Question 4 Explain the concept of the likelihood function and the maximum likelihood estimator, and briefly describe its properties.
Derive the expression for the loglikelihood function for the logit model.
(It is not expected here the students to solve maximization problem)

$$L(y_i|\theta) = \prod_{y_i=1} P(y_i=1) \cdot \prod_{y_i=0} (y_i=0) =$$

$$= \prod_{y_i=1} p_i \prod_{y_i=0} (1-p_i) =$$

$$= \prod_i p_i^{y_i} (1-p_i)^{1-y_i} =$$

$$= \prod_i \left(\frac{1}{1+e^{-(\beta_1+\beta_2 X_i)}}\right)^{y_i} \left(1 - \frac{1}{1+e^{-(\beta_1+\beta_2 X_i)}}\right)^{1-y_i}$$

$$\ln L(y|\theta) = \sum_i \ldots \rightarrow \max_{\beta_1,\beta_2}$$

$$\Rightarrow \hat{\beta}_{MLE}$$

$$L(y|\theta) = \prod f(y - X\beta)$$

Properties

If regularity conditions are satysfied:

1) $E(\hat{\theta}) \overset{n\to\infty}{=} \theta$

2) $\underset{n\to\infty}{\text{plim}}\ \hat{\theta} = \theta$

3) $\lim E(\tilde{\theta} - \theta)^2 \geq E(\hat{\theta} - \theta)^2 \quad \forall\ \tilde{\theta} \neq \hat{\theta}$

4) $\hat{\theta} \overset{d}{\sim} N(\theta, Var(\hat{\theta}))$

$\star$ $\widehat{g(\theta)} = g(\hat{\theta}) \quad \forall\ g$ - smooth function

$Var(\hat{\theta}) \geq \dfrac{1}{n \cdot I(\theta)}$ for unbiased $\hat{\theta}$

$Var(\hat{\theta}) \xrightarrow{n\to\infty} \dfrac{1}{I(\hat{\theta})}$

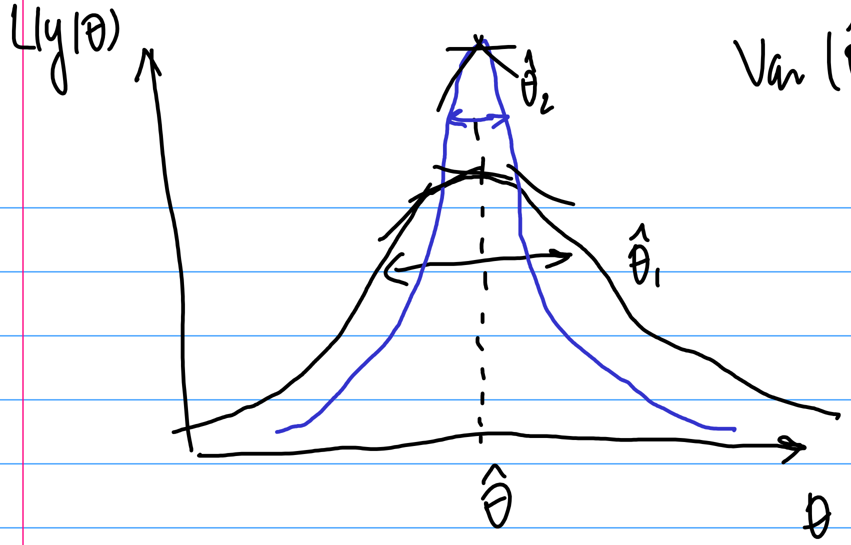
$$I(\hat{\theta}) = Var\left(\frac{\partial}{\partial\theta} \log f(x;\theta) \,\middle|\, \theta \right) =$$

$$= E\left(\left(\frac{\partial}{\partial\theta} \log f(x;\theta) \right)^2 \middle| \theta \right) - 0^2$$

$$E\left(\frac{\partial}{\partial\theta} \log f(x;\theta) \,\middle|\, \theta \right) =$$

$$= \int \frac{\frac{\partial}{\partial\theta} f(x,\theta)}{\cancel{f(x,\theta)}} \cdot \cancel{f(x;\theta)}\, dx =$$

$$= \frac{\partial}{\partial\theta} \int f(x,\theta)\, dx = \frac{\partial}{\partial\theta} 1 = 0$$

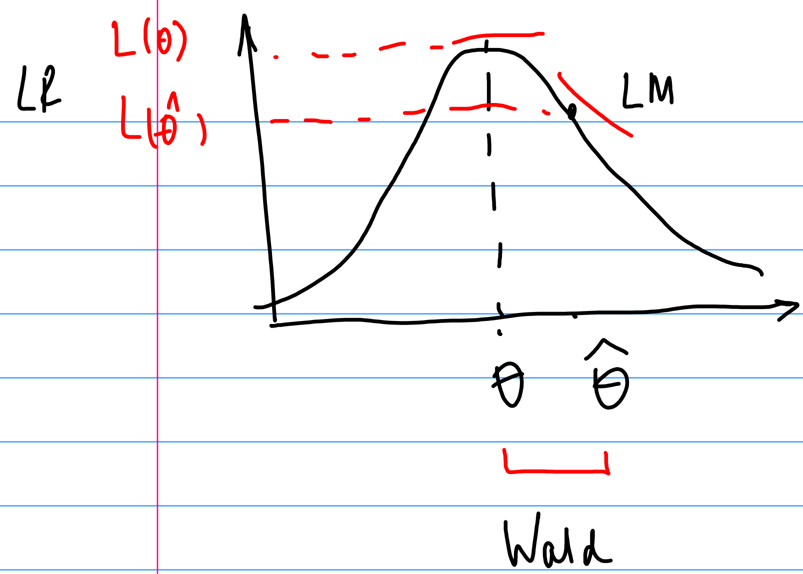


$$Var(\hat{\theta}_1) > Var(\hat{\theta}_2)$$

$$\frac{\partial^2 L}{\partial \theta^2}$$

$$Var(\hat{\theta}_{ML}) \sim \frac{1}{I(\theta)}$$

$$Var(\hat{\theta}_{ML}) = -\left[E \frac{\partial^2 L}{\partial \theta^2} \right]^{-1}$$



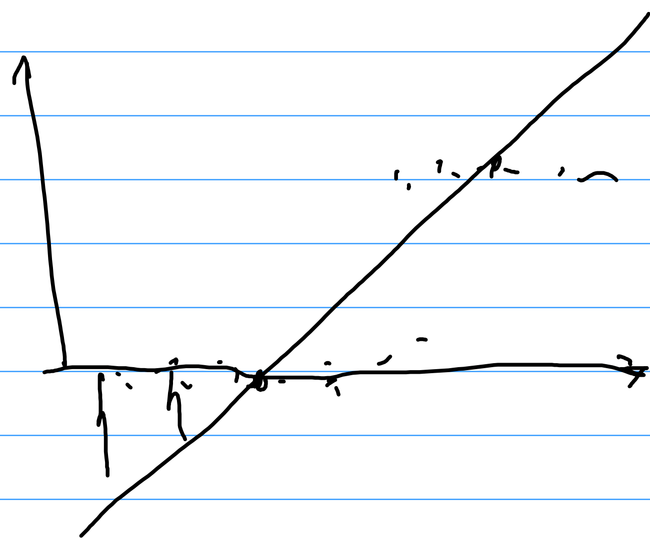
$H_0: \theta = \theta_0$

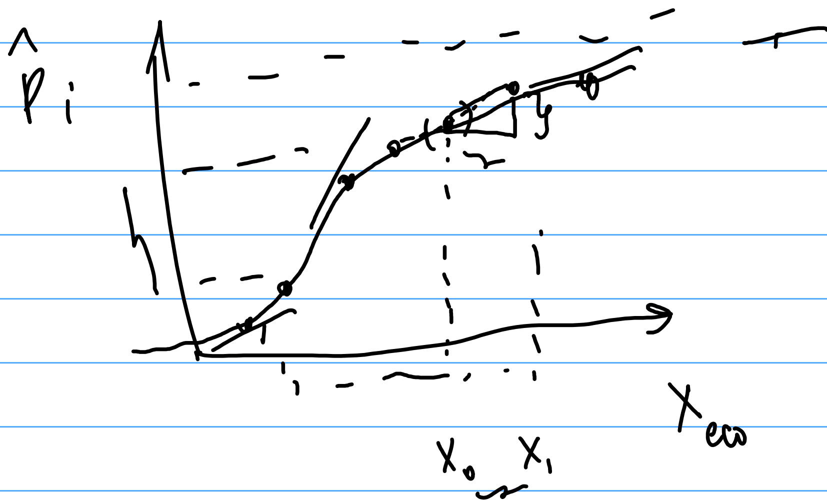
$$W = \frac{(\hat{\theta} - \theta_0)^2}{se^2(\hat{\theta})} \sim \chi^2_k$$

k – # of restrictions

$$\hat{\sigma}^2_{\hat{\theta}} = - \frac{1}{\left.\frac{\partial^2 \log L}{\partial \theta^2}\right|_{\theta = \theta_0}}$$

Q7





c) $H_0: \beta_3 = \dots = \beta_6 = 0$

$$LR = 2(\log L^{UR} - \log L^{R}) =$$

$$= \log \left(\frac{L^{UR}}{L^{R}}\right)^2 \sim \chi^2_4$$

$$LR^{obs} = 17,12$$

$$LR^{crit} = 9,5$$