Prep. 1

Q1 $y_i = \alpha + \beta X_i + u_i$, $E(u_i) = 0$,

i	1	2	3	4	5	6
X_i	1	2	3	4	5	6

$E(u_i^2) = \sigma^2$,

$E(u_i u_j) = 0$

$\hat{\beta}$ – OLS estimator

$Var(\hat{\beta}) = \sigma^2 / 17{,}5$

$\tilde{\beta}$ – alternative estimator

$\tilde{\beta} = \frac{1}{8}[y_6 + y_5 - y_2 - y_1]$

Compare $Var(\tilde{\beta})$ vs $Var(\hat{\beta})$

Sol

$$\tilde{\beta} = \frac{1}{8}[\cancel{\alpha} + \beta \underset{\substack{\parallel \\ 6}}{X_6} + u_6 +$$

$$+ \cancel{\alpha} + \beta \cdot 5 + u_5 +$$

$$- \cancel{\alpha} - \beta \cdot 2 - u_2 +$$

$$- \cancel{\alpha} - \beta - u_1]$$

$$= \frac{1}{8}(8\beta + u_6 + u_5 - u - u_1) =$$

$$= \beta + \frac{1}{8}(u_6 + u_5 - u_2 \quad u_1)$$

$$E(\tilde{\beta}) = E\left(\frac{1}{8}[y_6 + y_5 - y_2 - y_1]\right) =$$

$$= E\left(\beta + \frac{1}{8}(u_6 + u_5 - u_2 - u_1)\right) =$$

$$= \beta + \frac{1}{8} E(u_6 + u_5 - u_2 - u_1) = \{E(u_i) = 0\} =$$

$$= \beta \quad \Rightarrow \quad \text{unbiased}$$

$$Var(\tilde{\beta}) = Var\left(\beta + \frac{1}{8}(u_6 + u_5 - u_2 - u_1)\right) =$$

$$\frac{1}{64} Var(u_6 + u_5 - u_2 - u_1) = \{E(u_i u_j) = 0 \underset{i \neq j}{}\}$$

$$\frac{1}{64}(Var(u_6) + \ldots) =$$

$$\frac{1}{64} \cdot 4 \cdot \sigma^2 = \frac{\sigma^2}{16}$$

$$Var(\tilde{\beta}) > Var(\hat{\beta})$$

$$\frac{\sigma^2}{16} > \frac{\sigma^2}{17{,}5}$$

Q2 $\underbrace{y_t = \beta x_t + u_t}_{\text{TRUE}}$, $t = \overline{1, T}$, $E(u_t) = 0$,

$E(u_t) = \sigma^2$,

$E(u_i u_j) = 0$, $i \neq j$

Obtain $\hat{\beta}$ – as estimator

Show linearity & unbiasedness

OLS: $RSS \to \min\limits_{\hat\beta}$ $\qquad \hat\beta = (X'X)^{-1}X'y$

$$RSS = \sum \hat{u}_i^2 = \sum (y_i - \hat\beta X_i)^2 \qquad \hat\beta = \frac{\sum X_i y_i}{\sum X_i^2}$$

$$\frac{\partial RSS}{\partial \hat\beta} = \sum -2x_i(y_i - \hat\beta X_i) = 0$$

$$\sum X_i y_i - \hat\beta \sum X_i^2 = 0$$

$$\hat\beta_{OLS} = \frac{\sum X_i y_i}{\sum X_i^2}$$

1) Linearity in y: $\hat\beta = \sum w_i y_i$

$$\hat\beta_{OLS} = \frac{\sum X_i y_i}{\sum X_i^2} = \sum \frac{X_i y_i}{\sum X_i^2} = \sum \underbrace{\frac{X_i}{\sum X_i^2}}_{w_i} \cdot y_i$$

$$= \sum w_i y_i$$

2) Unbiasedness:

$$\hat\beta = \sum w_i (\beta X_i + u_i) = \beta \underbrace{\sum w_i X_i}_{=1} + \sum w_i u_i =$$

$$= \beta \cdot \underbrace{\sum \frac{X_i}{\sum X_i^2} \cdot X_i}_{=1} + \sum w_i u_i = \beta + \sum w_i u_i$$

$$E(\hat{\beta}) = E(\beta + \sum W_i u_i) = \beta + E(\sum W_i u_i) =$$

$$= \beta + \sum E(W_i u_i) = \{ W_i - \text{deterministic} \} =$$

$$= \beta + \sum W_i \cdot \underbrace{E(u_i)}_{=0} = \beta$$

Q3 Which criteria are used to compare stat. estimators

TRUE Equation 1) $y_i = \beta X_i + u_i$

Method 2) $\hat{\beta}_{OLS} = \dfrac{\sum X_i y_i}{\sum X_i^2}$ – OLS

Properties 3)

Unbiasedness $E(\hat{\beta}) = \beta$

Consistency $\text{plim}_{n \to \infty} \hat{\beta} = \beta$

Efficiency (in class LUE) $Var(\hat{\beta}) \le Var(\tilde{\beta})$, $\forall \tilde{\beta} \in C_{LUE}$

For stoch. regr. $E(\hat{\beta}) \neq \beta$ not necessary

Only if $X \perp u$

Cond. Unbiasedness $E(\hat{\beta} | X) = \beta$

Q4 $y = \beta_1 X^{\beta_2}$ — 1) Show fun has constant elasticity
— 2) How to estimate it

Sol $\log y = \log \beta_1 + \beta_2 \log X \quad (*)$

$\dfrac{dy/y}{dx/x}$ — elasticity

Differentiate $(*)$:

$$\frac{1}{y}\frac{dy}{dx} = 0 + \beta_2 \cdot \frac{1}{X}$$

$$\frac{dy/y}{dx/x} = \beta_2 \Rightarrow \text{constant elasticity}$$

1) $\log y_i = \log \beta_1 + \beta_2 \log X_i + u_i$

$y_i = \beta_1 \cdot X_i^{\beta_2} \cdot u_i^*$ — multi. u_i

$\Rightarrow$ model linear in log.

$\Rightarrow$ use OLS

2) $y_i = \beta_1 \cdot X_i^{\beta_2} + u_i$ - add. u_i

$\Rightarrow$ NLS (numerically minimised RSS by $\hat{\beta}_1, \hat{\beta}_2$)