HJB Examples

(1) $J = \frac{1}{2}\int_{t_0}^{t_f} (x^2 + u^2)\, dt$

s.t. $\dot{x} = -x^3 + u$

$x(0) = x_0$

integrent of cost. fun.

▷ $H = \frac{1}{2}(x^2+u^2) + \lambda(-x^3+u)$ ← state eq.

$\frac{\partial H}{\partial u} = u + \lambda = 0$

$u = -\lambda = \left(-\frac{\partial V}{\partial x}\right)$

$H_{opt} = \frac{1}{2}x^2 - \frac{1}{2}\lambda^2 - x^3\lambda$

$$\boxed{\begin{aligned} &\frac{\partial V}{\partial t} + H_{opt} = 0 \\ &H_{opt} = \min_{u\in\Omega}(H) \\ &= \min_{u\in\Omega}(L + \lambda^T f) \end{aligned}}$$

HJB Equation:

$\frac{\partial V}{\partial t} - \frac{1}{2}\left(\frac{\partial V}{\partial x}\right)^2 - \left(\frac{\partial V}{\partial x}\right)x^3 + \frac{1}{2}x^2 = 0$ ← PDE

$V(t_f, x_f) = 0$

↳ diff. to solve PDE

⇒ assumption $t_f \to \infty$ ⇒ $\frac{\partial V}{\partial t} = 0$, $\forall t$,

⇒ $V(t_0) = V(t_f) = 0$

$\left(\frac{dV}{dx}\right)^2 + 2\left(\frac{dV}{dx}\right)x^3 - x^2 = 0$ ← ODE

↳ approximate closed-form solution

$V(x) = a_0 + a_1 x + \frac{a_2 x^2}{2!} + \frac{a_3 x^3}{3!} + \frac{a_4 x^4}{4!}$

$\frac{dV}{dx} = a_1 + a_2 x + \frac{a_3 x^2}{2!} + \frac{a_4 x^3}{3!}$

(powers ⇒ $a_0 = a_1 = a_3 = 0$, $a_2 = 1$, $a_4 = -6$

⇒ (X)

$\lambda = \frac{dV}{dx} = x - x^3$

$u = -\lambda = -x + x^3$ ⇒ closed loop dynamics

$\dot{x} = x^3 - x - x^3 = -x$

(3) $J = \frac{1}{2}[x_1(2) - 4]^2 + \frac{1}{2}[x_2(2) - 2]^2 + \frac{1}{2}\int_0^2 u^2\,dt$

$x(0) = [1 \;\; 2]$ $\qquad\qquad \dot{x}_1 = x_2$

$x_1(2) = 0$

$x_2(2)$ is free $\qquad\qquad \dot{x}_2 = u$

▷ 1) HJB : $\frac{\partial V}{\partial t} + H = 0$

$H = L + \lambda^T f$

$= \frac{1}{2}u^2(t) + [\lambda_1 \;\; \lambda_2]\begin{bmatrix} x_2 \\ u \end{bmatrix}$

$= \frac{1}{2}u^2(t) + \lambda_1 x_2 + \lambda_2 u$

2)

station eq. $\left\{ \frac{\partial H}{\partial u} = 0 \right.$ $\qquad u^*(t) + \lambda_2(t) = 0$

$u^*(t) = -\lambda_2(t)$

3) $H^* = \frac{1}{2}\lambda_2^2 + \lambda_1 x_2 - \lambda_2^2 = \lambda_1 x_2 - \frac{1}{2}\lambda_2^2$

$\frac{\partial H^*}{\partial x_1} = 0$

$\frac{\partial H^*}{\partial x_2} = \lambda_1$

$\frac{\partial H^*}{\partial \lambda_1} = x_2$

$\frac{\partial H^*}{\partial \lambda_2} = -\lambda_2$

4) st. eq. $\left\{ \begin{array}{ll} \dot{x}_1 = x_2 & \Rightarrow \\ \dot{x}_2 = u = -\lambda_2 & \Rightarrow \end{array} \right.$

$\dot{x}_1 = \frac{1}{2}c_1 t^2 - c_2 t + c_3$

$x_1 = \frac{1}{6}c_1 t^3 - \frac{1}{2}c_2 t^2 + c_3 t + c_4$

$\dot{x}_2 = c_1 t - c_2$

$x_2 = \frac{1}{2}c_1 t^2 - c_2 t + c_3$

cost. eq. $\left\{ \begin{array}{ll} \dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 & \Rightarrow \lambda_1 = c_1 \\ \dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 & \Rightarrow \dot{\lambda}_2 = -c_1 \end{array} \right.$

$\lambda_2 = -c_1 t + c_2$

$u = \frac{15}{4}t - 5$

$\Uparrow$

$\lambda_2 = -\frac{15}{4}t + 5$

$\Uparrow$

5) $x_1(0) = 1 = c_4$

$x_2(0) = 2 = c_3$

$x_1(2) = 0 = \frac{1}{6}c_1 \cdot 2^3 - \frac{1}{2}c_2 \cdot 2^2 + \underset{2}{\underset{\parallel}{c_3}} \cdot 2 + \underset{1}{\underset{\parallel}{c_4}} = \frac{8}{6}c_1 - 2c_2 + 5 \Rightarrow c_1 = \frac{15}{4}$

$c_2 = 5$

$\Uparrow$

$x_2(2)$ is free

$\frac{\partial S}{\partial x_2} - \lambda_2 \Big|_{t_f} = 0 \qquad (x_2 - 2) - \lambda_2 = 0$

$\lambda_2(t_f) = x_2(t_f) - 2 =$ $\qquad\qquad 4c_1 - 3c_2 = 0$

$-c_1 \cdot 2 + c_2 = \frac{1}{2}c_1 \cdot 2^2 - c_2 \cdot 2 + 2 - 2 \Rightarrow$

IV. Discrete-time OC

① FOC general case

$$J = \underbrace{\phi(x_n, N)}_{\text{terminal cost}} + \underbrace{\sum_{k=i}^{N-1} L_k(x_k, u_k)}_{\text{running cost}}$$

final time — initial time

$$J \to \begin{cases} \text{min Time} & \phi = N \quad L = 0 \\ \text{min fuel} & \phi = 0 \quad L = |u_k| \\ \text{min energy} & \phi = 0 \quad L = u_k^T u_k \\ \text{LQ} & \phi = \frac{1}{2} x_N^T S_N x_N \quad L = \frac{1}{2}(x_k^T Q x_k + u_k^T R u_k) \end{cases}$$

$$\boxed{\begin{aligned} &\min_{x,u} \; J_i(x,u) \\ &\text{s.t.} \quad x_{k+1} = f_k(x_k, u_k) \end{aligned}} \quad , x_i \text{ given}$$

FOC: augmented cost (~ Lagrangian fun.) — original cost f. — l.c. of const. f.

$$J_i'(x,u,\lambda) = J_i(x,u) + \sum_{k=i}^{N-1} \lambda_k^T \left[f_k(x_k,u_k) - x_{k+1} \right]$$

$$= \phi(x_n, N) + \sum_{k=i}^{N-1} \left[\underbrace{L_k(x_k,u_k) + \lambda_{k+1}^T (f_k(x_k,u_k)}_{} - x_{k+1}) \right]$$

$$\boxed{H_k(x_k, u_k, \lambda_{k+1}) = L_k(x_k,u_k) + \lambda_{k+1}^T f_k(x_k,u_k)}$$

(control) Hamiltonian

$$= \phi(x_n, N) + \sum_{k=i}^{N-1} \left[H_k(\cdot) - \lambda_{k+1}^T x_{k+1} \right]$$

$$= \underbrace{\phi(x_n, N) - \lambda_N^T x_N}_{\text{@ term. time}} + \underbrace{H_i(\cdot)}_{\text{@ initial time}} + \underbrace{\sum_{k=i+1}^{N-1} \left[H_k(\cdot) - \lambda_k^T x_k \right]}_{\text{@ running Time}}$$

$$\boxed{\nabla J_i' = 0}$$

$$dJ_i' = (\nabla_{x_N}\phi - \lambda_N)^T dx_N + (\nabla_{x_i} H_i)^T dx_i + \sum_{k=i+1}^{N-1} (\nabla_{x_k} H_k - \lambda_k)^T dx_k +$$

$$\sum_{k=i}^{N-1} (\nabla_{u_k} H) du_k + \sum_{k=i+1}^{N} (\nabla_{\lambda_k} H_{k-1} - x_k)^T d\lambda_k$$

$d J_i' = 0 \Rightarrow$

state eqs.

FOC of optimality

$$x_{k+1} = \nabla_{\lambda_{k+1}} H_k = f_k(x_k, u_k),\ k = \overline{i, N-1}$$

$$\lambda_k = \nabla_{x_k} H_k = \nabla_{x_k} L_k + \underbrace{\nabla_{x_k}(\lambda^T f_k)}_{= \nabla_{x_k} f_k \cdot \lambda_{k+1},\ k=\overline{i,N-1}}$$

$$0 = \nabla_{u_k} H_k = \nabla_{u_k} L_k + \nabla_{u_k} f_k \cdot \lambda_{k+1},\ k = \overline{i, N-1}$$

$$0 = \nabla_{x_i} H_i \cdot dx_i$$ (until specified that the state is fixed whole product should $= 0$)

$$0 = (\nabla_{x_N} \phi - \lambda_N)^T dx_N$$ ↳ for fixed x_i $dx_i = 0 \Rightarrow x_i = r_i$ (given)

$$\lambda^T f = \sum \lambda_i \cdot f_i$$

$$\nabla(\lambda^T f) = \nabla(\sum \lambda_i f_i) = \sum \lambda_i \nabla f_i = \underbrace{[\nabla f_1\ \nabla f_2 \dots]}_{\nabla f} \cdot \lambda$$

transpose of Jacobian matrix

fixed x_N: $dx_N = 0$, $\boxed{x_N = r_N}$

free x_N: $dx_N \neq 0$, $\boxed{\lambda_N = \nabla_{x_N} \phi}$

2 Linear systems, Quadratic cost:

$$J = \frac{1}{2} x_N^T \underset{\geq 0}{S_N} x_N + \frac{1}{2} \sum_{k=0}^{N-1} [x_k^T \underset{\geq 0}{Q} x_k + u_k^T \underset{> 0}{R} u_k]$$

$$H_k = \underbrace{\frac{1}{2}(x_k^T Q x_k + u_k^T R u_k)}_{L_k(x_k, u_k)} + \lambda_{k+1}^T \underbrace{(A x_k + B u_k)}_{f_k(x_k, u_k)}$$

$x_{k+1} = A x_k + B u_k$ ← state eq. $f_k(x_k, u_k)$

$\lambda_k = Q x_k + A^T \lambda_{k+1}$ ← costate eq. $\nabla_{x_k} L_k + \nabla_{x_k} f_k \cdot \lambda_{k+1}$

$0 = R u_k + B^T \lambda_{k+1}$ ← stationarity eq. $\nabla_{u_k} L_k + \nabla_{u_k} f_k \cdot \lambda_{k+1}$

$x_0 = r_0$ ← initial condition

$(\lambda_N - S_N x_N) dx_N = 0$ ↗ $x_N = r_N$ ↘ $\lambda_N = S_N x_N$

final cond. (state given) or (costate λ of fin. time given as a lin. transf. of x_N state of fin. time.)

$u_k = -\underset{> 0}{R^{-1}} B^T \lambda_{k+1}$ → state eq. ⇓

dyn. system w 2 (co)state var

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A & -B R^{-1} B^T \\ Q & A^T \end{bmatrix} \cdot \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix} \Rightarrow$$ TWO-POINT boundary value problem (BVP)

w in. cond. $x_0 = r_0$

$x_N = r_N$ or $\lambda_N = S_N x_N$

2.1 TWO-POINT BVP

(state given)

$$\begin{bmatrix} x_{k+1} \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix} \begin{bmatrix} x_k \\ \lambda_{k+1} \end{bmatrix}$$

$x_0 = r_0$

$x_N = r_N$

SHOOTING:

Idea – determine everything at one of the "ends" of interval. make a guess for another and see if it hits bound. cond. after solving the system

$$\begin{bmatrix} x_N \\ \lambda_0 \end{bmatrix} = \underbrace{\begin{bmatrix} A & -BR^{-1}B^T \\ Q & A^T \end{bmatrix}^N}_{M} \cdot \begin{bmatrix} x_0 \\ \lambda_N \end{bmatrix}$$

In linear case

$$x_N = M_{11} \underset{r_0}{\overset{\shortparallel}{x_0}} + M_{12} \lambda_N = r_N$$

$\Downarrow$ determine λ_N

$$\lambda_N = M_{12}^{-1}(r_N - M_{11} \cdot r_0)$$

$\Downarrow$ det. λ_k at any point

$$\lambda_k = Q x_k + A^T \lambda_{k+1}$$

$\Downarrow$ control signal at any point

$$u_k = -R^{-1}B^T \lambda_{k+1}$$

$\boxed{i.s.}$ assume $Q = 0$:

$J = \sum u_k^T R u_k$ ... min energy control

$$x_{k+1} = Ax_k - BR^{-1}B^T\lambda_{k+1}$$

$$\lambda_k = A^T\lambda_{k+1}$$

$\Downarrow$

$$\lambda_k = (A^T)^{N-k} \cdot \lambda_N$$

$\Downarrow$

$$\lambda_{k+1} = (A^T)^{N-k-1} \lambda_N$$

$$\lambda_{k+1} = -(A^T)^{N-k-1} G_{0,N,R}^{-1}(r_N - A^N x_0)$$

inverse of G?

dif. between desired state & state reached w/o control

$$\boxed{u_k = -R^{-1}B^T\lambda_{k+1} = \dots}$$

$$x_{k+1} = Ax_k - \underbrace{BR^{-1}B^T(A^T)^{N-k-1}\lambda_N}_{\text{fictitious input}}$$

$$x_k = A^k x_0 - \sum_{i=0}^{k-1} A^{k-1-i} BR^{-1}B^T(A^T)^{N-i-1}\lambda_N$$

← convolution of all external (fict.) inputs

@ N:

$$r_N = x_N = A^N \underset{r_0}{\overset{\shortparallel}{x_0}} - \underbrace{\sum_{i=0}^{N-1} A^{N-1-i} BR^{-1}B^T(A^T)^{N-i-1}}_{G_{0,N,R}} \lambda_N \Rightarrow \lambda_N = -G_{0,N,R}^{-1}(r_N - A^N x_0)$$

Reachability gramian (when G exists)

$$G = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k$$ must be > 0 if system is reachable

$$G_{0,N} = \sum_{k=0}^{N-1} A^k B B^T (A^T)^k$$

$R^{-1} \Rightarrow G_{0,N,R}$ can't improve rank $\Rightarrow G$ is invertible as system is reachable!

CAYLEY - HAMILTON THM:

if we consider matrix $A \in \mathbb{R}^{n \times n}$ of size

for $k > n-1$ $\quad A^k = \sum_{i=0}^{k-1} \alpha_i A^i$

sum of lower powers of matrix A

2.2 TWO - POINT BVP

(free fin. state)

$$x_{k+1} = Ax_k - BR^{-1}B^T \lambda_{k+1}$$

$$\lambda_k = Q x_k + A^T \lambda_{k+1}$$

$$u_k = -R^{-1}B^T \lambda_{k+1}$$

$$x_0 = r_0$$

$$S_N x_N = \lambda_N$$

boundary cond.: initial cond. + final cond.: state and co-state are related in linear fashion

assume same linear relation fin. cond. holds throughout the control horizon

$S_k \cdot x_k = \lambda_k$ ← sweep assumption (generalization)

$x_{k+1} \xrightarrow{\text{state}} x_{k+1} = Ax_k - BR^{-1}B^T S_{k+1} x_{k+1}$

$$x_{k+1} = (I + BR^{-1}B^T S_{k+1})^{-1} Ax_k$$

$\lambda_k \xrightarrow{\text{co-state}} S_k x_k = Qx_k + A^T S_{k+1} x_{k+1}$

$$S_k x_k = Q x_k + A^T S_{k+1} (I + BR^{-1}B^T S_{k+1})^{-1} Ax_k \quad | : x_k \Rightarrow \text{for any } x_k$$

$$\boxed{S_k = Q + A^T S_{k+1} (I + BR^{-1}B^T S_{k+1})^{-1} A}$$

difference Riccati equation

$S_N \to S_{N-1} \to \dots$

$u_k = -R^{-1}B^T S_{k+1} x_{k+1} = -R^{-1}B^T S_{k+1} (Ax_k + Bu_k)$

! state feedback

$$\boxed{u_k = \underbrace{-(I + R^{-1}B^T S_{k+1} B)^{-1} R^{-1} B^T S_{k+1} A}_{K_k \text{ - Kalman gain}} x_k} \Rightarrow u_k = f(x_k)$$

! state feedback

ex.: **Scalar system**

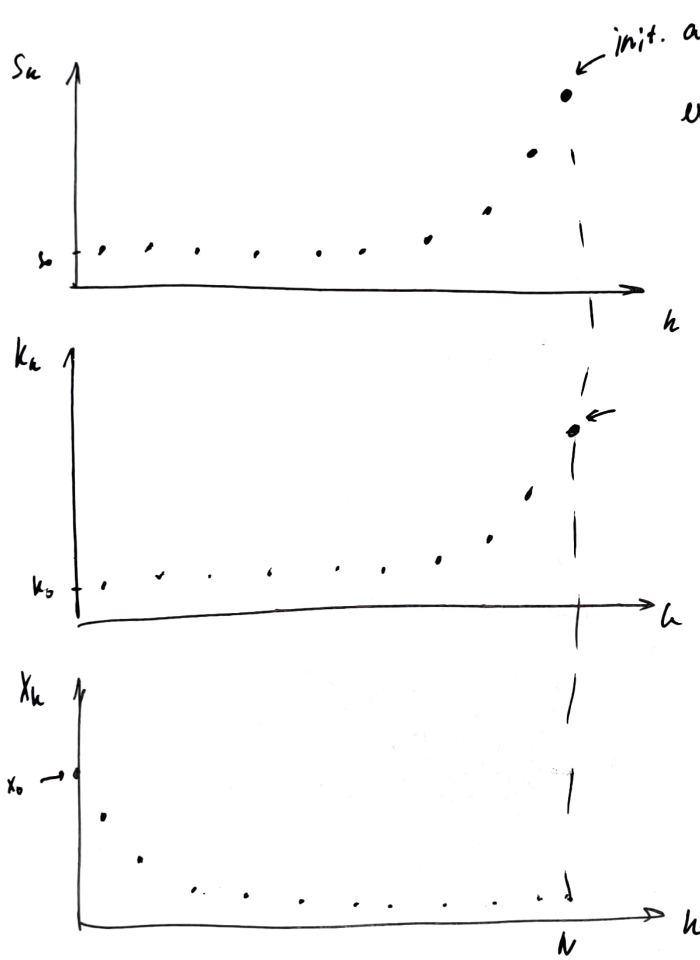
$$x_{k+1} = a x_k + b u_k$$

$$J = \frac{1}{2} s_N x_N^2 + \frac{1}{2} \sum_{k=0}^{N-1} \left[q x_k^2 + r u_k^2 \right]$$

$$S_k = q + \frac{a^2 S_{k+1}}{1 + \frac{b^2 S_{k+1}}{r}} \Leftarrow \text{diff. Ric. equation}$$

$\Downarrow$

$$S_k = q + \frac{a^2 r S_{k+1}}{r + b^2 S_{k+1}}$$



evolve: if N is long enough $\Rightarrow$ for most of time remains constant

–11– same holds for time-var. state feedback gain

state var. doesn't necessary $\to 0$ at fin. time but if we want to then we penalize S_N term (penalize state at fin time to bring it closer to zero)

(2.3) **Steady state** (inf. horizon)

Question: why not use instead of time-var. K_k something constat like $K_\infty := \lim_{k \to \infty} K_k$

computed from $S_\infty := \lim_{k \to \infty} S_k$ $\leftarrow$ solution to dif. Ric. equation

$\Rightarrow K_\infty$ is only suboptimal for $N < \infty$ but optimal for $N = \infty$

$$J = \frac{1}{2} \sum_{k=0}^{\infty} [x_k^T Q x_k + u_k^T R u_k]$$

$\boxed{S_k = S_{k+1}} =: S_\infty$ ~~the~~ of the sol. to Ric. eq.

assume this value doesn't change from step to step

$$\boxed{S_\infty = Q + A^T S_\infty (I + B R^{-1} B^T S_\infty)^{-1} A}$$

Discrete-time ARE (not differential as before)

ex. in scalar case

$$S = q + \frac{a^2 s}{1 + \frac{b^2 s}{r}} = q + \frac{a^2 r s}{r + b^2 s}$$

$(r + b^2 s) s = (r + b^2 s) \cdot q + a^2 r s$ => ! Quadratic equation in S

=> has 2 (or 0) real solutions

but

$S_N \to S_{N-1} \to \dots \to S_\infty$

converges to S_∞ uniquely

? → is $\lim_{k \to -\infty} S_k$ bounded (converging to someting)? sys. (A, B) should be stabilizable

→ is $\lim_{k \to -\infty} S_k$ independent of S_N

auxiliary (fictitious) sys.

(A, r_Q) should be detectable

e.g. unstable A, $Q = 0$

$J = \frac{1}{2} \sum u_k^T R u_k$

$\min J = 0 \Rightarrow u_k = 0 \Rightarrow x_k$ - unstable

→ which solution of DARE to choose?

is the one that we choose stabilizes our closed-loop system

(relevant for $N = \infty$)

more generally

$Q = \begin{bmatrix} q_1 & & \\ & q_2 & \\ & & 0 \\ & & & \ddots \end{bmatrix} \geq 0$

can behave similarly

$\mathbb{R}^n$

Banach Space (n. + compl.)
- complete NVS

w.r.t. norm $\|\cdot\|_X$

e.g. for every Cauchy seq $\{x_n\}$ in X: $\lim_{n\to\infty} x_n = x$

X is Banach if a.o.i.

$$\sum_{n=1}^{\infty} \|v_n\|_X < \infty \Rightarrow \sum_{n=1}^{\infty} v_n \text{ conv in } X$$

Hilbert Space (⟨·⟩ + compl.)
- complete space w inner product

gen. Euclidean space

$\sum_{n=0}^{\infty} x_n$ is abs conv.:

$\sum_{n=0}^{\infty} \|x_n\| < \infty$

series converges in H in the sense that part. sums conv. to an element of H

NVS (norm)

1. $\|x\| \geq 0$, ...
2. $\|\alpha x\| = |\alpha| \, \|x\|$
3. $\|x+y\| \leq \|x\| + \|y\|$

Inner product Space. (⟨·⟩)

V over $\mathbb{R}$ or $\mathbb{C}$
vec sp. field

$\langle \cdot , \cdot \rangle : V \times V \to F$

1. $\langle x, y\rangle = \overline{\langle y, x\rangle}$
2. $\langle ax, y\rangle = a\langle x, y\rangle$
   $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$
3. $\langle x, x\rangle \geq 0$, ...

Metric space (dist.)

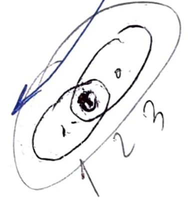
$d: M \times M \to \mathbb{R}$

1. $d(x,y) \geq 0$, ...
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$

locally convex Sp. (seminorm)

$p: V \to \mathbb{R}$

1. $p(x) \geq 0$
2. $p(\lambda x) = |\lambda| \, p(x)$
3. $p(x+y) \leq p(x) + p(y)$



TVS

Topological space (set)

(X, τ):
1. $\emptyset \in \tau$, $X \in \tau$
2. $\forall$ arbitrary $\cup \in \tau$
3. $\forall$ ~~...~~ fin. $\cap \in \tau$

Vector space (LC)
or linear

set V over F/field
op.
1) addition $V \times V \to V$
2) scalar multiplication $F \times V \to V$

that sat. 8 axioms

Manifold (многообразие)
- TS, that locally resembles ES near each point

A {1, 2, 3}

8 SA: {∅} {A} {1} {R}

{∅} {X} {1} {3} {1, 3}

T: {∅} {X}
{1, 2} {1, 2, 3}
{1} {2}

Principles of Optimal Control.

I. Non linear Optimization

Unconstrained non linear optimization.

$F(x)$ - scalar

$x^* = \arg\min_x F(x)$

Minima:

Strong: obj. fun. increases locally in all directions

x^* is a strong min of a function $F(x)$

if $\exists \delta > 0 : F(x^*) < F(x^* + \Delta x)$

for $\forall \Delta x : 0 < \|\Delta x\| \leq \delta$

Weak: obj. fun. remains same in some dir. and inc. loc. in oth.

x^* is a weak min of a function $F(x)$

if $\exists \delta > 0 : F(x^*) \leq F(x^* + \Delta x)$

for $\forall \Delta x : 0 < \|\Delta x\| \leq \delta$

Global: -//- $\delta = \infty$

FOC

If $F(x) \in C^2$

$$F(x + \Delta x) \approx F(x) + \Delta x^T g(x) + \frac{1}{2} \Delta x^T G(x) \Delta x + \dots$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad g = \left(\frac{\partial F}{\partial x}\right)^T = \begin{bmatrix} \partial F / \partial x_1 \\ \vdots \end{bmatrix}, \quad G = \begin{bmatrix} \partial^2 F / \partial x_1^2 & \dots & \partial^2 F / \partial x_1 \partial x_n \\ \vdots & \ddots & \vdots \\ & & \dots \end{bmatrix}$$

FOC: $\|\Delta x\| \ll 1$

- $\Delta x^T g(x)$ - ambig. of sign $\Rightarrow$ can only avoid cost decrease $F(x + \Delta x) < F(x)$ if $g(x^*) = 0$

$g(x^*) = 0$ - a nec. and suf. cond. for a point to be a stat. point

a nec. (not suf.) cond. for a point to be a min $\Rightarrow$? min ? max ? saddle

- if we set $g(x^*)=0$
  then $F(x^* + \Delta x) \approx F(x^*) + \underbrace{\frac{1}{2}\Delta x^T G(x^*) \Delta x}_{>0} + \dots$

  for a strong min for $\forall \Delta x$

  => which is suf. to ensure that

  $$F(x^* + \Delta x) > F(x^*)$$

  if $\Delta x \neq 0$ $G(x^*) > 0$ is suf. cond.
  (PD)
  pos. def mat.

- sec. order nec. cond. for a strong min
  is that $G(x^*) \geq 0$
  (PSD)

  ↳ high order terms in the expansion
  i.e. $\Delta x^T G(x^*) \Delta x = 0$ but 3^{rd} term in the Taylor ser. expansion is pos.

<u>Summary:</u> $g(x^*) = 0$
$G(x^*) > 0$ (suf.) or $G(x^*) \geq 0$ (nec.)

<u>Solution Methods</u>

- iterative algorithm
  - Given: an initial est. of the opt. value of $x \Rightarrow \hat{x}_k$
    and search director p_k
  - Find: $\hat{x}_{k+1} = \hat{x}_k + \alpha_k p_k$ for some scalar $\alpha_k \neq 0$
  - Q: p_k?
    α_k? ← line search
    x_0? ← sens. of ans. to the choice

- search direction:

$$F_{k+1} = F(\hat{x}_k + \alpha p_k) \approx F(\hat{x}_k) + \frac{\partial F}{\partial x}(\hat{x}_{k+1} - \hat{x}_k) =$$

$$= F_k + g_k^T (\alpha_k p_k)$$

assume $\alpha_k > 0$ =>
to ensure that func decreases ($F_{k+1} < F_k$) set $g_k^T p_k < 0$

p_k to sat. that provide descent directions
(steepest descent given by $p_k = -g_k$)

Summary: gradient search methods (first-order methods)
use est. upr. of

$$x_{k+1} = x_k - \alpha_k g_k$$

Line Search

- given a search dir. must decide how far to "step"

$x_{k+1} = x_k + \alpha_k p_k \Rightarrow ?\ \alpha$

α_k : $\min F(x_k + \alpha_k p_k)$

ex.
$$F = x_1^2 + x_1 x_2 + x_2^2$$

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad p_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_1 = \mathbf{x}_0 + \alpha p_0 = \begin{bmatrix} 1 \\ 1 + 2\alpha \end{bmatrix}$$

$$F = 1 + 1 \cdot (1+2\alpha) + (1 + 2\alpha)^2$$

$$\frac{\partial F}{\partial \alpha} = 2 + 2 \cdot (1+2\alpha) \cdot 2 = 0 \quad \Rightarrow \alpha^* = -3/4$$

$$\mathbf{x}_1 = [1 \quad -1/2]^T$$

1

Stochastic Optimal Control Problem

$T>0$

State Equation:

$$\begin{cases} dX(s) = b(s, X(s), a(s))\, ds + \sigma(s, X(s), a(s))\, dW(s) \ ; \ s \in [t,T) \\ X(t) = x \end{cases}$$

Assumptions: (Lipschits)

$b : [0,T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$

$\sigma : [0,T] \times \mathbb{R}^n \times \Lambda \to \mathbb{R}^{n \times m}$

b, σ - continuous

$b(\cdot,\cdot,a), \sigma(\cdot,\cdot,a) \leftarrow$ unif. cont. on bounded subsets of $[0;T] \times \mathbb{R}^n$ unif for $a \in \Lambda$

$|b(s,x,a) - b(s,y,a)| \leq C|x-y|$, $\forall s \in [0;T]; x,y \in \mathbb{R}^n, a \in \Lambda$

$\|\sigma(s,x,a) - \sigma(s,y,a)\| \leq C|x-y|$, -//-

$|b(s,x,a)| + \|\sigma(s,x,a)\| \leq C(1+|x|)$, -//-

Λ - complete separ. metric space (Polish)

W - std. Br. motion in $\mathbb{R}^n$

$a(\cdot)$ - control process

Cost functional:

$$J(t,x,a(\cdot)) = \mathbb{E}\left\{ \int_t^T e^{-\int_t^s c(X(r))\,dr} h(s, X(s), a(s))\, ds + e^{-\int_t^T c(X(r))\,dr} g(X(T)) \right\}$$

discounting; running cost; terminal cost

$c \geq 0$; c, h, g - cont. functions

Generalized Reference Probability Space:

$\mu = (\Omega, \mathcal{F}, \mathcal{F}_s^t, \mathbb{P}, W)$

Ω, $\mathcal{F}$: σ-field of meas. sp.; $\mathcal{F}_s^t$: filtrations

$(\Omega, \mathbb{P}, W)$ - compl. prob. space

$\mathcal{F}_s^t$ - right cont. compl. filtr.

↳ filt: $\mathcal{F}_{s_1}^t < \mathcal{F}_{s_2}^t$, $s_1 < s_2$

↳ compl.: $\forall \mathcal{F}_s^t$ cont. all $\mathbb{P}$null sets of $\mathcal{F}$

↳ right cont.: $\mathcal{F}_s^t = \bigcap_{r>s} \mathcal{F}_r^t$

Def.: BM: $W(t_2) - W(t_1)$ is indep. of $\mathcal{F}^t_{t_1}$; $t_2 > t_1$

$\mathcal{L}(W(t_2) - W(t_1)) = N_{0,(t_2-t_1)I}$, $t_2 > t_1$

W has cont. trajectories $\mathbb{P}$ a.s.

Nat. filtration: $\mathcal{F}^{t,0}_s = \sigma(W(r) : t \le r \le s)$

$\mathcal{F}^t_s$ = augmentation of $\mathcal{F}^t_s$ (by $\mathbb{P}$-null sets)

$= \sigma(\mathcal{F}^{t,0}_s, \underset{\substack{\uparrow \\ \mathbb{P}\text{-null sets}}}{N})$

If $\mathcal{F}^t_s$ is the nat. filtr. gen. by W and $W(t) = 0$ then GRPS μ is RPS

Opt. cont. problem:

- (Strong form.):

Fix GRPS μ

Admissible controls: $\mathcal{U}^\mu_t = \{ a(\cdot)\ [0,T] \times \Omega \to \Lambda :$ s.t.: $a(\cdot)$ is $\mathcal{F}^t_s$ - progressively measur.$\}$

Def: $a(\cdot)$ is progr. meas. if $\forall s \ge t$ $a(\cdot) : [t,s] \times \Omega \to \Lambda$ is $\mathcal{B}([t,s]) \times \mathcal{F}^t_s / \mathcal{B}(\Lambda)$ - measurable

Goal: Min $J(t,x,a(\cdot))$ over all $a(\cdot) \in \mathcal{U}^\mu_t$

- (Weak form.):

Admissible controls: $\mathcal{U}_t = \bigcup_\mu \mathcal{U}^\mu_t$ (union over all GRPS μ)

Goal: Min $J(t,x,a(\cdot))$ over all $a(\cdot) \in \mathcal{U}_t$

Norm

- absolute-value norm $\quad \|x\| = |x|$
- Euclidean norm $\quad \|x\|_2 := \sqrt{x_1^2 + \ldots + x_n^2}$

  $\|x\| := \sqrt{x^T x}$
- Manhattan norm $\quad \|x\|_1 := \sum |x_i|$
- p-norm $\quad \|x\|_p := \left(\sum |x_i|^p\right)^{1/p}$

$$\frac{\partial}{\partial x_n} \|\mathbf{x}\|_p = \frac{x_n |x_n|^{p-2}}{\|x\|_p^{p-1}}$$

$$\frac{\partial}{\partial X} \|X\|_p = \frac{X \circ |X|^{p-2}}{\|X\|_p^{p-1}}$$

$$\frac{\partial}{\partial x_n} \|X\|_2 = \frac{x_n}{\|X\|_2}$$

$$\frac{\partial}{\partial X} \|X\|_2 = \frac{X}{\|X\|_2}$$

Infinite-dimensional case:

$$\|x\|_p = \left(\sum_{i \in N} |x_i|^p\right)^{1/p}$$

$$\|f\|_{p,X} = \left(\int_X |f(x)|^p \, dx\right)^{1/p}$$

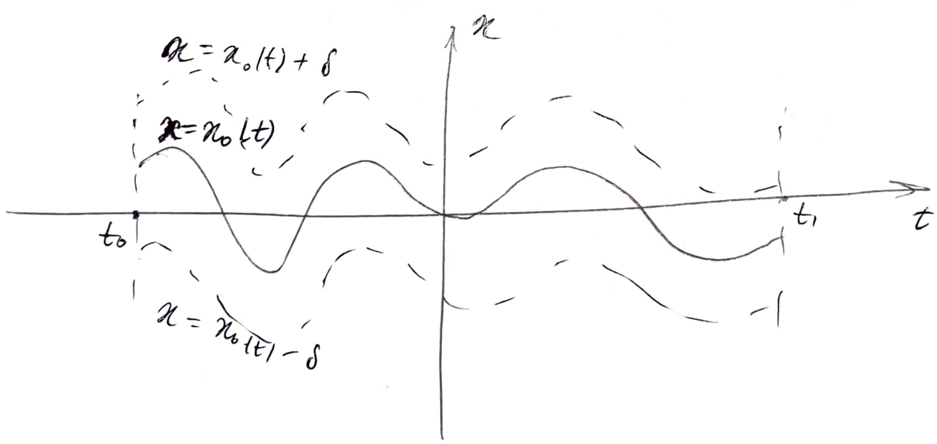
$$\boxed{\|x(t)\|_{2,x[t_0;t_1]} = \left(\int_{t_0}^{t_1} |x(t)|^2 \, dx\right)^{1/2}}$$

В норм. пр-ве для опр. окр-ти вводим метрику

- $\rho(x,y) = \|x - y\| \Rightarrow O_\delta(x_0) \subset X \doteq \{\forall x \in X : \rho(x, x_0) < \delta\}$

  $\Rightarrow$

  $\boxed{d = \|x_2(t) - x_1(t)\|}$

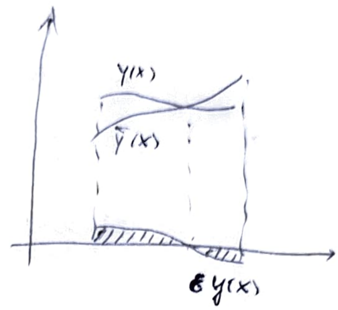


$$O_\delta(x_0(t)) = \{x(t) : \max_{t \in T} |x(t) - x_0(t)| < \delta\}$$

$$(t \in \underbrace{[t_0; t_1]}_{T})$$

Functional Derivative

$\delta y(x) = y(x) - \bar{y}(x)$



$J[f] = \int_a^b L(x, f(x), f'(x))\,dx \quad ; \quad L(x, f + \delta f, f' + \delta f'$

$$\delta J = \int_a^b \underbrace{\frac{\delta J}{\delta f(x)}}\, \delta f(x)\, dx$$

$$= \frac{\partial L}{\partial f} - \frac{d}{dx} \cdot \frac{\partial L}{\partial f'} \quad \leftarrow \quad \frac{\delta J}{\delta f(x)} \text{ funct. derivative w.r.t. } f \text{ at } x$$

$$\boxed{\frac{\delta F}{\delta \rho(z)} = \frac{\partial f}{\partial \rho} - \nabla \cdot \frac{\partial f}{\partial \nabla \rho}}$$

$$\delta F(\rho, \phi) = \left\langle \frac{\delta F(\rho(z))}{\delta \rho(z)}, \phi(x) \right\rangle$$

$$= \int \frac{\delta F}{\delta \rho}(x) \cdot \phi(x)\, dx \qquad \text{variation of } \rho$$

$$= \lim_{\varepsilon \to 0} \frac{F[\rho + \varepsilon \phi] - F[\rho]}{\varepsilon}$$

$$= \frac{d}{d\varepsilon} F[\rho + \varepsilon \phi] \Big|_{\varepsilon = 0}$$

$\frac{\delta F}{\delta \rho}$ - functional derivative

$\delta F(\rho, \phi)$ - functional differential (variation, first var.)

" $\phi = \delta \rho$ ϕ is change in ρ $\Rightarrow$ sim as calc. "

functional

$J : D \subset X \to R$

domain of J

adm. variation $h \in A$

$h \in X$:

class of adm. var.

$x + \varepsilon h \in D \quad \forall \varepsilon > 0$

Gâteaux variation (1st)

$\delta J(X, h)$, $x \in D$

(der. on LC-TVS)

Frechet curvative (der. on Banach sp.)

$A: V \to W$

metric spaces

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - Ah \|_W}{\| h \|_V} = 0$$

funct.

sc.

$$\delta J(f, h) = \lim_{\varepsilon \to 0} \frac{J(f + \varepsilon h) - J(f)}{\varepsilon}$$

extremal

$x \in D$ of J

→ слабое решение ур-е Эйлера
$\delta J(x, h) = 0$, $\forall h \in A$

functional differential (1st)

$$\delta J(f, h) = \int \frac{\partial J}{\partial f}(x)\, h(x)\, dx$$

function

$f : R^n \to R$

direction vector

$h \in R^n$

directional derivative

$D_h f(x)$;

($D_h f(x)$; $f_h'(x)$; $\frac{\partial f(x)}{\partial t}$; $h \cdot \nabla f(x)$)

curve, tan vec of which at some point is h

- rate of change of a fun moving through x with velocity h

$$\nabla_h f(x) = \lim_{h \to 0} \frac{f(x + hv) - f(x)}{h}$$

critical point

$x \in R^n$ of f

dif. of a fun

$$dF = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} dp_i$$

$F(p_1, \ldots, p_n)$