Question 1. In the model

$$y = \beta x_t + u_t,\ t = 1, 2, ..., T$$

x_t is measured with error. Data is only available on x_t^*, where

$$x_t^* = x_t + v_t;\ t = 1, 2, ..., T$$

and $Eu_t = Ev_t = 0$, $E(u_t v_t) = E(x_t u_t) = E(x_t v_t) = 0$. y_t, x_t and x_t^* have zero means.

If $\hat{\beta}$ is the ordinary least squares estimator of β from regressing y_t on x_t^*, show that $\hat{\beta}$ is inconsistent.

$Var(u_t) = E(x^2) - E^2(x)$

$$\hat{\beta} = \frac{\sum x_i^* y_i}{\sum x_i^{*2}} = \frac{\sum (x_t + v_t)(\beta x_t + u_t)}{\sum (x_t + v_t)^2} =$$

$$= \frac{\left(\beta \sum x_t^2 + \sum x_t \cdot u_t + \beta \sum x_t \cdot v_t + \sum u_t v_t\right) \frac{1}{T-1}}{\left(\sum x_t^2 + \sum v_t^2 + 2 \sum x_t v_t\right) \cdot \frac{1}{T-1}}$$

$$\operatorname*{plim}_{t\to\infty} \hat{\beta} = \frac{\beta E(x_t^2) + \overbrace{E(x_t u_t)}^{=0} + \beta \overbrace{E(x_t v_t)}^{=0} + \overbrace{E(u_t v_t)}^{=0}}{E(x_t^2) + E(v_t^2) + 2\underbrace{E(x_t v_t)}_{0}}$$

$$= \beta \underbrace{\frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2}}_{0<\ \ <1} \Rightarrow \text{inconsist}$$

$\Rightarrow$ biased towards 0

$$E(\hat{\beta} - \beta) = \beta\left(\frac{\sigma_x^2}{\sigma_x^2 + \sigma_v^2} - 1\right) = \underbrace{\frac{-\sigma_v^2}{\sigma_x^2 + \sigma_v^2}}_{-1 < \ldots < 0} \beta$$

b) • $y_t^* = y_t + w_t$

$E\,w_t = 0 \qquad E(u_t w_t) = E(x_t w_t) = 0$

$$\hat{\beta} = \frac{\sum x_t \cdot y_t^*}{\sum x_t^2} =$$

$$= \frac{\sum x_t(\beta x_t + u_t + w_t)}{\sum x_t^2} = \dots$$

$y_t = \beta x_t + \underbrace{u_t - w_t}_{\varepsilon_t}$

$\Rightarrow \hat{\beta}$ as - consistent

$$\text{plim}\,\hat{\beta} = \beta \frac{E(x_t^2) + \overbrace{E(x_t \cdot u_t)}^{=0} + \overbrace{E(x_t w_t)}^{=0}}{E(x_t^2)}$$

$= \beta \quad \Rightarrow \quad$ consistent

Question 2 Briefly explain what is instrumental variable estimation. Consider the model

$$Y_t = \beta X_t + u_t\,;\ t = 1, 2, \ldots, T. \qquad (1)$$

What is instrumental variable estimator for this model? Is it consistent? Give an example of a model where instrumental variable estimation is an improvement on ordinary least squares (OLS). Explain why IV estimation is superior to OLS in this case.

Give an example of a model where instrumental variable estimation is an improvement on ordinary least squares (OLS).

Explain why IV estimation is superior to OLS in this case.

$E(u_t) = 0 \qquad E(u_t^2) = \sigma^2 \qquad E(u_t u_s) = 0$

$cov(x_t, u_t) \neq 0$, x_t, z_t - has 0 mean

z_t - instrument

$\nearrow$ $cov(x_t, z_t) \neq 0$ relevance

$\searrow$ $cov(z_t, u_t) = 0$ exogeneity

1) $\hat{X}_t = \hat{\alpha} z_t \qquad \hat{\alpha} = \dfrac{\sum x_t \cdot z_t}{\sum z_t^2}$

2) $y_t = \beta \hat{X}_t + u_t$

$$\hat{\beta} = \frac{\sum \hat{X}_t y_t}{\sum \hat{X}_t^2} = \frac{\hat{\alpha} \sum z_t y_t}{\hat{\alpha}^2 \sum z_t^2} =$$

$$= \frac{\sum z_t y_t}{\sum_{i_t} z}$$