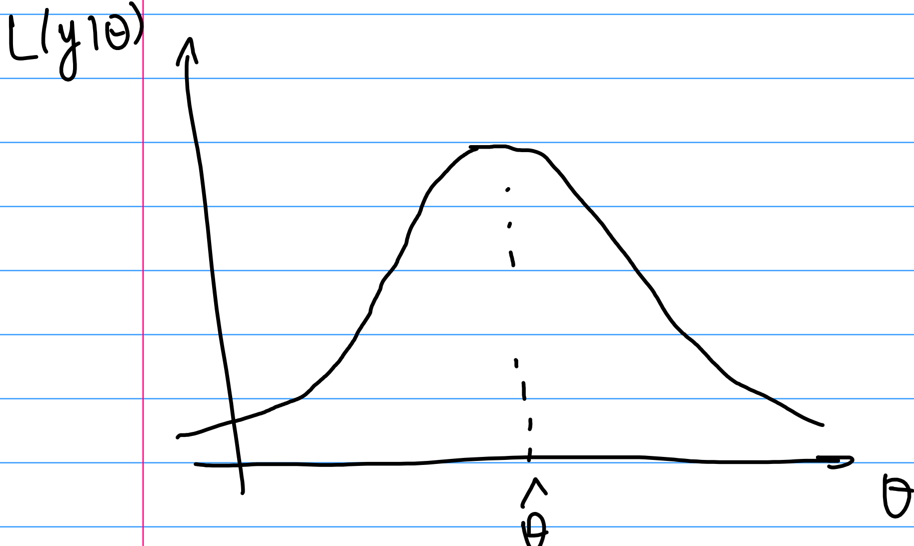
$$L(y_1,\dots,y_n|\theta) = \prod_{y_i=1} p_i \prod_{y_i=0} (1-p_i) =$$

$$= \prod_i p_i^{y_i} (1-p_i)^{1-y_i}$$

$$= \prod_i \left(\frac{1}{1+e^{-\underbrace{\beta_1+\beta_2 x_i}_{K}}}\right)^{y_i} \cdot \left(1 - \frac{1}{1+e^{-(\beta_1+\beta_2 x_i)}}\right)^{1-y_i}$$

$$\ln L(y|\theta) = -\sum_i y_i \cdot \ln(1+e^{-K}) - \sum_i (1-y_i) \cdot \ln\left(1 - \frac{1}{1+e^{-K}}\right)$$

$$\longrightarrow \max_{\beta_1,\beta_2} \Rightarrow \hat{\beta}_{MLE}$$



Properties:

1) $E(\hat{\theta}) \overset{n\to\infty}{=} \theta$

2) $\operatorname*{plim}_{n\to\infty} \hat{\theta} = \theta$

3) $\lim E(\tilde{\theta} - \theta)^2 \geq E(\hat{\theta} - \theta)^2 \quad \forall \tilde{\theta} \neq \hat{\theta}$

4) $\hat{\theta} \overset{d}{\sim} N(\theta, Var(\hat{\theta}))$

5) $\widehat{g(\theta_{ML})} = g(\hat{\theta}_{ML}) \quad \forall\ g$ - smooth fun

$$Var(\hat{\theta}) \geq \frac{1}{n \cdot I(\theta)} \qquad \hat{\theta} \text{ - unbiased}$$

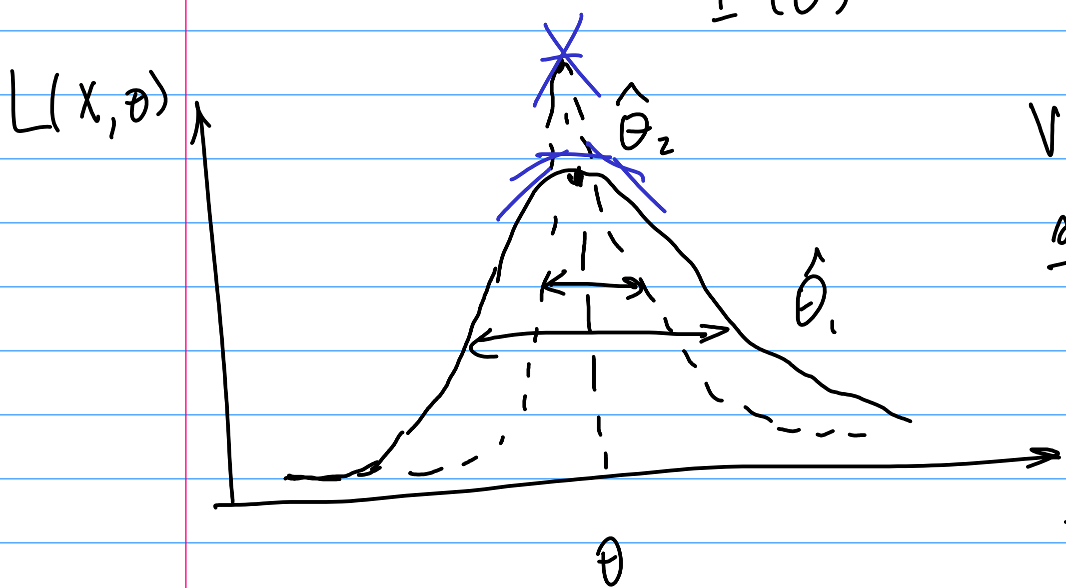
$$I(\theta) = Var\left(\frac{\partial}{\partial\theta} \log f(x, \theta) \,\Big|\, \theta \right) =$$

$$= E\left(\left(\frac{\partial}{\partial\theta} \log f(x, \theta)\right)^2 \,\Big|\, \theta \right) - 0^2$$

$$E\left(\frac{\partial}{\partial\theta} \log f(x, \theta) \,\Big|\, \theta \right) =$$

$$= \int \frac{\partial/\partial\theta\, f(x, \theta)}{f(x, \theta)} \cdot f(x, \theta)\, dx =$$

$$\mathrm{Var}(\hat{\theta}_{ML}) \xrightarrow{n\to\infty} \frac{1}{I(\hat{\theta})}$$



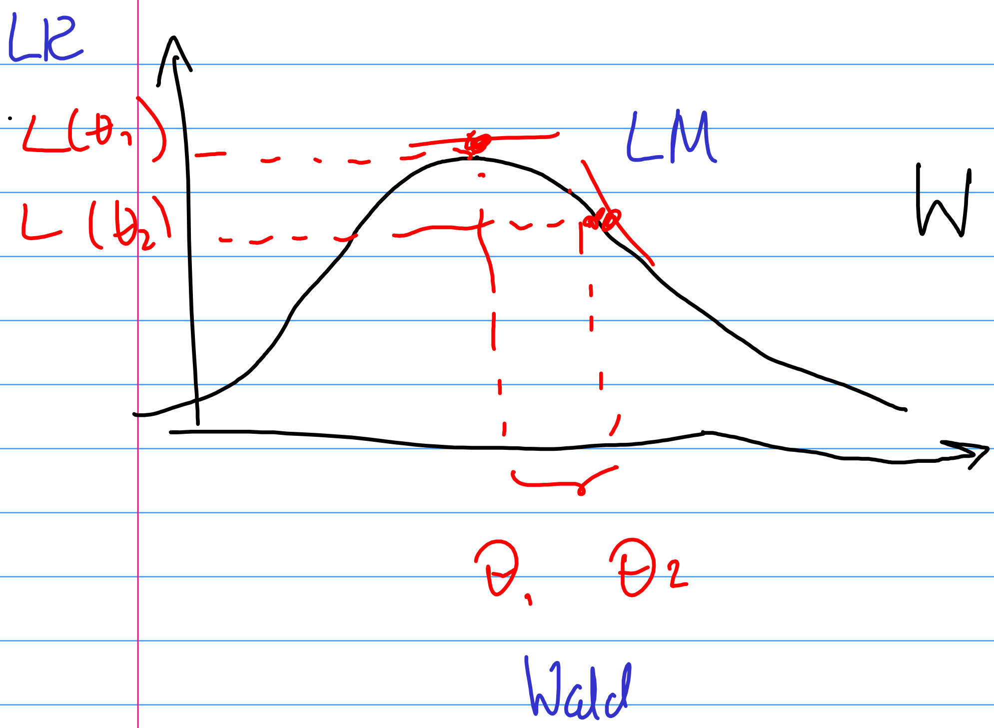
$$\mathrm{Var}(\hat{\theta}_2) > \mathrm{Var}(\hat{\theta}_1)$$

$$\frac{\partial^2 L}{\partial\theta^2} < \frac{\partial^2 L}{\partial\theta^2}$$

$$I(\hat{\theta}_2) < I(\hat{\theta}_1)$$

$$\mathrm{Var}(\hat{\theta}_{ML}) = I^{-1}(\hat{\theta}_{ML}) = -\frac{1}{E\left(\frac{\partial^2 L}{\partial\theta^2}\right)}$$

Question 5 Let the probability density function of a random variable X be $f(x, \theta)$. Explain the procedure to use the Wald test for testing the null $H_0 : \theta = \theta_0$.

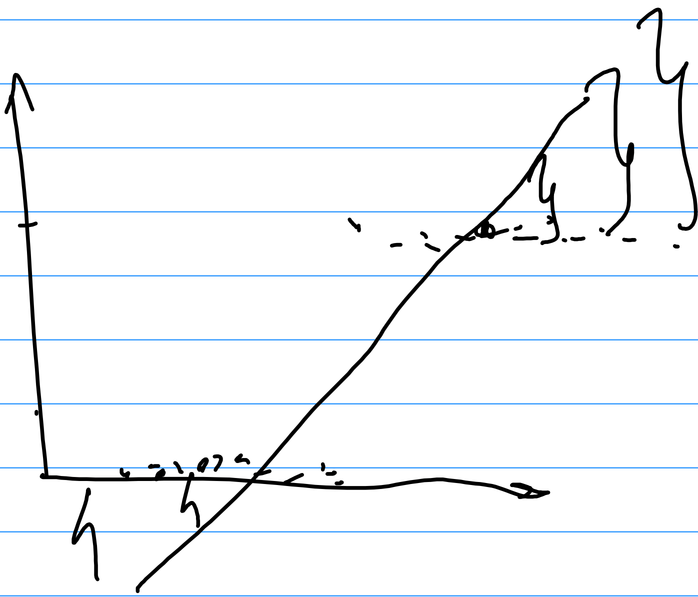


$$W = \frac{(\hat{\theta} - \theta_0)^2}{\hat{Var}(\hat{\theta}_{ML})} \sim \chi^2_k$$

k - # restr.

$k = 1$

$$\hat{Var}(\hat{\theta}_{ML}) = \frac{1}{\hat{I}(\hat{\theta})} = - \frac{1}{\left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta = \theta_0}}$$



$$= \frac{\partial}{\partial\theta} \int f(x,\theta)\, dx = \frac{\partial}{\partial\theta} 1 = 0$$

$$H_0: \beta_3 = \dots = \beta_6 = 0$$

$$LR = 2\left(\log L^{UR} - \log L^{R}\right) =$$

$$= \log\left(\frac{L^{UR}}{L^{R}}\right)^2 \sim \chi^2_4$$