$$\text{DLM} \to \text{Geometrically} \to \text{PAM}, \text{AEM}, \text{ECM}$$

$$\text{DLM} \to \text{Polynomial}$$

(1) PAM model : $y_t^* = \alpha + \beta x_t + \varepsilon_t$

PA hypothesis $y_t - y_{t-1} = (1-\lambda)(y_t^* - y_{t-1})$

(2) Adaptive expectation model (AEM)

$[y_t = \alpha + \beta \cdot x_t^* + \varepsilon_t]$

$0<\lambda<1$ AE hypothesis $x_t^* - x_{t-1}^* = (1-\lambda)(x_t - x_{t-1}^*)$

x_t^* – unobserved (expected) value of x_t

$\lambda \approx 0$ – fast revision $x_t^* = (1-\lambda)\cdot x_t + \lambda \cdot x_{t-1}^*$

$\lambda \approx 1$ – slow revision $= (1-\lambda)(x_t + \lambda x_{t-1} + \lambda^2 x_{t-2} + \dots)$

$$= (1-\lambda)\cdot \sum_{j=0}^{\infty} \lambda^j x_{t-j}$$

ARDL$(0,\infty)$: $y_t = \alpha + \beta \cdot \sum w_j x_{t-j} + \varepsilon_t$, $w_j = (1-\lambda)\lambda^j$

$\Rightarrow$ in lag form

ARDL$(1,0)$: $y_t = \alpha_0 + \beta_0 \cdot x_t + \lambda y_{t-1} + v_t$

$\alpha_0 = \alpha(1-\lambda)$

$\beta_0 = \beta(1-\lambda)$

$v_t = \varepsilon_t - \lambda \varepsilon_{t-1}$

SR: $\beta w_0 X_t = \beta(1-\lambda)$

LR: $\dfrac{\beta_0}{1-\lambda} = \beta$

$\beta \cdot \Sigma w_j = \beta \cdot \Sigma (1-\lambda)\lambda^j$

AEM: $y_t = \alpha + \beta \cdot x_t^e + \varepsilon_t$

$$x_t^e = E[x_{t+1} \mid I_t]$$

$$\Delta x_t^e = (1-\lambda)\underbrace{(x_t - x_{t-1}^e)}$$

Δx_t^e – revision of expectations

$$x_t - x_{t-1}^e = x_t - E[x_t \mid I_{t-1}]$$

↳ forecast error

$$x_t^e - x_{t-1}^e = (1-\lambda)(x_t - x_{t-1}^e)$$

$$x_t^e - \lambda x_{t-1}^e = (1-\lambda) x_t$$

$$\underbrace{(1-\lambda L)} x_t^e = (1-\lambda) x_t$$

$$x_t^e = (1-\lambda L)^{-1}(1-\lambda) x_t =$$

$$\frac{1}{1-\lambda L}$$

$$= (1-\lambda)\sum_{k=0}^{\infty} \lambda^k L^k x_t = (1-\lambda)\Sigma \lambda^k x_{t-k}$$

Estimation:

1) ARDL$(0,\infty)$: $y_t = \beta_1 + \beta_2 (1-\lambda) X_t + \beta_2 \cdot (1-\lambda)\lambda \cdot X_{t-1} + \ldots + u_t$

↳ Approximately ARDL$(0,s)$

$y_t = \beta_1 + \beta_2 (1-\lambda) X_t + \beta_2 (1-\lambda)\lambda X_{t-1} + \ldots + \beta_2 \cdot (1-\lambda)\lambda^s X_{t-s} + u_t$

1) fix $\hat{\lambda}$

2) fit $y \mid [(1-\lambda) X_t + (1-\lambda)\lambda X_{t-1} \ldots]$

$\Rightarrow \hat{\beta}_2 \Rightarrow$ RSS

3) change $\hat{\lambda}$

4) repeat 1–3 until RSS min

2) $y_t = \beta_1 + \beta_2 (1-\lambda) X_t + \beta_2 \cdot (1-\lambda)\lambda \cdot X_{t-1} + \ldots + u_t$

$\Rightarrow$ ARDL$(1,0)$:

$y_t = (1-\lambda)\beta_1 + \beta_2 (1-\lambda) X_t + \lambda y_{t-1} + \underbrace{u_t - \lambda u_{t-1}}$

1) Fit $y \mid X_t, y_{t-1}$

Error Correction Model (ECM)

$$y_t^* = \alpha + \beta x_t + \varepsilon_t$$

$\hookrightarrow$ unobserved (equilibrium) level of y_t

EC Hypothesis:

$$y_t - y_{t-1} = (1-\gamma)(y_t^* - y_{t-1}^*)$$
$$+ (1-\lambda)(y_{t-1}^* - y_{t-1})$$

Δy_t^* – change of equilibrium y

$y_{t-1}^* - y_{t-1}$ – previous disequilibrium

If $\gamma = \lambda \Rightarrow$ PAM

$$y_t - y_{t-1} = (1-\gamma)\left[\beta(x_t - x_{t-1}) + (\varepsilon_t - \varepsilon_{t-1})\right]$$
$$+ (1-\lambda)(\alpha + \beta x_{t-1} + \varepsilon_{t-1} - y_{t-1})$$

ARDL(1,1) $\hookrightarrow$ $y_t = \alpha_0 + \beta_0 x_t + \beta_1 x_{t-1} + \lambda y_{t-1} + v_t$

$\alpha_0 = (1-\lambda)\alpha$ SR: $(1-\gamma)\beta$

$\beta_0 = (1-\gamma)\beta$ LR: $\dfrac{\beta_0 + \beta_1}{1-\lambda} = \beta$

$\beta_1 = (\gamma - \lambda)\beta$

$v_t = (1-\gamma)\varepsilon_t + (\gamma - \lambda)\varepsilon_{t-1}$