Question 4 [25 marks]

The plot of this question is based on a real story that happened at the thesis defense at ICEF many years ago. A student in her thesis was studying the dependence of final econometrics grade E_i (in %) on average homework grade H_i (also in %) using sample data on performance ICEF students,

$$E_i = \beta_1 + \beta_2 H_i + u_i \quad \textbf{(1)}$$

but she mixed up the variables in the final equation and calculated using OLS a regression

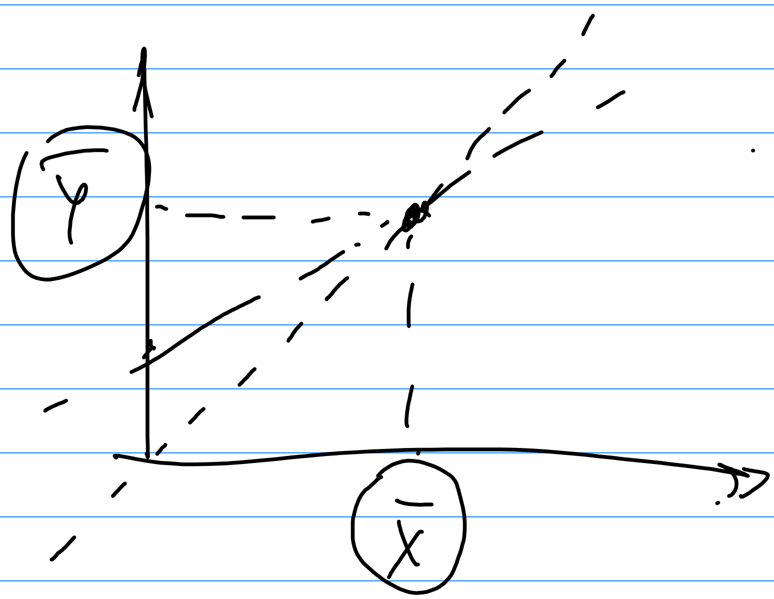
$$\hat{H}_i = \hat{\gamma}_1 + \hat{\gamma}_2 E_i \qquad \textbf{(2)}$$

The student had to correct her mistake during the defence. She decided to simply express the variable H_i from equation (1)

$$H_i = -\frac{\beta_1}{\beta_2} + \frac{1}{\beta_2} E_i - \frac{1}{\beta_2} u_i \ \textbf{(3)}$$

and use the value $\hat{\gamma}_2$ as an estimator of $\frac{1}{\beta_2}$ from which she could obtain an estimate of β_2. To round off the story, it is enough to say that the student managed to successfully defend her diploma. We wish you the same success in answering our questions.

For theoretical discussion let us assume that the values of H_i are assumed to be stochastic (say that they are drawn randomly from a population with variance σ_H^2) and disturbance term u satisfies the following assumptions $\mathrm{E}u = 0$, $\mathrm{E}u^2 = \sigma_u^2$, $\mathrm{E}(u_s u_t) = 0, t \neq s$ and $\mathrm{E}(Hu) = 0$.



TSS = ESS + RSS

(1) $$Y = \beta_1 + \beta_2 X + u$$

(2) $$X = \alpha_2 Y + v \Rightarrow Y = \frac{X}{\alpha_2} - \frac{v}{\alpha_2}$$

$$\alpha_2 = \frac{\bar{X}}{\bar{Y}}$$

$$\beta_1 + \beta_2 X + u = \frac{X}{\alpha_2} - \frac{v}{\alpha_2}$$

$$X = -\frac{\beta_1}{\beta_2} + \frac{Y}{\beta_2} - \frac{u}{\beta_2}$$

$$= \alpha_2 Y + v$$

$$-\alpha_2 Y + \frac{Y}{\beta_2} = +\frac{\beta_1}{\beta_2} + \frac{u}{\beta_2} + v$$

$$Y = \frac{\beta_1/\beta_2 + u/\beta_2 + v}{1/\beta_2 - \alpha_2}$$

$$\frac{X}{\alpha_2} - \beta_2 X = \beta_1 + u + \frac{v}{\alpha_2}$$

$$X\left(\frac{1}{\alpha_2} - \beta_2\right) = \beta_1 + u + \frac{v}{\alpha_2}$$

$$X = \frac{\beta_1 + u + \frac{v}{\alpha_2}}{\frac{1}{\alpha_2} - \beta_2}$$

$$\alpha_2 = \bar{X}/\bar{Y} = \frac{\beta_1 + \bar{u} + \bar{v}/\alpha_2}{1/\alpha_2 - \beta_2} \cdot \frac{1/\beta_2 - \alpha_2}{\beta_1/\beta_2 + \bar{u}/\beta_2 + \bar{v}}$$

$$\operatorname{plim} \alpha_2 = \frac{\beta_1 \cdot \left(1/\beta_2 - \alpha_2\right)}{\left(1/\alpha_2 - \beta_2\right) \beta_1/\beta_2} =$$

$$= \frac{1/\beta_2 - \alpha_2}{1/\alpha_2 \beta_2 - 1} = \frac{\alpha_2 - \alpha_2^2 \beta_2}{1 - \alpha_2 \beta_2} = \alpha_2$$

$$\alpha_2 = \bar{X}/\bar{Y} = \frac{\beta_1 + \bar{u} + \bar{v}/\alpha_2}{1/\alpha_2 - \beta_2} \cdot \frac{1/\beta_2 - \alpha_2}{\beta_1/\beta_2 + \bar{u}/\beta_2 + \bar{v}}$$

$$E(\alpha_2) = \frac{\beta_1 + E(\bar{u}) + E(\bar{v})/\alpha_2}{1/\alpha_2 - \beta_2} \cdot \frac{1/\beta_2 - \alpha_2}{\beta_1/\beta_2 + \frac{E(\bar{u})}{\beta_2} + E(\bar{v})}$$

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1) $$\hat{\gamma}_2 = \frac{\mathrm{Cov}(E, H)}{\mathrm{Var}(E)} = \frac{\mathrm{Cov}(\beta_1 + \beta_2 H + u, H)}{\mathrm{Var}(\beta_1 + \beta_2 H + u)} =$$

$$= \frac{\beta_2 \mathrm{Var}(H) + \mathrm{Cov}(u, H)}{\beta^2 \mathrm{Var}(H) + \mathrm{Var}(u) + 2\beta_2 \mathrm{cov}(u, H)}$$

$$\mathrm{plim}\, \hat{\gamma}_2 = \frac{\beta_2 \sigma_H^2}{\beta_2^2 \cdot \sigma_H^2 + \sigma_u^2} = \frac{1}{\beta_2} \left(\frac{1}{1 + \frac{1}{\beta_2^2} \frac{\sigma_u^2}{\sigma_H^2}} \right)$$

A attendance of seminars

$$\hat{\gamma}_2^{IV} = \frac{Cov(H, A)}{Cov(E, A)} = \frac{Cov(H, A)}{Cov(\beta_1 + \beta_2 \cdot H + u, A)} =$$

$$= \frac{Cov(H, A)}{\beta_2 Cov(H, A) + Cov(u, A)}$$

$$plim\ \hat{\gamma}_2^{IV} = \frac{cov(H, A)}{\beta_2 cov(H, A)} = \frac{1}{\beta_2}$$

SECTION B. Answer **ONE** question from this section (**4 OR 5**).

4. [30 marks] A researcher investigating the shadow economy (illegal economy) using international cross-section data for 40 countries hypothesizes that consumer expenditure on shadow goods and services, q, is related to total consumer expenditure, z, by the relationship

$$q = \alpha + \beta z + v \qquad (1)$$

where v is a disturbance term which satisfies the Gauss-Markov conditions. Both variables q and z are measured with error, and from the meaning of variables of this model q is part of z, so any error in the estimation of q affects the estimate of z **by the same amount**. Hence

$$y_i = q_i + w_i$$

and

$$x_i = z_i + w_i$$

where y_i is the estimated value of q_i, x_i is the estimated value of z_i, and w_i is the measurement error **affecting both variables** in observation i. It is assumed that the expected value of w is zero and that v and w are distributed independently of z and of each other. Note since shadow expenditure is a component of total consumer expenditure, β will lie between 0 and 1.

a) bias (β)

$$y - w = \alpha + \beta(x - w) + v$$

$$y = \alpha + \beta x + \underbrace{v + (1-\beta)w}_{u}$$

$$\hat{\beta}_{OLS} = \frac{\mathrm{Cov}(x,y)}{\mathrm{Var}(x)} = \frac{\mathrm{Cov}(x, \alpha + \beta x + u)}{\mathrm{Var}(x)} =$$

$$= \beta + \frac{\mathrm{Cov}(x,u)}{\mathrm{Var}(x)}$$

$$\mathrm{plim}\, \mathrm{Cov}(x,u) = \mathrm{plim}\, \mathrm{Cov}(z+w, v + (1-\beta)w) =$$

$$\mathrm{plim}(\mathrm{Cov}(z,v))^{=0} + (1-\beta)\,\mathrm{plim}\, \mathrm{Cov}(z,w)^{=0} +$$

$$\mathrm{plim}\, \mathrm{Cov}(w,v)^{=0} + (1-\beta)\,\mathrm{plim}\, \mathrm{Var}(w) = (1-\beta)\sigma_w^2$$

$$\text{plim } Var(x) = \text{plim } Var(z + w) = \sigma_z^2 + \sigma_w^2$$

$$\text{plim } \hat{\beta}_{OLS} = \beta + (1-\beta)\frac{\sigma_w^2}{\sigma_z^2 + \sigma_w^2}$$

$>0 \qquad >0$

(b) $\text{plim } \hat{\beta}_{OLS} = c_1 \beta + c_2 \cdot 1$

$$c_1 + c_2 = 1$$

$$\text{plim } \hat{\beta}_{OLS} = \frac{\sigma_z^2}{\sigma_z^2 + \sigma_w^2} \cdot \beta + \frac{\sigma_w^2}{\sigma_z^2 + \sigma_w^2} \cdot 1$$