# Cointegrated T.S.:

1) Same order of integration (e.g. $I(1)$)
2) $\exists$ lin. comb. of these t.s. st. it is stationary

Problem 1. $X_t, y_t$ – $I(1)$:

$$y_t = \alpha + y_{t-1} + \varepsilon_t$$

$$X_t = \beta + X_{t-1} + V_t$$

$\varepsilon_t, V_t$ – WN, unrelated

What happens if $y / X$ is estimated?

$$y_t = \pi_0 + \pi_1 X_t + w_t$$

If $X, y$ aren't related

$\Rightarrow H_0: \pi_1 = 0$ (true)

$\Rightarrow y_t = \pi_0 + w_t$

But $y_t$ is $I(1) \Rightarrow$

Hence, $w_t$ will be $I(1)$

$\Rightarrow w_t$ violates GM assumptions $\Rightarrow$

LM's results are misleading

# ECM

Consider ADL(1,1) ; $X_t, y_t$ − I(1)

$$y_t = \alpha_1 + \alpha_2 \cdot y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$y_t - y_{t-1} = \alpha_1 + \alpha_2 y_{t-1} - y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$\Delta y_t = \alpha_1 - (1-\alpha_2) y_{t-1} + \alpha_3 X_t - \alpha_3 X_{t-1} + \alpha_3 X_{t-1}$$

$$+ \alpha_4 X_{t-1} + u_t$$

$$\Delta y_t = \alpha_1 - (1-\alpha_2) y_{t-1} + \alpha_3 \Delta X_t + (\alpha_3 + \alpha_4) X_{t-1} + u_t$$

$$\Delta y_t = \alpha_3 \Delta X_t - (1-\alpha_2)\left[ y_{t-1} - \frac{\alpha_1}{1-\alpha_2} - \frac{\alpha_3+\alpha_4}{1-\alpha_2} X_{t-1} \right] + u_t$$

ECM: $$\Delta y_t = \alpha_3 \Delta X_t - \pi \big[ \underbrace{y_{t-1} - \beta_1 - \beta_2 X_{t-1}}_{\varepsilon_{t-1}} \big] + u_t$$

$\pi = (1-\alpha_2)$ − adjusment coef.

$$\beta_1 = \frac{\alpha_1}{1-\alpha_2} \qquad \beta_2 = \frac{\alpha_3 + \alpha_4}{1-\alpha_2}$$

$$y_t = \alpha_1 + \alpha_2 \cdot y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$(1-\alpha_2)\bar{y} = \alpha_1 + (\alpha_3 + \alpha_4)\bar{X}$$

$$\bar{y} = \beta_1 + \beta_2 \bar{X}$$

Estimation :

1) Est. regr $Y | X \Rightarrow \hat{\varepsilon}_t$ - residuals

2) Est. $\Delta Y = \alpha_1 + \alpha_2 \Delta X + \underline{\alpha_3 \hat{\varepsilon}_{t-1}} + u_t$

Granger Causality

(1) $Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \dots + \alpha_m Y_{t-m} +$

$\beta_1 X_{t-1} + \beta_2 X_{t-2} + \dots + \beta_m X_{t-m} + u_t$

$H_0 : \beta_1 = \dots = \beta_m = 0$

If $H_0$ is rejected $\Rightarrow$ X Granger causes Y

(2) $X_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \dots + \alpha_m Y_{t-m} +$

$\beta_1 X_{t-1} + \beta_2 X_{t-2} + \dots + \beta_m X_{t-m} + u_t$

$H_0 : \alpha_1 = \dots = \alpha_m = 0$

If $H_0$ is rejected $\Rightarrow$ Y Granger causes X

Granger Causality $\neq$ Causality

$\downarrow$

in-sample fitting

# VAR(m)

$$y_t = A_1 y_{t-1} + \dots + A_m y_{t-m} + B X_t + \epsilon_t$$

$y_t$: vector endog. regr.

$A_1$: matrix of coef.

$X_t$: vector exog. regr.

$\epsilon_t$: vector of innovat.

e.g. VAR(1)

$y_t$ - inflation rate

$x_t$ - unemployment rate

$$\begin{cases} y_t = \phi_{11} y_{t-1} + \phi_{12} x_{t-1} + u_t \\ x_t = \phi_{21} y_{t-1} + \phi_{22} x_{t-1} + v_t \end{cases}$$

In Matrix form:

$$z_t = \phi z_{t-1} + w_t$$

$$z_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} \quad \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \quad w_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

Problem 3:

ADL(2,1), where $X_t, Y_t$ - $I(1)$, in logarithms

(1) $Y_t = \alpha_1 + \alpha_2 Y_{t-1} + \alpha_3 Y_{t-2} + \underbrace{\alpha_4}_{\text{SR elasticity coefficient}} X_t + \alpha_5 X_{t-1} + u_t$

a) $\bar{Y}(1 - \alpha_2 - \alpha_3) = \alpha_1 + (\alpha_4 + \alpha_5)\bar{X}$

$$\bar{Y} = \frac{\alpha_1}{1-\alpha_2-\alpha_3} + \frac{\alpha_4+\alpha_5}{1-\alpha_2-\alpha_3}\bar{X}$$

LR elasticity coefficient

b) (1) − Lag(1)

(1) $Y_t = \alpha_1 + \alpha_2 Y_{t-1} + \alpha_3 Y_{t-2} + \alpha_4 X_t + \alpha_5 X_{t-1} + u_t$

$\hookrightarrow$ $\Delta Y_t = \alpha_2 \Delta Y_{t-1} + \alpha_3 \Delta Y_{t-2} + \alpha_4 \Delta X_t$

$+ \alpha_5 \Delta X_{t-1} + \underbrace{u_t - u_{t-1}}$