A simple approximation algorithm for WIS based on the approximability in k-partite graphs

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Abstract

In this note, simple approximation algorithms for the weighted independent set problem are presented with a performance ratio depending on $\Delta(G)$. These algorithms do not improve the best approximation algorithm known so far for this problem but they are of interest because of their simplicity. Precisely, we show how an optimum weighted independent set in bipartite graphs and a ρ-approximation of WIS in k-partite graphs respectively allows to obtain a $\frac{2}{\Delta(G)}$-approximation and a $\frac{k}{\Delta(G)}\rho$-approximation in general graphs. Note that the ratio $\frac{2}{\Delta(G)}$ is the best bound known for the particular cases $\Delta(G) = 3$ or $\Delta(G) = 4$.

Keywords: Graph algorithms; approximation algorithms; combinatorial optimization; Coloring; Weighted Independent Set; k-partite graphs.

1 Introduction

In the *Maximum Weighted Independent Set problem* (WIS, for short), we are given a simple, connected, undirected and loop-free graph $G = (V, E)$ on n vertices, with maximum degree $\Delta(G)$. Each vertex v in V has a positive weight $w(v) \geq 0$. For a subset $S \subseteq V$ of vertices, we denote by $w(S) = \sum_{v \in S} w(v)$ the sum of the weights of the elements in S. The goal of WIS is to find an independent set S (that is a subset of pairwise non-adjacent vertices) in G that maximizes $w(S)$. When the weight of each vertex is equal to one, this problem is usually called *Maximum Independent Set problem* (IS, for short).

WIS is known to be **NP-hard** in general graphs when $\Delta(G) \geq 3$, but also for certain classes of graphs (see Garey and Johnson [5]). On the other hand, for a variety of graphs from both practical and theoretical frameworks, this problem is polynomial, see Gondran and Minoux [6]. For instance, this is the case of bipartite graphs. Very recently Alekseev and Lozin, [1] provided a complete characterization of the (p, q)-colorable graphs for which WIS is polynomial. We recall that a graph is (p, q)-colorable if it can be partitioned into at most p cliques and q independent sets.

Because WIS is one of the most important problems in combinatorial optimization from both a practical and a theoretical point of view, many approximation results have been obtained by several authors (see, for instance Hochbaum [8], and Demange and Paschos [4]). Very recently,

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Sakai et al. [12] studied the behavior of several greedy strategies on WIS. In particular, they proved that one of these greedy algorithms is a $\frac{1}{\Delta(G)}$-approximation and that this ratio is tight. This algorithm selects, as long as the current graph G_i is not empty, a vertex v maximizing $\frac{w(v)}{d_{G_i}(v)+1}$ (where $d_{G_i}(v)$ is the degree of v in the current graph G_i), and then deletes v and its neighborhood from the current graph.

We shall show how a maximum weight independent set in bipartite graphs and a ρ-approximation of WIS in k-partite graphs respectively allows to obtain a $\frac{2}{\Delta(G)}$-approximation and a $\frac{k}{\Delta(G)}\rho$-approximation in general graphs. In order to build a k-partite graph from a given graph, we use the notion of coloring, that is a partition of the vertices into independent sets (see, Paschos [11], for a survey on the approximability of the coloring problem and de Werra and Hansen [3], for some relations between independent set and coloring). Hochbaum [8], has exploited this notion of coloring to obtain a $\frac{2}{\Delta(G)}$-approximation of WIS, but in a complete different way. The algorithm of [8] is based on a preprocessing due to Nemhauser and Trotter [10] which provides two disjoint subgraphs, with an independent set is one of them; then it computes on the other subgraph a coloring from which it selects the best independent set which is finally added to the first subgraph. Observe that, when $\Delta(G)$ is small (i.e., $\Delta(G) = 3, 4$), the approximation ratio $\frac{2}{\Delta(G)}$ is the best known result until today. More recently, Halldórsson and Lau [7], proposed an algorithm which consists in partitioning G into at most $\lceil\frac{\Delta(G)+1}{3}\rceil$ subgraphs G_i of degree at most 2; then, for each G_i, an optimum weighted independent set in G_i is computed and finally, the best of these solutions is returned. The performance ratio of this algorithm is $1/\lceil(\Delta(G) + 1)/3\rceil$.

Our algorithm computes, for every k-partite graph built on the coloring, an optimum or an approximate solution; then it returns the best one as a solution of the initial problem.

2 The Algorithm

`Algorithm 1`

1 Find a coloring $\mathcal{S} = (S_1, \ldots, S_\ell)$ by using a polynomial-time algorithm A;

2 For any $1 \leq i_1 < \ldots < i_k \leq \ell$ do

  2.1 Find an approximate independent set $S_{i_1,\ldots,i_k}$ of the k-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ using an approximation algorithm B;

3 Return $S = argmax\{w(S_{i_1,\ldots,i_k}) : \ 1 \leq i_1 < \ldots < i_k \leq \ell\}$;

This algorithm is polynomial as soon as the algorithm B runs in polynomial time (k being a constant not depending on the instance size). Remark that we cannot modify the step 2.1 by finding an optimal independent set $S_{i_1,\ldots,i_k}$ on the k-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ for each $i_1 < \ldots < i_k$ since such an algorithm does not run in polynomial time even for $k = 3$ (see the characterization of the complexity of WIS in (p, q)-colorable graphs in [1]).

Theorem 2.1 *If algorithm B is a ρ-approximation of* WIS *in k-partite graphs, then* `Algorithm 1` *is an $\frac{k}{\ell}\rho$-approximation for* WIS *in general graphs.*

Proof. Let $I = (G, w)$ be an instance of WIS and let S^* be an optimal solution with value $opt(I) = w(S^*)$. We set $S_i^* = S^* \cap S_i$ for $i \leq \ell$ where $\mathcal{S} = (S_1, \ldots, S_\ell)$ is the coloring provided by algorithm A. Let B be an algorithm which yields a ρ-approximation of WIS in k-partite graphs.

For any $i_1, \ldots, i_k$ with $i_k > \ldots > i_1$, the following key result holds:

$$w(S) \geq \rho \sum_{j=1}^{k} w(S_{i_j}^*) \tag{1}$$

In order to see that, just remark that $S_{i_1}^* \cup \ldots \cup S_{i_k}^*$ is an independent set in the k-partite graph induced by $S_{i_1} \cup \ldots \cup S_{i_k}$ (we denoted by G', this graph); since $S_{i_1,\ldots,i_k}$ is a ρ-approximation on G', we get $w(S_{i_1,\ldots,i_k}) \geq \rho\ opt(G') \geq \rho(w(S_{i_1}^*) + \ldots + w(S_{i_k}^*))$.

Summing up inequalities (1) for all $i_1, \ldots, i_k$ such that $1 \leq i_1 < \ldots < i_k \leq \ell$, we obtain:

$$\frac{\ell(\ell-1)\ldots(\ell-k+1)}{k(k-1)\ldots 2}\ w(S) \geq \rho \sum_{i=1}^{\ell} \frac{(\ell-1)\ldots(\ell-k+1)}{(k-1)\ldots 2}\ w(S_i^*) \tag{2}$$

Actually, when summing the inequalities (1), the term $w(S)$ appears exactly as many times as the number of choices of k elements among ℓ, and each $w(S_i^*)$ appears as many times as the number of choices of $k-1$ elements among $\ell - 1$. Finally, since $\sum_{i=1}^{\ell} w(S_i^*) = w(S^*)$, the result follows. ■

Since, we can always assume that each connected component of G is not a $K_{\Delta(G)+1}$ (WIS is polynomial on such a component), we can easily obtain a coloring using at most $\Delta(G)$ colors by using Brooks theorem, [2] and the constructive proof of Lovasz [9]. Moreover, there exists an exact algorithm for WIS in bipartite graphs, [1]. Thus, using Theorem 2.1 with $k = 2, \rho = 1$ and the inequality $\ell \leq \Delta(G)$, we deduce:

Corollary 2.2 `Algorithm 1` *is a $\frac{2}{\Delta(G)}$-approximation for* WIS.

This theorem becomes interesting when having some good bounds of the approximability of WIS in k-partite graphs. For instance, using Corollary 2.2, we obtain the bound $\rho = \frac{2}{3}$ for tripartite graphs; unfortunately, this does not allow to improve the bound $\frac{2}{\Delta(G)}$ (since $k = 3$ and $\rho = \frac{2}{3}$). Thus, one should improve the bound of $\frac{2}{3}$ in order to improve the best performance ratio of $\frac{2}{\Delta(G)}$.

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