Kinematics & Calculus

Discussion

constant acceleration

Calculus is an advanced math topic, but it makes deriving two of the three equations of motion much simpler. By definition, acceleration is the first derivative of velocity with respect to time. Take the operation in that definition and reverse it. Instead of differentiating velocity to find acceleration, integrate acceleration to find velocity. This gives us the velocity-time equation. If we assume acceleration is constant, we get the so-called first equation of motion [1].

$$a = \frac{dv}{dt}$$

$$dv = a\ dt$$

$$\int_{v_0}^{v} dv = \int_{0}^{t} a\ dt$$

$$v - v_0 = at$$

$$v = v_0 + at \quad [1]$$

Again by definition, velocity is the first derivative of position with respect to time. Reverse the operation in the definition. Instead of differentiating position to find velocity, integrate velocity to find position. This gives us the position-time equation for constant acceleration, also known as the second equation of motion [2].

$$v = \frac{ds}{dt}$$

$$ds = v\ dt$$

$$ds = (v_0 + at)\ dt$$

$$\int_{s_0}^{s} ds = \int_{0}^{t} (v_0 + at)\ dt$$

$$s - s_0 = v_0 t + ½at^2$$

$$s = s_0 + v_0 t + ½at^2 \quad [2]$$

Unlike the first and second equations of motion, there is no obvious way to derive the third equation of motion (the one that relates velocity to position) using calculus. We can't just reverse engineer from the definitions. We need to play a rather sophisticated trick.

The first equation of motion relates velocity to time. We essentially derived it from this derivative…

$$\frac{dv}{dt} = a$$

The second equation of motion relates position to time. It came from this derivative…

$$\frac{ds}{dt} = v$$

The third equation of motion relates velocity to position. By logical extension, it should come from a derivative that looks like this…

$$\frac{dv}{ds} = ?$$

But what does this equal? Well nothing by definition, but like all quantities it does equal itself. It also equals itself multiplied by 1. We'll use a special version of 1 ($\frac{dt}{dt}$) and a special version of algebra (algebra with infinitesimals). Look what happens when we do this. We get a derivative equal to acceleration ($\frac{dv}{dt}$) and another equal to the inverse of velocity ($\frac{dt}{ds}$).

$$\frac{dv}{ds} = \frac{dv}{ds}\ 1$$

$$\frac{dv}{ds} = \frac{dv}{ds}\frac{dt}{dt}$$

$$\frac{dv}{ds} = \frac{dv}{dt}\frac{dt}{ds}$$

$$\frac{dv}{ds} = a\ \frac{1}{v}$$

Next step, separation of variables. Get things that are similar together and integrate them. Here's what we get when acceleration is constant…

$$\frac{dv}{ds} = a\frac{1}{v}$$

$$v\ dv = a\ ds$$

$$\int_{v_0}^{v} v\ dv = \int_{s_0}^{s} a\ ds$$

$$½(v^2 - v_0^{\ 2}) = a(s - s_0)$$

$$v^2 = v_0^{\ 2} + 2a(s - s_0) \quad [3]$$

Certainly a clever solution, and it wasn't all that more difficult than the first two derivations. However, it really only worked because acceleration was constant — constant in time and constant in space. If acceleration varied in any way, this method would be uncomfortably difficult. We'd be back to using algebra just to save our sanity. Not that there's anything wrong with that. Algebra works and sanity is worth saving.

$$v = v_0 + at \quad [1]$$

$$+$$

$$s = s_0 + v_0t + ½at^2 \quad [2]$$

$$=$$

$$v^2 = v_0^2 + 2a(s - s_0) \quad [3]$$

constant jerk

The method shown above works even when acceleration isn't constant. Let's apply it to a situation with an unusual name — constant jerk. (No lie, that's what it's called.) Jerk is the rate of change of acceleration with time.

$$j = \frac{da}{dt}$$

This makes jerk the first derivative of acceleration, the second derivative of velocity, and the third derivative of displacement.

$$j = \frac{da}{dt} = \frac{d^2v}{dt^2} = \frac{d^3s}{dt^3}$$

The SI unit of jerk is the meter per second cubed.

$$\left[\text{m/s}^3 = \frac{\text{m/s}^2}{\text{s}}\right]$$

An alternate unit is the g per second.

$$\left[\frac{\text{g}}{\text{s}} = \frac{9.80665\ \text{m/s}^2}{\text{s}} = 9.80665\ \text{m/s}^3\right]$$

Jerk is not just some wise ass physicists response to the question, "Oh yeah, so what do you call the third derivative of displacement?" Jerk is a meaningful quantity.

The human body comes equipped with sensors to sense acceleration and jerk. Located deep inside the ear, integrated into our skulls, lies a series of chambers called the labyrinth. Part of this labyrinth is dedicated to our sense of hearing (the cochlea) and part to our sense of balance (the vestibular system). The vestibular system comes equipped with sensors that detect angular acceleration (the semicircular canals) and sensors that detect linear acceleration (the otoliths). We have two otoliths in each ear — one for detecting acceleration in the horizontal plane (the utricle) and one for detecting acceleration in the vertical place (the saccule). Otoliths are our own built in accelerometers.

The word otolith comes from the Greek οτο (*oto*, ear) and λιθος (*lithos*, stone). Each of our four otoliths consists of a hard bone-like plate attached to a mat of sensory fibers. When the head accelerates, the plate shifts to one side, bending the sensory fibers. This sends a signal to the brain saying "we're accelerating." Since gravity also tugs on the plates, the signal may also mean "this way is down." The brain is quite good at figuring out the difference between the two interpretations. So good, that we tend to ignore it. (Sight, sound, smell, taste, touch — where's balance in this list?) We ignore it until something changes in an unusual, unexpected, or extreme way.

I've never been in orbit or lived on another planet. Gravity always pulls me down in the same way. Standing, walking, sitting, lying — it's all quite sedate. Now let's hop in a roller coaster or engage in a similarly thrilling activity like downhill skiing, Formula One racing, or cycling in Manhattan traffic. Acceleration is directed first one way, then another. You may even experience brief periods of weightlessness or inversion. These kinds of sensations generate intense mental activity, which is why we like doing them. They also sharpen us up and keep us focused during possibly life ending moments, which is why we evolved this sense in the first place. Your ability to sense jerk is vital to your health and well being. Jerk is both exciting and necessary.

Constant jerk is easy to deal with mathematically. As a learning exercise, let's derive the equations of motion for constant jerk. You are welcome to try more complicated jerk problems if you wish.

Jerk is the derivative of acceleration. Undo that process. Integrate jerk to get acceleration. That's the first time I've ever said that. I propose we call this the zeroeth equation of motion for constant jerk. The reason why will be apparent after we finish the next derivation.

$$j = \frac{da}{dt}$$

$$da = j\ dt$$

$$\int_{a_0}^{a} da = \int_{0}^{t} j\ dt$$

$$a - a_0 = jt$$

$$a = a_0 + jt \quad [0]$$

Acceleration is the derivative of velocity. Integrate acceleration to get velocity. We've done this process before. We called the result the velocity-time relationship or the first equation of motion when acceleration was constant. We should give it a similar name. This is the first equation of motion for constant jerk.

$$a = \frac{dv}{dt}$$

$$dv = a\ dt$$

$$dv = (a_0 + jt)\ dt$$

$$\int_{v_0}^{v} dv = \int_{0}^{t} (a_0 + jt)\ dt$$

$$v - v_0 = a_0t + ½jt^2$$

$$v = v_0 + a_0t + ½jt^2 \quad [1]$$

Velocity is the derivative of displacement. Integrate velocity to get displacement. We've done this before too. The resulting displacement-time relationship will be our second equation of motion for constant jerk.

$$v = \frac{ds}{dt}$$

$$ds = v\,dt$$

$$ds = (v_0 + a_0t + \tfrac{1}{2}jt^2)\,dt$$

$$\int_{s_0}^{s} ds = \int_0^t (v_0 + a_0t + \tfrac{1}{2}jt^2)\,dt$$

$$s - s_0 = v_0t + \tfrac{1}{2}a_0t^2 + \tfrac{1}{6}jt^3$$

$$s = s_0 + v_0t + \tfrac{1}{2}a_0t^2 + \tfrac{1}{6}jt^3 \quad [2]$$

Please notice something about these equations. When jerk is zero, they all revert back to the equations of motion for constant acceleration. Zero jerk means constant acceleration, so all is right with the world we've created. (I never said constant acceleration was realistic. Constant jerk is equally mythical. In hypertextbook world, however, all things are possible.)

Where do we go next? Should we try to remold the velocity-displacement relationship (the third equation of motion) in our new realm of constant jerk?

$$v = v_0 + a_0t + \tfrac{1}{2}jt^2 \quad [1]$$

+

$$s = s_0 + v_0t + \tfrac{1}{2}a_0t^2 + \tfrac{1}{6}jt^3 \quad [2]$$

=

$$v = f(s) \quad [3]$$

How about an acceleration-displacement relationship (the fourth equation of motion)?

$$a = a_0 + jt \quad [1]$$

+

$$s = s_0 + v_0t + \tfrac{1}{2}a_0t^2 + \tfrac{1}{6}jt^3 \quad [2]$$

=

$$a = f(s) \quad [4]$$

I don't even know if these can be worked out algebraically. I doubt it. Look at that scary cubic equation for displacement. That can't be our friend. At the moment, I can't be bothered. I don't know if working this out would tell me anything interesting. I do know I've never needed a third or fourth equation of motion for constant jerk — not yet. I leave this problem to the mathematicians of the world.

This is the kind of problem that distinguishes physicists from mathematicians. A mathematician wouldn't necessarily care about the physical significance and just might thank the physicist for an interesting challenge. A physicist wouldn't necessarily care about the answer unless it turned out to be useful, in which case the physicist would certainly thank the mathematician for being so curious.

constant nothing

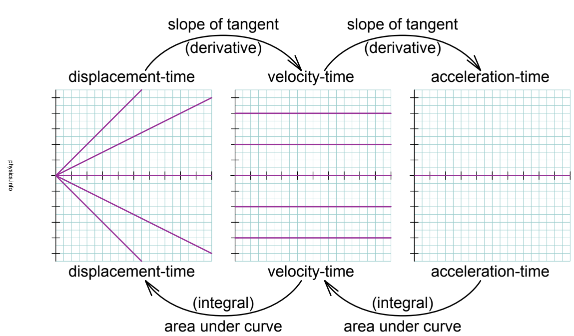
This page in this book isn't about motion with constant acceleration, or constant jerk, or constant snap, crackle or pop. It's about the general method for determining the quantities of motion (displacement, velocity, and acceleration) with respect to time and each other for any kind of motion. The procedure for doing so is either differentiation (finding the derivative)…

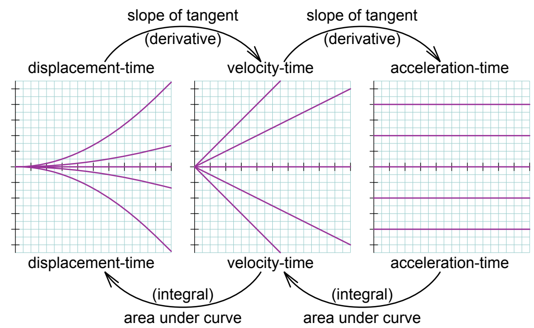
- The derivative of displacement with time is velocity ($v = ds/dt$).
- The derivative of velocity with time is acceleration ($a = dv/dt$).

or integration (finding the integral)…

- The integral of acceleration over time is velocity ($v = \int a\,dt$).
- The integral of velocity over time is displacement ($s = \int v\,dt$).

Here's the way it works. Some characteristic of the motion of an object is described by a function. Can you find the derivative of that function? That gives you another characteristic of the motion (or maybe it gives you something only a mathematician would love). Can you find its integral? That gives you a different characteristic (or maybe a load of nonsense that only a mathematician would love). Repeat either operation as many times as necessary (probably no more than twice). Then apply the techniques and concepts you learned in calculus and related branches of mathematics to extract more meaning — range, domain, limit, asymptote, minimum, maximum, extremum, concavity, inflection, analytical, numerical, exact, approximate, and so on. I've added some important notes on this to the summary for this topic.




slope of tangent
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slope of tangent
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displacement-time
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area under curve
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area under curve