I Monte Carlo for Stoch. OC.

① Optimal stopping

Markov process / chain $(X_n)_{n=0}^{N}$ on state space D : $(\Omega, \mathcal{F}_n, \mathbb{P})$

for any bounded function φ:

$\mathbb{E}(\varphi(X_{n+t}, \dots, X_N) \mid \mathcal{F}_n) = \mathbb{E}(\varphi(X_{n+t}, \dots X_N) \mid X_n)$

$\sup_\tau \mathbb{E}(f(\tau, X_\tau))$; τ is a $(\mathcal{F}_n)$ - stopping time if $\{\tau = n\} \in \mathcal{F}_n$

(*) American option price $\sup_\tau \mathbb{E}_{\mathbb{Q}}(e^{-r\tau} h(S_\tau))$

int. rate (cont. compounded)

martingale measure, payoff, stock price process

$(e^{-rt} S_t)$ is $\mathbb{Q}$-martingale

Bermudan option $\underset{0}{t_0} < t_1 < \dots < t_{N-1} < \underset{T}{t_N}$

can only be exercised at one of those times

$\Rightarrow \sup_\tau \mathbb{E}_{\mathbb{Q}}(e^{-r\tau} h(S_\tau))$

$\tau \in \{t_0, \dots, t_N\}$

$\left.\begin{array}{l} X_n = S_{t_n} \\ f(n,x) = e^{-rt_n} h(x) \end{array}\right] \Rightarrow$ optimal stopping (1)

Value function:

$V_n(x) = \sup_{n \le \tau \le N} \mathbb{E}\left[f(\tau, X_\tau) \mid X_n = x\right]$

$\tau : (\mathcal{F}_k^n)$ - stopping time, $k = n, n+1, \dots N$

$\mathcal{F}_k^n = \sigma\{X_n, X_{n+1}, \dots, X_k\}$

Goal:

find $V_n(x)$ for $0, \dots, N$

find optimizers $\tau^*_{n,x}$

Assumption: f is bounded

THM: V_n, $n = 0, \dots, N$ satisfy DPP: $V_N(x) = f(N,x)$

$V_n(x) = \max(f(n,x), \mathbb{E}[V_{n+1}(X_{n+1}) \mid X_n =$

and $\tau^*_{n,x} = \min\{n \le k \le N : X_k \in S_k\}$ when $S_k = \{x \in D : V_k(x) = f(k,x)\}$ is optimal for $V_n(x)$

(2) Numerical methods:

- approx. of (X_n) by a finite state markov chain
  - ↳ uniform grid
  - ↳ clustering of states
  - ↳ quantisation → low accuracy at low values... (Gilles Pages)
    → dimentionality problem

- non-parametric statistics
  ↳ approxim. $E(V_{n+1}(X_{n+1}) \mid X_n = x) = C_n(x)$ $= \sum_{i=1}^{k} \alpha_i \varphi_i(x)$ ← i.e. approx.

  (Tsitsiklis - Van Roy, 2000)
  maths

  (Longstaff - Schwartz, 2001)
  economic, am. option

  → regress now / regress later
  → value iteration / performance iteration
  → convergence ($n \sim \exp(u)$ ∄ $k/n \to 0$
  → dual methods (finding upper-bounds)

(3) Contin. - time stochastic control:

(W_t) - d_1 - dimentional BM

$\mathcal{U}$ - cont., closed subset of $\mathbb{R}^m$

$$\begin{cases} dX_t = \underbrace{\mu(X_t, u_t)}_{u} dt + \underbrace{\sigma(X_t, u_t)}_{\sigma} dW_t \\ X_0 = x \end{cases}$$

u_t ← control

e.g. $\Rightarrow dX_t = u_t\, dt + \sigma\, dW_t$

$X_t \in \mathbb{R}^d$

e.g. $E\left[X_T^2 + \int_0^T c \cdot u_t^2\, dt\right] \to \min$

X_T^2 ← dist. from target

$c \cdot u_t^2$ ← cost of control

Assumptions: μ, σ - Lipschitz, i.e. there is $L > 0$ s.t. $|\mu(x,u) - \mu(x',u')|$
$\leq L(|x-x'| + |u-u'|)$

(or differentiabity)

$|\sigma(x,u) - \sigma(x',u')|$
$\leq L(|x-x'| + |u-u'|)$

:)

Stoch. control is admissible $\mathcal{U}_{ad}$

$(u_t) \in \mathcal{U}_{ad}$ iff (u_t) is $(\mathcal{F}^W_t)$ - adapted

$$\mathbb{E} \int_0^T |u_t|^2 \, dt < \infty$$

We have: $\mathbb{E}\left[\sup_{0 \le t \le T} X_t^2 \right] \le C \left(1 + x^2 + \mathbb{E}\left[\int_0^T u_t^2 \, dt \right] \right)$

start. state

Eq. or norms

if dim is higher

or norms

$\left(X_t^{x,(u_t)} \right)_{t \ge 0}$

We want to assess the control

$$J(t,x,(u_s)) = \mathbb{E}\left[\int_0^{T-t} f(t+s, X_s, u_s)\, ds + g(X_{T-t}) \right] \longrightarrow \min$$

$X_s^{x,(u_t)}$

f, g - given functions

e.g. $f(t,x,u) = e u^2$

$g(x) = x^2$

<u>Assumption</u>: $|f(t,x,u)| \le C(1 + |x|^2)$

$|g(x)| \le C(1+|x|^2)$

<u>Value functions</u>: $V(t,x) = \inf_{(u_t) \in \mathcal{U}_{ad}} J(t,x,(u_s))$

$\hookrightarrow V(0,x)$?

<u>THM</u>: $V(t,x) \le C(1+|x|^2)$

<u>THM (DPP)</u>: $\forall\, t < T,\ x,\ h \le T-t$ (Touzi (2013))

$$V(t,x) = \inf_{(u_t) \in \mathcal{U}_{ad}} \mathbb{E}\left[\int_0^{h} f(t+s, X_s, u_s)\, ds + V(t+h, X_h) \right]$$

HJB equation

In the above formula I take control $u_s = u$, $s \geq 0$ for some fixed $u \in \mathcal{U}$

$$V(t,x) \leq \mathbb{E}\left[\int_0^h f(t+s, X_s, u)\,ds + V(t+h, X_h)\right]$$

$$\mathbb{E}\left[-\int_0^h f(t+s, X_s, u)\,ds\right] \leq \underbrace{\mathbb{E}\left[V(t+h, X_h) - V(t,x)\right]}$$

$$= \mathbb{E}\left[\int_0^h V' \dots ds\right]$$

$$= \mathbb{E}\Big[\int_0^h \{V_t(t+s, X_s) + V_x(t+s, X_s)\,\mu(X_s, u) +$$

$$+ \frac{1}{2} V_{xx}(t+s, X_s)\,\sigma^2(X_s, u)\}\,ds$$

$$+ \underbrace{\int_0^h V_x(t+s, X_s)\,\sigma(X_s, u)\,dW_s}_{\mathbb{E}(\cdot) = 0}\Big]$$

$$\mathbb{E}\left[-\frac{1}{h}\int_0^h f(t+s, X_s, u)\,ds\right] \leq \mathbb{E}\Big[\underbrace{\frac{1}{h}\int_0^h \mathcal{L}^u v(t+s, X_s)\,ds}\Big]$$

$$\mathbb{E}\left[\frac{1}{h}\int_0^h \{V_x \mu + \frac{1}{2} V_{xx}\sigma^2 + V_t\}\,ds\right]$$

$\downarrow h \to 0$ $\qquad\qquad\qquad\qquad$ $\downarrow h \to 0$

$$-f(t,x,u) \leq V_x(t,x)\,\mu(t,x) + \frac{1}{2} V_{xx}(t,x)\,\sigma^2(t,x) + V_t(t,x)$$

for any control u

$$-f(\cdot) \leq V_x \mu + \frac{1}{2} V_{xx}\sigma^2 + V_t$$

$$0 \leq \inf_u \{f(\cdot) + V_x \mu + \frac{1}{2} V_{xx}\sigma^2 + V_t\}$$

For any $\varepsilon > 0$ there is $h > 0$ and $u \in \mathcal{U}$ s.t.:

$$V(t,x) \geq \mathbb{E}\left[\int_0^h f(t+s, X_s, u_s)\,ds + V(t+h, X_h)\right] - \varepsilon$$

[THM] Assume V is $C^{1,2}$, f is continuous, then

$$V_t + \inf_{u \in \mathcal{U}} \{\underbrace{V_x}_{y}\,\mu(x,u) + \frac{1}{2}\underbrace{V_{xx}}_{z}\,\sigma^2(x,u) + f(t,x,u)\} = 0$$

Verification theorem: $u^*(t,x)$

(Pham)
for intuition

$$G(t,x,u,y,z) = y \cdot \mu(x,u) + \frac{1}{2} z \sigma^2(x,u) + f(t,x,u)$$

$$H(t,x,y,z) = \inf_{u \in U} G(t,x,u,y,z)$$

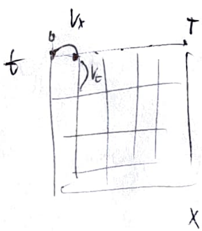
$$V_t + H(t, V, V_x, V_{xx}) = 0$$

$$\begin{cases} V(T,x) = g(x) \\ V_t + H(t,V,V_x,V_{xx}) = 0 \end{cases} \Rightarrow \text{find } V$$

$$V(t,x) \stackrel{?}{=} \mathbb{E}[\ldots]$$

Feynman-Kac formula

Solving PDE : approx. derivatives



$$V_x(t,x) = \lim_{h \to 0} \frac{V(t,x+h) - V(t,x)}{h}$$

e.g. $S_t^1, \ldots, S_t^d$ → stock prices

control $\pi_t = (\pi_t^1, \ldots, \pi_t^d)$ - the number of shares

$$X_t = \pi_t \cdot S_t = \sum_{i=1}^{d} \pi_t^i S_t^i$$

→ utility maximization $\mathbb{E}[X_T]$
↓
$\mathbb{E}[u(X_T)]$

→ hedging :

option with payoff $h(S_T)$

$\mathbb{E}[\varphi(X_T^\pi - h(X_T))] \to \min_\pi$

measures dissatisfaction from loss ($\varphi(y) = y^2$) or $\varphi(y) = y_-$

value payoff

II ML for SOC

$$V_n(x) = \max\left(f(n,x),\ \mathbb{E}\left(V_{n+1}(X_{n+1}) \mid X_n = x\right)\right)$$

$X, Y \in \mathbb{R}^d \times \mathbb{R}^d$

ν_{XY} – the joint law

$$\mathbb{E}\left[F(X,Y) \mid X\right] = \varphi(X)$$

$$F \in L^2(\mathbb{R}^{d\times d}, \nu_{XY}) = L^2_{\nu_{XY}}$$

$$\varphi \in L^2(\mathbb{R}^d, \nu_X) = L^2_{\nu_X} \qquad L^2_{\nu_X} \subset L^2_{\nu_{XY}}$$

$$\|\psi\|_{L^2_{\nu_{XY}}} = \left(\mathbb{E}\left[\psi^2(X,Y)\right]\right)^{1/2}$$

$$\langle \psi, \eta \rangle_{L^2_{\nu_{XY}}} = \mathbb{E}\left[\psi(X,Y)\, \eta(X,Y)\right]$$

Lemma φ is an orthogonal projection of F onto $L^2_{\nu_X} \subset L^2_{\nu_{XY}}$

Proof: We have to prove that for any $\psi \in L^2_{\nu_X}$

$$\langle F - \varphi, \psi \rangle_{L^2_{\nu_{XY}}} = 0$$

$$\langle F - \varphi, \psi \rangle_{L^2_{\nu_{XY}}} = \mathbb{E}\left[\left(F(X,Y) - \varphi(X)\right)\psi(X)\right] =$$

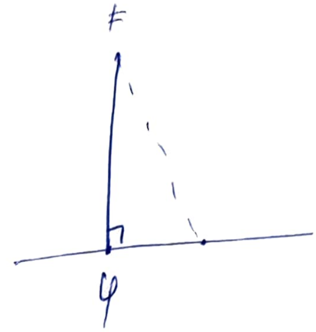
$$= \mathbb{E}\left[\mathbb{E}\left(\left(F(X,Y) - \varphi(X)\right)\psi(X) \mid X\right)\right] =$$

$$= \mathbb{E}\left[\underbrace{\mathbb{E}\left[\left(F(X,Y) - \varphi(X)\right) \mid X\right]}_{\substack{=0 \\ \text{since} \\ \mathbb{E}[F(X,Y) \mid X] = \varphi(X)}} \psi(X)\right] = 0 \qquad \square$$

Corollary:

φ is a solution to the convex opt. problem:

$$\begin{cases} \|F - \varphi\|_{L^2_{\nu_{XY}}} \longrightarrow \min \\ \varphi \in L^2_{\nu_X} \end{cases}$$



Least Squares Monte Carlo
(Regression MC)

Let $(X^m, Y^m)_{m=1}^{M}$ i.i.d. with law ν_{XY}

$(X, Y)^{1\dots M}$

$(X^{1\dots M}, Y^{1\dots M})$

mass in one point ....

Let $\nu^M_{XY} = \frac{1}{M}\sum_{m=1}^{M} \delta_{(X^m, Y^m)}$

$\nu^M_X = \frac{1}{M}\sum_{m=1}^{M} \delta_{X^m}$

We will consider L^2 spaces with respect to those measures

$$\|\psi\|_{\nu^M_{XY}} = \left(\frac{1}{M}\sum_{m=1}^{M} \psi^2(X^m, Y^m)\right)^{1/2} \iff E(\dots)$$

$$\|\psi\|_{\nu^M_X} = \left(\frac{1}{M}\sum_{m=1}^{M} \psi^2(X^m)\right)^{1/2} \iff E(\dots)$$

Take $\varphi_1 \dots \varphi_n \in L^2_{\nu_X}$ which are linearly independent (basis functions)

$$\begin{cases} \|F - \varphi\|_{L^2_{V^M_{XY}}} \to \min \\ \varphi \in \mathrm{lin}(\varphi_1, \dots, \varphi_k) \end{cases}$$

The solution will be denoted

$OLS(F, \mathcal{L}, (X,Y)^{1\dots M})$ $\quad \mathcal{L} = \mathrm{lin}(\varphi_1, \dots, \varphi_k)$

or

$OLS(F, (X,Y)^{1\dots M})$

<u>Algorithm</u>

- generate $(X^m, Y^m) \sim V_{XY}$;
- compute $\hat{\alpha} = \underset{\alpha \in \mathbb{R}^k}{\mathrm{argmin}} \sum_{m=1}^{M} \left(\sum_{j=1}^{k} \alpha_j \varphi_j(X^m) - F(X^m, Y^m) \right)^2$
- output: $\sum_{j=1}^{k} \hat{\alpha}_j \varphi_j$

<u>Remark 1</u> Linear regression of $F(X^m, Y^m)$ on $\varphi_1(X^m), \dots, \varphi_k(X^m)$

<u>Remark 2</u> This is a linear program $\Leftrightarrow$ system of linear equations

Original problem:

cond. expectation

$$\begin{cases} \|F - \varphi\|_{L^2_{V_{XY}}} \to \min \\ \varphi \in L^2_{V_{XY}} \end{cases} \underset{\text{approx. error}}{\leftrightsquigarrow} \begin{cases} \|F - \varphi\|_{L^2_{V_{XY}}} \to \min \\ \varphi \in \mathrm{lin}(\varphi_1, \dots, \varphi_k) \end{cases} \underset{\text{MC error}}{\leftrightsquigarrow} \begin{cases} \|F - \varphi\|_{L^2_{V^M_{XY}}} \to \min \\ \varphi \in \mathrm{lin}(\varphi_1, \dots, \varphi_k) \end{cases}$$

Optimal stopping time problem

$(X_n)_{n=0}^{N}$ - Markov process on $D \subseteq \mathbb{R}^d$

$$V_n(x) = \sup_{n \le \tau \le N} \mathbb{E}\left[f(\tau, X_\tau) \mid X_n = x \right]$$

optimal stopping time $\tau_n^* = \min\{ n \le j \le N : f(j, x_j) = V_j(x_j) \}$

DPP: $V_n(x) = \max\left(f(n, x), \mathbb{E}(V_{n+1}(X_{n+1}) \mid X_n = x) \right)$

Notation:

$X^n_{n\in N} = (X_n^n, \dots X_N^n)$ - one

$X^{1:M}_{n:N} = (X^1_{n:N}, \dots X^M_{N:N})$ - whole collection

$X^m_{n+} := X^m_{n:(n+1)}$ - cond. expectation

$OLS(F, X^{1:M}_{n:N}) = OLS(F(\underbrace{X^{1:M}_n}_{X}, \underbrace{X^{1:M}_{n+1,N}}_{Y}))$

Recall that we assumed $\|f\|_\infty - \beta$

$T_\beta \eta = (\eta \vee (-\beta)) \wedge \beta$ - truncation operator

Value iteration algorithm ~~algorithm~~ (Tsitsiklis, Van Roy, 2000)

Generate M independent trajectories $(X^{1:M}_{0:N})$ with $X_0^m = x$

$\hat{V}^M_N(x) = f(N, x)$

For $n = N-1, \dots, 0$

where $\hat{\alpha}^n = \operatorname{argmin}_{\alpha \in \mathbb{R}^k} \sum_{m=1}^{M} (\underbrace{V^M_{n+1}(X^m_{n+1})}_{Y} - \sum \alpha_j \varphi_j(\underbrace{X^m_n}_{X}))^2$

$\hat{C}^M_n = OLS(\hat{V}^m_{n+1}, X^{1:M}_{n+}) = \sum_{j=1}^{k} \hat{\alpha}^n_j \varphi_j(x)$

$\downarrow$ truncation

$\mathbb{E}[V_{n+1}(X_{n+1}) | X_n]$

$\hat{C}^M_n = T_\beta \hat{C}^M_n$

$\downarrow$ update

$\hat{V}^M_n(x) = \max(f(n,x), \hat{C}^M_n(x))$

End for

Performance iteration (Longstaff - Schwartz, 2001)
(policy)

Generate M independent trajectories $(X^{1:M}_{0:N})$ with $X_0^m = x$

$\hat{V}^M_N(x) = f(N, x)$

$\hat{C}^M_j \le f(j, x_j)$

For $n = N-1, \dots, 0$

$\hat{\tau}_{n+1}(X_{n+1:N}) = \min\{n+1 \le j \le N : \hat{V}^M_j(X_j) = f(j, x_j)\}$

$\hat{S}_{n+1}(X_{n+1:N}) = f(\hat{\tau}_{n+1}(X_{n+1:N}), X_{\hat{\tau}_{n+1}(X_{n+1:N})})$

$\Rightarrow$ $\hat{C}_h^m = OLS(\hat{S}_{n+1}, X_{n:N}^{1:M})$

$\hat{\hat{C}}_h^m = T_\beta \hat{C}_h^m$

$\hat{V}_h^m(x) = \max(f(n,x), \hat{C}_h^m)$

End for

$\hat{S}_h(X_{n:N}) = \{\hat{C}_h^m(X_n) \le f(n,X_n)\}\, f(n,X_n)$

or

$\{\hat{C}_h^m(X_n) > f(n,X_n)\}\, \hat{S}_{n+1}(X_{n+1,\dots,N})$

Convergence

$\hat{V}_h^{k,m}$ ← Basis φ ← MC trajectories

⟶ values obtained by the algorithm

$\hat{V}_h^k \to \approx \hat{V}_h^{k,\infty}$

[Thm] Cleveland, Landertz, Protter, 2002

$(\varphi_j)_{j=1}^\infty$ - seq. of functions total in $L^2_{\mathcal{L}(X_n)} = L^2_{X_n}$ $\forall n$

dense, infinite sequence

$\oplus$ linearly independent

1) $\hat{V}_h^h \xrightarrow{L^2} V_n$ when $k \to \infty$

2) additional assumption (about $\hat{\alpha}^m$)

$\hat{V}_h^{m,k} \to \hat{V}_h^k$ a.s (almost sure convergence)

Interplay between k & M: [Glasserman 2004 AAP]

$\sum_{j=1}^k \beta_j \varphi_j$, biggest MC error? $\|\beta\| \le 1$ ← projection errors

X_n → sampling of BM $\Rightarrow$ $k = O(\log M)$ exponential

X_n → sampling of GBM $\Rightarrow$ $k = O(\sqrt{\log M})$

Stentoft (2004)

$\hat{V}_n^{n,k} \to V_n$ in prob if $\frac{K^3}{M} \to 0$

Egloff (2005) AAP

$\hat{V}_n^{m,k} \longrightarrow V_n$ in L^2 if $\frac{k}{M} \to 0$

Gobet, Turkedjiev (2016) Mathematics of Computation

Györfi, Kohler, Krzyżak, Walk (2002)

Let $\|\varphi\|_{n,m} = \left(\frac{1}{M}\sum_{m=1}^{M} \varphi^2(X_n^m)\right)^{1/2}$

and

$$\varepsilon^2_{n,k} = \inf_{\varphi \in \mathrm{lin}(\varphi_1,\dots\varphi_L)} \| T_\beta \varphi - C_n \|^2_{L^2_{X_m}} \quad \leftarrow \text{approx. error}$$

where $C_n(x) = \mathbb{E}\left[V_{n+1}(X_{n+1}) \mid X_n = x\right]$

$\beta = \|f\|_\infty$

- $\hat{V}_N^M = VV_n = f(N_i) \Rightarrow \|\hat{V}_n^m - V_n\|^2_{n,M} = 0$

regression approximation of $C_{N-1}(x)$

emp. norm

- $\mathbb{E}\|\hat{C}^M_{N-1} - C_{N-1}\|^2_{N-1,} \leq 2\,\mathbb{E}\underbrace{\|\hat{V}^M_n - V_n\|^2_{N,M}}_{=0} + \underbrace{\frac{8\beta^2 k}{M}}_{\substack{\text{MC error} \\ \text{of comp. projection}}} + \varepsilon^2_{n-1,k}$

rough est. of var

approx. error

True norm

- $\mathbb{E}\|\hat{C}^M_{N-1} - C_{N-1}\|^2_{L^2_{X_{N-1}}} \leq 2\,\mathbb{E}\|\hat{C}^M_{N-1} - C_{N-1}\|^2_{N-1,M} + C \cdot \frac{\beta(K+1)\log(3^1 M)}{M}$

- $(V_{N-1} = \max(f, C_{N-1}))$

- $\mathbb{E}\|\hat{V}^M_{N-1} - V_{N-1}\|^2_{L^2_{X_{N-1}}} \leq \mathbb{E}\|\hat{C}^M_{N-1} - C_{N-1}\|^2_{L^2_{X_{N-1}}}$

$$\mathbb{E}\|\hat{V}^{M}_{N-1} - V_{N-1}\|^2_{N-1,M} \le \mathbb{E}\|V^{M}_{N-1} - V_{N-1}\|^2_{L^2_{X_{N-1}}}$$

$$\mathbb{E}\|\hat{V}^{M}_{n} - V_n\|^2_{L^2_{X_n}} \le 4\,\mathbb{E}\|\hat{V}^{M}_{n+1} - V_{n+1}\|_{L^2_{X_{n+1}}} + R(K,M)$$

$$\le 4^{N-n} R(K,M)$$

$$R(K,M) = O\left(\frac{K \cdot \log(3M)}{M}\right)$$

<u>Non-parametric</u>

Let G be a set of functions with values in $[0;B]$

bound

and let $Z_1, \dots$ i.i.d with distribution D

For any M and any $\varepsilon > 0$

$$\mathbb{P}\left[\sup_{g \in G}\left|\frac{1}{M}\sum_{m=1}^{M} g(z_m) - \int g(z)\nu(dz)\right| > \varepsilon\right]$$

$$\le 8\,\mathbb{E}\left[N_1\left(\frac{\varepsilon}{8}, G, \|\cdot\|_{L^2_{\nu_M}}\right)\right] e^{-\frac{M\varepsilon^2}{128B}}$$

where $\nu_M = \frac{1}{M}\sum_{m=1}^{M}\delta_{z_m}$

$$G = \{T_\beta \varphi : \varphi \in \mathrm{lin}(\varphi_1, \dots, \varphi_K)\}$$

$N_1(\varepsilon, G, \|\cdot\|)$ ← set

ε - covering number

$$N_1 \le 3\left(\frac{2eB}{\varepsilon}\log\frac{3eB}{\varepsilon}\right)^K$$

Dual - Methods

(M_n) - martingale wrt $(\mathcal{F}_n)$

Optional sampling theorem

if τ is a bounded stopping time, then

$$\mathbb{E}[M_\tau] = M_0$$

Fix a martingale (M_n) s.t. $M_0 = 0$

For any stopping time τ

$$\mathbb{E}[f(\tau, X_\tau)] - M_0 = \mathbb{E}[f(\tau, X_\tau) - M_\tau] \leq$$

$$\mathbb{E}[\sup_n (f(n, X_n) - M_n)]$$

$$\sup_\tau \mathbb{E}[f(\tau, X_\tau)] \leq \inf_{\substack{(M_n)\text{-martingale} \\ M_0 = 0}} \mathbb{E}[\sup_n (f(n, X_n) - M_n)]$$

process

martingale

sup.

average

$\Rightarrow$ MC est of upper bound

What will equality?

$$M_0^* = 0$$

$$M_{n+1}^* = M_n^* + V_{n+1}(X_{n+1}) - \mathbb{E}[V_{n+1}(X_{n+1}) \mid X_n]$$

$\leftarrow$ martingale $\mathcal{SS}$

and

since X_n is random variable

$$Z_n = \operatorname*{ess\,sup}_{n \leq \tau \leq N} \mathbb{E}[f(\tau, X_n) \mid X_n]$$

Then

$$Z_n = V_n(X_n),$$

$$Z_n \geq f(n, X_n)$$

$$\mathbb{E}[\sup_n (f(n, X_n) - M_n^*)] = \mathbb{E}[\sup_n \underbrace{(Z_n - M_n^*)}_{Y_n \leftarrow \text{decreasing process}}] = \mathbb{E}[\sup_n Y_n] = Y_0 = \underbrace{V_0(X_0)}_{-M_0} = V_0(X_0)$$

We show that $Y_{n+1} \leq Y_n$:

$$Y_{n+1} - Y_n = Z_{n+1} - M_n^* - (Z_n - M_n^*) = (Z_{n+1} - Z_n) - (M_{n+1}^* - M_n^*)$$

$$= Z_{n+1} - Z_n - \Big[\underbrace{V_{n+1}(X_{n+1})}_{Z_{n+1}} - \mathbb{E}(V_{n+1}(X_{n+1}) \mid X_n) \Big]$$

$$= \mathbb{E}(V_{n+1}(X_{n+1}) \mid X_n) - \underbrace{V_n(X_n)}_{= \max(f,\ \mathbb{E}(V_{n+1}(X_{n+1}) \mid X_n))} \leq 0$$

Generate M' trajectories of the process $X_{0:N}^{1:M'}$
(new from ones used for estimation)

for $m = 1, \dots, M'$
$M_0^m = 0$

for $n = 0, 1, \dots, N-1$

Generate $\xi_1, \xi_2, \dots, \xi_J \sim (X_{n+1} \mid X_n^m)$

$$M_{n+1}^m = M_n^m + \hat{V}_{n+1}^m (X_{n+1}^m) - \frac{1}{J} \sum_{j=1}^{J} \hat{V}_{n+1}^n (\xi_j)$$

conditional exp.
$J \uparrow \Rightarrow$ close to a martingale

Endfor

$y^m = \max \{ f(n, X_n^m) - M_n^m : n = 0, \dots N \}$

Output: $\frac{1}{M'} \sum_{m=1}^{M} y^m$ – dual bound (upper bound
+ lower bound
a value function itself)

$V_{n+1}^{M} \in \mathcal{F}_{n+1}$

project on $\text{lin}(\varphi_1(X_n), \dots, \varphi_k(X_n)) \in \mathcal{F}_n$

$$
\begin{array}{ccc}
V_{n+1}^{M} \text{ or } & \xrightarrow{\text{projection}} & \sum \hat{\alpha}_j^{n+1} \varphi_j(X_{n+1}) \\
\downarrow \text{ comp cond exp. } \mathbb{E}(\dots \mid \mathcal{F}_n) & & \downarrow \mathbb{E}(\dots \mid X_n) \\
\xrightarrow{\quad} & & ✓
\end{array}
$$

projecting on $\varphi = \text{lin}(\varphi_1, \dots, \varphi_k)$

$$\mathbb{E}(\varphi_j(X_{n+1}) \mid X_n) = \hat{\varphi}_j^{n}(X_n)$$

$$\left[\hat{\alpha} = \left(\langle \varphi_j, \varphi_i \rangle \right)_{i,j=1}^{k} \right]^{-1} \sum \langle \varphi_j \hat{V}_{n+1} \rangle$$

unknown => approx. using MC

comp. explicitly

$$A^{-1} \frac{1}{M} \sum_{m=1}^{M} \varphi_j(X^m) \hat{V}_{n+1}(X^m)$$

(III) MC in SOC

Probabilistic representation of solutions to PDE's

$$\begin{cases} V_t + \mathcal{L} V + F(t, x, V, V_x) = 0 & \text{on } [0, T) \times \mathbb{R}^d \\ V(T, x) = g(x) \end{cases}$$

$$\mathcal{L} V = \mu(x) V_x + \frac{1}{2} \operatorname{Tr}(V_{xx} \sigma \sigma^T)$$

is a generator of SDE

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t$$

Assume that V is $C^{1,2}$.

Then we can apply Ito's formula

$$V(T, X_T^{x,t}) - V(t, \underbrace{X_t^{x,t}}_{x}) = \int_t^T [\underbrace{V_t(s, X_s^{x,t}) + \mathcal{L}V(s, X_s^{x,t})}_{-F(s, X_s^{x,t}, V(\cdot), \sigma^T V_x(\cdot))}] ds$$

$$+ \int_t^T \sigma^T V_x(s, X_s^{t,x})\, dW_s$$

$$Y_s = V(s, X_s^{t,x})$$

$$Z_s = \sigma^T(X_s^{t,x}) V_x(s, X_s^{t,x})$$

$$Y_T - Y_s = \int_s^T - F(\dots)\, dv + \int_s^T Z_v\, dW_v$$

$$Y_s = Y_T + \int_s^T F(v, X_v, \underbrace{V(v, X_v)}_{Y_v}, \underbrace{\sigma^T V_x(v, X_v)}_{Z_v})\, dv - \int_s^T Z_v\, dW_v$$

Backward process →

$$Y_s = \overset{g(X_T)}{Y_T} + \int_s^T F(v, X_v, Y_v, Z_v)\, du - \int_s^T Z_u\, dW_u \quad \forall s \le T$$

Forw. process →

$$dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t$$

Equation with 2 unknowns (Y, Z) given terminal condition $g(X_T)$ and driver F

If $F = 0$ $\Rightarrow$ martingale representation theorem

If $F \neq 0$: fix Z_v, recount dv_t, iterate, hope it converges

Def: A solution to FBSDE is a pair of processes (Y_s, Z_s)

- adapted to (F_t^W)
- $\mathbb{E} \int_0^T |Z_s|^2 \, ds < \infty$
- $\mathbb{E} \sup_t |Y_t|^2 < \infty$, (Y_s) is continuous

Thm: [El Karoui, Hamadane, Matoussi]

→ μ, σ are Lipschitz ~~ceffe~~

→ $F(t, x, y, z)$ is Lipschitz in y, z uniformly in t, x

→ $|F(t, x, 0, 0)| \leq C(1 + |x|^p)$ polynomial... $p \geq 1/2$

Then there exists a unique solution $\mathcal{L}$ to FBSDE

There are measurable functions $\eta, \gamma(t, x)$ s.t.

$$Y_t = \eta(t, X_t), \quad Z_t = \gamma(t, X_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$V_t \qquad\qquad \sigma^T V_x$$

functions of state

$\Rightarrow$ mark. property

$\Rightarrow$ Markovian Solution (also work for non-Markovian fin. model!)

$$\begin{cases} dX_s^{t,x} = \mu(X_s^{t,x}, u_s)\,dt + \sigma(X_s^{t,x})\,dW_t \\ X_t^{t,x} = x \end{cases}$$

$(u_\cdot) \in \mathcal{U}_{ad}$ – admissible control $(u_t \in \mathcal{U})$

$$J(t,x,(u_\cdot)) = \mathbb{E}\left[\int_t^T f(s, X_s^{t,x}, u_s)\,ds + g(X_T^{t,x})\right]$$

Value function: $V(t,x) = \inf_{(u_\cdot) \in \mathcal{U}_{ad}} J(t,x,(u_\cdot))$

V satisfies HJB equation

$$V_t + \inf_{u \in \mathcal{U}} \left\{ f(t,x,u) + V_x \mu(x,u) + \frac{1}{2} \mathrm{Tr}\left(V_{xx} \sigma\sigma^T(x)\right) \right\} = 0$$

$$V_t + \underbrace{\frac{1}{2}\mathrm{Tr}(V_{xx}\sigma\sigma^T)}_{\text{generator}} + H(t,x,V,V_x) = 0$$

$$H(t,x,y,z) = \inf_{u \in \mathcal{U}} \left\{ f(t,x,u) + z\,\mu(x,u) \right\}$$

$\longrightarrow$ Hamiltonian

If σ is uniformly elliptic then we can define

$$\underbrace{F(t,x,y,z)}_{\text{driver of BSDE}} = H(t,x,y,\sigma^{-1}z)$$

This PDE is linked to the FBSDE

$$dX_t = \sigma(X_t)\,dW_t$$

$$Y_t = g(X_T) + \int_t^T F(\cdot\,\cdot)\,ds - \int_t^T Z_s\,dW_s$$

Trick: pick a drift $\hat{\mu}$ (best guess of opt. strategy)

$$V_t + \frac{1}{2}\mathrm{Tr}(V_{xx}\sigma\sigma^T) + \hat{\mu}(x) V_x + \underbrace{\inf_{u \in \mathcal{U}} \left\{ f(t,x,u) + V_x(\mu(x,u) - \hat{\mu}(x)) \right\}}_{f} = 0$$

Numerical Solutions for FBSDEs

FBSDE

$$\begin{cases} dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t \\ Y_t = g(X_1) + \int_t^1 F(s, X_s, Y_s, Z_s)\,ds - \int_t^1 Z_s\,dW_s \end{cases}$$

Use Euler scheme for (X_t):

$t_i = \frac{i}{N}$, $i = 0, \dots, N$; $\Delta = \frac{1}{N}$

$$X^N_{t_{i+1}} = X^N_{t_i} + \mu(X^N_{t_i})\cdot\Delta + \sigma(X^N_{t_i})\,\Delta W_i \quad ; \quad \Delta W_i = W_{t_{i+1}} - W_{t_i}$$

From BSDE:

$$Y_{t_i} = Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} F(s, X_s, Y_s, Z_s)\,ds - \int_{t_i}^{t_{i+1}} Z_s\,dW_s$$

$$\mathbb{E}[\,\cdot \mid \mathcal{F}_{t_i}] =: \mathbb{E}_{t_i}[\,\cdot\,] \leftarrow \text{cond. expectation}$$

$$Y_{t_i} = \mathbb{E}_{t_i}[Y_{t_{i+1}}] + \int_{t_i}^{t_{i+1}} \underbrace{\mathbb{E}_{t_i}[F(s, X_s, Y_s, Z_s)]}_{F(t_i, X_{t_i}, Y_{t_i}, Z_{t_i})}\,ds - 0$$

$$Y_{t_i} = \mathbb{E}_{t_i}[Y_{t_{i+1}}] + \Delta \cdot F(t_i, X_{t_i}, Y_{t_i}, \underset{?}{Z_{t_i}})$$

Multiply by incr. of BM:

$$\underbrace{\mathbb{E}_{t_i}[Y_{t_i}\cdot \Delta W_i]}_{=0} = \mathbb{E}_{t_i}[Y_{t_{i+1}}\,\Delta W_i] + \overbrace{\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} F(s, X_s, Y_s, Z_s)\,ds \cdot \Delta W_i\right]}^{\mathcal{F}_{t_i}\text{-measurable} \Rightarrow \approx 0}$$

$$\underbrace{- \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s\,dW_s \cdot \Delta W_i\right]}$$

$$\mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s\,dW_s \cdot \int_{t_i}^{t_{i+1}} dW_s\right] = \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s\,ds\right] \approx \mathbb{E}_{t_i}[\Delta \cdot Z_{t_i}]$$

$$\boxed{0 = \mathbb{E}_{t_i}[Y_{t_{i+1}}\,\Delta W_i] - \Delta Z_{t_i}}$$

$$\begin{cases} Y^N_{t_i} = \mathbb{E}_{t_i}[Y^N_{t_{i+1}}] + \Delta \cdot F(t_i, X^N_{t_i}, Y^N_{t_i}, Z^N_{t_i}) \\ Z_{t_i} = \mathbb{E}_{t_i}\left[Y_{t_{i+1}} \frac{\Delta W_i}{\Delta}\right] \\ Y^N_1 = g(X^N_1) \end{cases}$$

$\hookrightarrow$ numerical scheme

$$\begin{cases} Y_{t_i}^N = \mathbb{E}_{t_i}[Y_{t_{i+1}}^N] + \Delta \cdot F(t_i, X_{t_i}^N, Y_{t_i}^N, Z_{t_i}^N) \\ Z_{t_i}^N = \mathbb{E}_{t_i}\left[Y_{t_{i+1}}^N \cdot \frac{\Delta W_i}{\Delta} \right] \\ Y_1^N = g(X_1^N) \end{cases}$$

★ $\frac{\Delta W_i}{\Delta} \sim N(0, \frac{1}{\Delta}) = N(0, N)$

$Y_{t_i}^N$ and $Z_{t_i}^N$ are functions $X_{t_i}^N$

So we will estimate those functions

$$Y_{t_{i+1}}^N \in \sigma(X_{t_{i+1}}^N, \dots, X_{t_N}^N) \leftarrow \text{only source of randomness}$$

$$\mathbb{E}\left[Y_{t_{i+1}}^N \frac{\Delta W_i}{\Delta} \mid \mathcal{F}_{t_i} \right] = \mathbb{E}[\dots \mid X_{t_i}^N]$$

all functions Lipschitz

$$\sup_{0 \le t \le 1} \mathbb{E}|X_t - X_t^N|^2 \le \frac{C}{N} \leftarrow \text{const. from Lipschitz}$$

$$\mathbb{E}\left[\sup_{t \in [t_i, t_{i+1})} |X_t - X_{t_i}|^2 \right] \le C/N \quad \text{for } i = 0, \dots, N-1$$

Let (Y_t, Z_t) - the solution to FBSDE

Def. $\bar{Z}_{t_i}^N = \frac{1}{\Delta} \mathbb{E}_{t_i}\left[\int_{t_i}^{t_{i+1}} Z_s \, ds \right] \leftarrow$ "best approx" of Z process is average

THM 1 [Zhang 2004]:

$$\max_i \sup_{t_i \le t \le t_{i+1}} \mathbb{E}|Y_t - Y_{t_i}|^2 + \sum_{i=0}^{N-1} \mathbb{E}\left[\int_{t_i}^{t_{i+1}} |Z_s - \bar{Z}_{t_i}^N|^2 ds \right] \le C/N$$

▷ when driver F is Lipschitz $\Rightarrow$ Y can't grow too fast... □

by Mart. representation theorem:

$$Y_{t_{i+1}}^N = \mathbb{E}_{t_i}[Y_{t_{i+1}}^N] + \int_{t_i}^{t_{i+1}} Z_s^N dW_s$$

$$Y_s^N = Y_{t_i}^N - \underbrace{(s - t_i)}_{\Delta} F(t_i, X_{t_i}^N, Y_{t_i}^N, Z_{t_i}^N) + \int_{t_i}^{s} Z_v^N dW_v, \quad \text{for } s \in [t_i, t_{i+1}]$$

THM 2:

$$\sup_{0\le t\le 1} \mathbb{E}|Y_t - Y_t^N|^2 + \mathbb{E}\left[\int_0^1 |Z_t^N - Z_t|^2 dt\right] \le C/N$$

Denote by $\hat{\mathbb{E}}_{t_i}[\cdot]$ an approx of $\mathbb{E}_{t_i}[\cdot]$

s.t.: $\hat{\mathbb{E}}_{t_i}[\cdot] \perp \sigma(W_s : t_i \le s \le 1)$ ← approx. is independent from BM (we can't use old trajectories)

$$\begin{cases} \mathbb{E}_{t_i}[Y^N_{t_{i+1}}] = \tilde{y}_i^N(X^N_{t_i}) \\ y_i^N(x) = \mathbb{E}\left[y_{i+1}^N(x + \mu(x)\Delta + \sigma(x)\xi)\right], \quad \xi \sim N(0,\Delta) \end{cases}$$

Let $(\hat{Y}^N_{t_i}, \hat{Z}^N_{t_i})$ be the solution using $\hat{\mathbb{E}}_{t_i}[\cdot]$

THM 3:

$$\|\hat{Y}^N_{t_i} - Y^N_{t_i}\|_{L^2} \le \frac{C}{N} \max_n \left\{ \|(\hat{\mathbb{E}}_{t_n} - \mathbb{E}_{t_n})\cdot(\hat{Y}^N_{t_{n+1}})\|_{L^2} + \|(\hat{\mathbb{E}}_{t_n} - \mathbb{E}_{t_n})(\hat{Y}^N_{t_{n+1}} \Delta W_i)\|_{L^2} \right\}$$

* instead of simulating new MC trajecties, whe can use regression MC and get a bit of bias.

Variance reduction techinique, since $\mathbb{E}_{t_i}[\mathbb{E}_{t_i}(Y_{t_{i+1}})\frac{\Delta W_i}{\Delta}] = 0$

$$\mathbb{E}_{t_i}\left[Y_{t_{i+1}} \frac{\Delta W_i}{\Delta}\right] = \mathbb{E}_{t_i}\left[\underbrace{\underset{\Downarrow}{(Y_{t_{i+1}} - \mathbb{E}_{t_i}(Y_{t_{i+1}}))}\,\underset{\Downarrow}{\frac{\Delta W_i}{\Delta}}}_{\text{doesn't explode}}\right]$$

$Var \sim \Delta$ $\quad$ $Var \sim \frac{1}{\Delta}$ ⇓ explodes

doesn't explode

[Warin, Touzi, Pham]

$F(t, x, V_t, V_x, V_{xx})$

↳ 2nd order BSDEs [Soner]

↳ simulate GBM

↳ [Kharroubi, Pham] BSDEs with non-positive jumps

↳ poisson to get rid of V_{xx}

↳ [Warin, Pham] finite difference methods

$V_t + F(t, x, V, V_x, V_{xx}) = 0$ ⇒ can't simulate

$V_t + \mathcal{L}V + \hat{F}(t, x, V, V_x, V_{xx}) = 0$ ⇒ can simulate

$= F - \mathcal{L}V$

solve w/o V_{xx}

correct for V_{xx}

hope it converges

$V(t,x) = \Psi(V(t+\Delta, \cdot))$ - fin. dif. method

$V(t,x) = \mathbb{E}\left[V(t+\Delta, X_\Delta) \cdot \underbrace{\quad}_{\text{something... Int. by part...}} \mid X_t = x \right]$

$f(x) = \mathbb{E}[F(\xi)] \quad , \xi \sim N(0,1)$

$f_x = \mathbb{E}[F(\xi)\xi]$

$f'(x) = \mathbb{E}[F'(\xi)] = \int F'(x)\varphi(x)\,dx$

① Stoch. Optimal Control Problem

$T>0$ St. eq: $\begin{cases} dX(s) = b(s, X(s), a(s))ds + \sigma(\cdot)\, dW(s) \\ X(t) = x \end{cases}$ $s\in(t,T)$

$t\in[0;T]$

Assumptions (guarantee the solution of ODE (SDE))

$b: [0;T]\times\mathbb{R}^n\times\Lambda \to \mathbb{R}^n$

$\sigma: [0;T]\times\mathbb{R}^n\times\Lambda \to \mathbb{R}^{n\times m}$

b, σ - continuous

$b(\cdot), \sigma(\cdot)$ — unif. cont. on bounded subsets of $[0;T]\times\mathbb{R}^n$, $\forall a\in\Lambda$

$|b(s,x,a) - b(s,y,a)| \le C|x-y|$ $s\in[0;T]$, $x,y\in\mathbb{R}^n$, $a\in\Lambda$

$\|\sigma(s,x,a) - \sigma(s,y,a)\| \le C|x-y|$ –//–

$|b(s,x,a)| + \|\sigma(s,x,a)\| \le C(1+|x|)$ –//–

Λ - Polish space

W - st. B.M in $\mathbb{R}^m$

$a(\cdot)$ - control process

Cost functional

$$J(t,x;a(\cdot)) = \mathbb{E}\Big\{\int_t^T e^{-\int_t^s c(X(r))dr} L(s,X(s),a(s))ds + e^{-\int_t^T c(X(r))dr} g(X(T))\Big\}, \quad c\ge 0$$

discounting; running cost; term. cost

c, L, g - continuous

Generalized Reference Prob. Space

$\mu = (\Omega, F, \mathcal{F}_s^t, \mathbb{P}, W)$; $(\Omega, \mathcal{F}, \mathbb{P})$ - complete ~~measurable~~ probability space

(Ω: field of measurable space)

$\mathcal{F}_s^t$ - complete filtration (right continuous)

- $\mathcal{F}_{s_1}^t \subset \mathcal{F}_{s_2}^t$, $s_1 < s_2$ (filtration)
- each $\mathcal{F}_s^t$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$ (complete)
- $\mathcal{F}_s^t = \bigcap_{r>s} \mathcal{F}_r^t$ (right-cont. filtration)

W is $\mathcal{F}_s^t$ wiener process in $\mathbb{R}^m$

- $W(t_2) - W(t_1)$ is independent of $\mathcal{F}_{t_1}^t$, $t_2 > t_1$
- $\mathcal{L}(W(t_2) - W(t_1)) = N_{0,(t_2-t_1)I}$
- W has cont. trajectories $\mathbb{P}$ a.s.

Natural Filtration: $\mathcal{F}_s^{t,0} = \sigma(W(r): t\le r\le s)$ - σ-field gen. by WP.

$\mathcal{F}_s^t$ = augmentation of $\mathcal{F}_s^t$ by $\mathbb{P}$ null sets

$= \sigma(\mathcal{F}_s^{t,0}, N)$ ← $\mathbb{P}$ null sets

If the $\mathcal{F}_s^t$ is the natural filtration generated by W and if $W(t) = 0$, then a GRPS μ is called a RPS

OCP (strong formulation):

Fix GRPS μ

Find admissible controls $\mathcal{U}_t^\mu = \{ a(\cdot)\ [0,T] \times \Omega \longrightarrow \Lambda \}$

$a(\cdot)$ is $\mathcal{F}_s^t$ - progressively measurable:

$a(\cdot) : [t,s] \times \Omega \longrightarrow \Lambda$ is $\mathcal{B}([t,s]) \times \mathcal{F}_s^t / \mathcal{B}(\Lambda)$ - measurable

$s > t$

$$\boxed{\text{Goal: } \min_{a(\cdot) \in \mathcal{U}_t^\mu} J(t, x, a(\cdot))}$$

OCP (Weak formulation): easier to proof DPP with weaker form.

Admissible controls: $\mathcal{U}_t = \bigcup_\mu \mathcal{U}_t^\mu$ - union over all GRPS μ

$$\boxed{\text{Goal: } \min_{a(\cdot) \in \mathcal{U}_t} J(t, x, a(\cdot))}$$

↳ Admissible controls: $\mathcal{U}_t = \bigcup_\mu \mathcal{U}_t^\mu$ - union over all RPS μ (by WD)

State eq.: $X(\cdot)$ is a solution