Cointegration

1) Same order of integration (e.g. $I(1)$)

2) $\exists$ stationary lin. combination of these time series

Problem 1. $X_t = \alpha + X_{t-1} + \varepsilon_t$

$y_t = \beta + y_{t-1} + v_t$

ε_t, v_t - unrelated WN

What happens if $y_t | X_t$ estimated?

$y_t = \pi_0 + \pi_1 X_t + V_t$

X, y aren't related: $\Rightarrow$

$H_0: \pi_1 = 0$ is true

$y_t = \pi_0 + V_t$

But y_t - $I(1)$ $\Rightarrow$

Hence V_t will $I(1)$,

which violates GM assumptions

$\Rightarrow$ misleading results

ECM

ADL(1, 1) where y_t, X_t - $I(1)$

$$y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$y_t - y_{t-1} = \alpha_1 + \alpha_2 y_{t-1} - y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$\Delta y_t = \alpha_1 - (1 - \alpha_2) y_{t-1} + \alpha_3 X_t - \alpha_3 X_{t-1} + \alpha_3 X_{t-1}$$
$$+ \alpha_4 X_{t-1} + u_t$$

$$\underline{\underline{\Delta y_t}} = \alpha_1 - (1 - \alpha_2) y_{t-1} + \underline{\underline{\alpha_3 \Delta X_t}} +$$
$$(\alpha_3 + \alpha_4) X_{t-1} + u_t$$

$$\Delta y_t = \alpha_3 \Delta X_t - (1 - \alpha_2)\left[y_{t-1} - \frac{\alpha_1}{1-\alpha_2} - \frac{\alpha_3 + \alpha_4}{1 - \alpha_2} X_{t-1} \right] + u_t$$

ECM: $\Delta y_t = \alpha_3 \Delta X_t - \pi \left[\underbrace{y_{t-1} - \beta_1 - \beta_2 X_{t-1}}_{\varepsilon_t} \right] + u_t$

$$\beta_1 = \frac{\alpha_1}{1 - \alpha_2} \qquad \beta_2 = \frac{\alpha_3 + \alpha_4}{1 - \alpha_2}$$

π - adjustment coefficient

$$y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$\bar{y}(1 - \alpha_2) = \alpha_1 + (\alpha_3 + \alpha_4)\bar{X}$$

$$\bar{y} = \beta_1 + \beta_2 \bar{X}$$

Estimating ECM:

1) Est. $y_t | X_t \Rightarrow \hat{\varepsilon}_t$ - residuals

2) Est. $\Delta y_t \mid \Delta X_t, \hat{\varepsilon}_{t-1}$

Granger Causality

(1) $$y_t = \alpha_0 + \alpha_1 y_{t-1} + \dots + \alpha_m y_{t-m} + \beta_1 X_{t-1} + \dots + \beta_m X_{t-m} + \varepsilon_t$$

if $H_0: \beta_1 = \dots = \beta_m = 0$ is rejected $\Rightarrow$ X Granger causes y

(2) $$X_t = \alpha_0 + \alpha_1 y_{t-1} + \dots + \alpha_m y_{t-m} + \beta_1 X_{t-1} + \dots + \beta_m X_{t-m} + \varepsilon_t$$

if $H_0: \alpha_1 = \dots = \alpha_m = 0$ is rejected $\Rightarrow$ y Granger causes X

Granger causality $\neq$ causality

↓

in-sample fitting

VAR(m)

$$y_t = A_1 y_{t-1} + \dots + A_m y_{t-m} + BX_t + u_t$$

vector of end. variables

matrix of coef.

vector of exog. variables

vector of innovat.

e.g. VAR(1)

y_t - inflation rate

x_t - unemployment rate

$$\begin{cases} y_t = \phi_{11} y_{t-1} + \phi_{12} x_{t-1} + \varepsilon_t \\ x_t = \phi_{21} y_{t-1} + \phi_{22} x_{t-1} + v_t \end{cases}$$

Matrix Form:

$$Z_t = \phi Z_{t-1} + W_t$$

$$Z_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} \quad \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \quad W_t = \begin{pmatrix} \varepsilon_t \\ v_t \end{pmatrix}$$

Problem 3:

ADL(2,1), y_t, X_t – $I(1)$, in logarithms

(1) $y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 y_{t-2} + \alpha_4 X_t +$

$+ \alpha_5 X_{t-1} + \varepsilon_t$ SR elasticity

a) $\overline{y}(1 - \alpha_2 - \alpha_3) = \alpha_1 + (\alpha_4 + \alpha_5)\overline{X}$

$$\overline{y} = \frac{\alpha_1}{1-\alpha_2-\alpha_3} + \underbrace{\frac{\alpha_4 + \alpha_5}{1-\alpha_2-\alpha_3}}_{\text{LR elasticity}} \overline{X}$$

b) (1) – lag(1)

$$y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 y_{t-2} + \alpha_4 X_t + \alpha_5 X_{t-1} + \varepsilon_t$$

$$\Delta y_t = \alpha_2 \Delta y_{t-1} + \alpha_3 \Delta y_{t-2} + \alpha_4 \Delta X_t + \alpha_5 \Delta X_{t-1} + \varepsilon_t - \varepsilon_{t-1}$$