Question 1. In the model

$$y_t = \beta x_t + u_t,\ t = 1, 2, \ldots, T$$

x_t is measured with error. Data is only available on x_t^*, where

$$x_t^* = x_t + v_t;\ t = 1, 2, \ldots, T$$

and $Eu_t = Ev_t = 0$, $E(u_t v_t) = E(x_t u_t) = E(x_t v_t) = 0$. y_t, x_t and x_t^* have zero means.

If $\hat{\beta}$ is the ordinary least squares estimator of β from regressing y_t on x_t^*, show that $\hat{\beta}$ is inconsistent.

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$E(u_t) = 0$$

$$E(u_t^2) = \sigma^2$$

$$E(u_t^2) - E^2 u_t$$

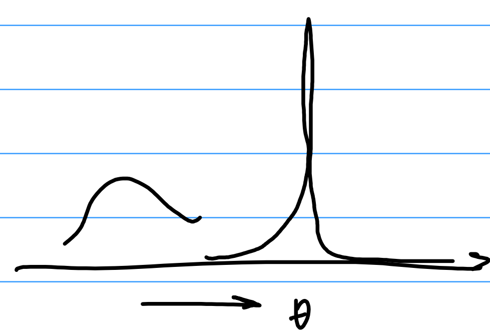
$$\hat{\beta} = \frac{\sum X_t^* \cdot y_t}{\sum X_t^{*2}} = \frac{\sum (X_t + V_t)(\beta X_t + u_t)}{\sum (X_t + V_t)^2} =$$

$$= \frac{\left(\beta \sum X_t^2 + \sum X_t \cdot u_t + \beta \sum V_t \cdot X_t + \sum u_t V_t\right)\frac{1}{T-1}}{\left(\sum X_t^2 + \sum V_t^2 + 2\sum X_t V_t\right)\frac{1}{T-1}}$$

$$\operatorname{plim} \hat{\beta} = \frac{\beta E(X_t^2) + \underbrace{E(\cancel{X_t u_t})}_{=0} + \beta \underbrace{E(\cancel{V_t X_t})}_{=0} + \underbrace{E(\cancel{u_t V_t})}_{=0}}{E(X_t^2) + E(V_t^2) + 2\underbrace{E(\cancel{X_t V_t})}_{0}}$$

$$= \beta \underbrace{\frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2}}_{\text{bias} > 0} \Rightarrow \text{inconsistent}$$

$$0 < \frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2} < 1$$



$\Rightarrow$ biased towards 0

b) $y_t^* = y_t + w_t$

$E w_t = 0 \qquad E(u_t w_t) = E(x_t w_t) = 0$

$$\hat{\beta} = \frac{\sum x_t \cdot y_t^*}{\sum x_t^2} = \frac{\sum x_t (y_t + w_t)}{\sum x_t^2} =$$

$$= \frac{\left(\beta \sum x_t^2 + \sum x_t u_t + \sum x_t w_t\right) \frac{1}{T-1}}{\sum x_t^2 \quad \frac{1}{T-1}}$$

$$p\lim \hat{\beta} = \frac{\beta E(x_t^2) + \underbrace{E(\cancel{x_t u_t})}_{=0} + \underbrace{E(\cancel{x_t w_t})}_{=0}}{E(x_t^2)} = \beta$$

Question 2 Briefly explain what is instrumental variable estimation. Consider the model

$$Y_t = \beta X_t + u_t;\ t = 1, 2, ..., T.$$

What is instrumental variable estimator for this model? Is it consistent? Give an example of a model where instrumental variable estimation is an improvement on ordinary least squares (OLS). Explain why IV estimation is superior to OLS in this case.

Give an example of a model where instrumental variable estimation is an improvement on ordinary least squares (OLS).

Explain why IV estimation is superior to OLS in this case.

$E u_t = 0 \quad E u_t^2 = \sigma^2 \quad E u_t u_s = 0$

$cov(x_t, u_t) \neq 0 \quad \Rightarrow$ endogeneity

z_t – instrument:

$cov(x_t, z_t) \neq 0$ relevance

$cov(u_t, z_t) = 0$ exogeneity

1) $\hat{X}_t = \hat{\alpha} \cdot z_t \qquad \hat{\alpha} = \dfrac{\sum x_t z_t}{\sum z_t^2}$

2) $y_t = \hat{\beta}_{IV} \cdot \hat{X}_t \leftarrow$

$$\hat{\beta}_{IV} = \frac{\sum y_t \cdot \hat{X}_t}{\sum \hat{X}_t^2} = \frac{\hat{\alpha} \sum y_t \cdot z_t}{\hat{\alpha}^2 \sum z_t^2} =$$

$$= \frac{\sum y_t \cdot z_t}{\sum x_t z_t} = \frac{\beta \sum x_t z_t + \sum u_t z_t}{\sum x_t \cdot z_t}$$

$$= \beta + \frac{\sum u_t z_t}{\sum x_t z_t}$$

$$\text{plim } \hat{\beta}_{IV} = \beta + \frac{E(u_t z_t)}{E(x_t z_t)}$$

exogeneity

relevance

	$\hat{\beta}_{IV}$	$\hat{\beta}_{OLS}$
$cov(X, \varepsilon) \neq 0$	unbiased, cons.	biased, inconsistent
$cov(X, \varepsilon) = 0$	unbiased, cons.	unbiased, cons.
	ineff.	eff.

H_0: no endogeneity, $\hat{\beta}_{OLS}$ – consistent

H_a: $\hat{\beta}_{OLS}$ – inconsistent

$$(\hat{\beta}_{IV} - \hat{\beta}_{OLS})^T \left(\hat{V}(\hat{\beta}_{IV}) - \hat{V}(\hat{\beta}_{OLS})\right)^{-1} (\hat{\beta}_{IV} - \hat{\beta}_{OLS}) \sim \chi^2$$