А.Д. (1)

$\alpha = (1 \ 0 \ldots 0)$. $\qquad m$ - active

$\alpha = (\alpha_1 \ \alpha_2 \ \ldots \ \alpha_n)$ $\qquad m+1 - \phi$

$$e^{Qt} = I + Qt + \frac{1}{2!}Q^2t^2 + \ldots$$

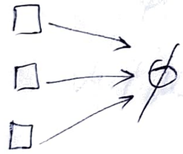
$$\det[e^{Qt}] = e^{tr[Q]\cdot t}$$

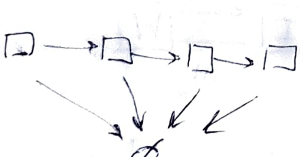
$$Q = \begin{pmatrix} -\mu_{11} & \mu_{12} & \ldots & \\ & & \ddots & \\ 0 & 0 & \ldots & 0 \end{pmatrix} \qquad \sum_j \mu_{ij} = 0 \quad \forall i$$

$$Q = \left(\begin{array}{c|c} T & t \\ \hline 0\,0\,0 & 0 \end{array}\right) \qquad \boxed{t = -T\mathbb{1}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

чтобы замкнулось

1)  Erlang dist

2)  Hypoexponential dist

3)  Cox dist

1)
$$\left(\begin{array}{ccccc|c} -\mu & \mu & \ldots & & 0 & 0 \\ 0 & -\mu & \mu & & & \vdots \\ \vdots & & & & \vdots & \vdots \\ 0 & \ldots & & -\mu & \mu & 0 \\ & & & & -\mu & \mu \\ \hline 0 & & & & 0 & 0 \end{array}\right) \qquad t = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu \end{pmatrix}$$

$$\underset{n\times n}{T} = -\mu I + \mu B, \quad \text{где} \quad B = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}$$

$$e^{Tx} = e^{-\mu I x} \cdot e^{\mu B} \qquad B^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & & \ddots & 1 \\ & & & 0 \\ 0 & & & 0 \end{pmatrix}$$

$$e^{\mu B} = I + \mu B + \frac{\mu^2}{2!} B^2 + \dots \qquad B^n = 0$$

$$e^{\mu B} = \begin{pmatrix} 1 & \mu & \mu^2/2! & \dots & \mu^{n-1}/(n-1)! \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \mu^2/2! \\ & & & \ddots & \mu \\ 0 & & & & 1 \end{pmatrix}$$

$f(x) = \alpha \cdot e^{Tx} \cdot t$ — плотность

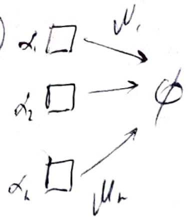
$$(1\ 0 \dots 0) \begin{pmatrix} 1 & & \mu^{n-1}/(n-1)! \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu \end{pmatrix} = e^{-\mu x} \cdot \frac{\mu^{n-1}}{(n-1)!} \mu =$$

$$= \frac{\mu^n}{(n-1)} e^{-\mu x}$$

Erlang dist

$$1 - F(x) = P(\xi > x) = \sum_{i=1}^{n} \alpha_i (e^{Tx})_{ij} = \alpha e^{Tx} \mathbb{1}_{n\times 1}$$

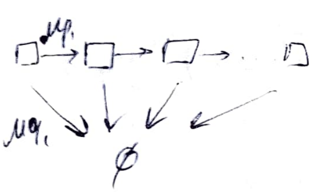
$$S(x) = \alpha e^{Tx} \underset{\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}}{1_{n\times 1}} \qquad f(x) = -S'(x) = -\alpha e^{Tx} \cdot \underbrace{T \cdot 1}_{-t} = \alpha e^{Tx} t$$

2) 

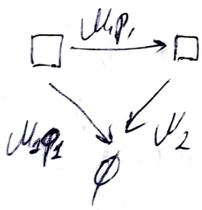
$$f(x) = \sum_{j=1}^{n} \alpha_j \mu_j e^{-\mu_j x}$$

$$Q = \left(\begin{array}{cccc|c} -\mu_1 & 0 & \cdots & & \mu_1 \\ & -\mu_2 & 0 & \cdots & \mu_2 \\ & & \ddots & & \vdots \\ & & & -\mu_n & \mu_n \\ \hline 0 & \cdots & & 0 & 0 \end{array}\right)$$

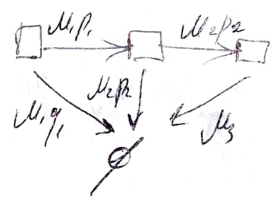
$$f(x) = (\alpha_1 \ldots \alpha_n) \begin{pmatrix} e^{-\mu_1 x} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{-\mu_n x} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$$

3) 

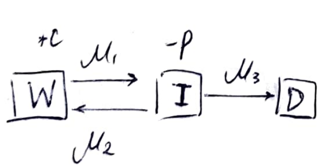
$$Q = \left(\begin{array}{cccc|c} -\mu_1 & \mu_1 p_1 & \cdots & & \mu_1 q_1 \\ & -\mu_2 & \cdots & & \\ & & & & \\ \hline 0 & & & 0 & 0 \end{array}\right) \qquad p_1 + q_1 = 1$$

e.g. 1) 

$\mu_1 \neq \mu_2$

2) 

$\mu_1 \neq \mu_2 \neq \mu_3$

3) 

$$\begin{array}{c} W \\ I \\ D \end{array} \left(\begin{array}{cc|c|c} -\mu_1 & \mu_1 & 0 & \cancel{0} \\ \mu_2 & -\mu_2-\mu_3 & \mu_3 & \\ \hline 0 & \cdots & 0 & \cancel{0} \end{array}\right)$$

$$-c(-T)_{11} + p(-T)_{12} = 0$$

$\uparrow$ шур. прод.

$\delta I - T$

$$f(x) = (\alpha_1 \ldots \alpha_2) \begin{pmatrix} e^{-\mu_1 x} & e^{\mu_1 x} \\ e^{\mu_2 x} & e^{(-\mu_2-\mu_1)x} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu_3 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_1 e^{-\mu_1 x} + \alpha_2 \cdot e^{\mu_2 x} \\ \alpha_1 e^{\mu_1 x} + \alpha_2 \cdot e^{(-\mu_1-\mu_2)x} \end{pmatrix} \begin{pmatrix} 0 \\ \mu_3 \end{pmatrix} =$$

$1 \times 2 \qquad 2 \times 1$

$$= \alpha_1 \mu_3 e^{\mu_1 x} + \alpha_2 e^{(-\mu_1 - \mu_2)x} \cdot \mu_3$$

Laplace transform

$$\int_0^\infty e^{-sx} \alpha e^{Tx} t \, dx = \alpha (sI - T)^{-1} t$$

$$\alpha \int_0^\infty e^{-sIx} e^{Tx} dx \, t = \alpha (sI - T)^{-1} t$$

$$E[\xi] = -(-1) \alpha (sI - T)^{-2} t \Big|_{s=0} =$$

$$= \alpha (-T)^{-2} t = -\alpha (-T)^{-1} \cdot 1$$

$$-Tt = 1$$

$$E[\xi^k] = (-1)^k k$$

$$\alpha (sI - T)^{-k-1} t \Big|_{s=0} = (-1)^{k+1} k! \, \alpha (-T)^{-k} (-T)^{-1} (-T1)$$

$$= (-1)^k k! \, \alpha (-T)^{-k} \mathbb{1}$$

$$(-\alpha - b \quad a \quad b)$$

$$E_{ij} = E_i [\underset{\text{time}}{x_j}] = (-T)^{-1}_{ij}$$

$$p(i \to k) = \frac{T_{ik}}{-T_{ii}}$$

Среднее время нахождения в сост j до поглощения (выйдя из i)

$$\begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ \vdots & \dots & & \end{pmatrix}$$

$$i \neq j \quad E_{ij} = \sum_{k \neq j} \frac{T_{ik}}{-T_{ii}} E_{kj} \iff E(-T)^{-1} = I$$

$$\iff \sum_k E_{ik} E_{kj} = 0 \quad i \neq j$$

$$E_{ii} = -\frac{1}{T_{ii}} + \sum_{k \neq i} \frac{T_{ik}}{-T_{ii}} E_{ki}$$

$$(-T_{ii}) E_{ij} + \sum_{k \neq i} (-T)_{ik} E_{kj} = 0$$

$$-\sum_k E_{ik} T_{ki} = 1$$

$$(-T_{ii}) E_{ii} + \sum_{k \neq i} (-T)_{ik} E_{ki} = 1$$

$$Q = \begin{pmatrix} -3/12 & 9/14 & 6/7 \\ 5/2 & -11/2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \alpha = (1/2 \quad 1/2)$$

$\lambda_1 = -1 \qquad \lambda_2 = -6$

$\lambda_1 + \lambda_2 = tr(Q)$

$\lambda_1 \cdot \lambda_2 = det(Q)$

пр. лев. собств. вектора:

$\pi = a(t_{21}, \lambda_1 - t_{11}) = a(7/2, \ 1/2)$

$v = b\begin{pmatrix} t_{12} \\ \lambda - t_{11} \end{pmatrix} = b\begin{pmatrix} 9/24 \\ 9/2 \end{pmatrix}$

$\langle \pi_1, v \rangle = 1 \quad \Rightarrow \quad ab = 2/5$

$P_1 = \begin{pmatrix} \pi_1 v_1 & \pi_2 v_1 \\ \pi_1 v_2 & \pi_2 v_2 \end{pmatrix}$ — проекторы $\Rightarrow \begin{pmatrix} 9/10 & 9/70 \\ 7/10 & 1/10 \end{pmatrix}$

$P_2 = \begin{pmatrix} \pi_2 v_2 & -\pi_2 v_1 \\ -\pi_1 v_2 & \pi_1 v_1 \end{pmatrix}$

$I = P_1 + P_2$

$P_1^2 = P_2^2 = I$

$P_1 P_2 = P_2 P_1 = 0$

$Q = \lambda_1 P_1 + \lambda_2 P_2$

$f(Q) = f(\lambda_1) \cdot P_1 + f(\lambda_2) \cdot P_2 \qquad T^{-k} = (-1)^k (T_1 + 6^{-k} T_2)$

$E[\xi^k] = (-1)^k k! \, \alpha (-T)^{-k} \mathbb{1}$

$f(x) = \alpha e^{Tx} t \qquad e^{Tx} = e^{-x} \begin{pmatrix} 9/10 & 9/10 \\ 7/10 & 1/10 \end{pmatrix} + e^{-6x}(\ldots)$

разложение степени

$e^{Ax} = I + \underbrace{\lambda_1 T_1 + \lambda_2 T_2}_{A} + \frac{1}{2}(\lambda_1 T_1 + \lambda_2 T_2)^2 + \ldots = e^{\lambda_1 x} T_1 + e^{\lambda_2 x} T_2$

$\underbrace{\lambda_1^2 T_1^2}_{I} + \underbrace{2\lambda_1\lambda_2 T_1 T_2}_{=0} + \lambda_2^2 T_2^2$

* спектральное разл. матрицы ...

$X \sim (\alpha, T_1)$ $\leftarrow n\times n$ — фаз. мера (PH-representation)

$Y \sim (\beta, T_2)$ $\leftarrow m\times m$

$\min[X,Y]$ — $(\alpha \otimes \beta,\ T_1 \oplus T_2)$

$\max[X,Y]$ — $\left((\alpha\otimes\beta, 0, 0),\ \begin{pmatrix} \underset{mn\times mn}{T_1 \oplus T_2} & \underset{mn\times n}{I \oplus t^{(1)}} & \underset{mn\times m}{t^{(2)} \otimes I} \\ 0 & T_1 & 0 \\ 0 & 0 & T_2 \end{pmatrix} \right)$

$(mn+n+m)\times(mn+n+m)$

$\sum_{j=1}^{N} X_j \sim (\alpha \otimes \beta,\ \overset{mn\times mn}{T \otimes I_m + t\alpha \otimes S})$

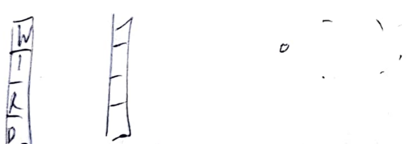
$X \sim (\alpha, T)$ PH

$N \sim (\beta, S)$ (PH) дискр

Tiele equation

H W

о 18 мочек



$\overset{\text{пол.}}{P\ddot{a}_{x:\overline{t}|}} - \overset{\text{дожит.}}{SA^{\ \ 1}_{x:\overline{t}|}} = \overset{\text{что уплачено}}{SA_{x+t:\overline{n-t}|}} - \overset{\text{что получим}}{P\ddot{a}_{x+t:\overline{n-t}|}}$

$\underbrace{{}_tP_x\, v^t}$

actuarial accumulation

акт. накопление

⊕ если $\mu_{21} \neq 0$ (8×8)

до 1 см. 4×4

до 2 см. 8×8

$(2\cdot 2\cdot 2\cdot 2)$

$*$ 5.3.20

Мартингалы

$p = 1/2$ $\qquad\qquad$ $p = 1/2 + 3/1000$

$p(A \text{ win}) = 10/11$ $\qquad \approx 0{,}95$

$p(B \text{ win}) = 1/11$ $\qquad \approx 0{,}05$

$S_n = X_1 + \ldots + X_n$

$p = 1/2 \Rightarrow S_n$ – martingale $\qquad E(S_{n+1} | F_n) = S_n \Rightarrow 10/11, \ 1/11$

$\left(\frac{q}{p}\right)^{S_n}$ – martingale?

↳ осн. по теореме Дуба $\Rightarrow p_1$ и p_2

$E X_\tau = E X_0$

$t B(t) - 1/3 \, B^3(t)$ – martingale?

gamma-process pr.
neg. gauss process
brown. pr.

$E e^{rX} = (1+r) E X$

$\varkappa(R) = 0$

$e^{sX(t)}$ – март. Вальда

$e^{sX(t) - \varkappa(s)t}$ ← всегда мартингал

$e^{RX(t)}$ – martingale $(s = 0)$

$e^{\sigma B(T(t))}$ ← субординатор (обычн. торгов (накопленный))

а не BS

А.Д (2)

$$\begin{tikzcd} \boxed{1} \arrow[r, "p\mu_1"] \arrow[dr, "q\mu_1"'] & \boxed{2} \arrow[d, "\mu_2"] \\ & \varnothing \end{tikzcd}$$

1) $I = T_1 + T_2$ 2) $T_1 T_2 = T_2 T_1 = 0$

3) $T_1^2 = T_1$ $T_2^2 = T_2$

$$T = \begin{pmatrix} -\mu_1 & p\mu_1 \\ 0 & -\mu_2 \end{pmatrix} \begin{pmatrix} q\mu_1 \\ \mu_2 \end{pmatrix}$$

$$T = -\mu_2 \begin{pmatrix} 0 & p\mu_1/(\mu_1 - \mu_2) \\ 0 & 1 \end{pmatrix} - \mu_1 \begin{pmatrix} 1 & -p\mu_1/(\mu_1 - \mu_2) \\ 0 & 0 \end{pmatrix}$$

↑ разложение единицы

$$\pi = a(t_{21}, \dots) = a(0, \mu_1 - \mu_2) \qquad ab = \frac{1}{(\mu_2 - \mu_1)^2}$$

$$v = b\begin{pmatrix} t_{12} \\ -\lambda_2 t_{11} \end{pmatrix} = b\begin{pmatrix} p\mu_1 \\ \mu_1 - \mu_2 \end{pmatrix}$$

$$T_1 = \begin{pmatrix} \pi_1 v_1 & \pi_2 v_1 \\ \pi_1 v_2 & \pi_2 v_2 \end{pmatrix}$$

$$T_2 = \begin{pmatrix} \pi_2 v_2 & -\pi_2 v_1 \\ -\pi_1 v_2 & \pi_1 v_1 \end{pmatrix}$$

$$e^{Tx} = e^{\overset{-\mu_2}{\overset{\parallel}{\lambda_1}} x} \cdot \underbrace{T_1}_{\parallel} + e^{\overset{-\mu_1}{\overset{\parallel}{\lambda_2}} x} \cdot \underbrace{T_2}_{\parallel} = \dots$$

$$\begin{pmatrix} 0 & p\mu_1/(\mu_1 - \mu_2) \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -p\mu_1/(\mu_1 - \mu_2) \\ 0 & 0 \end{pmatrix}$$

$$f(x) = (1\ 0)\begin{pmatrix} e^{-\mu_1 x} & (e^{-\mu_2 x} - e^{-\mu_1 x}) \frac{p\mu_1}{\mu_1 - \mu_2} \\ 0 & e^{-\mu_2 x} \end{pmatrix} \begin{pmatrix} q\mu_1 \\ \mu_2 \end{pmatrix}$$

$$f(x) = e^{-\mu_1 x} q\mu_1 + \frac{p\mu_1 \mu_2}{\mu_2 - \mu_1}(e^{-\mu_1 x} - e^{-\mu_2 x})$$

(★) $\quad f(x) = -p\mu e^{-\mu x}(1 + \mu x) \quad$ (при $\mu_1 = \mu_2$)

пусть $\mu_1 = \mu_2 = \mu$

$$e^{-\mu x} q\mu + \mu_2 \frac{p\mu}{\underbrace{\tilde{\mu} - \mu}_{-\Delta\mu}} \left(e^{-(\mu - \overbrace{\tilde{\mu}}^{\Delta\mu})x} - 1\right) e^{-\mu x}$$

$$\mu e^{-\mu x}\left(1 - \frac{e^{-\Delta\mu x} - 1}{-\Delta\mu}\right) = \underbrace{\mu e^{-\mu x}}_{0/0} (q + p\mu x)$$

по лопиталю

$$\exp\begin{pmatrix} -\mu & p\mu \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} e^{-\mu x} & e^{-\mu x}(a+bx) \\ 0 & e^{-\mu x} \end{pmatrix} \Leftarrow p\mu x \qquad \begin{pmatrix} -\mu & -a\mu + b \\ 0 & -\mu \end{pmatrix} \Downarrow$$

жорд. клетка

$$\begin{pmatrix} -\mu & p\mu \\ 0 & -\mu \end{pmatrix}\begin{pmatrix} -\mu & p\mu \\ 0 & -\mu \end{pmatrix} = \left(\begin{array}{c|c} & * \; -2p\mu^2 \\ \hline & \end{array} \right)$$

$$\left(e^{-\mu x}(a+bx)\right)'' = \left(-a\mu e^{-\mu x} + b e^{-\mu x} - b\mu x e^{-\mu x}\right)' \Big|_{x=0}$$

$$= a\mu^2 - 2b\mu \quad \Rightarrow \quad \begin{cases} a\mu^2 - 2b\mu = -2p\mu^2 \\ -a\mu + b = p\mu \end{cases}$$

$$\Downarrow \quad \begin{matrix} a = 0 \\ b = p\mu \end{matrix}$$

$$f(x) = (1\ 0)\begin{pmatrix} e^{-\mu x} & p\mu x e^{-\mu x} \\ 0 & e^{-\mu x} \end{pmatrix}\begin{pmatrix} q\mu \\ \mu \end{pmatrix} = q\mu e^{-\mu x} + p\mu^2 x e^{-\mu x}$$

CLT для порядк. статистик

$T_1, \dots, T_n$ $\qquad S(x)$

$n = 100, 000$

$T_{\lfloor \alpha n \rfloor}$

$\alpha = 0,9$ (90% умерло)

$T_{1;n} < T_{2;n} < \dots < T_{n;n}$

1ый умер

Thm: $\left(T_\alpha - \underset{\text{квантиль}}{G(\alpha)}\right) \underbrace{\sqrt{\frac{n}{\alpha(1-\alpha)}}}_{\text{поправка}} f(G(\alpha)) \sim N(0,1)$, $n \to \infty$

Ex. $n = 100\,000$ $\quad \mu = 0,005$
$\alpha = 0,99$ $\quad b = 0,05$

с $p = 0,99$ $\quad$ 99% в группе умрут

$$S(x) = e^{-\int_0^x \mu(u)du} = e^{-(0,1e^{0,05x} - 0,1)} = 0,01$$

$$\mu(x) = 0,005 \cdot e^{0,05x} \qquad \Rightarrow x = 77,025$$

$$G(0,99) = 77,025$$

$$\xi = (x - G(\alpha)) \sqrt{\frac{n}{\alpha(1-\alpha)}} \overset{S'(x)}{f}(G(\alpha)) = 2,33$$

$\Rightarrow x = 77,33$ $\square$

$X_1, \dots, X_n$ – i.i.d. $F(x)$

$X_{1:n} \leq \dots \leq X_{n:n}$

$F_{k:n}(x) = P(X_{k:n} \leq x)$

$f_{k:n}(x) = C_n^{k-1} \overset{<}{F(x)^{k-1}} \overset{=}{(n-k+1) f(x)} \overset{>}{(1-F(x))^{n-k}} = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} (1-F(x))^{n-k} f(x)$

$$F_{k:n}(x) = P(X_{k:n} \le x) = \sum_{m=k}^{n} C_n^m F(x)^m (1-F(x))^{n-m}$$

$$\sum_{m=k}^{n} C_n^m x^m (1-x)^{n-m} = \int_0^x \frac{n!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} dt$$ производные равны:

$$\sum_{m=k}^{n} \frac{n!}{m!(n-m)!} \left(m x^{m-1}(1-x)^{n-m} - x^m (n-m)(1-x)^{n-m-1}\right) =$$

$$\frac{n!}{(k-1)!(n-k)!} t^{k-1} (1-x)^{n-k} - x^k \frac{n!}{k!(n-k-1)!} (1-x)^{n-k-1}$$

$$+ \frac{n!}{k!(k+1)!(n-k-1)!} (k+1)! \, x^k (1-x)^{n-k+1} - \ldots$$

(все сокращается

$$\int_0^{F(x)} \frac{n!}{(k-1)!(n-k)!} t^{k-1} (1-t)^{n-k} dt$$

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} (1-F(x))^{n-k} f(x)$$

Пусть $F(x) = x$, $x \in [0,1]$:

$$M_{k:n} = \int_0^{\infty} \frac{n!}{(k-1)!(n-k)!} t^k (1-t)^{n-k} dt = \frac{n!}{(k-1)!(n-k)!} B(k+1, n-k+1)$$

$$= \frac{n!}{(k-1)!(n-k)!} \cdot \frac{k!(n-k)!}{(n+1)!} = \frac{k}{n+1} \to \alpha$$

$$\sigma^2[X_{k:n}] = \frac{k(n-k+1)}{(n+1)^2(n+2)} \to \frac{\alpha(1-\alpha)}{n}$$ при $n \to \infty$, $k \to \alpha n$

$U^1, \dots, U^n$ – i.i.d. $U[0,1]$

$(U_{1:n} \leqslant U_{2:n} \leqslant \dots \leqslant U_{n:n})$

$X_1, \dots, X_{n+1}$ – i.i.d $\quad Exp(1) \quad S_k = \sum_{j=1}^{k} X_j$

$(U_{1:n}, \dots, U_{n:n}) \sim \left(\frac{S_1}{S_{n+1}}, \frac{S_2}{S_{n+2}}, \dots, \frac{S_n}{S_{n+1}} \right)$ лемма + зам. перем.

$P(U_{k:n} - \alpha \leqslant x) = P\left(\frac{S_k}{S_{n+1}} - \alpha \leqslant x\right) = P(\underbrace{S_k - x S_{n+1} - \alpha S_{n+1}}_{\eta_k} \leqslant 0) = \phi\left(\frac{n\sigma x}{\sqrt{\alpha n(1-\sigma x-\alpha)^2 + n(1-\alpha)(\sigma x+\alpha)^2}}\right)$ Гаусс

$E\eta_k = \underbrace{k}_{\alpha n}(1-x-\alpha) - (x+\alpha)\underbrace{(n-k)}_{(1-\alpha)n} = \cancel{\dots} = \cancel{\dots} - nx$

$n(\alpha - \alpha x - \alpha^2 - x \quad -\alpha + \alpha + \alpha x + \alpha^2)$

$\sigma^2(\eta) = \alpha n(1-x-\alpha)^2 + \quad n(1-\alpha)(x+\alpha)^2$

$\sigma^2 = \frac{\alpha(1-\alpha)}{2}$

$$\approx \phi\left(\frac{\sqrt{n} \cdot x \cdot \sqrt{\alpha(1-\alpha)}}{\sqrt{\alpha n(1-\alpha)^2 + n(1-\alpha)\alpha^2}}\right) = \phi\left(\frac{x\sqrt{n}\sqrt{\alpha(1-\alpha)}}{\sqrt{n}\sqrt{\alpha(1-\alpha)}\sqrt{\alpha+1-\alpha}}\right) = \phi(x)$$

А.Д. (3)

KP1	20
KP2	20
D2	30
FE	30

CLT для пор. ст-к

$X_1 \dots X_n$

$X_{1:n} < \dots < X_{n:n}$

(*) для крайних ст-ка экстр. значений

$X_\alpha = X_{\lfloor n\alpha \rfloor : n}$

$$(X_\alpha - G(\alpha)) \sqrt{\frac{n}{\alpha(1-\alpha)}} f(G(\alpha)) \sim N(0,1)$$

. если $X_1, \dots, X_n \sim U[0,1]$

$$\Rightarrow G(\alpha) = \alpha \quad f(G(\alpha)) = 1$$

$$\Rightarrow (X_\alpha - \alpha) \sqrt{\frac{n}{(1-\alpha)\alpha}} \sim N(0,1)$$

e.g. $\mu(x) = \mu e^{\beta}$ $\quad \mu = 0{,}005 \quad \beta = 0{,}08 \quad \Rightarrow \approx 0{,}7$

$\triangleright$ $F_X(x) = Y$ $\qquad P(Y < x) = P(F(X) < x) = P(X < F^{-1}(x)) =$

$$= F(F^{-1}(x)) = x$$

$F(X_1) \dots F(X_n) \sim Unif[0,1]$

$$(F(X_\alpha) - \alpha) \sqrt{\frac{n}{(1-\alpha)\alpha}} \sim N(0,1)$$

$$F(X_\alpha) \overset{\text{Taylor}}{=} F(G(\alpha)) + (X_\alpha - G(\alpha)) f(G(\alpha)) + o(1)$$

$\Rightarrow \dots$

$\blacksquare$

$$\int_0^\infty \frac{\sin\alpha x \, \sin\beta x}{x}\, dx = \frac{1}{2} \ln \frac{\alpha+\beta}{|\alpha-\beta|}$$

типа Фруллани
(не расходится) $\quad \nexists \lim f(\infty)$
$\alpha = \beta$

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx \qquad \text{интеграл Фруллани}$$

$$\overset{*}{=} (f(0) - f(a)) \ln b/a$$

$f(0) < \infty$

$f(\infty) < \infty$

$$(*)\ I = \int_0^\infty \frac{1}{x} \int_{bx}^{ax} f'(z)\, dz = \int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx$$

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \ln b/a \qquad z = xy$$

$$(*)\ I = \int_0^\infty \int_b^a f'(xy)\, dy =$$

$$= \int_b^a dy/y \int_0^\infty f'(xy)\, dxy = -\int_a^b dy/y\, (f(\infty) - f(0))$$

$$= (f(0) - f(\infty)) \ln b/a$$

$e^{\sigma B(t)}$ — GBM

$e^{\sigma B(\Gamma(t)) + c\cdot\Gamma(t)}$

$\Gamma(t)$ — субордин., процесс Леви с неуб. траект.

$\Gamma(t)$ — Gamma process

$\Delta\Gamma(t) \sim$ Gamma

$$p(x,t) = \frac{\alpha^{\beta t}}{\Gamma(\beta t)} x^{\beta t - 1} e^{-\alpha x}$$

$$\varphi_t(\theta) = E e^{i\theta X_t} = \frac{\alpha^{\beta t}}{\Gamma(\beta t)} \int_{-\infty}^{\infty} x^{\beta t - 1} e^{\overbrace{i\theta x - \dots}^{= -(i\theta - \alpha)x = y}} dx$$

$$= \frac{\alpha \beta}{\Gamma(\beta t)} \cdot \frac{1}{(\alpha - it)^{\beta t}} \int \frac{y}{\dots} e^{-y} \, dy =$$

$$= \frac{\alpha_\beta t}{\Gamma(\theta - \beta)\Gamma(\beta t)} \int_{\infty}^{\infty} y^{\beta t - 1} e^{i\theta} \, dy = \left(1 - \frac{i\theta}{2}\right)^{-\beta t}$$

Levy - process subordinator

$$E e^{i\theta X_t} = e^{-t \cdot \Psi(\theta)}$$

$\Downarrow$

нер. и смещ. $\psi(\theta) = \beta \ln\left(1 - \frac{i\theta}{2}\right)$

символ процесса Леви

$s < t$, X_s, $X_t - X_s$

$$E e^{i\theta X_t} = E e^{i\theta X_s}$$

$$e^{-E(t|\theta)} = E e^{i\theta X_t} = e^{-s\psi(t)} \cdot e^{-(t-s)\psi(t)} =$$

$$= E e^{itX_s} \; E e^{i\theta t(X_t - X_s)}$$

BM: $E e^{itB_t} = E e^{-\frac{\delta^2 t^2}{2}}$

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{i\theta x} e^{-x^2/2\sigma^2} = e^{-\left(\delta^2\theta^2/2\right)t}$$

$$B_t = B_s + (B_t - B_s)$$

$$E \{e^{itB_t}\} = e^{-\delta^2 t^2/2 \cdot t} = e^{-\delta\theta^2/2 \cdot s} \; \underbrace{e^{-\delta^2\theta^2/2 \cdot (t-s)}}_{= E e^{i\theta(B_t - B_s)}}$$

$$\psi(\theta) = \beta \ln\left(1 - \frac{i\theta}{2}\right) = \int_0^\infty (1-e^{i\theta x}) \frac{e^{-\alpha x}}{x}\, dx$$

$f(x) = e^{-x}$

$$= \int_0^\infty \frac{e^{-\alpha x} - e^{-\alpha x + i\theta x}}{x}\, dx$$

$a = -\alpha$

$b = -\alpha + i\theta$

$$= \ln \frac{\alpha - i\theta}{\alpha} = \ln\left(1 - \frac{i\theta}{2}\right)$$

$$\psi(\theta) = \beta \ln(1 - i\theta/\alpha) = \beta \int_0^\infty (1 - e^{i\theta x}) \boxed{\frac{e^{-\alpha x}}{x}}\, dx$$

веро. сильна

* если x^α $\Rightarrow 0 < \alpha < 2 \Rightarrow$ stable

Lévy measure ("плотность" меры Леви)

$\Rightarrow$ tempered stable process

(устойчивый процесс)

из-за e в числ.

похоже не $CP(\lambda\ldots)$

$\alpha = 2 \Rightarrow BM$

$0 < \alpha \le 1 \Rightarrow$ Var конечна

$\alpha > 1 \Rightarrow$ Var ∞

для $B(T(t))$ Var конечна

$$S(t) = \sum_{j=1}^{N(t)} Y_j$$

$$E e^{i\theta S(t)} = e^{-\lambda \int_0^\infty (1 - e^{i\theta x}) f(x)\, dx}$$

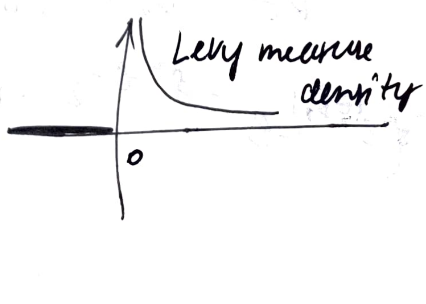
$\updownarrow$

$$\beta \int_0^\infty (1 - e^{i\theta x}) \frac{e^{-\alpha x}}{x}\, dx$$

нельзя отнорм.

(?) $\int_0^\infty \frac{e^{-\alpha x}}{x} = \infty$

расход. возле 0 (если нет e, то расх. везде)



$e^{\sigma B(t) + ct}$

$\sum |B_{t_{k+1}} - B_{t_k}|$

Variation
(вариация)

$|f| = \sup_{\delta} \sum_k |f(t_{k+1}) - f(t_k)|$

δ ↑ $\forall$ разбиение

для Gamma-Process вариация конечна

$E e^{i\theta [\sigma B(\Gamma(t)) + cT(t)]}$

tower property ⇒ $E[E[e^{i\theta[\dots]} | \Gamma(t)]] =$

$= E[e^{i(\theta^2\sigma^2/2 + c\theta)\Gamma(t)}] =$

$= \beta \ln \left(1 - \frac{i(c\theta + \frac{i\theta^2\sigma^2}{2})}{\alpha}\right) =$

$= \beta \ln \left(1 - \frac{i\theta c}{\alpha} + \frac{\sigma^2\theta^2}{2\alpha}\right) =$

$= \beta \ln \left(1 - \frac{i\theta}{\alpha^{(1)}}\right) + \beta \ln \left(1 + \frac{i\theta}{\alpha^{(2)}}\right)$

$\alpha^{(1,2)} = \left(\sqrt{\frac{c^2}{4\alpha} - \frac{\sigma^2}{2\alpha}} \pm \frac{c}{2\alpha}\right)^{-1}$

$1 - i\theta\left(\frac{1}{\alpha^{(1)}} - \frac{1}{\alpha^{(2)}}\right) + \frac{\theta^2}{\alpha^{(1)}\alpha^{(2)}} \dots = \frac{c^2}{4\alpha} + \frac{\sigma^2}{2\alpha} - \frac{c^2}{4\alpha^2}$

$$Y(t) = \sigma B(T(t)) + c\,T(t) = G_1(t) - G_2(t)$$

$$\beta \int_0^\infty \frac{1-e^{itx}}{x} e^{-\alpha x}\,dx \qquad \beta \int_0^\infty \frac{1-e^{-itx}}{x} e^{-\alpha x}\,dx$$

(*) $$S(t) = \sum_{j=1}^{N_t} Y_j$$

$$E\left[e^{itS(t)}\right] = e^{-\lambda \int_0^\infty (1-e^{i\theta x}) f(x)\,dx}$$

CP отв. только за больш. прыжки

GP учит. беск. большое число беск. малых прыж.

$X \sim (T_1^1, \alpha^1)$

$Y \sim (T_1^2, \alpha^2)$

$Z = \min[X, Y]$

$S(x) = P(Z > x) = P(X > x) \cdot P(Y > x) =$

$= \alpha^1 e^{T^1 x} \mathbb{1}_n \otimes \alpha^2 e^{T^2 x} \mathbb{1}_m =$

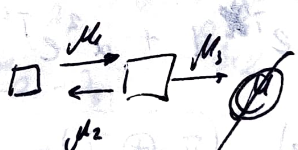
$= (\alpha^1 \otimes \alpha^2)(e^{T^1 x} \otimes e^{T^2 x}) \mathbb{1}_{m \times n}$

$\qquad \qquad ||$

$e^{(T_1 \oplus T_2)x} \quad = Z \sim (\alpha^1 \otimes \alpha^2 ; T_1 \oplus T_2)$

$T_1 \oplus T_2 = T_1 \otimes I_m + I_n \otimes T_2$

$\alpha^1 = \alpha^2 = (1, 0)$



$+$ [redacted] $\begin{pmatrix} -\mu_1 & \mu_1 \\ \mu_2 & -\mu_2 - \mu_3 \end{pmatrix} \begin{pmatrix} 0 \\ \mu_3 \end{pmatrix}$

$T_2 = \begin{pmatrix} -\mu_1 & \mu_1 \\ \mu_2 & -\mu_2 - \mu_3 \end{pmatrix}$

$T_1 = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 - \lambda_3 \end{pmatrix}$

$\underset{n \times n}{A} \otimes \underset{m \times n}{B} = \begin{pmatrix} a_{11} B & \cdots & a_{1n} B \\ \vdots & & \vdots \\ a_{n1} B & \cdots & a_{nn} B \end{pmatrix}$

$$T_1 \oplus T_2 = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_3 & -\lambda_2-\lambda_3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -\mu_1 & \mu_1 \\ \mu_3 & -\mu_2-\mu_3 \end{pmatrix} =$$

$$= \begin{pmatrix} -\lambda_1 & 0 & \lambda_1 & 0 \\ 0 & -\lambda_1 & 0 & \lambda_1 \\ \lambda_3 & 0 & -\lambda_2-\lambda_3 & 0 \\ 0 & \lambda_3 & 0 & -\lambda_2-\lambda_3 \end{pmatrix} + \begin{pmatrix} -\mu_1 & \mu_1 & 0 & 0 \\ \mu_3 & -\mu_2-\mu_3 & 0 & 0 \\ 0 & 0 & -\mu_1 & \mu_1 \\ 0 & 0 & \mu_3 & -\mu_2-\mu_3 \end{pmatrix} =$$

$$= \begin{pmatrix} -\lambda_1-\mu_1 & \mu_1 & \lambda_1 & 0 \\ \mu_3 & -\lambda_1-\mu_2-\mu_3 & 0 & \lambda_1 \\ \lambda_3 & 0 & -\lambda_2-\lambda_3-\mu_1 & \mu_1 \\ 0 & \lambda_3 & \mu_3 & -\mu_2-\mu_3-\lambda_2-\lambda_3 \end{pmatrix}$$

$S_Z(x)$

$$P(\max[X,Y]>x) = 1-(1-S(x))(1-S(y)) =$$

$$= 1-\left(1-\alpha^1 e^{T^1 x}\mathbb{1}_n\right)\otimes\left(1-\alpha^2 e^{T^2 x}\mathbb{1}_m\right) =$$

$$= \cancel{1}-\cancel{1}+\alpha^1 e^{T^1 x}\mathbb{1}+\alpha^2 e^{T^2 x}\mathbb{1} -$$

$$-(\alpha^1\otimes\alpha^2)\underbrace{e^{(T_1\oplus T_2)x}}_{= e^{T_1 x}\otimes e^{T_2 x}}\mathbb{1}_{mn}$$

$$= \alpha^1\otimes\alpha^2 e^{\hat{T}x}\mathbb{1}_{nm}$$

$$= \alpha^1 e^{T_1 x}1_n + \alpha^2 e^{T_2 x}1_m - \alpha^1\otimes\alpha^2 e^{(T_1\oplus T_2)x}\mathbb{1}_{mn} \quad (\star)$$

$$e^{\hat{T}x} = \begin{pmatrix} \underset{nm\times nm}{e^{(T_1+T_2)x}} & \underset{nm\times n}{\left(e^{T_1 x}\otimes I_m - e^{(T_1\oplus T_2)x}\right)1_m} & \overbrace{\left(I_n\otimes e^{T_2 x} - e^{(T_1\oplus T_2)x}\right)}^{nm\times nm}\mathbb{1}_n \\ 0_{n\times nm} & e^{T_1 x} & \underset{nm\times m}{0} \\ 0_{m\times nm} & \underset{n\times n}{0_{m\times n}} & \underset{m\times m}{e^{T_2 x}} \end{pmatrix}$$

$$(\alpha^1 \otimes \alpha^2, 0, 0) \cdot e^{\hat{T}x} \cdot \mathbb{1}_{nm} \qquad (\star)$$

$nm \quad n \quad m$

$$\hat{T} = \begin{pmatrix} T_1 \oplus T_2 & T_1 \otimes t^m & t^n \otimes T_2 \\ 0 & T_1 & 0 \\ 0 & 0 & T_2 \end{pmatrix}$$

$$\cancel{T_1 \otimes I_n} - \cancel{T_1 \otimes I_n} - I_n \otimes \underbrace{T_2 1}_{=-t_2} = I_n \otimes t^2$$

$$\cancel{I_1 \otimes T_2 - I_n T_2} - I_1 \otimes I_1$$

$$t \sim \begin{pmatrix} T & t \\ 0 & 0 \end{pmatrix} \qquad t = -T \cdot 1$$

$I \otimes t \qquad t \otimes I$ ← T? I?

$$\hat{T} = \left(\begin{array}{c|cc|cc} & 0 & 0 & 0 & 0 \\ & -\lambda_1\mu_3 & \lambda_1\mu_3 & 0 & 0 \\ T_1 \otimes T_2 & 0 & 0 & -\mu_1\lambda_3 & \mu_1\lambda_3 \\ 4\times4 & \lambda_2\mu_3 & (-\lambda_2-\lambda_3)\mu_3 & \mu_2\lambda_3 & (-\mu_2-\mu_3)\lambda_3 \\ \hline 0_{2\times4} & -\lambda_1 & \lambda_1 & \multicolumn{2}{c}{0_{4\times2}} \\ & \lambda_3 & -\lambda_2-\lambda_3 & & \\ \hline 0_{2\times4} & \multicolumn{2}{c|}{0_{2\times2}} & -\mu_1 & \mu_1 \\ & & & \mu_1 & -\mu_2-\mu_3 \end{array}\right)$$

$f(x) = \alpha e^{T_1 \otimes T_2} t$

t когда 1ый умрет

$f(x) = \alpha e^{\hat{T}} t$

t когда 2ой умрет ?

$\hat{t}$

$\mu_1 = 0,1 \qquad \lambda_1 = 0,3$

$\mu_2 = 0,3 \qquad \lambda_2 = 0,2 \qquad \Rightarrow f(x)$

$\mu_3 = 0,3 \qquad \lambda_3 = 0,1$

$$E\xi = \alpha(-T)^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

$(1,0,0,0)$

$$E\xi = \int_0^\infty S(x)\,dx$$

$$\| $$

$$\alpha e^{Tx}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$$

$$E\xi = \int_0^\infty x\, f(x)\,dx$$

$$\| $$

$$\alpha e^{Tx}\cdot t$$

$$t = \begin{pmatrix}0\\0\\0\\0\\0\\0{,}3\\0\\0{,}1\end{pmatrix}$$

$$P(\max[T^1, T^2] > x) = 1 - (1 - S^1(x))(1 - S^2(y))$$

$$= 1 - P(T^1 < x)P(T^2 < x) =$$

$$= 1 - (1 - (\alpha^1 e^{T^1 x} 1)) \otimes (1 - \alpha^2 e^{T^2 x} 1) =$$

$$= \alpha^1 e^{T^1 x} 1_n + \alpha^2 e^{T^2 x} 1_m - \alpha_1 \otimes \alpha_2 e^{T_1 \otimes T_2 x} 1_{n \times m}$$

$$e^{Tx} = \left(\begin{matrix} e^{T_1 \oplus T_2 x} & (e^{T_1 x} \otimes I_m - e^{T_1 \oplus T_2 x}) 1_m & (I_n \otimes e^{T_2 x} - e^{(T_1 \oplus T_2)x} \\ 0 & T_1 & 0 \\ 0 & 0 & T_2 \end{matrix}\right)$$

$$(\alpha^1 \otimes \alpha^2, 0, 0) \cdot e^{Tx} \cdot \mathbb{1}_{nm} =$$

$$= \alpha_1 e^{T_1 x} \mathbb{1}_n + \alpha_2 e^{T_2 x} \mathbb{1}_m - \alpha_1 \otimes \alpha_2 \, e^{T_1 \oplus T_2 x} \mathbb{1}_{mn}$$

$t_1 \otimes I_m \quad (-I \otimes T_2) +$

$$\begin{pmatrix} T_1 \oplus T_2 & \cancel{T_1 \otimes I_m} & -t_1 + T_2 \\ 0 & T_1 & 0 \\ 0 & 0 & T_2 \end{pmatrix}$$

производная имеет вид T

$$\begin{tikzcd}
\boxed{1} \arrow[r, "\lambda p"] \arrow[dr, "\lambda q"'] & \boxed{2} \arrow[r, "\lambda p"] \arrow[d, "\lambda q"] & \boxed{3} \arrow[dl, "\lambda"] \\
& \phi &
\end{tikzcd}$$

$Cox(3)$

$$\begin{pmatrix} -\lambda & \lambda p & 0 & \lambda q \\ 0 & -\lambda & \lambda p & \lambda q \\ 0 & 0 & -\lambda & \lambda \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} e^{-\lambda x} & (a_{12} + b_{12}x + c_{12}x^2) e^{-\lambda x} & \ldots \\ 0 & e^{-\lambda x} & \ldots \\ 0 & 0 & e^{-\lambda x} \end{pmatrix}$$

А.П. (4)

РН

Реј.

Марк.

Резервы:

$$\pi = \frac{S\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$$

$\alpha' + \alpha'' S$

$\beta' + \beta'' \pi'$

V_t $\qquad$ $V_n = S$

x $\qquad$ $x+n$

$$\pi' = \frac{1}{1-\beta''}\left[\pi + \frac{\alpha' + \alpha'' S}{\bar{a}_{x:\overline{n}|}} + \beta'\right]$$

$$e^{r(t)\Delta t}\left[V_t + (\overset{\text{gross premium}}{\pi'} \underbrace{- \beta' - \beta''\pi'}_{\text{net premium}})\Delta t\right] = \mu_{x+t}\,\Delta t\, S + \underbrace{(1-\mu_{x+t}\,\Delta t)\, V_{t+\Delta t}}_{\shortparallel}$$

$\shortparallel$

$(1 + r(t)\Delta t)$

$$V_{t+\Delta t} - \mu_{x+t}\,\Delta t\, V_t$$

$V_n = S$ ← гран. усл.

$$\Rightarrow \frac{V_{t+\Delta t} - V_t}{\Delta t} = (\pi' - \beta' - \beta''\pi) + \mu_{x+t} V_t - \mu_{x+t} S + r V_t$$

$$\frac{dV_t}{dt} = (\pi' - \beta' - \beta''\pi) + r V_t - \mu_{x+t}(S - V_t)$$

$V_0 = -\alpha' - \alpha'' S$

$V_n = S$

$$\frac{dV_t}{dt} - (r(t) + \mu_{x+t}) V_t = \pi' - \beta' - \beta''\pi' - \mu_{x+t} S$$

$$e^{\int_t^n (r(s) + \mu_{x+s}) ds} [-//-] = e^{\int_t^n (r(s) + \mu_{x+s}) ds} [-//-]$$

$$\int_t^n ds \frac{d}{dt} \left[e^{\int_t^n (r(s) + \mu_{x+s}) ds} V_t \right] = \int e^{\dots} -//-$$

$$\underset{V_n = S}{S} - e^{\int_t^n (r(s) + \mu_{x+s}) ds} V_t = \int_t^n ds\, e^{\int_s^n (r(u) + \mu_{x+u}) du} (\pi' - \beta' - \beta''\pi') -$$

$$- S \int_t^n \mu_{x+s}\, e^{\int_s^n (r(u) + \mu_{x+u}) du}$$

$$V_t = \int_t^n e^{-\int_t^s (r(u) + \mu_{x+u}) du} (-\pi'(1 - \beta'') + \beta' + \mu_{x+s} \cdot S) ds$$

$$+ S e^{\int_t^n (r(u) + \mu_{x+u}) du} \qquad S \cdot \bar{A}^{\;1}_{x:\bar{n}|}$$

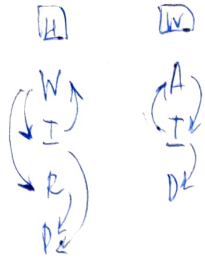
$$V_0 = -\alpha' - \alpha'' S \Rightarrow \pi' = \frac{1}{1-\beta''} \left[\frac{\alpha' + \alpha'' S + S e^{-\int_0^n (r(u) + \mu_{x+u}) du} + \dots}{\int_0^n e^{-\int_0^s r(u) + \mu_{x+u} du} ds} \right]$$

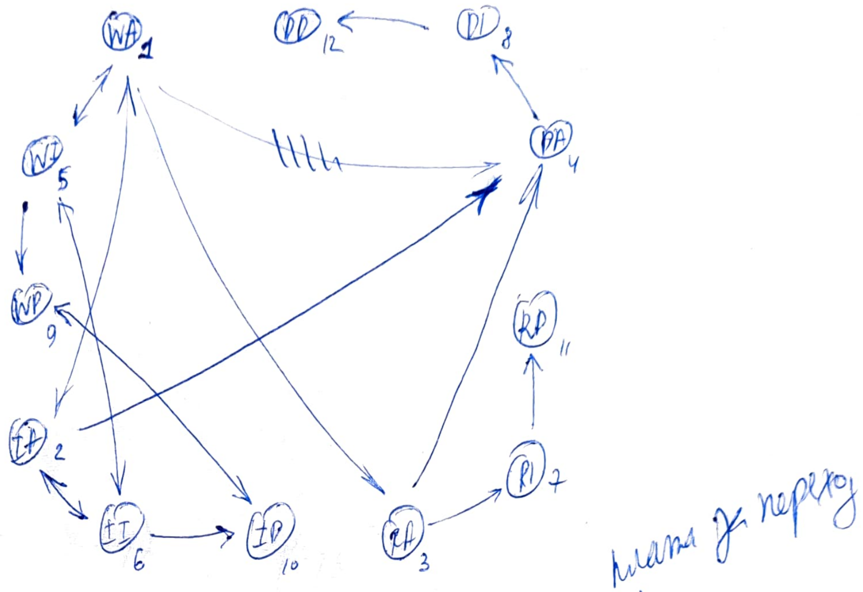
$$S \cdot \bar{A}^{\;1}_{x:\bar{n}|} \quad \dots + S \int_0 \mu_{x+u}\, e^{-\int_0^u (r(v) + \mu_{x+v}) dv} du$$

$$\bar{a}_{x:\bar{n}|} \qquad \int_0^n e^{-\int_0^u r(s) ds} S(u) du$$

$$\bar{A}^{\;1}_{x:\bar{n}|} = \int_0^n e^{-\int_0^u r(v) dv} \mu_{x+u} S(u) du = \int_0^n e^{-\int_0^u r(v) dv} S(u) du =$$

$$= \int_0^n e^{-\int_0^u (r(v) + \mu_{x+v}) dv} du \qquad = \int_0^n e^{-\int_0^u (r(v) + \mu_{x+v}) dv} du$$





$$\begin{pmatrix} Z_1(t) \\ \vdots \\ Z_{11}(t) \end{pmatrix}$$

сколько придется запл. по полисам, $i = \overline{1,11}$ (без полисов для DD)

$E[Z_i(t)] = V_i(t)$

$E[Z_i^2(t)] = V_i^{(2)}(t)$

плата за переход

пост. W

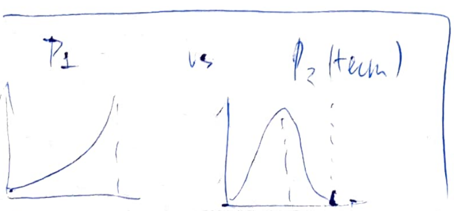
$$e^{k r(t)\Delta t}\left(Z_i(t) + c_i \Delta t\right)^k = \sum_j \mu_{ij}\Delta t\left(b_{ij} + Z_j(t)\right)^k + \left(1 - \sum_j \mu_{ij}\Delta t\right) Z_i(t+\Delta t)^k$$

или $- p_i \Delta t$

вып. пенсии

(*) решение для $k=2$ зав. от $k=1$ (для E) (Var)

$$(W,A):\ e^{r\Delta t}\left(V_1(t) + c\Delta t\right) = \mu_{12}\left(b + V_2(t)\right) + \mu_{13}\left(d + V_3(t)\right) + \mu_{15}\left(b + V_5(t)\right) + \left(1 - \mu_{12}\Delta t - \mu_{13}\Delta t - \mu_{15}\Delta t\right)V_1(t+\Delta t)$$



$$\frac{dV_1(t)}{dt} = c + r V_1(t) + \mu_{12}\left(V_1(t) - V_2(t) - b\right) + \mu_{13}\left(V_1(t) - V_3 - d\right) + \mu_{15}\left(V_1(t) - V_5(t) - b\right)$$

Мартингалы

$\geq$ ← sub-martingale

$$E\,(M(t)\,|\,F_s) = M(s)\ , \quad s<t$$

$\leq$ ← super-martingale

$$X_i = \begin{cases} +1, & p=1/2 \\ -1, & q=1/2 \end{cases}$$

$$S_n = \sum_{j=1}^{n} X_j - n(p-q) \quad , p+q=1$$

← martingale

$$M_n = \left(q/p\right)^{S_n}$$

$$E\left[\left(q/p\right)^{S_{n+1}} \,|\, X_1 \dots X_n\right] = E\left[\left(q/p\right)^{S_n + X_{n+1}} \,|\, X_1 \dots X_n\right] =$$

$$= \left(q/p\right)^{S_n} \cdot E\left[\left(q/p\right)^{X_{n+1}}\right] = \left(q/p\right)^{S_n} \cdot \underbrace{\left(\left(q/p\right)^{1} \cdot p + \left(q/p\right)^{-1} \cdot q\right)}_{=1} \Rightarrow \text{martingale}$$

$$E\,(M_\tau) = E\,(M_0) \qquad \{\tau \leq n\} \in F_n$$

↑ стоп. время

$$\tau = \inf\{n : X_n \in A\}$$

$p < 1/2 + 0{,}0027$

$T = \inf \{ n : S_n = -100 \text{ or } S_n = 1000 \}$

$E[(q/p)^{S_T}] = 1$

$\psi (q/p)^{1000} + (1-\psi)(q/p)^{-100} = 1$

$q = 1/2 - \varepsilon$

$p = 1/2 + \varepsilon$

ε	0,001	0,003	...
ψ	0,98...	0,6988	...

$M(t) = B^2 - t$ ← Martingale

$M(t) = 3t \cdot B(t) - B^3(t)$ ← Martingale

$E[M(t+\Delta t) \mid F_t] = E[3(t+\Delta t)B(t+\Delta t) - B^3(t+\Delta t) \mid F_t]$

$= 3E[(t+\Delta t) \cdot B(t+\Delta t) \mid F_t] - E[B^3(t+\Delta t) \mid F_t]$

$= 3t\, E[B(t+\Delta t) \mid F_t] + 3E[\Delta t \cdot B(t+\Delta t) \mid F_t] - E[B^3(t+\Delta t) \mid F_t]$

$= 3t B_t + 3\Delta t B_t -$

BM:

$B(t)$ - ind. stationary increments

$B(t)-B(s) \sim N(0, \underbrace{t-s}_{=\sigma^2})$

Wald Martingal (верно для всех процессов Леви)

$e^{\lambda B(t) - \lambda^2/2\, t}$ ← март.

$e^{\lambda B(t) - \varkappa(\lambda) t}$ ← март.

↑ ~~[illegible]~~ ПФК Леви

$E[B_\tau^2 - \tau] = 0$

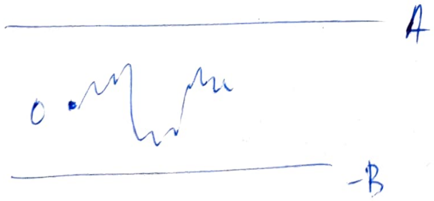
$\Downarrow$

$E\tau = A^2$

$$E e^{\lambda L(t)} = e^{\varkappa(\lambda) t}$$

$$E e^{isL(t)} = e^{-t\psi(s)}$$

← симв. процесса Леви



$\tau = \inf\{t : B(t) = A \text{ or } B(t) = -B\}$

$\psi = P(B(\tau) = A),\ 1 - \psi = P(B(\tau) = -B)$

$E\tau,\ E\tau^2$

$E[e^{\lambda B(\tau) - \lambda^2/2\, \tau}] = 1$

$\lambda^2/2 = s \qquad \lambda = \sqrt{2s}$

$E[e^{-s\tau}] = \frac{1}{\cosh(\sqrt{2s} A)}$

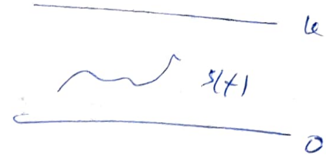
$\frac{1}{2}(e^{\lambda A} + e^{-\lambda A}) E e^{-\lambda^2/2\, \tau} = 1$

$\Rightarrow E\tau = A^2 \qquad E\tau^2 = 5A^4/3$

$e^{sL(t)-\varkappa(s)t} \Rightarrow e^{RL(t)}$ – Martingale

$S(t) = u + \mu t + \sigma B(t) = u - L(t) = R$

$\varkappa(s) = \ln E e^{sL(t)} = -\mu s + \frac{\sigma^2 s}{2}$



$$1 = \psi^a(u) + (1-\psi^a(u))\, e^{R(u-a)}$$

$$\psi^a(u) = \frac{e^{-2\mu/\sigma^2 u} - e^{-2\mu/\sigma^2 a}}{1 - e^{-2\mu/\sigma^2 a}}$$

$$\psi^a(u) = \frac{(1 e^{R(u-a)})e^{-Ru}}{(e^{Ru} - e^{R(u-a)})e^{-Ru}}$$

$Lu = 0$

генератор $u(0) = 1,\ u(a) = 0$

$Lu(x) = \mu\, d/dx\, u + \sigma^2/2\, d^2/dx\, u \quad \big|\ e^{-2\mu/\sigma^2 x}$

$Lu = \lim_{t\to 0}\left[\frac{1}{\sqrt{2\pi t}\,\sigma}\int_{-\infty}^{\infty} u(x+y)\, e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}}\, dy - u(x)\right]$

$Lu(x) = -\mu\, {}^{2\mu}/_{\sigma^2}\, e^{-2\mu/\sigma^2 x} + \sigma^2/2\ {}^{4\mu^2}/_{\sigma^4}\, e^{-2\mu/\sigma^2 x}$

$Lu = -1 \quad u(0) = u(a) = 0 \Rightarrow E\tau$

$Lu = -(1-\psi(x)),\ u(0) = u(1) = 0 \Rightarrow E[\tau \mid X_\tau = A] = E[\tau \mathbb{1} R(t)]/(1-\psi(a))$

A. g. (5)

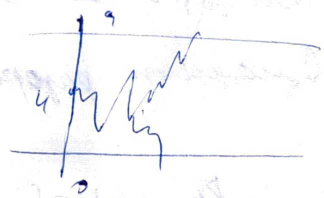
$E_u(\tau_a^+ \mid \tau_a^+ < \tau_0^-)$

$S(t) = u + ct - \sum_{j=1}^{N(t)} y_j$

$\ln E e^{\theta S(t)}$

$X_u = P_u(T = \tau_a^+) = \frac{\varphi(u)}{\varphi(a)} = \frac{1-\psi(u)}{1-\psi(a)}$

$\psi = P_u(T = \tau_0^-) = \frac{\psi(u) - \psi(a)}{1-\psi(a)}$



$$E_u\left[e^{-\theta \tau_a^+} \mathbb{1}_{\{\tau_a^+ < \tau_0^-\}}\right] = \frac{W^{(\theta)}(u)}{W^{(\theta)}(a)}$$

Scale function

$L(t) = \sum y_i - ct$

CGF: $\frac{1}{t} \ln E e^{\theta S(t)} =$

$E e^{i\theta X(t)} = e^{-t\psi(\theta)}$

$= \left[-c\theta + \ln E e^{\sum_{j=1}^{N(t)} y_j}\right]$

$= -c\theta + \ln E e^{\lambda t(E(e^{\theta y}) - 1)}$

$E\left[E\left(e^{\theta \sum_{j=1}^{N(t)} y_j} \mid N(t) = n\right)\right] =$

$= e^{\lambda(E[e^{\theta y}] - 1)}$

сильные мен
↳ $\varkappa(\theta) = -c\theta + \lambda\left[E[e^{\theta y}] - 1\right]$

$E\left((E[e^{\theta y}])^n\right) = \sum_{j=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} E(e^{\theta y})^n$

$\varkappa(t) = 0$

решили ур-е Лундберга

$\lambda E[y](1+\theta) = \lambda(E[e^{\theta y}] - 1)$

$= e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t\, E(e^{\theta y}))^n}{n} = e^{\lambda t[E[e^{\theta y}] - 1]}$

$E[e^{\theta y}] = 1 + E(y)(1+\theta)$

=> можно из них построить процесс Вальда

$e^{\theta X(t) - \varkappa(\theta) t} \Leftrightarrow e^{R X(t)}$

↑ март. Вальда . $\varkappa(\theta) = 0$ R – решение ур-я Лундберга

$L(t) = \sum_{j=1}^{N(t)} \xi(t) - ct = u + \xi$

$e^{R L(t) - 0}$

$$1 = \psi(u) E[e^{\overset{u+\xi}{\overset{\shortparallel}{R L(\tau)}}} \mid \tau_0 < \infty]$$

$\tau = \inf\{t : L(t) \geq u\}$.

↑ момент разорения

$$1 = e^{Ru} \psi(u) E[e^{R\xi} \mid \tau_0 < \infty]$$

$$\psi(u) = e^{-Ru} \frac{1}{E[e^{R\xi} \mid \tau < \infty]} \leq e^{-Ru}$$

$E_u[(\tau_a^+)^n \mid \tau_a^+ < \tau_0]$ → диф. ур. I
→ scale function II
→ подбор мартингала III

I ↓ оператор (полугруппы)

$T_t f(x) = E_x f(x + X(t))$

$T_t \cdot T_s = T_{t+s}$

$L = \lim_{t \to t} \frac{1}{t} (T_t f - f)$ ↑ оператор скорости средн. от положения гаус. в момент t

↑ генератор $X(t) = x + \mu t + \sigma W(t)$

$$L f(x) = \lim_{t \to 0} \frac{1}{t} \left(\frac{1}{\sqrt{2\pi t}\,\sigma} \int_{-\infty}^{\infty} e^{-\frac{(y-\mu t)^2}{2\sigma^2 t}} f(x+y)\, dy - f(x) \right) =$$

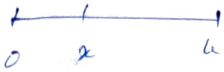
↑ дисперсия растет со временем ↑ ядро

$\overset{\shortparallel}{f(x) \int_{-\infty}^{\infty} \ldots dy} = 1$

$f(x) + f'(x)(y - \mu t) + f'(x)\mu t + \frac{1}{2} f''(x)(y - \mu t)^2 + \frac{1}{2} f''(x)[2\mu t y - \mu^2 t^2]$, где $(y-\mu t)^2 \to \sigma^2 t$, $f'(x)\mu t \to o(t)$, $[\ldots] \to o(t)$

$\boxed{= \mu f'(x) + \frac{\sigma^2}{2} f''(x)}$

$Lf = 0$



$f(0)=0 \quad f(\tau=1)$

$Lf = 1$ ← среднее время выхода

$f(0)=f(a)=0$

$$\underbrace{E[\tau\, \mathbb{1}\{\tau_a^+ < \tau_0^-\}]}_{\phi_a(x)}$$ ← среднее время выхода при условии $\tau_a^+ < \tau_0^-$

$Lf = -\phi_a(x)$

$f(a) = f(0) = 0$

для процесса со скачками:

$$Lf(x) = cf'(x) + \lambda\left[\int_0^\infty f(x-y) f_y(y)\,dy - f(x)\right]$$

(II) $$E_u\left[e^{-q\tau_a^+}\, \mathbb{1}\{\tau_a^+ < \tau_0^-\}\right] = \frac{W^{(q)}(u)}{W^{(q)}(a)}$$

$W^0(x) = \dfrac{\phi(x)}{\varkappa'(0)}$ — 1ая ф-я шкалирования

$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(u)\,du$ — 2ая ф-я шкалирования

Def: $$\boxed{\int_0^\infty e^{-\theta x} W^{(q)}(x)\,dx = \frac{1}{\varkappa(\theta) - q}}$$

$$E e^{\theta B(t)} = e^{\frac{\sigma^2\theta^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma^2\theta^2}{2}} e^{\sigma x\theta} e^{-\frac{x^2}{2\sigma^2}}\,dx = e^{\frac{\sigma^2\theta^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sigma} e^{-\frac{1}{2}(x/\sigma - \sigma)^2}\,dx = e^{\frac{\sigma^2\theta^2}{2}}$$

$\Rightarrow \quad x(\theta) = \dfrac{\sigma^2\theta^2}{2}$

$F^{-1}\left[\dfrac{1}{\frac{\sigma^2\theta^2}{2} - 9}\right] = \sqrt{\dfrac{2}{9\sigma^2}} \sinh \sqrt{\dfrac{2}{9\sigma^2}}\, x$

$\Rightarrow \quad x(\theta) = \dfrac{\sigma^2\theta^2}{2} + \mu\theta$

$F^{-1}\left[\dfrac{1}{\mu\theta + \frac{\sigma^2\theta^2}{2} - 9}\right] = \dfrac{2}{\sqrt{2 \cdot 9\sigma^2 + \mu^2}}\, e^{-\frac{\mu x}{\sigma^2}} \sinh\left(\dfrac{x}{\sigma^2}\sqrt{2 \cdot 9\sigma^2 + \mu^2}\right)$

$T = T_b^{+} - \mu T_b$

$E_u T = ?$

$\sigma^2/2\, f'' = -1 \qquad f = \dfrac{x(a-x)}{\sigma^2} \qquad f(0) = f(1) = 0$

$\mu f' + \dfrac{\sigma^2}{2} f'' = -1 \qquad y(t) = x(t) - \mu t$

$f(0) = f(1) = 0 \qquad E y(t) = x(t) - \mu t$

$E B(t) = u \qquad x(t) = \mu t + \sigma B(t)$

u

$a\, \phi_u(a) \qquad \phi_u(a) = u/a$

Lemma $y(t) = B^2(t) - t$

$y(t) = \lambda^2(t) - \frac{\mu t}{\sigma^2}$

$y(0) = u$

$(a-u)^2 p(u) + u^2 (1 - \varphi(\varepsilon)) = \frac{\mu}{\sigma^2} E(T)$

$ET = \frac{1}{\sigma^2} \left[(a-u)^2 \frac{u}{a} + u^2 \left(\frac{u-a}{a} \right) \right]$

$= \frac{1}{\sigma^2} u(a-u) \left(\frac{a-u+u}{a} \right) = \frac{u(a-u)}{\sigma^2}$

$ET = \frac{u(a-u)}{\sigma^2}$

$\sigma^2 B^2(t) - \sigma^2 t$

$\sigma^2 ET = (a-u)^2 \frac{u}{a} + u^2 \left(1 - \frac{u}{a}\right)$

$y(t) = \sigma B(t)$

$T = t_\varepsilon^+ \wedge t_\delta^-$

$ET = \frac{u(a-u)}{\sigma^2}$

$\frac{\sigma^2}{2} f'' = -1$

$f(0) = f(0) = 0$

e.d

$y(t) = B^2 - t$

$ET = \phi_u (a-u)^2 + (1 - \varphi_a(u)) u^2$

$Et = \frac{u(a-u)}{\sigma^2}$

$B^2(t) - t$ a martingale

$E[(B(t) - B(s) + B(s))^2 - (t-s) - s \mid F_s] = B(t) - B(s) \ldots - t/s$

$= \ldots$

1

$$E_u\left[e^{-qt}\mathbb{1}_{\{\tau_a^+<\tau_0^-\}}\right]=\frac{W^{(q)}(u)}{W^{(q)}(a)}$$

$\frac{W(u)}{W(a)}$, $\cancel{a}$, $q=0 \Rightarrow \frac{u}{a}$

$-\left(\frac{W(u)}{W(a)}\right)'_q \quad q=0 \Rightarrow$

$$E_u\left[\tau_a^+ \mid \tau_a^+<\tau_0^-\right]=\frac{x(a^2-x^2)}{3a\sigma^2}$$

2 $\begin{cases} \sigma^2/2\, f'' = -x/a \\ f(0)=f(a)=0 \end{cases}$

$$E\tau=\varphi_a(u)(a-u)^2+(1-\varphi_a(u))u^2$$

$$\Rightarrow E\tau=\frac{u(a-u)}{\sigma^2}$$

---

ex 5.2.7

$S(t)=u+\mu t+\sigma B(t)=u-L(t)$

$L(t)=-\mu t+\sigma B(t)$

$\Downarrow$

$-\mu\theta+\frac{\sigma^2}{2}\theta^2=0$

$\Downarrow$

$\theta=\mu/2\sigma^2 \Rightarrow \overset{\text{мартингал}}{E e^{\theta L(\tau)}}=\varphi_a(u)e^{\overset{L(t)=a}{(u-a)\theta}}+(1-\varphi_a(u))e^{u\theta}$

$$\boxed{W^{(q)}(x)=\sqrt{\frac{2}{q\sigma^2}}\sinh\sqrt{\frac{2q}{\sigma^2}}\,x}$$

$\begin{cases} \sigma^2/2\, f''+\mu f'=1 \\ f(0)=0,\ f(a)=0 \end{cases}$ $\Updownarrow$ (5.2.12) 1 знак

$$E\tau=\frac{1}{\mu}\left[(a-u)\varphi_a(u)-u\psi_a(u)\right]$$

$$E_a\left[e^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^-<\tau_a^+\}}\right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)} Z^{(q)}(a)$$

$q \to 0 \Rightarrow \psi(x)$

(5.2.20)

(5.2.23) $$W^{(q)}(x) = \frac{2}{\sqrt{2q\sigma^2+\mu^2}} e^{-\frac{\mu x}{\sigma^2}} \sinh\left(\frac{x}{\sigma^2}\sqrt{2q\sigma^2+\mu^2}\right)$$

со сносом

$$Z^{(q)}(x) = 1 + \int_0^x W^{(q)}(s)\,ds$$

x без сноса

$$\begin{cases} \sigma^2/2\, f'' + \mu f' = -\psi^q(x) \quad 1-\phi^q(x) \\ f(0)=0, \ f(a)=0 \end{cases}$$

$\Rightarrow$ условное Exp

$\Rightarrow \Sigma = Exp$

(A)

$q + E B\tau) + B(\tau_q) = a \qquad \tau = \inf\{t;\ B_t = \varepsilon\}$

$= E(B)$

с 2 границами можно ост.

с 1 не всегда

$E(\sqrt{t}) < \infty$ когда можно ост.

$E\left(\frac{\Sigma_1^?}{q+\sqrt{t}}\right) < \infty$ можно остановить

(Замена меры: (при помощи м-ле Кальдо)

$\tilde{\mu}(dx) = e^{Z} \mu(dx)$

$e^{\theta X(t) - \varkappa(\theta) t}$, $e^{\dots} = \phi(q)$, $\varkappa(\theta) = q$

найти θ : $\varkappa(\theta) = q$ => решение $\phi(q)$

Радон-Никодим — вес новой меры по старой мере

(5.2.17) чтобы найти конст.
способ Винера-Коши?
мерт. Кекма-Витте

Задача 5.2.10

$B^2(t) - t$

компенсатор

$E\left[(B(s) + B(t) - B(s))^3 \mid \mathcal{F}_s \right]$ $B^3(t) - 3tB(t)$

от 2 степ.

$\Delta^3 \Rightarrow 0$ (симметрич.)

$- 3sB(s) - 3s(\Delta) - 3(t-s)B(t)$

А.Д. (6)

Cl Model: $L(t) = \sum y_j - ct$, $y_j \sim Exp(\mu)$

$\rho = \frac{\lambda}{c\mu} > 1$; при $q > 0$ $E[e^{-q\tau(u)} \mathbb{1}_{\{\tau(u) < \infty\}}] =$

$= e^{-\Phi(q) u} \left(1 - \frac{\Phi(q)}{\mu}\right)$

где $\Phi(q) = \theta_q = \frac{1}{2}\left[\mu - \frac{\lambda}{c} - \frac{q}{c} + \sqrt{\left(\mu - \frac{\lambda}{c} - \frac{q}{c}\right)^2 + 4\frac{q\mu}{c}}\right]$

— это пол. решение ур-я $\lambda\left(\frac{\mu}{\mu - \theta_q} - 1\right) - c\theta_q = q$

символ процесса Леви

▷ $k(\theta) = \lambda\left(\frac{\mu}{\mu - \theta} - 1\right) - c\theta$ ← пр. ф-я пр.

$k(\theta_q) = q$

$\min k(\theta) = \theta_{\min} = \mu - \sqrt{\lambda\mu/c}$

при $q > 0 \geq k(\theta_{\min}) = \theta_{\min} c(\rho^{1/2} - 1)$

март. Вальда $e^{\theta L(t) - k(\theta) t}$

где $\theta = \Phi(q)$; ост. в случ. момент

$\mu > \lambda/c$ $c = \frac{\lambda}{\mu}(1+\theta)$ $\mu - \lambda/c = R$

$E_u[\tau | \tau < \infty] = \frac{1}{\mu - \lambda/c} \cdot \frac{1}{c}(\lambda/c\, u + 1)$

$\psi(u) = P(\tau < \infty) = \frac{\lambda}{c\mu} e^{-(\mu - \lambda/c) u}$

$Var_u[\tau | \tau < \infty] = \frac{1}{(\mu - \lambda/c)^2} \frac{1}{c^2}\left(2\frac{\lambda}{c}\mu u + \frac{\lambda}{c} + \mu\right)$

Опр. Гаус. процесс.

$X(t) = B(t) + \mu t$ $\qquad \mu > 0$

$t_s = \inf\{t : X(t) = s\}$ $\qquad s \geq 0$

$F(x,s) = P(t_s < x) = \Phi\left(-\frac{s}{\sqrt{x}} + \mu\sqrt{x}\right) + e^{2\mu s}\Phi\left(-\frac{s}{\sqrt{x}} - \mu\sqrt{x}\right)$

$f(x,s) = \frac{s}{\sqrt{2\pi}\, x^{3/2}} e^{\mu s - \frac{1}{2}(s^2/x + \mu^2 x)}$

$\mu = 0 \qquad (X(t) = B(t))$

$t_s \sim \frac{1}{\xi^2}, \quad \xi \sim N(0, \frac{1}{s^2})$

время до им. выйти на барьер – обратный процесс

$t_s \sim \frac{1}{\xi^2}$

$F = 2\Phi\left(-\frac{s}{\sqrt{x}}\right)$

$P\left(\frac{1}{\xi^2} < x\right) = 2P\left(\underset{N(0,1)}{\xi \cdot s} < -\frac{1 \cdot s}{\sqrt{x}}\right) = 2\Phi\left(-\frac{s}{\sqrt{x}}\right)$

$Z(t) = e^{\theta X(t) - \chi(\theta) \cdot t}$ $\qquad \frac{1}{t} \ln E\left[e^{\theta B(t) + \theta \mu t}\right]$

$\chi(\theta) = q$

$E\left[e^{i\theta t_s}\right] = e^{-\psi(\theta) s}$

$\psi(\theta) = \sqrt{-2i\theta + \mu^2} - \mu$

$-\frac{x^2}{2} + \mu x = \theta \Rightarrow x^2 - 2\mu x - 2\theta = 0$

$x = \mu + \sqrt{\mu^2 + 2\theta}$

$e^{(\theta X(t) + \psi(\theta))t}$

$E[e^{i\theta X(t)}] = e^{-\psi(\theta)s}$

? $Y(s)$ — субордин. с (переход) леви (a, b, Π)
{тройкой}

$$\Pi(dx) = \frac{1}{\sqrt{2\pi}\, x^{3/2}} e^{-\mu^2 x/2} \mathbb{1}\{x > s\}\, dx$$

$$E\left[e^{\lambda \overset{=s}{X(t)} - \underbrace{(\lambda^2/2 + \mu\lambda)}_{= -\theta} t}\right] = 1$$

- мартин. Вальда

$\lambda^2/2 + \mu\lambda = -\theta$

$X(\tau_s) = s$

$E[e^{\theta \tau_s}] = e^{-\lambda s}$, где $\frac{\lambda^2}{2} + \mu\lambda = -\theta$

$\lambda = -\mu + \sqrt{\mu^2 + 2\theta i}$

$\Downarrow$

$E[e^{i\theta\tau_s}] = e^{-s\psi(\theta)}$

$\Uparrow$ по опр. символ процесса Леви

$\psi(\theta) = -\mu + \sqrt{\mu^2 + 2i\theta}$

$$\int_0^\infty (e^{-ut}-1) z^{-1-\alpha} dz = -\frac{u}{\alpha}\int_0^\infty e^{-u} z^{-\alpha} dz =$$

$$= -\Gamma(-\alpha) u^{\alpha}$$

символ Леви $\psi(\theta) = i\mu\theta + \frac{\sigma^2\theta^2}{2} + \int_{-\infty}^{\infty} (1 - e^{i\theta x} + i\theta x \mathbb{1}_{\{|x|\le 1\}}) \Pi(dx))$

$\uparrow$ снос $\uparrow$ броун. движ. (σ^2)

(у субординаторов $\sigma^2 = 0$)

(просто вверх)

если $\mathbb{1}$ => чистый Pois

$\int_{-\infty}^{\infty} \Pi(dx) = \infty$ $(\lambda F(dx))$

(если конечен => и леви с CPois)

$\int_{-\infty}^{\infty} |x| \Pi(dx) < \infty$

(процесс с огр. вариацией)

$\int_{-\infty}^{\infty} x^2 \Pi(dx) < \infty$

$\uparrow$ процесс Леви

$$E[e^{i\theta\sum_{j=1}^{N(t)} Y_j}] = e^{-\lambda(\int (1-e^{i\theta x})\Pi dx}$$

$$EE[e^{\theta\sum_{j=1}^{N(t)} Y_j} | N(t) = k] = E[(e^{\theta Y})^k] =$$

$$= \sum_{k=1}^{\infty} e^{-\lambda t} \frac{\lambda t^{k-1}}{k!} (E e^{\theta Y})^k = e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t E[e^{\theta Y}])^k}{k!}$$

$$= e^{\lambda t(\int_0^\infty (e^{i\theta x}-1) F dx)}$$

$$E\left[e^{i\theta\psi(t)}\right] = e^{\cancel{\ \ } - t\,\psi(\theta)} \Rightarrow$$

$$\Rightarrow \psi(\theta) = \lambda \int_0^\infty (1 - e^{i\theta x}) F(dx)$$

$$\pi(dx) = \lambda F(dx)$$ ← велич. страх. треб.

⇓ Comp. Pois

$$\pi(dx) = \frac{1}{\sqrt{2\pi}} x^{3/2} e^{-\mu^2 x/2} \mathbb{1}_{\{x>0\}}$$

$$\int_0^\infty (1 - e^{i\theta x}) \pi(dx) = \sqrt{\mu^2 - 2i\theta} - \mu$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{(1 - e^{i\theta x})}{x^{3/2}} e^{-\mu^2 x/2} dx = \sqrt{\mu^2 - 2i\theta} - \mu$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{(e^{-\mu^2 x/2} - e^{i\theta x - \mu^2/2} + 1 - 1)}{x^{3/2}} dx = \sqrt{\mu^2 - 2i\theta} - \mu$$

⇒ 5.5.36

Рекуррентные процессы

$X \sim Pois(\lambda)$

$E[X_t] = \lambda t$

$Var[X_t] = \lambda t$

$\Rightarrow$ среднее экспон. длина распр.

$Y_t = \left\lfloor \frac{N_t}{2} \right\rfloor \qquad E(Y_t) \neq \frac{\lambda t}{2}$

$Var(Y_t) \neq \frac{\lambda t}{2}$

а) тер. вер.

б) DH-метод

в) преобр. Лапласа

$$\underset{\varphi(z,t)}{E[z^{N(t)}]} = \sum_{j=0}^{\infty} z^k P(N(t) = k) = \sum_{j=0}^{\infty} z^k \Big(P(X_1 + \ldots + X_n < t) - P(X_1 + \ldots + X_{n+1} < t) \Big)$$

Рекуррентные потоки

$S(t) = \sum_{j=1} Y_j$

$N(t) = \sup_n \{ Y_1 + \ldots + Y_n < t \}$

$Y_1, Y_2 \ldots \geq 0$ — i.i.d. н.о.р.

если $Y \sim \exp \Rightarrow Pois$

$$\hat{\varphi}(z,t) = \int e^{-st} E[z^{N(t)}]\, ds =$$

$$= \frac{1}{s} \sum_{j=0}^{\infty} z^k \left(\hat{f}(s)^k - \hat{f}^{k+1}(s) \right) = \frac{1}{s}(1 - \hat{f}(s)) \sum_{j=0}^{\infty} z^k \hat{f}^k(s)$$

$$\circledast \int_0^\infty e^{-st} \lambda e^{-\lambda t} dt = \left(\frac{\lambda}{s+\lambda}\right) \Rightarrow \hat{f} = \left(\frac{\lambda}{s+\lambda}\right)^2 \qquad = \frac{1 - \hat{f}(s)}{1 - z\hat{f}(s)} \cdot \frac{1}{s}$$

$\hat{}$ преобр. Лапл. от Σ exp.

$$\left(\int_0^\infty e^{-st} \; E[z^{N(t)}]\,dt\right)'_z\Big|_{z=1} =$$

$$= \int_0^\infty e^{-st} \; E[N(t)]\,dt = \hat{m}(s)$$

$$\hat{f}(s) = \left(\frac{\lambda}{\lambda+s}\right)^2$$

$$\text{Inv. Lap}\left[\frac{\lambda^2}{s^2(\lambda^2+2\lambda s)}\right]$$

$$m = \underbrace{\frac{\lambda t}{2}}_{\left[\frac{\lambda t}{3}\right]} - \underbrace{\frac{1}{4}}_{\text{зл. член const}} + \underbrace{\frac{e^{-2\lambda t}}{4}}_{\text{поправка}}$$

$\left\lfloor\frac{\lambda t}{3}\right\rfloor$; зл. член const ← поправка

$$m = -\frac{1}{3} + \frac{\lambda t}{3} + \frac{e^{-3\lambda t/2}\,\sin\left(\frac{1}{2}\sqrt{3}\lambda t\right) + \sqrt{3}\cos\left(\frac{1}{2}\sqrt{3}\lambda t\right)}{3\sqrt{3}}$$

Твеке 1

РИ ~~1~~ 2

порезн. ином 1

CL на пон. врем. fin. Time win 1

Gamma Variance 1

A1. (7)

(I) (a) $N^{(2)}(t) = \frac{1}{2}\left[N(t) - \mathbb{1}\{N(t) \text{ odd}\}\right]$

$E\, N^{(2)}(t) = \frac{1}{2}\left[\lambda t - P(N_{odd})\right] = \frac{1}{2}\left(\lambda t - e^{-\lambda t}\frac{1}{2}(e^{\lambda t} - e^{-\lambda t})\right) = \frac{\lambda t}{2} - \frac{1}{4} + \frac{1}{4}(1 - e^{-2\lambda t})$

$P(N_{odd}) = \cancel{e^{-\lambda t}} \sum_{k\ odd} \frac{(\lambda t)^k}{k!} = \frac{1}{2}(e^{+\lambda t} - e^{-\lambda t})$

$\sum \frac{(\lambda t)^k}{k!} - \sum_{k\ even} \frac{(\lambda t)^k}{k!} + \sum \frac{(\lambda t)^k}{k!}$

$\int_0^\infty e^{-st} E[z^{N^{(2)}(t)}]\,dt = \sum_j z^j \left[F^{*j}(t) - F^{*(j+1)}(t)\right]$

$F(t) = Exp(\lambda) \cdot Exp(\lambda)$

$\phi(z,s) = \frac{1}{s}\sum_j z^j\left[\hat{f}^j(s) - \hat{f}^{j+1}(s)\right] = \frac{1-\hat{f}(s)}{s}\sum z^j \hat{f}^j(s) = \frac{1-\hat{f}(s)}{s(1-z\hat{f}(s))}$

$\phi'(z,s) = \dots\ \dots = \frac{\hat{f}(s)}{s(1-\hat{f}(s))} = \frac{\lambda^2/(\lambda+s)^2}{s(1-\lambda^2/(\lambda+s)^2)} = \frac{\lambda^2}{s(s+2s\lambda)}$

$\int_0^\infty e^{-st}\lambda e^{-\lambda t}\,dt = \left(\frac{\lambda}{\lambda+s}\right)$

$\hat{f}(s) = \frac{\lambda^2}{(\lambda+s)^2}$

$\text{Inv Laplace}\left[\frac{\lambda^2}{s^2(s+2\lambda)}\right] = -\frac{1}{4} + \frac{\lambda t}{2} + \frac{1}{4}e^{-2\lambda t}$

(b) • $N^{(2)}(t) = \int_0^t \cancel{\ldots}\ v(x)\,dx$

$v(x) = \alpha e^{(T+t\cdot\alpha)x}t \qquad F \sim (\alpha, T)$

$\underbrace{\begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}}_{T=}\underbrace{\begin{pmatrix} 0 \\ \lambda \end{pmatrix}}_{t=} \qquad \alpha = (1, 0)$

$x \cdot \lambda^2 \cdot e^{-\lambda x} = (1\ 0)\, e^{\begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}x}\begin{pmatrix} 0 \\ \lambda \end{pmatrix}$

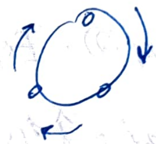
$\mu^n x^{n-1} e^{-\mu x} = (1\,0\ldots0)\, e^{Tx}\begin{pmatrix} 0 \\ \vdots \\ \lambda \end{pmatrix} \qquad T = \begin{pmatrix} -\lambda & \lambda & & 0 \\ & -\lambda & \lambda & \\ 0 & & \ddots & \\ & & & -\lambda \end{pmatrix} \qquad t = \begin{pmatrix} 0 \\ \vdots \\ \lambda \end{pmatrix}$

$$\begin{pmatrix}-\lambda & \lambda\\ 0 & -\lambda\end{pmatrix} + \begin{pmatrix}0\\ \lambda\end{pmatrix}(1\ \ 0) = \begin{pmatrix}-\lambda & \lambda \\ \lambda & -\lambda\end{pmatrix}$$

$$e^{\begin{pmatrix}-\lambda&\lambda\\ \lambda&-\lambda\end{pmatrix}\cdot x} = \frac12\begin{pmatrix}1+e^{-2\lambda x} & 1-e^{-2\lambda x}\\ 1-e^{-2\lambda x} & 1+e^{-2\lambda x}\end{pmatrix}$$

$$E N^{(2)}(t) = \int_0^t \frac{\lambda}{2}(1-e^{-2\lambda x})dx = \frac{\lambda t}{2} - \frac14 + \frac14 e^{-2\lambda x}$$

$$E N^{(3)}(t) = E\left\lfloor \frac{N(t)}{3}\right\rfloor = -\frac13 + \frac{\lambda t}{3} + e^{-\frac{3t\lambda}{2}}\left(\frac13\cos\left(\frac{\sqrt3}{2}t\lambda\right) + \frac{\sqrt3}{9}\sin\left(\frac{\sqrt3}{2}t\lambda\right)\right)$$

$T+t\alpha$ <=> 

(I) $$E\left[\left(N^{(2)}(t)\right)^2\right] = \frac14\Big[\underbrace{E\,N(t)^2}_{=E_1} - 2\underbrace{E\,N(t)\,\mathbb{1}_{\{N(t)\text{ odd}\}}}_{=E_2} + \underbrace{P(N(t)-\text{odd})}_{=E_3}\Big]$$

$E_1 = \lambda t + \lambda^2 t^2$

$E_3 = \frac12(e^{\lambda t} - e^{-\lambda t})e^{-\lambda t} = \frac12(1-e^{-2\lambda t})$

$E_2 = e^{-\lambda t}\sum_{n \text{ odd}} n\frac{(\lambda t)^n}{n!} = \lambda t\, e^{-\lambda t}\sum_{n=\text{even}}\frac{(\lambda t)^n}{n!} = \lambda t e^{-\lambda t}\frac12(1+e^{-2\lambda t})$

А.Д. (7)

ergodic

1) $X(t) = t - \sum_{j=1}^{N(t)}$

$\lambda \mu_1 < 1$

$\dots < \infty$

$E[e^{-\theta B}] = \int_0^\infty E[e^{-\theta \tau_x}] F(dx) = \hat{F}[\phi(\theta)]$

$C = \lambda_1 \mu_1 (1+t)$

$e^{\overset{\phi(t)}{\theta} X(t) - \overset{\theta}{\varkappa(\theta)} t}$

$e^{\phi(x) X(t) - \theta t}$

$\mu_1 = E[X_1]$

$E[e^{\phi(\theta) X(\tau_x)^{=x} - \theta\tau}] = 1$

$(B \mid X_1 = x) \sim \tau_x$

мартингал Вальда

$E[e^{\phi(\theta)x - \theta\tau_x}] = 1$

в виде спек. спектр. отриц.

$e^{-\phi(\theta)x}$ $\quad E[e^{-\theta\tau_x}] = e^{-\phi(\theta)x}$

символ: $\frac{1}{t} \ln E e^{\theta X(t)} = \varkappa(\theta)$

$[E e^{\theta t} [E e^{-\theta \sum_{j=1}^{N(t)} y_j} \mid N(t) = k]] =$

$= \theta + \lambda \int_0^\infty (e^{-\lambda x} - 1) F(dx)$

$= e^{\theta t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} (E e^{-\theta y})^k =$

$= e^{\theta t} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t\, E e^{-\theta y})^k}{k!}$

$\varkappa(\theta) = \eta$

$\theta + \lambda \int_0^\infty \mu e^{-\mu x} (e^{-\theta x} - 1) dx = \eta$

$\theta + \frac{\lambda\mu}{\mu + \theta} - \lambda = \eta \quad \overset{\theta>0}{\Rightarrow}{?} \quad \phi(x)$

$\hat{F}(\phi(u)) = \frac{\mu}{\phi(u) + \mu} \qquad \hat{F}(s) = \int_0^\infty \mu e^{-sx} e^{-\mu x} dx = \frac{\mu}{s+\mu}$

$\uparrow$

дифр. 2 рода

$\tau_x = \inf\{t : X(t) = \infty\}$

$(B \mid X_1 = 2) \simeq \tau_x$

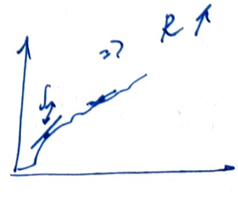
$E[e^{-\theta\beta}] = \int_0^\infty e^{-\theta \tau_x} F(dx) = \hat{F}(\phi(\theta))$

$\quad\quad\quad\quad\quad\quad\quad\quad \overset{\shortparallel}{e^{-\phi(\theta)x}}$

$\Downarrow$

$\mathrm{Var}[B] = \frac{\lambda + \mu}{(\mu - \lambda)^3}$

$E[B] = \frac{1}{\mu - \lambda}$



R ↑ ⇒? сист. более устойчива

2) $U(t) = u + \underset{\shortparallel}{\overset{(1+\theta)\lambda\mu_1}{ct}} - \overbrace{\sum_{j=1}^{N(t)} y_j}^{=S(t)}$

$(1+\theta) \int_0^t \frac{S(u)}{u} du \quad \leftarrow$ адаптивная модель страхования (скор роста расх. пропорц. до пред. мом.)

$S(u)/u = \sum_{j=1}^{N(t)} y_j / u$

$$E[e^{RV}] = 1 + (1+\theta) E[y]$$

$$E\left[\frac{e^{\gamma y}}{1+(1+\theta) y\gamma}\right] = 1 \qquad \gamma \geq R$$

$$L(t) = \sum_{j=1}^{N(t)} y_j - (1+\theta)\int_0^t \frac{\sum_{j=1}^{N(u)} y_i}{u} du$$

$$(1+\theta)\sum_{j=1}^{N(t)} \int \frac{du}{u} \mathbb{1}\{T_j \leq u \leq t\}$$

$$\Downarrow$$

$$\int_{T_j}^t du/u = \ln t/T_j$$

$$L(t) = \sum_{j=1}^{N(t)} y_j (1 - (1+\theta)\ln t/T_j)$$

$$W_j \sim \underbrace{y_j (1 + (1+\theta)\ln \theta_j)}_{\text{фиктивные длины страх. треб.}}$$

$$\{T_{j}/t\} \sim U^{(k)}$$

pois $\xrightarrow{1}$ $5^{\infty}+$ $t = 15$

сколько все вместе прош. входит - порядк. стат.

$$N(t) = k$$

$$\left(\frac{T^{(1)}}{t}, \frac{T^{(2)}}{t}, \ldots, \frac{T^{(k)}}{t}\right) \sim$$

$$\sim (U^{(1)}, U^{(2)} \ldots U^{(k)})$$

$$V(t) = \sum_{j=1}^{N(t)} W_j$$

$$E[e^{rW}] = E[e^{r y_j} e^{\ln \theta_j (1+\theta) y}] =$$

$$= E[e^{ry} \theta^{r(1+\theta)y}] =$$

$$= E[e^{ry} \int_0^1 e^{(1+\theta) y t} dt]$$

3) Модель дробового шума: выстрелы

$$S(t) = E\left[\sum_{j=1}^{N(t)} \underbrace{e^{-\lambda(t-T_j)}}_{\text{затухн. шума}} y_j \;\middle|\; N(t) = k\right]$$

суммарн. шум в уселях

$$= -\lambda t\left(1 - \frac{T_j}{t}\right) \sim U^{(k)}$$, т.к. в каком порядке суммирую не важно;

$$E[S(t) | N(t) = k] = k \cdot E[e^{-\lambda t(1-U)} y] =$$

$$= k \underbrace{E[y]}_{\mu} \int_0^1 e^{-\lambda t \theta} d\theta = \frac{k\mu}{\lambda t}[1 - e^{-\lambda t}]$$

$$, E[K] = \lambda t$$

$$E[S(t)] = \frac{\lambda \mu}{\lambda}(1 - e^{-\lambda t})$$

ур. фон. шума (стационарн. уровень)

4) $S_n = Y_1 + \ldots + Y_n$

$N(t) = \max\{n : S_n < t\}$ recurrent point process

$N^{(2)}(t) = \lfloor \frac{N(t)}{2} \rfloor$

$\mathrm{Var}\, N^{(2)}(t)$

$P(N(t) = n)$

свертка

$$E[z^{N(t)}] = \sum_{k=0}^{\infty} z^k \underbrace{\left(F^{*k}(t) - F^{*k+1}(t)\right)}_{P(N(t)=k)}$$

$$\int_0^\infty e^{-st} E[z^{N(t)}]dt = \int_0^\infty e^{-st} \sum_{k=0}^\infty z^k(-\ldots-)\,dt$$

$$= \sum_{k=0}^\infty z^k\left(\frac{\hat{f}(s)^k}{s} - \frac{\hat{f}(s)^{k+1}}{s}\right) = \cancel{\frac{\hat{f}(s)(1-\hat{f}(s))}{s}} \frac{(1-\hat{f}(s))}{s}\sum_{k=0}^\infty z^k \hat{f}(s)^k =$$

$$= \frac{1-\hat{f}(1)}{s(1-z\hat{f}(s))}$$

$\downarrow$

$$p(\;\;, z)\Big|_{z=1} = \frac{\hat{f}(s)}{s(1-\hat{f}(s))}$$

$\Downarrow$

$$\hat{m}(s) = \int_0^\infty e^{-st} E[N(t)]dt$$

$$E[N(N-1)] = \sum_{k=0}^\infty k(k+1)\left(F^{*k}(t) - F^{*k+1}(t)\right) =$$

$$= \frac{\hat{f}(s)(1-\hat{f}(s))}{s}\sum k(k+1)\hat{f}^{k+1} =$$

$$= \frac{\hat{f}(s)(1-\hat{f}(s))}{s}\left(\frac{1}{1-\hat{f}(s)}\right)^k = \frac{2\hat{f}(s)}{s(1-\hat{f}(s))^2}$$

$$E[N(t)(N(t)-1)] - E(N) - E(N(t))^2 = \mathrm{Var}(N(t))$$

A.L. (8)

- рек.
- бурь мен
- автобусы
...

Bayes vs Credibility Theory (Buhlman)

Pois(θ)

$\theta \in U(1, 3)$

Years	Claims
1	1
2	2
3	1

← полис

$i = 1, \dots, T$ $\quad j = 1, \dots, n$

$\{\theta_i, \{X_{ij}\}\}$ $\quad \theta_i$ – пар-р неоднородности

(1) cond. ind. for fixed θ

(2) $Cov(X_{ij}, X_{ij'}) \geq 0$

$Cov(X_{ij}, X_{ij'})$

$$P(B) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \prod_{i=1}^{n} \frac{\theta^{x_i}}{x_i!} e^{-\theta}$$

$$c_1 \theta^{\alpha + X_T - 1} e^{-\theta(\beta + n)}$$

$$\alpha' = \alpha + X_T \qquad \beta' = \beta + n$$

$$E[\theta \mid X] = \frac{\alpha + X_T}{\beta + n}$$

$$\boxed{Z = \frac{n}{n+\beta}}$$ ← оптимальное значение минимизирующее погрешность

← чему мы верим берем из априорного

$$(1 - Z)\frac{\alpha}{\beta} + Z \bar{X}$$

↑ уровень доверия (пар-р Бульмана)

Double Poisson $\leftrightarrow$ Poisson - Gamma Model

$\theta \in G$ $\quad P(N=k \mid \theta) = \frac{\theta^k}{k!} e^{-\theta}$

$Cov(X_{ij}, X_{ij}) = E[Var(\mu \mid t)]$ $\overset{!}{=} 0$

$$Z = \frac{3}{3 + \frac{2}{1/3}} = \frac{1}{3}$$

$E[\theta]$

$Var[\theta]$

$\frac{4}{3} \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} = 1,7778$ Bayes $1,7542$

$1 \cdot \frac{1}{7} + 2 \cdot \frac{6}{7} = 1,8571$ $1,85054$

$$\frac{10}{10 + \frac{2}{1/3}} = 5/8$$

1
2
2
1
3
1
4
0
1
2

17

$\frac{17}{10} \cdot \frac{5}{8} + 2 \cdot \frac{3}{8} = 1,8125$ $1,80263$

EX Vaxh

$$\mathrm{Cov}(X_{ij}, X_{ij}) = E[\underbrace{\mathrm{cov}(X_{ij}, X_{ij} \mid \sigma_1)}_{\mathrm{Var}(X_{ij} \mid \sigma_1)}] +$$

$$+ \mathrm{Cov}[E(\underbrace{X_i \mid \sigma_1) \cdot E(X_2 \mid \sigma_2}_{=0})]$$

$$P(N^{(m)}(t) = k) =$$

4.6.16. $$\int_0^\infty \frac{(\theta t)^k}{k!} e^{-\theta t} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta t} d\theta =$$

$$= \frac{1}{t^{k+\alpha-1}} \cdot \frac{1}{t} e^{-\beta\theta} \frac{\beta^\alpha t^k}{k! \Gamma(\alpha)} \int_0^\infty (t\theta)^{k+\alpha-1} e^{-\theta t} d\theta =$$

$$\int_0^\infty \underbrace{u^{k+\alpha-1} e^{-u} du}_{= \Gamma(\alpha + k)}$$

$$= \frac{\Gamma(\alpha+k)}{\Gamma(\alpha) \cdot k!} \frac{\beta^\alpha t^k}{t^{k+\alpha}} = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha) \cdot k!} \cdot \left(\frac{\beta}{t}\right)^\alpha \overset{?}{=} \left(\frac{\beta}{\beta+t}\right)^\alpha$$

....

random
$\theta \lambda(t)$

for Double Poisson

$\mathrm{Var}[N(t)] > E[N(t)]$

(по ф-ле полн. дисперсии)

А.Д. (9)

Kelly Betting

Ставки по Келли

$\circ \xrightarrow{1/2} 80\%$, $\circ \xrightarrow{1/2} 60\%$

$80 - 60 = 20\%$ $\quad S_n : 20\% \quad \forall n$

$S\, 1.8^{n/2} \cdot 0.4^{n/2} = S \cdot (1.8^{n/2} \cdot 0.4^{n/2}) = S \cdot 0.72^{n/2} \longrightarrow 0$

Принцип Беллмана

$x_{t+1} = x_t + (x_t - u_t)$

$x_T = 0 \qquad x_0 = S$

$\left(\sum_{t=1}^{T} x_t \to \max \right)$ $\quad \boxed{E[\log x_T] \to \max}$

$F_0(x) = \ln x$

$F_1(x) = \sup_{0 \le u \le x} [\theta_1 F(x+u) + (1-\theta_1) F_0(x-u)] =$

$= \sup_{0 \le p \le 1} [\theta_1 \ln x(1+p) + (1-\theta_1) \ln x(1-p)] =$

$= \ln x + \sup_{0 \le p \le 1} [\theta_1 \ln(1+p) + (1-\theta_1) \ln(1-p)]$

$\circ \xrightarrow{p}$, $\circ \xrightarrow{q}$

$p > q \ \forall n$

$u = p x$

(Kelly betting)

$\theta_i = p_{T+1-i}$

$\dfrac{\theta_1}{1+p} = \dfrac{1-\theta_1}{1-p}$

$\theta_1 - \theta_1 p = 1 + p - \theta_1 - \theta_1 p$

$\hat{p}_j = 2\theta_j - 1$

$F_2(x) = \ln x + \theta_1 \ln(1+\hat{p}_1) + (1-\theta_1) \cdot \ln(1-\hat{p}_1) + \theta_2 \cdot \ln(1-\hat{p}_2) + (1-\theta_2) \cdot \ln(1-\hat{p}_2)$

$F_T(x) = \ln x + \sum_{j=1}^{T} p_j \ln(2p_j) + (1-p_j) \ln(2(1-p_j))$

$= \ln x + T \cdot \ln 2 - \sum_{j=1}^{T} H(p_j)$

$\uparrow$ Shannon

$H(p) = -p \log p - q \cdot \log q$

Задача о разборчивой невесте $1/e$ - days

собрать, взять лучшего после

с вер. $1/e$ можно получить лучшего

6.17.

$1-\beta$ $F = 0$

β max

$\beta = 1/2$ $g(x) = \frac{2}{x^3}$, $x \geq 1$

2 случая незав. и один. распр.

=> Евр. парето выбрать первое, пот. дальше

$\sqrt{\beta/(1-\beta)}$

$$F(1) = \int_0^\infty \max[y, \beta F(1)]\, g(y)\, dy =$$

$$\beta F(1) + \int_0^\infty \max[y - \beta F(1), 0]\, g(y)\, dy =$$

$$\beta F(1) + \int_{\beta F(1)}^\infty (y - \beta F(1))\, g(y)\, dy$$

$$(1-\beta) F(1) = \int_{\beta F(1)}^\infty (y - \beta F(1))\, g(y)\, dy$$

$$(1-\beta) F(1) = \int_{\beta F(1)}^\infty \dots \Rightarrow \frac{1}{\sqrt{\beta(1-\beta)}}$$

$$\int_1^\infty \Rightarrow \text{противоречие}$$

$$F(x,t) = \sup_u \left[C(x,u,t)\cdot \Delta t + e^{-\alpha \Delta t} F(x + a(x,u,t)\Delta t,\ t+\Delta t) \right]$$

$$F(x,t) = \sup_u \left[C(\cdot)\cdot \Delta t + (1-\alpha\Delta t)\left(F(x,t) + \frac{\partial F}{\partial t}\cdot \Delta t + \frac{\partial F}{\partial x} a(x,u,t)\Delta t\right)\right]$$

$$\sup_u \left[C(x,u,t) - \alpha F + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\cdot a \right] = 0$$

e.g.: $C(x,u,t) = \sqrt{u}$ $\quad a(x,u,t) = \beta x - u$

$$\sup_u \left[\sqrt{u} - \alpha F + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\cdot(\beta x - u) \right] = 0$$

$$F(x,t) = \sqrt{x}\cdot \phi(t) \qquad \phi(T) = 0$$

$$\cancel{\frac{1}{2\sqrt{u}} - \alpha\sqrt{x}\,\phi(t) + \sqrt{x}\,\phi'(t) + \frac{\phi}{2\sqrt{x}}(\beta x}$$

$$\frac{1}{2\sqrt{u}} - \frac{\phi(t)}{2\sqrt{x}} = 0$$

$$\sqrt{x/u} = \phi(t) \qquad u = \frac{x}{\phi^2(t)}$$

$$F = x/\sqrt{u} \qquad c = \frac{\sqrt{x}}{\phi(t)}$$

$$\frac{\sqrt{x}}{\phi(t)}\left(\frac{1}{2} - \left(\alpha - \frac{1}{2}\beta\right)\cdot\phi^2 + \phi\phi'\right)$$

6.1.9 $\int_0^T e^{-\alpha t}\sqrt{u(t)}\,dt \longrightarrow \max$

$\dot{x}(t) = \beta x(t) - u(t) \qquad x(T)=0$

$dx = (\beta x(t) - u(t))\,dt$

$F(x,t) = \sup_u \left[\sqrt{u}\,\Delta t + \underbrace{e^{-\alpha\Delta t}}_{(1-\alpha\Delta t)} F(x + \underbrace{(\beta x(t) - u(t))\Delta t}_{dx},\ t+\Delta t)\right]$

$F(x,t) = \sup_u \left[\sqrt{u}\,\Delta t + (1-\alpha\Delta t)F(x,t) + \frac{\partial F}{\partial t}\Delta t + \frac{\partial F}{\partial x}(\beta x(t) - u(t))\Delta t\right]$

$0 = \sup_u \left[\sqrt{u} - \alpha F + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}(\beta x - u)\right] = \sup_u \left[\sqrt{u} - \alpha\sqrt{x}\cdot f + \sqrt{x}\cdot f' + \frac{1}{2\sqrt{x}}f(\beta x - u)\right]$

$F(x,t) = \sqrt{x}\,f(t)$

$\frac{\partial F}{\partial t} = \sqrt{x}\,f', \quad \frac{\partial F}{\partial x} = \frac{1}{2\sqrt{x}}f$

$\xi'_u = \frac{1}{2\sqrt{u}} - \frac{1}{2\sqrt{x}}f = 0$

$u^* = \frac{x}{f^2}$

$\frac{\sqrt{x}}{f} - \alpha\sqrt{x}f + \sqrt{x}f' + \frac{1}{2\sqrt{x}}\left(\beta x - \frac{x}{f^2}\right) = 0 \quad \Big| : \sqrt{x}$

$\frac{1}{f} - \alpha f + f' + \frac{f}{2}\beta - \frac{1}{2f} = 0$

$\frac{1}{f}\left[1 - \alpha f^* + ff' + f^2\frac{3}{2} - \frac{1}{2}\right] = 0$

$\frac{1}{f}\left[1/2 - (\alpha - 1/2)\beta\cdot f^* + f\cdot f'\right] = 0$

$f^2 = v$

$(f^2)'/2 - 2f\cdot f'\cdot 1/2 - ff'$

$$\frac{v'}{2} - (\alpha - \frac{1}{2}\beta) v + \frac{1}{2} = 0$$

$$\frac{v'}{2} = (\alpha - \frac{1}{2}\beta) \cdot v$$

$$\int \frac{d v}{v} = 2 \int (\alpha - \frac{1}{2}\beta)\, dt \cdot v = \lambda^2$$

$$\Rightarrow v = e^{2(\alpha - 1/2)\beta t} + C \cdot e^{(2\alpha - \beta)t} + c(t) = 0$$

$$\Rightarrow \quad C = \frac{1}{2\alpha - \beta}$$

$$v = \frac{1}{2\alpha - \beta} e^{(2\alpha - \beta)t} = \frac{1}{2\alpha - \beta}$$

$$\lambda^2(t) = \frac{1 - e^{(2\alpha - \beta)(T - t)}}{2\alpha - \beta}$$

6.1.2 bang - bang control

$$C = \sum_{t=0}^{h-1} u_t$$

$x_t - u_t$

$0 \le u_t \le x_t$

$\theta \cdot 100\%$

$$x_t = a(x_t, u_t) = x_t + \theta(x_t - u_t)$$

$$s = h - t$$

$$F_s(x) = \max_{0 \le u \le x} [u + F_{s-1}(x + \theta(x - u))]$$

$$F_0(x) = 0$$

$$F_1(x) = \max_{0 \le u \le x} [u + F_0(x + \theta(x - u))] = \max_{0 \le u \le x} [u + 0] = x$$

$$F_2 = \max_{0 \le u \le x} [u + F_1(x + \theta(x - u))] = \max_{0 \le u \le x} [u + x + \theta(x - u)] =$$

$$\max x = 0 \vee x \Rightarrow \quad = \max_{0 \le u \le x} [(1+\theta)x, 2x] = \max [1 + \theta, 2]\, x = \beta_2 x$$

$$F_3(x) = \max_u \left[u + 2(\lambda + \theta(x-u)) \right] =$$

$$= \max_u \left[2(1+\theta), 3 \right] x = \beta_3 x$$

$$F_s(x) = \max \left[(1+\theta) \cdot \beta_{s-1}, 1 + \beta_{s-1} \right] x = \beta_s x$$

$$\beta_s = \begin{cases} s, & s \leq s^* \\ (1+\theta)^{s-s^*}, & s \geq s^* \end{cases} \qquad \Leftarrow \quad (1+\theta)\beta_{s-1} > 1 - \beta_{s-1}$$