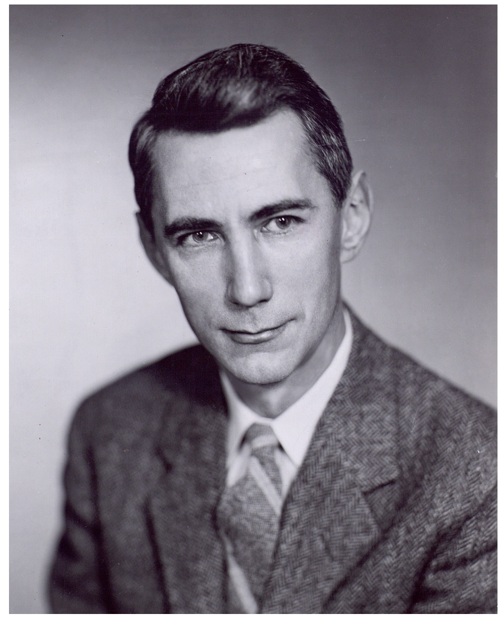
Chapter 33

Communicating without errors

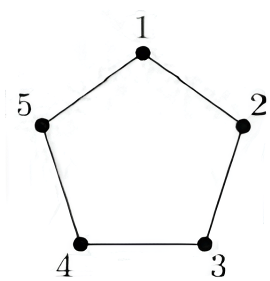
In 1956 Claude Shannon, the founder of information theory, posed the following very interesting question:

> *Suppose we want to transmit messages across a channel (where some symbols may be distorted) to a receiver: What is the maximum rate of transmission such that the receiver may recover the original message without errors?*



Claude Shannon

Let us see what Shannon meant by "channel" and "rate of transmission." We are given a set V of symbols, and a message is just a string of symbols from V. We model the channel as a graph $G = (V, E)$ where V is the set of symbols, and E the set of edges between unreliable pairs of symbols, that is, symbols which may be confused during transmission. For example, communicating over a phone in everyday language, we connnect the symbols B and P by an edge since the receiver may not be able to distinguish them. Let us call G the *confusion graph.* The 5-cycle C_5 will play a prominent role in our discussion. In this example, 1 and 2 may be confused, but not 1 and 3, etc. Ideally we would like to use all 5 symbols for transmission, but since we want to communicate error-free we can if we only send single symbols use only one letter from each pair that might be confused. Thus for the 5-cycle we can use only two different letters (any two that are not connected by an edge). In the language of information theory, this means that for the 5-cycle we achieve an information rate of $\log_2 2 = 1$ (instead of the maximal $\log_2 5 \approx 2.32$). It is clear that in this model, for an arbitrary graph $G = (V, E)$, the best we can do is to transmit symbols from a largest independent set. Thus the information rate, when sending single symbols, is $log_2\alpha(G)$, where $\alpha(G)$ is the *independence number* of G.



Let us see whether we can increase the information rate by using larger strings in place of single symbols. Suppose we want to transmit strings of length 2. The strings ulu2 and vlv, can only be confused if one of the following three cases holds:

- $u_1 = v_1$ and u_2 can be confused with v_2,
- $u_2 = v_2$ and u_1 can be confused with v_1, or
- $u_1 \neq v_1$ can be confused and $u_2 \neq v_2$ can be confused.

In graph-theoretic terms this amounts to considering the product $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$. $G_1 x G_2$ has the vertex set $V_1 \times V_2 = \{(u_1, u_2) : u_1 \in V_1, u_2 \in V_2\}$, with $(u_1, u_2) \neq (v_1, v_2)$ connected by an edge if and only if $u_i = v_i$ or $u_i v_i \in E$ for $i = 1, 2$. The confusion graph for strings of length 2 is thus $G^2 - G \times G$, the product of the confusion graph G for single symbols with itself. The information rate of strings of length 2 *per symbol* is then given by

$$\frac{\log_2 \alpha(G^2)}{2} = \log_2 \sqrt{\alpha(G^2)}.$$

Now, of course, we may use strings of any length n. The n-th confusion graph $G^n = G \times G \times \cdots \times G$ has vertex set $V^n = \{(u_1, \cdots, u_n) : u_i \in V\}$ with $(u_1, \cdots, u_n) \neq (v_1, \cdots, v_n)$ being connected by an edge if $u_i = v_i$ or $u_i v_i \in E$ for all i. The rate of information per symbol determined by strings of length n is

$$\frac{\log_2 \alpha(G^n)}{n} = \log_2 \sqrt[n]{\alpha(G^n)}.$$

What can we say about $\alpha(G^n)$? Here is a first observation. Let $U \subseteq V$ be a largest independent set in G, $|U| = \alpha$. The an vertices in G^n of the form $(u_1, \cdots, u_n)$, $u_i \in U$ for all i, clearly form an independent set in G^n. Hence

$$\alpha(G^n) \geq \alpha(G)^n$$

and therefore

$$\sqrt[n]{\alpha(G^n)} \geq \alpha(G),$$

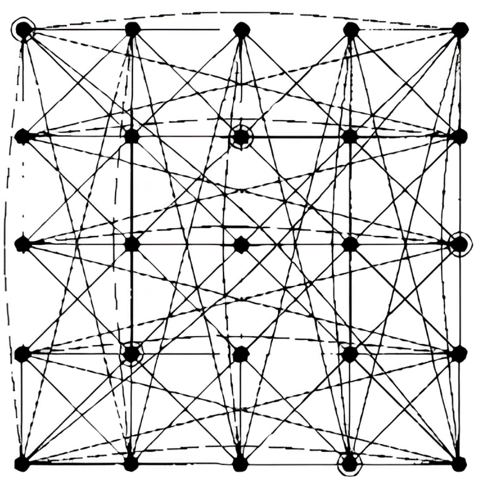
meaning that we never decrease the information rate by using longer strings instead of single symbols. This, by the way, is a basic idea of coding theory: By encoding symbols into longer strings we can make error-free communi- cation more efficient. Disregarding the logarithm we thus arrive at Shannon's fundamental definition: The *zero-error capacity* of a graph G is given by

$$\Theta(G) := \sup_{n \geq 1} \sqrt[n]{\alpha(G^n)},$$

and Shannon's problem was to compute $\Theta(G)$, and in particular $\Theta(C_5)$.

Let us look at C_5. So far we know $\alpha(C_5) = 2 \leq \Theta(C_5)$. Looking at the 5-cycle as depicted earlier, or at the product $C_5 x C_5$ as drawn on the left, we see that the set $\{(1,1), (2,3), (3,5), (4,2), (5,4)\}$ is independent in C_5^2. Thus we have $\alpha(C_5^2) \geq 5$. Since an independent set can contain only two vertices from any two consecutive rows we see that $\alpha(C_5^2) = 5$. Hence, by using strings of length 2 we have increased the lower bound for the capacity to $\Theta(C_5) \geq \sqrt{5}$.

So far we have no upper bounds for the capacity. To obtain such bounds we again follow Shannon's original ideas. First we need the dual definition of an independent set. We recall that a subset $C \subseteq V$ is a *clique* if any two vertices of C are joined by an edge. Thus the vertices form trivial



The graph $C_5 \times C_6$.

cliques of size 1, the edges are the cliques of size 2, the triangles are cliques of size 3, and so on. Let C be the set of cliques in G. Consider an arbitrary probability distribution $x = (x, : v \in V)$ on the set of vertices, that is, $x_v \geq 0$ and $\sum_{v \in V} x_v = 1$. To every distribution x we associate the "maximal value of a clique"

$$\lambda(x) = \max_{C \in \mathcal{C}} \sum_{v \in C} x_v.$$

and finally we set

$$\lambda(G) = \min_x \lambda(x) = \min_x \max_{C \in \mathcal{C}} \sum_{v \in C} x_v.$$

To be precise we should use inf instead of min, but the minimum exists because $\lambda(x)$ is continuous on the compact set of all distributions. Consider now an independent set $U \subseteq V$ of maximal size $\alpha(G) = \alpha$. Associated to U we define the distribution $x_U = (x_v : v \in V)$ by setting $x_v = \frac{1}{\alpha}$ if $v \in U$ and $x_v = 0$ otherwise. Since any clique contains at most one vertex from U, we infer $\lambda(x_U) = \frac{1}{\alpha}$, and thus by the definition of $\lambda(G)$

$$\lambda(G) \leq \frac{1}{\alpha(G)} \quad \text{or} \quad \alpha(G) \leq \lambda(G)^{-1}.$$

What Shannon observed is that $\lambda(G)^{-1}$ is, in fact, an upper bound for all $\sqrt[n]{\alpha(G^n)}$, and hence also for $\Theta(G)$. In order to prove this it suffices to show that for graphs G, H

$$\lambda(G \times H) = \lambda(G)\lambda(H) \tag{1}$$

holds, since this will imply $\lambda(G^n) = \lambda(G)^n$ and hence

$$\begin{aligned} \alpha(G^n) &\leq \lambda(G^n)^{-1} = \lambda(G)^{-n} \\ \sqrt[n]{\alpha(G^n)} &\leq \lambda(G)^{-1}. \end{aligned}$$

To prove (1) we make use of the duality theorem of linear programming (see [I]) and get

$$\lambda(G) = \min_x \max_{C \in \mathcal{C}} \sum_{v \in C} x_v = \max_y \min_{v \in V} \sum_{C \ni v} y_C, \tag{2}$$

where the right-hand side runs through all probability distributions $y = (y_C : C \in \mathcal{C})$ on $\mathcal{C}$.

Consider $G \times H$, and let x and x' be distributions which achieve the minima, $\lambda(x) = \lambda(G)$, $\lambda(x') = \lambda(H)$. In the vertex set of $G \times H$ we assign the value $z_{(u,v)} = x_u x'_v$ to the vertex (u,v). Since $\sum_{(u,v)} z_{(u,v)} = \sum_u x_u \sum_v x'_v, = 1$, we obtain a distribution. Next we observe that the maximal cliques in $G \times H$ are of the form $C \times D = \{(u,v) : u \in C, v \in D\}$ where C and D are cliques in G and H, respectively. Hence we obtain

$$\begin{aligned} \lambda(G \times H) \leq \lambda(z) &= \max_{C \times D} \sum_{(u,v) \in C \times D} z_(u,v) \\ &- \max_{C \times D} \sum_{u \in C} x_u \sum_{v \in D} x'_v = \lambda(G)\lambda(H) \end{aligned}$$