Prep. 2

Q 1

$$Var(y_t) = Var(\hat{y}_t) + Var(\hat{u}_t) \qquad (*)$$

$$y_t = \alpha + \beta X_t + u_t$$

$$TSS = ESS + RSS$$

$$\Updownarrow$$

$$\underbrace{Var(y_t)}_{\text{Total var}} = \underbrace{Var(\hat{y}_t)}_{\text{Explained var}} + \underbrace{Var(\hat{u}_t)}_{\text{Unexplained var}} \quad \Big| \cdot T$$

$$R^2 = \frac{ESS}{TSS} = \frac{Var(\hat{y}_t) \cdot \cancel{T}}{Var(y_t) \cdot \cancel{T}}$$

— explained share of variance

$$Var(y_t) = \frac{\Sigma (y_t - \bar{y})^2}{T}$$

$$TSS = \Sigma (y_t - \bar{y})^2$$

$$R^2 = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS} =$$

$$= 1 - \frac{Var(\hat{u}_t) \cdot T}{Var(y_t) \cdot T}$$

$ESS = 0 \Rightarrow R^2 = 0$

(e.g. naive model on a constant)

$$\Rightarrow R^2 \in [0; 1]$$

$RSS = 0 \Rightarrow R^2 = 1$

$$y_t = \hat{y}_t + \hat{u}_t$$

$$Var(y_t) = Var(\hat{y}_t + \hat{u}_t) =$$

$$= Var(\hat{y}_t) + Var(\hat{u}_t) + 2\underbrace{Cov(\hat{y}_t, \hat{u}_t)}_{? = 0}$$

$$Cov(\hat{y}_t, \hat{u}_t) = Cov(\hat{\beta}_0 + \hat{\beta}_1 \cdot X_t, \hat{u}_t) =$$

$$= Cov(\underbrace{\hat{\beta}_0}_{const}, \hat{u}_t) + Cov(\underbrace{\hat{\beta}_1}_{const} \cdot X_t, \hat{u}_t) =$$

$$= 0 + \hat{\beta}_1 \underbrace{Cov(X_t, \hat{u}_t)}_{=0} = 0$$

(from normal equations)

(Q2) $$y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + \varepsilon_i$$

$$Var(\hat{\beta}_2) = \frac{\sigma_\varepsilon^2}{\sum (X_{2i} - \bar{X}_2)^2} \cdot \frac{1}{1 - r^2_{X_2, X_3}}$$

(a) Which 4 factor allows obtain more precise estimators for $\hat{\beta}_2$?

1) $\sigma_\varepsilon^2 \downarrow$

$$Var(\hat{\beta}_i) = \frac{\sigma_\varepsilon^2}{TSS_i \cdot (1 - R_i^2)}$$

$TSS_i \uparrow$

R_i^2 – aux. regression $X_i \mid X_{-i}$

$TSS_i = N \cdot Var(X_i)$

2) $N \uparrow$

3) $Var(X_i) \uparrow$

4) $(1 - R_i^2) \uparrow \Rightarrow R_i^2 \downarrow$

$\Rightarrow$ no multicollinearity

(b) $\beta_3 = 0$ – true value

$\Rightarrow$ Estimated reg. has irrelevant var.

$$y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + \varepsilon_i$$

How it affect on

unbiasedness and precision of $\hat{\beta}_2$

Omitted variable bias (X_3) $\nearrow$ $\beta_3 \neq 0$

$\searrow$ $r_{X_2, X_3} \neq 0$

if $r_{X_2, X_3} = 0 \Rightarrow$ no bias

if $\beta_3 = 0 \Rightarrow$ no bias

Precision : if $r_{X_2, X_3} \neq 0 \Rightarrow$

$$\text{Var}(\hat{\beta}_2^{\,EST}) = \frac{\sigma_\varepsilon^2}{\sum (X_{2i} - \bar{X}_2)^2} \cdot \underbrace{\frac{1}{1 - r^2_{X_2, X_3}}} \geq \text{Var}(\hat{\beta}_2^{\,TRUE}) = \frac{\sigma_\varepsilon^2}{\sum (X_{2i} - \bar{X}_2)^2}$$

(c) Multicollinearity ?

Q3

$$y_i = \beta X_i + \varepsilon_i \qquad \{(y_i, X_i)\} \text{ i.i.d.}$$

Stoch   no AC

$$E(\varepsilon_i) = 0$$

$$\Uparrow$$

$$E(\varepsilon_i | X) = 0$$

$$Var(\varepsilon_i) = \sigma^2$$

$$Cov(X_i, \varepsilon_i) = E(X_i \varepsilon_i) - E(X_i)E(\varepsilon_i) = E(X_i \varepsilon_i) = 0$$

$$\Leftarrow \begin{cases} E(\varepsilon_i) = 0 \\ + \\ Cov(X_i, \varepsilon_i) = 0 \end{cases} \Leftarrow \left[\rho_{X_i, \varepsilon_i} = 0 \right]$$

a) Convergence in probability:

$$\operatorname*{plim}_{n\to\infty} X_n = X, \qquad X_n \xrightarrow{p} X$$

$$\forall \varepsilon \qquad \lim_{n\to\infty} p(|X_n - X| > \varepsilon) = 0$$

b) Convergence in distribution

$$X_n \xrightarrow{d} X$$

$$\lim_{n\to\infty} F_n(x) = F(x)$$

c) $$\operatorname{plim} \hat{\beta} = \operatorname{plim} \frac{\Sigma X_i y_i}{\Sigma X_i^2} =$$

$$\text{plim} \frac{\sum x_i(\beta x_i + \varepsilon_i)}{\sum x_i^2} =$$

$$\text{plim}\left(\beta + \frac{\sum x_i \varepsilon_i}{\sum x_i^2}\right) =$$

$$= \beta + \text{plim} \frac{\sum x_i \varepsilon_i}{\sum x_i^2} =$$

$$\text{plim}\ \bar{x} \to E(X)$$

$$\text{plim} \frac{\sum x}{n} \to \mu$$

$$= \beta + \frac{\text{plim}\ \sum x_i \varepsilon_i / n}{\text{plim}\ \sum x_i^2 / n} =$$

$$= \beta + \frac{E(x_i \varepsilon_i)^{=0}}{E(x_i^2)} = \beta$$

Q4

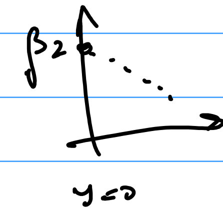
wages, educ., years of exp.

$$W_i = \beta_0 + \beta_1 E_i + \underline{\underline{\beta_2}}\, y_i + \underline{\underline{\beta_3}} \cdot y_i^2 + u_i$$

$\frac{dy/y}{dx/x}$ 1%, β%. $\{(W_i, E_i, y_i)\}$, i.i.d. $\frac{dW}{dy} = \beta_2 + \beta_3 \cdot 2y$ <0

$\frac{dy}{100 \cdot dx/x} = \beta \cdot \frac{1}{100}$ 1%, $\frac{1}{100}\beta$

$E(u_i) = 0$ $Var(u_i) = \sigma^2$

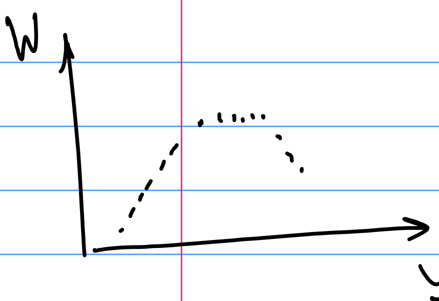


$u_i \perp E_i$, $u_i \perp y_i$

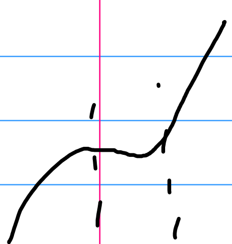
$\frac{100\, dy/y}{dx/x}$ 1↑, 100 β%.
" β·100

a) Rationale of both y and y²?

Signs of β_2 and β_3?



Assumes diminishing return of y on W



$\Rightarrow$ $\beta_3 < 0$

$\beta_2 > 0$

b) Test that y affects W

$H_0: \beta_2 = \beta_3 = 0$

$H_a: \beta_2 \neq 0$ or $\beta_3 = 0$

$$F = \frac{(RSS_R - RSS_{UR})/2}{RSS_{UR}/n-4} \overset{H_0}{\sim} F_{2,\,n-4}$$