Definition.

A **group** is a set G with an operation $\cdot$ such that:

1. G is closed under the operation.
2. The operation is associative.
3. There is an identity element.
4. Every element has an inverse.

Corollary.

The identity element is unique in any group.

Remark.

This result follows from the definition of groups.

Example.

Consider the group $(Z, +)$, the set of integers under addition. The identity element is 0, and each element $z \in Z$ has an inverse $-z$.

Exercise 1.

Prove the following theorem about groups.

Theorem.

No group can be written down as the union of two of its proper subgroups.

Solution.

We provide a proof by contradiction.

Proof.

Assume two proper subgroups of group G, call them H_1 and H_2.
Assume $H_1 \cup H_2 = G$. We know that $e \in H_1 \cap H_2$, so $e \in H_1 \cup H_2$.
Let $h_1 \in G$ such that $h_1 \in H_1 \setminus H_2$.
Since $H_1 \neq H_2 \neq G \neq \emptyset$, there exists h_1 only in $H_1 \setminus H_2$. Similarly, $h_2 \in G$ exists such that $h_2 \in H_2 \setminus H_1$.
If $h_1 h_2 \in H_1$, then $h_2 \in H_1$, which contradicts our assumption. Therefore, $H_1 \cup H_2 \neq G$.