Lecture 1
Risk Theory

VaR $\quad VaR_\lambda(X_T), \quad \lambda \in (0,1)$

VaR – the λ quantile of the law of X_T with opposite sign

$VaR_\lambda(X_T) = -q_\lambda(X_T)$

$q_\lambda(X_T) = \inf(x : P(X_T \leq x) \geq \lambda)$

i.e. $F_{X_T}(q_\lambda) = \int_{-\infty}^{q_\lambda} p_{X_T}(z)\,dz \approx \lambda$

If $X_T \geq 0$ then $q_\lambda \geq 0$ and $VaR_\lambda < 0$ (there is no risk)

If $F_T(a) = P(X_T \leq a)$ the d.f. of X_T is cont. then $q_\lambda(x)$ is the solution to the equation $F_T(x) = \lambda$

Hermite – Ito – Wick polynomials

$N(\vec{0}, B) \quad B = \begin{bmatrix} 1 & & \rho_{ij} \\ & \ddots & \\ \rho_{ij} & & 1 \end{bmatrix} \quad \rho_{ij} = Cov(X_i, X_j)$

$|\rho_{ij}| \leq 1$

if $\sum_{i;j\neq i} |\rho_{ij}| < 1$

then $\det B > 0 \qquad B = Cov(X)$

Lemma 1 $\quad$ If $\sum_{j:j\neq i} |\rho_{ij}| < \infty$

and $D = \max_i \sum_{j\neq i} |\rho_{ij}|$

$\|B - I\|_\infty < D$

$\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \quad \|\vec{a}\|_\infty = \max_i |a_i|$

$$\|\underbrace{(B-I)}_{C}\,\bar{a}\|_\infty \leq D\|\bar{a}\|_\infty \quad \Rightarrow \quad \|C\|_\infty = D$$

$$\|(C a)_i\| = |\sum_j c_{ij} a_j| \leq \sum_j |c_{ij}||a_j| \leq \|a\|_\infty \sum_{j:\, j\neq i} |c_{ij}| \leq D\|a\|_\infty$$

Н $\|B - I\| < 1$

$$B^{-1} = (I + (B-I))^{-1} = I + (B-I) + (B-I)^2 + \dots$$

$$\|B^{-1}\|_\infty \leq 1 + D + D^2 \dots = \frac{1}{1-D}$$

$\bar{X} = (x_1, \dots, x_d)$ $\quad N(0, \bar{B})$ $\quad$ after normalization

$$B = \begin{bmatrix} 1 & & \rho_{ij} \\ & \ddots & \\ \rho_{ij} & & 1 \end{bmatrix}$$

$d \approx 30-40$
размерность
вектора $\tilde{X}$

$$D = \max_i \sum_{j:\, j\neq i} \underbrace{|\rho_{ij}|}_{<1} \Rightarrow \det B > 0 \ ; \|B - I\|_\infty = D < 1$$

$$p_B(\bar{X}) = \frac{e^{-\frac{(B^{-1}\bar{x}, \bar{x})}{2}}}{(2\pi)^{d/2}\sqrt{\det B}}$$

$$\varphi_B(\bar{t}) = E\, e^{(\bar{t}, \bar{x})} = e^{-\frac{(B\bar{t}, \bar{t})}{2}}$$

$X \sim N(0,1) \quad d = 1$

$y = \varphi(X), E y = 0, \ E y^2 < \infty$

$(y_1, y_2) = E\, y_1 y_2$

$$E\,\varphi(x) = \int_{-\infty}^{+\infty} \varphi(x)\, p(x)\, dx$$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$L^2(\mathbb{R}, p(x)) = \{\varphi(x) : \int_{-\infty}^{\infty} \varphi\, p\, dx = 0, \ \int_{-\infty}^{\infty} \varphi^2 p\, dx = \|\varphi\|_2^2$$

$1, x, x^2 \dots x^n$

$$\int_{\mathbb{R}} x^{2k-1} p(x)\, dx = 0 \qquad \int_{\mathbb{R}} x^{2k} p(x)\, dx = 1 \cdot 3 \cdots (2k-1) = (2k-1)!!$$

Hermit's polynomials (gaussian)

$H_n(x)$, $n = 0, 1, \dots$

$\int_{\mathbb{R}} H_n(x) H_m(x) p(x)\, dx = 0 \quad m \neq n$

normalisation

$\int H_n^2 p(x) dx = 1 \quad \{H_n\}$ orthonom. basis

$H_n(x) = x^n + \dots$

Находим с пом. ортогонализации

$H_n(x)$, $n = 0, 1 \dots$

$1 \quad x \quad x^2 \quad x^3 \quad \in \quad L^2(\mathbb{R}, p(x))$

$1 \perp x \perp x^2 - c \perp x^3 - c_1 x$

$\boxed{x^2 - 1} \quad \boxed{x^3 - 3x}$

$\int (x^2 - c)\, p(x)\, dx = 0$

Th 1 $\quad e^{ax - a^2/2} = 1 + (ax - a^2/2) + \frac{1}{2!}(ax - \frac{a^2}{2})^2 + \dots$

$= \sum_{n=0}^{\infty} \frac{a^n}{n!} H_n(x)$

$H_n(x) = x^n + \dots$

$(H_n, H_n) = \int_{\mathbb{R}} H_n H_n\, p(x)\, dx = \delta_{n,m} \cdot n!$

$\int_{\mathbb{R}} (H_n/\sqrt{n!})^2 p(x)\, dx = 1$

$E\left(\sum_{n=0}^{\infty} \frac{a^n}{n!} H_n(x) \sum_{m=0}^{\infty} \frac{b^m}{m!} H_m(x)\right) = e^{ab}$

$E\, H_n(x) H_m(x) = 0, \; m \neq n; \quad E\, H_n^2(x) = n!$

ортогональны

$\Rightarrow$ ортонорм. система

Th2

a) $H_{n+1}(x) = x \cdot H_n(x) - n H_{n-1}(x)$

b) $H_n'(x) = n H_{n-1}(x)$

$\triangleright$ $e^{ax - a^2/2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} H_n(x)$

(a) dif. H over a

(b) dif H over x

$L^2(\mathbb{R}, p(x))$

$\varphi(x), x \in \mathbb{R}$ $\qquad \int_{\mathbb{R}} \varphi\, p\, dx = 0 \qquad \int \varphi^2 p\, dx = \|\varphi\|_p^2 < \infty$

$H_n(X) = :X^n:$ $\qquad$ nth ~~[illegible]~~ Wick's degree

$d/dx :X^n: = n :X^{n-1}:$ $\qquad :X^0: = 1$

$d^n/dx^n :X^n: = n!$ $\qquad :X^1: = x$

$:X^2: = x^2 - 1$

$E :X^n::X^m: = \delta_{mn} n!$

С гаусс. случ. величина + диагр. Фейнмана

Lection 2

$L^2(\Omega, P_B)$

$L^2(\Omega, P_I)$

2 Hilb. space ① все элементы эквивалентны are equiv. iff ② $\|Y\|_B \asymp \|Y\|_I$

грубо эквивалентны

$C^- \|Y\|_I \le \|Y\|_B \le C^+ \|Y\|_I$ (отношение двух норм закл. между 2 конст)

$B \neq I \Rightarrow L^2_B \not\sim L^2_I$

$L^2(\Omega, P_B)$

$Y = \varphi(\bar{X}) \qquad \bar{X} \sim N(0, B)$

$E(Y) = 0 \qquad E(Y^2) = \|Y\|_B^2 < \infty$

$|\varphi| \le c$

$$\varphi(x_1, \dots x_d) = \sum \overbrace{\left(\frac{:x_1^{k_1}:}{\sqrt{k_1!}} \cdots \frac{:x_d^{k_d}:}{\sqrt{k_d!}} \right)}^{\cap\, \bar{L}_1^2(\Omega, P_B)} c_{k_1} \dots c_{k_d}$$

разложение по ортонорм. базису

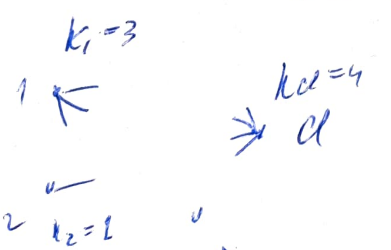
Всякую функцию можно разл. по пол. Эрм.

$$E\, :x_1^{k_1}: \, :x_2^{k_2}: \dots :x_d^{k_d}: = \begin{cases} 0, & k_1 + \dots + k_d = 1 \pmod 2 \\ \sum\limits_{G_\Gamma} \left(\prod\limits_{i,j \in G_\Gamma} \rho_{ij} \right), & \text{else} \end{cases}$$

фейнм. диагр.

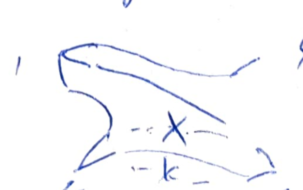
$X_1 \sim N(0,1)$

$$e^{aX_1 - a^2/2} = \sum_{n=0}^{\infty} \frac{a^n}{n!} :X_1^n:$$



каждый из полиномов

Eg:



т.к. отдел. петли

$G_\Gamma = \rho_{14}\rho_{13}\rho_{12}\rho_{23}$

$k_1 = 3$

$k_2 = 2$

$k_3 = 2$

$k_4 = 1$

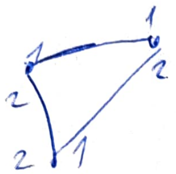
Eg. 1. $E :X_1^{k_1}: :X_2^{k_2}: = \begin{cases} 0 & k_1 \neq k_2 \\ k_1!\rho^{k_1} & k_1 = k_2 \end{cases}$

$B = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

$\Leftarrow \quad \rightarrow$

$E :X_1^{k_1}: :X_2^{k_2}: = \delta_{k_1 k_2} k_1! \rho^{k_1}$

Eg. 2. $:X_1^2: :X_2^2: :X_3^2: = 8\rho_{12}\rho_{23}\rho_{13}$



$e^{\rho_{12}a_1a_2 + \rho_{13}a_1a_3 + \rho_{23}a_2a_3}$

$= \dots \bigcirc \frac{a_1^2}{2!} \frac{a_2^2}{2!} \frac{a_3^2}{2!}$

$\dots \frac{(x+y+z)^3}{3!} = \frac{1}{3!} x! y! z! \frac{3!}{1!1!1!} \dots$

$\dots e^{\frac{(\rho_{12}a_1a_2 \dots)^3}{3!}} \dots a_1^2 a_2^2 a_3^2 \cdot 1$

Th. 9

$\left| E :X_1^{k_1}: \dots :X_d^{k_d}: \right| = \left| \sum_{G} \left(\prod_{(i,i')} \rho_{ii'} \right) \right| \le \rho^{N/2} \sqrt{k_1!} \dots \sqrt{k_d!}$

If $\rho = \max_i \sum_{j: i \neq j} |\rho_{ij}| < 1$

$E \frac{:X_1^{k_1}:}{\sqrt{k_1!}} \dots \frac{:X_d^{k_d}:}{\sqrt{k_d!}} \le \rho^{\frac{k_1 \dots k_d}{2}}$

$d = 2 \quad \rho = Cov(X_1, X_2)$

$E \frac{:X_1^{k_1}:}{\sqrt{k_1!}} \frac{:X_2^{k_2}:}{\sqrt{k_2!}} = \delta_{k_1 k_2} \rho^{k_1}$

④ $S = S_1 + S_2$, где $S_1 = pS$, $S_2 = qS$, $p + q = 1$

A) $S_1(1 + 0{,}04 + 0{,}02\xi_1)$

B) $S_2^{(1)} = \begin{cases} -1 & \text{w.p. } 1-10^{-4} \\ 1200 & \text{w.p. } 10^{-4} \end{cases}$

можно аппроксимировать

$\tilde{S}_2 = S_B(1 + a_2 + b_2 \cdot \xi_2)$

где $a_2 = 1200 \cdot \frac{1}{10^4} - 1 \cdot \left(1 - \frac{1}{10^4}\right) \approx -0{,}88$

$b_2 = \sqrt{1200^2 \cdot \frac{1}{10^4} + 0{,}999 - 0{,}88^2} \approx 12$

а) поскольку доходность второй стратегии в среднем ниже первой, то, очевидно, что для $\max \tilde{S}$ необходимо выбрать $p=1$, $q=0$

б) проверим вероятность убытка в случае выбора стратегии из п. а)

$P(S(1{,}04 + 0{,}02\xi_1) < S) = P(\xi_1 < -2) =$

$= 0{,}02227 < 0{,}05$

т.е. стратегия оптимальна при таком условии.

Если бы доходность лотереи в среднем выше, чем доходность стратегии A, был бы смысл использовать смешанную стратегию

(I)

$$\begin{cases} F_{n+1} = F_n + F_{n-1} \\ F_0 = 1,\ f_1 = 1 \end{cases}$$

$$F_n = C_1 \lambda_1^n + C_2 \lambda^n$$

$n=0$, $n=1$ $\begin{cases} C_1 + C_2 = 1 \\ C_1\lambda_1 + C_2\lambda_2 = 1 \end{cases}$

$F_n = \lambda^n$ $\qquad \lambda^{n+1} = \lambda^n + \lambda^{n-1}$

$$\lambda^2 - \lambda - 1 = 0$$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

част. решение

(II) Теория перколяций

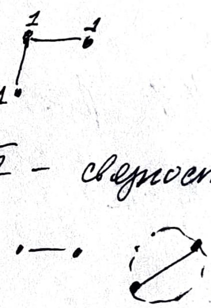
$\mathbb{Z}^2$ $\qquad x \in \mathbb{Z}^2$

$x = (x_1, x_2)$

$\mathcal{E}(x) = \begin{cases} 1 & p \\ 0 & q = 1-p \end{cases}$

1 – связность

$\sqrt{2}$ – связность



$p_{crit} \neq 0{,}33$ $\qquad$ океан 0 vs океан 1

$p_{crit} = 0{,}66$

(III) $Y_{n+1} = \rho Y_n + \sigma \xi_n$, $\quad \xi_n \sim N(0,1)$

$$Y_{n+1} = \rho(\rho Y_{n-1} + \sigma \xi_{n-1}) + \sigma \xi_n = \ldots =$$

$$= \sigma \xi_n + \rho \sigma \xi_{n-1} + \rho^2 \sigma \xi_{n-2} + \ldots$$

$$\mathrm{Var}(Y_{n+1}) = \sigma^2 (1 + \rho + \rho^2 \ldots) = \frac{\sigma}{1-\rho^2}$$

$$E(Y_n, Y_{n-t}) = b(t)$$

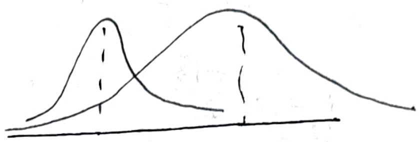
$$P(X_n > h, X_{n-1} > h, \ldots, X_{n-k+1} > h\} \leq P\left\{\sum_{n-k+1}^{n} X_i > kh\right\}$$

(IV) $\boxed{\text{Дз 2 \#1}}$

$$\xi_A^{(1)} = \xi_A^{(0)} (1 + (0{,}04 + 0{,}01\, \xi_1))$$

$$\xi_B^{(1)} = \xi_B^{(0)} (1 + (0{,}06 + 0{,}8\, \xi_2))$$

$\xi_1, \xi_2 \sim N(0,1)$



Критерий? $A-(p)$, $B-(1-p)$

$P(X > 10^4) = 0{,}95$

если $p = 1 \Rightarrow P(\xi_A^{(1)} > 10^4) > 0{,}95$

...

(V) $\boxed{\text{Дз 2 \#2}}$

$X_1, \dots, X_n$ н.ф. $f(x)$, $x \geq 0$

$\{\tau = k\} = \sigma(X_1, \dots, X_2)$

(Moser)

$EX_\tau = \max$

$P(X_\tau = M_n) = \max$

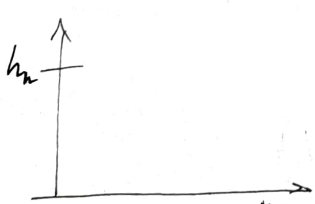
$M_n = \max(X_1, \dots, X_n)$

$X_1 \dots X_n$, $EX_1 < \infty$

$C_n = \max\limits_{\tau} EX_\tau$, $C_n = F(C_{n-1})$?

$C_1 = EX_1 = a$ $\quad C_2 = F(a)$ $\quad C_3 = F(F(a))$

$\rightarrow p(x), x \geq 0$



- $X_1 > h_n$ stop $\tau = 1$

$$\left(\frac{p(x)}{\int_{h_n}^{\infty} p(x)dx} \cdot I_{[h_n, \infty)}(x)\right) \quad ; \int_{h_n}^{\infty} p(x)dx < 1$$

$p_{h_n}(x)$

$$E[X_1 \mid X_1 > h_n] = \int_{h_n}^{\infty} x\, p_{h_n}(x)\, dx = \int_{h_n}^{\infty} x p(x) dx \Big/ \int_{h_n}^{\infty} p(x) dx$$

IX $\boxed{\text{Дз 2 \#4}}$

$$\bar{X}_{n+1} = A\bar{X}_n$$

$$\bar{X}_n = \lambda^n \vec{\xi}$$

$$(A - \lambda I)\xi = 0$$

хар-е ур-е

Thm. λ_{max} всегда $\exists (>0)$

X $\boxed{\text{Дз 2 \#1}}$ $S = 10^4$

x_1, x_2, x_3 ; $x_1 + x_2 + x_3 = N$ $\quad p_1 + p_2 + p_3 = 1$

$\tilde{X}_1 = x_1(1 + \overbrace{a_1 + b_1\xi_1}^{v_1})$ $\quad \tilde{X}_2 = (X_2(1 + a_2 + b_2\xi_2))$ $\quad \tilde{X}_3 = \_\_$

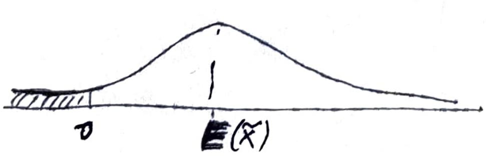
$a_1 = 0{,}04$ $\quad a_2 = 0{,}06$ $\quad a_3 = 0{,}1$

$b_1 = 0{,}01$ $\xi_1 \sim N(0,1)$ $\quad b_2 = 0{,}1$ $\quad b_3 = 0{,}3.$

$$P\{\tilde{X}_1 + \tilde{X}_2 + X_3 < S\}$$

$$\tilde{X}_1 + \tilde{X}_2 + \tilde{X}_3 = X_1(\ldots) + \ldots \sim N$$

$$P\{\underbrace{a_1 \overset{p_1}{\cancel{X_1}} + a_2 \overset{p_2}{\cancel{X_2}} + a_3 \overset{p_3}{\cancel{X_3}}}_{>0} + b_1 \overset{p_2}{\cancel{X_2}}\xi_2 + \ldots < 0\}$$



(IV) (XI) Ветв. процессы (Galton – Watson)

p_0 p_1 p_2 ... p_n

$X_0 = 1$ X_1 X_2 ...

$X_{n+1} = \xi_1 + \ldots + \xi_{X_n}$ ξ_i i.i.d.

$\xi: \ \varphi_\xi(t) = E e^{it\xi} = \int_R e^{itx} p(x)\,dx$ $\boxed{\varphi_\xi(0) = 1}$

$\xi > 0 \quad \psi_\xi(\lambda) = E e^{-\lambda\xi} = \int_0^\infty e^{-\lambda x} p(x)\,dx$

$\xi \in [0 \ldots n] \quad \varphi(z) = E z^\xi = p_0 + p_1 z + \ldots p_n z^n + \ldots$, $|z| \le 1$

(ν) $0 \quad 1 \quad \ldots \quad n \ldots$
$q_0 \quad q_1 \quad \quad q_n$

$F_\nu(z) = E(z^\nu) = q_0 + q_1 z + \ldots + q_s z^s$

(ξ) $0 \quad 1 \quad \ldots$
$p_0 \quad p_1$

$E z^{\xi_i} = \Phi_\xi(z)$

$S = \xi_1 + \ldots + \xi_\nu$

$E z^S = E z^{\xi_1 + \ldots + \xi_\nu} = F_\nu(\Phi_\xi(z))$

$X_1 < h_n \qquad P(X_1 < h_n) = \int_0^{h_n} p(x)\,dx$

$$C_n = \max_{h_n} \{ P(X_1 > h_n) \cdot E(X_1 \mid X_1 > h_n) + P(X_1 < h_n) \cdot C_{n-1} \}$$

$$E(X_1 \mid X_1 > h_n) = \frac{\int_{h_n}^{\infty} x\, p(x)\,dx}{\int_{h_n}^{\infty} p(x)\,dx}$$

$$= \max_{h_n} \{ \underbrace{\int_{h_n}^{\infty} x\,p(x)\,dx + \int_0^{h_n} p(x)\,dx \cdot C_{n-1}}_{\Downarrow \; f' = 0} \}$$

$$-h_n \cdot \cancel{p(h_n)} + C_{n-1}\, \cancel{p(h_n)} = 0$$

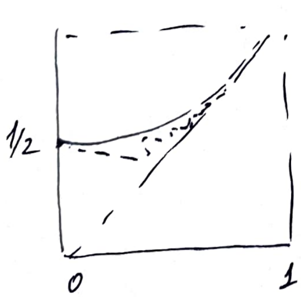
$$\boxed{C_{n-1} = h_n}$$

$$C_n = \int_{C_{n-1}}^{\infty} x\,p(x)\,dx + C_{n-1} \int_0^{C_{n-1}} p(x)\,dx \quad \Rightarrow \quad C_n = F(C_{n-1})$$

- $p(x) = \begin{cases} 1 & x \in (0,1) \\ 0 & x \notin (0,1) \end{cases}$

$$C_n = \frac{1 - C_{n-1}^2}{2} + C_{n-1}^2 = \frac{1 + C_{n-1}^2}{2}$$

$$C_n = f(C_{n-1}); \quad f(x) = \frac{1+x^2}{2}.$$



$$C_1 = 1/2 \qquad C_2 = \frac{1 + (1/2)^2}{2} = \frac{5}{8}$$

$$C_n = 1 - \frac{c_0}{n^\alpha} + \frac{c_1}{n^{\alpha+1}} + \dots$$

$$\boxed{\frac{1}{(n+1)^\alpha} = \frac{1}{n^\alpha}\left(1 + \frac{1}{n}\right)^{-\alpha}}$$

$$= 1 - \frac{c_0}{(n+1)^\alpha} \dots = \frac{1}{2} + \frac{1}{2}\left(1 + \frac{c_0}{n^\alpha} + \dots\right)^2 \Rightarrow C_n = 1 - 2/n + O(1/n^2)$$

$E \, Max(X_1, \dots, X_n) = \widetilde{C}_n$

$P(X_n < x) = F(x) = \int_0^x p(x)\,dx$

$M_n = max(x_1, \dots, x_n)$

$F_{M_n}(a) = P(max(x_1, \dots, x_n) < a) \overset{!!}{=} P(X_1 < a, \dots, X_n < a) =$

$= P(X_1 < a)^n = F^n(a)$

$\underline{E M_n = n \int_0^\infty a \, p(a) F^{n-1}(a) da}$ $\quad p_{M_n}(x) = n F^{n-1}(a) p(a)$

e.g. $X_1 \sim Unif([0,1])$

$F_{M_n}(x) = F^n(a) = a^n$

$p_{M_n}(x) = n a^{n-1}$

$E M_n = \int_0^1 a \, p_{M_n}(a) \, da = n \int_0^1 a^n \, da = \frac{n}{n+1} = 1 - \frac{1}{n+1} =$

$= 1 - \frac{1}{n} + O(1/n^2)$

$E M_n = 1 - \frac{1}{n} + \dots$

$E x_t = 1 - \frac{2}{n} + \dots$

$\frac{1 - E X_t}{1 - E M_n} = 2$