# 支持向量机

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$$h_{\vec{w},b}(\vec{x}) = \mathrm{sign}(\vec{w} \cdot \vec{x} + b) \tag{0.1}$$

$$\hat{\gamma}^{(i)} = y^{(i)}(\vec{w} \cdot \vec{x}^{(i)} + b) \tag{0.2}$$

对 $\vec{w}$ 与 $b$ 缩放不会改变 $h_{\vec{w},b}(\vec{x})$ 的实际结果。

$$\hat{\gamma} = \underset{i=1,\ldots,n}{min} \frac{\hat{\gamma}^{(i)}}{\|\vec{w}\|} \tag{0.3}$$

$$\begin{aligned} max_{\vec{w},b} \quad & \gamma \\ s.t. \quad & y^{(i)}(\vec{w} \cdot \vec{x}^{(i)} + b) \geq \gamma, \quad i = 1, \ldots, n \end{aligned} \tag{0.4}$$

由于 $\vec{w}$ 与 $b$ 可以任意缩放，令 $\gamma = 1$。

$$\begin{aligned} min_{\vec{w},b} \quad & \frac{1}{2}\|\vec{w}\|^2 \\ s.t. \quad & y^{(i)}(\vec{w} \cdot \vec{x}^{(i)} + b) \geq 1, \quad i = 1, \ldots, n \end{aligned} \tag{0.5}$$

拉格朗日函数，其中 $\alpha_i >= 0$。

$$\mathscr{L}(\vec{w}, b, \vec{\alpha}) = \frac{1}{2}\|\vec{w}\|^2 - \sum_{n}^{i=1} \alpha_i \left[ y^{(i)}(\vec{w} \cdot \vec{x}^{(i)} + b) - 1 \right] \tag{0.6}$$

这里跳一步，直接转化为对偶问题：

$$\underset{\vec{\alpha}}{max} \, \underset{\vec{w},b}{min} \mathscr{L}(\vec{w}, b, \vec{\alpha}) \tag{0.7}$$

$$\nabla_{\vec{w}} \mathscr{L}(\vec{w}, b, \vec{\alpha}) = \vec{w} - \sum_{i=1}^{n} \alpha_i y^{(i)} \vec{x}^{(i)} = 0 \tag{0.8}$$

$$\frac{\partial}{\partial b} \mathscr{L}(\vec{w}, b, \vec{\alpha}) = \sum_{i=1}^{n} \alpha_i y^{(i)} = 0 \tag{0.9}$$

将等式0.8与等式0.9代入等式0.6得到：

$$\mathscr{L}(\vec{\alpha}) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} y^{(i)} y^{(j)} \alpha_i \alpha_j \langle \vec{x}^{(i)}, \vec{x}^{(j)} \rangle \tag{0.10}$$

$$\begin{aligned} max_{\vec{\alpha}} \quad & \mathscr{L}(\vec{\alpha}) = \sum_{i=1}^{n} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{n} y^{(i)} y^{(j)} \alpha_i \alpha_j \langle \vec{x}^{(i)}, \vec{x}^{(j)} \rangle \\ s.t. \quad & a_i \geq 0, i = 1, \ldots, n \\ & \sum_{i=1}^{n} \alpha_i y^{(i)} = 0 \end{aligned} \tag{0.11}$$

$$\begin{aligned} \vec{w}^* &= \sum_{i=1}^{n} \alpha_i y^{(i)} \vec{x}^{(i)} \\ b^* &= -\frac{1}{2} (max_{i:y^{(i)}=-1} \vec{w}^* \cdot x^{(i)} + min_{i:y^{(i)}=1} \vec{w}^* \cdot x^{(i)}) \end{aligned} \tag{0.12}$$

$$\begin{aligned} min_{\vec{w},b} \quad & \frac{1}{2} \|\vec{w}\|^2 + C \sum_{i=1}^{n} \xi_i \\ s.t. \quad & y^{(i)} (\vec{w} \cdot \vec{x}^{(i)} + b) \geq 1, \quad i = 1, \ldots, n \\ & \xi_i \geq 0, \quad i = 1, \ldots, n \end{aligned} \tag{0.13}$$

$$\mathscr{L}(\vec{w}, b, \vec{\xi}, \vec{\alpha}, \vec{r}) = \frac{1}{2} \|\vec{w}\|^2 + C \sum_{i=1}^{n} \xi_i - \sum_{n}^{i=1} \alpha_i \left[ y^{(i)} (\vec{w} \cdot \vec{x}^{(i)} + b) - 1 \right] - \sum_{i=1}^{n} r_i \xi_i \tag{0.14}$$