A Continuous-Time Model for the Securities Market

Self-Financing Portfolio and No-Arbitrage

Suppose that the set of time epochs consists of the closed interval $[0, T]$, where $T < \infty$ and the time $t = 0$ denotes the current time. We consider a financial market in which there are available $n + 1$ financial securities, the one numbered 0 being a risk-free security (for example, default- free money-market account) and the others numbered $1, 2, ..., n$ being risky securities (for example, stocks), where $n \geq 1$. Let $S_i(t)$ denote the time-t price of security i, where the current prices $S_i(0)$ are known by all the investors in the market. The security price process denoted by $\{S(t); 0 \leq t \leq T\}$ (or $\{S(t)\}$ for short), where

$$S(t) = (S_0(t), S_1(t), ..., S_n(t))', 0 \leq t \leq T$$

is a vector-valued stochastic process in continuous time. $\theta_i(t)$ denotes the number of security i possessed at time t, $0 \leq t \leq T$, and $\theta(t) = (\theta_0(t), \theta_1(t), ..., \theta_n(t))'$ is the portfolio at that time. Also, $d_i(t)$ denotes the dividend rate paid at time t by security i, while the cumulative dividend paid by security i until time t is denoted by $D_i(t) = \int_0^t d_i(s)ds$.

Throughout this chapter, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with filtration $\{\mathcal{F}_t; 0 \leq t \leq T\}$, where $\mathcal{F}_t$ denotes the information about security prices available in the market at time t. For example, $\mathcal{F}_t$ is the smallest σ-field generated from $\{S(u); u \leq t\}$. However, $\mathcal{F}_t$ can include any information as far as the time-t prices of the securities can be known based on the information. That is, the prices $S_i(t)$ are measurable with respect to $\mathcal{F}_t$.

Recall that, when determining the portfolio $\theta(t)$, we cannot use the future information about the price processes. This restriction has been formulated by predictability of the portfolio process in the discrete-time setting. As we know, an adapted process $\{X(t)\}$ is predictable if $X(t)$ is left- continuous in time t. Hence, as for the discrete-time case, we assume that while the price process $\{S(t)\}$ and the dividend processes $\{d_i(t)\}$ in the continuous-time securities market are adapted to the filtration $\{\mathcal{F}_t\}$, the portfolio process $\{\theta(t)\}$ is predictable with respect to $\{\mathcal{F}_t\}$. The value process $\{V(t)\}$ in the continuous-time setting is similar to the one given in the discrete-time case. However, in order to define self-financing portfolios, recall that, in order to derive the desired continuous- time model from the discrete-time counterpart, it is enough to replace the difference by a differential and the sum by an integral.

For a portfolio process $\{\theta(t)\}$, let

$$V(t) = \sum_{i=0}^{n} \theta_i(t)S_i(t), 0 \leq t \leq T. \ (14.1)$$

The process $\{V(t)\}$ is called the value process. Note the difference between (14.1) and (6.2). In particular, in the continuous-time case, the dividend rates do not appear in the portfolio value (14.1).

Now, note that Equation (6.6) can be rewritten as

$$dV(t) = \sum_{i=0}^{n} \theta_i(t)\{dS_i(t) + d_i(t)dt\}, 0 \leq t \leq T. \ (14.2)$$

As in Section 6.2, let us define the time-t gain obtained from security i by

$$G_i(t) = S_i(t) + D_i(t), i = 0, 1, \ldots, n,$$

or, in the differential form,

$$dG_i(t) = dS_i(t) + d_i(t)dt,$$

Definition

A portfolio process $\{\theta(t)\}$ is said to be self-financing if the time-t portfolio value V(t) is represented by $V(t) = V(0) + \sum_{i=0}^{n} \int_0^t \theta_i(s)\mathrm{d}S_i(s), \quad 0 \leqslant t \leqslant T.$

A contingent claim X is said to be attainable if there exists some self-financing trading strategy $\{\theta(t); 0 \leqslant t \leqslant T\}$, called the replicating portfolio, such that $V(T) = X$. Hence, the attainable claim X is represented as

$X = V(0) + \sum_{i=0}^{n} \int_0^T \theta_i(t)\mathrm{d}G_i(t) \quad (*)$

for some self-financing portfolio process $\{\theta(t)\}$. In this case, the portfolio process is said to generate the contingent claim X.

The definition of arbitrage opportunities is unchanged in the continuous-time setting. That is, an arbitrage opportunity (a risk-free way of making profit) is the existence of some self-financing trading strategy $\{\theta(t)\}$ such that (a) $V(0) = 0$, and (b) $V(T) \geqslant 0$ almost surely (a.s.) and $V(T) > 0$ with positive probability. For the securities market model to be sensible from the economic standpoint, there cannot exist any arbitrage opportunities.

The no-arbitrage pricing theorem is also unchanged in the continuous-time case. That is, for a given contingent claim X, suppose that there exists a replicating trading strategy $\{\theta(t)\}$ given by $(*)$. If there are no arbitrage opportunities in the market, V(0) is the correct value of the contingent claim X.