Question 3. (25 marks)
(a) (10 marks) □ Explain what is meant by a stationary time series and what is non-stationary time series. How to understand if a time series is stationary?

A process $\{y_t\}$ is (covariance) stationarity if its mean and variance exist and do not depend on time and its covariance is a function of distance in time only (not location). Violation of either these requirements renders the process non-stationary.

To understand the properties of a time series, it is enough to test the series for stationarity using the Dickey-Fuller test in the form

$$\Delta X_t = (\beta - 1)X_{t-1} + \gamma t + u_t$$

The hypothesis $H_0: \beta = 1$ is tested against $H_1: \beta < 1$, for which the t-statistic of the coefficient at X_{t-1} is compared with special ADF critical t-values. If the null hypothesis is not rejected, the series is non-stationary. The hypothesis $H_0: \gamma = 0$ is tested separately, for which the t-statistic of the coefficient of t is compared with the conventional critical values of the t-distribution. If rejected, then the series is also non-stationary.

□ What is detrending of the time series? What is differencing of the time series? Explain what do you understand by difference-stationary and trend-stationary time series. What is the difference in the impact of random shocks on the difference-stationary and trend-stationary time series?

Detrending is the procedure of removing time trend from the series.
If after removing the trend from a non-stationary series the resulting variable becomes stationary, then the variable is called **trend-stationary.**

The transformation in which the lagged values are subtracted from the original values of the time series and the differences are obtained is called differeencing.
If a non-stationary process can be transformed into a stationary process by differencing, then the series is said to be **difference-stationary** or **integrated of order 1, I(1).**.

For trend-stationary variables shocks are **transitory**.
For difference-stationary variables shocks have a **permanent** effect

For example in time trend $X_t = \alpha + \beta t + u_t$ shocks u_t at each point in time are determined independently of the previous ones $\mathrm{Cov}(u_t, u_{t-s}) = 0$.
On the contrary, in a random walk $X_t = X_t + u_t$, shocks accumulate $X_t = X_0 + u_1 + u_2 + ... + u_{t-1} + u_t$ and continue to influence the behavior of the series at future times

□ Demonstrate that time trend $X_t = \alpha_0 + \alpha_1 t + u_t$ is a trend-stationary time series (*no need to prove that this series is non-stationary*). We assume $E[u_t]) = 0$, $\mathrm{Var}(u_t) = \sigma^2$ and $E[u_t u_s] = 0 \; \forall \; s \neq t$.

After detrending $Z_t = X_t - \alpha_1 t = \alpha_0 + u_t$ new time series Z_t becomes stationary:

$$E[Z_t] = E[\alpha_0 + u_t] = \alpha_0$$

$$\mathrm{Var}(Z_t) = \mathrm{Var}(\alpha_0 + u_t) = \sigma^2$$

$$\mathrm{Cov}(Z_t, Z_{t-s}) = E[Z_t - E[Z_t]][Z_{t-s} - E[Z_{t-s}]] = E[u_t u_{t-s}] = 0.$$

This means that Z_t has constant mean and variance for all *t*, and covariance is zero for $s \neq 0$. It implies that the series is trend-stationary.

□ Demonstrate that random walk $X_t = X_{t-1} + u_t$ is a difference-stationary time series (*no need to prove that this series is non-stationary*). The same assumptions about u_t.

Subtract X_{t-1} from both sides of $X_t = X_{t-1} + u_t$ to get

$$\Delta X_t = X_t - X_{t-1} = u_t.$$

It can easily be checked that $E[\Delta X_t] = E[u_t] = 0$, $\text{Var}(\Delta X_t) = \text{Var}(u_t) = \sigma^2$ and $\text{Cov}(\Delta X_t, \Delta X_{t-1}) = \text{Cov}(u_t, u_{t-1}) = 0 \;\forall\; s \neq t$. This means that ΔX_t is stationary. This implies that X_t is difference-stationary.

(b) (7 marks) Consider the following non-stationary process

$$y_t = \gamma_0 + \gamma_1 t + u_t, \text{ with } u_t = \rho u_{t-1} + \varepsilon_t \qquad (1)$$

and ε_t i.i.d. $(0, \sigma^2)$.

□ Explain the source(s) of non-stationarity of y_t. Indicate at what values of the parameters the process (1) turns out to be difference stationary and at what trend stationary.

Here there are two possible sources of non-stationarity: the presence of the deterministic trend, when $\gamma_1 \neq 0$ and the presence of a unit root, when $\rho = 1$.

$\{u_t\}$ and therefore $\{y_t\}$ is an AR(1) process, which is (trend) stationary only if $|\rho| < 1$, and may exhibit unit root when $\rho = 1$.

$\{y_t\}$ is nonstationary, because $E(y_t) = \gamma_0 + \gamma_1 t$ depends on time when $\gamma_1 \neq 0$.
If $\rho = 1$, there is a more serious non-stationarity problem as that indicates the presence of a unit root.

Taking $\rho = 1$ we obtain

$$y_t = \gamma_0 + \gamma_1 t + u_{t-1} + \varepsilon_t \text{ where } u_{t-1} = u_{t-2} + \varepsilon_{t-1}$$

So

$$y_t = \gamma_0 + \gamma_1 t + u_{t-2} + \varepsilon_t + \varepsilon_{t-1} \text{ where } u_{t-2} = u_{t-3} + \varepsilon_{t-2}$$

Repeating this process as many times as needed, we get

$$y_t = \gamma_0 + \gamma_1 t + u_0 + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1$$

Here $y_0 = \gamma_0 + \gamma_1 0 + u_0$, so $u_0 = y_0 - \gamma_0$ and so

$$y_t = y_0 + \gamma_1 t + \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1$$

Evaluating variance

$$\text{Var}(y_t) = \text{Var}(\varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2} + \ldots + \varepsilon_1) = \text{Var}(\varepsilon_t) + \text{Var}(\varepsilon_{t-1}) + \text{Var}(\varepsilon_{t-2}) + \ldots + \text{Var}(\varepsilon_1) = t\sigma^2$$

we can see that variance of this process will depend on time as well, and there will be strong dependence.

□ Invesigate the implications of detrending the process (1) under assumption $|\rho| < 1$.

Removing linear trend $\gamma_0 + \gamma_1 t$ from $y_t = \gamma_0 + \gamma_1 t + \rho u_{t-1} + \varepsilon_t$, obtain $z_t = \rho u_{t-1} + \varepsilon_t$ where $|\rho| < 1$. This is autoregressive process, which is asymptotically stationary.

□ Investigate the consequences of applying the differencing transformation to the process (1) under assumption $\rho = 1$.

Now $y_t = \gamma_0 + \gamma_1 t + u_{t-1} + \varepsilon_t$, and $u_{t-1} = u_{t-2} + \varepsilon_{t-1}$ so lagged equation is $y_{t-1} = \gamma_0 + \gamma_1(t-1) + u_{t-2} + \varepsilon_{t-1}$, subtracting them obtain ain $\Delta y_t = y_t - y_{t-1} = \gamma_1 + u_{t-1} - u_{t-2} + \varepsilon_t - \varepsilon_{t-1}$ and $u_{t-1} - u_{t-2} = \varepsilon_{t-1}$, so finally $\Delta y_t = \gamma_1 + \varepsilon_t$. This process is certainly stationary, so the original process (1) is difference stationary.

(c) (8 marks) □ Show that you can rewrite the model (1) as

$$\Delta y_t = \beta_0 + \beta_1 t + \beta_2 y_{t-1} + \varepsilon_t. \qquad \textbf{(2)}$$

Clearly indicate the one-to-one relation between $(\gamma_0, \gamma_1, \rho)$ and $(\beta_0, \beta_1, \beta_2)$.

To obtain the testing equation, observe that we need to subtract from $y_t = \gamma_0 + \gamma_1 t + u_t$ the lagged equation $y_{t-1} = \gamma_0 + \gamma_1(t-1) + u_{t-1}$, multiplied by ρ

$$\rho y_{t-1} = \rho\gamma_0 + \rho\gamma_1(t-1) + \rho u_{t-1}$$

to yield

$$y_t - \rho y_{t-1} = \gamma_0(1-\rho) + \rho\gamma_1 + \gamma_1(1-\rho)t + \underbrace{u_t - \rho u_{t-1}}_{\varepsilon_t}.$$

(as $u_t = \rho u_{t-1} + \varepsilon_t$). By rewriting we get

$$\Delta y_t = \underbrace{\gamma_0(1-\rho) + \rho\gamma_1}_{\beta_0} + \underbrace{\gamma_1(1-\rho)}_{\beta_1}t + \underbrace{(\rho-1)}_{\beta_2}y_{t-1} + \varepsilon_t$$

□ How to use equation (2) to test time series (1) for stationarity.

For this purpose one can use representation

$$\Delta y_t = \gamma_1 + \gamma_2 t + \gamma_3 y_{t-1} + v_t \qquad (2)$$

from previous part.

To test for difference stationarity one should test null hypothesis $H_0 : \gamma_3 = 0$ against $H_1 : \gamma_3 < 0$ using ADF t test using the test statistic $\frac{\hat{\gamma}_3}{\mathrm{SE}(\hat{\gamma}_3)}$ and the DF critical values (**with trend and constant**)

To test for trend stationarity one should test null hypothesis $H_0 : \gamma_2 = 0$ against $H_1 : \gamma_2 \neq 0$ using conventional t test.

If $\rho < 1$ while $\gamma_1 \neq 0$ then we call the process **trend stationary**. Observe, we would be left with

$$y_t = \beta_0 + \beta_1 t + \rho y_{t-1} + \varepsilon_t$$

here y_t has a deterministic trend with short run fluctuations represented by a stationary AR(1).

In this case $\rho = 1$ and we call the process **difference stationary**, or integrated of order 1, I(1).
Actually we would be left with

$$\Delta y_t = \gamma_1 + \varepsilon_t, \text{ or}$$

$$y_t = \gamma_1 + y_{t-1} + \varepsilon_t$$

a random walk with drift. Realize that if $\rho = 1$ we get $\beta_2 = 0$ and $\beta_1 = 0$!