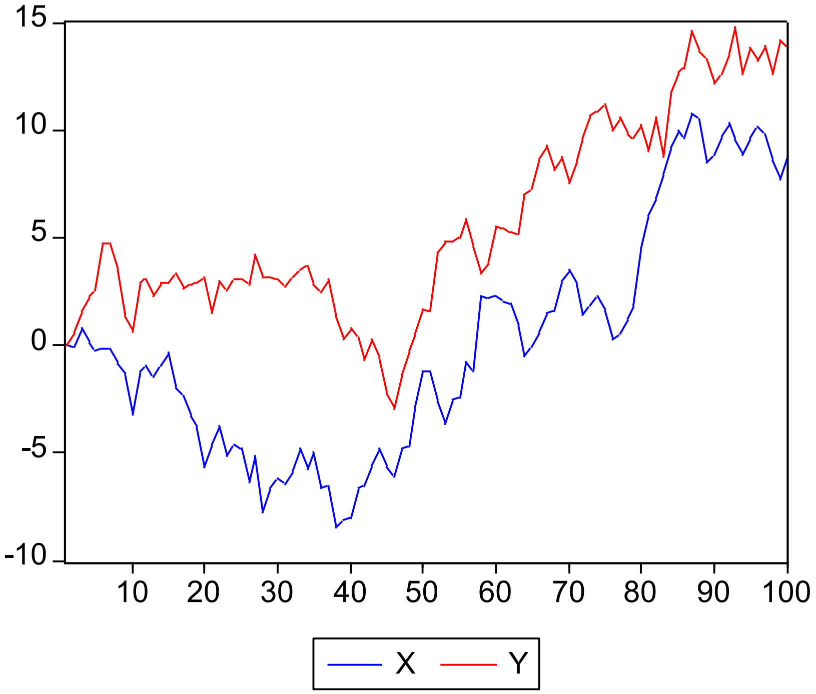
Elements of Econometrics.
Lecture 21.
Testing for Non-Stationarity.

FCS, 2022-2023

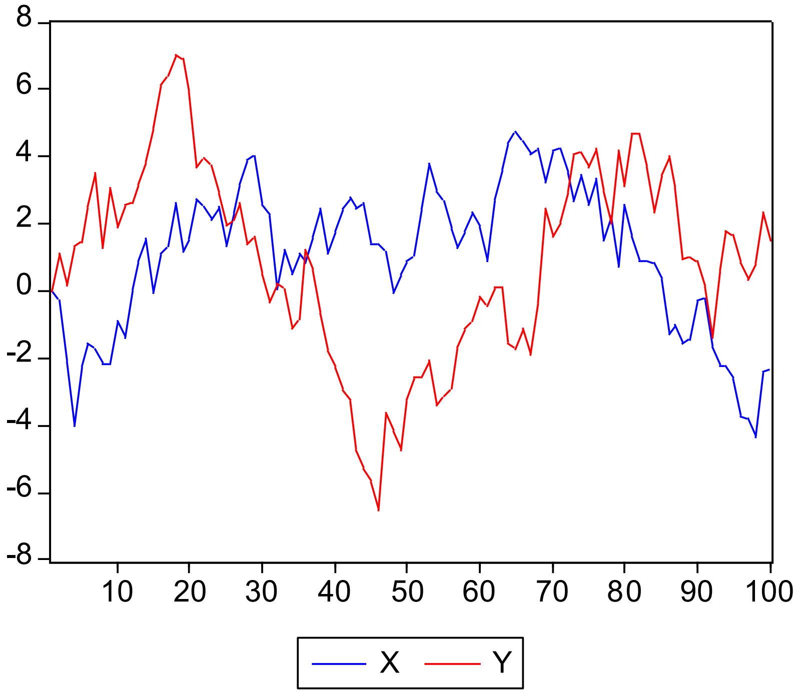
Regressions with Nonstationary Data

Monte Carlo experiment: $X(0)=Y(0)=0$; $X(t)=X(t\text{-}1)+\varepsilon_t$; $Y(t)=Y(t\text{-}1)+\mu_t$ (ε_t and μ_t are white noises with $\sigma^2_\varepsilon=\sigma^2_\mu=1$), T=100.

Sample 1:



Sample 2:



Spurious Regression Example

Dependent Variable: Y Method: Least Squares

Sample 1:

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	5.634	0.227	24.822	0.0000
X	**0.750**	**0.042**	17.835	0.0000

R-squared	0.764	Mean dependent var	5.702
S.D. dependent var	4.652	S.E. of regression	2.259
Sum squared resid	504.71	F-statistic	318.08
Durbin-Watson stat	**0.284**	Prob(F-statistic)	0.0000

Sample 2:

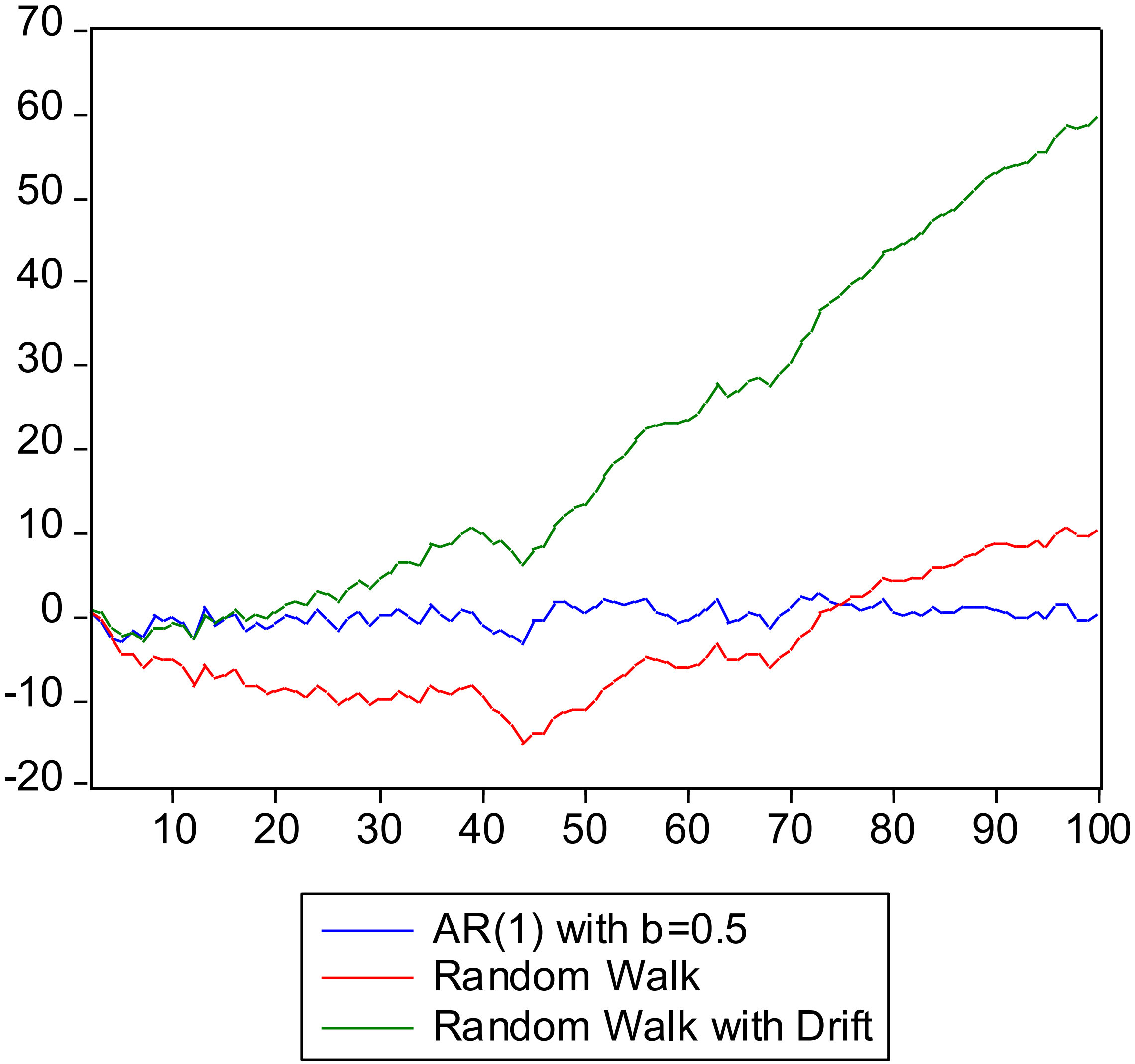
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	1.052	0.324	3.248	0.0016
X	**-0.158**	**0.136**	-1.160	0.2488

R-squared	0.014	Mean dependent var	0.884
S.D. dependent var	2.904	S.E. of regression	2.899
Sum squared resid	823.6	F-statistic	1.346
Durbin-Watson stat	**0.136**	Prob(F-statistic)	0.249

Pay attention to the Durbin-Watson Statistic: it indicates incorrect specification.
Disturbance term is a Random Walk ($u_t=Y_t-\beta_1-0*X_t$).

STATIONARY AND NONSTATIONARY SERIES

X(0)=0; u=N(0;1) – the same for all series. Drift=0.5.



DETECTING NONSTATIONARITY: AUTOCORRELATION FUNCTION

$$Y_t = \theta_0 + \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + ... + \theta_p Y_{t-p} + \phi_1 \varepsilon_t + \phi_2 \varepsilon_{t-1} + ... + \phi_{q+1} \varepsilon_{t-q}$$

Autocorrelation function

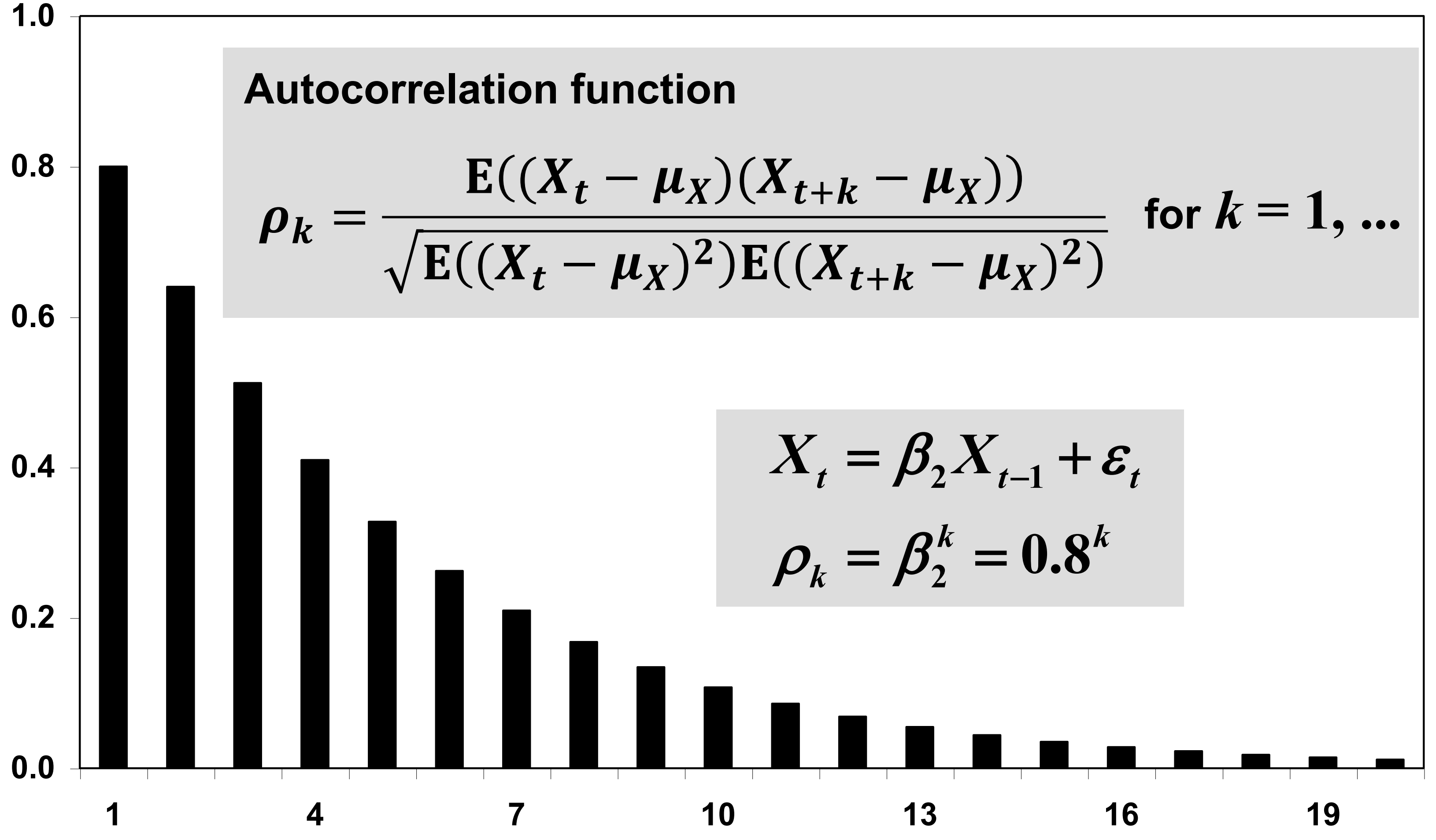
$$\rho_k = \frac{E\left((X_t - \mu_X)(X_{t+k} - \mu_X)\right)}{\sqrt{E\left((X_t - \mu_X)^2\right)E\left((X_{t+k} - \mu_X)^2\right)}} \quad \text{for } k = 1, ...$$

Autocorrelation function of an AR(1) process $X_t = \beta_2 X_{t-1} + \varepsilon_t \quad \rho_k = \beta_2^k$

Autocorrelation function of an MA(1) process $X_t = \varepsilon_t + \alpha_2 \varepsilon_{t-1} \quad \rho_1 = \dfrac{\alpha_2}{1 + \alpha_2^2}$

For example, the autocorrelation function for an AR(1) process $X_t = \beta_2 X_{t-1} + \varepsilon_t$ is $\rho_k = \beta_2^k$ if $\beta_2 < 1$ and the process is stationary. For MA(1) $\rho_k=0$ for all k>1.

DETECTING NONSTATIONARITY: CORRELOGRAM



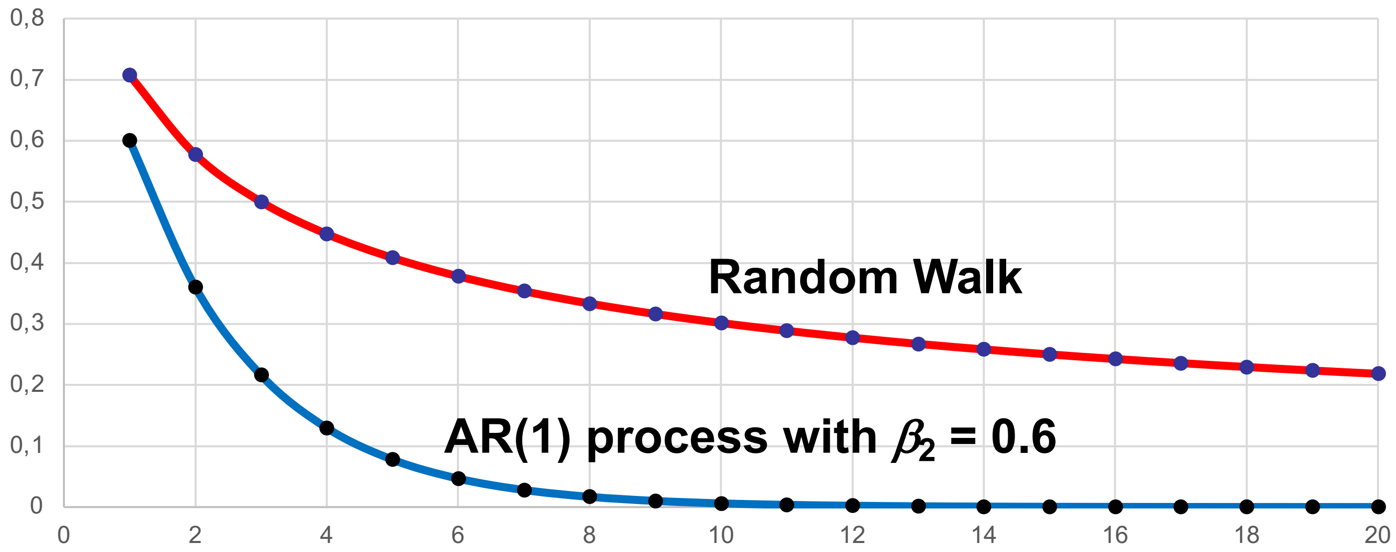
Correlogram of an AR(1) process

The figure shows the correlogram for the AR(1) process with $\beta_2 = 0.8$.

DETECTING NONSTATIONARITY: CORRELOGRAM

Autocorrelation function

$$\rho_k = \frac{\mathrm{E}((X_t - \mu_X)(X_{t+k} - \mu_X))}{\sqrt{\mathrm{E}((X_t - \mu_X)^2)\mathrm{E}((X_{t+k} - \mu_X)^2)}} \quad \text{for } k = 1, \ldots$$



AR(1) process with β_2 = 0.6

$$X_{t+i} = \beta_2 X_{t+i-1} + \varepsilon_{t+i}$$

$$\rho_k = \beta_2^k = 0.6^k$$

Random Walk

$$X_{t+i} = X_{t+i-1} + \varepsilon_{t+i}$$

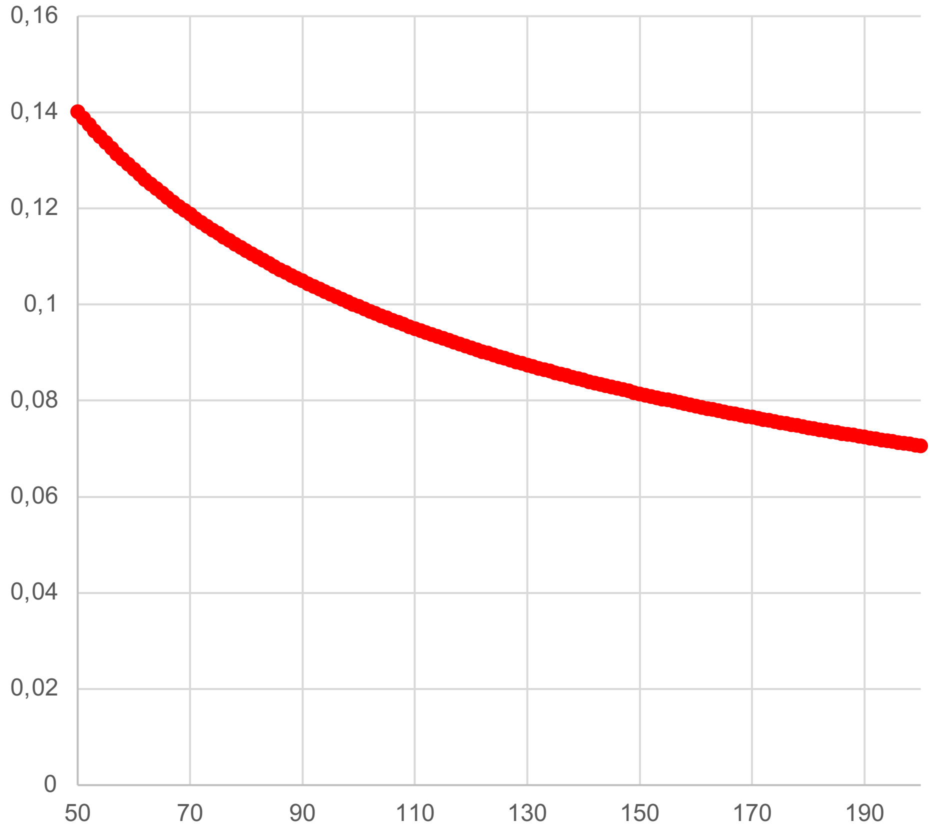
$$\rho_k = \sqrt{t/(t+k)}$$

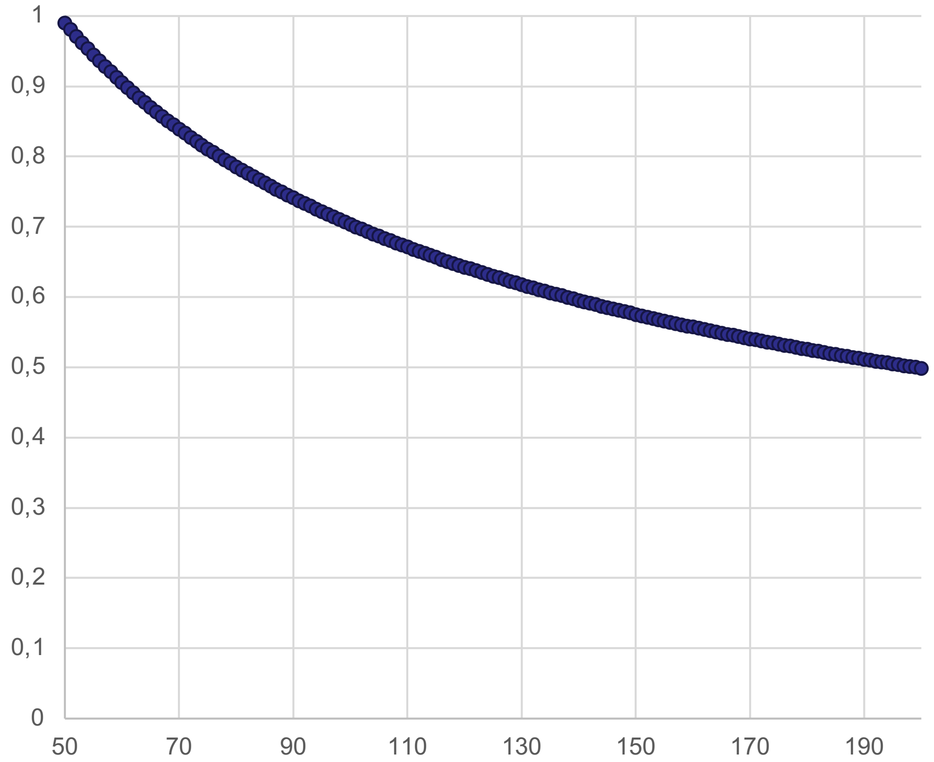
$$\rho_k = \sqrt{1/(1+k)} \quad (if \;\; t = 1)$$

Correlograms of an AR(1) process with β_2 = 0.6 and of the Random Walk for k=1,…,20.

GRAPHICAL TECHNIQUES FOR DETECTING NONSTATIONARITY

$$X_t = X_{t-1} + \varepsilon_t$$





Correlogram of a random walk (for k=51,…,200; t=1 (left) and t=50 (right))

In the case of nonstationary processes, the theoretical autocorrelation coefficients are not defined (since there may be no unique expectations and pop. variances) but for some processes one may be able to obtain an expression for E(r_k), the expected value of the sample autocorrelation coefficients.

Partial Autocorrelations

Partial Autocorrelations (PAC)

The partial autocorrelation at lag *k* is the regression coefficient on Y_{t-k} when Y_t is regressed on a constant, $Y_{t-1}, Y_{t-2}, \dots, Y_{t-k}$. This is a partial correlation since it measures the correlation of values that are *k* periods apart after removing the correlation from the intervening lags.

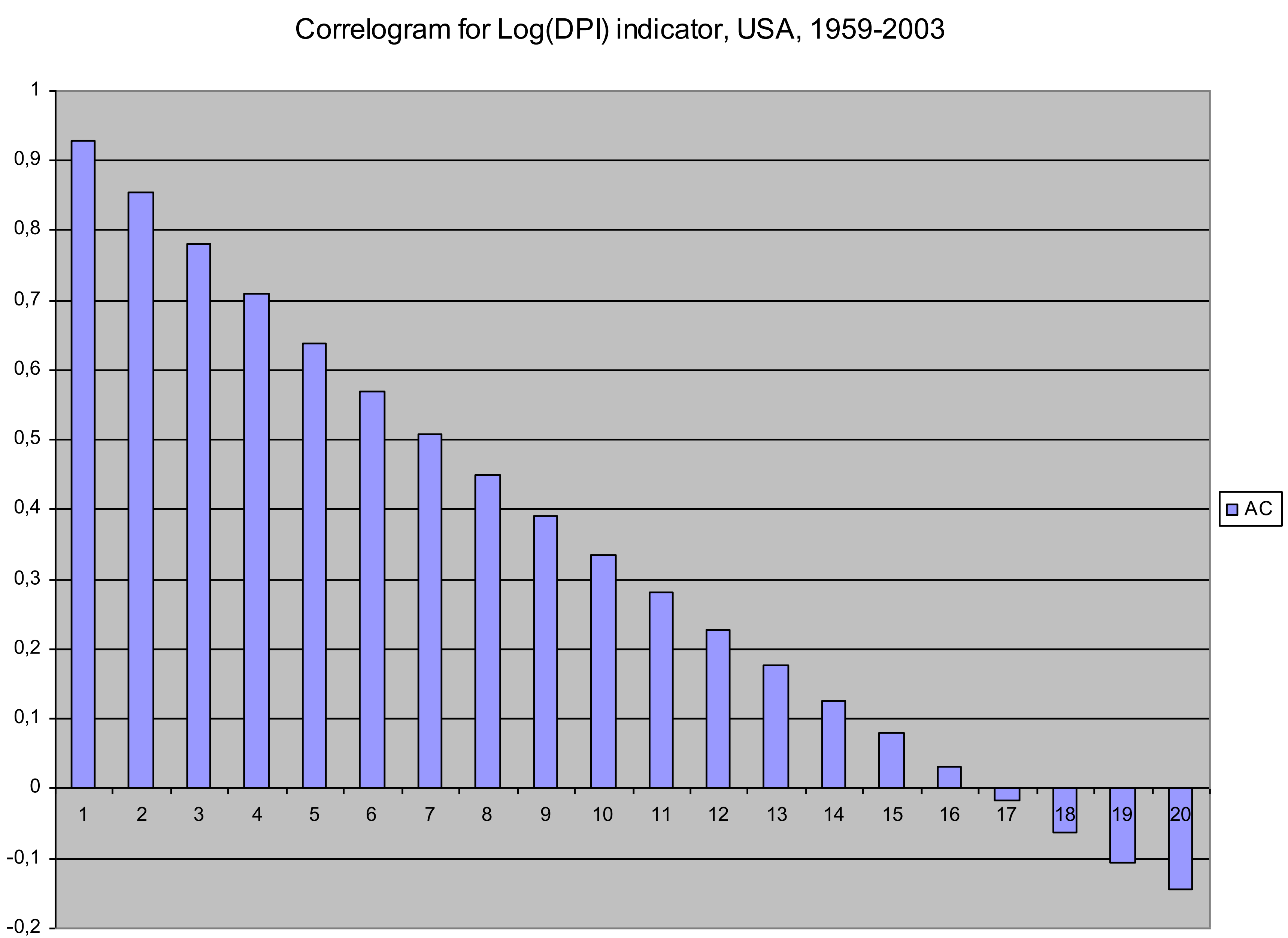
If the pattern of autocorrelation is one that can be captured by an autoregression of order less than *k*, then the partial autocorrelation at lag *k* will be close to zero.

Autocorrelation and Partial Autocorrelation Coefficients for Log(DPI), USA, 1059-2003 (EViews)

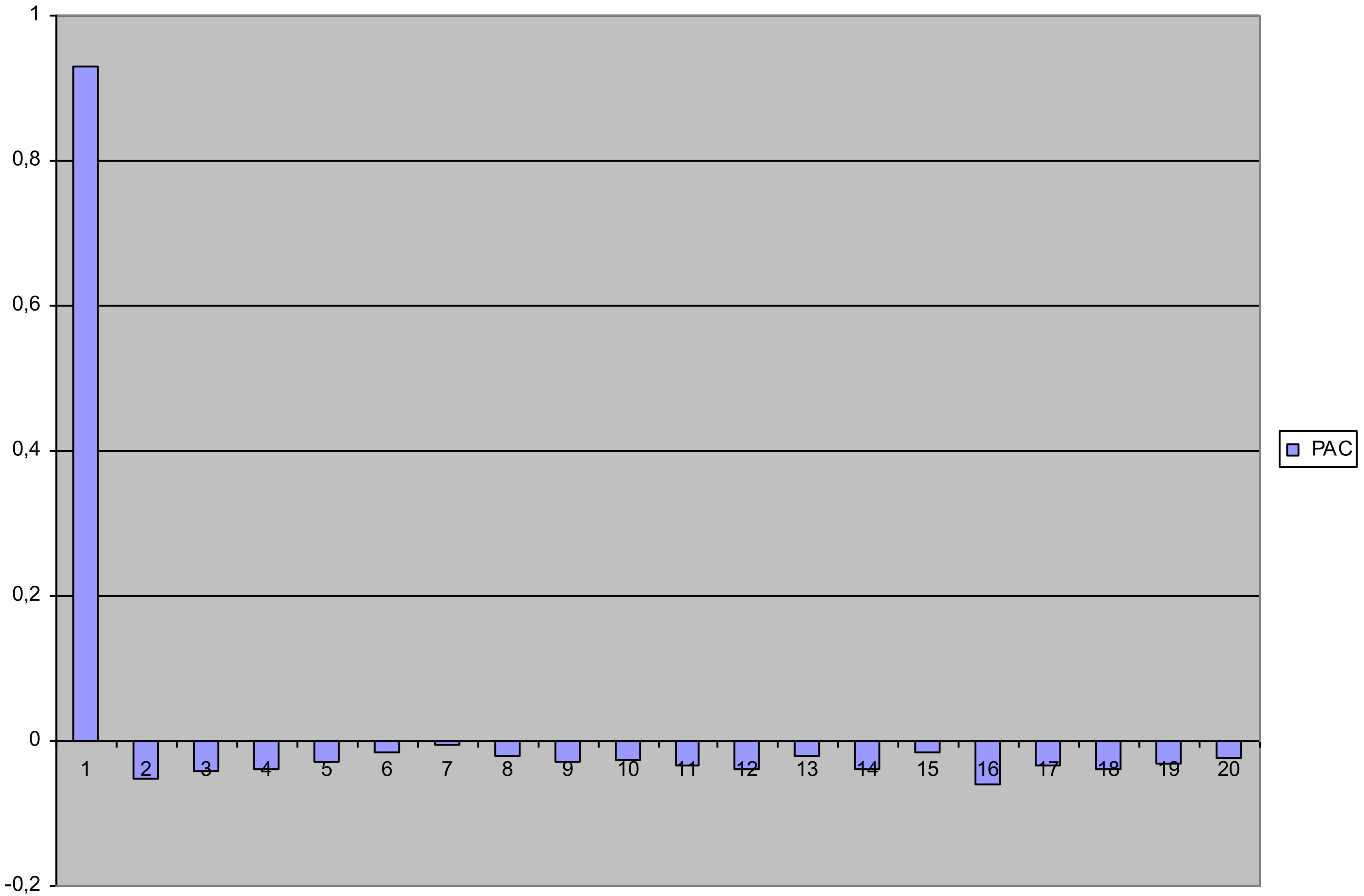
Sample: 1959 2003 Included observations: 45

k	AC	PAC	Q-Stat	Prob
1	0.929	0.929	41.474	0.000
2	0.856	-0.051	77.497	0.000
3	0.782	-0.042	108.31	0.000
4	0.709	-0.038	134.25	0.000
5	0.638	-0.029	155.77	0.000
6	0.570	-0.016	173.40	0.000
7	0.508	-0.006	187.76	0.000
8	0.449	-0.020	199.26	0.000
9	0.391	-0.029	208.24	0.000
10	0.336	-0.026	215.05	0.000
11	0.281	-0.034	219.98	0.000
12	0.227	-0.039	223.29	0.000
13	0.176	-0.022	225.35	0.000
14	0.126	-0.038	226.43	0.000
15	0.079	-0.016	226.87	0.000
16	0.031	-0.061	226.94	0.000
17	-0.016	-0.035	226.96	0.000
18	-0.062	-0.038	227.26	0.000
19	-0.105	-0.031	228.15	0.000
20	-0.144	-0.023	229.91	0.000

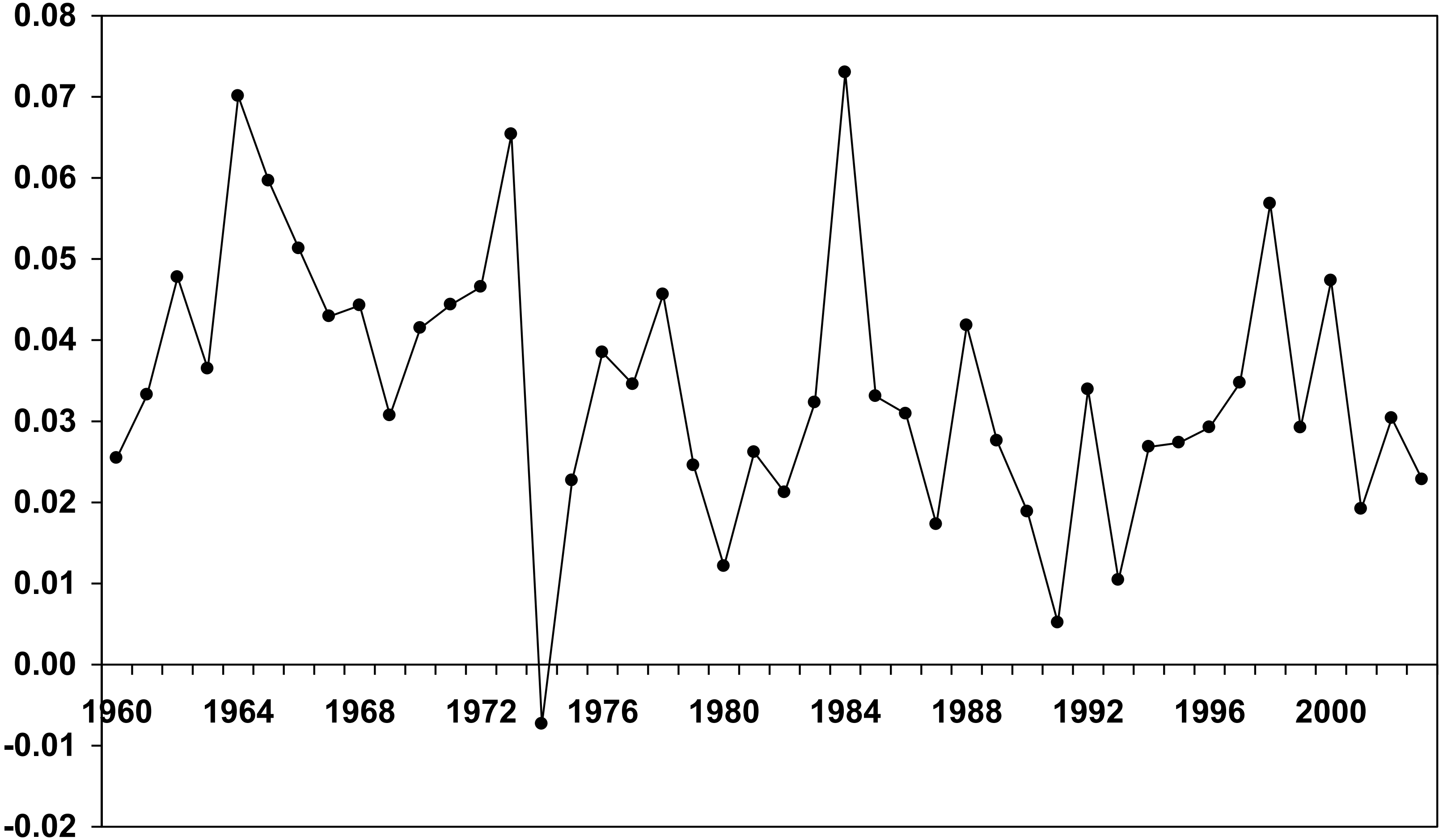
AC is Autocorrelation coefficient; PAC is Partial Autocorrelation coefficient. The Q-statistic at lag *k* is a test statistic for the null hypothesis that there is no autocorrelation up to order *k* (asymptotic χ^2 distribution, df=k)


Correlogram for Log(DPI) indicator, USA, 1959-2003
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-0,1
-0,2
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
AC

Partial Autocorrelation Coefficient, Log(DPI), USA, 1959-2003

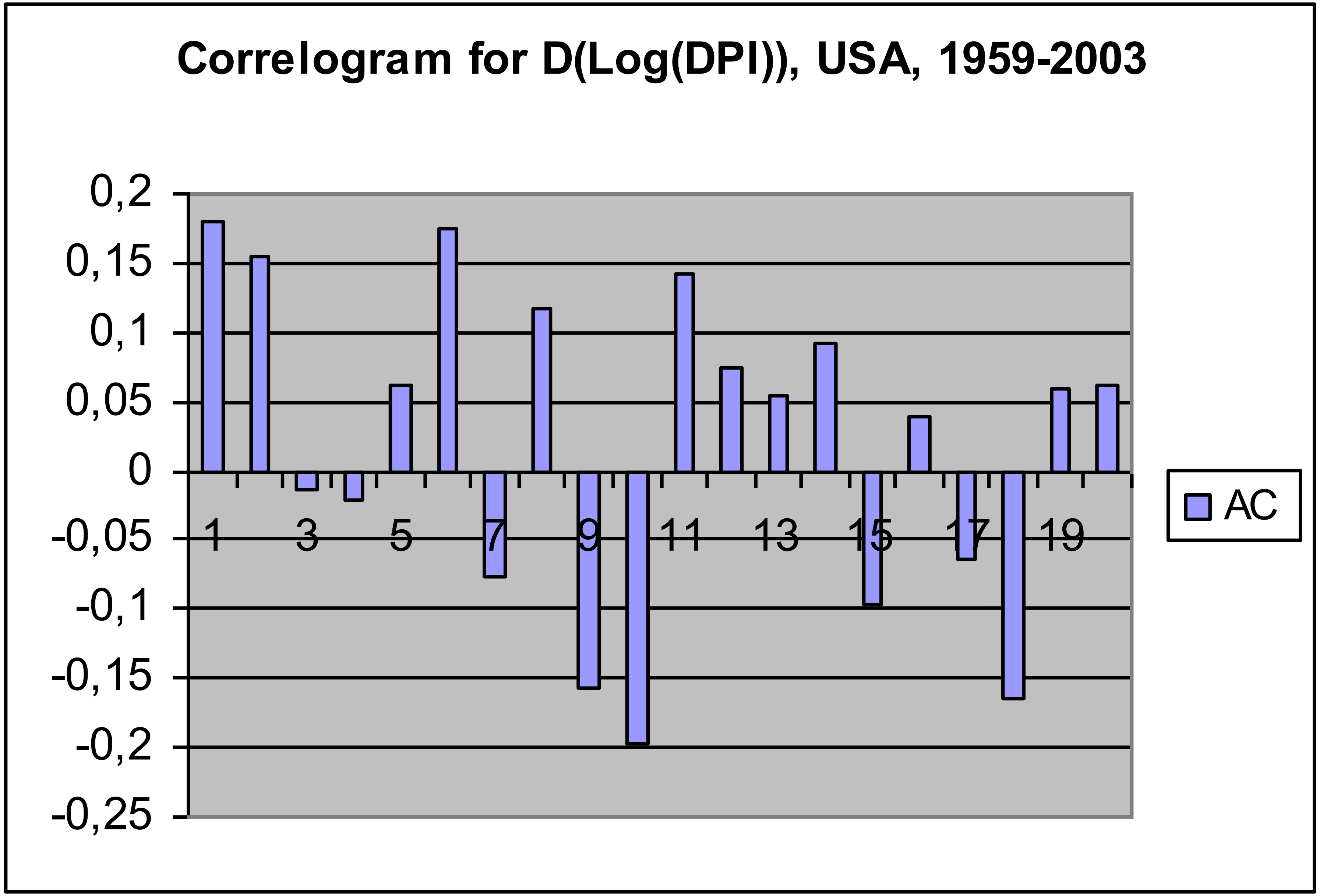


GRAPHICAL DETECTION OF NONSTATIONARITY



First difference of *LGDPI*

This figure shows the differenced series, which appears to be stationary around a mean annual growth rate of between 2 and 3 percent.


Correlogram for D(Log(DPI)), USA, 1959-2003
0,2
0,15
0,1
0,05
0
-0,05
-0,1
-0,15
-0,2
-0,25
1
3
5
7
9
11
13
15
17
19
AC

FORMAL TESTS FOR NONSTATIONARITY

General model $$Y_t = \beta_1 + \beta_2 Y_{t-1} + \delta t + \varepsilon_t$$

Transformed: $$Y_t - Y_{t-1} = b_1 + (b_2 - 1) Y_{t-1} + dt$$

Alternatives $$-1 < \beta_2 < 1 \quad \text{or} \quad \beta_2 = 1$$
$$\delta = 0 \quad \text{or} \quad \delta \neq 0$$

Case (a): Stationary AR(1)	$\beta_1 = *$	$\lvert\beta_2\rvert < 1$	$\delta = 0$
Case (b): Random Walk	$\beta_1 = 0$	$\beta_2 = 1$	$\delta = 0$
Case (c): Random Walk with Drift	$\beta_1 \neq 0$	$\beta_2 = 1$	$\delta = 0$
Case (d): Stationary AR(1) around Trend	$\beta_1 = *$	$\lvert\beta_2\rvert < 1$	$\delta \neq 0$
Case (e): Random Walk around Trend	$\beta_1 = *$	$\beta_2 = 1$	$\delta \neq 0$

There are five cases. β_1 = * means β_1 is unrestricted. Case (e) will be excluded. Cases (a) and (b) do not include time trend, cases (c) and (d) include trend. Availability of trend will be used to distinguish between these cases.

TESTS OF NONSTATIONARITY: INTRODUCTION

General model $$Y_t = \beta_1 + \beta_2 Y_{t-1} + \delta t + \varepsilon_t$$

Alternatives $$-1 < \beta_2 < 1 \text{ or } \beta_2 = 1$$

$$\delta = 0 \text{ or } \delta \neq 0$$

Case (e): Random Walk around Trend $\quad \beta_1 = * \quad \beta_2 = 1 \quad \delta \neq 0$

Excluded, as well as $|\beta_2| > 1$.

$$Y_t = \beta_1 + Y_{t-1} + \delta t + \varepsilon_t$$

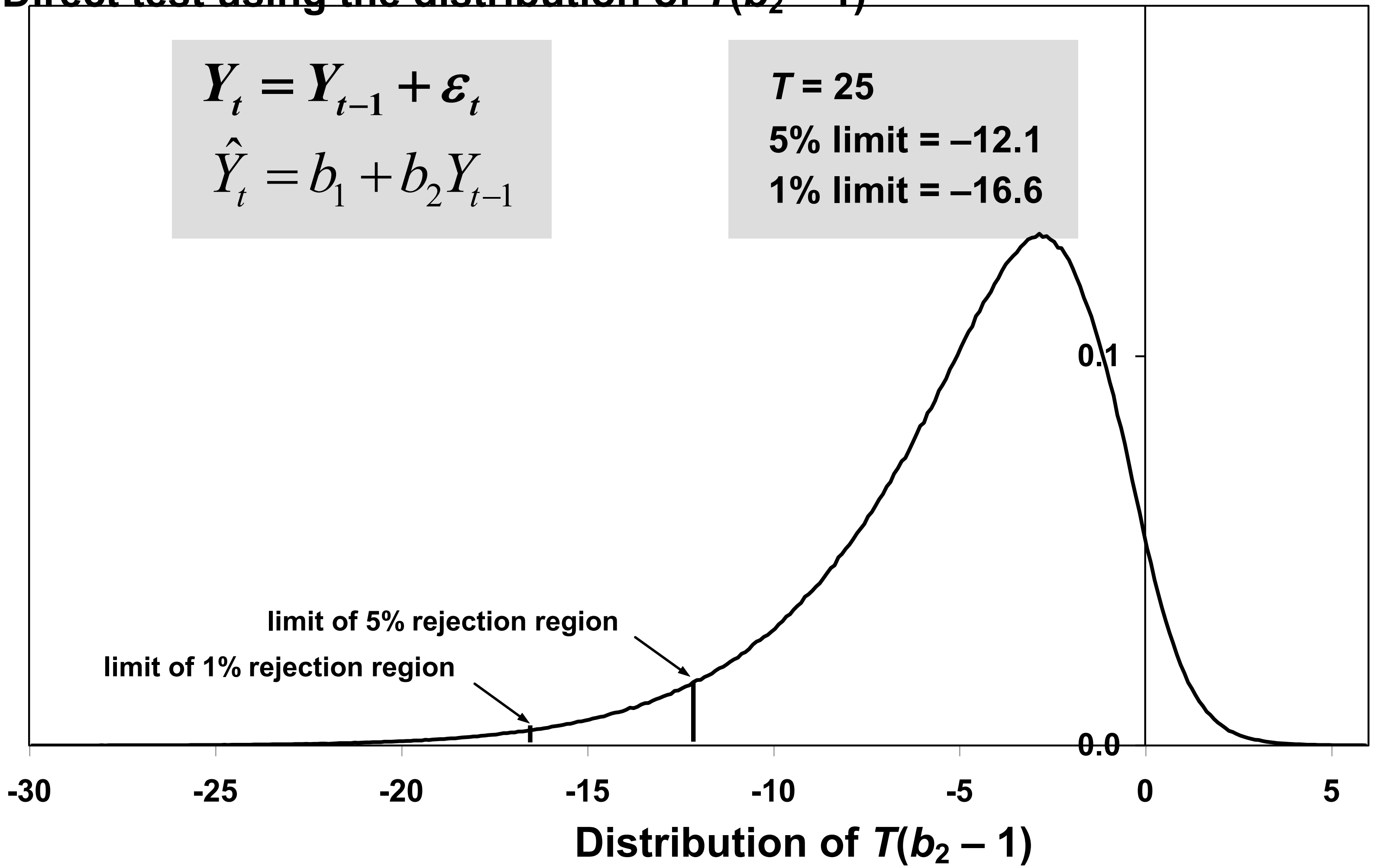
$$Y_t = 2\beta_1 + 2\delta t - \delta + Y_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$Y_t = t\beta_1 + \frac{t(t+1)}{2}\delta + Y_0 + \sum_{s=1}^{t} \varepsilon_s$$

Lagging and substituting *t* times, we can express Y_t in terms of the initial Y_0, the innovations, and a convex quadratic expression for *t*. It is not reasonable to suppose that any time series process can be characterized as a convex quadratic function of time.

TESTS OF NONSTATIONARITY: UNTRENDED DATA. DICKEY-FULLER TESTS.

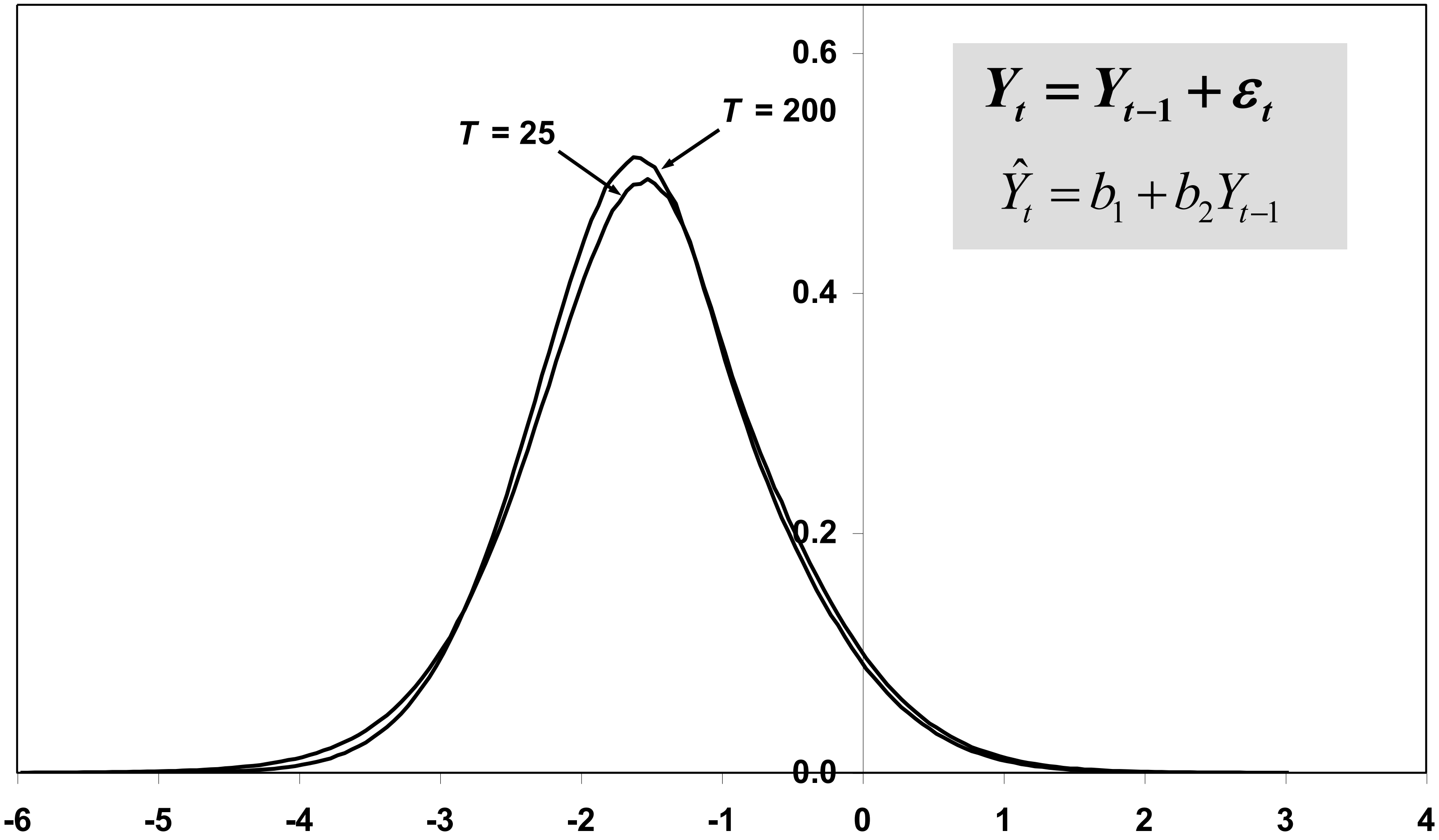
Direct test using the distribution of $T(b_2 - 1)$



H_0: $\beta_2 = 1$ and H_1: $\beta_2 < 1$. The statistic $T(b_2$-1) has a limiting distribution under H_0. The figure shows the rejection regions for one-sided 5 percent (limit –12.1) and 1 percent (limit –16.6) tests for T = 25. The Table A.7 is available in Dougherty, Edition 5.

TESTS OF NONSTATIONARITY: UNTRENDED DATA

t test



Distribution of OLS *t* statistic

The second test proposed by Dickey and Fuller uses the conventional OLS *t* statistic for H_0: $\beta_2 = 1$. The H_1: $\beta_2 < 1$ (one-sided test). The distribution of the *t* statistic, established by simulation, is nonstandard (the Table A.6 is available in Dougherty, Ed. 5.

TESTS OF NONSTATIONARITY: UNTRENDED DATA

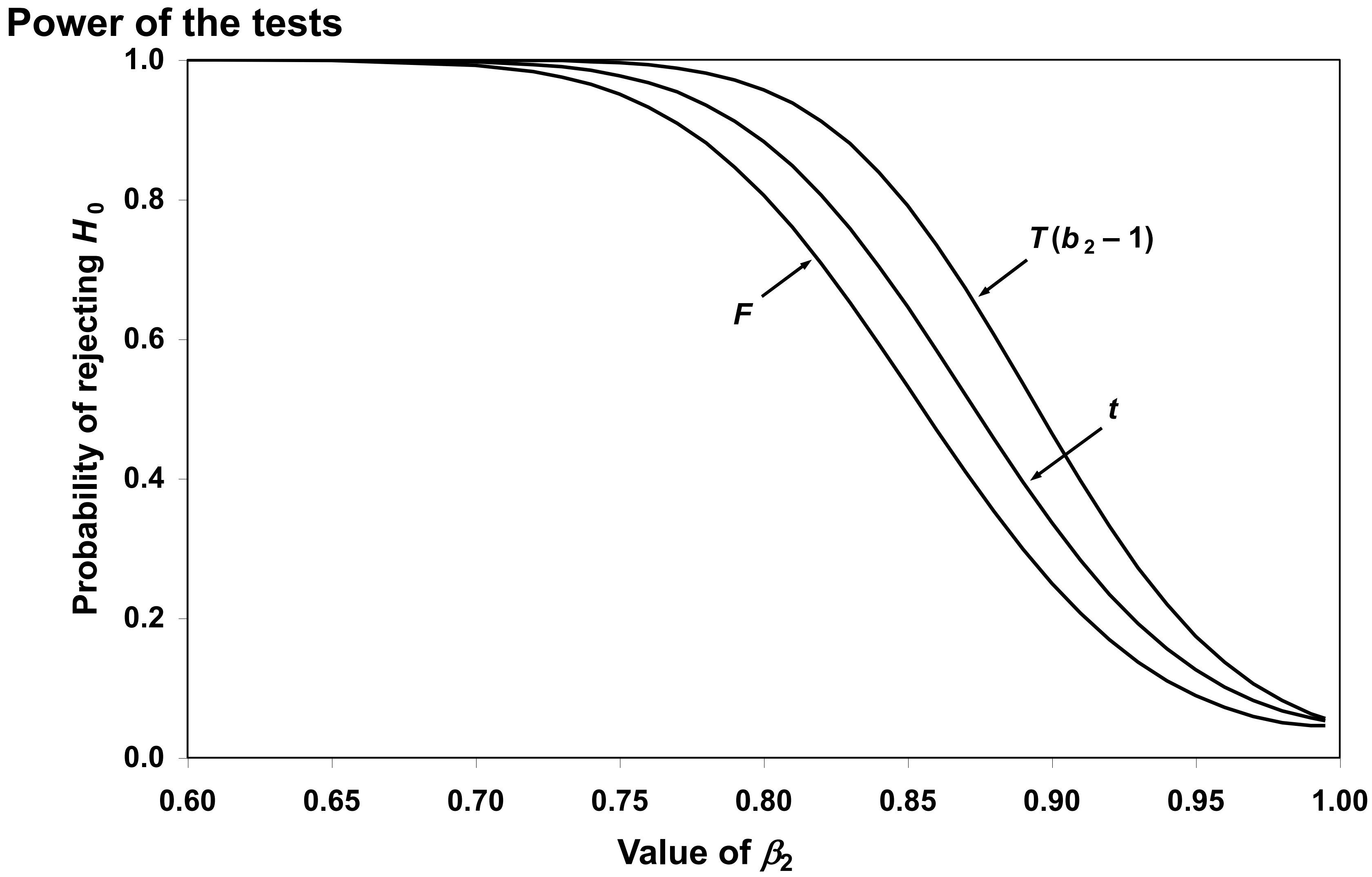
***F* test**

$$\boldsymbol{Y_t = Y_{t-1} + \varepsilon_t}$$
$$\hat{Y}_t = b_1 + b_2 Y_{t-1}$$

$$H_0: \beta_1 = 0 \text{ and } \beta_2 = 1$$
$$H_1: \text{unrestricted model}$$

The third Dickey–Fuller test (F-test) exploits the fact that, under H_0, the model is a restricted version of that under H_1 with two restrictions: $\beta_2 = 1$ and, also, $\beta_1 = 0$.The OLS *F* statistic is constructed in the usual way, but its distribution is not standard (the Table A.8 is available in Dougherty, Ed.5).

TESTS OF NONSTATIONARITY POWER, UNTRENDED DATA



The power of the test with $T(b_2 - 1)$ is greatest for all values of $\beta_2 < 1$. The *t* test is next, and the *F* is test least powerful (since being two-sided while the first two are one-sided).

TESTS OF NONSTATIONARITY: TRENDED DATA

General model $$Y_t = \beta_1 + \beta_2 Y_{t-1} + \delta t + \varepsilon_t$$

Alternatives $$-1 < \beta_2 < 1 \text{ or } \beta_2 = 1$$

$$\delta = 0 \text{ or } \delta \neq 0$$

Case (c)	$\beta_1 \neq 0$	$\beta_2 = 1$	$\delta = 0$

$$Y_t = \beta_1 + Y_{t-1} + \varepsilon_t$$

Case (d)	$\beta_1 = *$	$\lvert\beta_2\rvert < 1$	$\delta \neq 0$

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \delta t + \varepsilon_t$$

Trended data: random walk with drift (Case (c)), or stationary process around deterministic trend (Case (d))? We fit the general model (d), and test H_0: $\beta_2 = 1$ using as a test statistic either $T(b_2 - 1)$ or the t statistic for b_2.

F test: we test the composite hypothesis H_0: $\beta_2 = 1$, $\delta = 0$.

Critical values for the tests are given in Dougherty, Ed.5, Tables A.6-A.8.

TESTS OF NONSTATIONARITY: UNTRENDED DATA

Augmented Dickey–Fuller tests

Second-order autoregressive process

Necessary condition for stationarity: $\left|\beta_2 + \beta_3\right| < 1$

$$\boldsymbol{Y_t = \beta_1 + \beta_2 Y_{t-1} + \beta_3 Y_{t-2} + \varepsilon_t}$$

$$Y_t - Y_{t-1} = \beta_1 + \beta_2 Y_{t-1} - Y_{t-1} + \beta_3 Y_{t-1} - \beta_3 Y_{t-1} + \beta_3 Y_{t-2} + \varepsilon_t$$

$$= \beta_1 + \left(\beta_2 + \beta_3 - 1\right) Y_{t-1} - \beta_3 \left(Y_{t-1} - Y_{t-2}\right) + \varepsilon_t$$

$$\Delta Y_t = \beta_1 + \left(\beta_2 + \beta_3 - 1\right) Y_{t-1} - \beta_3 \Delta Y_{t-1} + \varepsilon_t$$

$$= \beta_1 + \left(\beta_2^* - 1\right) Y_{t-1} + \beta_3^* \Delta Y_{t-1} + \varepsilon_t$$

$$\boldsymbol{\beta_2^* = \beta_2 + \beta_3} \qquad \beta_3^* = -\beta_3 \qquad \boldsymbol{\Delta Y_{t-1} = Y_{t-1} - Y_{t-2}}$$

Under the null hypothesis H_0: $\beta_2^* = 1$, the process is nonstationary. H_0 is tested as if the coefficient of Y_{t-1} is significantly different from 0. Under the null hypothesis, the estimator of β_2^* is superconsistent and the test statistics $T(b_2^* - 1)$, t, and F have the same distributions, and therefore critical values, as before.

TESTS OF NONSTATIONARITY: UNTRENDED DATA

Necessary condition for stationarity:

$$\left|\beta_2 + \beta_3 + ... + \beta_{p+1}\right| < 1$$

Augmented Dickey–Fuller tests

General autoregressive process

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + ... + \beta_{p+1} Y_{t-p} + \varepsilon_t$$

$$\Delta Y_t = \beta_1 + \left(\beta_2^* - 1\right) Y_{t-1} + \beta_3^* \Delta Y_{t-1} + ... + \beta_{p+1}^* \Delta Y_{t-p} + \varepsilon_t$$

$$\beta_2^* = \beta_2 + ... + \beta_{p+1}$$

Under the null hypothesis of non-explosive nonstationarity, the test statistics $T(b_2^* - 1)$, t, and F asymptotically have the same distributions and critical values as before.

In practice, the *t* test is particularly popular and is generally known as the augmented Dickey–Fuller (ADF) test.

ADF TESTS: DETERMINING NUMBER OF LAGS

Augmented Dickey–Fuller tests

How to determine *p*? Two main approaches.

1) **The model is fitted with some $p = p_{max}$ and t test is performed on the coefficient of $\Delta Y_{t-p\max}$. If not significant, dropped. Then two last terms, F test, if not significant, dropped, etc.**

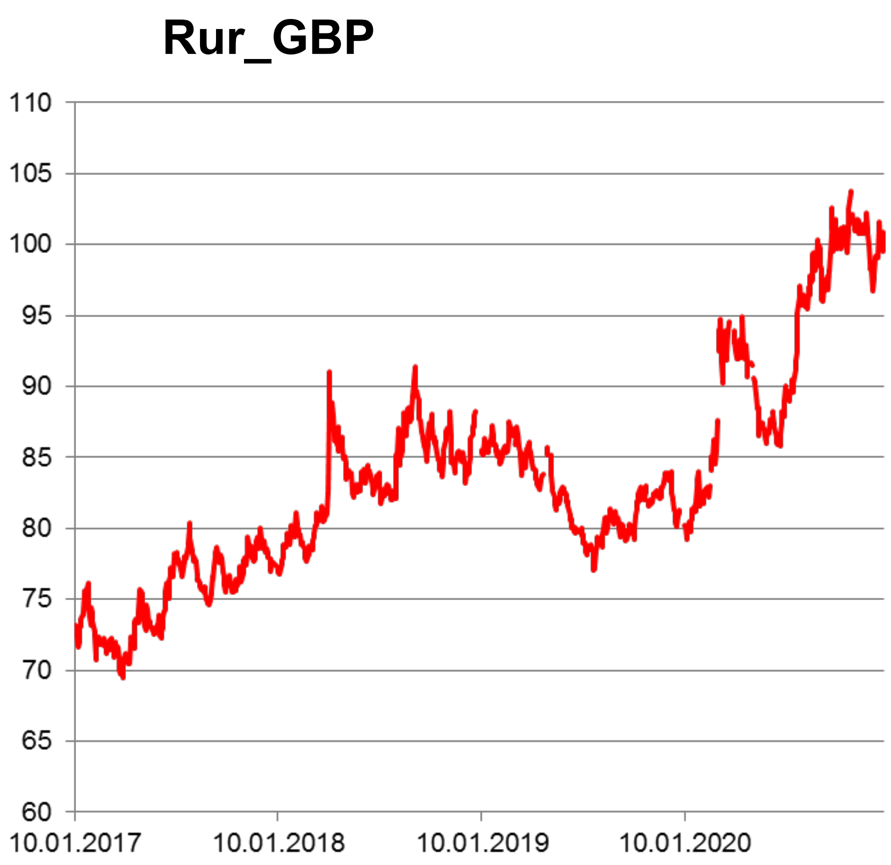
2) **Information Criteria, Schwarz (SIC) or Akaike (AIC), minimum of:**

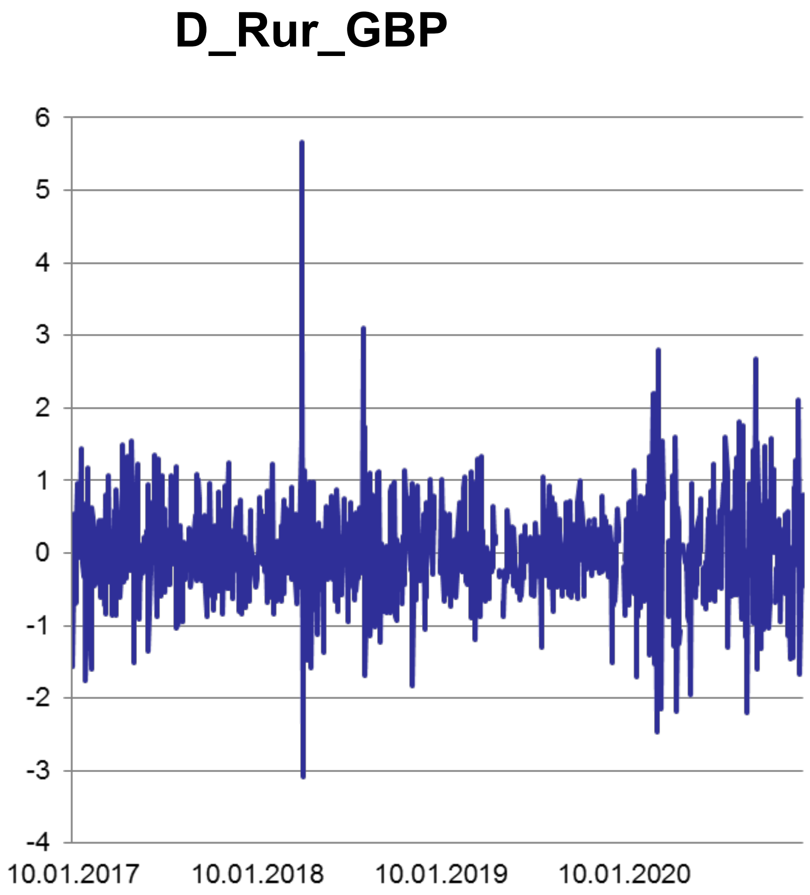
$$SIC = \log\left(\frac{RSS}{T}\right) + \frac{k \log T}{T} = \log\left(\frac{RSS}{T}\right) + \frac{(p+2)\log T}{T}$$

$$AIC = \log\left(\frac{RSS}{T}\right) + \frac{2k}{T}$$

ADF TESTS OF NONSTATIONARITY, AN EXAMPLE:

THE RuR/GBP EXCHANGE RATE, 10.01.2017-01.01.2021





The RuR/GBP (left) looks like random walk with or without drift, while D_RuR/GBP (right) could be stationary.

THE RuR/GBP EXCHANGE RATE, 10.01.2017-01.01.2021: Correlograms in Eviews for the levels and 1st differences.

Date: 02/14/21 Time: 17:41
Sample: 1/10/2017 1/01/2021
Included observations: 984

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.979	0.979	946.64	0.000
		2	0.968	0.228	1873.2	0.000
		3	0.958	0.065	2780.9	0.000
		4	0.950	0.086	3674.9	0.000
		5	0.942	0.020	4554.1	0.000
		6	0.935	0.035	5420.7	0.000
		7	0.926	-0.015	6272.3	0.000
		8	0.918	0.009	7110.5	0.000
		9	0.912	0.048	7938.2	0.000
		10	0.907	0.041	8757.1	0.000
		11	0.901	0.012	9566.4	0.000
		12	0.894	-0.017	10364.	0.000
		13	0.888	-0.002	11152.	0.000
		14	0.882	0.016	11929.	0.000
		15	0.879	0.095	12703.	0.000
		16	0.874	-0.023	13469.	0.000
		17	0.870	0.018	14229.	0.000
		18	0.867	0.048	14983.	0.000
		19	0.862	-0.030	15730.	0.000
		20	0.857	-0.016	16469.	0.000
		21	0.851	-0.027	17199.	0.000
		22	0.846	-0.004	17920.	0.000
		23	0.838	-0.057	18629.	0.000
		24	0.831	-0.030	19326.	0.000
		25	0.819	-0.115	20006.	0.000
		26	0.811	0.005	20672.	0.000
		27	0.804	0.044	21328.	0.000
		28	0.798	0.024	21975.	0.000
		29	0.791	-0.007	22611.	0.000
		30	0.784	-0.007	23237.	0.000
		31	0.775	-0.060	23849.	0.000
		32	0.768	0.024	24450.	0.000
		33	0.763	0.027	25044.	0.000
		34	0.756	-0.027	25628.	0.000
		35	0.749	0.004	26202.	0.000
		36	0.742	-0.026	26765.	0.000

Date: 02/14/21 Time: 17:43
Sample: 1/10/2017 1/01/2021
Included observations: 969

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.034	0.034	1.1490	0.284
		2	-0.060	-0.061	4.6053	0.100
		3	-0.061	-0.057	8.2555	0.041
		4	0.004	0.004	8.2675	0.082
		5	0.067	0.060	12.612	0.027
		6	-0.048	-0.056	14.848	0.021
		7	-0.015	-0.004	15.064	0.035
		8	-0.057	-0.056	18.265	0.019
		9	0.016	0.013	18.526	0.030
		10	-0.017	-0.030	18.826	0.043
		11	-0.045	-0.042	20.789	0.036
		12	-0.006	-0.005	20.821	0.053
		13	-0.005	-0.006	20.847	0.076
		14	-0.029	-0.043	21.677	0.085
		15	0.040	0.045	23.244	0.079
		16	-0.023	-0.032	23.762	0.095
		17	0.003	0.003	23.771	0.126
		18	0.071	0.070	28.691	0.052
		19	0.001	-0.007	28.693	0.071
		20	0.010	0.010	28.797	0.092
		21	0.013	0.026	28.971	0.115
		22	0.011	0.002	29.099	0.142
		23	0.052	0.054	31.794	0.105
		24	-0.009	-0.009	31.869	0.130
		25	-0.034	-0.029	33.045	0.130
		26	-0.076	-0.060	38.740	0.052
		27	0.034	0.032	39.870	0.053
		28	-0.020	-0.037	40.268	0.063
		29	-0.052	-0.038	43.021	0.045
		30	-0.031	-0.028	43.996	0.048
		31	-0.030	-0.022	44.893	0.051
		32	-0.019	-0.035	45.252	0.060
		33	0.001	-0.004	45.252	0.076
		34	0.015	0.013	45.476	0.090
		35	-0.006	-0.008	45.513	0.110
		36	0.005	-0.009	45.536	0.133

ADF TESTS OF NONSTATIONARITY, AN EXAMPLE:

THE RuR/GBP EXCHANGE RATE, 10.01.2017-01.01.2021

Null Hypothesis: RUR_GBP has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=21)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-2.032856	0.5820
Test critical values:	1% level	-3.967547	
	5% level	-3.414458	
	10% level	-3.129363	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(RUR_GBP)
Method: Least Squares
Date: 02/14/21 Time: 17:02
Sample (adjusted): 1/11/2017 1/01/2021
Included observations: 969 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
RUR_GBP(-1)	-0.010181	0.005008	-2.032856	0.0423
C	0.756539	0.368665	2.052106	0.0404
@TREND("1/10/2017")	0.000241	0.000128	1.884975	0.0597

RuR/GBP is found to be nonstationary (random walk) since the null hypothesis of unit root is not rejected.

ADF TESTS OF NONSTATIONARITY, AN EXAMPLE:

THE RuR/GBP EXCHANGE RATE, 10.01.2017-01.01.2021

Null Hypothesis: D(RUR_GBP) has a unit root
Exogenous: Constant, Linear Trend
Lag Length: 0 (Automatic - based on SIC, maxlag=21)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-29.73858	0.0000
Test critical values:	1% level	-3.967697	
	5% level	-3.414531	
	10% level	-3.129407	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(RUR_GBP,2)
Method: Least Squares
Date: 02/14/21 Time: 17:11
Sample (adjusted): 1/12/2017 1/01/2021
Included observations: 954 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
D(RUR_GBP(-1))	-0.965065	0.032452	-29.73858	0.0000
C	0.010068	0.043299	0.232522	0.8162
@TREND("1/10/2017")	4.00E-05	7.50E-05	0.532883	0.5942

D_RuR/GBP is found to be stationary since the null hypothesis of unit root is rejected.

ADF TESTS OF NONSTATIONARITY, AN EXAMPLE:

THE RuR/GBP EXCHANGE RATE, 10.01.2017-01.01.2021.

Null Hypothesis: D(RUR_GBP) has a unit root
Exogenous: None
Lag Length: 0 (Automatic - based on SIC, maxlag=21)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-29.70524	0.0000
Test critical values:	1% level	-2.567388	
	5% level	-1.941155	
	10% level	-1.616476	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(RUR_GBP,2)
Method: Least Squares
Date: 02/14/21 Time: 17:20
Sample (adjusted): 1/12/2017 1/01/2021
Included observations: 954 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
D(RUR_GBP(-1))	-0.963449	0.032434	-29.70524	0.0000

No trend, no intercept. D_RuR/GBP is found to be stationary since the null hypothesis of unit root is rejected.