Martingale and Ito's Integral

Def

right-cont. $\{F_t\}_{t\geq 0}$

$$F_t = F_{t+} = \bigcap_{n\geq 1} F_{t+1/n} \quad \forall t\geq 0$$

Lemma: if τ is optional and $c>0$-const $\Rightarrow \tau+c$ is a stopping time

$$\{t\wedge\beta\leq t\} = \{\tau\leq t\}\cup\{\beta\leq t\}$$

page 4: HW ~~proof~~ ...

$$\int_G X\, dP = \int_G E(X|g)\, dP$$

Prop:

W_t – F_t – BM and τ F_t-stop. time

$\Rightarrow$ $W_{t+\tau} - W_\tau$ is $\perp$ to F_t

X_t, Y_t, Z_t – mg

mg: $\to X_t Z_t - \langle X, Z\rangle$ (+)

mg: $\to Y_t Z_t - \langle Y, Z\rangle$ (1)

$\Rightarrow$ mg.: $(\langle X,Z\rangle_t + \langle Y,Z\rangle_t) + [\quad]$

Lebesgue – Stieltjes – gen. of R+L integrals

$$\frac{dC_t}{C_t} = \left(r(t)\,dt + m\frac{dS_t}{S_t} + m\,r(t)\,d(t) \right) =$$

$$= \left((1+m)\,r_t\,dt + m\frac{dS_t}{S_t} \right)$$

$$d\ln C_t = \frac{dC_t}{C_t} - \frac{1}{2}\frac{(dC_t)^2}{C_t^2} =$$

$$= (1+m)\,r_t\,dt + m\frac{dS_t}{S_t} - \frac{1}{2}m^2\sigma_t^2\,dt \left(+ \frac{m\cdot\sigma_t^2\,dt}{2} \right)$$

$$\left\{ (1+m)\,r_t\,dt + m\frac{dS_t}{S_t} + \left(\frac{m}{2} - \frac{m^2}{2}\right)\sigma_t^2\,dt \right.$$

$$(dC_t)^2 = C_t^2\left(\left((1+m)\,r_t\,dt + m\,(\mu_t\,dt + \sigma_t\,dW_t\right)\right)$$

$$= C_t^2\left(\left((1+m)r_t + m\mu_t\right)dt + m\,\sigma_t\,dW_t\right)^2$$

$$= C_t^2\,m^2\,\sigma_t^2\,dt$$

$$\int \ln C_t = \int (1+m)\,r_t\,dt + \int m\frac{dS_t}{S_t} - \int \frac{1}{2}m^2\sigma_t^2\,dt$$

$$\log C_t - \log C_0 = m(\log S_t - \log S_0) + m\cdot\int \frac{(1+m)}{m} r_t\,dt$$

$$- m\frac{1}{2}\int m\,\sigma_t^2\,dt$$

$$C_t = C_0\cdot\left(\frac{S_t}{S_0}^{m}\right)\cdot\exp\left(a\int r_t\,dt + b\int \sigma^2(t)\,dt\right)^m$$

$$\exp\left(\log\left(\frac{S_t}{S_0}\right)^m\right)$$

$$a = \frac{1}{m} + 1$$

$$b = -\frac{1}{2}m \quad [\,b?: \tfrac{1}{2} - \tfrac{1}{2}m$$

⑤ $dS_t = S_t\,(\mu(t)\,dt + \sigma(t))\,dW_t\ , \quad S_0 = \xi$

$$\frac{dS_t}{S_t} = \mu(t)\,dt + \sigma(t)\,dW_t$$

$$\cancel{\int d\ln S_t = \int \mu(t)\,dt + \sigma(t)\,dW_t}$$

$$d\ln S_t = \frac{dS_t}{S_t} - \frac{1}{2}\sigma^2\frac{S_t^2}{S_t^2}\,dt =$$

$$\frac{dS_t}{S_t} = $$

$$= \mu(t)\,dt + \sigma(t)\,dW_t - \frac{1}{2}\sigma^2\,dt$$

$$= \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)\,dW_t$$

$$\int d(\ln S_t) = \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)\,dW_s$$

$$\exp(\log S_t - \log S_0) = \exp\left\{\int(\quad)\,ds + \int(\quad)\,dW\right.$$

⑦ $\int \sim \int$

SDE
Lecture 2.

Pseudo - random variables. $X_{n+1} = (aX_n + b) \pmod c$

$\{X_n\} \subseteq \{0, 1, \dots, c-1\}$

$X_0 \in \{0, \dots, c-1\}$

$a, b, c \in \mathbb{Z}^+$

$U_n = X_n / c \in (0,1)$

$\{U_n\}$ mimics a seq. of r.v.

$\{\tilde{U}_n\}$ are i.i.d. $U(0,1)$

$\{X_i\}$ seq. of r.v.

Its elements are ind. if $P(X_{n_1} < x_{n_1}, X_{n_2} < x_{n_2}, \dots X_{n_n} < x_{n_n})$

$= \prod_{i=1}^{n} P(X_{ni} \le x_{ni})$

$\{X_n\}_{n \ge 1} \xrightarrow{a.s.} X$ if

$P(\lim_{n\to\infty} \dots$

- $P(\{w : \lim_{n\to\infty} X_n(w) = X(w)\}) = 1$

- $\{X_n\}, \{F_{X_n}\}, F_X$

  $C(F_X) = \{x \in \mathbb{R} ; F_X(x) \text{ is continuous}\}$

  $X_n \xrightarrow{d} X$ iff $F_{X_n}(x) \xrightarrow[n\to\infty]{} F_X(x), \ x \in C(F_X)$

- $X_n \xrightarrow[n\to\infty]{a.s.} X \Rightarrow X_n \xrightarrow[n\to\infty]{P} X \Rightarrow X_n \xrightarrow[n\to\infty]{d} X$

  $\Uparrow$

  $X_n \xrightarrow{L^p} X$

period

by Jens. ineq. $\rightarrow \left(\frac{1}{n}\sum f(u_i)\right)^2 \leq \frac{1}{N}\sum f^2(u)$

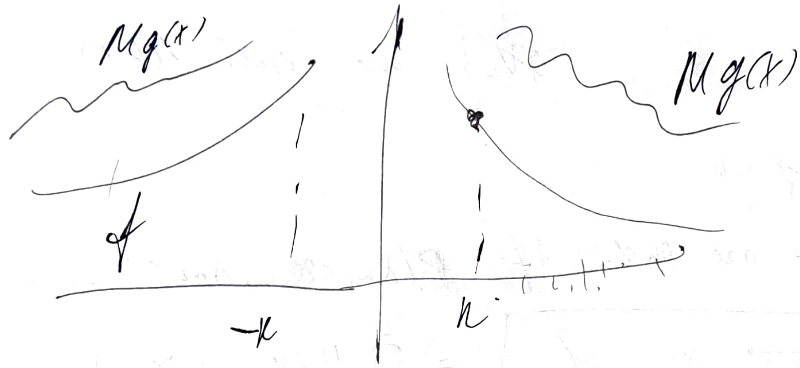
$\Rightarrow E\left(\frac{1}{n}\sum f(u)\right)^2 \leq E\left(\frac{1}{N}\sum f^2 u\right)$

by Jens. ineq.

$\overset{i.i.d.}{=} E[f(u_i)^2]$

$f(x) = O(g(x))$, $x \to a$ for $0 \leq a \leq \infty$

f is bounded from above g



$\{X_i\}_{i\geq 1}$ seq of r.v. i.i.d

$E(X)$, $E(X)^2 \leq \infty$

$E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \underbrace{E(X_i)}_{\mu}\right)^2\right] =$

$X'u = 0$

$X'(Y - X\beta) = 0$

$X'y - X'X\beta = 0$

$X'y = X'X\beta$

$= \frac{1}{n^2}\left[\sum_{i=1}^{n} X_i\right]^2 - \frac{2}{N}\mu\sum_{i=1}^{N} X_i + \mu^2 =$

$= E\left[\frac{1}{N^2}\left[\sum X_i^2 + 2\sum_{i<j} X_i X_j\right] - \frac{2}{N}\mu\sum x_i + \mu^2\right]$

$= \frac{1}{N^2}E\left[\sum x_i^2\right] + \frac{2}{N^2}n\cdot\mu^2 - \frac{2}{N}n\cdot\mu^2 + \mu^2 =$

$= \frac{1}{N}\cdot E[x_i^2] + \frac{2}{N}\cdot\mu^2 - \mu^2 \leq \frac{1}{N}\operatorname{Var}(X_i)$

SDE

Lecture 1

Pseudo-rand. numbers:

Def. (Seq. of unif distr. r. numbers over $[0,1]$.)
A seq. $\{X_n\}_{n\geq 1} \subseteq [0,1]$ is a seq of r.n. if there exists a p.s. (Ω, F, P), a seq. $\{U_n\}_{n\geq 1}$ of i.i.d. r.v. with unif. dist $U[0,1]$ and $w \in \Omega$ s.t. $X_n = U_n(w)$, $\forall n \geq 1$.

Pseudo-rand. generators cons. of deterministic alg. producing a seq. $\{U_n\}_{n\geq 1}$ of $[0,1]$ - real numbers which mimics the stat. properties of a seq. of ran. num. ($U[0,1]$ - dist. and mut. ind.)
Most algorithms include a linear congruential pseudo-random number generator:

$$X_{n+1} = (aX_n + b)(\text{mod } c) \qquad a, c \in \mathbb{Z}^+ \quad b \in \mathbb{Z}^+ \cup \{0\}$$

$\Rightarrow$ $U_n = \frac{X_n}{c}$ seem to be unif distr. on $[0,1]$

$\{X_n\} \in \{0, \dots, c-1\}$, $n = 0, 1 \dots$
$\{\tilde{U}_n\}$ i.i.d. $U(0,1)$
$\{X_i\}$ seq. of r.v.

$c > 0$ - the "modulus"
$0 < a < m$ - the "multiplier"
$0 \leq b < m$ - The "increment"
$0 \leq X_0 < m$ - the "seed" or start value

Properties
- consider $k > 1$ the first time that
  $X_k = (aX_0 + b)(\text{mod } c)$
  after this time the seq. repeats itself $\Rightarrow$ k - period
- a congruential seq. which producing $m-1$ distinct values for any seed X_0 is said to have full period

sufficient conditions for a full period seq. are:

- In case $b=0$ and c is a prime number then $\{X_n\}$ has full period if either
  - $a^{m-1} - 1$ is a multiple of m or
  - $a^j - 1$, $\forall j \in \{1, \dots, m-2\}$ are not multipliers of m

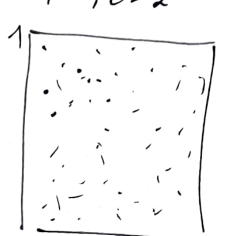
- In case $b \neq 0$, $\{X_n\}$ has full period in each of the following cases:
  - (i) The GCD of b and c is 1: $GCD(b, c) = 1$
  - (ii) Every prime number that divide c also divides $a-1$
  - (iii) $a-1$ is divisible by 4 if c is divisible by 4
  - (iv) c is of the form 2^k, b is odd and $a = 4l + 1$ for $l \in \mathbb{N}$

Example:
- RANDU .. $a = 2^{16} + 3$, $c = 2^{31}$, $b = 0$
- IBM 360... $a = 7^5$, $c = 2^{31} - 1$, $b = 0$

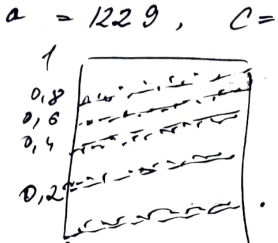
A test for independence

→ Plot successive pairs (U_{2n-1}, U_{2n}) $\forall n \geq 1$ on the unit square these points lie on c different straight lines of slope a/c and a large number of them should fairly evenly fill the unit square

$a = 7^5$, $c = 2^{31} - 1$, $b = 0$



$a = 1229$, $c = 2048$, $b = 1$

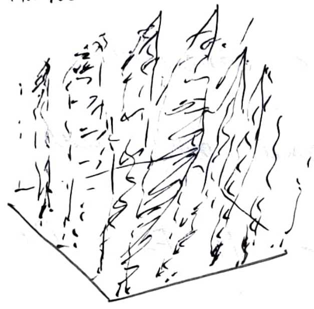


! The proceeding test is useful in eliminating glaringly biased generators, but does not guarantee an unbiased generator

(e.g.) RANDU: $X_{n+1} = 65539 \pmod{2^{31}}$

appears relatively unbiased, but successive triplets satisfy

$X_{n+2} = (6X_n - 9X_{n+1}) \pmod{2^{31}}$



The inverse transform method:

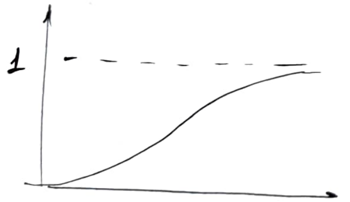
$\forall X \sim \mu$ The c.d.f of X is given by

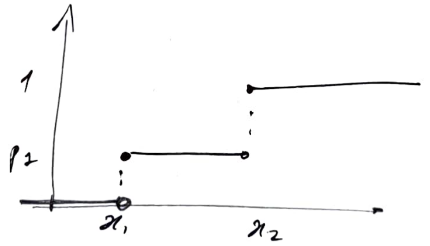
$F_X(X) = \mu((-\infty, x)) = \int_{(-\infty, x)} \mu(dy)$

non-decreasing function and càd làg functions

(right continuous with left limits = continue à droite limite à gauche)

s.t.: $\lim_{x \to \infty} F_X(X) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$





F_X admits a generalized inverse G_X which is defined by

$G_X(U) = \inf\{x \in \mathbb{R} : F_X(x) \geq U\}$, where $u \in (0,1)$

Properties of a gen. inv. function:

- G_X is a $\mathbb{R}$-valued function defined on $(0,1)$ non-decreasing and càd làg, s.t.

$G_X(u) \leq x \iff u \leq F_X(x)$, $\forall u \in (0,1)$, $x \in \mathbb{R}$

- if F_x is strictly increasing cont. function, then G_x corresponds to the classical inverse function F_x^{-1} of F_x
- if in addition G_x is cont. differentiable

$$G_x'(U) = \frac{1}{F_x'(G_x(U))}, \quad \forall U \in (0,1)$$

If μ admits a pos. density function f_μ then G_x satisfy the above condition

<u>Prop.</u> $\forall$ r.v. $U \sim U(0,1)$ and any real r.v. $X \sim \mu$ whose cum. distr. function is F_x, the real r.v. $G_x(U) \sim \mu$

<u>Proof</u> We know than $\{G_x(U) \leq x\} = \{U \leq F_x(x)\}$

$$\Rightarrow \quad \mathbb{P}(G_x(U) \leq x) = \mathbb{P}(U \leq F_x(x)) = F_x(x)$$

<u>Example:</u>

SP(1)

$(\Omega, F, \mathbb{P})$ - prob. space, • Ω - sample space

• F - σ-algebra

1) $\Omega \in F$

2) $A \in F \Rightarrow \Omega \setminus A \in F$

3) $A_1, \dots A_n \dots \in F$

$\bigcup_{i=1}^{\infty} A_i \in F$

Be

$\# \Omega = 2^n$

$\# F = 2^{\# \Omega}$

(power set)

(D) σ-algebra, that includes all closed (minimal) $[\alpha, \beta]$ or (α, β) open intervals is called Borel σ-algebra

• $\mathbb{P}$ - prob. measure

1) $\mathbb{P}\{\Omega\} = 1$

2) $A_1, A_2 \in F$

$P\{\cup A_i\} = \sum \mathbb{P}(A_i)$

r.v. - measurable function $\xi : \Omega \to \mathbb{R}$

s.t. $\forall B \in \mathcal{B}(\mathbb{R}) : \xi^{-1}(B) \in F$

R. function: $X : T \times \Omega \to \mathbb{R}$

if $\forall t \in T$: $X(t, \cdot)$ - r.v. on $(\Omega, F, \mathbb{R})$
$\quad \overset{\parallel}{X_t}$

e.g. $T = \mathbb{R}_+ \Rightarrow$ r. function is rand. process

$T = \mathbb{N} (\mathbb{Z})$ discr.

$T = \mathbb{R}_+ (\mathbb{R})$ cont.

$T = \mathbb{R}_+^n \Rightarrow$ r. field

Trajectory (=path) $T \to \mathbb{R}$

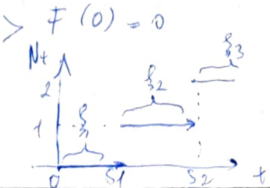
X_t fix ω

Finite - dimentional dist. $(X_{t_1}, \dots, X_{t_n})$, $t_1, \dots, t_n \in \mathbb{R}$

Renewal process (counting) : $S_0 = 0$, $S_n = S_{n-1} + \xi_n$

$\xi_1, \dots, \xi_n$ - i.i.d. > 0 a.s. $\Leftrightarrow F(0) = 0$

$N_t = \arg\max_k \{S_k \le t\}$



S_1, S_2 - renewal times

ξ_1, ξ_2 - interarrival times

$(\Leftarrow)$ $\{S_n > t\} = \{N_t < n\}$

$\uparrow$ renewal time $\quad\quad$ $\uparrow$ # of events occured

$S_n = \sum_{i=1}^{n} \xi_i$

1) <u>Convolution</u> $\quad X \perp\!\!\!\perp Y, \; X \sim F_X, \; Y \sim F_Y \Rightarrow \underbrace{F_X * F_Y}$

Conv. in terms of distr. funct.

$$F_{X+Y}(x) = \int_{\mathbb{R}} F_X(x-y)\, dF(y)$$

$\uparrow$ Stilties Integral

2) $X \sim P_X, \; Y \sim P_Y \Rightarrow \underbrace{P_X * P_Y}_{\text{conv. in t. of densities}}$

$$P_{X+Y}(x) = \int_{\mathbb{R}} P_X(x-y)\, P_Y(y)\, dy$$

$$F^{n*} = \underbrace{F * \dots * F}_{n}$$

1) $F^{n*}(x) \le F^{n}(x)$ $\quad$ if $F(0) = 0 \iff P(X > 0) = 1$

$\triangleright$ $\xi_1, \dots, \xi_n$ - i.i.d. $\sim F$

$\{\xi_1 + \dots + \xi_n \le x\} \subset \{\xi_1 \le x, \dots, \xi_n \le x\}$

$$\underbrace{P\{\xi_1 + \dots + \xi_n \le x\}}_{= F^{n*}(x)} \le \prod_{k=1}^{n} \underbrace{P\{\xi_k \le x\}}_{F(x)}$$

2) $F^{n*}(x) \ge F^{(n+1)*}(x)$

$\{\xi_1 + \dots + \xi_n \le x\} \supset \{\xi_1 + \dots + \xi_{n+1} \le x\}$

3) + commutativity

4) + associativity

<u>Thm:</u> $S_n = S_{n-1} + \xi_n$, ξ_1, ξ_2 - i.i.d. F, $F(0) = 0$

$\uparrow$ renewal process

(i) $U(t) = \sum_{n=1}^{\infty} F^{n*}(t) < \infty$ (converge)

(ii) $\mathbb{E} N(t) = U(t)$

$\triangleright$ $\mathbb{E} N_t = \mathbb{E}[\#\{n : S_n \le t\}] = \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{1}\{S_n \le t\}\right] = \sum_{n=1}^{\infty} P\{S_n \le t\} = \sum F^{n*}(t)$ $\square$

Laplace transform: $f: \mathbb{R}_+ \to \mathbb{R}$

$$\mathcal{L}_f(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

1) if f - density function of ξ then $\mathcal{L}_f(s) = \mathbb{E}[e^{-s\xi}]$

2) f_1, f_2 : $\mathcal{L}_{\underbrace{f_1 * f_2}_{\text{i.e. } f \text{ of dens.}}}(s) = \mathcal{L}_{f_1}(s) \cdot \mathcal{L}_{f_2}(s)$

3) F - distr. funct., $F(0) = 0$, $p = F'$ $\Rightarrow$

$$\mathcal{L}_F(s) = \frac{\mathcal{L}_p(s)}{s}$$

$$\triangleright \int_{\mathbb{R}_+} F(x) \frac{d(e^{-sx})}{s} \overset{IBP}{=} \underbrace{-\cancel{\frac{F(x)e^{-sx}}{s}}\Big|_0^\infty}_{\to 0} + \frac{1}{s}\int_{\mathbb{R}_+} p(x) e^{-sx}\, dx \quad \square$$

e.g. 1) $\mathcal{L}_{x^n}(s) = \int_{\mathbb{R}_+} x^n \frac{d(e^{-sx})}{s} = \frac{n}{s}\int_{\mathbb{R}_+} x^{n-1} e^{-sx}\, dx = \frac{n}{s} \cdot \frac{n-1}{s} \cdots \frac{2}{s} \int_{\mathbb{R}_+} e^{-sx} dx$

$$= \frac{n!}{s^n}$$

2) $\mathcal{L}_{e^{ax}}(s) = \dfrac{1}{s-a}$ if $a < s$

$$\mathbb{E} N_t = U(t) = \sum_{n=1}^\infty F^{n*}(t) \underset{\text{assoc. property}}{=} F(t) + \underbrace{\left(\sum_{n=1}^\infty F^{n*}\right)}_{U} * F(t)$$

$$U = F + U \underset{\text{distr.}}{\circledast} F = F + U \underset{\text{dens.}}{\circledast} p \quad \text{if } F' = p$$

$$\int_{\mathbb{R}} U(x-y)\, dF(y) = \int_{\mathbb{R}} U(x-y)\, p(y)\, dy$$

$$\mathcal{L}_U(s) = \underbrace{\mathcal{L}_F(s)}_{= \frac{\mathcal{L}_p(s)}{s}} + \mathcal{L}_U(s) \cdot \mathcal{L}_p(s)$$

$$\Rightarrow \boxed{\mathcal{L}_U(s) = \frac{\mathcal{L}_p(s)}{s(1 - \mathcal{L}_p(s))}}$$

1. $F \to \mathcal{L}_p$
2. $\mathcal{L}_p \to \mathcal{L}_U$
3. $\mathcal{L}_U \to$ guess U

Example: $S_n = S_{n-1} + \xi_n$

$\xi_1, \xi_2, \dots$ – $p(x) = \frac{e^{-x}}{2} + e^{-2x}$, $x > 0$

$E N_t$ – ?

1) $\mathcal{L}_p(s) = \frac{1}{2} \mathcal{L}_{e^{-x}}(s) + \mathcal{L}_{e^{-2x}}(s) =$

$$= \frac{1}{2(s+1)} + \frac{1}{s+2} = \frac{3s+4}{2(s+1)(s+2)}$$

2) $\mathcal{L}_u = \frac{\mathcal{L}_p(s)}{s(1-\mathcal{L}_p(s))} = \frac{3s+4}{s^2(2s+3)}$

3) Guess inv. Lapl. transform:

$$\mathcal{L}_u(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{2s+3}$$

$A = 4/3$

$B = 1/9$

$C = -2/9$

$$u(t) = \frac{4}{3} t + \frac{1}{9} - \frac{1}{9} \cdot e^{-3/2 t}$$

Limit thms for renewal processes

$S_n = S_{n-1} + \xi_n$; $\xi_1, \xi_2, \dots$ – i.i.d > 0

Thm 1 $\mu = E\xi_1 < \infty$

Then $\frac{N_t}{t} \xrightarrow[t\to\infty]{} \frac{1}{\mu}$ a.s. (convergence)

⊛ SLLN: $\frac{\xi_1 + \dots + \xi_n}{n} \xrightarrow[n\to\infty]{} \mu$ a.s. (analogue for ren process)

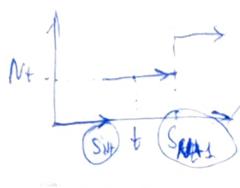
Thm 2 $\sigma^2 = Var\, \xi_1 < \infty$ Then $Z_t = \frac{N_t - t/\mu}{\sigma\sqrt{t}/\mu^{3/2}} \xrightarrow[t\to\infty]{d} N(0,1)$

(analogue of CLT)

$$P\{Z_t \le x\} \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

⊛ CLT: $\frac{\xi_1 + \dots + \xi_n - n\cdot\mu}{\sigma\sqrt{n}} \xrightarrow{d} N(0,1)$

1▷ $S_{N_t} \le t \le S_{N_t+1}$



$\Rightarrow \frac{N_t}{S_{N_t+1}} \le \frac{N_t}{t} \le \frac{N_t}{S_{N_t}}$

$$\lim_{t\to+\infty} \frac{N_t}{S_{N_t}} = \lim_{n\to\infty} \frac{N_t}{S_n} = \frac{1}{\mu}$$

$$\lim_{t\to+\infty} \frac{N_t}{S_{N_t+1}} = \underbrace{\lim \frac{N_t}{N_{t+1}}}_{=1} \cdot \underbrace{\lim \frac{N_{t+1}}{S_{N_t+1}}}_{=1/\mu} = \frac{1}{\mu} \quad \Rightarrow \quad \square$$

2▷ $$\mathbb{P}\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} \to \Phi(x), \quad x \in \mathbb{R}$$

$$\mathbb{P}\{\underset{n}{S_n} < \underbrace{n\mu + \sigma\sqrt{n}\,x}_{t}\} \to \Phi(x)$$

$$\mathbb{P}\{N_t \geq n\}$$

$n\mu \approx t \ : \ n \approx t/\mu$

$$n = t/\mu - \frac{\sigma\sqrt{n}}{\mu} \cdot x = t/\mu - \frac{\sigma\sqrt{t}}{\mu^{3/2}}\, x$$

$$\mathbb{P}\{Z_t \geq -x\} \to \Phi(x)$$

$$\mathbb{P}\{Z_t \leq x\} = 1 - \mathbb{P}\{Z_t > x\} \to 1 - \Phi(-x) = \Phi(x)$$

Quantpie

① Arithmetic BM.

$$dX_t = \underbrace{\mu dt}_{\text{drift}} + \underbrace{\sigma dB_t}_{\text{diffusion term}}$$

$$\int_0^T dX_t = \int_0^T \mu dt + \int_0^T \sigma dB_t$$

$$X_T - X_0 = \mu(T-0) + \sigma(B_T - B_0)$$

$$X_T = X_0 + \mu T + \sigma B_T$$

$\boxed{B_T \sim N[0,T]}$ $\quad B_T - B_0 \sim N[0,T]$

$$X_T = X_0 + \mu T + \sigma B_T \Rightarrow X_T \sim N[X_0 + \mu T, \sigma^2 T]$$

$$Cov[X_T, X_S] = \sigma^2 Cov[B_T, B_S] = \sigma^2 \min(T, S)$$

★ cov. of 2 BM is the lenght of the overlapping time

② Geometric BM

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t \Rightarrow dX_t^2 = \sigma^2 X_t^2 dt$$

drift and diffusion
are proportional to
the value of the process

$$Y_t = \ln X_t$$

$$dY_t = d\ln X_t = \frac{dX_t}{X_t} - \frac{1}{2}\frac{dX_t^2}{X_t^2}$$

$$d\ln X_t = \mu dt + \sigma dB_t - 1/2\, \sigma^2 dt$$

$$d\ln X_t = (\mu - \sigma^2/2)\, dt + \sigma\, dB_t$$

$$\int_0^T d\ln X_t = \int_0^T (\mu - \sigma^2/2)\, dt + \int_0^T \sigma\, dB_t$$

$$\ln X_T - \ln X_0 = (\mu - \sigma^2/2)\, T + \sigma B_T$$

$$X_T = X_0\, e^{(\mu - \sigma^2/2) + \sigma B_T}$$

$$= e^{\underbrace{\ln X_0 + \ldots + \ldots}_{=Z}}$$

$$X_T = e^Z \sim LN\left[\ln X_0 + (\mu - \frac{\sigma^2}{2})T,\ \sigma^2 T\right]$$

$$Z \sim N[0,1] \qquad E[e^{tZ}] = e^{t^2/2}$$

$$E[e^{t(a+\sigma Z)}] = e^{at + \frac{(t\sigma)^2}{2}}$$

$$\frac{C_t}{C_0} = \left(\frac{S_t}{S_0}\right)^m \exp\left(a\int z(s)\,ds + b\int \sigma^2\,ds\right)^m$$

$$\log C_t - \log C_0 = m\left(\log\left(\frac{S_t}{S_0}\right) + \left(a\int z\,ds + b\cdot\int \sigma^2 ds\right)\right)$$

$$\log C_s \Big|_0^t$$

$$\int_0^t \frac{dC_s}{C_s} = \cancel{m\int_0^t \frac{dS_s}{S_s}} + m$$

$$\int_0^t \frac{dC_s}{C_s} = m \cdot \int_0^t \frac{dS_s}{S_s} + m\left(a\int z\,ds + b\int \sigma^2\,ds\right)$$

$$\frac{dC_t}{C_t} = m\left(\frac{dS_t}{S_t} + \underbrace{\left(a\,z(t) + b\cdot\sigma^2(t)\right)}\right)$$

$$dC_t = C_t \cdot m\left(\quad\right)$$

$\delta \sim \delta$

$$\frac{dC_t}{C_t} = (r(t)\,dt + m\left(\frac{dS_t}{S_t} + r,\ dt\right))$$

$$\underbrace{\int_0^t d \log C_t}_{\log C_t - \log C_0 =} = \underbrace{\int_0^t r(s)\,ds}_{} + \underbrace{\int_0^t m(\dots)\,ds}_{}$$

$$\downarrow$$

$$\int m\left(\frac{ds_t}{S_t},\ dt\right) + \int m(\ \ ,\ dt)$$

$$\int m(\mu\,dt + \sigma\,dW,\ dt)$$

$$\int f\,dg = g \cdot f - \int g\,df \qquad \int m(\mu$$

$$\int m\left(\frac{ds_t}{S_t},\ dt\right) dt =$$

$$\left(\frac{S_t}{S_0}\right)^h \qquad (e^{x})^m = e^{x \cdot m}$$

$$\int m(\dots)\,dt = t \cdot m - \int t\,dm$$

$$\int m(\mu\,dt + \sigma\,dW,\ dt)\,dt$$

$$\int m(\mu\,dt,\ dt)\,dt + \int m(\sigma\,dW,\ dt)\,dt$$

$$\frac{dS_t}{S_t} = \mu(t)\,dt + \sigma(t)\,dW_t$$

$$d(\log(S_t)) = \frac{dS_t}{S_t} - \frac{1}{2}\frac{\sigma^2 p^2\,dt}{p^2}$$

$$= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dW$$

$$\int dS_t = S_t\left(\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dW_t\right)$$

$$\cancel{S_t = \int_0^t S_t(\ldots)\,dt}$$

$$\cancel{= \int_0^t S_t\left(\mu_t - \frac{\sigma^2}{2}\right)dt} + \int_0^t S_t\,\sigma\,dW_t$$

$$\int_0^t \frac{dS_t}{S_t} = \int_0^t\left(\mu - \sigma^2/2\right)dt + \int_0^t \sigma\,dW_t$$

$$\log S_t\Big|_0^t = \int_0^t\left(\mu - \frac{\sigma^2}{2}\right)dt + \int_0^t \sigma\,dW_t$$

$$\log S_t - \log S_0 = \left[\int \quad\right] + \log S_0$$

$$S_t = S_0 \exp\left[\quad\right]$$

$$dS_t = d\left(\xi \exp[\quad]\right) = \xi \cdot S_t \cdot d\left[\int_0^t (\dots)ds + \int_0^t \sigma(s) dW_s\right]$$

$$\left(\int x\, dt\right)' = x\, dt$$

$$\frac{\partial\left(\int x\, dx\right)}{\partial x}$$

$$d\left[\int_0^t (\quad)ds + \int_0^t \sigma(s)\, dW_s\right] =$$

$$= [\dots]\, dt + [\dots]\, dW + \frac{1}{2}[\dots]\, dt$$

$$\frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial W} dW + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} dt$$

$$\left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\, dW + \frac{1}{2}\sigma^2\, dt$$

$F = f(g(x))^{t}$

$$dF = \frac{df}{dg} \cdot \frac{dg}{dx}$$

$(dF)^2$

$$\frac{dz}{ds} \cdot \frac{du}{dx}$$

$$\frac{d^2 k}{ds^2} = \cancel{\frac{d\left(\frac{df}{dy} \cdot \frac{dg}{dx}\right)}{dx}} \cdot \frac{df}{dg}$$

$$\int m \frac{dS_t}{S_t} = \int m \left(\frac{S_t (\mu(t)\, dt + \sigma(t)\, dW_t)}{S_t} \right)$$

$$\boxed{\int m \frac{dS_t}{S_t} + \frac{1}{2} \int \sigma_s^2 \, ds}$$

$$\cancel{m \int_0^t \frac{dS_s}{S_s}}$$

$$\int m \frac{dS_t}{S_t} = \int m\, \mu(t)\, dt + \int m\, \sigma(t)\, dW_t)$$

$$dX_t = \frac{\xi - X_t}{1-t}\,dt + dW_t \quad , \; X_0 = 0$$

$$X_t = Y_t(1-t)$$

$$d(Y_t(1-t)) = \frac{\xi - Y_t(1-t)}{(1-t)}\,dt + dW_t \quad , \; Y_0 = 0$$

$$dX_t = \frac{\xi}{1-t}\,dt - Y_t\,dt + dW_t$$

$$-\frac{X}{1-t}\,dt + dW_t$$

$$\cancel{dX_t} = dY_t \cdot (1-t) - \cancel{d}Y_t\,dt =$$

$$= \frac{\xi}{1-t}\,dt - \cancel{Y_t\,dt} + dW_t$$

~~dY_t (1-t) Y_t~~ dY_t

$$dY_t = \frac{\xi}{(1-t)^2}\,dt + \frac{1}{1-t}\,dW_t$$

$$\int_0^t dY_s = \int \frac{\xi}{(1-s)^2}\,ds + \int \frac{1}{1-s}\,dW_s$$

$$Y_t = \frac{X_t}{1-t} \Rightarrow X_t = (1-t)\int_0^t \frac{\xi}{(1-s)^2}\,ds + \int \frac{1-t}{1-s}\,dW_s$$

$$= (1-t)\,\xi\left(-\frac{t}{1-t}\right) + \int \frac{1-t}{1-s}\,dW_s = \xi t$$

$$X_t = t\xi + \int_0^t \frac{1-t}{1-s} \, dW_s \, , \quad t \in [0,1]$$

$$0 + \int \dots ds + \int \cdot \, dW_t$$

(1) X_t is cont. and adapted to F_t

(2) $P(X_0 = 0) = 1$

(3) $|X_t| \leq |t\xi| + \int_0^t \left| \frac{1-t}{1-s} \right| dW_s$

$E|X_t| \leq E|t\xi|$

$E|X_t| \leq t\xi$ ~~...~~

$$E\left[\int \left| \frac{\xi - X_s}{1-s} \right| ds + \int_0^t |1|^2 ds \right]$$

$$= \int \frac{\xi - E[|X_s|]}{1-s} ds + t$$

$$\leq \int \frac{\xi - t\xi}{1-s} ds + t < \infty$$

(4) $X_t = 0 + \int \quad ds + \int \quad dW_t$

$$\xi \sim N(m, \sigma^2)$$

$$dX_t = \frac{\xi - X_t}{1-t}\,dt + dW_t \quad , \quad X_0 = 0$$

$$X_t = Y_t(1-t)$$

$$d(Y_t(1-t)) = \frac{\xi - Y_t(1-t)}{(1-t)}\,dt + dW_t \quad , \quad Y_0 = 0$$

$$dY_t(1-t) - \cancel{Y_t\,dt} = \frac{\xi}{1-t}\,dt - \cancel{Y_t\,dt} + dW_t$$

$$dY_t = \frac{\xi}{(1-t)^2}\,dt + \frac{1}{1-t}\,dW_t$$

$$\int_0^t dY_s = \int_0^t \frac{\xi}{(1-s)^2}\,ds + \int_0^t \frac{1}{1-s}\,dW_s$$

$$Y_t = \int_0^t \frac{\xi}{(1-s)^2}\,ds + \int_0^t \frac{1}{1-s}\,dW_s \quad , \quad Y_t = \frac{X_t}{1-t}$$

$$X_t = (1-t)\cdot\xi\cdot\left(\frac{t}{1-t}\right) + \int_0^s \frac{1-t}{1-s}\,dW_s$$

$$X_t = t\xi + \int_0^t \frac{1-t}{1-s}\,dW_s \quad , \quad t \in [0,1]$$

Show uniqueness

(1) $X_t = 0 + \int_0^t \frac{\xi - X_s}{1-s} ds + \int_0^t dW_s$

is cont. and adapted to F_t

(2) $P(X_0 = 0) = 1$

(3) $|X_t| \leq |t\xi| + \int_0^t \frac{1-t}{1-s} dW_s$

$E[|X_t|] \leq E[|t\xi|]$

$E[|X_t|] \leq t\, E[|\xi|]$, $\xi \sim N(m, \sigma^2)$

$E[|X_t|] \leq mt$

$$E\left[\int_0^t \frac{\xi - X_s}{1-s} ds + \int_0^t ds\right] =$$

$$= \int_0^t \frac{m - E[|X_s|]}{1-s} ds + t \leq$$

$$= \int_0^t \frac{m - mt}{1-s} ds + t < \infty$$

(4) $X_t = 0 + \int_0^t \frac{\xi - X_s}{1-s} ds + \int_0^t dW_s$

then $(X_t^1 ; t \geq 0)$ and $(X_t^2 ; t \geq 0)$ are indistinguishable $\iff P(X_t^1 = X_t^2,\ t \geq 0) = 1$

$X_t = t\xi + \int_0^t \frac{1-t}{1-s} dW_s$ is Gaussian

$X_t \sim N\left(tm, \sigma^2 + \frac{(1-t)^2}{(1-t)} t\right) \Rightarrow X_t \sim N(tm, \sigma^2 + t(1-t))$

$$\overbrace{C_0 \cdot \exp\left\{ \int_0^t \left(m\mu_s - m \frac{\sigma_s^2}{2} \right) ds + m \int_0^t \sigma_s \, dW_s \right\}}^{\left(S_t / S_0 \right)^m}$$

$$\exp\left\{ \underline{\int_0^t \left(m - \frac{1}{2} m^2 \right) \sigma_s^2 \, ds} + \underline{\int_0^t r_s (m+1)} \, ds \right\}$$

$C_0 \cdot \exp$

$$\cancel{\exp(m) = \exp(\log e^m)}$$

$$\cancel{\exp(m \cdot \int) = \exp(\int)^m}$$

$$\cancel{e^{ma} = (e^a)^m}$$

p.82

$$\begin{cases} dX = gX\,dW \\ X(0) = 1 \end{cases}$$

$$X(t) = e^{-1/2 \int_0^t g^2 ds + \int_0^t g\,dW}$$

$$Y(t) :=> e^{Y(t)}$$

$$dX = d\left(e^{Y(t)}\right) = e^{Y(t)} \cdot d(Y(t)) \cdot dx$$

$\cancel{dX = 1}$

$$u(X) = e^X$$

$$F(x,t) = e^Y \qquad \frac{\partial F}{\partial y} dY + \frac{1}{2}\frac{\partial F^2}{\partial y}(dY)^2$$

$$\underbrace{\frac{\partial F}{\partial W} dW + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial W^2} dt}_{\cancel{\frac{\partial F}{\partial Y} dY}}$$

$$= e^Y \cdot (dY$$

$$(dY)^2 = \underset{\underset{0}{\parallel}}{a_1 dt^2} + \underset{\underset{0}{\parallel}}{a_2 dt\,dW} + g^2 \underset{dt}{(dW^2)}$$

(

$p_0 + \int_0^t \mu E(p(s))ds =$

$p_0 + \int_0^t \mu E\left(p_0 e^{\sigma W_s + (\mu - \frac{\sigma^2}{2})t}\right) ds =$

$= p_0 + \int_0^t \mu \cdot p_0 e^{(\mu t - \sigma^2/2 t)} \cdot \underbrace{E(e^{\sigma W_s})}_{e^{\sigma^2/2\, t}} ds$

$\cancel{E(e^X) = e^{E(X)}}$

$p_0 + \int_0^t \mu p_0 e^{\mu t} ds$

$p_0 + \cancel{\mu} p_0 e^{\mu t} \Big|_0^t$

$E[e^{2\sigma W_t}] = e^{2\sigma 0 + \frac{4\sigma^2 t}{2}}$

$\int_0^t e^{2\sigma W_s} ds =$

$E[X]$

$= \int x f(x) dx$

$\int x dF(x)$

$$y = f(at + bz) = at + bt\,z \sim N[at,\ \sigma^2 t^2]$$

$$E[e^y] = e^{E[y] + 1/2\,Var[y]}$$

$$X_T = e^{\ln X_0 + (\mu - \sigma^2/2)T + \sigma B_T} = e^y$$

$$y \sim N[\ln X_0 + (\ldots)T,\ \sigma^2 T]$$

$$y \sim N[E[y],\ Var[y]]$$ ~~Delete~~

$$E[X_T] = E[e^y] = e^{\ln X_0 + (\mu - \sigma^2/2)T + 1/2\,\sigma^2 T} = X_0 e^{\mu T}$$

$$Var[X_T] = E[X_T^2] - E[X_T]^2$$

$$E[X_T^2] = E[e^{2y}] = e^{E[2y] + 1/2\,Var(2y)} =$$

$$e^{2E[y] + 2Var[y]} = e^{2(\ln X_0 + \ldots) + 2\sigma^2 T} = \ldots = X_0^2 e^{2\mu T + \sigma^2 T}$$

$$Var[X_T] = X_0^2 e^{2\mu T + \sigma^2 T} - X_0^2 e^{2\mu T}$$

$$= X_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

$$Cov(X_T, X_S) = E(X_T X_S) - E(X_T)E(X_S)$$

$$= X_0^2 e^{\mu(T+S) + \sigma^2 S} - X_0^2 e^{\mu(T+S)}$$