Answer all questions from this section

1. Consider the following regression model

$$Y_t = \beta_0 + \beta_1 X_t + \underbrace{\beta_2 t} + \varepsilon_t, \qquad t = 1, ..., T. \qquad (1)$$

Both $\{Y_t\}_{t=1}^T$ and $\{X_t\}_{t=1}^T$ are trend stationary processes. The errors $\{\varepsilon_t\}_{t=1}^T$ are independent random variables with zero mean and constant variance.

(a) **(4 marks)** Discuss the concept "trend stationarity" and contrast it to the concept "difference stationarity". In your answer make sure you also explain what stationarity means.

(b) **(4 marks)** Provide a clear interpretation of the parameter β_1. You are told that you can obtain the estimator for β_1 using "detrended" variables only. Discuss this statement.

a) $\hat{y}_t = \underbrace{\hat{\alpha} + \hat{\beta} t}$

b) $y^*_t = y_t - \hat{\alpha} - \hat{\beta} t$

$X^*_t = X_t - \hat{\gamma} - \hat{\varphi} t$

$y_t = \beta_0 + \underbrace{\beta_1} X_t + \underset{\overset{\|}{\beta_2 t + \varepsilon_t}}{u_t}$

By Frisch-Waugh Thm

$$\left.\begin{array}{l} y_t | t \Rightarrow y^*_t \\ X_t | t \Rightarrow X^*_t \end{array}\right\} \Leftrightarrow \text{equivalent to adding } t$$

2. Consider the following ADL(1,1) model relating the crime rate in a particular province, $crime_t$, to the clear-up rate (percentage of crimes resulting in a conviction)

$$crime_t = \alpha + \underbrace{\rho crime_{t-1}} + \delta_1 clearup_t + \delta_2 clearup_{t-1} + u_t, \text{ with } |\rho| < 1$$

where $\underbrace{u_t \text{ is white noise,}}$ an i.i.d. innovation that is uncorrelated to anything in the past.

(a) **(2 marks)** Briefly indicate whether OLS will provide unbiased and consistent parameter estimators.

(b) **(2 marks)** Derive the long run relationship between *crime* and *clearup*.

(c) **(4 marks)** Rewrite the model above in terms of an error correction model (ECM) and interpret its coefficients.

a) OLS is consistent since u_t - WN

OLS is biased because of $L crime_t$

b) $\tilde{crime} = \alpha + \rho \tilde{crime} + \delta_1 \tilde{clearup} + \delta_2 \tilde{clearup}$

$$\tilde{crime} = \frac{\alpha}{1-\rho} + \frac{\delta_1 + \delta_2}{1-\rho} \tilde{clearup}$$

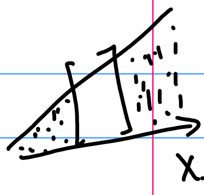
c) $\Delta CR_t \mid \underbrace{\Delta CL_t}_{SR}, \underbrace{u_{t-1}}_{LR}$

$CR_{t-1} - \psi_1 - \psi_2 CL_{t-1}$

...

$$\Delta CR_t = \underbrace{\delta_1 \Delta CL_t}_{\text{SR effect}} + \underbrace{(\rho - 1)}_{\text{speed of adj}} \cdot \underbrace{\left(CR_{t-1} - \frac{\alpha}{1-\rho} - \frac{\delta_1 + \delta_2}{1-\rho} \cdot CL_{t-1} \right)}_{\text{LR relationship}} + u_t$$

GQ => WLS



White => HC-se

$\hat{\varepsilon}_t^2 \mid X_i, X_i^2, X_i X_j$

3. Consider the simple linear regression model

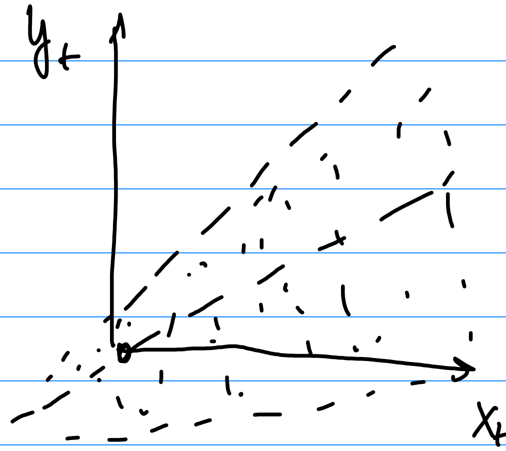
$$(1) \qquad Y_i = \beta X_i + u_i, \qquad i = 1, ..., n.$$

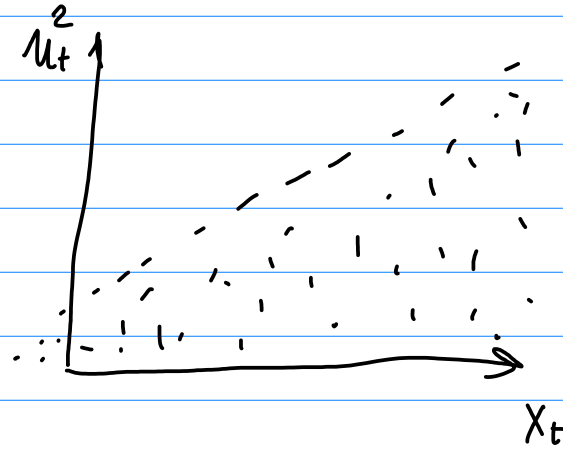
We assume that the errors $\{u_i\}_{i=1}^n$ are independent normal random variables with zero mean. The regressor $\{X_i\}_{i=1}^n$ is non-stochastic (fixed under repeated sampling). You suspect that the errors exhibit heteroskedasticity.

(a) **(3 marks)** Explain what we mean by the concept of heteroskedasticity. Enhance your answer with the help of a graphical illustration.

(b) **(5 marks)** Suppose you want to test $H_0 : \beta = 1$ against $H_1 : \beta > 1$. Discuss how you would conduct this test based on the OLS estimator, recognizing the presence of heteroskedasticity. Please provide a detailed answer.

a) $\mathrm{Var}(u_i) = E(u_i^2) \neq \sigma_u^2$





b) $H_0: \beta = 1$

$H_a: \beta > 1$

$$t = \frac{\hat{\beta} - 1}{[se(\hat{\beta})]} \sim t_{n-1}$$

HC(robust) estimators of se

$|t^{obs}| > t_{n-1, 1-\alpha}$ => reject H_0

$$\widehat{\mathrm{Var}}^{HC}(\hat{\beta}) = \frac{\sum X_i^2 \hat{u}_i^2}{(\sum X_i^2)^2}$$

4. Consider the simple linear regression model

$$(1) \qquad Y_i = \beta_0 + \beta_1 X_i + u_i, \qquad i = 1, ..., n.$$

We assume that the errors $\{u_i\}_{i=1}^n$ are independent random variables with zero mean. The regressor $\{X_i\}_{i=1}^n$ is non-stochastic (fixed under repeated sampling). Under these conditions, the OLS estimator for β_1, $\hat{\beta}_1$, is unbiased. (You are not asked to derive $\hat{\beta}_1$).

(a) **(2 marks)** Explain the concept of unbiasedness of an estimator.

(b) **(4 marks)** Let us consider two other estimators for the slope β_1 :

$$\hat{\beta}_1^{\circ} = \frac{\sum_{i=1}^n (Z_i - \bar{Z}) Y_i}{\sum_{i=1}^n (Z_i - \bar{Z}) \underbrace{X_i}}; \quad \hat{\beta}_1^{*} = \frac{\sum_{i=1}^n (Z_i - \bar{Z}) Y_i}{\sum_{i=1}^n (Z_i - \bar{Z}) \underbrace{Z_i}}$$

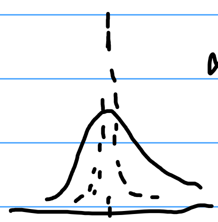
where $Z_i = \sqrt{X_i}$ for all i and $\bar{Z} = \frac{1}{n}\sum_{i=1}^n Z_i$. Please indicate whether $\hat{\beta}_1^{\circ}$ and $\hat{\beta}_1^{*}$ are unbiased estimators for β_1. Clearly show your derivations.

(c) **(2 marks)** Briefly indicate how you would choose between the three estimators, $\hat{\beta}_1$, $\hat{\beta}_1^{\circ}$ and $\hat{\beta}_1^{*}$.

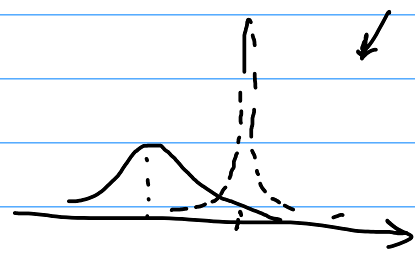
c) $\hat{\beta}_1$ vs $\hat{\beta}_1^{o}$ by Var (by GMT $\hat{\beta}_1$)

$\hat{\beta}_1$ vs $\hat{\beta}_1^{*}$ by MSE

a) $E(\hat{\beta}) = \beta \qquad (\lim E(\hat{\beta}) = \beta)$



β



$\beta^* \to \beta$

$\sum Z_i - \sum \bar{Z}$
$= \sum Z_i - n \frac{\sum Z_i}{n} = 0$

b) $$\hat{\beta}_1^{o} = \frac{\sum (Z_i - \bar{Z}) Y_i}{\sum (Z_i - \bar{Z}) X_i} = \beta_0 \frac{\overbrace{\sum (Z_i - \bar{Z})}^{=0}}{\sum (Z_i - \bar{Z}) \cdot X_i} + \beta_1 \frac{\sum (\cancel{Z_i - \bar{Z}}) \cdot X_i}{\sum (\cancel{Z_i - \bar{Z}}) \cdot X_i} + \frac{\sum (Z_i - \bar{Z}) u_i}{\sum (Z_i - \bar{Z}) X_i}$$

$$= \beta_1 + \frac{\sum (Z_i - \bar{Z}) u_i}{\sum (Z_i - \bar{Z}) X_i}$$

$$E(\hat{\beta}_1^{o}) = \beta_1 + \frac{\sum (Z_i - \bar{Z}) \overbrace{E(u_i)}^{=0}}{\sum (Z_i - \bar{Z}) X_i} = \beta_1$$

$$\hat{\beta}_1^* = \frac{\sum (z_i - \bar{z}) y_i}{\sum (z_i - \bar{z}) z_i} = \beta_0 \frac{\overbrace{\sum (z_i - \bar{z})}^{=0}}{\sum (z_i - \bar{z}) \cdot z_i}$$

$$+ \beta_1 \frac{\sum (z_i - \bar{z}) \cdot \overbrace{x_i}^{= z_i^2}}{\sum (z_i - \bar{z}) \cdot z_i} + \frac{\sum (z_i - \bar{z}) u_i}{\sum (z_i - \bar{z}) z_i}$$

$$E(\hat{\beta}_1^*) = \beta_1 \cdot \frac{\sum (z_i - \bar{z}) z_i^2}{\sum (z_i - \bar{z}) \cdot z_i} + \frac{\sum (z_i - \bar{z}) \overbrace{E(u_i)}^{=0}}{\sum (z_i - \bar{z}) \cdot z_i}$$

$$= \beta_1 \cdot \frac{\sum (z_i - \bar{z}) \cdot z_i^2}{\sum (z_i - \bar{z}) \cdot z_i} \neq \beta_1$$

5. Consider the OLS estimator for β in the linear regression model

$$Y_i = \beta X_i + \varepsilon_i, \qquad i = 1, ..., n,$$

where $\{(Y_i, X_i)\}_{i=1}^n$ form an i.i.d. sample from a population and the errors are drawn from an **unknown** distribution with mean zero and variance σ^2. You are told that X_i and ε_i are **uncorrelated** (not necessarily independent therefore!).

(a) **(4 marks)** Discuss the importance of convergence in probability. $\lim_{n \to \infty} P(|\hat{\beta} - \beta| > \varepsilon) = 0$

(b) **(4 marks)** Discuss the importance of convergence in distribution.

$$\hat{\beta} \xrightarrow{P} \beta$$

$$\lim_{n \to \infty} F_n(\hat{\beta}) = F(\hat{\beta})$$

$$= N(\beta, se(\hat{\beta}))$$