Changing the "cut rule" to "remove after loosing the k-th son"

Problem 1. *Let us redefine the rule used by the Fibonacci heap algorithm for disconnecting a node from it's tree. We disconnect a node from it's tree when for the first time it looses it's k-th son. Prove that if k is a fixed number then every tree in the Fibonacci heap has degree bounded by $O(\log(n))$, where n is the size of the tree.*

Answer: We will prove that the size of the tree is exponentially large in the degree of the root. Let S_l denote the minimum size of a tree whose root's degree is l, when applying the new rule (notice that this S_l is different for different values of k). We are looking for a number $c > 1$ for which we can prove $S_l \geq c^l$ (for $k = 2$ it is proven in the book that c can be chosen to be ϕ).
First we generalize the lemma used in the book.

Lemma 1. *Let $y_1, y_2, ..., y_l$ be the sons of some root tree, then the degree of y_i which we denote by d_i is at least $i - k$.*

Proof. similar to the book (strait forward). □

We will leave the value of c to be fixed later, for now we rely on the fact that c will be a fixed constant greater than 1 (of course we will have to verify that it when we will fix c). Using the lemma we derive:

$$S_l \geq 1 + k + \sum_{i=k+1}^{l} S_i = 1 + k + \sum_{i=1}^{l-k} S_i$$

We will prove that $S_l \geq c^l$ by induction on l. We will now use the induction step:

$$S_l \geq 1 + k + \sum_{i=1}^{l-k} S_i \geq 1 + k + \sum_{i=1}^{l-k} c^i \geq 1 + k + \frac{c^{l-k+1} - c}{c - 1}$$

The last inequality uses the formula for the sum of a geometric finite sequence. To prove the induction step we have to show that:

$$1 + k + \frac{c^{l-k+1} - c}{c - 1} \geq c^l$$

(for every $l \geq 1$), but this depends on the value of c. Now we have to choose c in a clever way so that the inequality holds. It is enough to fix c so the following two inequalities holds (summing them will give the above inequality):

1. $1 + k - \frac{c}{c-1} \geq 0$
2. $\frac{c^{l-k+1}}{c-1} \geq c^l$ (or equivalently $\frac{c^{-k+1}}{c-1} \geq 1$)

We choose $c = 1 + 1/k$ so the first inequality holds as equality. Now we have to verify that the second equality is satisfied by our choice. After substituting $c = 1 + 1/k$ the second inequality becomes $c^{k-1} < k$.

$$c^{k-1} < c^k = (1 + 1/k)^k < e < 3 \leq k$$

(we assume $k \geq 3$ as the case $k = 2$ is handled in the book)

Remarks: notice that $c > 1$ as required. We need also to check the base of the induction $l = 1$, in this case $S_1 \geq 2 > c$.