on r. There all background quantities are functions of r, and the vertical wavenumber $k_\perp^2 \to \ell(\ell+1)/r^2$ for angular dependence given by the spherical harmonic $Y_{\ell m}(\theta, \phi)$.

The density, pressure and velocity perturbations will be written as

$$
\begin{aligned}
\delta\rho(\boldsymbol{x},t) &= \delta\rho(z)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp - i\omega t} && (1.164)\\
\delta P(\boldsymbol{x},t) &= \delta P(z)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp - i\omega t} && (1.165)\\
\delta\boldsymbol{v}(\boldsymbol{x},t) &= \delta\boldsymbol{v}(z)e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp - i\omega t} && (1.166)\\
& && (1.167)
\end{aligned}
$$

where we have assumed a plane wave form in the x and y directions, but retained the z-dependence in the coefficients out front. We will solve for the coefficients $\delta\rho(z)$, $\delta P(z)$ and $\delta\boldsymbol{v}(z)$ and then plug them back into these expressions and take the real part to get the physical answer.

The wave (angular) frequency is ω and the horizontal wavenumber is $\boldsymbol{k}_\perp = \boldsymbol{e}_x k_x + \boldsymbol{e}_y k_y$. The associated wavelengths are $\lambda_x = 2\pi/k_x$ and $\lambda_y = 2\pi/k_y$. The horizontal position vector is $\boldsymbol{x}_\perp = \boldsymbol{e}_x x + \boldsymbol{e}_y y$, so the dot product $\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp = k_x x + k_y y$. Derivatives with respect to horizontal coordinates just bring down factors of ik, that is $\partial/\partial x \to ik_x$ and $\partial/\partial y \to ik_y$.

The linearized continuity and Euler equations were already derived for sound waves, and are written

$$
\begin{aligned}
\frac{\partial\delta\rho}{\partial t} &+ \boldsymbol{\nabla}\cdot(\rho\delta\boldsymbol{v}) = 0 && (1.168)\\
\rho\frac{\partial\delta\boldsymbol{v}}{\partial t} &= -\boldsymbol{\nabla}\delta P - \boldsymbol{e}_z g\delta\rho. && (1.169)
\end{aligned}
$$

The only change we've made is to add in the buoyancy force $-g\delta\rho$ in the $\boldsymbol{e}_z$ direction. Plugging in the assumed dependence on x, y and t, and writing out the vertical and horizontal Euler equations then gives

$$
\begin{aligned}
-i\omega\delta\rho &+ \frac{d\rho}{dz}\delta v_z + \rho\frac{d\delta v_z}{dz} + i\boldsymbol{k}_\perp\cdot\delta\boldsymbol{v}_\perp = 0 && (1.170)\\
-i\omega\rho\delta v_z &= -\frac{d\delta P}{dz} - g\delta\rho && (1.171)\\
-i\omega\rho\delta\boldsymbol{v}_\perp &= -i\boldsymbol{k}_\perp\delta P. && (1.172)
\end{aligned}
$$

This is 4 equations but 5 variables, $\delta\rho$, δP, δv_x, δv_y and δv_z.

An equation of state is needed to relate the pressure and density perturbation. It's a bit more complicated for a stratified atmosphere. Our basic assumption is that the fluid motion is adiabatic, i.e. the entropy of each fluid element is constant as the fluid element moves around. Since $s \propto \ln(P/\rho^\gamma)$, for adiabatic index γ, then $Ds/Dt = 0$ can be written *at linear order* as

$$\begin{aligned}
& \frac{D}{Dt}\left(\frac{P(z)+\delta P(\boldsymbol{x},t)}{(\rho(z)+\delta\rho(\boldsymbol{x},t))^\gamma}\right) \\
= \; & \frac{\partial}{\partial t}\left(\frac{P(z)+\delta P(\boldsymbol{x},t)}{(\rho(z)+\delta\rho(\boldsymbol{x},t))^\gamma}\right) + \delta\boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{P(z)+\delta P(\boldsymbol{x},t)}{(\rho(z)+\delta\rho(\boldsymbol{x},t))^\gamma}\right) \\
\simeq \; & \frac{\delta\dot{P}}{\rho^\gamma} - \gamma\frac{P\delta\dot{\rho}}{\rho^{\gamma+1}} + \delta v_z \frac{1}{\rho^\gamma}\frac{dP}{dz} - \delta v_z \gamma \frac{P}{\rho^{\gamma+1}}\frac{d\rho}{dz} = 0. \qquad (1.173)
\end{aligned}$$

Use $c^2 = \gamma P/\rho$ and multiplying by ρ^γ/c^2 gives the desired result

$$\begin{aligned}
\delta\dot{\rho} &= \frac{\delta\dot{P}}{c^2} + \delta v_z\left(\frac{\rho}{\gamma P}\frac{dP}{dz} - \frac{d\rho}{dz}\right) \\
&= \frac{\delta\dot{P}}{c^2} - \rho\delta v_z\left(\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz}\right) \\
&= \frac{\delta\dot{P}}{c^2} + \rho\frac{N^2}{g}\delta v_z, \qquad (1.174)
\end{aligned}$$

where we have used the definition of the Brunt-Vaisalla frequency

$$N^2 = -g\left(\frac{d\ln\rho}{dz} - \frac{1}{\gamma}\frac{d\ln P}{dz}\right). \qquad (1.175)$$

It will be convenient to define the "vertical displacement" ξ_z by $\delta v_z \equiv -i\omega\xi_z$. We can then get rid of the time derivatives in the equation, which becomes $-i\omega$ factors, to find

$$\delta\rho = \frac{\delta P}{c^2} + \rho\frac{N^2}{g}\xi_z. \qquad (1.176)$$

We will use this equation to eliminate $\delta\rho$ in favor of δP and ξ_z.

We can also eliminate the horizontal velocity by using the horizontal Euler equation to find

$$\delta\boldsymbol{v}_\perp = \frac{\boldsymbol{k}_\perp}{\omega}\frac{\delta P}{\rho}. \qquad (1.177)$$

Eliminating $\delta\rho$, $\delta\boldsymbol{v}_\perp$, and using $\delta v_z = -i\omega\xi_z$, the radial Euler and continuity equations become

$$\left(N^2 - \omega^2\right)\xi_z = -\frac{1}{\rho}\frac{d\delta P}{dz} - \frac{g}{c^2}\frac{\delta P}{\rho} \tag{1.178}$$

$$\frac{\delta P}{\rho c^2} + \left(\frac{N^2}{g} + \frac{d\ln\rho}{dz}\right)\xi_z + \frac{d\xi_z}{dz} - \frac{k_\perp^2}{\omega^2}\frac{\delta P}{\rho} = 0. \tag{1.179}$$

These are the equations we have been aiming toward. They are two coupled equations, each first order in z, for the two variables δP and ξ_z. Given the solution for those two variables, we can plug back in to find all the other variables. In general we can solve a boundary value problem to find the eigenvalue for the frequency ω which satisfies the proper boundary conditions. However, we will not pause to do that here as it's a lot of work.

Instead, we will try to understand the "plane wave" or WKB solution of these equations. In this case we assume a z-dependence

$$\xi_z(z) = \xi_z e^{ik_z z} \tag{1.180}$$

$$\delta P(z) = \delta P e^{ik_z z}, \tag{1.181}$$

where ξ_z and δP are now constant. We can simplify the equation further by assuming that the vertical wavelength, $2\pi/k_z$, is very small so that the wavenumber k_z is very large. In this limit, $k_z \gg g/c^2$ and $k_z \gg N^2/g + d\ln\rho/dz$ and the equations simplify to

$$\left(N^2 - \omega^2\right)\xi_z \simeq -\frac{1}{\rho}\frac{d\delta P}{dz} \tag{1.182}$$

$$\frac{\delta P}{\rho}\left(\frac{1}{c^2} - \frac{k_\perp^2}{\omega^2}\right) + \frac{d\xi_z}{dz} = 0. \tag{1.183}$$

Plugging in the plane wave assumption then gives the algebraic equations

$$\left(N^2 - \omega^2\right)\xi_z \simeq -ik_z\frac{\delta P}{\rho} \tag{1.184}$$

$$\frac{\delta P}{\rho}\left(\frac{1}{c^2} - \frac{k_\perp^2}{\omega^2}\right) + ik_z\xi_z = 0. \tag{1.185}$$

Solving for ξ_z in the second equation and plugging it into the first equation gives

$$\frac{\delta P}{\rho}\left[ik_z + \left(N^2 - \omega^2\right)\left(-\frac{1}{ik_z}\right)\left(\frac{1}{c^2} - \frac{k_\perp^2}{\omega^2}\right)\right] = 0 \tag{1.186}$$

or

$$\frac{\delta P}{\rho}\left[k_z^2 - \left(N^2-\omega^2\right)\left(\frac{k_\perp^2}{\omega^2}-\frac{1}{c^2}\right)\right] \;=\; 0. \tag{1.187}$$

In order for a solution to exist requires the argument of the parenthesis to be zero. This is our dispersion relation, written as $k_z = k_z(k_\perp, \omega)$ as

$$k_z^2 \;=\; \left(N^2-\omega^2\right)\left(\frac{k_\perp^2}{\omega^2}-\frac{1}{c^2}\right). \tag{1.188}$$

Let's discuss how to use this dispersion relation.

The solution is "propagating", meaning oscillation in z as e.g. $\cos(k_z z)$, only if k_z is real or $k_z^2 > 0$. The region of the atmosphere where the solution propagates is called the "propagation zone", it is basically where the wave lives. In regions where $k_z^2 < 0$, k_z is imaginary and the solution decays exponentially and is small; this is called the "evanescent zone."

How can we make k_z^2 positive? There are two ways, corresponding to the two different types of waves. Acoustic waves require high frequencies so that $\omega^2 > N^2$ and $\omega^2 > c^2 k_\perp^2$. In this high frequency limit, the dispersion relation becomes

$$\omega^2 \;\simeq\; c^2\left(k_r^2 + k_\perp^2\right), \tag{1.189}$$

as we've seen before. These waves have both group and phase velocity given by the sound speed c, and they are longitudinal waves. The second way to make $k_z^2 > 0$ is to assume $\omega^2 < N^2$ and $\omega^2 < c^2 k_\perp^2$. These are called "internal gravity waves" (not to be confused with gravitational waves in General Relativity). In the low-frequency limit, their dispersion relation is

$$\omega^2 \;\simeq\; N^2\left(\frac{k_\perp^2}{k_r^2}\right) \tag{1.190}$$

and this requires that $k_r \gg k_h$. These waves have a maximum frequency near the Brunt, and can have very low frequencies as $k_\perp/k_r$ becomes small. If we plug this dispersion relation back into the equations, we will find that they are *transverse* waves, with $\boldsymbol{k}\cdot\delta\boldsymbol{v} \simeq 0$, and small density perturbations $\delta\rho$. As is clear by the central role of the Brunt, these waves are restored by buoyancy.

A brief note on observations. The best observed oscillating star is our Sun, where thousands of acoustic waves ("p-modes") are observed through the vertical motion and light fluctuations they induce at the surface. No internal gravity waves ("g-modes") are observed for the Sun, since they can only propagate in radiative zones where $N^2 > 0$, and they evanesce at the convective surface layer of the Sun where $N^2 < 0$. G-modes are commonly observed in other stars though, e.g. in some white dwarfs.

But why do we observe oscillation modes at all? A wine glass does not ring on its own, you have to tap it to excite its oscillation modes. In stars, oscillation modes can be excited by turbulent convection in convection zones (e.g. the turbulent outflow from jet engines creates very loud sound waves), and also through "heat engines" in ionization zones, where some of the escaping heat energy can be converted to mechanical energy in waves.

So far we've focused on neutral gases. Now let's consider new effects that come in to ionized gases.

1.16 magnetohydrodynamics (MHD)

In an ionized gas, in addition to the pressure and gravity forces there is the Lorentz force due to electric and magnetic fields. Recall we use CGS units here. For a single particle of charge q and velocity $\boldsymbol{v}$, the Lorentz force is

$$\boldsymbol{F} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v} \times \boldsymbol{B}, \tag{1.191}$$

where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ the magnetic field. In fluid dynamics, we deal with forces per unit volume. Let's say there are now many species, labeled by the index i, and each with number density n_i, charge q_i and bulk (not random thermal) velocity $\boldsymbol{v}_i$. Then the Lorentz force per unit volume is

$$\boldsymbol{f} = \sum_i q_i n_i \boldsymbol{E} + \sum_i \frac{q_i}{c} n_i \boldsymbol{v}_i \times \boldsymbol{B} \equiv \rho_q \boldsymbol{E} + \frac{\boldsymbol{J} \times \boldsymbol{B}}{c}, \tag{1.192}$$

where the charge density is $\rho_q = \sum_i q_i n_i$ and the current density is $\boldsymbol{J} = \sum_i q_i n_i \boldsymbol{v}_i$.

In an ionized gas, on long length and timescales, at non-relativistic speeds, it is typically the case that ρ_q and $\boldsymbol{E}$ are very small and can be ignored compared to the magnetic force. The Lorentz force then becomes

$$\boldsymbol{f} \simeq \frac{\boldsymbol{J} \times \boldsymbol{B}}{c}. \tag{1.193}$$

The reason is that any charge imbalances create strong electric fields to try to try to decrease the charge imbalance. This can be quantified. If you are familiar with a bit of plasma physics, on timescales long compared to the plasma oscillation frequency, and lengthscales large compared to the Debye length, the plasma tends to have small charge imbalance and electric field. [6]

Next we need to understand how to evaluate the magnetic force, i.e. what do we plug in for current. It turns out that for non-relativistic motion, the current can be related to the field. Ampere's equation is

$$\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{4\pi}{c}\boldsymbol{J} \simeq 0, \tag{1.194}$$

where we can ignore the left hand side if the variations are slow compared to light cross timescales in the region of interest. When this is valid, the current and magnetic field are related:

$$\boldsymbol{J} \simeq \frac{c}{4\pi}\boldsymbol{\nabla} \times \boldsymbol{B}. \tag{1.195}$$

This form has charge conservation,

$$\frac{\partial \rho_q}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0, \tag{1.196}$$

built in, as we consider slow motions for which the term $\partial\rho_q/\partial t$ can be ignored, and hence $\boldsymbol{\nabla} \cdot \boldsymbol{J} \simeq 0$, the current is divergenceless.

Given eq.1.195, the Lorentz force then becomes

$$\boldsymbol{f} \simeq \frac{\boldsymbol{J} \times \boldsymbol{B}}{c} = \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}. \tag{1.197}$$

It turns out this expression can be rewritten in a more intuitive form using the no monopoles condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0. \tag{1.198}$$

We find

$$\boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \frac{1}{2}\boldsymbol{\nabla} B^2 - \boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B}, \tag{1.199}$$

[6] One exception is the "charge separation electric field" in ionized, hydrostatic objects. For instance, in a hydrogen gas, protons are heavier than electrons, and would tend to diffusively settle below them. But this would create a strong electric field. The end result is that an electric field $E \sim m_p g/e$ is needed to support the protons.

giving Lorentz force

$$\boldsymbol{f} \simeq \underbrace{-\boldsymbol{\nabla}\left(\frac{B^2}{8\pi}\right)}_{\text{magnetic pressure force}} + \overbrace{\frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi}}^{\text{magnetic tension force}} . \qquad (1.200)$$

The first term has a similar form as the gas pressure force, where the "magnetic pressure" is $B^2/8\pi$. The second term has a similar form as tension forces on strings: $f = \tau dy/dx$, where τ is the tension, y is the displacement from equilibrium, and x is the distance along the string. Pressure and tension forces give rise to new types of waves than the sound wave in a neutral gas. In particular, there are waves that rely on magnetic pressure which are similar to sound waves, and there are new waves that rely solely on magnetic tension.

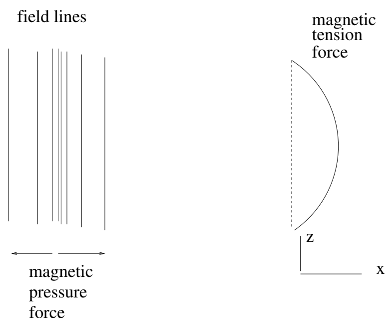

Figure 1.15: Separation of the Lorentz force into magnetic pressure and tension. The magnetic pressure term points from regions of high density of field lines to regions of low density. The magnetic tension term can be pictured similar to the tension force for a string.

Including the magnetic force, the Euler equation becomes

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}\right) = -\boldsymbol{\nabla}\left(P + \frac{B^2}{8\pi}\right) + \frac{\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B}}{4\pi} + \rho\boldsymbol{g}, \qquad (1.201)$$

where $P_{\rm tot} = P + B^2/8\pi$ is the total pressure.

Given that we have now added three additional variables ($\boldsymbol{B}$) to the fluid equations, we need equations for their time evolution. First, the magnetic field must remain divergence free:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0, \qquad (1.202)$$

the no monopoles condition. Second, Faraday's law gives the time evolution:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\boldsymbol{\nabla} \times \boldsymbol{E}, \tag{1.203}$$

where the negative sign is referred to as "Lenz's law." But what is $\boldsymbol{E}$? The simplest possible case to consider is to relate current to electric field with Ohm's law. But we must be careful! The standard form only applies in the rest frame of the fluid, as a pure magnetic field in one frame gives rise to an electric field in another frame in relative motion. This generation of electric fields due to motion is called "induction". Ohm's law then takes on the form

$$\boldsymbol{E}_{\rm rest} = \frac{\boldsymbol{J}}{\sigma} = \boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}, \tag{1.204}$$

where $\boldsymbol{E}_{\rm rest}$ is the electric field in the rest frame of the fluid, $\boldsymbol{E}$ and $\boldsymbol{B}$ are measured in a frame moving at velocity $\boldsymbol{v}$ with respect to the rest frame, and σ is the conductivity that appears in Ohm's law.

Plugging back into Faraday's law we find an evolution equation for the magnetic field ("the induction equation"):

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times \left(c\frac{\boldsymbol{J}}{\sigma}\right) \\ &= \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times \left(\frac{c^2}{4\pi\sigma}\boldsymbol{\nabla} \times \boldsymbol{B}\right). \end{aligned} \tag{1.205}$$

The quantitiy $\eta \equiv c^2/4\pi\sigma$ is called "the magnetic diffusivity", and has the units of a diffusion coefficient. If we let η be constant, and use the vector identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B} = \boldsymbol{\nabla}\left(\boldsymbol{\nabla} \cdot \boldsymbol{B}\right) - \nabla^2 \boldsymbol{B} = -\nabla^2 \boldsymbol{B}, \tag{1.206}$$

we find the more conventional form of the induction equation, including the motional EMF term and the Ohmic diffusion term:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{1.207}$$

Let's ignore the first term on the right hand side for the moment. The equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta \nabla^2 \boldsymbol{B} \tag{1.208}$$

has the standard form of a diffusion equation. Using dimensional analysis, with B varying over lengthscale L, the timescale the field changes on is

$$t \sim \frac{L^2}{\eta}, \tag{1.209}$$

the same as in any diffusion process. Ohmic diffusion and decay can cause magnetic fields of stars and planets to weaken over time, if there is no dynamo to build them back up.

Aside on the diffusion equation: We've talked about the random walk, and how in a diffusion process the diffusing quantity travels a distance $x = \sqrt{\eta t}$, where η is the diffusion coefficient. Now let's derive an exact solution of the diffusion equation to show this.

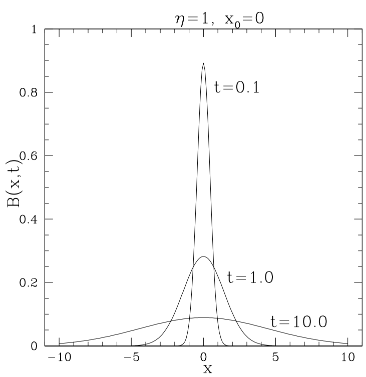

Figure 1.16: Solution of the diffusion equation for the magnetic field with delta function initial condition.

Consider one component $B(x, t)$ of the magnetic field, in one spatial dimension. The diffusion equation becomes

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}. \tag{1.210}$$

To solve, Fourier transform in space:

$$B(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} B(k,t) e^{ikx}. \tag{1.211}$$

Plugging the Fourier expansion in, and projecting out the wavenumber k gives the ODE

$$\frac{\partial B(k,t)}{\partial t} = -\eta k^2 B(k,t), \tag{1.212}$$

which is easily solved as

$$B(k,t) = B(k,0)e^{-\eta k^2 t}. \tag{1.213}$$

The general time-dependent solution is then

$$B(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} B(k,0)e^{ikx-\beta k^2 t}. \tag{1.214}$$

The integration constants $B(k,0)$ are set by the initial condition at $t=0$. If the initial condition is $B(x,0)=\delta(x-x_0)$, a Dirac delta function centered on $x=x_0$, then

$$\delta(x-x_0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0)} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} B(k,0)e^{ikx} \tag{1.215}$$

so that $B(k,0) = e^{-ikx_0}$. Plugging back in, and using the "complete the square" to do the integral, we get the final answer:

$$\begin{aligned} B(x,t) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x_0)-\beta k^2 t} \\ &= e^{-(x-x_0)^2/4\eta t} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-\eta t \left(k - \frac{i(x-x_0)}{2\eta t}\right)^2\right] \\ &= \frac{e^{-(x-x_0)^2/4\eta t}}{(4\pi\eta t)^{1/2}}. \end{aligned} \tag{1.216}$$

The result shows a Gaussian centered on x_0, but with width $\sqrt{4\eta t}$ that spreads in time. The denominator shows that the height must decrease as the width spreads.

So if field lines are initially tightly packed in a certain region, diffusion will cause the number of field lines in that region to decrease, as they spread out.

Q: If this field diffusion occurs in a fluid, can fluid motion be ignored? What force is changing as the fluid diffuses?

A: If the system was initially in force balance and then the field diffused, you are changing the magnetic pressure gradient force. In order for this force to remain nearly in balance with the gas pressure force, the gas must then move as well to change its density.

Lastly, what if oppositely directed magnetic fields are placed right next to each other? In this simple example in which only diffusion acts on the field, the oppositely directed fields will "annihilate" each other in a process called "magnetic reconnection." Consider an field at time $t = 0$ given by $B_z(x, 0) = -B_0 + 2B_0\Theta(x)$, which has value $-B_0$ for $x < 0$ and $+B_0$ for $x > 0$. Evolving this configuration forward in time using the diffusion equation gives the numerical result in Figure 1.17.

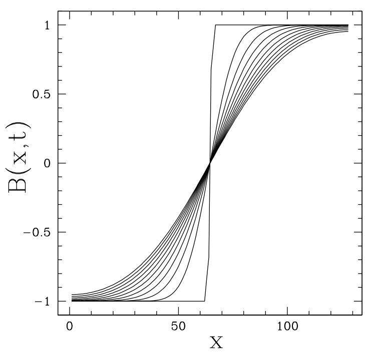

Figure 1.17: A field which is initially $B = -1$ for $x < 0$ and $B = +1$ for $x > 0$ evolves under pure magnetic diffusion. Notice the field near the interface becomes much smaller over time. The "annihilation" of the magnetic field moves away from the interface, as shown by the lines at different times. The distance the flux can diffuse in a time t is $x(t) \sim (\eta t)^{1/2}$, where η is the magnetic diffusion coefficient arising from the finite Ohmic resistivity.

Now include the first term as well. Which one dominates, first or second term? Again using dimensional analysis, the ratio of the first to the second term is

$$\frac{vB/L}{\eta B/L^2} = \frac{vL}{\eta} \equiv R_m, \tag{1.217}$$

which looks like the Reynolds number. This quantity, relating to diffusion and advection of magnetic fields, is called the "Magnetic Reynolds number."

When $R_m << 1$, diffusion is faster than advection, and the field motion is higly resistive. When $R_m \gg 1$, diffusion can be ignored except in very thin regions (diffusion in those small regions, which can destroy fields, is often called "magnetic reconnection").

When magnetic diffusion can be ignored, there is a rather profound way of thinking of the magnetic field lines as "stuck to" the fluid.

1.16.1 magnetic flux freezing

In this section we present the derivation that magnetic field lines are "frozen in" to the fluid when diffusion is ignored.

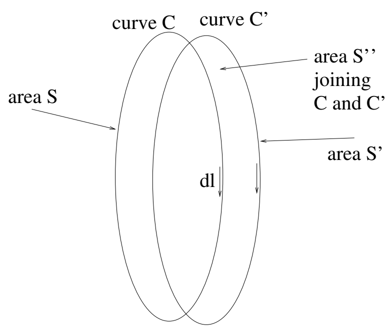

Figure 1.18: Surfaces bounding the motion of a fluid element. The curve C surrounds the area S at time t. A short time Δt later, this same fluid is bounded by the curve C' and surface S'. The two surfaces are connected by an area element called S''.

Let the magnetic field evolve according to

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{1.218}$$

where $\boldsymbol{v}$ is the fluid velocity.

The flux through surface S, bounded by curve C, at time t is defined as

$$\Phi(S,t) = \int_S dA \boldsymbol{n} \cdot \boldsymbol{B}(\boldsymbol{x},t), \tag{1.219}$$

where dA is the area element for S and $\boldsymbol{n}$ is the unit normal. At a later time $t + \Delta t$, the area S encircling the fluid has moved to S', now encircled by the curve C'. The flux through this latter area is

$$\Phi(S', t+\Delta t) \;=\; \int_{S'} dA\boldsymbol{n}\cdot\boldsymbol{B}(\boldsymbol{x}, t+\Delta t). \tag{1.220}$$

Expand this expression to first order in Δt:

$$\begin{aligned}\Phi(S', t+\Delta t) \;&\simeq\; \int_{S'} dA\boldsymbol{n}\cdot\boldsymbol{B}(\boldsymbol{x}, t) + \Delta t\int_{S'} dA\boldsymbol{n}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\\ &\simeq\; \int_{S'} dA\boldsymbol{n}\cdot\boldsymbol{B}(\boldsymbol{x}, t) + \Delta t\int_{S} dA\boldsymbol{n}\cdot\frac{\partial\boldsymbol{B}}{\partial t},\end{aligned} \tag{1.221}$$

where in the second equality, we ignored the difference between S and S', as that would be higher order in Δt. Next, the magnetic flux through the "endcap" surface connecting S and S' is, to first order in Δt,

$$\Phi(S'', t) \;=\; \int_{S''} (d\boldsymbol{\ell}\times\boldsymbol{v}\Delta t)\cdot\boldsymbol{B}(\boldsymbol{x}, t), \tag{1.222}$$

since $d\boldsymbol{\ell}\times\boldsymbol{v}\Delta t$ is the outward pointing area element. Since the magnetic field is divergence free, the flux out of any closed surface is zero. Hence, using the *outward* pointing normal vectors, the outward flux is zero. But the way we've defined the bounding curve C in figure 1.18, the surface integration over S is using the inward pointing normal, and we need to keep that sign in mind. Hence, at time t,

$$\Phi(S, t) \;=\; \Phi(S', t) + \Phi(S'', t). \tag{1.223}$$

Given this intermediate result, what is the change in flux through the fluid element during it's evolution? We find

$$\begin{aligned}\Phi(S', t+\Delta t) - \Phi(S, t) \;&=\; (\Phi(S', t) + \Delta t\int_S dA\boldsymbol{n}\cdot\frac{\partial\boldsymbol{B}}{\partial t}) - (\Phi(S', t) + \Phi(S'', t))\\ &=\; \Delta t\int_S dA\boldsymbol{n}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \int_{S''}(d\boldsymbol{\ell}\times\boldsymbol{v}\Delta t)\cdot\boldsymbol{B}(\boldsymbol{x}, t)\\ &=\; \Delta t\int_S dA\boldsymbol{n}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \Delta t\int_{S''} d\boldsymbol{\ell}\cdot(\boldsymbol{v}\times\boldsymbol{B})\\ &=\; \Delta t\int_S dA\boldsymbol{n}\cdot\left(\frac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})\right)\to 0\end{aligned} \tag{1.224}$$

as $\Delta t\to 0$, so that higher order terms can be neglected, and we used the induction equation to make the quantity in parenthesis vanish.

This concludes the proof of frozen in flux.

Deep thoughts: The ability to visualize "magnetic field lines" as "frozen into the fluid" is a huge conceptual tool in MHD. In fact, since currents have disapeared from the equations, we only have to think about the what the field is doing. Hence the magnetic field is the primary variable in MHD, and currents only need be considered when including deviations from ideal MHD behavior, e.g. resistivity.

The magnetic flux problem in star formation: If magnetic fields of size $B_0 \sim \mu$ G are frozen into interstellar gas clouds they would seem to completely inhibit star formation! If a cloud of size $R_0 \sim 1$ pc collapse to a size $\simeq 1\ R_\odot$, the flux freezing condition implies that

$$\Phi \;\simeq\; B_0 R_0^2 = BR^2 \tag{1.225}$$

so that the magnetic field in the Sun today would be

$$B \;=\; B_0 \left(\frac{R_0}{R}\right)^2 \simeq 10^8 \text{ G}. \tag{1.226}$$

The (average) field in the Sun is, however, estimated to be closer to $B \sim 1$ G, so apparently some non-MHD effect (e.g. resistivity) has gotten rid of magnetic flux.

When was the field separated from the gas? It had to happen *before* the star formed, not after, since the ratio of magnetic to gas pressure forces would otherwise be

$$\frac{B^2/4\pi R}{P/R} \;\simeq\; \frac{B^2}{4\pi P} \sim \frac{(10^8 \text{ G})^2 R_\odot^4}{GM_\odot^2} \sim 1, \tag{1.227}$$

i.e. gas and magnetic pressure forces would be comparable, and would have frustrated gravity in trying to get the cloud to collapse.

Interestingly, the above ratio

$$\frac{B^2R^4}{GM^2} \;=\; \frac{\Phi^2}{GM^2} = \left(\frac{\Phi}{G^{1/2}M}\right)^2 \tag{1.228}$$

involves the flux and the mass. If flux freezing is valid, then both these quantities are constant, even as the cloud is collapsing. Hence this "mass-to-flux ratio" is a constant of the motion, and is a key parameter in understanding the conditions under which stars can form.

Generation of neutron star magentic fields: There are two camps for this. One camp believes that the $R_0 \sim 10^9$ cm and $M \simeq 1\ M_\odot$ iron core of the massive star, which collapses to the neutron star has a "seed" magentic field which is amplified by flux freezing during collapse to a $R = 10^6$ cm neutron star. The second camp believes that a "dynamo" can operate in convection zones during the supernova, in which convective fluid motions, combined with flux freezing, stretch and compress magnetic fields. In this case, "equipartition" fields of size $B^2/8\pi \sim \rho v^2$ are reached, and everything depends on the eddy velocities, v.

Q1: If the precollapse Fe core had $B_0 \sim 10^6$ G, what would the neutron star magnetic field be?

A1:

$$B = B_0 \left(\frac{R_0}{R} \right)^2 \simeq 10^{12} \text{ G.} \tag{1.229}$$

This is about right, so the precollapse field must be orders of magnitude larger than in the Sun. We do see white dwarfs with fields this large, even up to $B_0 \sim 10^9$ G, so maybe this mechanism is possible.

Q2: If a neutron star has a convection zone which carries an energy 10^{53} erg out in $\simeq 10$ s, and the density is $\rho \sim 10^{14}$g cm^{-3}, then what "equipartition" magnetic field is produced?

A2: The luminosity is $L = 10^{52}$ erg, giving a flux $F = L/4\pi R^2 = 10^{39}$ erg cm^{-2} s^{-1}. The eddy velocity is then $v \sim (F/\rho)^{1/3} = 10^8$ cm s^{-1}. The equipartition field is $B \sim 10^{16}$ G!! This is comparable to the high fields inferred for magnetars. The main problem with this mechanism is that convection tends to create fields on small (scale height) sizes, not a dipole field with lengthscale of order the size of the star.

1.16.2 mhd waves (slow, Alfven, fast)

The introduction of the magnetic force gives rise to three waves in mhd, as opposed to just the sound wave in neutral fluids. These wave speeds then appear in a wide variety of other physics, such as new types of shocks, new critical points in winds, etc.

Let's consider the simplest case of a background state with $\boldsymbol{v} = 0$, and ρ and P constant. Let the background field be constant as well, with $\boldsymbol{B} = B\boldsymbol{e}_z$. Ignore gravity. It gets tiring writing the subscript "O" on background quantities, so let's skip it. All perturbations have a δ (the Eulerian perturbation) in front of them.

Since the background is uniform in space and time, again we can fourier transform in space and time, i.e. plug in the plane wave form, e.g. $\delta P(\boldsymbol{x}, t) = \delta P \exp(i\boldsymbol{k} \cdot \boldsymbol{x} - i\omega t)$ for all perturbations, where now δP is just a constant. Space and time derivatives can then be replaced by $\partial/\partial t \to -i\omega$ and $\boldsymbol{\nabla} \to i\boldsymbol{k}$.

Let's review how to perturb the equations, keeping only first order terms in the perturbations, and ignoring background terms and nonlinear terms. The continuity equation is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) \quad = \quad 0, \tag{1.230}$$

with linearized form

$$\frac{\partial \delta\rho}{\partial t} + \rho \boldsymbol{\nabla} \cdot \delta \boldsymbol{v} \quad = \quad 0, \tag{1.231}$$

and after plugging in plane waves,

$$\omega \frac{\delta\rho}{\rho} \quad = \quad \boldsymbol{k} \cdot \delta \boldsymbol{v}. \tag{1.232}$$

The momentum equation is

$$\rho \left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} \right) \quad = \quad -\boldsymbol{\nabla} \left(P + \frac{B^2}{8\pi} \right) + \frac{\boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B}}{4\pi}, \tag{1.233}$$

with linearized form

$$\frac{\partial \delta \boldsymbol{v}}{\partial t} \quad = \quad -\boldsymbol{\nabla} \left(\delta P + \frac{\delta \boldsymbol{B} \cdot \boldsymbol{B}}{4\pi} \right) + \frac{\delta \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{B}}{4\pi}, \tag{1.234}$$

and after plugging in plane waves

$$-\omega \delta \boldsymbol{v} \quad = \quad -\boldsymbol{k} \left(\delta P + \frac{\delta \boldsymbol{B} \cdot \boldsymbol{B}}{4\pi} \right) + \frac{\boldsymbol{k} \cdot \boldsymbol{B} \delta \boldsymbol{B}}{4\pi}. \tag{1.235}$$

The adiabatic EOS is

$$P \propto \rho^\gamma, \tag{1.236}$$

with perturbed form

$$\delta P = c^2 \delta\rho, \tag{1.237}$$

where $c^2 = \gamma p/\rho$ is the adiabatic sound speed.

The no monopoles condition is

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{1.238}$$

with linearized form

$$\boldsymbol{\nabla} \cdot \delta\boldsymbol{B} = 0 \tag{1.239}$$

and after plugging in plane waves

$$\boldsymbol{k} \cdot \delta\boldsymbol{B} = 0. \tag{1.240}$$

Lastly, the induction equation is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{1.241}$$

with linearized form

$$\frac{\partial \delta\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\delta\boldsymbol{v} \times \boldsymbol{B}) \tag{1.242}$$

and after plugging in plane waves

$$-\omega\delta\boldsymbol{B} = \boldsymbol{k} \times (\delta\boldsymbol{v} \times \boldsymbol{B}) = \boldsymbol{k} \cdot \boldsymbol{B}\delta\boldsymbol{v} - \boldsymbol{k} \cdot \delta\boldsymbol{v}\boldsymbol{B}. \tag{1.243}$$

The tedious part is over. Now we need to solve these algebraic equations to find the dispersion relation relating frequency to wavenumber $\boldsymbol{k}$. To start, use the EOS to eliminate $\delta\rho$ in favor of δP. The continuity equation then becomes

$$\boldsymbol{k} \cdot \delta\boldsymbol{v} = \omega\frac{\delta P}{\rho c^2}. \tag{1.244}$$

This expression can in turn be used to eliminate δP from the Euler equation:

$$\omega^2 \delta \boldsymbol{v} \;=\; \boldsymbol{k}\left[c^2 \boldsymbol{k}\cdot\delta\boldsymbol{v} + \omega\frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi\rho}\right] - \omega\frac{\boldsymbol{k}\cdot\boldsymbol{B}}{4\pi\rho}\delta\boldsymbol{B}. \tag{1.245}$$

Next, eliminate $\delta\boldsymbol{B}$ in the momentum equation using the induction equation, to get our final equation involving only $\delta\boldsymbol{v}$:

$$\begin{aligned}\omega^2 \delta \boldsymbol{v} \;=\; & \boldsymbol{k}c^2\boldsymbol{k}\cdot\delta\boldsymbol{v} + \frac{\boldsymbol{k}}{4\pi\rho}\left[B^2\boldsymbol{k}\cdot\delta\boldsymbol{v} - \boldsymbol{k}\cdot\boldsymbol{B}\boldsymbol{B}\cdot\delta\boldsymbol{v}\right] \\ & - \frac{\boldsymbol{k}\cdot\boldsymbol{B}}{4\pi\rho}\left[\boldsymbol{B}\boldsymbol{k}\cdot\delta\boldsymbol{v} - \delta\boldsymbol{v}\boldsymbol{k}\cdot\boldsymbol{B}\right].\end{aligned} \tag{1.246}$$

To solve this equation, first we need the two dot products $\boldsymbol{k}\cdot\delta\boldsymbol{v}$ and $\boldsymbol{B}\cdot\delta\boldsymbol{v}$. These are found by dotting with $\boldsymbol{k}$ and $\boldsymbol{B}$, respectively. To write the results compactly, define the three frequencies

$$\omega_s^2 \;=\; c^2k^2 \tag{1.247}$$

$$\omega_B^2 \;=\; \frac{k^2B^2}{4\pi\rho} \tag{1.248}$$

$$\omega_A^2 \;=\; \frac{(\boldsymbol{k}\cdot\boldsymbol{B})^2}{4\pi\rho}. \tag{1.249}$$

Dotting eq.1.246 with $\boldsymbol{k}$ gives, after some algebra

$$(\omega^2 - \omega_s^2 - \omega_B^2)\boldsymbol{k}\cdot\delta\boldsymbol{v} \;=\; -k^2\frac{\boldsymbol{k}\cdot\boldsymbol{B}}{4\pi\rho}\boldsymbol{B}\cdot\delta\boldsymbol{v}. \tag{1.250}$$

Dotting eq.1.246 with $\boldsymbol{B}$ gives, after some algebra

$$\omega^2\boldsymbol{B}\cdot\delta\boldsymbol{v} \;=\; c^2\boldsymbol{k}\cdot\boldsymbol{B}\boldsymbol{k}\cdot\delta\boldsymbol{v}. \tag{1.251}$$

To find the dispersion relation, we must combine eq.1.250 and 1.251. But careful! There is a subtlety in how this is done, because you can have both $\boldsymbol{k}\cdot\delta\boldsymbol{v}\neq 0$ or $\boldsymbol{k}\cdot\delta\boldsymbol{v} = 0$ solutions. Let's do the two cases in turn.

Alfven waves

Alfven waves have $\boldsymbol{k}\cdot\delta\boldsymbol{v} = 0$, i.e. they are *incompressible*, or *transverse* waves, since their motion is perpedicular to the direction of propagation ($\boldsymbol{k}$). Further, they have no motion along field lines, $\boldsymbol{B}\cdot\delta\boldsymbol{v} = 0$, by eq.1.251. From the continuity and EOS, this also implies that $\delta\rho = \delta P = \boldsymbol{B} \cdot \delta\boldsymbol{B} = 0$, i.e. no density perturbations, or perturbations to gas or magnetic pressure. So thermodynamics doesn't enter into these waves as there is no compression. Going back to the momentum and induction equations, these simplifications lead to

$$\begin{aligned} \omega\delta\boldsymbol{v} &= -\frac{\boldsymbol{k}\cdot\boldsymbol{B}}{4\pi\rho}\delta\boldsymbol{B} && (1.252) \\ \omega\delta\boldsymbol{B} &= -\boldsymbol{k}\cdot\boldsymbol{B}\delta\boldsymbol{v}, && (1.253) \end{aligned}$$

i.e. the only force is magnetic tension, and the field only evolves due to gradients in the transverse velocity along field lines. Combining eq.1.252 and 1.253, we find

$$\omega^2\delta\boldsymbol{v} = -\frac{\boldsymbol{k}\cdot\boldsymbol{B}}{4\pi\rho}(-)\boldsymbol{k}\cdot\boldsymbol{B}\delta\boldsymbol{v}, \tag{1.254}$$

and, canceling $\delta\boldsymbol{v}$, we find

$$\omega^2 = \omega_A^2 = \frac{(\boldsymbol{k}\cdot\boldsymbol{B})^2}{4\pi\rho}. \tag{1.255}$$

So these wave have large frequency when $\boldsymbol{k} \parallel \boldsymbol{B}$, and small when perpendicular.

Since magnetic tension is the only force in play, Alfven waves are often pictured as "plucking" the magnetic field lines, as one plucks the strings of a guitar. The velocity and magnetic field perturbations are transverse to the field lines, and parallel to each other (for a linearly polarized wave).

slow and fast waves

These waves have $\boldsymbol{k}\cdot\delta\boldsymbol{v} \neq 0$, and hence involve fluid compression and thermodynamics to some extent. Combining eq.1.250 and 1.251, we find the dispersion relation

$$\omega^4 - \omega^2(\omega_s^2 + \omega_B^2) + \omega_s^2\omega_A^2 = 0. \tag{1.256}$$

In general, this gives a quadratic equation to solve for $\omega = \omega(\boldsymbol{k})$. But if we assume that either gas pressure dominates magnetic pressure ($P \gg B^2/4\pi$), or the opposite ($P \ll B^2/4\pi$), then the solution is simple.

In this limit, consider the high frequency solution, called the *fast wave*. For large frequencies, the first two terms in eq.1.256 dominate, and the non-trivial solution is

$$\omega_{\rm fast}^2 = \omega_s^2 + \omega_B^2 = \left(c^2 + \frac{B^2}{4\pi\rho} \right) k^2. \tag{1.257}$$

So the fast wave is like the hydrodynamic sound wave, but it is restored by *both* gas and magnetic pressure.

The low frequency solution in this limit is found by balancing the 2nd and 3rd terms, giving the *slow wave*:

$$\omega_{\rm slow}^2 \simeq \frac{\omega_s^2 \omega_A^2}{\omega_s^2 + \omega_B^2}. \tag{1.258}$$

When gas pressure dominates, $\omega_s^2 \gg \omega_B^2$, the slow wave has the same frequency as the Alfven wave. When magnetic pressure dominates, the frequency becomes

$$\omega_{\rm slow}^2 \simeq \frac{\omega_s^2 \omega_A^2}{\omega_B^2} = c^2 (\boldsymbol{k} \cdot \hat{\boldsymbol{B}})^2, \tag{1.259}$$

a wave that travels at the gas sound speed, but along field lines!

The slow and fast waves in general can be longitudinal, meaning that their fluid velocity has a component along the direction of propagation. In the gas pressure dominated limit, the slow mode has only a small longitudinal motion, and it's mostly transverse, just like the Alfven wave.

How do waves transport energy through space? Let's compute their *group velocity* to see how they differ.

1.16.3 aside on the group velocity

We mentioned phase velocity already. If a quantity $f(\boldsymbol{x}, t) \propto \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega t)$, then the phase is constant when

$$\frac{d\boldsymbol{x}}{dt} = \frac{\omega}{\boldsymbol{k}}. \tag{1.260}$$

A separate concept is how fast a *wave packet* confined in space and time, carries its energy from one place to another. Suppose the Fourier representation of f is

$$f(\boldsymbol{x}, t) = \int \frac{d^3k}{(2\pi)^3} f(\boldsymbol{k}) \exp\left[i\boldsymbol{k}\cdot\boldsymbol{x} - i\omega\left(\boldsymbol{k}\right) t \right], \tag{1.261}$$

where each Fourier component oscillates at its own frequency $\omega(\boldsymbol{k})$. For a finite wave packet, there will be a small range of wavenumbers $\Delta\boldsymbol{k}$ around a central wavenumber $\boldsymbol{k}_0$. When the width is small compared to the mean, we can expand

$$\omega(\boldsymbol{k}) \simeq \omega(\boldsymbol{k}_0) + \frac{\partial\omega}{\partial\boldsymbol{k}}(\boldsymbol{k}-\boldsymbol{k}_0) + ... \equiv \omega(\boldsymbol{k}_0) + \boldsymbol{v}_g \cdot (\boldsymbol{k}-\boldsymbol{k}_0) + ... \tag{1.262}$$

where we have defined the *group velocity*

$$\boldsymbol{v}_g = \frac{\partial\omega}{\partial\boldsymbol{k}}. \tag{1.263}$$

Eq.1.261 then takes the form

$$\begin{aligned} f(\boldsymbol{x},t) &= \exp\left[i\boldsymbol{k}_0\cdot\boldsymbol{x} - i\omega\left(\boldsymbol{k}_0\right)t\right] \\ &\times \int \frac{d^3k}{(2\pi)^3} f(\boldsymbol{k}) \exp\left[i(\boldsymbol{k}-\boldsymbol{k}_0)\cdot(\boldsymbol{x}-\boldsymbol{v}_g t)\right]. \end{aligned} \tag{1.264}$$

The integral now has an overall phase factor set by the phase velocity of the central wavenumber, but also a factor which propagates at the group velocity $\boldsymbol{v}_g$. It is the latter that determines now fast the wave packet moves in space. This can be shown explicitly for simply functions $f(\boldsymbol{k})$, such as a Gaussian.

propagation of MHD waves

Given a dispersion relation $\omega = \omega(\boldsymbol{k})$, the group velocity

$$\boldsymbol{v}_g = \frac{\partial\omega(\boldsymbol{k})}{\partial\boldsymbol{k}} \tag{1.265}$$

is the vector giving the direction and speed at which energy transported by the wave. Recall this is distinct from the *phase velocity*, which is the speed at which crests/troughs travel.

First consider the ordinary hydrodynamic sound wave with dispersion relation

$$\omega = ck = c\sqrt{\boldsymbol{k}\cdot\boldsymbol{k}}. \tag{1.266}$$

The group velocity is

$$\boldsymbol{v}_g = c\frac{\boldsymbol{k}}{k} = c\hat{\boldsymbol{k}}. \tag{1.267}$$

This wave travels at the gas sound speed along the direction of the wavenumber. There is no preferred direction in space, aside from $\hat{k}$.

Next consider the Alfven wave with $\omega = V_A \boldsymbol{k} \cdot \hat{\boldsymbol{B}}$, where $V_A = B/\sqrt{4\pi\rho}$ is the Alfven speed. The group velocity is

$$\boldsymbol{v}_g = V_A \hat{\boldsymbol{B}}, \tag{1.268}$$

i.e. the wave travels at the Alfven speed, but along field lines, no matter direction of $\boldsymbol{k}$! In the gas pressure dominated limit, the slow mode is identical to the Alfven mode.

In either the gas or magnetic pressure dominated limit, the fast mode has dispersion relation $\omega = (c^2 + V_B^2)^{1/2} k^2$, where $V_B = B^2/(4\pi\rho)$. Notice that this differs from the Alfven wave in that it's not $\boldsymbol{k} \cdot \boldsymbol{B}$ that enters, but rather kB. This wave is gain isotropic, ignoring the small terms in the dispersion relation.

Another good example of an anisotropic wave is the slow mode in the magnetic pressure dominated limit. The dispersion relation is $\omega = c\boldsymbol{k} \cdot \hat{\boldsymbol{B}}$, giving group velocity

$$\boldsymbol{v}_g = c\hat{\boldsymbol{B}}, \tag{1.269}$$

i.e. the wave travels at the gas sound speed, but along the magnetic field.

Q: Consider a region of space with constant density and pressure, threaded by a uniform magnetic field along the $\boldsymbol{e}_z$ direction. A sudden perturbation excites a spectrum of slow, fast and Alfven waves, with wavenumbers distributed uniformly in all directions. Describe how the three types of waves propagate outward in space.

A: Isotropic waves travel in the direction of their $\boldsymbol{k}$ vector, and hence spread out in three dimensional space, with wave energy flux decreasing as $1/r^2$. This applies to the fast mode in either gas or magnetic pressure dominated limits, and the slow mode in the gas pressure dominated limit.

The Alfven wave is an example of a wave that travels only along field lines, and hence spreads out only in one spatial dimension. Generally then, Alfven wave fluxes decrease more slowly from the source. It is as if you "pluck the field lines."

Bibliography

G. B. Arfken and H. J. Weber. *Mathematical methods for physicists 6th ed.* 2005.

J. Binney and S. Tremaine. *Galactic Dynamics: Second Edition*. Princeton University Press, 2008.

S. Chandrasekhar. Stochastic Problems in Physics and Astronomy. *Reviews of Modern Physics*, 15:1–89, January 1943. doi: 10.1103/RevModPhys.15.1.

H. Goldstein, C. Poole, and J. Safko. *Classical mechanics*. 2002.

L. D. Landau and E. M. Lifshitz. *Mechanics*. 1969.

L. D. Landau and E. M. Lifshitz. *Statistical physics. Pt.1, Pt.2*. 1980.

J. Marion, S. Thornton, and R. W. Robinett. *Classical Dynamics of Particles and Systems, 3rd ed.* Harcourt Brace Jovanovich, 1988.

C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. 1973.

C. D. Murray and S. F. Dermott. *Solar system dynamics*. 1999.

F. Reif. *Fundamentals of Statistical and Thermal Physics*. Waveland Press, 1965.

G. B. Rybicki and A. P. Lightman. *Radiative Processes in Astrophysics*. June 1986.

J. J. Sakurai. *Modern quantum mechanics*. 1985.

L. I. Schiff. *Quantum mechanics*. 1968.

S. Seager. *Exoplanets*. 2011.