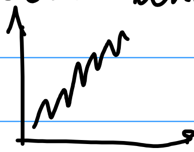
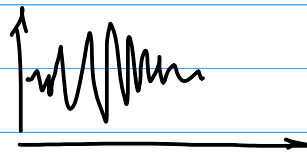
Non-Stationary Time Series

Weak Stationary / Covariance stationary:

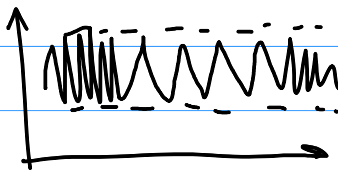
1. $E(X_t) = const$



2. $Var(X_t) = const$



3. $Cov(X_t, X_{t-s}) = f(s)$



Stationary Processes:

1. AR(1) for finite samples

$$X_t = \beta_1 \cdot X_{t-1} + \varepsilon_t$$

$$0 < \beta_1 < 1$$

2. MA(1), MA(2)

$$X_t = \varepsilon_t + \alpha_1 \varepsilon_{t-1}$$

$$X_t = \varepsilon_t + \alpha_1 \varepsilon_{t-1} + \alpha_2 \cdot \varepsilon_{t-2}$$

Not-Stationary:

1. Trend-Stationary (time trend)

$$X_t = \alpha + \beta t + \varepsilon_t$$

2. RW

$$X_t = X_{t-1} + \varepsilon_t$$

RW with drift

$$X_t = \alpha + X_{t-1} + \varepsilon_t$$

Problem 3. $X_t = \alpha + \beta t + \varepsilon_t$

1) $E(X_t) = \alpha + \beta t$

Problem 4. $X_t = X_{t-1} + \varepsilon_t$

1) $E(X_t) = E(X_{t-1} + \varepsilon_t) =$

$E(X_0 + \varepsilon_1 + \ldots + \varepsilon_t) = E(X_0)$

if X_0 r.v. $E(X_t) = \mu$

2) $Var(X_t) = Var(X_0 + \varepsilon_1 + \ldots + \varepsilon_t) =$

$= \sigma^2_{X_0} + t \cdot \sigma^2_{\varepsilon}$

$\Rightarrow$ X_t - non-stationary

Problem 5: AR(1) for finite sample

and X_0: $E(X_0) = 0$

$$Var(X_0) = \frac{1}{1-\beta_2^2} \sigma_\varepsilon^2$$

$$X_t = \beta_2 X_{t-1} + \varepsilon_t =$$

$$= \beta_2 (\beta_2 X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \ldots = \beta_2^t \cdot X_0 + \beta_2^{t-1} \varepsilon_1 + \ldots + \beta_2 \varepsilon_{t-1} + \varepsilon_t$$

1) $E(X_t) = \beta_2^t \cdot E(X_0)$

if $E(X_0) = 0 \Rightarrow 1^{st}$ prop. not violated

2) $Var(X_t) = \beta_2^{2t} \sigma_{X_0}^2 + \beta_2^{2t-2} \sigma_\varepsilon^2 + \ldots + \beta_2^2 \sigma_\varepsilon^2 + \sigma_\varepsilon^2$

$$= \beta_2^{2t} \sigma_{X_0}^2 + \frac{1-\beta_2^{2t}}{1-\beta_2^2} \sigma_\varepsilon^2 = \ldots$$

if $Var(X_0) = \frac{1}{1-\beta_2^2} \sigma_\varepsilon^2$

$$\ldots = \beta_2^{2t} \cdot \frac{1}{1-\beta_2^2} \sigma_\varepsilon^2 + \frac{1-\beta_2^{2t}}{1-\beta_2^2} \cdot \sigma_\varepsilon^2 = \frac{1}{1-\beta_2^2} \sigma_\varepsilon^2$$

3) $X_t = \beta_2 X_{t-1} + \varepsilon_t$

$$X_{t+s} = \beta_2 X_{t+s-1} + \varepsilon_t = \beta_2^s X_t + \beta_2^{s-1} \cdot \varepsilon_{t+1} + \ldots + \beta_2 \varepsilon_{t+s-1} + \varepsilon_{t+s}$$

$$Cov(X_t, X_{t+s}) = \beta_2^s Var(X_t) + Cov(X_t, \beta_2^{s-1} \cdot \varepsilon_{t+1} + \ldots + \beta_2 \varepsilon_{t+s-1} + \varepsilon_{t+s})$$

$$= \beta_2^s \cdot \sigma_X^2 + 0$$

Problem 7. MA(1) :

$$X_t = \varepsilon_t + \alpha_1 \cdot \varepsilon_{t-1}$$

1. $E(X_t) = 0$

2. $Var(X_t) = \sigma_\varepsilon^2 + \alpha_1 \cdot \sigma_\varepsilon^2$

3. $s = 1$: $Cov(X_t, X_{t+1}) = Cov(\varepsilon_t + \alpha_1 \cdot \varepsilon_{t-1}, \varepsilon_{t+1} + \alpha_1 \cdot \varepsilon_t) =$

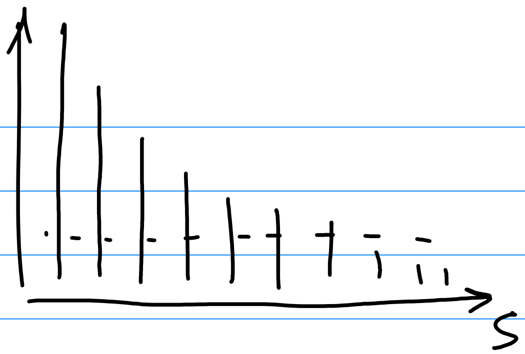
$= \alpha_1 \cdot \sigma_\varepsilon^2$

$s > 1$: $Cov(X_t, X_{t+s}) = 0$

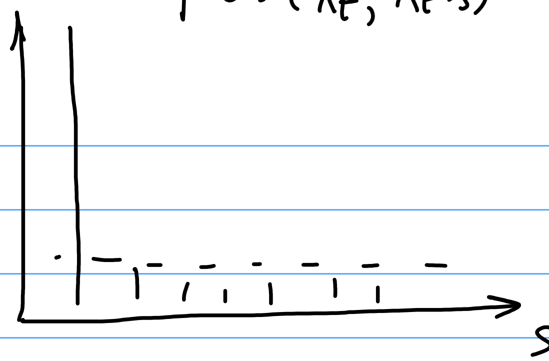
$$X_t = \beta_1 X_{t-1} + \varepsilon_t$$

$\hat{Corr}(X_t, X_{t-s})$ $\qquad\qquad$ $p\hat{Corr}(X_t, X_{t-s})$

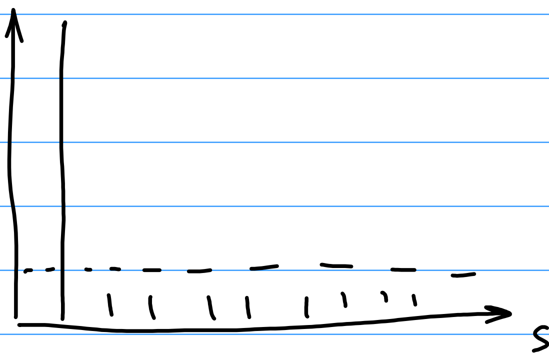
AR(1) ACF



PACF



MA(1) ACF



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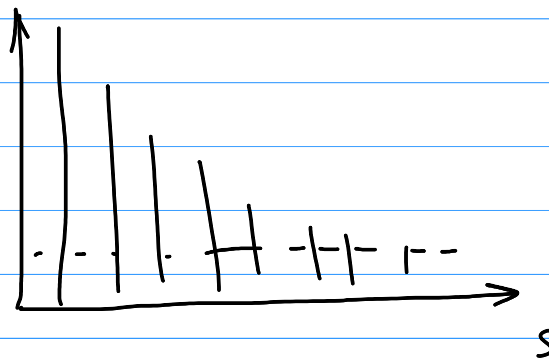


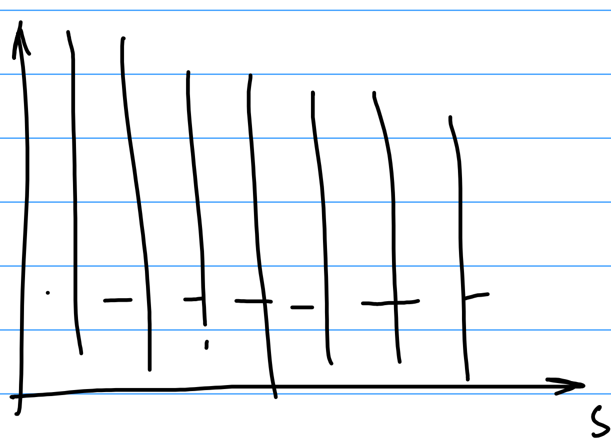
ACF

$X_t, \underbrace{X_{t-1}}_{\text{fix}}, X_{t-2}$

PACF

ACF

non. stat.



1) ACF : $X_t = \varepsilon_t + \alpha_1 \varepsilon_{t-1}$

$Var(X_t) = \sigma_\varepsilon^2 + \alpha_1 \cdot \sigma_\varepsilon^2$

$s = 1$ $Cov(X_t, X_{t+1}) = \alpha_1 \cdot \sigma_\varepsilon^2$

$s > 1$: $Cov(X_t, X_{t+s}) = 0$

$$Corr(X_t, X_{t+s}) = \begin{cases} \dfrac{\alpha_1 \cdot \sigma_\varepsilon^2}{(1+\alpha_1)\sigma_\varepsilon^2} = \dfrac{\alpha_1}{1+\alpha_1}, & s = 1 \\ 0, & s > 1 \end{cases}$$

2) ACF : $X_t = \beta_2 X_{t-1} + \varepsilon_t$

$$\rho_k = corr(X_t, X_{t+k}) = \frac{\beta_2^k \sigma_\varepsilon^2}{1-\beta_2^2} \Big/ \frac{\sigma_\varepsilon^2}{1-\beta_2^2} = \beta_2^k$$