MLE and Binary Choice Models

Problem 1. p - prob of occurence

n - # of obs

m - # of occurencies

$$L(\theta \mid T_1, \dots, T_n) = f(\theta \mid T_1) \cdot \ldots \cdot f(\theta \mid T_n)$$

$$\ln(\theta \mid T_1, \dots, T_n) = \sum f(\theta \mid T_i)$$

$$L(p \mid n, m) = \frac{n!}{m!\,(n-m)!} p^m (1-p)^{n-m}$$

$$\log L(p \mid n, m) = \log C_m^n + m \log p + (n-m)\log(1-p)$$

$$\frac{\partial l}{\partial p} = \frac{m}{p} - \frac{n-m}{1-p} = 0$$

$$m(1-p) = p(n-m)$$

$$\hat{p} = \frac{m}{n}$$

$$\frac{\partial^2 l}{\partial p^2} = -\frac{m}{p^2} - \frac{n-m}{(1-p)^2} < 0 \Rightarrow \text{maximum}$$

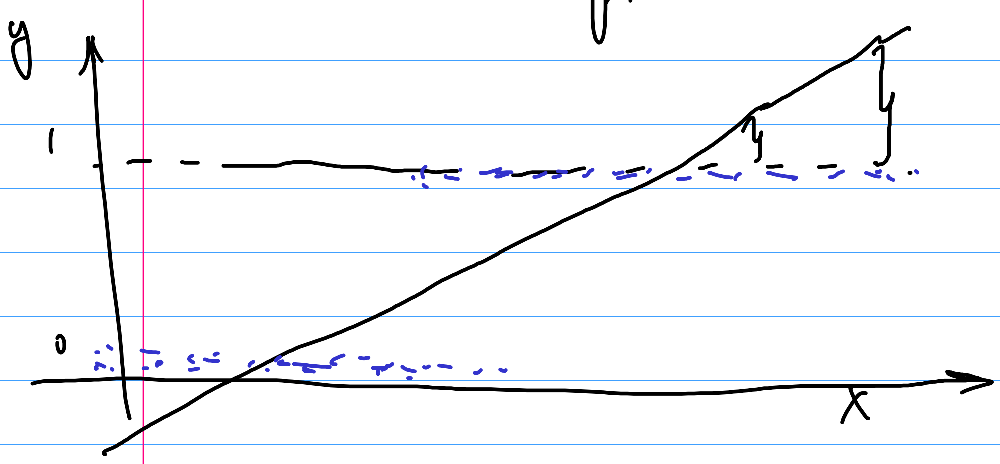
Binary Choice Models

$$y_i = \begin{cases} 0 \\ 1 \end{cases}$$

Linear Probability Model

$$p_i = P(y_i = 1) = \beta_1 + \beta_2 \cdot x_i$$

$$\hat{\beta}_1 = -0{,}3 \qquad \hat{\beta}_2 = 0{,}1$$



Disadvantages

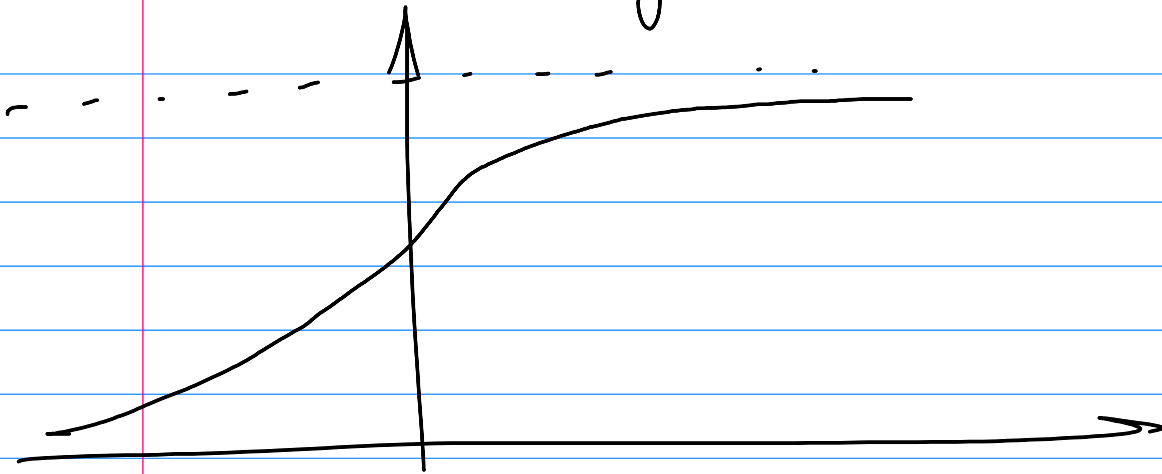
1. P_i can be < 0 and > 1
2. Non-linear effect of x on p

   1 → 2

   100 → 101

3. Heteroscedasticity (+ depending on β)

Logit Model



$$P(y_i=1) = F(z_i) = \frac{1}{1+e^{-z_i}}$$

$$z_i = \beta_1 + \beta_2 \cdot x$$

$\hat{\beta}_1 = -9$ $\hat{\beta}_2 = 0,5$

Interpretation:

a) $x_i = 15$ $\hat{P}(y_i = 1) = \dfrac{1}{1+e^{-(-9+0,5\cdot 15)}}$

$= 0,18$

b) marginal effect

$$\frac{dP(y_i=1)}{dx} = \frac{e^{-(\beta_1+\beta_2 x)}}{\left(1+e^{-(\beta_1+\beta_2 x)}\right)^2} \cdot \beta_2$$

$X = 15$

$$\frac{d\hat{P}(y_i=1)}{dx} = \frac{e^{-(-9+0{,}5\cdot 15)}}{\left(1+e^{-(-5+0{,}5\cdot 15)}\right)^2} \cdot 0{,}5 = 0{,}07$$

$X = 100$

$$\frac{d\hat{P}(y_i=1)}{dx} = 8 \cdot 10^{-19}$$

In practice

- marginal effect for average obs
- average marginal effect:

$$\frac{\sum \text{m.e. for } X_i}{n}$$

☆ if x is binary

$$\hat{P}(y_i=1) = \frac{1}{1+e^{-(\beta_1+\beta_2 X)}}$$

$$\hat{\beta}_1 = -2 \quad \hat{\beta}_2 = 2$$

$$\Delta \hat{p} = \hat{p}(x_i = 1) - \hat{p}(x_i = 0) = \frac{1}{1+e^0} - \frac{1}{1+e^2} = 0{,}38$$

Likelihood Ratio Test

(compare R and UR)

$$LR = -2(\ln L_R - \ln L_{UR}) \sim \chi^2_q$$

$$\left\{ = \ln\left(\frac{L_{UR}}{L_R}\right)^2 \right\}$$

q - # linear restrictions

- Pseudo-R^2 (R^2 McFadden)

$$\text{Pseudo } R^2 = 1 - \frac{\ln L}{\ln L_0} \in [0,1]$$

↳ Likelihood for naive model (on a const)

- fraction of correctly predicted outcomes
- correlation between outcomes and predicted probs.

Problem 2 $\quad y_i = 1$ if hired

	Model 1	Model 2	Model 3
x	–	0,49	0,49
gender	–	–	0,15
black	–	–	−0,32
Const	−0,32	−1,02	−0,9
ln L	−68	−62	−61

a) Model 1: $\text{Pseudo } R^2 = 1 - \frac{\ln L}{\ln L_0} =$

$$= 1 - \frac{-68}{-68} = 0$$

Model 2: $\text{Pseudo } R^2 = 1 - \frac{62}{68} = 0,09$

Model 3: $\text{Pseudo } R^2 = 1 - \frac{61}{68} = 0,1$

b) $LR = -2(\ln L_R - \ln L_{UR}) =$

$$-2(-68 + 61) = 14 \sim \chi^2_3$$

$\alpha = 5\%$ $\qquad \chi^2_{crit} = 7,81$

c) $LR = -2(-62+61) = 2 \sim \chi^2_2$

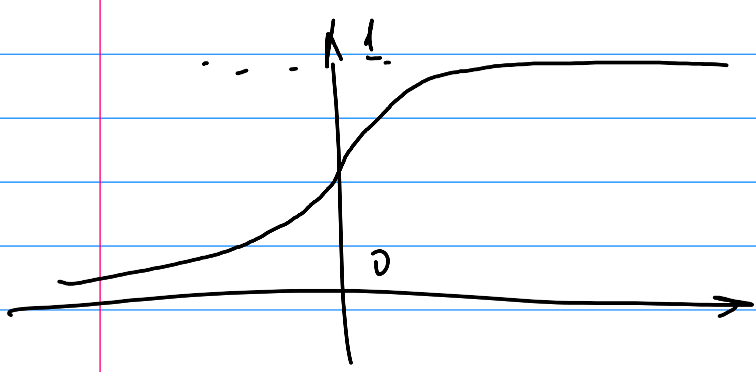
$\alpha = 5\%$ $\qquad \chi^2_{crit} = 5{,}99$

d) $\bar{X} = 1{,}4$

$$P(y_i = 1) = \frac{1}{1+e^{-z_i}}$$

$$\frac{d\hat{P}}{dx} = \frac{e^{-(-1{,}02+0{,}49\cdot 1{,}4)}}{\left(1+e^{-(-1{,}02+0{,}45\cdot 1{,}4)}\right)^2} \cdot 0{,}49 = 0{,}12$$

Probit Model



$$P(y_i = 1) = \Phi(z_i)$$

$$z_i = \beta_1 + \beta_2 x_{2i} + \ldots + \beta_k x_{ki}$$

$$\frac{\partial P}{\partial x_j} = \Phi'(z_i)\,\beta_j = \frac{1}{\sqrt{2\pi}} e^{-\frac{(\beta_1+\beta_2 x_{2i}+\ldots\beta_k x_{ki})^2}{2}} \beta_j$$