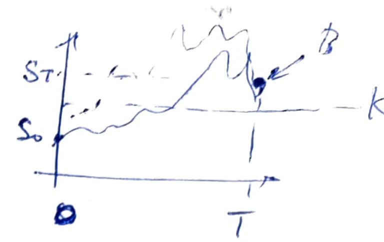
Lecture 1

1) - call
- put
- Barrier
- asian

2) - amer
- europ

1) (Call) Barrier option

Payoff = $f(S_t)$, $(S_t)_{t\in[0,T]}$



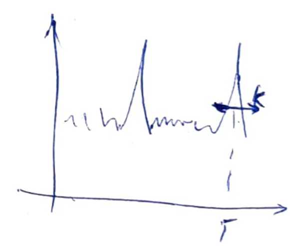
Asian option payoff = f of trajectory

$(A_T - K)^+$

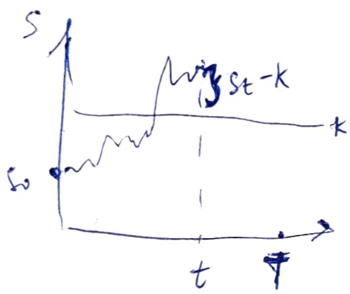
↑ average of prices of S

A price of electricity has jumps

⇒ by taking averages it can be normalized



American option (can be exercised at any time before mat.)



Problem
1) Price
2) Hedging

Am. 3) Exercise? optimal stopping
+ add optimisation problem

2) Principles of financial models

Article: 1
Levy process-driven mean reverting electricity price model;

a• value of time $X_T = X_t(1+r(T-t))$

↑ interest (annual) rate

If we assume S_0 fixed

(5?) $q = \frac{S_0 e^{rT} - S^-}{S^+ - S^-} \in]0;1[$ → this is true iff $S^- < e^{rT} S_0 < S^+$

↖ risk-neutral probability

THE IDEA: $X_T = e^{r(T-t)} \cdot X_t$ ⇒ $X_t = e^{-r(T-t)} X_T$

↑ discounted value of X_T

if this is not true we can always invest and produce money for free (if $S_0 e^{rt} > S^+$)

b• risk-neutral pricing

	T	0		T
Bank		1		e^{rt}
Asset	$S_0 = (p S^+ + (1-p) S^-) e^{-rT}$ (fair) ← risk-neutral price			$S_T = \begin{cases} S^+ & p \in]0;1[\\ S^- & 1-p \end{cases}$

0	T
Option $C := e^{-rt} \cdot (qC^+ + (1-q)C^-)$	$C_T = \begin{cases} (S^+ - K)^+ \\ (S^- - K)^+ \end{cases}$

$S_0 \to q$ [risk-neutral probability]

$\Downarrow$

C_0 [risk-neutral price]

$$= f(\underbrace{S_0, r, T, K}_{\text{observable}}, \underbrace{S^+, S^-}_{\text{params}})$$

$\Downarrow$

choosing by calibration

to make a model realistic

- p – real world prob.

When working with TS, we are estimating p

- q – is pricing measure (=> risk-neutral measure for pricing derivatives)

- No-arbitrage rule

Implication:

knowing $X_T = Y_T$ will be equal => implies $X_t = Y_t$ equal now, $\forall t \le T$ (otherwise we can buy cheaper one => ...)

ex

$X_t = S_t + P_t$

$Y_t = C_t + \underbrace{K \cdot e^{-r(T-t)}}_{\text{bank}}$

$X_T = \max\{S_T, K\}$

$Y_T = \max\{S_T - K, 0\} + K = X_T$

N.A. Principle =>

$$\boxed{S_t + P_t = C_t + K \cdot e^{-r(T-t)}}$$

obs. obs.

Put-Call parity

should satisfy, else we can arbitrate and make money for free

	0	T
Bank	1	e^{rt}
Asset	S_0	$S = \begin{cases} S^+ \\ S^- \end{cases}$
Price	$C_0 = ?$	$C_T = \begin{cases} (S^+ - K) = C^+ \\ (S^- - K) = C^- \end{cases}$

s.t.

$V = \alpha S + \beta B$ ← bank account

$\boxed{V_T = C_T}$ → find replicating strategy (α, β), so the equality holds.

strategy

$$\Leftrightarrow \begin{cases} \alpha S^+ + \beta e^{rt} = C^+ & \leftarrow \text{price increase} \\ \alpha S^- + \beta e^{rt} = C^- & \leftarrow \text{p. decrease} \end{cases} \Rightarrow \begin{cases} \bar{\alpha} = \dfrac{C^+ - C^-}{S^+ - S^-} \\ \bar{\beta} = \dots \end{cases}$$

$\boxed{V_T = C_T}$ $\Rightarrow$ no-arbitrage price $C_0 := \bar{\alpha} S_0 + \bar{\beta} \cdot 1 = V_0$. ☆

if price $\neq C_0$ market has an arbitrage opportunity $\Rightarrow$ $C_0 = f(\underbrace{S_0, T, K, r}, S^+, S^-)$

N.A. principle gives hedging solution for bank

☆ C_0 price for an option equal to money V_0 that needs to be invested by bank strategy

To find V_0 we have lin. system $\{ \dots$ and cond. $V_T = C_T$ $\Leftrightarrow$ like Cauchy pr.

☆☆ risk-neutral price - the expectation of the pay off

Incomplete Markets

	0	T	
Bank	1	e^{rt}	
Asset	S_0	$S_T = \begin{cases} S^1 \\ S^2 \\ S^3 \end{cases}$	q_1 q_2 $1-q_1-q_2$
option	C_0		

• R-N $S_0 = e^{-rT}(q_1 S^1 + q_2 S^2 + (1-q_1-q_2) S^3)$ 1 eq 2 unkn. $\Rightarrow$ inf r-n measures (probabilities) $\Rightarrow$ incomplete model (because mart. measure isn't unique)

• N-A $V_T = C_T$

$$\begin{cases} \alpha S^1 + \beta B = C^1 \\ \alpha S^2 + \beta \cdot B = C^2 \\ \alpha S^3 + \beta B = C^3 \end{cases}$$

in general $\nexists (\alpha, \beta)$

not solvable $\underbrace{C_1, C_2, C_3}_{\text{three conditions}}$ and only 2 params (d.o.f.) $\Rightarrow$ $\nexists$ N.A. price

$\Rightarrow$ incomplete model

To complete market another derivative is ~~needed~~ needed

$(\Omega, \mathcal{F}, P)$

$\uparrow$ σ-algebra (information)

$\nwarrow$ prob. measure

1) Measurability (deterministic dependance)

$X : \Omega \longrightarrow \mathbb{R}^d$

m.v. $(X \in M) \in \mathcal{F}$ $\quad \mathcal{M} = \mathcal{B}$

$\sigma(X) \leftarrow \sigma$ - algebra about X

Problem: measurability $\quad X \in m\mathcal{F} \leftrightarrow \sigma(X) \subseteq \mathcal{F}$

Assume $\quad Y \in m\,\sigma(X) \iff Y = f(X)$

$\quad f \in m\mathcal{B}$

X, Y independent w.r.t P might become dependent

if we change the p. measure

Lecture 2

① $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

$$P(A|\sigma(B))(\omega) = \begin{cases} P(A|B), & \text{if } \omega\in B \\ P(A|B^c), & \text{if } \omega\in B^c \end{cases}$$

② $\mathcal{G} = \sigma(Y)$

$Y(\Omega) = \{y_1, \ldots, y_n\}$ or $\{y_k\}$ $k\in\mathbb{N}$

and all elements are unions of partitions of space

$E[X|Y]^{\omega} = E[X|\sigma(Y)](\omega) = E[X \mid \underbrace{Y = Y(\omega)}_{\text{event}}]$

③ $W = \boxed{E[X|\mathcal{G}] = Z}$ ← r.v. s.t.

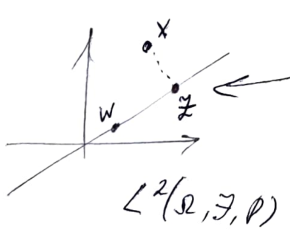
general σ-algebra

1) $Z \in m\mathcal{G}$

2) $E[ZW] = E[XW]$ $\forall W\in m\mathcal{G}$

not possible for Borel algebra

$L^2(\Omega, \mathcal{F}, P)$: $E[X^2] < +\infty$ (like Euclidian space for inf. dimentions)



Interpretation of ②: orthogonality

$\langle Z - X, W\rangle_{L^2} = E[(Z-X)\cdot W] = 0$

Prop.

1) $E[X] = E[Z]$

2) $X\in m\mathcal{G} \Rightarrow X = E[X|\mathcal{G}]$ → best case est.

3) $X, \mathcal{G}$ independent w.r.t. P $\Rightarrow E[X|\mathcal{G}] = E[X]$ ⇒ the worst expectation we can get for X

4) $E[XY|\mathcal{G}] = Y E[X|\mathcal{G}]$, $Y\in m\mathcal{G}$

5) $\mathcal{G}\subseteq\mathcal{H}$ knowledge constr. by smaller σ-al

$E[E[X|\mathcal{G}]|\mathcal{H}] =$

$= E[E[X|\mathcal{H}]|\mathcal{G}] =$

$= E[X|\mathcal{G}]$

ex $X(\Omega) = \{x_k\}_{k \in \mathbb{N}}$

$Y(\Omega) = \{y_n\}_{n \in \mathbb{N}}$

if $P(X = x_k | Y)$ is constant $\forall k$

$\iff X, Y$ ind w.r.t. P

Martingales: $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n \in \mathbb{N}})$

$(X_n)_{n \in \mathbb{N}}$ filtration $\to \mathcal{F}_n \le \mathcal{F}_{n+1} \dots \le \mathcal{F} \ \forall n$

$\mathcal{F}_n^X = \sigma(X_1, \dots, X_n) \leftarrow$ natural filtration of X

smallest filtration w.r.t. which X is adapted

$X = (X_n)_{n \in \mathbb{N}}$

if $X_k \in m\mathcal{F}_n \ \forall n$ then X is adapted to $(\mathcal{F}_n)_{n \ge 1}$

MG: $X = (X_n)_{n \ge 1}$ on $(\Omega, \mathcal{F}, P, (\mathcal{F}_n))$

if $X_n = E[X_{n+1} | \mathcal{F}_n] \ \forall n$

MG property depends on prob. measure P (not on trajectories!)

Remarks: 1) X is MG $\Rightarrow X$ adapted

2) $X_n = E[X_{n+1} | \mathcal{F}_n] =$

$= E[E[X_{n+2} | \mathcal{F}_{n+1}] | \mathcal{F}_n]$

$= E[X_{n+2} | \mathcal{F}_n]$

$\Rightarrow X_n = E[X_{n+k} | \mathcal{F}_n], \ \forall k \ge 1$

3) $E[X_n] = E[X_{n+k}] = E[X_0]$

ex $(Y_k)_{k \ge 1}$

$(\Omega, \mathcal{F}, P, (\mathcal{F}_n^X))$ ← nat. filtr.

Y_k i.i.d. Ber(p) $\Rightarrow Y_k = \begin{cases} 1, & p \\ -1, & 1-p \end{cases}$

$X_n = \sum^n Y_k$

is X MG? $E[X_{n+1} | \mathcal{F}_n^X] = E[\underbrace{X_n}_{\in m\mathcal{F}_n^X} + Y_{n+1} | \mathcal{F}_n^X] =$

$= X_n + \underbrace{E[Y_{n+1} | \mathcal{F}_n^X]}_{\text{independ.}} = X_n + E[Y_{n+1}] = X_n + 2p - 1$

$p = 1/2$

If $p > 1/2$ $\quad X_n \le E[X_{n+1} \mid \mathcal{F}_n^X]$ $\quad X$ is a sub-mg.

$\left(X_n \ge E[X_{n+1} \mid \mathcal{F}_n^X] \Rightarrow \text{super-mg} \right)$

Doob's Decomposition ($\Rightarrow$ Ito formula)

X adapted process

$X_n = M_n + A_n$ $\quad \forall n \in \mathbb{N} \cup \{0\}$

drift (gives direction)

M_n: mg

A_n: predictable

$A_0 = 0$

$A_n \in m\mathcal{F}_{n-1}$ $\quad \forall n \ge 1$

- $X_{n+1} - X_n = \underbrace{M_{n+1} - M_n}_{} + A_{n+1} - A_n$ $\quad \forall n$

$E[X_{n+1} \mid \mathcal{F}_n] - X_n = 0 + A_{n+1} - A_n$ $\quad \Rightarrow$ $A: \begin{cases} A_0 = 0 \\ A_{n+1} = A_n + E[X_{n+1} \mid \mathcal{F}_n] - X_n \end{cases}$

is uniq. defined

ex $\quad M_n = X_n - n(2p-1)$

$A_n = n(2p-1)$

Market Model

$t_0 \qquad t_N$

$0 \qquad T$

$t_k = k \cdot \frac{T}{N}$, $\quad k = \overline{0,N}$

Bank $\begin{cases} B_0 = 1 \\ B_{n+1} = B_n(1+r) \end{cases}$

$\Rightarrow B_n = (1+r)^n$

price at t_n

$S^1, \dots S^d$ - assets

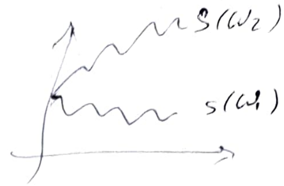
$\begin{cases} S^i_{n+1} = S^i_n (1 + \mu^i_{n+1}) \\ S^i_0 > 0 \end{cases}$

rates of return

$\mu = (\mu_n)_{n \ge 0}$ s.p. on $(\Omega, \mathcal{F}, P, (\mathcal{F}^\mu))$

strategy : $(\alpha_n, \beta_n)_{n=\overline{1,n}}$

α is d - dimentional



$$V_n = \underbrace{\alpha_n \cdot S_n}_{\sum_{i=1}^{d} \alpha_n^i S_n^i} + \beta_n \cdot B_n$$

$\omega_1, \omega_2 \in \Omega$

1) (α, β) are predictable $\Leftrightarrow$

$(\alpha_n, \beta_n) \in m \mathcal{F}_{n-1}, \forall n \geq 1$

2) self - financing

$V_n = \alpha_{n+1} S_n + \beta_{n+1} B_n \quad , \forall n$ (can't draw money from outside)

$A = \{(\alpha, \beta)$ predictable + self - financing $\}$

$\hookrightarrow$ set of admissible strategies

discounted

Rem : $\tilde{V}_n = \alpha_{n+1} \tilde{S}_n + \beta_{n+1} \cdot 1$

$\tilde{V}_{n+1} = \alpha_{n+1} \tilde{S}_{n+1} + \beta_{n+1} \cdot 1$

$\Downarrow$

$\underbrace{\tilde{V}_{n+1} - \tilde{V}_n}_{\Delta \tilde{V}} = \underline{\underline{\alpha_{n+1}}} \underbrace{(\tilde{S}_{n+1} - \tilde{S}_n)}_{\Delta \tilde{S}}$

self - fin $\Rightarrow \underset{\text{value ch.}}{\Delta V}$ dep. only on $\underset{\text{price ch.}}{\underline{\Delta S}}$

- knowing V_0 and pred. α $\Rightarrow (\alpha, \beta)$

Model is correct?

DEF Arbitrage opportunity : $(\alpha, \beta) \in \underline{A}$ s.t.

1) $V_0^{(\alpha,\beta)} = 0$

2) $V_N^{(\alpha,\beta)} \geq 0$ a.s.

3) $P(V_N^{(\alpha,\beta)} > 0) > 0$

REM: $Q \sim P$ equivalent measures (e.g. sure events has eq. measures)

Model is arbitrage-free => good model
if arb. do not exist

DEF Martingale Measure - prob. measure $Q \sim P$ s.t.

$\tilde{S}$ is a Q - mg

discounted
=> time change effect is removed

$\Rightarrow \tilde{S}_n = E^Q[\tilde{S}_{n+1} \mid \mathcal{F}_n]$

risk-neutral measure

1° Fund. Theor. of asset pricing

* A model is arbitrage-free iff a mg measure exists

Prop. $(\alpha, \beta) \in A$ and Q - mg. measure

then $\tilde{V}^{(\alpha,\beta)}$ is a Q-mg

$$E^Q[\tilde{V}_{n+1} - \tilde{V}_n \mid \mathcal{F}_n] = E^Q[\alpha_{n+1}(\tilde{S}_{n+1} - \tilde{S}_n) \mid \mathcal{F}_n] \overset{\bullet}{=}$$

$$= \alpha_{n+1} \underbrace{E^Q[\tilde{S}_{n+1} - \tilde{S}_n \mid \mathcal{F}_n]}_{=0} = 0$$

=> a martingale condition

N.A. principle: (α,β), $(\alpha',\beta') \in A$ and Q mg measure exists

then if $V_N^{(\alpha,\beta)} = V_N^{(\alpha',\beta')}$

then $V_n^{(\alpha,\beta)} = V_n^{(\alpha',\beta')}$, $\forall n \leq N$

$$\tilde{V}_n^{(\alpha,\beta)} = E^Q[\tilde{V}_N^{(\alpha,\beta)} \mid \mathcal{F}_n] =$$

$$= E^Q[\tilde{V}_N^{(\alpha',\beta')} \mid \mathcal{F}_n] = \tilde{V}_n^{(\alpha',\beta')}$$

Proof to 1°:

1° ~~⟺~~ (⇐) Assume $\exists Q$: if (α,β) is an arbitrage

$$0 = \tilde{V}_0^{(\alpha,\beta)} = E^Q[\underbrace{\tilde{V}_N^{(\alpha,\beta)}}] > 0$$

- nonneg. ≥ 0
- str. pos w. str. pos p

(⇒) impossible => arb. doesn't exist

(=>) ...

European option: X r.v.

ex call-option $X = (S_N - k)^+$ payoff

$= \varphi(S_N)$

1) risk-neutral pricing. $E^Q\left[\frac{X}{B_N}\right]$ not unique (because Q mg measure is not unique)

2) N.A. pricing => interval of prices (R.N. price $\in$ int)

Lecture 3

X - price

Assume : $\exists Q$ market measure

1) $E^Q[\frac{X}{B_n}]$ initial price

2) $A_X^+ = \{(\alpha,\beta) \in A \mid V_n^{(\alpha,\beta)} \geq X\}$

$A_X^- = \{\dots\}$

$A_X^+ \; ; \; A_X^-$

$V_0^{(\alpha,\beta)} \leq$ "price" $< V_0^{(\alpha,\beta)}$

$\forall (\alpha,\beta) \in A_X^- \qquad \forall (\alpha,\beta) \in A_X^+$

Lemma: $\sup\limits_{(\alpha,\beta)\in A_X^-} V_0^{(\alpha,\beta)} \leq E^Q[\frac{X}{B_N}] \leq \inf\limits_{(\alpha,\beta)\in A_X^+} V_0^{(\alpha,\beta)}$ $\forall$ Q mg meas.

If $(\alpha,\beta) \in A_X^+$: $E^Q[\frac{X}{B_N}] \leq E^Q[\tilde{V}_N^{(\alpha,\beta)}] = V_0^{(\alpha,\beta)}$

Hedging: find $(\alpha,\beta) \in A$ s.t. $V_n^{(\alpha,\beta)} = X$ if payoff is replicable

$\Rightarrow (\alpha,\beta) \in A_X^+ \cap A_X^-$

strategy is super and sub repl.

$\Rightarrow$ price is equal to risk - neutral price

Complete model: $\forall X$ is replicable
(euro. option)

$\downarrow$

II fund. theory of asset pricing : a model is complete iff the mg measure is unique

Binominal model : $d=1$

$B_n = (1+r)^n$ $\qquad\qquad (\Omega, \mathcal{F}, P)$

$$\begin{cases} S_0 > 0 \\ S_{n+1} = S_n(1+\mu_{n+1}) \end{cases}$$

$1+\mu_n \sim \mathrm{Ber}(u, d, p)$, $0 < d < u$

if $\mu_1 \dots \mu_n$ - i.i.d $\Rightarrow$ P is uniq. def. :

$P(S_n = u^k d^{n-k} S_0) = \binom{n}{k} p^k (1-p)^{n-k}$

Prop.: $\exists! Q$ iff $d < 1+r < u$

Proof: $E^Q[\tilde{S}_{n+1} \mid \mathcal{F}_n] = \tilde{S}_n$?

$E^Q[S_{n+1} \mid \mathcal{F}_n] = S_n(1+r)$

$E^Q[\cancel{S_n}(1+\mu_{n+1}) \mid \mathcal{F}_n] = \cancel{S_n}(1+r)$

$E^Q[1+\mu_{n+1} \mid \mathcal{F}_n] = 1+r$

$u\,Q(1+\mu_{n+1} = u \mid \mathcal{F}_n) + d(1 - Q(\dots)) = 1+r$

$Q(1+\mu_{n+1} = u \mid \mathcal{F}_n) = \dfrac{1+r-d}{u-d} = q$

ind. $\qquad \in \,]0,1[$

$\Uparrow$

if $\exists Q \Rightarrow \mu_1, \dots, \mu_n$ - iid

$\Downarrow$

$Q(1+\mu_{n+1} = u)$

Call - option T, K

$$C_0 = E^Q\left[\frac{(S_T - K)^+}{(1+r)^N}\right] = \frac{1}{(1+r)^N} \sum_{k=0}^{n} (u^k d^{N-k} S_0 - K)^+ \cdot \overbrace{Q(S_T = u^k d^{N-k} S_0)}^{\binom{N}{k} q^k (1-q)^{N-k}}$$

Calibration $u, d, r, \overset{\text{fixed}}{N}, \cancel{p}$

1) statistical (past)

2) implicit (future)

$\delta = \frac{T}{N} << 1$

0 T

$1 + r = e^{R\delta}$ R - annual rate

$(\bar{S}_n)_{n=\overline{1,M}}$ - TS - realization of s. process $\Rightarrow \bar{\mu}, \bar{\sigma}^2$

$S_T = S_0 e^{\mu t} \Rightarrow \mu := \frac{1}{T} \lg \frac{S_T}{S_0} = \frac{1}{T} \lg \prod_{n=0}^{N-1} (1 + \mu_{n+1})$

$= \frac{1}{T} \sum_{n=0}^{N-1} \lg (1 + \mu_{n+1})$

$E^p[\mu] = \underbrace{\frac{1}{T} N}_{=\frac{1}{\delta}} (p \ln u + (1-p) \lg d)$ ← theor. exp.

$\begin{cases} \delta \bar{\mu} = p \lg u + (1-p) \lg d \\ \delta \bar{\sigma}^2 = \dots \end{cases} \Rightarrow \begin{aligned} u &= e^{\bar{\sigma}\sqrt{\delta} + O(\delta)} \\ d &= e^{-\bar{\sigma}\sqrt{\delta} + O(\delta)} \end{aligned}, \quad \delta \to 0^+$

$\bar{\mu} \to$ negligible

$var^p \left(\sum^{n-1} \lg (1 + \mu_{n+1}) \right) \overset{ind.}{=} N \cdot var (\lg (1 + \mu_1)) = N p (1-p) (\lg \frac{u}{d})^2$

$\Rightarrow \quad u = e^{\sigma\sqrt{\delta}} = \frac{1}{d}$

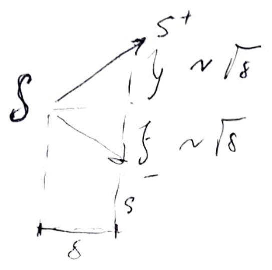
σ - volatility

$\boxed{u_n = e^{\sigma\sqrt{\frac{T}{N}}}}$

by Taylor exp. $u \sim 1 + \sigma\sqrt{\delta} + O(\delta)$

$d \sim 1 - \sigma\sqrt{\delta} + O(\delta)$

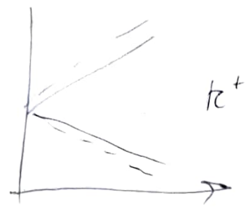
$\frac{S_{t+\delta} - S_t}{\delta} \approx \frac{\sqrt{\delta}}{\delta} \to \infty \Rightarrow$ not differentiable process

$\Rightarrow$ 

$\sigma, R \Rightarrow \quad u_N = e^{\sigma\sqrt{\delta_N}}$

Rem 1. $S_N^{(max)} = u_N^N S_0 = e^{\sigma N\sqrt{\frac{T}{N}}} \xrightarrow[N\to\infty]{} \infty$

$S_N^{(min)} \to 0$



K^+ is covered

B.S. formula

$C_0 = E[(S_0 e^X - K)^+] = \int_{\mathbb{R}} (S_0 e^x - K)^+ \Gamma(\cdot, x)\,dx$

$C_0^{(N)} \xrightarrow[N\to\infty]{} C_0$

$E[\varphi(X^{(N)})] \xrightarrow[N\to\infty]{} E[\varphi(X)] \quad \forall \varphi \in C_b$

Rem 2. $d_N < 1 + r_N < u_N$

$e^{-\sigma\sqrt{\delta}} < e^{R\delta} < e^{\sigma\sqrt{\delta}}$

$-\sigma < R\sqrt{\delta} < \sigma$

$\downarrow N \to \infty \quad \Rightarrow$ model is arb.-free

0

conv. weakly to norm. d. value

Rem 3. $S_T^{(N)} = S_0 \prod_{k=1}^{N} (1 + \mu_k^{(N)}) = S_0 e^{X^{(N)}} \xrightarrow{CLT} S_0 e^X, \; X \sim N\left(\left(R - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$

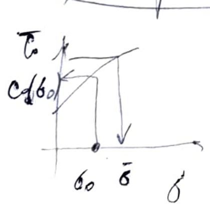
$X^{(N)} = \sum \log(1 + \mu_k^{(N)})$

i.i.d

B.S. formula:

$f(\underbrace{S_0, T, R, K}_{\text{observable}}, \underbrace{\sigma}_{?})$ $\qquad f'_\sigma > 0$

2) implicit calibration



$\bar\sigma$ implied volatility (BS-implied volatility)

$\bar\sigma = \bar\sigma(S_0, T, K, R)$

Lecture 4

BS formula $e^{-RT} \cdot E[(S_0 e^{x} - K)^+]$

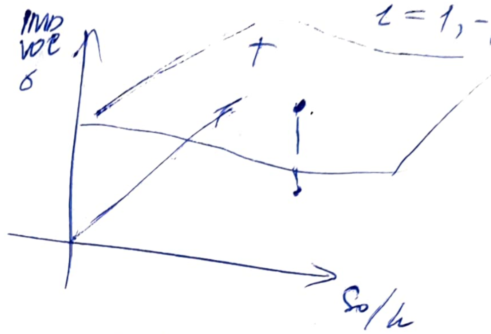
$$= \underbrace{C_{BS}(R, T, \tfrac{S_0}{K}, \sigma)}_{F(\sigma)}$$

$\frac{S_0}{K}$: moneyness

gaussian, payoff

$\int_{\mathbb{R}} \Gamma(y,t)\varphi(y)\,dy$

Fix R: $\quad \sigma_i \leftarrow \bar{C}(T_i; k_i)$ $\quad i = 1, .., M$



<u>American options</u>: $(X_n)_{n=\overline{0,N}}$; adapted; $X_n \geq 0$, $\forall n$

$X_n \in m\mathcal{F}_n$

e.x.: call: $X_n = (S_n - K)^+$

Pb 1) Pricing $\Pi_n \geq X_n$

2) Hedging: (for European: find $(\alpha,\beta) \in \mathcal{A}$, s.t. $V_n^{(\alpha,\beta)} = X$ (payoff))

— Find $(\alpha,\beta) \in \mathcal{A}$, s.t.: $V_n^{(\alpha,\beta)} = X_n$, $\forall n$

but $(\alpha,\beta) \nexists$

(even in complete market models), bcs

$\tilde{V}^{(\alpha,\beta)}$ is a Q-mg $\quad X_n$ - is generic process

+ Find $(\alpha,\beta) \in \mathcal{A}$ s.t. $V_n^{(\alpha,\beta)} \geq X_n$ $\forall n$

3) Optimal early exercise: $\boxed{\text{early exercise strategy}}$

stopping time $\tau: \Omega \to \{1, ..., N\}$ r.v.

s.t. $\underbrace{(\tau = n)}_{\{\omega\in\Omega \mid \tau(\omega) = n\}} \in \mathcal{F}_n$, $\forall n$

$(X_\tau)(\omega) = X_{\tau(\omega)}(\omega)$, $\omega \in \Omega$ $\quad$ r.V. — a payoff if exercise $\tau(\omega)$ str.

$$\tilde{X}_t = \frac{X_\tau}{B_\tau}$$

$$\sup_{\tau \in \mathcal{T}} E^Q[\tilde{X}_\tau] = E^Q[\tilde{X}_{\tau_0}]$$

τ_0 is optimal if given τ

$A_x^+ = \{(\alpha,\beta) \in A \mid V_n^{(\alpha,\beta)} \geq X_n \ \forall n\}$

$A_x^- = \{(\alpha,\beta) \in A \mid \exists \tau \in \mathcal{T} \text{ s.t. } V_\tau^{(\alpha,\beta)} \leq X_\tau\}$

$$\sup_{(\alpha,\beta) \in A_x^-} V_0^{(\alpha,\beta)} \leq \text{price } H_0 \leq \inf_{(\alpha,\beta) \in A_x^+} V_0^{(\alpha,\beta)}$$

If $\underset{\text{sell}}{H_0} > \underset{\text{buy}}{V_0^{(\alpha,\beta)}}$ with $(\alpha,\beta) \in A_x^+$

$\Rightarrow$ arbitrage, because of A_x^+

If $\underset{\text{buy}}{H_0} < \underset{\text{sell}}{V_0^{(\alpha,\beta)}}$ for some $(\alpha,\beta) \in A_x^+$

$\Rightarrow$ arbitrage

Lemma: if Q exists then $\sup_{\tau \in \mathcal{T}} E^Q[\tilde{X}_\tau] = H_0$

any price is N.A price

▷ based on optional sampling theorem

a) if $(\alpha,\beta) \in A_x^-$

$$E^Q[\tilde{V}_\tau^{(\alpha,\beta)}] \leq E^Q[\tilde{X}_\tau] \leq \underbrace{\sup_{\tau_1 \in \mathcal{T}} E^Q[\tilde{X}_{\tau_1}]}$$

is independent of (α,β)

by OST $\tilde{V}_0^{(\alpha,\beta)} = E^Q[\tilde{V}_\tau^{(\alpha,\beta)}]$ because of this time τ is s.time

$\Rightarrow \ldots \Rightarrow \sup_{(\alpha,\beta) \in A_x^-} V_0^{(\alpha,\beta)} \leq H_0$

$\tilde{H}_n$ - price process

DD: $\tilde{H} = M + A$ (predictable) ← s.t. $A_0 = 0$, $A\downarrow$

mg s.t. $\tilde{H} = M_0$

• M_N = european option

By completeness ⇒ $\tilde{V}_n^{(\alpha,\beta)} = M_n \ \forall n$ goal: $(\alpha,\beta) \in A_X^+ \cap A_X^-$

$\exists (\alpha,\beta) \in A$

s.t. $\tilde{V}_N^{(\alpha,\beta)} = M_N$ ⇒ $\tilde{V}_N^{(\alpha,\beta)} = M_n \geq \tilde{H}_n \geq \tilde{X}_n$

⇒ strategy is superreplicating (A_X^+)

• $A_n = \sum \left(E^Q [\tilde{H}_k | \mathcal{F}_{k-1}] - \tilde{H}_{k-1} \right)$

$$\tilde{H}_n = \max \{ \tilde{X}_n , \underbrace{E^Q(\tilde{H}_{n+1} | \mathcal{F}_n)}_{\tilde{E}_n} \}$$

$\tau_0 = \min \{ n | \tilde{X}_n \geq \tilde{E}_n \}$ $\tau_1 = \min \{ n | \tilde{X}_n > \tilde{E}_n \}$

$A_n = 0$, $n \leq \tau_0$

$\tilde{V}_{\tau_0}^{(\alpha,\beta)} = M_{\tau_0} = \tilde{H}_{\tau_0} = \tilde{X}_{\tau_0}$ ⇒ ... A_X^-

5) If $(\alpha,\beta) \in \mathcal{A}_x^+$ take $\tau \in \mathcal{T}$

$$\tilde{X}_\tau \leq \tilde{V}_\tau^{(\alpha,\beta)}$$

$$\underbrace{E^Q[\tilde{X}_\tau] \leq E^Q[\tilde{V}_\tau^{(\alpha,\beta)}] \underset{\text{by OST}}{=} \tilde{V}_0^{(\alpha,\beta)}}_{\text{upper bound doesn't dep on } \tau}$$

$$\Rightarrow \underbrace{\sup_{\tau \in \mathcal{T}} E^Q[\tilde{X}_\tau]}_{\text{N.A. price}} \leq \inf_{(\alpha,\beta) \in \mathcal{A}_x^+} \tilde{V}_0^{(\alpha,\beta)}$$

Theory: If the market is complete ($\exists ! Q$)

then $\exists (\alpha,\beta) \in \mathcal{A}_x^+ \cap \mathcal{A}_x^-$:

i) $V_n^{(\alpha,\beta)} \geq X_n \quad \forall n$;

ii) $\exists \tau_0 \in \mathcal{T}$ s.t. $V_{\tau_0}^{(\alpha,\beta)} = X_{\tau_0}$

iii) $V_0^{(\alpha,\beta)} = \sup_{\tau \in \mathcal{T}} E^Q[\tilde{X}_\tau]$

$\underset{\text{OST}}{=} E^Q[\tilde{V}_{\tau_0}^{(\alpha,\beta)}] = E^Q[\tilde{X}_{\tau_0}]$ $\Rightarrow \tau_0$ is optimal

Ex. (S_n) - adapted process

$\tau' = \max\{n \mid S_n \notin [a,b]\} \notin \mathcal{T}_0$

$\tau = \min\{n \geq 0 \mid S_n \notin [a,b]\}$

τ is a stopping time

$$(\tau = n) = \underbrace{(S_0 \in [a,b])}_{\mathcal{F}_0} \cap \dots \cap \underbrace{(S_n \notin [a,b])}_{\mathcal{F}_n}$$

* Doob's decomposition
* Snell envelope (= smallest super-mg above...)

$$H_n = \begin{cases} X_N & \text{if } n = N \\ \max\{\tilde{X}_n, \underbrace{E^Q[\tilde{H}_{n+1} \mid \mathcal{F}_n]}_{\tilde{E}_n}\} \end{cases} \qquad \tilde{H}_n \geq \tilde{X}_n, \quad \tilde{H}_n \geq E^Q[\tilde{H}_{n+1} \mid \mathcal{F}_n]$$

Stochastic Processes

1) Probability space $(\Omega, \mathcal{F}, \mathbb{P})$

Gen. theory   Be

- Ω - sample space $(1, 0)$ $\#\Omega = 2^n$
- $\mathcal{F}$ - σ-algebra $\mathcal{F}$ - powerset $\#\mathcal{F} = 2^{\#\Omega} = 2^{2^n}$

  1) $\Omega \in \mathcal{F}$

  2) $A \in \mathcal{F} \Rightarrow \Omega \setminus A \in \mathcal{F}$

  3) $A_1 \dots A_n \in \mathcal{F}$

  $\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$[0, 1]$

$[0, 1]$

$[\alpha, \beta]$, $(\alpha, \beta]$, (α, β), $\{\beta\}$ $\Rightarrow$ Borel σ-algebra

$0 \quad \alpha \quad \beta \quad 1$

- $\mathbb{P}$ - prob. measure $\mathbb{P}\{1\} = p$, $\mathbb{P}\{0\} = 1-p$ $\mathbb{P}\{[\alpha, \beta]\} = \beta - \alpha$

  1) $\mathbb{P}\{\Omega\} = 1$

  2) $A_1, \dots A_n \in \mathcal{F}$

  3) $\mathbb{P}\{\sqcup A_i\} = \sum \mathbb{P}\{A_i\}$

$\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$

Random variable is a measurable function $\xi : \Omega \to \mathbb{R}$ s.t.

$\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \in \mathcal{F}$

T - time

$X : T \times \Omega \to \mathbb{R}$ - random function

if $\forall t \in T : X(t, \cdot)$ - random variable on $(\Omega, \mathcal{F}, \mathbb{P})$

$T = \mathbb{N}(\mathbb{Z}) \longleftarrow$ $T = \mathbb{R}_+ \; (\mathbb{R})$

r.p. w.d.t $\Rightarrow$ random (stoch.) process (c.t.)

$T = \mathbb{R}_+^n$

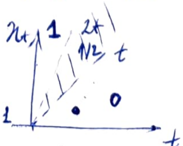
$\Rightarrow$ rand. (stoch.) field

Trajectory (path) $T \to \mathbb{R}$ fix w

any s.p. at any fixed time point is a random variable

Finite - dim. distr. $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$, $t_1, \dots, t_n \in \mathbb{R}$

e.g. $X_t = \xi \cdot t$ $\xi = \begin{cases} 1 : & 1/2 \\ 2 : & 1/2 \end{cases}$



$$\mathbb{P}\{X_{t_1} \le x_1, X_{t_2} \le x_2\} = \begin{cases} 0, & \min\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right) < 1 \\ 1/2, & -//- \in [1; 2) \\ 1, & -//- \ge 2 \end{cases}$$

(e.g.) s.p. $X_t = \sin t\, \xi_1 + \cos t\, \xi_2$, $t \ge 0$, ξ_1, ξ_2 – i.i.d. $\sim N(0,1)$

assume $\{A, B\} \in \Omega$ A for ξ_1, B for ξ_2

$X_t = A \sin t + B \cos t$

if $A^2 + B^2 = 0$ => the trajectory of X_t is horizontal line $X_t = 0$ $\forall t$

if $A^2 + B^2 \ne 0$ => $X_t = A \sin t + B \cos t = \sqrt{A^2+B^2}\left(\sin t \frac{A}{\sqrt{A^2+B^2}} + \cos t \cdot \frac{B}{\sqrt{A^2+B^2}}\right) =$

$= \sqrt{A^2+B^2}\left(\sin t \cos\left(\arccos \frac{A}{\sqrt{A^2+B^2}}\right) + \cos t \cdot \sin\left(\arccos\left(\frac{A}{\sqrt{A^2+B^2}}\right)\right)\right) =$

$= \sqrt{A^2+B^2}\left(\sin\left(t + \arccos \frac{A}{\sqrt{A^2+B^2}}\right)\right)$

2) Renewal Process

$S_0 = 0$

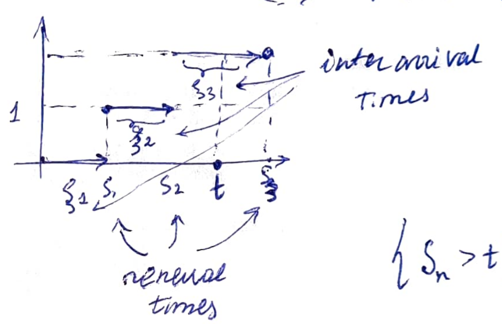
$S_n = S_{n-1} + \xi_n$, $\xi_1, \dots, \xi_n$ – i.i.d. > 0 a.s.

★ $= S_{n-2} + \xi_{n-1} + \xi_n = \sum^n \xi_k$ <=> $\mathbb{P}\{\xi_1 > 0\} = 1$

<=> $F(0) = 0$

Counting Process

$N_t = \arg\max_k \{S_k \le t\}$



$F \to \mathbb{E} N_t$

$\{S_n > t\} = \{N_t < n\}$

e.g. $\{S_3 > t\} = \{N_t < \overset{2}{n}\}$

renewal time #3 after t

before time t less than 2 events occured