Cointegrated time series

1) Order of integration is the same (e.g. $I(1)$)

2) $\exists$ lin. combination of these t.s. s.t. it is stationary

Problem 1.
$$y_t = \alpha + y_{t-1} + \varepsilon_t$$
$$x_t = \beta + x_{t-1} + \nu_t$$
ε_t, ν_t — W.N. unrelated

What if $y_t \mid x_t$ is estimated?

$$y_t = \pi_0 + \pi_1 \cdot x_t + \omega_t$$

$H_0: \pi_1 = 0$ (y_t, x_t aren't related)

if H_0 is true

$\Rightarrow$ $y_t = \pi_0 + \omega_t$

y_t — $I(1)$ process

$\Rightarrow$ Hence, w_t — nonstationary

$\Rightarrow$ GM assumptions are violated $\Rightarrow$ misleading results

ECM

Consider $ADL(1,1)$:

$$y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$y_t, X_t - I(1)$$

$$y_t - y_{t-1} = \alpha_1 + \alpha_2 y_{t-1} - y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$\Delta y_t = \alpha_1 - (1-\alpha_2) y_{t-1} + \underbrace{\alpha_3 X_t - \alpha_3 X_{t-1}}$$

$$+ \alpha_3 X_{t-1} + \alpha_4 X_{t-1} + u_t$$

$$\Delta y_t = \alpha_1 - (1-\alpha_2) y_{t-1} + (\alpha_3 + \alpha_4) X_{t-1}$$

$$+ \alpha_3 \Delta X_t + u_t$$

$$\Delta y_t = \alpha_3 \Delta X_t - (1-\alpha_2)\left[y_{t-1} - \frac{\alpha_1}{1-\alpha_2} - \frac{\alpha_3+\alpha_4}{1-\alpha_2} X_{t-1} \right] + u_t$$

$$\Delta y_t = \alpha_3 \Delta X_t - \pi \left[\underbrace{y_{t-1} - \beta_1 - \beta_2 X_{t-1}}_{\text{cointegration relation}} \right] + u_t$$

$\pi = (1-\alpha_2)$ - adjustment coefficient

$$\beta_1 = \frac{\alpha_1}{1-\alpha_2} \quad ; \quad \beta_2 = \frac{\alpha_3+\alpha_4}{1-\alpha_2}$$

$$y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 X_t + \alpha_4 X_{t-1} + u_t$$

$$\bar{y}(1-\alpha_2) = \alpha_1 + (\alpha_3+\alpha_4)\bar{X}$$

Estimation of ECM

1. Est. $y | X$ $\Rightarrow$ $\hat{u}_t$ - residuals
2. Est $\Delta y_t = \Pi_1 \hat{u}_{t-1} + \Pi_2 \Delta X_t + \Pi_0 + u_t$

Granger Causality

(1) $y_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \dots \alpha_m y_{t-m} +$

$+ \beta_1 X_{t-1} + \beta_2 X_{t-2} + \dots \beta_m X_{t-m} + \varepsilon_t$

(2) $X_t = \alpha_0 + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \dots \alpha_m y_{t-m} +$

$+ \beta_1 X_{t-1} + \beta_2 X_{t-2} + \dots \beta_m X_{t-m} + \varepsilon_t$

for (1) $H_0: \beta_1 = \dots = \beta_m = 0$

if H_0 is rejected $\Rightarrow$ X_t Granger causes y_t

for (2) $H_0: \alpha_1 = \dots = \alpha_m = 0$

if H_0 is rejected $\Rightarrow$ y_t Granger causes X_t

Granger Causality $\neq$ causality

$\downarrow$

about in-sample fitting

VAR (m)

$$y_t = A_1 y_{t-1} + \dots + A_m \cdot y_{t-m} + B X_t + \varepsilon_t$$

y_t ↑ vector endogenous variables

A_1 ↑ matrix of coef.

X_t ↗ vector of exogenous variables

ε_t ↑ vector of innovations

E.g. VAR (1)

y_t - inflation rate

x_t - unemployment rate

$$\begin{cases} y_t = \phi_{11} y_{t-1} + \phi_{12} x_{t-1} + u_t \\ x_t = \phi_{21} y_{t-1} + \phi_{22} x_{t-1} + v_t \end{cases}$$

Matrix form:

$$z_t = \phi z_{t-1} + w_t$$

$$z_t = \begin{pmatrix} y_t \\ x_t \end{pmatrix} \quad \phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix} \quad w_t = \begin{pmatrix} u_t \\ v_t \end{pmatrix}$$

Problem 3.

ADL(2,1) $\qquad y_t, X_t$ — $I(1)$, in logaritms

$$y_t = \alpha_1 + \alpha_2 y_{t-1} + \alpha_3 y_{t-2} + \alpha_4 X_t + \alpha_5 X_{t-1} + u_t$$

SR elasticity

(1) $\bar{y}(1-\alpha_2-\alpha_3) = \alpha_1 + (\alpha_4+\alpha_5)\bar{X}$

$$\Rightarrow \quad \bar{y} = \frac{\alpha_1}{1-\alpha_2-\alpha_3} + \frac{\alpha_4+\alpha_5}{1-\alpha_2-\alpha_3}\bar{X}$$

LR elasticity