MLE and Binary Choice

I. MLE

Problem 1 $\quad p$ – prob. of an event

n – sample size

m – # event occurences

$$L(\theta \mid T_1, \dots, T_n) = f(\theta \mid T_1) \cdot \ldots \cdot f(\theta \mid T_n)$$

$$\log L(\theta \mid T_1, \dots, T_n) = \sum \ln f(\theta \mid T_i)$$

$$L(p \mid n, m) = \frac{n!}{m!\,(n-m)!} p^m (1-p)^{n-m}$$

$$\log L(p \mid n, m) = \log \frac{n!}{m!\,(n-m)!} + m \log p + (n-m) \log (1-p)$$

$$\frac{d\ell(p \mid n, m)}{dp} = 0 + \frac{m}{p} + \frac{n-m}{1-p} = 0$$

$$m(1-p) = (n-m)\,p$$

$$\hat{p} = m/n$$

$$\frac{d^2 \ell}{dp^2} = -\frac{m}{p^2} - \frac{n-m}{(1-p)^2} < 0$$

$\Rightarrow \hat{p} = m/n$ gives us max

II. Binary Choice Models

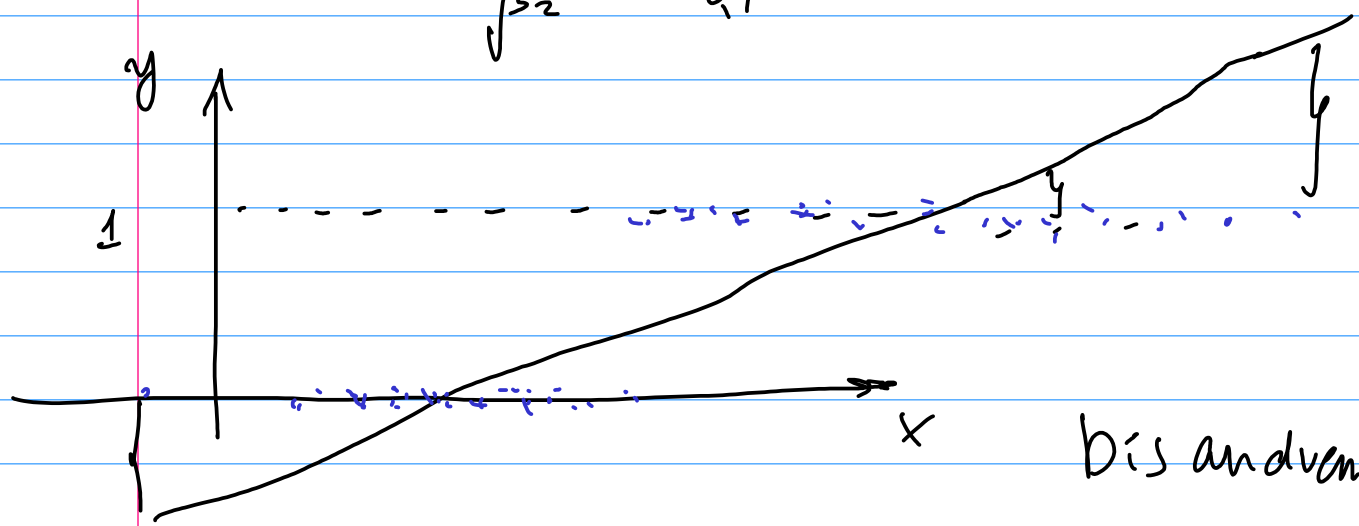
$$y_i = \begin{cases} 0, & \text{else} \\ 1, & \text{if event occurs} \end{cases}$$

Linear prob model

$$p_i = P(y_i = 1) = \beta_1 + \beta_2 \cdot x_i$$

$\hat{\beta}_1 = -0,3$

$\hat{\beta}_2 = 0,1$



$1 \to 2$

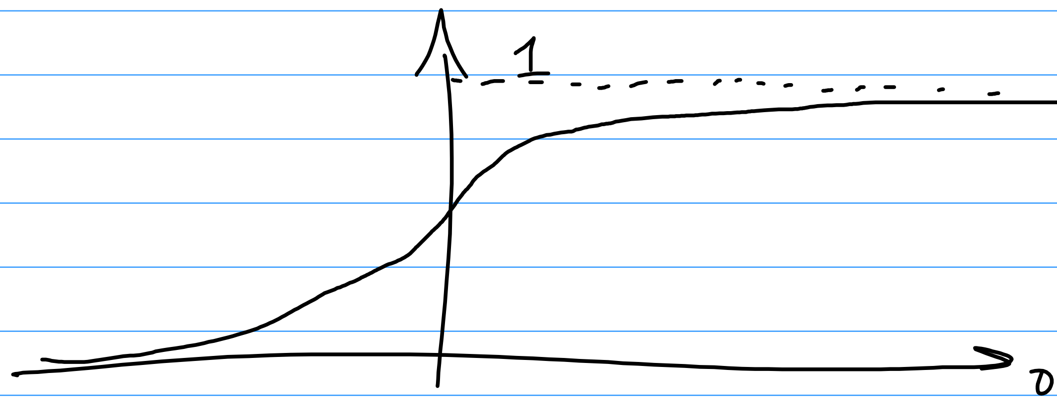
$100 \to 101$

Disadvantages

1. Prob can be negative or > 1
2. Change is not linear
3. Heteroscedasticity (also dep. on β)

Logit and Probit

Logit :



$$p(y_i = 1) = F(z_i) = \frac{1}{1 + e^{-z_i}}$$

$$z_i = \beta_1 + \beta_2 \cdot x_i$$

$$\hat{\beta}_1 = -9 \qquad \hat{\beta}_2 = 0{,}5$$

Interpretation : $\quad x = 15$

a) $$P(y_i = 1) = \frac{1}{1 + e^{-(-9 + 0{,}5 \cdot 15)}} = 0.18$$

b) $$\frac{d\, P(y_i = 1)}{dx} = \frac{e^{-(\beta_1 + \beta_2 \cdot x)}}{\left(1 + e^{-(\beta_1 + \beta_2 x)}\right)^2} \cdot \beta_2$$

$$\frac{d\hat{p}(y_i=1)}{dx} = \frac{e^{-(-9+0,5\cdot 15)}}{\left(1+e^{-(-9+0,5\cdot 15)}\right)^2}\cdot 0,5 =$$

$$= 0,07$$

for student who st. 15 h.

an additional h of study will increase p by 0,07

* for $x = 100$

$$\frac{d\hat{P}(y_i=1)}{dx} = 0,0,\ldots\ldots 08$$

In practice

- marginal effect for $\bar{x}$
- average marginal effect

$$\frac{\sum m.e \text{ for } x_i}{n}$$

A If X is binary

$$\hat{p}(X=1) - \hat{p}(X=0) =$$

$$= \frac{1}{1+e^0} - \frac{1}{1+e^2} = 0{,}38$$

$$P(y_i=1) = \frac{1}{1+e^{-(\beta_1+\beta_2 X)}}$$

$$\hat{\beta}_1 = -2$$

$$\hat{\beta}_2 = +2$$

Likelihood Ratio

(for comparing

R vs UR)

$$LR = -2(\ln L_R - \ln L_{UR}) \sim \chi^2_q$$

q – # restrictions

a) Pseudo R^2 (McFadden R^2)

$$\text{Pseudo } R^2 = 1 - \frac{\ln L}{\ln L_0}$$

likelihood for model that only contains constant

b) fraction of correctly predicted outcomes

c) correlation between outcomes and predicted probabilities

	Model 1	Model 2	Model 3
X	–	0,49	0,49
gender	–	–	0,15
black	–	–	–0,32
Constant	–0,32	–1,02	–0,9
log L	–68	–62	–61
pseudo R^2	0	0,09	0,1

a) $$\text{Pseudo } R^2 = 1 - \frac{\ln L}{\ln L_0}$$

Model 2 $$1 - \frac{62}{68} = 0{,}09$$

Model 3 $$1 - \frac{61}{68} = 0{,}1$$

b) $$LR = -2(\ln L_R - \ln L_{UR}) =$$

$$= -2(-68 + 61) = 14 \sim \chi_3$$

$\alpha = 5\%$ $\quad \chi^{crit} = 7{,}81$

$\Rightarrow$ H_0 is rejected

c) $LR = -2(-62 + 61) = 2 \sim \chi^2_2$

$\alpha = 5\%$ $\qquad \chi^{crit} = 5,95$

$\Rightarrow$ H_0 is not rejected

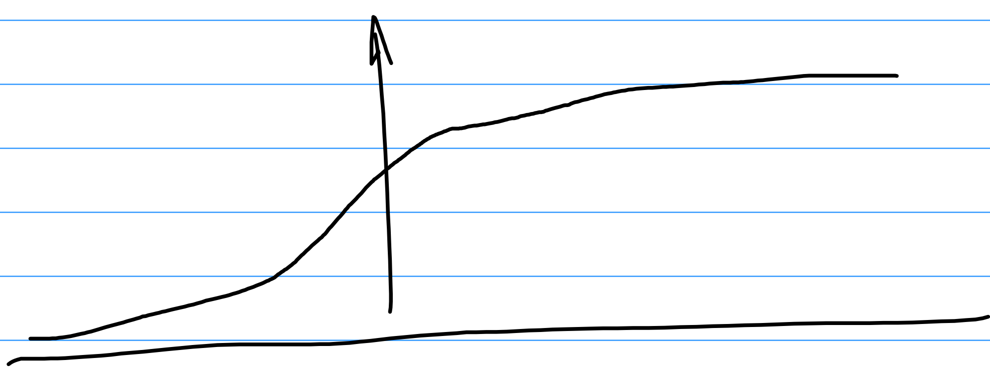
d) $\bar{x} = 1,4$

$$P(y_i = 1) = \frac{1}{1 + e^{-z_i}}$$

$$\frac{d\hat{p}}{dx} = \frac{e^{-(-1,02 + 0,49 \cdot 1,4)}}{\left(1 + e^{-(-1,02 + 0,49 \cdot 1,4)}\right)^2} \cdot 0,49 =$$

$$= 0,12$$

Probit Model



$$P(y_i = 1) = \Phi(z_i) \qquad z_i = \beta_1 + \beta_2 \cdot x_{1i} + \dots + \beta_k x_{ki}$$

$$\frac{\partial p(y_i = 1)}{\partial x_j} = \Phi'(z_i) \cdot \beta_j' =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\beta_1 + \beta_2 \cdot x_{1i} + \dots + \beta_k x_{ki})^2}{2}} \cdot \beta_j'$$