This paper is not to be removed from the Examination Halls

UNIVERSITY OF LONDON EC2020 ZB

BSc degrees and Diplomas for Graduates in Economics, Management, Finance and the Social Sciences, the Diplomas in Economics and Social Sciences

Elements of Econometrics

Thursday, 18 May 2017 : 14:30 to 17:30

Candidates should answer **EIGHT** of the following **TEN** questions: **ALL** of the questions in Section A (8 marks each) and **THREE** questions from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Extracts from statistical tables are given after the final question on this paper.

Graph paper is provided at the end of this question paper. If used, it must be detached and fastened securely inside the answer book.

A calculator may be used when answering questions on this paper and it must comply in all respects with the specification given with your Admission Notice. The make and type of machine must be clearly stated on the front cover of the answer book.

If more questions are answered than requested, only the first answers attempted will be counted.

PLEASE TURN OVER

SECTION A

Answer all questions from this section

1. Discuss the consequences of omitting relevant variables.

   In your answer consider the following empirical study attempting to estimate the social benefits of an increase in public spending on education. Using data from a cross-section survey of employees, the following regression is estimated:

   $$\log W_i = \alpha_1 + \alpha_2 S_i + \varepsilon_i,$$

   where W_i is the hourly wage rate and S_i is the number of years of schooling completed of employee i. The coefficient of S_i is found to be positive and strongly significant. What can be concluded from this? In your answer discuss the interpretation of α_2, and discuss whether we are likely to obtain an unbiased (consistent) estimator of it when using OLS. Rigour of your answer will be rewarded. **(8 marks)**

2. Suppose you are given a random sample $X_1, ..., X_n$ from the exponential distribution:

   $$f(x) = \lambda \exp(-\lambda x), \quad x > 0,\ \lambda > 0.$$

   According to this distribution $E(X_i) = 1/\lambda$ and $Var\,(X_i) = 1/\lambda^2$ for $i = 1, ..., n$.

   Show that the maximum likelihood estimator for λ is $1/\bar{X}$ where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Is the estimator unbiased and/or consistent? Prove your claims. **(8 marks)**

3. Consider the following non-stationary process:

   $$y_t = \gamma_0 + \gamma_1 t + u_t, \text{ with } u_t = \rho u_{t-1} + \varepsilon_t$$

   and ε_t i.i.d. $(0, \sigma^2)$. Indicate (with explanation) the source(s) of non-stationarity of y_t. Show that you can rewrite the model as:

   $$\Delta y_t = \beta_0 + \beta_1 t + \beta_2 y_{t-1} + \varepsilon_t.$$

   Clearly indicate the one-to-one relation between $(\gamma_0, \gamma_1, \rho)$ and $(\beta_0, \beta_1, \beta_2)$. What name do we give the non-stationary process y_t when $\rho = 1$? Briefly indicate how to test whether this is the case. **(8 marks)**

4. Explain the RESET test as a general test for functional form misspecification and discuss the drawbacks and advantages of this test.

   In your answer consider the following multiple linear regression model:

$$y_i = \gamma_1 + \gamma_2 x_{2i} + \gamma_3 x_{3i} + u_i \qquad\qquad i = 1, ..., n,$$

   where x_{2i} and x_{3i} are exogenous variables known to affect $E(y_i)$. **(8 marks)**

5. Consider the following regression model:

$$Y_i = \beta_0 + \beta_1 \frac{1}{X_i} + u_i, \qquad i = 1, ..., n.$$

   We assume that the errors $\{u_i\}_{i=1}^n$ are independent normal random variables with zero mean and variance σ^2/X_i^2. The regressor, $1/X_i,$ is non-stochastic with positive sample variability. You are interested in testing the hypothesis $H_0 : \beta_0 = 0$ against $H_1 : \beta_0 \neq 0$. You are advised to use the BLUE estimator of β_0 for this purpose.

   Discuss how you would obtain the BLUE estimator of β_0 (note, you are not asked to derive this estimator).

   Give two reasons why you would prefer using the BLUE estimator for β_0 instead of the OLS estimator $\hat{\beta}_{0,OLS} = \bar{Y} - \hat{\beta}_{1,OLS}\overline{(1/X)}$ when testing this hypothesis, where $\bar{Y} = \frac{1}{n}\sum Y_i$ and $\overline{(1/X)} = \frac{1}{n}\sum \frac{1}{X_i}$. **(8 marks)**

SECTION B

Answer three questions from this section.

6. Let us consider a model for the sale price of Monet paintings. The data we have contains the sale prices, widths, and heights of 430 Monet paintings, which sold at auction for prices ranging from \$10,000 to \$33 million. A linear regression provided the following results:

$$\widehat{\ln \text{Price}}_i = \underset{(0.612)}{-8.427} + \underset{(0.091)}{1.334} \ln \text{Area}_i - \underset{(0.128)}{0.165} \text{Aspect Ratio}_i$$
$$N = 430, R^2 = 0.336,$$

where Area=Width$\times$Height and Aspect Ratio=Height/Width. The standard errors are given in parentheses.

(a) Test the joint significance of the regression. Discuss its relation to the goodness of fit measure: R^2. **(5 marks)**

(b) You want to test the hypothesis that auction prices are inelastic with respect to area. Specifically, you are asked to test $H_0 : \beta_{\ln \text{Area}} = 1$ against $H_0 : \beta_{\ln \text{Area}} > 1$. Perform this test, clearly indicating the test statistic and the rejection rule. In view of your answer, indicate whether the p-value of the test is bigger or smaller than 5%.**(5 marks)**

(c) A friend points out that you should be worried about the presence of heteroskedasticity. Explain the concept of heteroskedasticity and discuss the properties of the OLS estimator in the presence of heteroskedasticity. **(5 marks)**

(d) Discuss how you should modify the test in (b) if it is known that the variance of the error term is given by

$$\sigma_i^2 = \exp\left(\gamma_0 + \gamma_1 \ln \text{Area}_i + \gamma_2 \text{Aspect Ratio}_i\right),$$

where $(\gamma_0, \gamma_1, \gamma_2)$ are unknown parameters. You are told to use the FGLS estimator. Explain this estimator clearly. **(5 marks)**

7. In this question we look at a large data set on weekly hours worked by women having at least two children. Consider the following specification for labour supply, estimated by OLS

$$\begin{aligned} \widehat{\textit{hours}} &= \underset{(6.503)}{-17.347} - \underset{(.116)}{2.242}\textit{kids} + \underset{(.035)}{.938}\,\textit{educ} + \underset{(.453)}{2.089}\textit{age} \underset{(.008)}{-.028}\,\textit{age}^2 - \underset{(.005)}{.075}\,\textit{nonmomi} \\ n &= 31,857 \text{ and } R^2 = .052, \end{aligned} \tag{7.1}$$

where *kids* is the total number of children, *educ* is the years of schooling, *age* is the woman's age in years, *nonmomi* is income from sources other than the mother's wage income, and *hours* is the hours worked per week. The standard errors are given in parentheses.

(a) Interpret the coefficient of *kids* and discuss its statistical significance. **(3 marks)**

(b) It is commonly thought that the decision to have more children is correlated with unobserved factors that affect labour supply. What name do we give such a problem in econometrics, and what consequences will this have for the properties of the OLS estimator? Support your answer using a simple model. **(5 marks)**

(c) Consider the variables *multi2nd* and *samesex*, which are binary variables indicating whether the second birth was for multiple babies and whether the first two children are of the same gender. What properties do we need to assume for us to be able to use *multi2nd* and *samesex* as instruments for *kids*? Are these assumptions reasonable and can we test (if so how) these requirements? **(6 marks)**

You may assume that *educ, age* and *nonmomi* can be treated as being exogenous.

(d) The following IV results were obtained in Stata:

$$\begin{aligned} \widehat{\textit{hours}} &= \underset{(7.020)}{-16.828} - \underset{(.134)}{2.504}\textit{kids} + \underset{(.115)}{.916}\,\textit{educ} + \underset{(.460)}{2.105}\textit{age} \underset{(.008)}{-.028}\,\textit{age}^2 - \underset{(.007)}{.076}\,\textit{nonmomi} \\ n &= 31,857 \text{ and } R^2 = .052. \end{aligned} \tag{7.2}$$

Discuss how these results can be obtained using Two Stage Least Squares (2SLS) and briefly discuss how you would test whether the results in (7.1) and (7.2) are significantly different from each other. **(6 marks)**

8. Let us consider an analysis on recidivism (probability of re-arrest) among a group of young men in California who have at least one arrest prior to 1986. The dependent variable, *arr86,* is equal to unity if the man was arrested at least once during 1986, and zero otherwise.

	OLS A	OLS B	Logit A	Logit B	Logit B Marginal Effect
pcnv	$-.152$ $(.021)$	$-.162$ $(.021)$	$-.880$ $(.122)$	$-.901$ $(.120)$	$-.176$ $(.023)$
avgsen	$.005$ $(.006)$	$.006$ $(.006)$	$.027$ $(.035)$	$.031$ $(.034)$	$.006$ $(.007)$
tottime	$-.003$ $(.005)$	$-.002$ $(.005)$	$-.014$ $(.028)$	$-.010$ $(.027)$	$-.002$ $(.005)$
ptime86	$-.023$ $(.005)$	$-.022$ $(.005)$	$-.140$ $(.031)$	$-.127$ $(.031)$	$-.025$ $(.006)$
qemp86	$-.038$ $(.005)$	$-.043$ $(.005)$	$-.199$ $(.028)$	$-.216$ $(.028)$	$-.042$ $(.005)$
black	$.170$ $(.024)$	–	$.823$ $(.117)$	–	–
hispan	$.096$ $(.021)$	–	$.522$ $(.109)$	–	–
constant	$.380$ $(.019)$	$.380$ $(.019)$	$-.464$ $(.095)$	$-.169$ $(.084)$	
R^2	$.068$	$.047$			
$\log L$			-1512.35	-1541.24	

pcnv is the proportion of prior arrests that led to a conviction, *avgsen* is the average sentence served from prior convictions, *tottime* is the months spent in prison since age 18 prior to 1986, *ptime86* is months spent in prison in 1986, *qemp86* is the number of quarters the man was legally employed in 1986, while *black* and *hispan* are two race dummies (*white* the excluded dummy). The standard errors are reported in parentheses.

(a) When estimating the parameters by OLS, we are using the Linear Probability Model. Why might you then report heteroskedasticity-robust standard errors? **(2 marks)**

(b) Using the OLS B results, what is the estimated effect on the probability of arrest if *pcnv* goes from 0.25 to 0.75 holding everything else constant? **(4 marks)**

(question continues on next page)

(c) It is argued that the linear probability model is not appropriate for explaining the binary variable $arr86$ and a logit regression model has been estimated. Explain how the Logit estimates are obtained. **(5 marks)**

Hint: You may recall, that for the Logit model A, we will specify:

$$\Pr(arr86_i = 1) = \Lambda\left(\beta_0 + \beta_1 \textit{pcnv} + \beta_2 \textit{avgsen} + ... + \beta_6 \textit{black} + \beta_7 \textit{hispan}\right),$$

where $\Lambda(z) = \frac{1}{1+\exp(-z)}$.

(d) Using the Logit model results, discuss whether *black* and *hispan* are jointly significant. Clearly indicate the null and alternative hypothesis, the test statistic and the rejection rule. **(3 marks)**

(e) An important distinction between the two approaches is that the marginal effect of *pcnv* on the probability of re-arrest is constant for the LPM unlike the marginal effect using the logit analysis. What this means for instance is that the estimated effect on the probability of arrest if *pcnv* goes from 0.25 to 0.75 will depend on the other characteristics.

i. Explain how the marginal effects evaluated at the mean values of the explanatory variables (reported in the last column) were obtained. Give a brief comment as to how they compare to the marginal effect of the associated LPM. **(3 marks)**

ii. Using the Logit B results, how would you obtain the estimated effect on the probability of arrest if *pcnv* goes from 0.25 to 0.75 for a *white* man, with characteristics *avgsen=1, tottime=1, ptime86=0 and qemp86=2.* A clear explanation of what calculations are required is sufficient. **(3 marks)**

9. Let us consider a distributed lag model:

$$y_t = \beta_0 + \delta_0 z_t + \delta_1 z_{t-1} + ... + \delta_q z_{t-q} + \beta_1 x_t + u_t\,, \qquad t = 1, ..., T,$$

where u_t is independent of x_t, z_t, z_{t-1}, ..., and z_{t-q} with zero mean and constant variance. For simplicity, we will assume that there is no autocorrelation in the errors.

(a) Explain the concept of autocorrelation and indicate **for the above model** what the consequence of autocorrelation for our OLS estimator would be. **(4 marks)**

Let us consider the example of the effects of tax policy on the U.S. fertility rates. Let *gfr* denote the number of children born per 1,000 women aged 15-44, pe denotes the real value of the personal tax exemption, and $ww2$ and $pill$ are dummy variables (WW II, availability of the birth control pill). Using annual data, the following OLS results were obtained:

$$\begin{aligned}\widehat{gfr}_t \;=\; & \underset{(3.30)}{92.50} + \underset{(.126)}{.089} pe_t - \underset{(.153)}{.004} pe_{t-1} + \underset{(.165)}{.007} pe_{t-2} + \underset{(.154)}{.018} pe_{t-3} + \underset{(.105)}{.014} pe_{t-4} \\ & - \underset{(11.03)}{21.48} ww2_t - \underset{(3.94)}{31.25} pill_t, \qquad R^2 = .537, T = 68.\end{aligned} \tag{9.1}$$

The usual standard errors are reported in parentheses.

(b) A novice in econometrics argues that the result indicates that the effect of tax policy on the US fertility rate is ineffective because of the insignificance of the distributed lag coefficients (the coefficients on pe_t and its lags). Explain why he/she could be wrong. In your answer you may discuss what the short and long run effect of the tax policy are according to these results **(5 marks)**

(c) Your friend argues that only two lags should have been included. Using the same sample, he obtains the following result:

$$\begin{aligned}\widehat{gfr}_t \;=\; & \underset{(3.27)}{92.52} + \underset{(.119)}{.101} pe_t - \underset{(.147)}{.011} pe_{t-1} + \underset{(.119)}{.033} pe_{t-2} \\ & - \underset{(10.13)}{22.95} ww2_t - \underset{(3.76)}{30.83} pill_t, \qquad R^2 = .536, T = 68.\end{aligned} \tag{9.2}$$

Test for the joint significance of the third and fourth lag. Clearly indicate H_0 and H_1, the test statistic, the rejection rule and interpret your results. **(4 marks)**

(d) Your result in (c) may be affected by the presence of autocorrelation. Discuss how you would conduct the Breusch-Godfrey test for the presence of autocorrelation in (9.2). Clearly indicate H_0 and H_1, the test statistic, the rejection rule and interpret your results. Briefly indicate what you might want to do to try and remove the autocorrelation. **(7 marks)**

10. Consider the model:

$$y_t = \alpha_1 y_{t-1} + u_t,\ t = 1, ..., T,$$

where $y_0 = 0$. $E(u_t) = 0$, $E(u_t^2) = \sigma^2$ and $E(u_t u_s) = 0$ when $s \neq t$, for all $s, t = 1, ..., T$

(a) Discuss what we mean by the concept of stationarity (more precisely "covariance stationarity") and indicate under what condition $\{y_t\}_{t=1}^{T}$ will be stationary. **(3 marks)**

(b) Discuss the Dickey-Fuller procedure used to test for the presence of a unit root in the above model. Clearly indicate the null and alternative hypothesis, test statistic and rejection rule. **(5 marks)**

(c) Consider a slight variation of the above model:

$$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + v_t,\ t = 1, ..., T,$$

where $y_0 = 0$. $E(v_t) = 0$, $E(v_t^2) = \sigma^2$ and $E(v_t v_s) = 0$ when $s \neq t$, for all $s, t = 1, ..., T$ What do we call such a process? Discuss what problem you will have when conducting your test as in (b). **(3 marks)**

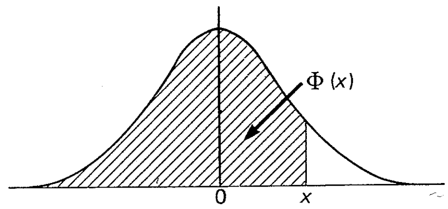
(d) Instead of conducting the Dickey-Fuller procedure, you are told to apply the Augmented DF test. Indicate how you would conduct the test for the presence of a unit root here. Derivation of the test equation will be required for full marks. **(5 marks)**

(e) What are the potential problems associated with performing a regression with $I(1)$ variables? In your answer explain what it means for a variable to be $I(1)$. **(4 marks)**

END OF PAPER

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

The function tabulated is $\Phi(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,dt$. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean and unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with zero mean and unit variance is symmetric about zero.



x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·00	0·5000	0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·97725
·01	·5040	·41	·6591	·81	·7910	·21	·8869	·61	·9463	·01	·97778
·02	·5080	·42	·6628	·82	·7939	·22	·8888	·62	·9474	·02	·97831
·03	·5120	·43	·6664	·83	·7967	·23	·8907	·63	·9484	·03	·97882
·04	·5160	·44	·6700	·84	·7995	·24	·8925	·64	·9495	·04	·97932
0·05	0·5199	0·45	0·6736	0·85	0·8023	1·25	0·8944	1·65	0·9505	2·05	0·97982
·06	·5239	·46	·6772	·86	·8051	·26	·8962	·66	·9515	·06	·98030
·07	·5279	·47	·6808	·87	·8078	·27	·8980	·67	·9525	·07	·98077
·08	·5319	·48	·6844	·88	·8106	·28	·8997	·68	·9535	·08	·98124
·09	·5359	·49	·6879	·89	·8133	·29	·9015	·69	·9545	·09	·98169
0·10	0·5398	0·50	0·6915	0·90	0·8159	1·30	0·9032	1·70	0·9554	2·10	0·98214
·11	·5438	·51	·6950	·91	·8186	·31	·9049	·71	·9564	·11	·98257
·12	·5478	·52	·6985	·92	·8212	·32	·9066	·72	·9573	·12	·98300
·13	·5517	·53	·7019	·93	·8238	·33	·9082	·73	·9582	·13	·98341
·14	·5557	·54	·7054	·94	·8264	·34	·9099	·74	·9591	·14	·98382
0·15	0·5596	0·55	0·7088	0·95	0·8289	1·35	0·9115	1·75	0·9599	2·15	0·98422
·16	·5636	·56	·7123	·96	·8315	·36	·9131	·76	·9608	·16	·98461
·17	·5675	·57	·7157	·97	·8340	·37	·9147	·77	·9616	·17	·98500
·18	·5714	·58	·7190	·98	·8365	·38	·9162	·78	·9625	·18	·98537
·19	·5753	·59	·7224	·99	·8389	·39	·9177	·79	·9633	·19	·98574
0·20	0·5793	0·60	0·7257	1·00	0·8413	1·40	0·9192	1·80	0·9641	2·20	0·98610
·21	·5832	·61	·7291	·01	·8438	·41	·9207	·81	·9649	·21	·98645
·22	·5871	·62	·7324	·02	·8461	·42	·9222	·82	·9656	·22	·98679
·23	·5910	·63	·7357	·03	·8485	·43	·9236	·83	·9664	·23	·98713
·24	·5948	·64	·7389	·04	·8508	·44	·9251	·84	·9671	·24	·98745
0·25	0·5987	0·65	0·7422	1·05	0·8531	1·45	0·9265	1·85	0·9678	2·25	0·98778
·26	·6026	·66	·7454	·06	·8554	·46	·9279	·86	·9686	·26	·98809
·27	·6064	·67	·7486	·07	·8577	·47	·9292	·87	·9693	·27	·98840
·28	·6103	·68	·7517	·08	·8599	·48	·9306	·88	·9699	·28	·98870
·29	·6141	·69	·7549	·09	·8621	·49	·9319	·89	·9706	·29	·98899
0·30	0·6179	0·70	0·7580	1·10	0·8643	1·50	0·9332	1·90	0·9713	2·30	0·98928
·31	·6217	·71	·7611	·11	·8665	·51	·9345	·91	·9719	·31	·98956
·32	·6255	·72	·7642	·12	·8686	·52	·9357	·92	·9726	·32	·98983
·33	·6293	·73	·7673	·13	·8708	·53	·9370	·93	·9732	·33	·99010
·34	·6331	·74	·7704	·14	·8729	·54	·9382	·94	·9738	·34	·99036
0·35	0·6368	0·75	0·7734	1·15	0·8749	1·55	0·9394	1·95	0·9744	2·35	0·99061
·36	·6406	·76	·7764	·16	·8770	·56	·9406	·96	·9750	·36	·99086
·37	·6443	·77	·7794	·17	·8790	·57	·9418	·97	·9756	·37	·99111
·38	·6480	·78	·7823	·18	·8810	·58	·9429	·98	·9761	·38	·99134
·39	·6517	·79	·7852	·19	·8830	·59	·9441	·99	·9767	·39	·99158
0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·9772	2·40	0·99180

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
2·40	0·99180	2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918
·41	·99202	·56	·99477	·71	·99664	·86	·99788	·01	·99869	·16	·99921
·42	·99224	·57	·99492	·72	·99674	·87	·99795	·02	·99874	·17	·99924
·43	·99245	·58	·99506	·73	·99683	·88	·99801	·03	·99878	·18	·99926
·44	·99266	·59	·99520	·74	·99693	·89	·99807	·04	·99882	·19	·99929
2·45	0·99286	2·60	0·99534	2·75	0·99702	2·90	0·99813	3·05	0·99886	3·20	0·99931
·46	·99305	·61	·99547	·76	·99711	·91	·99819	·06	·99889	·21	·99934
·47	·99324	·62	·99560	·77	·99720	·92	·99825	·07	·99893	·22	·99936
·48	·99343	·63	·99573	·78	·99728	·93	·99831	·08	·99896	·23	·99938
·49	·99361	·64	·99585	·79	·99736	·94	·99836	·09	·99900	·24	·99940
2·50	0·99379	2·65	0·99598	2·80	0·99744	2·95	0·99841	3·10	0·99903	3·25	0·99942
·51	·99396	·66	·99609	·81	·99752	·96	·99846	·11	·99906	·26	·99944
·52	·99413	·67	·99621	·82	·99760	·97	·99851	·12	·99910	·27	·99946
·53	·99430	·68	·99632	·83	·99767	·98	·99856	·13	·99913	·28	·99948
·54	·99446	·69	·99643	·84	·99774	·99	·99861	·14	·99916	·29	·99950
2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918	3·30	0·99952

The critical table below gives on the left the range of values of x for which $\Phi(x)$ takes the value on the right, correct to the last figure given; in critical cases, take the upper of the two values of $\Phi(x)$ indicated.

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
			0·9994		0·99990		0·99995
3·075		3·263		3·731		3·916	
	0·9990		0·9995		0·99991		0·99996
3·105		3·320		3·759		3·976	
	0·9991		0·9996		0·99992		0·99997
3·138		3·389		3·791		4·055	
	0·9992		0·9997		0·99993		0·99998
3·174		3·480		3·826		4·173	
	0·9993		0·9998		0·99994		0·99999
3·215		3·615		3·867		4·417	
	0·9994		0·9999		0·99995		1·00000

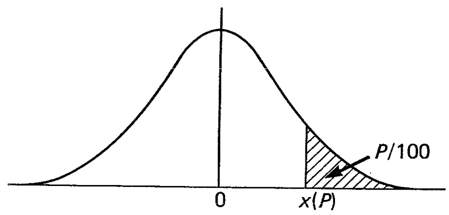
When $x > 3{\cdot}3$ the formula $1-\Phi(x) \doteqdot \dfrac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}\left[1-\dfrac{1}{x^2}+\dfrac{3}{x^4}-\dfrac{15}{x^6}+\dfrac{105}{x^8}\right]$ is very accurate, with relative error less than $945/x^{10}$.

TABLE 5. PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

This table gives percentage points $x(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{2\pi}}\int_{x(P)}^{\infty} e^{-\frac{1}{2}t^2}\,dt.$$

If X is a variable, normally distributed with zero mean and unit variance, $P/100$ is the probability that $X \geqslant x(P)$. The lower P per cent points are given by symmetry as $-x(P)$, and the probability that $|X| \geqslant x(P)$ is $2P/100$.



P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$
50	0·0000	5·0	1·6449	3·0	1·8808	2·0	2·0537	1·0	2·3263	0·10	3·0902
45	0·1257	4·8	1·6646	2·9	1·8957	1·9	2·0749	0·9	2·3656	0·09	3·1214
40	0·2533	4·6	1·6849	2·8	1·9110	1·8	2·0969	0·8	2·4089	0·08	3·1559
35	0·3853	4·4	1·7060	2·7	1·9268	1·7	2·1201	0·7	2·4573	0·07	3·1947
30	0·5244	4·2	1·7279	2·6	1·9431	1·6	2·1444	0·6	2·5121	0·06	3·2389
25	0·6745	4·0	1·7507	2·5	1·9600	1·5	2·1701	0·5	2·5758	0·05	3·2905
20	0·8416	3·8	1·7744	2·4	1·9774	1·4	2·1973	0·4	2·6521	0·01	3·7190
15	1·0364	3·6	1·7991	2·3	1·9954	1·3	2·2262	0·3	2·7478	0·005	3·8906
10	1·2816	3·4	1·8250	2·2	2·0141	1·2	2·2571	0·2	2·8782	0·001	4·2649
5	1·6449	3·2	1·8522	2·1	2·0335	1·1	2·2904	0·1	3·0902	0·0005	4·4172

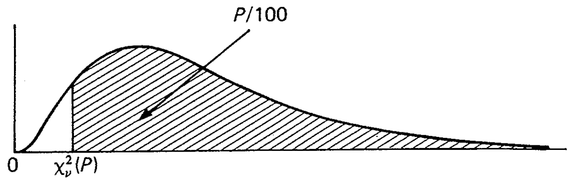
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \int_{\chi^2_\nu(P)}^{\infty} x^{\frac{1}{2}\nu-1}\, e^{-\frac{1}{2}x}\, dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu-1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

P	99·95	99·9	99·5	99	97·5	95	90	80	70	60
$\nu = 1$	$0{\cdot}0^{6}3927$	$0{\cdot}0^{5}1571$	$0{\cdot}0^{4}3927$	$0{\cdot}0^{3}1571$	$0{\cdot}0^{3}9821$	0·003932	0·01579	0·06418	0·1485	0·2750
2	0·001000	0·002001	0·01003	0·02010	0·05064	0·1026	0·2107	0·4463	0·7133	1·022
3	0·01528	0·02430	0·07172	0·1148	0·2158	0·3518	0·5844	1·005	1·424	1·869
4	0·06392	0·09080	0·2070	0·2971	0·4844	0·7107	1·064	1·649	2·195	2·753
5	0·1581	0·2102	0·4117	0·5543	0·8312	1·145	1·610	2·343	3·000	3·655
6	0·2994	0·3811	0·6757	0·8721	1·237	1·635	2·204	3·070	3·828	4·570
7	0·4849	0·5985	0·9893	1·239	1·690	2·167	2·833	3·822	4·671	5·493
8	0·7104	0·8571	1·344	1·646	2·180	2·733	3·490	4·594	5·527	6·423
9	0·9717	1·152	1·735	2·088	2·700	3·325	4·168	5·380	6·393	7·357
10	1·265	1·479	2·156	2·558	3·247	3·940	4·865	6·179	7·267	8·295
11	1·587	1·834	2·603	3·053	3·816	4·575	5·578	6·989	8·148	9·237
12	1·934	2·214	3·074	3·571	4·404	5·226	6·304	7·807	9·034	10·18
13	2·305	2·617	3·565	4·107	5·009	5·892	7·042	8·634	9·926	11·13
14	2·697	3·041	4·075	4·660	5·629	6·571	7·790	9·467	10·82	12·08
15	3·108	3·483	4·601	5·229	6·262	7·261	8·547	10·31	11·72	13·03
16	3·536	3·942	5·142	5·812	6·908	7·962	9·312	11·15	12·62	13·98
17	3·980	4·416	5·697	6·408	7·564	8·672	10·09	12·00	13·53	14·94
18	4·439	4·905	6·265	7·015	8·231	9·390	10·86	12·86	14·44	15·89
19	4·912	5·407	6·844	7·633	8·907	10·12	11·65	13·72	15·35	16·85
20	5·398	5·921	7·434	8·260	9·591	10·85	12·44	14·58	16·27	17·81
21	5·896	6·447	8·034	8·897	10·28	11·59	13·24	15·44	17·18	18·77
22	6·404	6·983	8·643	9·542	10·98	12·34	14·04	16·31	18·10	19·73
23	6·924	7·529	9·260	10·20	11·69	13·09	14·85	17·19	19·02	20·69
24	7·453	8·085	9·886	10·86	12·40	13·85	15·66	18·06	19·94	21·65
25	7·991	8·649	10·52	11·52	13·12	14·61	16·47	18·94	20·87	22·62
26	8·538	9·222	11·16	12·20	13·84	15·38	17·29	19·82	21·79	23·58
27	9·093	9·803	11·81	12·88	14·57	16·15	18·11	20·70	22·72	24·54
28	9·656	10·39	12·46	13·56	15·31	16·93	18·94	21·59	23·65	25·51
29	10·23	10·99	13·12	14·26	16·05	17·71	19·77	22·48	24·58	26·48
30	10·80	11·59	13·79	14·95	16·79	18·49	20·60	23·36	25·51	27·44
32	11·98	12·81	15·13	16·36	18·29	20·07	22·27	25·15	27·37	29·38
34	13·18	14·06	16·50	17·79	19·81	21·66	23·95	26·94	29·24	31·31
36	14·40	15·32	17·89	19·23	21·34	23·27	25·64	28·73	31·12	33·25
38	15·64	16·61	19·29	20·69	22·88	24·88	27·34	30·54	32·99	35·19
40	16·91	17·92	20·71	22·16	24·43	26·51	29·05	32·34	34·87	37·13
50	23·46	24·67	27·99	29·71	32·36	34·76	37·69	41·45	44·31	46·86
60	30·34	31·74	35·53	37·48	40·48	43·19	46·46	50·64	53·81	56·62
70	37·47	39·04	43·28	45·44	48·76	51·74	55·33	59·90	63·35	66·40
80	44·79	46·52	51·17	53·54	57·15	60·39	64·28	69·21	72·92	76·19
90	52·28	54·16	59·20	61·75	65·65	69·13	73·29	78·56	82·51	85·99
100	59·90	61·92	67·33	70·06	74·22	77·93	82·36	87·95	92·13	95·81

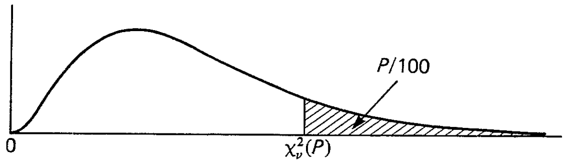
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})}\int_{\chi^2_\nu(P)}^{\infty} x^{\frac{1}{2}\nu-1}\,e^{-\frac{1}{2}x}\,dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu-1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

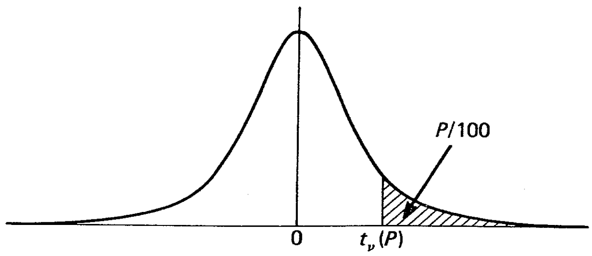
P	50	40	30	20	10	5	2·5	1	0·5	0·1	0·05
$\nu =$ 1	0·4549	0·7083	1·074	1·642	2·706	3·841	5·024	6·635	7·879	10·83	12·12
2	1·386	1·833	2·408	3·219	4·605	5·991	7·378	9·210	10·60	13·82	15·20
3	2·366	2·946	3·665	4·642	6·251	7·815	9·348	11·34	12·84	16·27	17·73
4	3·357	4·045	4·878	5·989	7·779	9·488	11·14	13·28	14·86	18·47	20·00
5	4·351	5·132	6·064	7·289	9·236	11·07	12·83	15·09	16·75	20·52	22·11
6	5·348	6·211	7·231	8·558	10·64	12·59	14·45	16·81	18·55	22·46	24·10
7	6·346	7·283	8·383	9·803	12·02	14·07	16·01	18·48	20·28	24·32	26·02
8	7·344	8·351	9·524	11·03	13·36	15·51	17·53	20·09	21·95	26·12	27·87
9	8·343	9·414	10·66	12·24	14·68	16·92	19·02	21·67	23·59	27·88	29·67
10	9·342	10·47	11·78	13·44	15·99	18·31	20·48	23·21	25·19	29·59	31·42
11	10·34	11·53	12·90	14·63	17·28	19·68	21·92	24·72	26·76	31·26	33·14
12	11·34	12·58	14·01	15·81	18·55	21·03	23·34	26·22	28·30	32·91	34·82
13	12·34	13·64	15·12	16·98	19·81	22·36	24·74	27·69	29·82	34·53	36·48
14	13·34	14·69	16·22	18·15	21·06	23·68	26·12	29·14	31·32	36·12	38·11
15	14·34	15·73	17·32	19·31	22·31	25·00	27·49	30·58	32·80	37·70	39·72
16	15·34	16·78	18·42	20·47	23·54	26·30	28·85	32·00	34·27	39·25	41·31
17	16·34	17·82	19·51	21·61	24·77	27·59	30·19	33·41	35·72	40·79	42·88
18	17·34	18·87	20·60	22·76	25·99	28·87	31·53	34·81	37·16	42·31	44·43
19	18·34	19·91	21·69	23·90	27·20	30·14	32·85	36·19	38·58	43·82	45·97
20	19·34	20·95	22·77	25·04	28·41	31·41	34·17	37·57	40·00	45·31	47·50
21	20·34	21·99	23·86	26·17	29·62	32·67	35·48	38·93	41·40	46·80	49·01
22	21·34	23·03	24·94	27·30	30·81	33·92	36·78	40·29	42·80	48·27	50·51
23	22·34	24·07	26·02	28·43	32·01	35·17	38·08	41·64	44·18	49·73	52·00
24	23·34	25·11	27·10	29·55	33·20	36·42	39·36	42·98	45·56	51·18	53·48
25	24·34	26·14	28·17	30·68	34·38	37·65	40·65	44·31	46·93	52·62	54·95
26	25·34	27·18	29·25	31·79	35·56	38·89	41·92	45·64	48·29	54·05	56·41
27	26·34	28·21	30·32	32·91	36·74	40·11	43·19	46·96	49·64	55·48	57·86
28	27·34	29·25	31·39	34·03	37·92	41·34	44·46	48·28	50·99	56·89	59·30
29	28·34	30·28	32·46	35·14	39·09	42·56	45·72	49·59	52·34	58·30	60·73
30	29·34	31·32	33·53	36·25	40·26	43·77	46·98	50·89	53·67	59·70	62·16
32	31·34	33·38	35·66	38·47	42·58	46·19	49·48	53·49	56·33	62·49	65·00
34	33·34	35·44	37·80	40·68	44·90	48·60	51·97	56·06	58·96	65·25	67·80
36	35·34	37·50	39·92	42·88	47·21	51·00	54·44	58·62	61·58	67·99	70·59
38	37·34	39·56	42·05	45·08	49·51	53·38	56·90	61·16	64·18	70·70	73·35
40	39·34	41·62	44·16	47·27	51·81	55·76	59·34	63·69	66·77	73·40	76·09
50	49·33	51·89	54·72	58·16	63·17	67·50	71·42	76·15	79·49	86·66	89·56
60	59·33	62·13	65·23	68·97	74·40	79·08	83·30	88·38	91·95	99·61	102·7
70	69·33	72·36	75·69	79·71	85·53	90·53	95·02	100·4	104·2	112·3	115·6
80	79·33	82·57	86·12	90·41	96·58	101·9	106·6	112·3	116·3	124·8	128·3
90	89·33	92·76	96·52	101·1	107·6	113·1	118·1	124·1	128·3	137·2	140·8
100	99·33	102·9	106·9	111·7	118·5	124·3	129·6	135·8	140·2	149·4	153·2

TABLE 10. PERCENTAGE POINTS OF THE t-DISTRIBUTION

This table gives percentage points $t_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2})}{\Gamma(\frac{1}{2}\nu)} \int_{t_\nu(P)}^{\infty} \frac{dt}{(1 + t^2/\nu)^{\frac{1}{2}(\nu+1)}}.$$

Let X_1 and X_2 be independent random variables having a normal distribution with zero mean and unit variance and a χ^2-distribution with ν degrees of freedom respectively; then $t = X_1/\sqrt{X_2/\nu}$ has Student's t-distribution with ν degrees of freedom, and the probability that $t \geqslant t_\nu(P)$ is $P/100$. The lower percentage points are given by symmetry as $-t_\nu(P)$, and the probability that $|t| \geqslant t_\nu(P)$ is $2P/100$.



The limiting distribution of t as ν tends to infinity is the normal distribution with zero mean and unit variance. When ν is large interpolation in ν should be harmonic.

P	40	30	25	20	15	10	5	2·5	1	0·5	0·1	0·05
$\nu = 1$	0·3249	0·7265	1·0000	1·3764	1·963	3·078	6·314	12·71	31·82	63·66	318·3	636·6
2	0·2887	0·6172	0·8165	1·0607	1·386	1·886	2·920	4·303	6·965	9·925	22·33	31·60
3	0·2767	0·5844	0·7649	0·9785	1·250	1·638	2·353	3·182	4·541	5·841	10·21	12·92
4	0·2707	0·5686	0·7407	0·9410	1·190	1·533	2·132	2·776	3·747	4·604	7·173	8·610
5	0·2672	0·5594	0·7267	0·9195	1·156	1·476	2·015	2·571	3·365	4·032	5·893	6·869
6	0·2648	0·5534	0·7176	0·9057	1·134	1·440	1·943	2·447	3·143	3·707	5·208	5·959
7	0·2632	0·5491	0·7111	0·8960	1·119	1·415	1·895	2·365	2·998	3·499	4·785	5·408
8	0·2619	0·5459	0·7064	0·8889	1·108	1·397	1·860	2·306	2·896	3·355	4·501	5·041
9	0·2610	0·5435	0·7027	0·8834	1·100	1·383	1·833	2·262	2·821	3·250	4·297	4·781
10	0·2602	0·5415	0·6998	0·8791	1·093	1·372	1·812	2·228	2·764	3·169	4·144	4·587
11	0·2596	0·5399	0·6974	0·8755	1·088	1·363	1·796	2·201	2·718	3·106	4·025	4·437
12	0·2590	0·5386	0·6955	0·8726	1·083	1·356	1·782	2·179	2·681	3·055	3·930	4·318
13	0·2586	0·5375	0·6938	0·8702	1·079	1·350	1·771	2·160	2·650	3·012	3·852	4·221
14	0·2582	0·5366	0·6924	0·8681	1·076	1·345	1·761	2·145	2·624	2·977	3·787	4·140
15	0·2579	0·5357	0·6912	0·8662	1·074	1·341	1·753	2·131	2·602	2·947	3·733	4·073
16	0·2576	0·5350	0·6901	0·8647	1·071	1·337	1·746	2·120	2·583	2·921	3·686	4·015
17	0·2573	0·5344	0·6892	0·8633	1·069	1·333	1·740	2·110	2·567	2·898	3·646	3·965
18	0·2571	0·5338	0·6884	0·8620	1·067	1·330	1·734	2·101	2·552	2·878	3·610	3·922
19	0·2569	0·5333	0·6876	0·8610	1·066	1·328	1·729	2·093	2·539	2·861	3·579	3·883
20	0·2567	0·5329	0·6870	0·8600	1·064	1·325	1·725	2·086	2·528	2·845	3·552	3·850
21	0·2566	0·5325	0·6864	0·8591	1·063	1·323	1·721	2·080	2·518	2·831	3·527	3·819
22	0·2564	0·5321	0·6858	0·8583	1·061	1·321	1·717	2·074	2·508	2·819	3·505	3·792
23	0·2563	0·5317	0·6853	0·8575	1·060	1·319	1·714	2·069	2·500	2·807	3·485	3·768
24	0·2562	0·5314	0·6848	0·8569	1·059	1·318	1·711	2·064	2·492	2·797	3·467	3·745
25	0·2561	0·5312	0·6844	0·8562	1·058	1·316	1·708	2·060	2·485	2·787	3·450	3·725
26	0·2560	0·5309	0·6840	0·8557	1·058	1·315	1·706	2·056	2·479	2·779	3·435	3·707
27	0·2559	0·5306	0·6837	0·8551	1·057	1·314	1·703	2·052	2·473	2·771	3·421	3·690
28	0·2558	0·5304	0·6834	0·8546	1·056	1·313	1·701	2·048	2·467	2·763	3·408	3·674
29	0·2557	0·5302	0·6830	0·8542	1·055	1·311	1·699	2·045	2·462	2·756	3·396	3·659
30	0·2556	0·5300	0·6828	0·8538	1·055	1·310	1·697	2·042	2·457	2·750	3·385	3·646
32	0·2555	0·5297	0·6822	0·8530	1·054	1·309	1·694	2·037	2·449	2·738	3·365	3·622
34	0·2553	0·5294	0·6818	0·8523	1·052	1·307	1·691	2·032	2·441	2·728	3·348	3·601
36	0·2552	0·5291	0·6814	0·8517	1·052	1·306	1·688	2·028	2·434	2·719	3·333	3·582
38	0·2551	0·5288	0·6810	0·8512	1·051	1·304	1·686	2·024	2·429	2·712	3·319	3·566
40	0·2550	0·5286	0·6807	0·8507	1·050	1·303	1·684	2·021	2·423	2·704	3·307	3·551
50	0·2547	0·5278	0·6794	0·8489	1·047	1·299	1·676	2·009	2·403	2·678	3·261	3·496
60	0·2545	0·5272	0·6786	0·8477	1·045	1·296	1·671	2·000	2·390	2·660	3·232	3·460
120	0·2539	0·5258	0·6765	0·8446	1·041	1·289	1·658	1·980	2·358	2·617	3·160	3·373
∞	0·2533	0·5244	0·6745	0·8416	1·036	1·282	1·645	1·960	2·326	2·576	3·090	3·291

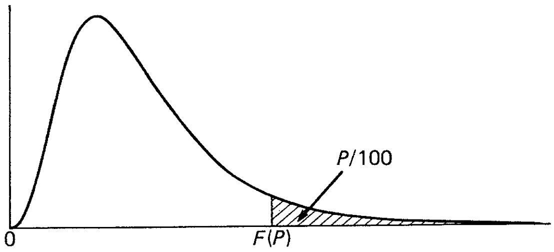
TABLE 12(*a*). 10 PER CENT POINTS OF THE *F*-DISTRIBUTION

The function tabulated is $F(P) = F(P|\nu_1, \nu_2)$ defined by the equation

$$\frac{P}{100} = \frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)}{\Gamma(\frac{1}{2}\nu_1)\,\Gamma(\frac{1}{2}\nu_2)}\,\nu_1^{\frac{1}{2}\nu_1}\,\nu_2^{\frac{1}{2}\nu_2}\int_{F(P)}^{\infty}\frac{F^{\frac{1}{2}\nu_1-1}}{(\nu_2+\nu_1 F)^{\frac{1}{2}(\nu_1+\nu_2)}}\,dF,$$

for $P = 10$, 5, 2.5, 1, 0.5 and 0.1. The lower percentage points, that is the values $F'(P) = F'(P|\nu_1, \nu_2)$ such that the probability that $F \leqslant F'(P)$ is equal to $P/100$, may be found by the formula

$$F'(P|\nu_1, \nu_2) = 1/F(P|\nu_2, \nu_1).$$

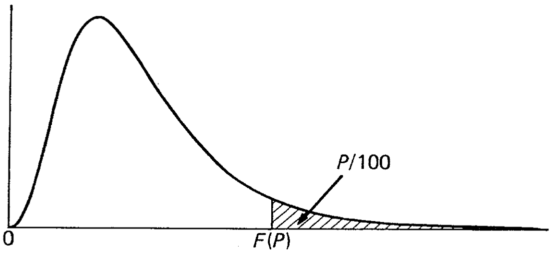


(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	60.19	60.71	62.00	63.33
2	8.526	9.000	9.162	9.243	9.293	9.326	9.349	9.367	9.392	9.408	9.450	9.491
3	5.538	5.462	5.391	5.343	5.309	5.285	5.266	5.252	5.230	5.216	5.176	5.134
4	4.545	4.325	4.191	4.107	4.051	4.010	3.979	3.955	3.920	3.896	3.831	3.761
5	4.060	3.780	3.619	3.520	3.453	3.405	3.368	3.339	3.297	3.268	3.191	3.105
6	3.776	3.463	3.289	3.181	3.108	3.055	3.014	2.983	2.937	2.905	2.818	2.722
7	3.589	3.257	3.074	2.961	2.883	2.827	2.785	2.752	2.703	2.668	2.575	2.471
8	3.458	3.113	2.924	2.806	2.726	2.668	2.624	2.589	2.538	2.502	2.404	2.293
9	3.360	3.006	2.813	2.693	2.611	2.551	2.505	2.469	2.416	2.379	2.277	2.159
10	3.285	2.924	2.728	2.605	2.522	2.461	2.414	2.377	2.323	2.284	2.178	2.055
11	3.225	2.860	2.660	2.536	2.451	2.389	2.342	2.304	2.248	2.209	2.100	1.972
12	3.177	2.807	2.606	2.480	2.394	2.331	2.283	2.245	2.188	2.147	2.036	1.904
13	3.136	2.763	2.560	2.434	2.347	2.283	2.234	2.195	2.138	2.097	1.983	1.846
14	3.102	2.726	2.522	2.395	2.307	2.243	2.193	2.154	2.095	2.054	1.938	1.797
15	3.073	2.695	2.490	2.361	2.273	2.208	2.158	2.119	2.059	2.017	1.899	1.755
16	3.048	2.668	2.462	2.333	2.244	2.178	2.128	2.088	2.028	1.985	1.866	1.718
17	3.026	2.645	2.437	2.308	2.218	2.152	2.102	2.061	2.001	1.958	1.836	1.686
18	3.007	2.624	2.416	2.286	2.196	2.130	2.079	2.038	1.977	1.933	1.810	1.657
19	2.990	2.606	2.397	2.266	2.176	2.109	2.058	2.017	1.956	1.912	1.787	1.631
20	2.975	2.589	2.380	2.249	2.158	2.091	2.040	1.999	1.937	1.892	1.767	1.607
21	2.961	2.575	2.365	2.233	2.142	2.075	2.023	1.982	1.920	1.875	1.748	1.586
22	2.949	2.561	2.351	2.219	2.128	2.060	2.008	1.967	1.904	1.859	1.731	1.567
23	2.937	2.549	2.339	2.207	2.115	2.047	1.995	1.953	1.890	1.845	1.716	1.549
24	2.927	2.538	2.327	2.195	2.103	2.035	1.983	1.941	1.877	1.832	1.702	1.533
25	2.918	2.528	2.317	2.184	2.092	2.024	1.971	1.929	1.866	1.820	1.689	1.518
26	2.909	2.519	2.307	2.174	2.082	2.014	1.961	1.919	1.855	1.809	1.677	1.504
27	2.901	2.511	2.299	2.165	2.073	2.005	1.952	1.909	1.845	1.799	1.666	1.491
28	2.894	2.503	2.291	2.157	2.064	1.996	1.943	1.900	1.836	1.790	1.656	1.478
29	2.887	2.495	2.283	2.149	2.057	1.988	1.935	1.892	1.827	1.781	1.647	1.467
30	2.881	2.489	2.276	2.142	2.049	1.980	1.927	1.884	1.819	1.773	1.638	1.456
32	2.869	2.477	2.263	2.129	2.036	1.967	1.913	1.870	1.805	1.758	1.622	1.437
34	2.859	2.466	2.252	2.118	2.024	1.955	1.901	1.858	1.793	1.745	1.608	1.419
36	2.850	2.456	2.243	2.108	2.014	1.945	1.891	1.847	1.781	1.734	1.595	1.404
38	2.842	2.448	2.234	2.099	2.005	1.935	1.881	1.838	1.772	1.724	1.584	1.390
40	2.835	2.440	2.226	2.091	1.997	1.927	1.873	1.829	1.763	1.715	1.574	1.377
60	2.791	2.393	2.177	2.041	1.946	1.875	1.819	1.775	1.707	1.657	1.511	1.291
120	2.748	2.347	2.130	1.992	1.896	1.824	1.767	1.722	1.652	1.601	1.447	1.193
∞	2.706	2.303	2.084	1.945	1.847	1.774	1.717	1.670	1.599	1.546	1.383	1.000

TABLE 12(*b*). 5 PER CENT POINTS OF THE *F*-DISTRIBUTION

If $F = \dfrac{X_1}{\nu_1}\Big/\dfrac{X_2}{\nu_2}$, where X_1 and X_2 are independent random variables distributed as χ^2 with ν_1 and ν_2 degrees of freedom respectively, then the probabilities that $F \geqslant F(P)$ and that $F \leqslant F'(P)$ are both equal to $P/100$. Linear interpolation in ν_1 and ν_2 will generally be sufficiently accurate except when either $\nu_1 > 12$ or $\nu_2 > 40$, when harmonic interpolation should be used.

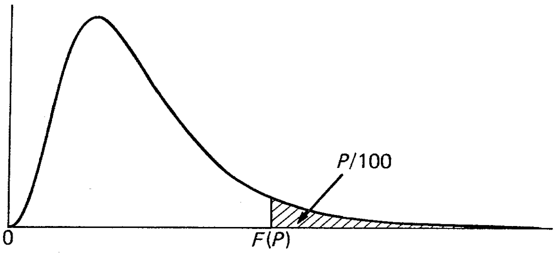


(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	161·4	199·5	215·7	224·6	230·2	234·0	236·8	238·9	241·9	243·9	249·1	254·3
2	18·51	19·00	19·16	19·25	19·30	19·33	19·35	19·37	19·40	19·41	19·45	19·50
3	10·13	9·552	9·277	9·117	9·013	8·941	8·887	8·845	8·786	8·745	8·639	8·526
4	7·709	6·944	6·591	6·388	6·256	6·163	6·094	6·041	5·964	5·912	5·774	5·628
5	6·608	5·786	5·409	5·192	5·050	4·950	4·876	4·818	4·735	4·678	4·527	4·365
6	5·987	5·143	4·757	4·534	4·387	4·284	4·207	4·147	4·060	4·000	3·841	3·669
7	5·591	4·737	4·347	4·120	3·972	3·866	3·787	3·726	3·637	3·575	3·410	3·230
8	5·318	4·459	4·066	3·838	3·687	3·581	3·500	3·438	3·347	3·284	3·115	2·928
9	5·117	4·256	3·863	3·633	3·482	3·374	3·293	3·230	3·137	3·073	2·900	2·707
10	4·965	4·103	3·708	3·478	3·326	3·217	3·135	3·072	2·978	2·913	2·737	2·538
11	4·844	3·982	3·587	3·357	3·204	3·095	3·012	2·948	2·854	2·788	2·609	2·404
12	4·747	3·885	3·490	3·259	3·106	2·996	2·913	2·849	2·753	2·687	2·505	2·296
13	4·667	3·806	3·411	3·179	3·025	2·915	2·832	2·767	2·671	2·604	2·420	2·206
14	4·600	3·739	3·344	3·112	2·958	2·848	2·764	2·699	2·602	2·534	2·349	2·131
15	4·543	3·682	3·287	3·056	2·901	2·790	2·707	2·641	2·544	2·475	2·288	2·066
16	4·494	3·634	3·239	3·007	2·852	2·741	2·657	2·591	2·494	2·425	2·235	2·010
17	4·451	3·592	3·197	2·965	2·810	2·699	2·614	2·548	2·450	2·381	2·190	1·960
18	4·414	3·555	3·160	2·928	2·773	2·661	2·577	2·510	2·412	2·342	2·150	1·917
19	4·381	3·522	3·127	2·895	2·740	2·628	2·544	2·477	2·378	2·308	2·114	1·878
20	4·351	3·493	3·098	2·866	2·711	2·599	2·514	2·447	2·348	2·278	2·082	1·843
21	4·325	3·467	3·072	2·840	2·685	2·573	2·488	2·420	2·321	2·250	2·054	1·812
22	4·301	3·443	3·049	2·817	2·661	2·549	2·464	2·397	2·297	2·226	2·028	1·783
23	4·279	3·422	3·028	2·796	2·640	2·528	2·442	2·375	2·275	2·204	2·005	1·757
24	4·260	3·403	3·009	2·776	2·621	2·508	2·423	2·355	2·255	2·183	1·984	1·733
25	4·242	3·385	2·991	2·759	2·603	2·490	2·405	2·337	2·236	2·165	1·964	1·711
26	4·225	3·369	2·975	2·743	2·587	2·474	2·388	2·321	2·220	2·148	1·946	1·691
27	4·210	3·354	2·960	2·728	2·572	2·459	2·373	2·305	2·204	2·132	1·930	1·672
28	4·196	3·340	2·947	2·714	2·558	2·445	2·359	2·291	2·190	2·118	1·915	1·654
29	4·183	3·328	2·934	2·701	2·545	2·432	2·346	2·278	2·177	2·104	1·901	1·638
30	4·171	3·316	2·922	2·690	2·534	2·421	2·334	2·266	2·165	2·092	1·887	1·622
32	4·149	3·295	2·901	2·668	2·512	2·399	2·313	2·244	2·142	2·070	1·864	1·594
34	4·130	3·276	2·883	2·650	2·494	2·380	2·294	2·225	2·123	2·050	1·843	1·569
36	4·113	3·259	2·866	2·634	2·477	2·364	2·277	2·209	2·106	2·033	1·824	1·547
38	4·098	3·245	2·852	2·619	2·463	2·349	2·262	2·194	2·091	2·017	1·808	1·527
40	4·085	3·232	2·839	2·606	2·449	2·336	2·249	2·180	2·077	2·003	1·793	1·509
60	4·001	3·150	2·758	2·525	2·368	2·254	2·167	2·097	1·993	1·917	1·700	1·389
120	3·920	3·072	2·680	2·447	2·290	2·175	2·087	2·016	1·910	1·834	1·608	1·254
∞	3·841	2·996	2·605	2·372	2·214	2·099	2·010	1·938	1·831	1·752	1·517	1·000

TABLE 12(d). 1 PER CENT POINTS OF THE F-DISTRIBUTION

If $F = \dfrac{X_1}{\nu_1}\Big/\dfrac{X_2}{\nu_2}$, where X_1 and X_2 are independent random variables distributed as χ^2 with ν_1 and ν_2 degrees of freedom respectively, then the probabilities that $F \geqslant F(P)$ and that $F \leqslant F'(P)$ are both equal to $P/100$. Linear interpolation in ν_1 or ν_2 will generally be sufficiently accurate except when either $\nu_1 > 12$ or $\nu_2 > 40$, when harmonic interpolation should be used.



(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	4052	4999	5403	5625	5764	5859	5928	5981	6056	6106	6235	6366
2	98·50	99·00	99·17	99·25	99·30	99·33	99·36	99·37	99·40	99·42	99·46	99·50
3	34·12	30·82	29·46	28·71	28·24	27·91	27·67	27·49	27·23	27·05	26·60	26·13
4	21·20	18·00	16·69	15·98	15·52	15·21	14·98	14·80	14·55	14·37	13·93	13·46
5	16·26	13·27	12·06	11·39	10·97	10·67	10·46	10·29	10·05	9·888	9·466	9·020
6	13·75	10·92	9·780	9·148	8·746	8·466	8·260	8·102	7·874	7·718	7·313	6·880
7	12·25	9·547	8·451	7·847	7·460	7·191	6·993	6·840	6·620	6·469	6·074	5·650
8	11·26	8·649	7·591	7·006	6·632	6·371	6·178	6·029	5·814	5·667	5·279	4·859
9	10·56	8·022	6·992	6·422	6·057	5·802	5·613	5·467	5·257	5·111	4·729	4·311
10	10·04	7·559	6·552	5·994	5·636	5·386	5·200	5·057	4·849	4·706	4·327	3·909
11	9·646	7·206	6·217	5·668	5·316	5·069	4·886	4·744	4·539	4·397	4·021	3·602
12	9·330	6·927	5·953	5·412	5·064	4·821	4·640	4·499	4·296	4·155	3·780	3·361
13	9·074	6·701	5·739	5·205	4·862	4·620	4·441	4·302	4·100	3·960	3·587	3·165
14	8·862	6·515	5·564	5·035	4·695	4·456	4·278	4·140	3·939	3·800	3·427	3·004
15	8·683	6·359	5·417	4·893	4·556	4·318	4·142	4·004	3·805	3·666	3·294	2·868
16	8·531	6·226	5·292	4·773	4·437	4·202	4·026	3·890	3·691	3·553	3·181	2·753
17	8·400	6·112	5·185	4·669	4·336	4·102	3·927	3·791	3·593	3·455	3·084	2·653
18	8·285	6·013	5·092	4·579	4·248	4·015	3·841	3·705	3·508	3·371	2·999	2·566
19	8·185	5·926	5·010	4·500	4·171	3·939	3·765	3·631	3·434	3·297	2·925	2·489
20	8·096	5·849	4·938	4·431	4·103	3·871	3·699	3·564	3·368	3·231	2·859	2·421
21	8·017	5·780	4·874	4·369	4·042	3·812	3·640	3·506	3·310	3·173	2·801	2·360
22	7·945	5·719	4·817	4·313	3·988	3·758	3·587	3·453	3·258	3·121	2·749	2·305
23	7·881	5·664	4·765	4·264	3·939	3·710	3·539	3·406	3·211	3·074	2·702	2·256
24	7·823	5·614	4·718	4·218	3·895	3·667	3·496	3·363	3·168	3·032	2·659	2·211
25	7·770	5·568	4·675	4·177	3·855	3·627	3·457	3·324	3·129	2·993	2·620	2·169
26	7·721	5·526	4·637	4·140	3·818	3·591	3·421	3·288	3·094	2·958	2·585	2·131
27	7·677	5·488	4·601	4·106	3·785	3·558	3·388	3·256	3·062	2·926	2·552	2·097
28	7·636	5·453	4·568	4·074	3·754	3·528	3·358	3·226	3·032	2·896	2·522	2·064
29	7·598	5·420	4·538	4·045	3·725	3·499	3·330	3·198	3·005	2·868	2·495	2·034
30	7·562	5·390	4·510	4·018	3·699	3·473	3·304	3·173	2·979	2·843	2·469	2·006
32	7·499	5·336	4·459	3·969	3·652	3·427	3·258	3·127	2·934	2·798	2·423	1·956
34	7·444	5·289	4·416	3·927	3·611	3·386	3·218	3·087	2·894	2·758	2·383	1·911
36	7·396	5·248	4·377	3·890	3·574	3·351	3·183	3·052	2·859	2·723	2·347	1·872
38	7·353	5·211	4·343	3·858	3·542	3·319	3·152	3·021	2·828	2·692	2·316	1·837
40	7·314	5·179	4·313	3·828	3·514	3·291	3·124	2·993	2·801	2·665	2·288	1·805
60	7·077	4·977	4·126	3·649	3·339	3·119	2·953	2·823	2·632	2·496	2·115	1·601
120	6·851	4·787	3·949	3·480	3·174	2·956	2·792	2·663	2·472	2·336	1·950	1·381
∞	6·635	4·605	3·782	3·319	3·017	2·802	2·639	2·511	2·321	2·185	1·791	1·000

Durbin–Watson test statistic d : 1% significance points of d_L and d_U.

n	$k'=1$ d_L	$k'=1$ d_U	$k'=2$ d_L	$k'=2$ d_U	$k'=3$ d_L	$k'=3$ d_U	$k'=4$ d_L	$k'=4$ d_U	$k'=5$ d_L	$k'=5$ d_U
15	0.81	1.07	0.70	1.25	0.59	1.46	0.49	1.70	0.39	1.96
16	0.84	1.09	0.74	1.25	0.63	1.44	0.53	1.66	0.44	1.90
17	0.87	1.10	0.77	1.25	0.67	1.43	0.57	1.63	0.48	1.85
18	0.90	1.12	0.80	1.26	0.71	1.42	0.61	1.60	0.52	1.80
19	0.93	1.13	0.83	1.26	0.74	1.41	0.65	1.58	0.56	1.77
20	0.95	1.15	0.86	1.27	0.77	1.41	0.68	1.57	0.60	1.74
21	0.97	1.16	0.89	1.27	0.80	1.41	0.72	1.55	0.63	1.71
22	1.00	1.17	0.91	1.28	0.83	1.40	0.75	1.54	0.66	1.69
23	1.02	1.19	0.94	1.29	0.86	1.40	0.77	1.53	0.70	1.67
24	1.04	1.20	0.96	1.30	0.88	1.41	0.80	1.53	0.72	1.66
25	1.05	1.21	0.98	1.30	0.90	1.41	0.83	1.52	0.75	1.65
26	1.07	1.22	1.00	1.31	0.93	1.41	0.85	1.52	0.78	1.64
27	1.09	1.23	1.02	1.32	0.95	1.41	0.88	1.51	0.81	1.63
28	1.10	1.24	1.04	1.32	0.97	1.41	0.90	1.51	0.83	1.62
29	1.12	1.25	1.05	1.33	0.99	1.42	0.92	1.51	0.85	1.61
30	1.13	1.26	1.07	1.34	1.01	1.42	0.94	1.51	0.88	1.61
31	1.15	1.27	1.08	1.34	1.02	1.42	0.96	1.51	0.90	1.60
32	1.16	1.28	1.10	1.35	1.04	1.43	0.98	1.51	0.92	1.60
33	1.17	1.29	1.11	1.36	1.05	1.43	1.00	1.51	0.94	1.59
34	1.18	1.30	1.13	1.36	1.07	1.43	1.01	1.51	0.95	1.59
35	1.19	1.31	1.14	1.37	1.08	1.44	1.03	1.51	0.97	1.59
36	1.21	1.32	1.15	1.38	1.10	1.44	1.04	1.51	0.99	1.59
37	1.22	1.32	1.16	1.38	1.11	1.45	1.06	1.51	1.00	1.59
38	1.23	1.33	1.18	1.39	1.12	1.45	1.07	1.52	1.02	1.58
39	1.24	1.34	1.19	1.39	1.14	1.45	1.09	1.52	1.03	1.58
40	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
45	1.29	1.38	1.24	1.42	1.20	1.48	1.16	1.53	1.11	1.58
50	1.32	1.40	1.28	1.45	1.24	1.49	1.20	1.54	1.16	1.59
55	1.36	1.43	1.32	1.47	1.28	1.51	1.25	1.55	1.21	1.59
60	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
65	1.41	1.47	1.38	1.50	1.35	1.53	1.31	1.57	1.28	1.61
70	1.43	1.49	1.40	1.52	1.37	1.55	1.34	1.58	1.31	1.61
75	1.45	1.50	1.42	1.53	1.39	1.56	1.37	1.59	1.34	1.62
80	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
85	1.48	1.53	1.46	1.55	1.43	1.58	1.41	1.60	1.39	1.63
90	1.50	1.54	1.47	1.56	1.45	1.59	1.43	1.61	1.41	1.64
95	1.51	1.55	1.49	1.57	1.47	1.60	1.45	1.62	1.42	1.64
100	1.52	1.56	1.50	1.58	1.48	1.60	1.46	1.63	1.44	1.65

n = number of observations

k' = number of explanatory variables

Durbin–Watson test statistic d : 5% significance points of d_L and d_U.

n	$k'=1$ d_L	$k'=1$ d_U	$k'=2$ d_L	$k'=2$ d_U	$k'=3$ d_L	$k'=3$ d_U	$k'=4$ d_L	$k'=4$ d_U	$k'=5$ d_L	$k'=5$ d_U
15	1.08	1.36	0.95	1.54	0.82	1.75	0.69	1.97	0.56	2.21
16	1.10	1.37	0.98	1.54	0.86	1.73	0.74	1.93	0.62	2.15
17	1.13	1.38	1.02	1.54	0.90	1.71	0.78	1.90	0.67	2.10
18	1.16	1.39	1.05	1.53	0.93	1.69	0.82	1.87	0.71	2.06
19	1.18	1.40	1.08	1.53	0.97	1.68	0.86	1.85	0.75	2.02
20	1.20	1.41	1.10	1.54	1.00	1.68	0.90	1.83	0.79	1.99
21	1.22	1.42	1.13	1.54	1.03	1.67	0.93	1.81	0.83	1.96
22	1.24	1.43	1.15	1.54	1.05	1.66	0.96	1.80	0.86	1.94
23	1.26	1.44	1.17	1.54	1.08	1.66	0.99	1.79	0.90	1.92
24	1.27	1.45	1.19	1.55	1.10	1.66	1.01	1.78	0.93	1.90
25	1.29	1.45	1.21	1.55	1.12	1.66	1.04	1.77	0.95	1.89
26	1.30	1.46	1.22	1.55	1.14	1.65	1.06	1.76	0.98	1.88
27	1.32	1.47	1.24	1.56	1.16	1.65	1.08	1.76	1.01	1.86
28	1.33	1.48	1.26	1.56	1.18	1.65	1.10	1.75	1.03	1.85
29	1.34	1.48	1.27	1.56	1.20	1.65	1.12	1.74	1.05	1.84
30	1.35	1.49	1.28	1.57	1.21	1.65	1.14	1.74	1.07	1.83
31	1.36	1.50	1.30	1.57	1.23	1.65	1.16	1.74	1.09	1.83
32	1.37	1.50	1.31	1.57	1.24	1.65	1.18	1.73	1.11	1.82
33	1.38	1.51	1.32	1.58	1.26	1.65	1.19	1.73	1.13	1.81
34	1.39	1.51	1.33	1.58	1.27	1.65	1.21	1.73	1.15	1.81
35	1.40	1.52	1.34	1.58	1.28	1.65	1.22	1.73	1.16	1.80
36	1.41	1.52	1.35	1.59	1.29	1.65	1.24	1.73	1.18	1.80
37	1.42	1.53	1.36	1.59	1.31	1.66	1.25	1.72	1.19	1.80
38	1.43	1.54	1.37	1.59	1.32	1.66	1.26	1.72	1.21	1.79
39	1.43	1.54	1.38	1.60	1.33	1.66	1.27	1.72	1.22	1.79
40	1.44	1.54	1.39	1.60	1.34	1.66	1.29	1.72	1.23	1.79
45	1.48	1.57	1.43	1.62	1.38	1.67	1.34	1.72	1.29	1.78
50	1.50	1.59	1.46	1.63	1.42	1.67	1.38	1.72	1.34	1.77
55	1.53	1.60	1.49	1.64	1.45	1.68	1.41	1.72	1.38	1.77
60	1.55	1.62	1.51	1.65	1.48	1.69	1.44	1.73	1.41	1.77
65	1.57	1.63	1.54	1.66	1.50	1.70	1.47	1.73	1.44	1.77
70	1.58	1.64	1.55	1.67	1.52	1.70	1.49	1.74	1.46	1.77
75	1.60	1.65	1.57	1.68	1.54	1.71	1.51	1.74	1.49	1.77
80	1.61	1.66	1.59	1.69	1.56	1.72	1.53	1.74	1.51	1.77
85	1.62	1.67	1.60	1.70	1.57	1.72	1.55	1.75	1.52	1.77
90	1.63	1.68	1.61	1.70	1.59	1.73	1.57	1.75	1.54	1.78
95	1.64	1.69	1.62	1.71	1.60	1.73	1.58	1.75	1.56	1.78
100	1.65	1.69	1.63	1.72	1.61	1.74	1.59	1.76	1.57	1.78

n = number of observations

k' = number of explanatory variables