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UNIVERSITY OF LONDON EC2020 ZA

BSc degrees and Diplomas for Graduates in Economics, Management, Finance and the Social Sciences, the Diplomas in Economics and Social Sciences

Elements of Econometrics

Thursday, 24 May 2018 : 14:30 to 17:30

Candidates should answer **EIGHT** of the following **TEN** questions: **ALL** of the questions in Section A (8 marks each) and **THREE** questions from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Extracts from statistical tables are given after the final question on this paper.

Graph paper is provided at the end of this question paper. If used, it must be detached and fastened securely inside the answer book.

A calculator may be used when answering questions on this paper and it must comply in all respects with the specification given with your Admission Notice. The make and type of machine must be clearly stated on the front cover of the answer book.

If more questions are answered than requested, only the first answers attempted will be counted.

PLEASE TURN OVER

SECTION A

Answer all questions from this section

1. We are interested in investigating the factors governing the precision of regression coefficients. Consider the model

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \varepsilon_i$$

with OLS parameter estimates $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\beta}_3$. Under the Gauss Markov assumptions, we have

$$Var\left(\hat{\beta}_2\right) = \frac{\sigma_\varepsilon^2}{\sum_{i=1}^n \left(X_{2i} - \bar{X}_2\right)^2} \times \frac{1}{1 - r_{X_2 X_3}^2},$$

where σ_ε^2 is the variance of ε and $r_{X_2X_3}$ is the sample correlation between X_2 and X_3.

(a) **(4 marks)** Provide four factors that help with obtaining more precise parameter estimates for, say, $\hat{\beta}_2$.

(b) **(4 marks)** In light of your answer to (a), discuss the concept of near multicollinearity. What consequences does its presence have when considering single and joint significance testing of our slope parameters?

2. Consider the linear regression model

$$Y_t = \beta_0 + \beta_1 X_t + \beta_2 Y_{t-1} + u_t, \qquad t = 1, ..., T$$

where the errors u_t are distributed independently of the regressors X_t and $|\beta_2| < 1$. You suspect that the, mean zero, errors exhibit autocorrelation.

(a) **(2 marks)** Explain what we mean by the concept of autocorrelation.

(b) Assume that u_t follows an AR(1) process.

i. **(3 marks)** Discuss, for the given model, the consequences for the ordinary least squares estimator. Support your answers with suitable arguments.

ii. **(3 marks)** Discuss how you would detect the presence of autocorrelation in the errors in this model. Clearly indicate the null and alternative hypothesis, the test statistic, and rejection rule.

3. For the population of men who grew up with disadvantaged backgrouns, let *poverty* be a dummy variable equal to one if a man is currently living below the poverty line, and zero otherwise. The variable *age* is age and *educ* is total years of schooling. Let *vocat* be an indicator equal to unity if a man's high school offered vocational training. Using a random sample of 850 men, you obtain

$$\widehat{\Pr\left(\textit{poverty=1}|\textit{educ,age,vocat}\right)} = \Lambda\left(0.453 - 0.016\ \textit{age} - 0.087\ \textit{educ} - 0.049\ \textit{vocat}\right)$$

where $\Lambda\left(z\right) = \exp\left(z\right)/\left(1+\exp\left(z\right)\right)$ is the logit function.

(a) **(5 marks)** It is argued that using the logit regression model is better than using the linear probability model when explaining the binary variable *poverty*. Discuss the benefits/drawback of using the logit regression model when trying to explain a binary variable.

(b) **(3 marks)** For a 40-year old man, with 12 years of education, what is the estimated effect of having vocational training available in high school on the probability of currently living in poverty?
Hint: Clarity of computations required is enough, no need to give an exact number.

4. The following model jointly determines monthly child support payments and monthly visitation rights for divorced couples with children:

$$\textit{support} = \alpha_1 + \alpha_2\ \textit{visits} + \alpha_3\ \textit{finc} + \alpha_4\ \textit{fremarr} + \alpha_5\ \textit{dist} + \varepsilon_1$$
$$\textit{visits} = \beta_1 + \beta_2\ \textit{support} + \beta_3\ \textit{mremarr} + \beta_4\ \textit{dist} + \varepsilon_2$$

We assume that children live with their mothers, so that fathers pay child support. Thus, the first equation is the father's "reaction function": it describes the amount of child support paid for any given level of visitation rights and the other exogenous variables *finc* (father's income), *fremarr* (binary indicator if father remarried), and *dist* (miles currently between the mother and father's residence). Similarly the second equation is the mother's reaction function: it describes visitation rights for a given amount of child support; *mremarr* is a binary indicator for whether the woman is remarried.

(a) **(3 marks)** Examine the identification of each structural equation.

(b) **(5 marks)** Your friend suggests you should implement the IV estimator to estimate the β parameters consistently. He tells you to use *finc* as instrument for *support.* Provide a critical discussion of this suggestion.

5. Consider the simple linear regression model

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

under the classical linear regression model assumptions, where X_i is fixed under repeated sampling. The usual OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased for their respective population parameters. Let $\tilde{\beta}_1$ be the estimator of β_1 obained by assuming the intercept is zero.

(a) **(4 marks)** Show that the **restricted least squares** estimator of β_1 is given by

$$\tilde{\beta}_1 = \frac{\sum_{i=1}^{n} X_i Y_i}{\sum_{i=1}^{n} X_i^2}.$$

(b) **(4 marks)** Find $E\left(\tilde{\beta}_1\right)$ in terms of the X_i, β_0 and β_1. Verify that $\tilde{\beta}_1$ is unbiased for β_1 when the population intercept is zero. Are there other cases where $\tilde{\beta}_1$ is unbiased?

SECTION B

Answer three questions from this section.

6. Let us consider the estimation of a hedonic price function for houses. The hedonic price refers to the implicit price of a house given certain attributes (e.g., the number of bedrooms). The data contains the sale price of 546 houses sold in the summer of 1987 in Canada along with their important features. The following characteristics are available: the lot size of the property in square feet (*lotsize*), the numbers of bedrooms (*bedrooms*), the number of full bathrooms (*bathrooms*), and a dummy indicating the presence of airconditioning (*airco)*.

   Consider the following ordinary least squares results

$$\begin{aligned} \widehat{\log(\textit{price})}_i &= \underset{\substack{(.232)\\ [.233]}}{7.094} + \underset{\substack{(.028)\\ [.028]}}{0.400} \log(\textit{lotsize})_i + \underset{\substack{(.015)\\ [.017]}}{0.078}\ \textit{bedrooms}_i + \\ &\quad \underset{\substack{(.023)\\ [.024]}}{0.216}\ \textit{bathrooms}_i + \underset{\substack{(.024)\\ [.023]}}{0.212}\ \textit{airco}_i, \qquad n = 546,\ RSS = 32.622 \end{aligned} \tag{6.1}$$

   The usual standard errors are in parentheses, the heteroskedasticity robust standard errors are in square brackets, and RSS measures the residual sum of squares.

   (a) **(5 marks)** Interpret the parameter estimates on $\log(\textit{lotsize})$, *bedrooms*, and *airco*. Briefly discuss the statistical significance of the results.

   (b) **(5 marks)** Suppose that lot size was measured in square metres rather than square feet. How would this affect the parameter estimates of the slopes and intercept? How would this affect the fitted values? *Note*: the conversion (approximate) $1m^2 = 10ft^2$.

   (c) **(5 marks)** We are interested in testing the hypothesis $H_0 : \beta_{\textit{bedrooms}} = \beta_{\textit{bathrooms}}$ against the alternative $H_A : \beta_{\textit{bedrooms}} \neq \beta_{\textit{bathrooms}}$. Discuss a test for this hypothesis that makes use of the following restricted regression result

$$\begin{aligned} \widehat{\log(\textit{price})}_i &= \underset{(.234)}{6.994} + \underset{(.282)}{0.408} \log(\textit{lotsize})_i + \underset{(.011)}{0.127}\ \textit{bbrooms}_i + \underset{(.024)}{0.215}\ \textit{airco}_i, \\ &\quad n = 546,\ RSS = 33.758 \end{aligned} \tag{6.2}$$

   where *bbrooms* $=$ *bedrooms* $+$ *bathrooms*. Clearly indicate the assumptions you are making for this test to be valid.

   (d) **(5 marks)** You are interested in testing for the presence of heteroskedasticity. Say you are told that the variance is increasing with $\log(\textit{lotsize})$. Discuss how you would test for the presence of heteroskedasticity. What is the name of the test you are proposing?

7. The following question concerns the effects of background characteristics and admission assessment scores on the performance of students in the final university examinations in a UK university. The following equation was estimated by Ordinary Least Squares:

$$
\begin{aligned}
\widehat{\textit{finalavg}} &= \underset{(2.78)}{53.89} + \underset{(.04)}{0.03}\ \textit{tst\_reas} + \underset{(.02)}{0.05}\ \textit{tst\_quan} + \underset{(.02)}{0.06}\ \textit{interview} \\
&\quad \underset{(.78)}{-0.04}\ \textit{indep} + \underset{(.66)}{0.67}\ \textit{male} + \underset{(.97)}{0.06}\ \textit{indep*male} \\
n &= 325;\ R^2 = .06, \qquad (7.1)
\end{aligned}
$$

where *finalavg* is the average finals score (the outcome), *tst_reas* and *tst_quan* are the pre-admission reasoning and quantitative test scores respectively, *interview* is the pre-admission interview score, *indep* indicates whether the student attended an independent school (1=yes, 0=no), and *male* indicates whether the student is male (1=yes, 0=no). The usual standard errors are in parentheses.

(a) **(5 marks)** We want to test whether gender has a significant impact on students' finals performance. Clearly indicating the null and the alternative hypothesis, provide the test statistic and the rejection rule. Discuss what information you would need to enable you to implement this test. You are expected to provide the assumptions which underlie your test.

(b) **(5 marks)** If we do not include the interaction term *indep*male* in our regression model, what are we implicitly assuming about the effect of gender and school background on finals performance?

(c) **(5 marks)** Suppose students who did not attend an independent school, attended a state school. Using the results in (7.1), provide the parameter estimates you would obtain if you had applied Ordinary Least Squares to the equation

$$
\begin{aligned}
\textit{finalavg} &= \beta_0 + \beta_1\ \textit{tst\_reas} + \beta_2\ \textit{tst\_quan} + \beta_3\ \textit{interview} \qquad (7.2) \\
&\quad \beta_4\ \textit{state} + \beta_5\ \textit{male} + \beta_6\ \textit{state*male} + \varepsilon,
\end{aligned}
$$

where *state* indicates whether the student attended a state school (1=yes, 0=no).

(d) **(5 marks)** Discuss any problem you may have in estimating the model if all males in your sample have attended an independent school prior to attending university. What name does this problem have and what can you do to mitigate this problem?

8. An OLS regression of y_t on x_t and x_{t-1} gives the following results (with the standard errors given in parentheses)

$$\widehat{y}_t = \underset{(2.30)}{8.88} + \underset{(3.01)}{5.07}x_t - \underset{(3.01)}{3.18}\ x_{t-1};\ R^2 = .095, T = 209 \qquad (8.1)$$

(a) **(4 marks)** What are the estimates of the short-run and long-run effect of x_t on y_t? Interpret these estimates.

(b) **(4 marks)** Test the hypothesis that a one unit increase in x results in a ten unit increase in y in the same year. Under what assumptions is this test valid?

Let e_t be the OLS residuals from the above regression. An OLS regression of e_t on e_{t-1} yields

$$e_t = \underset{(2.12)}{0.55} + \underset{(0.18)}{0.44}e_{t-1} + \underset{(2.18)}{2.16}x_t - \underset{(.99)}{1.09}x_{t-1};\ R^2 = .175, T = 208 \qquad (8.2)$$

(c) **(5 marks)** Using this result, test for evidence of autocorrelation, clearly indicating the null and alternative hypotheses, the test statistic, rejection rule and assumptions underlying the test. What name do we give this test?

(d) You are interested in testing whether the long-run effect of x_t on y_t is statistically significant.

i. **(4 marks)** Discuss how to reparameterize (8.1) to ensure that your regression output will provide you with a standard error for the long-run effect.

ii. **(3 marks)** Discuss the problem of implementing your test using the standard error obtained in (d)i. when you do find evidence of autocorrelation in (8.1). Briefly indicate how you proceed with your test.

9. Consider the model

$$\begin{aligned} y_t &= \alpha + \beta t + \varepsilon_t,\ t = 1, .., T \\ \varepsilon_t &= \rho \varepsilon_{t-1} + v_t,\ \text{and} \end{aligned} \tag{9.1}$$

v_t is an i.i.d.$(0, \sigma^2)$ innovation which is independent of the past. Let $|\rho| \leq 1.$

(a) **(4 marks)** What name do we give the ε_t process given above? Provide the condition(s) that ensures that ε_t is stationary? In your answer discuss what we mean by the concept of stationarity (more precisely "covariance stationarity").

(b) It will be important to distinguish between the above process for y_t being "trend stationary" as opposed to "difference stationary".

i. **(4 marks)** Explain these concepts clearly. Why is it important to distinguish between these two types of non-stationarity?

ii. **(2 marks)** Show that under the condition you provided in (a) that y_t is trend stationary.

iii. **(2 marks)** Show that if ε_t is difference stationary then y_t is difference stationary.

(c) **(4 marks)** Show that you can rewrite the above model in the following form

$$\Delta y_t = \gamma_1 + \gamma_2 t + \gamma_3 y_{t-1} + v_t. \tag{9.2}$$

Clearly indicate the relation between $(\gamma_1, \gamma_2, \gamma_3)$ and (α, β, ρ).

(d) **(4 marks)** What problem do you see here with using 9.2 to conducting the Dickey Fuller Test to distinguish between trend and difference stationarity when v_t exhibits autocorrelation? What solution do you suggest we adopt.

10. Let *math*10 denote the percentage of students at a high school receiving passing score on a standardized math test. We are interested in estimating the effect of per student spending on math performance. A simple model is

$$\textit{math}10_i = \beta_0 + \beta_1 \log(\textit{expend}_i) + \beta_2 \log(\textit{enroll}_i) + \beta_3\, \textit{poverty}_i + u_i \tag{10.1}$$

where, for each high school i, *poverty*$_i$ is the percentage of students living in poverty, *expend*$_i$ is the spending per student and *enroll*$_i$ the number of registered students. You may assume that this model satisfies all Gauss-Markov assumptions.

You are faced with the fact that data is unavailable on a key variable: *poverty*.

(a) **(5 marks)** Discuss the properties (unbiasedness and consistency) of the estimators when you drop the variable *poverty?* Explain your answers.

You do have information available on a closely related variable: the percentage of students eligible for the federally funded school lunch program, *lnchprg*$_i$. Let us consider using *lnchprg*$_i$ as a proxy for *poverty*$_i$.

(b) **(2 marks)** Briefly discuss why *lnchprg*$_i$ is a sensible proxy variable for the unobserved variable *poverty*$_i$.

(c) **(5 marks)** It is unlikely that *lnchprg*$_i$ is an ideal proxy, in the sense that there is an exact linear relationship between them, instead, we will assume that

$$\textit{poverty}_i = \alpha_0 + \alpha_1\, \textit{lnchprg}_i + v_i,\ \alpha_1 \neq 0 \tag{10.2}$$

Discuss the assumptions you need to make to enable consistent parameter estimators of β_1 and β_2 using your estimable equation

$$\textit{math}10_i = \gamma_0 + \gamma_1 \log(\textit{expend}_i) + \gamma_2 \log(\textit{enroll}_i) + \gamma_3\, \textit{lnchprg}_i + e_i,$$

Hint: Consider the relation between the γ and the β parameters and express e_i in terms of u_i and v_i.

(question continues on next page)

(d) The OLS results with and without *lnchprg*$_i$ as an explanatory variable are given by (standard errors in parentheses):

$$\widehat{\textit{math}10}_i = \underset{(26.72)}{-69.24} + \underset{(3.30)}{11.13} \log(\textit{expend}_i) + \underset{(0.615)}{0.022} \log(\textit{enroll}_i),$$
$$N = 428, R^2 = 0.0297$$

$$\widehat{\textit{math}10}_i = \underset{(24.99)}{-23.14} + \underset{(3.04)}{7.75} \log(\textit{expend}_i) - \underset{(0.58)}{1.26} \log(\textit{enroll}_i) - \underset{(0.036)}{0.324} \textit{lnchprg}_i$$
$$N = 428, R^2 = 0.1893$$

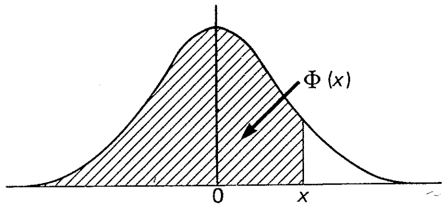
i. **(4 marks)** Interpret the coefficient on *lnchprg*. What does this parameter tell us regarding the parameter of interest β_3.

ii. **(4 marks)** Give an intuitive discussion explaining why the effect of expenditures on *math*10_i is lower in the regression where *lnchprg*$_i$ is included than where it is excluded.

END OF PAPER

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

The function tabulated is $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt$. $\Phi(x)$ is the probability that a random variable, normally distributed with zero mean and unit variance, will be less than or equal to x. When $x < 0$ use $\Phi(x) = 1 - \Phi(-x)$, as the normal distribution with zero mean and unit variance is symmetric about zero.



x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0·00	0·5000	0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·97725
·01	·5040	·41	·6591	·81	·7910	·21	·8869	·61	·9463	·01	·97778
·02	·5080	·42	·6628	·82	·7939	·22	·8888	·62	·9474	·02	·97831
·03	·5120	·43	·6664	·83	·7967	·23	·8907	·63	·9484	·03	·97882
·04	·5160	·44	·6700	·84	·7995	·24	·8925	·64	·9495	·04	·97932
0·05	0·5199	0·45	0·6736	0·85	0·8023	1·25	0·8944	1·65	0·9505	2·05	0·97982
·06	·5239	·46	·6772	·86	·8051	·26	·8962	·66	·9515	·06	·98030
·07	·5279	·47	·6808	·87	·8078	·27	·8980	·67	·9525	·07	·98077
·08	·5319	·48	·6844	·88	·8106	·28	·8997	·68	·9535	·08	·98124
·09	·5359	·49	·6879	·89	·8133	·29	·9015	·69	·9545	·09	·98169
0·10	0·5398	0·50	0·6915	0·90	0·8159	1·30	0·9032	1·70	0·9554	2·10	0·98214
·11	·5438	·51	·6950	·91	·8186	·31	·9049	·71	·9564	·11	·98257
·12	·5478	·52	·6985	·92	·8212	·32	·9066	·72	·9573	·12	·98300
·13	·5517	·53	·7019	·93	·8238	·33	·9082	·73	·9582	·13	·98341
·14	·5557	·54	·7054	·94	·8264	·34	·9099	·74	·9591	·14	·98382
0·15	0·5596	0·55	0·7088	0·95	0·8289	1·35	0·9115	1·75	0·9599	2·15	0·98422
·16	·5636	·56	·7123	·96	·8315	·36	·9131	·76	·9608	·16	·98461
·17	·5675	·57	·7157	·97	·8340	·37	·9147	·77	·9616	·17	·98500
·18	·5714	·58	·7190	·98	·8365	·38	·9162	·78	·9625	·18	·98537
·19	·5753	·59	·7224	·99	·8389	·39	·9177	·79	·9633	·19	·98574
0·20	0·5793	0·60	0·7257	1·00	0·8413	1·40	0·9192	1·80	0·9641	2·20	0·98610
·21	·5832	·61	·7291	·01	·8438	·41	·9207	·81	·9649	·21	·98645
·22	·5871	·62	·7324	·02	·8461	·42	·9222	·82	·9656	·22	·98679
·23	·5910	·63	·7357	·03	·8485	·43	·9236	·83	·9664	·23	·98713
·24	·5948	·64	·7389	·04	·8508	·44	·9251	·84	·9671	·24	·98745
0·25	0·5987	0·65	0·7422	1·05	0·8531	1·45	0·9265	1·85	0·9678	2·25	0·98778
·26	·6026	·66	·7454	·06	·8554	·46	·9279	·86	·9686	·26	·98809
·27	·6064	·67	·7486	·07	·8577	·47	·9292	·87	·9693	·27	·98840
·28	·6103	·68	·7517	·08	·8599	·48	·9306	·88	·9699	·28	·98870
·29	·6141	·69	·7549	·09	·8621	·49	·9319	·89	·9706	·29	·98899
0·30	0·6179	0·70	0·7580	1·10	0·8643	1·50	0·9332	1·90	0·9713	2·30	0·98928
·31	·6217	·71	·7611	·11	·8665	·51	·9345	·91	·9719	·31	·98956
·32	·6255	·72	·7642	·12	·8686	·52	·9357	·92	·9726	·32	·98983
·33	·6293	·73	·7673	·13	·8708	·53	·9370	·93	·9732	·33	·99010
·34	·6331	·74	·7704	·14	·8729	·54	·9382	·94	·9738	·34	·99036
0·35	0·6368	0·75	0·7734	1·15	0·8749	1·55	0·9394	1·95	0·9744	2·35	0·99061
·36	·6406	·76	·7764	·16	·8770	·56	·9406	·96	·9750	·36	·99086
·37	·6443	·77	·7794	·17	·8790	·57	·9418	·97	·9756	·37	·99111
·38	·6480	·78	·7823	·18	·8810	·58	·9429	·98	·9761	·38	·99134
·39	·6517	·79	·7852	·19	·8830	·59	·9441	·99	·9767	·39	·99158
0·40	0·6554	0·80	0·7881	1·20	0·8849	1·60	0·9452	2·00	0·9772	2·40	0·99180

TABLE 4. THE NORMAL DISTRIBUTION FUNCTION

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
2·40	0·99180	2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918
·41	·99202	·56	·99477	·71	·99664	·86	·99788	·01	·99869	·16	·99921
·42	·99224	·57	·99492	·72	·99674	·87	·99795	·02	·99874	·17	·99924
·43	·99245	·58	·99506	·73	·99683	·88	·99801	·03	·99878	·18	·99926
·44	·99266	·59	·99520	·74	·99693	·89	·99807	·04	·99882	·19	·99929
2·45	0·99286	2·60	0·99534	2·75	0·99702	2·90	0·99813	3·05	0·99886	3·20	0·99931
·46	·99305	·61	·99547	·76	·99711	·91	·99819	·06	·99889	·21	·99934
·47	·99324	·62	·99560	·77	·99720	·92	·99825	·07	·99893	·22	·99936
·48	·99343	·63	·99573	·78	·99728	·93	·99831	·08	·99896	·23	·99938
·49	·99361	·64	·99585	·79	·99736	·94	·99836	·09	·99900	·24	·99940
2·50	0·99379	2·65	0·99598	2·80	0·99744	2·95	0·99841	3·10	0·99903	3·25	0·99942
·51	·99396	·66	·99609	·81	·99752	·96	·99846	·11	·99906	·26	·99944
·52	·99413	·67	·99621	·82	·99760	·97	·99851	·12	·99910	·27	·99946
·53	·99430	·68	·99632	·83	·99767	·98	·99856	·13	·99913	·28	·99948
·54	·99446	·69	·99643	·84	·99774	·99	·99861	·14	·99916	·29	·99950
2·55	0·99461	2·70	0·99653	2·85	0·99781	3·00	0·99865	3·15	0·99918	3·30	0·99952

The critical table below gives on the left the range of values of x for which $\Phi(x)$ takes the value on the right, correct to the last figure given; in critical cases, take the upper of the two values of $\Phi(x)$ indicated.

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
			0·9994		0·99990		0·99995
3·075		3·263		3·731		3·916	
	0·9990		0·9995		0·99991		0·99996
3·105		3·320		3·759		3·976	
	0·9991		0·9996		0·99992		0·99997
3·138		3·389		3·791		4·055	
	0·9992		0·9997		0·99993		0·99998
3·174		3·480		3·826		4·173	
	0·9993		0·9998		0·99994		0·99999
3·215		3·615		3·867		4·417	
	0·9994		0·9999		0·99995		1·00000

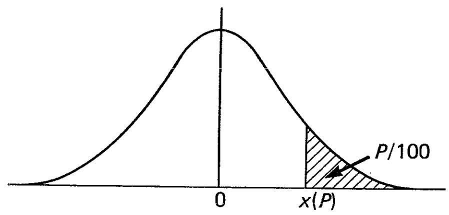
When $x > 3{\cdot}3$ the formula $1-\Phi(x) \doteqdot \frac{e^{-\frac{1}{2}x^2}}{x\sqrt{2\pi}}\left[1-\frac{1}{x^2}+\frac{3}{x^4}-\frac{15}{x^6}+\frac{105}{x^8}\right]$ is very accurate, with relative error less than $945/x^{10}$.

TABLE 5. PERCENTAGE POINTS OF THE NORMAL DISTRIBUTION

This table gives percentage points $x(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{2\pi}}\int_{x(P)}^{\infty} e^{-\frac{1}{2}t^2}\,dt.$$

If X is a variable, normally distributed with zero mean and unit variance, $P/100$ is the probability that $X \geqslant x(P)$. The lower P per cent points are given by symmetry as $-x(P)$, and the probability that $|X| \geqslant x(P)$ is $2P/100$.



P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$	P	$x(P)$
50	0·0000	5·0	1·6449	3·0	1·8808	2·0	2·0537	1·0	2·3263	0·10	3·0902
45	0·1257	4·8	1·6646	2·9	1·8957	1·9	2·0749	0·9	2·3656	0·09	3·1214
40	0·2533	4·6	1·6849	2·8	1·9110	1·8	2·0969	0·8	2·4089	0·08	3·1559
35	0·3853	4·4	1·7060	2·7	1·9268	1·7	2·1201	0·7	2·4573	0·07	3·1947
30	0·5244	4·2	1·7279	2·6	1·9431	1·6	2·1444	0·6	2·5121	0·06	3·2389
25	0·6745	4·0	1·7507	2·5	1·9600	1·5	2·1701	0·5	2·5758	0·05	3·2905
20	0·8416	3·8	1·7744	2·4	1·9774	1·4	2·1973	0·4	2·6521	0·01	3·7190
15	1·0364	3·6	1·7991	2·3	1·9954	1·3	2·2262	0·3	2·7478	0·005	3·8906
10	1·2816	3·4	1·8250	2·2	2·0141	1·2	2·2571	0·2	2·8782	0·001	4·2649
5	1·6449	3·2	1·8522	2·1	2·0335	1·1	2·2904	0·1	3·0902	0·0005	4·4172

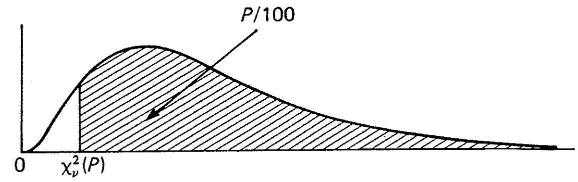
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \int_{\chi^2_\nu(P)}^{\infty} x^{\frac{1}{2}\nu - 1}\, e^{-\frac{1}{2}x}\, dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu - 1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

P	99·95	99·9	99·5	99	97·5	95	90	80	70	60
$\nu = 1$	0·0^{6}3927	0·0^{5}1571	0·0^{4}3927	0·0^{3}1571	0·0^{3}9821	0·003932	0·01579	0·06418	0·1485	0·2750
2	0·001000	0·002001	0·01003	0·02010	0·05064	0·1026	0·2107	0·4463	0·7133	1·022
3	0·01528	0·02430	0·07172	0·1148	0·2158	0·3518	0·5844	1·005	1·424	1·869
4	0·06392	0·09080	0·2070	0·2971	0·4844	0·7107	1·064	1·649	2·195	2·753
5	0·1581	0·2102	0·4117	0·5543	0·8312	1·145	1·610	2·343	3·000	3·655
6	0·2994	0·3811	0·6757	0·8721	1·237	1·635	2·204	3·070	3·828	4·570
7	0·4849	0·5985	0·9893	1·239	1·690	2·167	2·833	3·822	4·671	5·493
8	0·7104	0·8571	1·344	1·646	2·180	2·733	3·490	4·594	5·527	6·423
9	0·9717	1·152	1·735	2·088	2·700	3·325	4·168	5·380	6·393	7·357
10	1·265	1·479	2·156	2·558	3·247	3·940	4·865	6·179	7·267	8·295
11	1·587	1·834	2·603	3·053	3·816	4·575	5·578	6·989	8·148	9·237
12	1·934	2·214	3·074	3·571	4·404	5·226	6·304	7·807	9·034	10·18
13	2·305	2·617	3·565	4·107	5·009	5·892	7·042	8·634	9·926	11·13
14	2·697	3·041	4·075	4·660	5·629	6·571	7·790	9·467	10·82	12·08
15	3·108	3·483	4·601	5·229	6·262	7·261	8·547	10·31	11·72	13·03
16	3·536	3·942	5·142	5·812	6·908	7·962	9·312	11·15	12·62	13·98
17	3·980	4·416	5·697	6·408	7·564	8·672	10·09	12·00	13·53	14·94
18	4·439	4·905	6·265	7·015	8·231	9·390	10·86	12·86	14·44	15·89
19	4·912	5·407	6·844	7·633	8·907	10·12	11·65	13·72	15·35	16·85
20	5·398	5·921	7·434	8·260	9·591	10·85	12·44	14·58	16·27	17·81
21	5·896	6·447	8·034	8·897	10·28	11·59	13·24	15·44	17·18	18·77
22	6·404	6·983	8·643	9·542	10·98	12·34	14·04	16·31	18·10	19·73
23	6·924	7·529	9·260	10·20	11·69	13·09	14·85	17·19	19·02	20·69
24	7·453	8·085	9·886	10·86	12·40	13·85	15·66	18·06	19·94	21·65
25	7·991	8·649	10·52	11·52	13·12	14·61	16·47	18·94	20·87	22·62
26	8·538	9·222	11·16	12·20	13·84	15·38	17·29	19·82	21·79	23·58
27	9·093	9·803	11·81	12·88	14·57	16·15	18·11	20·70	22·72	24·54
28	9·656	10·39	12·46	13·56	15·31	16·93	18·94	21·59	23·65	25·51
29	10·23	10·99	13·12	14·26	16·05	17·71	19·77	22·48	24·58	26·48
30	10·80	11·59	13·79	14·95	16·79	18·49	20·60	23·36	25·51	27·44
32	11·98	12·81	15·13	16·36	18·29	20·07	22·27	25·15	27·37	29·38
34	13·18	14·06	16·50	17·79	19·81	21·66	23·95	26·94	29·24	31·31
36	14·40	15·32	17·89	19·23	21·34	23·27	25·64	28·73	31·12	33·25
38	15·64	16·61	19·29	20·69	22·88	24·88	27·34	30·54	32·99	35·19
40	16·91	17·92	20·71	22·16	24·43	26·51	29·05	32·34	34·87	37·13
50	23·46	24·67	27·99	29·71	32·36	34·76	37·69	41·45	44·31	46·86
60	30·34	31·74	35·53	37·48	40·48	43·19	46·46	50·64	53·81	56·62
70	37·47	39·04	43·28	45·44	48·76	51·74	55·33	59·90	63·35	66·40
80	44·79	46·52	51·17	53·54	57·15	60·39	64·28	69·21	72·92	76·19
90	52·28	54·16	59·20	61·75	65·65	69·13	73·29	78·56	82·51	85·99
100	59·90	61·92	67·33	70·06	74·22	77·93	82·36	87·95	92·13	95·81

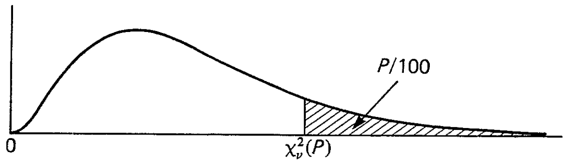
TABLE 8. PERCENTAGE POINTS OF THE χ^2-DISTRIBUTION

This table gives percentage points $\chi^2_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \int_{\chi^2_\nu(P)}^{\infty} x^{\frac{1}{2}\nu-1}\, e^{-\frac{1}{2}x}\, dx.$$

If X is a variable distributed as χ^2 with ν degrees of freedom, $P/100$ is the probability that $X \geqslant \chi^2_\nu(P)$.

For $\nu > 100$, $\sqrt{2X}$ is approximately normally distributed with mean $\sqrt{2\nu-1}$ and unit variance.



(The above shape applies for $\nu \geqslant 3$ only. When $\nu < 3$ the mode is at the origin.)

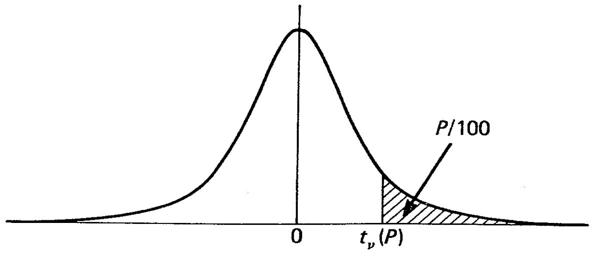
P	50	40	30	20	10	5	2·5	1	0·5	0·1	0·05
ν = 1	0·4549	0·7083	1·074	1·642	2·706	3·841	5·024	6·635	7·879	10·83	12·12
2	1·386	1·833	2·408	3·219	4·605	5·991	7·378	9·210	10·60	13·82	15·20
3	2·366	2·946	3·665	4·642	6·251	7·815	9·348	11·34	12·84	16·27	17·73
4	3·357	4·045	4·878	5·989	7·779	9·488	11·14	13·28	14·86	18·47	20·00
5	4·351	5·132	6·064	7·289	9·236	11·07	12·83	15·09	16·75	20·52	22·11
6	5·348	6·211	7·231	8·558	10·64	12·59	14·45	16·81	18·55	22·46	24·10
7	6·346	7·283	8·383	9·803	12·02	14·07	16·01	18·48	20·28	24·32	26·02
8	7·344	8·351	9·524	11·03	13·36	15·51	17·53	20·09	21·95	26·12	27·87
9	8·343	9·414	10·66	12·24	14·68	16·92	19·02	21·67	23·59	27·88	29·67
10	9·342	10·47	11·78	13·44	15·99	18·31	20·48	23·21	25·19	29·59	31·42
11	10·34	11·53	12·90	14·63	17·28	19·68	21·92	24·72	26·76	31·26	33·14
12	11·34	12·58	14·01	15·81	18·55	21·03	23·34	26·22	28·30	32·91	34·82
13	12·34	13·64	15·12	16·98	19·81	22·36	24·74	27·69	29·82	34·53	36·48
14	13·34	14·69	16·22	18·15	21·06	23·68	26·12	29·14	31·32	36·12	38·11
15	14·34	15·73	17·32	19·31	22·31	25·00	27·49	30·58	32·80	37·70	39·72
16	15·34	16·78	18·42	20·47	23·54	26·30	28·85	32·00	34·27	39·25	41·31
17	16·34	17·82	19·51	21·61	24·77	27·59	30·19	33·41	35·72	40·79	42·88
18	17·34	18·87	20·60	22·76	25·99	28·87	31·53	34·81	37·16	42·31	44·43
19	18·34	19·91	21·69	23·90	27·20	30·14	32·85	36·19	38·58	43·82	45·97
20	19·34	20·95	22·77	25·04	28·41	31·41	34·17	37·57	40·00	45·31	47·50
21	20·34	21·99	23·86	26·17	29·62	32·67	35·48	38·93	41·40	46·80	49·01
22	21·34	23·03	24·94	27·30	30·81	33·92	36·78	40·29	42·80	48·27	50·51
23	22·34	24·07	26·02	28·43	32·01	35·17	38·08	41·64	44·18	49·73	52·00
24	23·34	25·11	27·10	29·55	33·20	36·42	39·36	42·98	45·56	51·18	53·48
25	24·34	26·14	28·17	30·68	34·38	37·65	40·65	44·31	46·93	52·62	54·95
26	25·34	27·18	29·25	31·79	35·56	38·89	41·92	45·64	48·29	54·05	56·41
27	26·34	28·21	30·32	32·91	36·74	40·11	43·19	46·96	49·64	55·48	57·86
28	27·34	29·25	31·39	34·03	37·92	41·34	44·46	48·28	50·99	56·89	59·30
29	28·34	30·28	32·46	35·14	39·09	42·56	45·72	49·59	52·34	58·30	60·73
30	29·34	31·32	33·53	36·25	40·26	43·77	46·98	50·89	53·67	59·70	62·16
32	31·34	33·38	35·66	38·47	42·58	46·19	49·48	53·49	56·33	62·49	65·00
34	33·34	35·44	37·80	40·68	44·90	48·60	51·97	56·06	58·96	65·25	67·80
36	35·34	37·50	39·92	42·88	47·21	51·00	54·44	58·62	61·58	67·99	70·59
38	37·34	39·56	42·05	45·08	49·51	53·38	56·90	61·16	64·18	70·70	73·35
40	39·34	41·62	44·16	47·27	51·81	55·76	59·34	63·69	66·77	73·40	76·09
50	49·33	51·89	54·72	58·16	63·17	67·50	71·42	76·15	79·49	86·66	89·56
60	59·33	62·13	65·23	68·97	74·40	79·08	83·30	88·38	91·95	99·61	102·7
70	69·33	72·36	75·69	79·71	85·53	90·53	95·02	100·4	104·2	112·3	115·6
80	79·33	82·57	86·12	90·41	96·58	101·9	106·6	112·3	116·3	124·8	128·3
90	89·33	92·76	96·52	101·1	107·6	113·1	118·1	124·1	128·3	137·2	140·8
100	99·33	102·9	106·9	111·7	118·5	124·3	129·6	135·8	140·2	149·4	153·2

TABLE 10. PERCENTAGE POINTS OF THE t-DISTRIBUTION

This table gives percentage points $t_\nu(P)$ defined by the equation

$$\frac{P}{100} = \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu)} \int_{t_\nu(P)}^{\infty} \frac{dt}{(1+t^2/\nu)^{\frac{1}{2}(\nu+1)}}.$$

Let X_1 and X_2 be independent random variables having a normal distribution with zero mean and unit variance and a χ^2-distribution with ν degrees of freedom respectively; then $t = X_1/\sqrt{X_2/\nu}$ has Student's t-distribution with ν degrees of freedom, and the probability that $t \geqslant t_\nu(P)$ is $P/100$. The lower percentage points are given by symmetry as $-t_\nu(P)$, and the probability that $|t| \geqslant t_\nu(P)$ is $2P/100$.



The limiting distribution of t as ν tends to infinity is the normal distribution with zero mean and unit variance. When ν is large interpolation in ν should be harmonic.

P	40	30	25	20	15	10	5	2·5	1	0·5	0·1	0·05
$\nu = 1$	0·3249	0·7265	1·0000	1·3764	1·963	3·078	6·314	12·71	31·82	63·66	318·3	636·6
2	0·2887	0·6172	0·8165	1·0607	1·386	1·886	2·920	4·303	6·965	9·925	22·33	31·60
3	0·2767	0·5844	0·7649	0·9785	1·250	1·638	2·353	3·182	4·541	5·841	10·21	12·92
4	0·2707	0·5686	0·7407	0·9410	1·190	1·533	2·132	2·776	3·747	4·604	7·173	8·610
5	0·2672	0·5594	0·7267	0·9195	1·156	1·476	2·015	2·571	3·365	4·032	5·893	6·869
6	0·2648	0·5534	0·7176	0·9057	1·134	1·440	1·943	2·447	3·143	3·707	5·208	5·959
7	0·2632	0·5491	0·7111	0·8960	1·119	1·415	1·895	2·365	2·998	3·499	4·785	5·408
8	0·2619	0·5459	0·7064	0·8889	1·108	1·397	1·860	2·306	2·896	3·355	4·501	5·041
9	0·2610	0·5435	0·7027	0·8834	1·100	1·383	1·833	2·262	2·821	3·250	4·297	4·781
10	0·2602	0·5415	0·6998	0·8791	1·093	1·372	1·812	2·228	2·764	3·169	4·144	4·587
11	0·2596	0·5399	0·6974	0·8755	1·088	1·363	1·796	2·201	2·718	3·106	4·025	4·437
12	0·2590	0·5386	0·6955	0·8726	1·083	1·356	1·782	2·179	2·681	3·055	3·930	4·318
13	0·2586	0·5375	0·6938	0·8702	1·079	1·350	1·771	2·160	2·650	3·012	3·852	4·221
14	0·2582	0·5366	0·6924	0·8681	1·076	1·345	1·761	2·145	2·624	2·977	3·787	4·140
15	0·2579	0·5357	0·6912	0·8662	1·074	1·341	1·753	2·131	2·602	2·947	3·733	4·073
16	0·2576	0·5350	0·6901	0·8647	1·071	1·337	1·746	2·120	2·583	2·921	3·686	4·015
17	0·2573	0·5344	0·6892	0·8633	1·069	1·333	1·740	2·110	2·567	2·898	3·646	3·965
18	0·2571	0·5338	0·6884	0·8620	1·067	1·330	1·734	2·101	2·552	2·878	3·610	3·922
19	0·2569	0·5333	0·6876	0·8610	1·066	1·328	1·729	2·093	2·539	2·861	3·579	3·883
20	0·2567	0·5329	0·6870	0·8600	1·064	1·325	1·725	2·086	2·528	2·845	3·552	3·850
21	0·2566	0·5325	0·6864	0·8591	1·063	1·323	1·721	2·080	2·518	2·831	3·527	3·819
22	0·2564	0·5321	0·6858	0·8583	1·061	1·321	1·717	2·074	2·508	2·819	3·505	3·792
23	0·2563	0·5317	0·6853	0·8575	1·060	1·319	1·714	2·069	2·500	2·807	3·485	3·768
24	0·2562	0·5314	0·6848	0·8569	1·059	1·318	1·711	2·064	2·492	2·797	3·467	3·745
25	0·2561	0·5312	0·6844	0·8562	1·058	1·316	1·708	2·060	2·485	2·787	3·450	3·725
26	0·2560	0·5309	0·6840	0·8557	1·058	1·315	1·706	2·056	2·479	2·779	3·435	3·707
27	0·2559	0·5306	0·6837	0·8551	1·057	1·314	1·703	2·052	2·473	2·771	3·421	3·690
28	0·2558	0·5304	0·6834	0·8546	1·056	1·313	1·701	2·048	2·467	2·763	3·408	3·674
29	0·2557	0·5302	0·6830	0·8542	1·055	1·311	1·699	2·045	2·462	2·756	3·396	3·659
30	0·2556	0·5300	0·6828	0·8538	1·055	1·310	1·697	2·042	2·457	2·750	3·385	3·646
32	0·2555	0·5297	0·6822	0·8530	1·054	1·309	1·694	2·037	2·449	2·738	3·365	3·622
34	0·2553	0·5294	0·6818	0·8523	1·052	1·307	1·691	2·032	2·441	2·728	3·348	3·601
36	0·2552	0·5291	0·6814	0·8517	1·052	1·306	1·688	2·028	2·434	2·719	3·333	3·582
38	0·2551	0·5288	0·6810	0·8512	1·051	1·304	1·686	2·024	2·429	2·712	3·319	3·566
40	0·2550	0·5286	0·6807	0·8507	1·050	1·303	1·684	2·021	2·423	2·704	3·307	3·551
50	0·2547	0·5278	0·6794	0·8489	1·047	1·299	1·676	2·009	2·403	2·678	3·261	3·496
60	0·2545	0·5272	0·6786	0·8477	1·045	1·296	1·671	2·000	2·390	2·660	3·232	3·460
120	0·2539	0·5258	0·6765	0·8446	1·041	1·289	1·658	1·980	2·358	2·617	3·160	3·373
∞	0·2533	0·5244	0·6745	0·8416	1·036	1·282	1·645	1·960	2·326	2·576	3·090	3·291

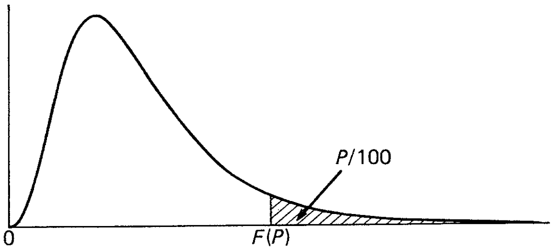
TABLE 12(*a*). 10 PER CENT POINTS OF THE *F*-DISTRIBUTION

The function tabulated is $F(P) = F(P|\nu_1, \nu_2)$ defined by the equation

$$\frac{P}{100} = \frac{\Gamma(\frac{1}{2}\nu_1+\frac{1}{2}\nu_2)}{\Gamma(\frac{1}{2}\nu_1)\,\Gamma(\frac{1}{2}\nu_2)}\,\nu_1^{\frac{1}{2}\nu_1}\,\nu_2^{\frac{1}{2}\nu_2}\int_{F(P)}^{\infty}\frac{F^{\frac{1}{2}\nu_1-1}}{(\nu_2+\nu_1 F)^{\frac{1}{2}(\nu_1+\nu_2)}}\,dF,$$

for $P = 10$, 5, 2·5, 1, 0·5 and 0·1. The lower percentage points, that is the values $F'(P) = F'(P|\nu_1, \nu_2)$ such that the probability that $F \leqslant F'(P)$ is equal to $P/100$, may be found by the formula

$$F'(P|\nu_1, \nu_2) = 1/F(P|\nu_2, \nu_1).$$

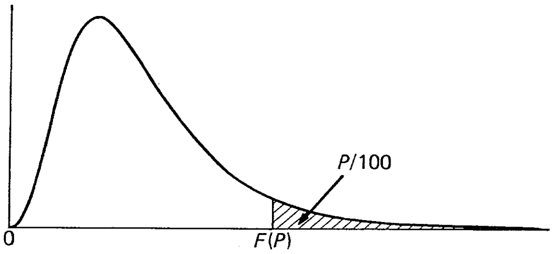


(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	39·86	49·50	53·59	55·83	57·24	58·20	58·91	59·44	60·19	60·71	62·00	63·33
2	8·526	9·000	9·162	9·243	9·293	9·326	9·349	9·367	9·392	9·408	9·450	9·491
3	5·538	5·462	5·391	5·343	5·309	5·285	5·266	5·252	5·230	5·216	5·176	5·134
4	4·545	4·325	4·191	4·107	4·051	4·010	3·979	3·955	3·920	3·896	3·831	3·761
5	4·060	3·780	3·619	3·520	3·453	3·405	3·368	3·339	3·297	3·268	3·191	3·105
6	3·776	3·463	3·289	3·181	3·108	3·055	3·014	2·983	2·937	2·905	2·818	2·722
7	3·589	3·257	3·074	2·961	2·883	2·827	2·785	2·752	2·703	2·668	2·575	2·471
8	3·458	3·113	2·924	2·806	2·726	2·668	2·624	2·589	2·538	2·502	2·404	2·293
9	3·360	3·006	2·813	2·693	2·611	2·551	2·505	2·469	2·416	2·379	2·277	2·159
10	3·285	2·924	2·728	2·605	2·522	2·461	2·414	2·377	2·323	2·284	2·178	2·055
11	3·225	2·860	2·660	2·536	2·451	2·389	2·342	2·304	2·248	2·209	2·100	1·972
12	3·177	2·807	2·606	2·480	2·394	2·331	2·283	2·245	2·188	2·147	2·036	1·904
13	3·136	2·763	2·560	2·434	2·347	2·283	2·234	2·195	2·138	2·097	1·983	1·846
14	3·102	2·726	2·522	2·395	2·307	2·243	2·193	2·154	2·095	2·054	1·938	1·797
15	3·073	2·695	2·490	2·361	2·273	2·208	2·158	2·119	2·059	2·017	1·899	1·755
16	3·048	2·668	2·462	2·333	2·244	2·178	2·128	2·088	2·028	1·985	1·866	1·718
17	3·026	2·645	2·437	2·308	2·218	2·152	2·102	2·061	2·001	1·958	1·836	1·686
18	3·007	2·624	2·416	2·286	2·196	2·130	2·079	2·038	1·977	1·933	1·810	1·657
19	2·990	2·606	2·397	2·266	2·176	2·109	2·058	2·017	1·956	1·912	1·787	1·631
20	2·975	2·589	2·380	2·249	2·158	2·091	2·040	1·999	1·937	1·892	1·767	1·607
21	2·961	2·575	2·365	2·233	2·142	2·075	2·023	1·982	1·920	1·875	1·748	1·586
22	2·949	2·561	2·351	2·219	2·128	2·060	2·008	1·967	1·904	1·859	1·731	1·567
23	2·937	2·549	2·339	2·207	2·115	2·047	1·995	1·953	1·890	1·845	1·716	1·549
24	2·927	2·538	2·327	2·195	2·103	2·035	1·983	1·941	1·877	1·832	1·702	1·533
25	2·918	2·528	2·317	2·184	2·092	2·024	1·971	1·929	1·866	1·820	1·689	1·518
26	2·909	2·519	2·307	2·174	2·082	2·014	1·961	1·919	1·855	1·809	1·677	1·504
27	2·901	2·511	2·299	2·165	2·073	2·005	1·952	1·909	1·845	1·799	1·666	1·491
28	2·894	2·503	2·291	2·157	2·064	1·996	1·943	1·900	1·836	1·790	1·656	1·478
29	2·887	2·495	2·283	2·149	2·057	1·988	1·935	1·892	1·827	1·781	1·647	1·467
30	2·881	2·489	2·276	2·142	2·049	1·980	1·927	1·884	1·819	1·773	1·638	1·456
32	2·869	2·477	2·263	2·129	2·036	1·967	1·913	1·870	1·805	1·758	1·622	1·437
34	2·859	2·466	2·252	2·118	2·024	1·955	1·901	1·858	1·793	1·745	1·608	1·419
36	2·850	2·456	2·243	2·108	2·014	1·945	1·891	1·847	1·781	1·734	1·595	1·404
38	2·842	2·448	2·234	2·099	2·005	1·935	1·881	1·838	1·772	1·724	1·584	1·390
40	2·835	2·440	2·226	2·091	1·997	1·927	1·873	1·829	1·763	1·715	1·574	1·377
60	2·791	2·393	2·177	2·041	1·946	1·875	1·819	1·775	1·707	1·657	1·511	1·291
120	2·748	2·347	2·130	1·992	1·896	1·824	1·767	1·722	1·652	1·601	1·447	1·193
∞	2·706	2·303	2·084	1·945	1·847	1·774	1·717	1·670	1·599	1·546	1·383	1·000

TABLE 12(*b*). 5 PER CENT POINTS OF THE *F*-DISTRIBUTION

If $F = \frac{X_1}{\nu_1} \Big/ \frac{X_2}{\nu_2}$, where X_1 and X_2 are independent random variables distributed as χ^2 with ν_1 and ν_2 degrees of freedom respectively, then the probabilities that $F \geqslant F(P)$ and that $F \leqslant F'(P)$ are both equal to $P/100$. Linear interpolation in ν_1 and ν_2 will generally be sufficiently accurate except when either $\nu_1 > 12$ or $\nu_2 > 40$, when harmonic interpolation should be used.

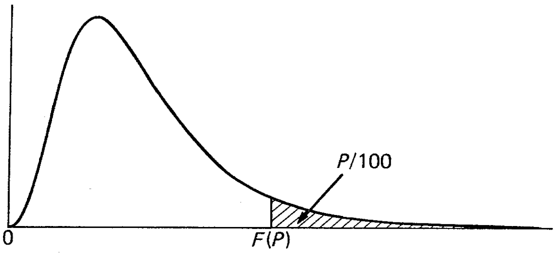


(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	161·4	199·5	215·7	224·6	230·2	234·0	236·8	238·9	241·9	243·9	249·1	254·3
2	18·51	19·00	19·16	19·25	19·30	19·33	19·35	19·37	19·40	19·41	19·45	19·50
3	10·13	9·552	9·277	9·117	9·013	8·941	8·887	8·845	8·786	8·745	8·639	8·526
4	7·709	6·944	6·591	6·388	6·256	6·163	6·094	6·041	5·964	5·912	5·774	5·628
5	6·608	5·786	5·409	5·192	5·050	4·950	4·876	4·818	4·735	4·678	4·527	4·365
6	5·987	5·143	4·757	4·534	4·387	4·284	4·207	4·147	4·060	4·000	3·841	3·669
7	5·591	4·737	4·347	4·120	3·972	3·866	3·787	3·726	3·637	3·575	3·410	3·230
8	5·318	4·459	4·066	3·838	3·687	3·581	3·500	3·438	3·347	3·284	3·115	2·928
9	5·117	4·256	3·863	3·633	3·482	3·374	3·293	3·230	3·137	3·073	2·900	2·707
10	4·965	4·103	3·708	3·478	3·326	3·217	3·135	3·072	2·978	2·913	2·737	2·538
11	4·844	3·982	3·587	3·357	3·204	3·095	3·012	2·948	2·854	2·788	2·609	2·404
12	4·747	3·885	3·490	3·259	3·106	2·996	2·913	2·849	2·753	2·687	2·505	2·296
13	4·667	3·806	3·411	3·179	3·025	2·915	2·832	2·767	2·671	2·604	2·420	2·206
14	4·600	3·739	3·344	3·112	2·958	2·848	2·764	2·699	2·602	2·534	2·349	2·131
15	4·543	3·682	3·287	3·056	2·901	2·790	2·707	2·641	2·544	2·475	2·288	2·066
16	4·494	3·634	3·239	3·007	2·852	2·741	2·657	2·591	2·494	2·425	2·235	2·010
17	4·451	3·592	3·197	2·965	2·810	2·699	2·614	2·548	2·450	2·381	2·190	1·960
18	4·414	3·555	3·160	2·928	2·773	2·661	2·577	2·510	2·412	2·342	2·150	1·917
19	4·381	3·522	3·127	2·895	2·740	2·628	2·544	2·477	2·378	2·308	2·114	1·878
20	4·351	3·493	3·098	2·866	2·711	2·599	2·514	2·447	2·348	2·278	2·082	1·843
21	4·325	3·467	3·072	2·840	2·685	2·573	2·488	2·420	2·321	2·250	2·054	1·812
22	4·301	3·443	3·049	2·817	2·661	2·549	2·464	2·397	2·297	2·226	2·028	1·783
23	4·279	3·422	3·028	2·796	2·640	2·528	2·442	2·375	2·275	2·204	2·005	1·757
24	4·260	3·403	3·009	2·776	2·621	2·508	2·423	2·355	2·255	2·183	1·984	1·733
25	4·242	3·385	2·991	2·759	2·603	2·490	2·405	2·337	2·236	2·165	1·964	1·711
26	4·225	3·369	2·975	2·743	2·587	2·474	2·388	2·321	2·220	2·148	1·946	1·691
27	4·210	3·354	2·960	2·728	2·572	2·459	2·373	2·305	2·204	2·132	1·930	1·672
28	4·196	3·340	2·947	2·714	2·558	2·445	2·359	2·291	2·190	2·118	1·915	1·654
29	4·183	3·328	2·934	2·701	2·545	2·432	2·346	2·278	2·177	2·104	1·901	1·638
30	4·171	3·316	2·922	2·690	2·534	2·421	2·334	2·266	2·165	2·092	1·887	1·622
32	4·149	3·295	2·901	2·668	2·512	2·399	2·313	2·244	2·142	2·070	1·864	1·594
34	4·130	3·276	2·883	2·650	2·494	2·380	2·294	2·225	2·123	2·050	1·843	1·569
36	4·113	3·259	2·866	2·634	2·477	2·364	2·277	2·209	2·106	2·033	1·824	1·547
38	4·098	3·245	2·852	2·619	2·463	2·349	2·262	2·194	2·091	2·017	1·808	1·527
40	4·085	3·232	2·839	2·606	2·449	2·336	2·249	2·180	2·077	2·003	1·793	1·509
60	4·001	3·150	2·758	2·525	2·368	2·254	2·167	2·097	1·993	1·917	1·700	1·389
120	3·920	3·072	2·680	2·447	2·290	2·175	2·087	2·016	1·910	1·834	1·608	1·254
∞	3·841	2·996	2·605	2·372	2·214	2·099	2·010	1·938	1·831	1·752	1·517	1·000

TABLE 12(*d*). 1 PER CENT POINTS OF THE *F*-DISTRIBUTION

If $F = \dfrac{X_1}{\nu_1}\Big/\dfrac{X_2}{\nu_2}$, where X_1 and X_2 are independent random variables distributed as χ^2 with ν_1 and ν_2 degrees of freedom respectively, then the probabilities that $F \geqslant F(P)$ and that $F \leqslant F'(P)$ are both equal to $P/100$. Linear interpolation in ν_1 or ν_2 will generally be sufficiently accurate except when either $\nu_1 > 12$ or $\nu_2 > 40$, when harmonic interpolation should be used.



(This shape applies only when $\nu_1 \geqslant 3$. When $\nu_1 < 3$ the mode is at the origin.)

$\nu_1 =$	1	2	3	4	5	6	7	8	10	12	24	∞
$\nu_2 =$ 1	4052	4999	5403	5625	5764	5859	5928	5981	6056	6106	6235	6366
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.40	99.42	99.46	99.50
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.23	27.05	26.60	26.13
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.55	14.37	13.93	13.46
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.05	9.888	9.466	9.020
6	13.75	10.92	9.780	9.148	8.746	8.466	8.260	8.102	7.874	7.718	7.313	6.880
7	12.25	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.620	6.469	6.074	5.650
8	11.26	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.814	5.667	5.279	4.859
9	10.56	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.257	5.111	4.729	4.311
10	10.04	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.849	4.706	4.327	3.909
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.539	4.397	4.021	3.602
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.296	4.155	3.780	3.361
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.100	3.960	3.587	3.165
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	3.939	3.800	3.427	3.004
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.805	3.666	3.294	2.868
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.691	3.553	3.181	2.753
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.593	3.455	3.084	2.653
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.508	3.371	2.999	2.566
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.434	3.297	2.925	2.489
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.368	3.231	2.859	2.421
21	8.017	5.780	4.874	4.369	4.042	3.812	3.640	3.506	3.310	3.173	2.801	2.360
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.258	3.121	2.749	2.305
23	7.881	5.664	4.765	4.264	3.939	3.710	3.539	3.406	3.211	3.074	2.702	2.256
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.168	3.032	2.659	2.211
25	7.770	5.568	4.675	4.177	3.855	3.627	3.457	3.324	3.129	2.993	2.620	2.169
26	7.721	5.526	4.637	4.140	3.818	3.591	3.421	3.288	3.094	2.958	2.585	2.131
27	7.677	5.488	4.601	4.106	3.785	3.558	3.388	3.256	3.062	2.926	2.552	2.097
28	7.636	5.453	4.568	4.074	3.754	3.528	3.358	3.226	3.032	2.896	2.522	2.064
29	7.598	5.420	4.538	4.045	3.725	3.499	3.330	3.198	3.005	2.868	2.495	2.034
30	7.562	5.390	4.510	4.018	3.699	3.473	3.304	3.173	2.979	2.843	2.469	2.006
32	7.499	5.336	4.459	3.969	3.652	3.427	3.258	3.127	2.934	2.798	2.423	1.956
34	7.444	5.289	4.416	3.927	3.611	3.386	3.218	3.087	2.894	2.758	2.383	1.911
36	7.396	5.248	4.377	3.890	3.574	3.351	3.183	3.052	2.859	2.723	2.347	1.872
38	7.353	5.211	4.343	3.858	3.542	3.319	3.152	3.021	2.828	2.692	2.316	1.837
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.801	2.665	2.288	1.805
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.632	2.496	2.115	1.601
120	6.851	4.787	3.949	3.480	3.174	2.956	2.792	2.663	2.472	2.336	1.950	1.381
∞	6.635	4.605	3.782	3.319	3.017	2.802	2.639	2.511	2.321	2.185	1.791	1.000

Durbin–Watson test statistic d : 1% significance points of d_L and d_U.

n	$k'=1$		$k'=2$		$k'=3$		$k'=4$		$k'=5$	
	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	0.81	1.07	0.70	1.25	0.59	1.46	0.49	1.70	0.39	1.96
16	0.84	1.09	0.74	1.25	0.63	1.44	0.53	1.66	0.44	1.90
17	0.87	1.10	0.77	1.25	0.67	1.43	0.57	1.63	0.48	1.85
18	0.90	1.12	0.80	1.26	0.71	1.42	0.61	1.60	0.52	1.80
19	0.93	1.13	0.83	1.26	0.74	1.41	0.65	1.58	0.56	1.77
20	0.95	1.15	0.86	1.27	0.77	1.41	0.68	1.57	0.60	1.74
21	0.97	1.16	0.89	1.27	0.80	1.41	0.72	1.55	0.63	1.71
22	1.00	1.17	0.91	1.28	0.83	1.40	0.75	1.54	0.66	1.69
23	1.02	1.19	0.94	1.29	0.86	1.40	0.77	1.53	0.70	1.67
24	1.04	1.20	0.96	1.30	0.88	1.41	0.80	1.53	0.72	1.66
25	1.05	1.21	0.98	1.30	0.90	1.41	0.83	1.52	0.75	1.65
26	1.07	1.22	1.00	1.31	0.93	1.41	0.85	1.52	0.78	1.64
27	1.09	1.23	1.02	1.32	0.95	1.41	0.88	1.51	0.81	1.63
28	1.10	1.24	1.04	1.32	0.97	1.41	0.90	1.51	0.83	1.62
29	1.12	1.25	1.05	1.33	0.99	1.42	0.92	1.51	0.85	1.61
30	1.13	1.26	1.07	1.34	1.01	1.42	0.94	1.51	0.88	1.61
31	1.15	1.27	1.08	1.34	1.02	1.42	0.96	1.51	0.90	1.60
32	1.16	1.28	1.10	1.35	1.04	1.43	0.98	1.51	0.92	1.60
33	1.17	1.29	1.11	1.36	1.05	1.43	1.00	1.51	0.94	1.59
34	1.18	1.30	1.13	1.36	1.07	1.43	1.01	1.51	0.95	1.59
35	1.19	1.31	1.14	1.37	1.08	1.44	1.03	1.51	0.97	1.59
36	1.21	1.32	1.15	1.38	1.10	1.44	1.04	1.51	0.99	1.59
37	1.22	1.32	1.16	1.38	1.11	1.45	1.06	1.51	1.00	1.59
38	1.23	1.33	1.18	1.39	1.12	1.45	1.07	1.52	1.02	1.58
39	1.24	1.34	1.19	1.39	1.14	1.45	1.09	1.52	1.03	1.58
40	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
45	1.29	1.38	1.24	1.42	1.20	1.48	1.16	1.53	1.11	1.58
50	1.32	1.40	1.28	1.45	1.24	1.49	1.20	1.54	1.16	1.59
55	1.36	1.43	1.32	1.47	1.28	1.51	1.25	1.55	1.21	1.59
60	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
65	1.41	1.47	1.38	1.50	1.35	1.53	1.31	1.57	1.28	1.61
70	1.43	1.49	1.40	1.52	1.37	1.55	1.34	1.58	1.31	1.61
75	1.45	1.50	1.42	1.53	1.39	1.56	1.37	1.59	1.34	1.62
80	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
85	1.48	1.53	1.46	1.55	1.43	1.58	1.41	1.60	1.39	1.63
90	1.50	1.54	1.47	1.56	1.45	1.59	1.43	1.61	1.41	1.64
95	1.51	1.55	1.49	1.57	1.47	1.60	1.45	1.62	1.42	1.64
100	1.52	1.56	1.50	1.58	1.48	1.60	1.46	1.63	1.44	1.65

n = number of observations

k' = number of explanatory variables

Durbin–Watson test statistic d : 5% significance points of d_L and d_U.

n	$k'=1$		$k'=2$		$k'=3$		$k'=4$		$k'=5$	
	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	1.08	1.36	0.95	1.54	0.82	1.75	0.69	1.97	0.56	2.21
16	1.10	1.37	0.98	1.54	0.86	1.73	0.74	1.93	0.62	2.15
17	1.13	1.38	1.02	1.54	0.90	1.71	0.78	1.90	0.67	2.10
18	1.16	1.39	1.05	1.53	0.93	1.69	0.82	1.87	0.71	2.06
19	1.18	1.40	1.08	1.53	0.97	1.68	0.86	1.85	0.75	2.02
20	1.20	1.41	1.10	1.54	1.00	1.68	0.90	1.83	0.79	1.99
21	1.22	1.42	1.13	1.54	1.03	1.67	0.93	1.81	0.83	1.96
22	1.24	1.43	1.15	1.54	1.05	1.66	0.96	1.80	0.86	1.94
23	1.26	1.44	1.17	1.54	1.08	1.66	0.99	1.79	0.90	1.92
24	1.27	1.45	1.19	1.55	1.10	1.66	1.01	1.78	0.93	1.90
25	1.29	1.45	1.21	1.55	1.12	1.66	1.04	1.77	0.95	1.89
26	1.30	1.46	1.22	1.55	1.14	1.65	1.06	1.76	0.98	1.88
27	1.32	1.47	1.24	1.56	1.16	1.65	1.08	1.76	1.01	1.86
28	1.33	1.48	1.26	1.56	1.18	1.65	1.10	1.75	1.03	1.85
29	1.34	1.48	1.27	1.56	1.20	1.65	1.12	1.74	1.05	1.84
30	1.35	1.49	1.28	1.57	1.21	1.65	1.14	1.74	1.07	1.83
31	1.36	1.50	1.30	1.57	1.23	1.65	1.16	1.74	1.09	1.83
32	1.37	1.50	1.31	1.57	1.24	1.65	1.18	1.73	1.11	1.82
33	1.38	1.51	1.32	1.58	1.26	1.65	1.19	1.73	1.13	1.81
34	1.39	1.51	1.33	1.58	1.27	1.65	1.21	1.73	1.15	1.81
35	1.40	1.52	1.34	1.58	1.28	1.65	1.22	1.73	1.16	1.80
36	1.41	1.52	1.35	1.59	1.29	1.65	1.24	1.73	1.18	1.80
37	1.42	1.53	1.36	1.59	1.31	1.66	1.25	1.72	1.19	1.80
38	1.43	1.54	1.37	1.59	1.32	1.66	1.26	1.72	1.21	1.79
39	1.43	1.54	1.38	1.60	1.33	1.66	1.27	1.72	1.22	1.79
40	1.44	1.54	1.39	1.60	1.34	1.66	1.29	1.72	1.23	1.79
45	1.48	1.57	1.43	1.62	1.38	1.67	1.34	1.72	1.29	1.78
50	1.50	1.59	1.46	1.63	1.42	1.67	1.38	1.72	1.34	1.77
55	1.53	1.60	1.49	1.64	1.45	1.68	1.41	1.72	1.38	1.77
60	1.55	1.62	1.51	1.65	1.48	1.69	1.44	1.73	1.41	1.77
65	1.57	1.63	1.54	1.66	1.50	1.70	1.47	1.73	1.44	1.77
70	1.58	1.64	1.55	1.67	1.52	1.70	1.49	1.74	1.46	1.77
75	1.60	1.65	1.57	1.68	1.54	1.71	1.51	1.74	1.49	1.77
80	1.61	1.66	1.59	1.69	1.56	1.72	1.53	1.74	1.51	1.77
85	1.62	1.67	1.60	1.70	1.57	1.72	1.55	1.75	1.52	1.77
90	1.63	1.68	1.61	1.70	1.59	1.73	1.57	1.75	1.54	1.78
95	1.64	1.69	1.62	1.71	1.60	1.73	1.58	1.75	1.56	1.78
100	1.65	1.69	1.63	1.72	1.61	1.74	1.59	1.76	1.57	1.78

n = number of observations

k' = number of explanatory variables