1 Метод обращения.

$F^{-1}(t) = \inf\{x, F(x) \geq t\}$, $X \sim F$

если $U \sim U(0,1)$, то $F^{-1}(U)$ имеет тот же закон, что и X

ex $F: t \in \mathbb{R} \to \sum p_i \mathbb{I}_{t \geq x_i}$

$F^{-1}: u \in [0;1] \to \sum x_i \mathbb{I}_{p_1+\dots+p_{j-1} \in u \leq p_1+\dots+p_j}$

ex $P(X > t) = \exp(-\lambda t)$

$F(t) = 1 - e^{-\lambda t}$

$F^{-1}(x) = -\frac{\log(1-x)}{\lambda}$

$-\frac{\log(1-x)}{\lambda} \sim \mathcal{E}(\lambda)$; $-\frac{\log(U)}{\lambda} \sim \mathcal{E}(\lambda)$, as $1-U \sim U[0,1]$

Pareto: $\frac{1}{(1-U)^{\alpha}} \sim Par(1,\alpha)$

$\frac{1}{U^{\alpha}} \sim Par(1,\alpha)$

Lemma: $U \sim U[0,1]$, then

$V = F_\mu(a) + (F_\mu(b) - F_\mu(a))U \sim U[F_\mu(a), F_\mu(b)]$

and

$F^{-1}(V) \sim \mathbb{P}(F^{-1}(V) \leq x) = P(X \leq x \mid a < X \leq b)$

Lemma: $U, V \sim U[0,1]$

$\left(\sqrt{-2\log(U)}\cos(2\pi V), \sqrt{-2\log(U)}\sin(2\pi V)\right) \sim N\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)$

Мет. выб. с отклонением

Пусть $\exists$ закон g, кот. легко мод-ся:

$\forall x \in \mathbb{R}^d \quad f(x) \leq k \cdot g(x)$

для некоторой k – const

Пусть $\alpha(x) = \dfrac{f(x)}{k \cdot g(x)}$ на $\{x;\ g(x) > 0\}$

Пусть $(X_n, U_n)_{n \geq 1}$: $X_n \sim g(x)$, $U_n \sim [0,1]$ $X_n \perp U_n$

Пусть $T = \inf\{n,\ U_n \leq \alpha(X_n)\}$

Пусть $X = X_T \Rightarrow X_T \sim f(x)$

(ex) $f(x) = \dfrac{e^{-x^3}}{Z} \mathbb{1}_{\{x \geq 1\}}$, где $Z = \int_1^{+\infty} e^{-x^3} dx$

– const, неизв., но её знач. не исп-ся

Пусть $g(x) = \dfrac{\mathbb{1}_{\{x \geq 1\}}}{x^2}$

$\uparrow$ с пом. мет обращения

$h: x \geq 1 \rightarrow x^2 e^{-x^3}$

$h'(x) = (2x - 3x^4) e^{-x^3} \leq 0 \quad \forall x \geq 1$

$\Rightarrow \forall x \geq 1 \quad h(x) \leq h(1) = e^{-1}$

$f(x) \leq \dfrac{1}{e} \cdot \dfrac{1}{Z} \cdot g(x)$

g(x) обр

Пусть $k = \dfrac{1}{eZ} \Rightarrow \alpha(x) = \dfrac{f(x)}{g(x)} = x^2 e^{-x^3+1}$

if $(u < y^2 \cdot \exp(-y^3+1)) \Rightarrow$ print(y)

(ex) $f(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$, $g(x) = \dfrac{e^{-|x|}}{2}$

$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}}\, e^{(-x^2/2 + |x|)} \le \sqrt{\frac{2e}{\pi}} = c$$

$\Rightarrow$ Rejection test $U > f(x) / c\, g(x)$

$$U > \sqrt{\frac{2}{\pi}}\, e^{-x^2/2 + |x| - 1/2} = e^{-(|x|-1)^2/2}$$

$z = -\log(U_1)$

if ($U_2 > e^{-(|z|-1)^2/2}$) then $\Rightarrow$ reject

else $\nearrow$ $1/2$ $X = -Z$

$\searrow$ $1/2$ $X = Z$

$$\int \frac{dx}{x} = \ln|x| + c \qquad (\log_a x)' = \frac{1}{x \ln a}$$

$$\int a^x dx = \frac{a^x}{\ln a} + c \qquad (a^x)' = a^x \cdot \ln a$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arctg}\frac{x}{a} + c \qquad (\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + c$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + c$$

Norm: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Bin: $f(x) = C_n^m p^m q^{n-m}$ $\quad E(X) = np \quad Var(X) = npq$

Pois: $f(x) = \frac{e^{-\lambda}\lambda^k}{k!} \quad E(x) = \lambda = Var(x)$

Exp: $f(x) = \lambda e^{-\lambda x} \quad F(x) = 1 - e^{-\lambda x} \quad E(X) = \frac{1}{\lambda} \quad Var(X) = \frac{1}{\lambda^2}$

Unif: $f(x) = \begin{cases} c, & x \in [a,b] \\ 0, & x \notin [a,b] \end{cases} \quad F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{a-b} & x \in [a,b) \\ 1 & x > b \end{cases}$

$$E(X) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

2 Monte Carlo

SLLN:

$\bar{X}_n := \frac{1}{n} \sum X_i \xrightarrow[n\to\infty]{} E[X]$

CLT:

$\frac{\sqrt{N}\left(\frac{1}{N}\sum X_i - E(X_1)\right)}{\sqrt{Var(X_1)}} \xrightarrow{d} N(0,1)$, where $Var(X) = E((X_1 - E(X_1))^2)$

Lemma

$I = \int_{[0,1]^d} f(u_1, \ldots u_n)\, du_1 \ldots du_d$

$I = E(f(U_1, \ldots, U_d)) = E(X)$

Fatou's Lemma:

If $\{X_n\}_n$ seq of r.v. s.t. $X_n \geq Y$ a.s. $(Y \in L^1)$

$E[\liminf_n X_n] \leq \liminf_n E[X_n]$

Big O (Bachman - Landau) notation

$f(x) = O(g(x))$, $x \to a$

if f is bounded from above by g

in the neigbourhood that is

- if $|a| < \infty$ $\exists\ 0 < M < \infty$, $\delta > 0$:

  $|f(x)| \leq M|g(x)|$ $\forall x \in R$ s.t. $|x - a| \leq \delta$

- if a is inf. $\exists\ 0 < M < \infty$, $k > 0$ s.t.

  $|f(x)| \leq M|g(x)|$ $\forall x \in R$ s.t. $|x| \geq k$

Conf. int:

for $\varepsilon = 1,96$ prob. that $\frac{\sum f(x_i)}{N}$

is in $\left\{ \int_0^1 f(x)dx \pm \frac{1.96\, \sigma_f}{\sqrt{N}} \right\} \approx 0,95$

σ_F can be approx. by $\hat{\sigma}_{F,N} = \frac{1}{N-1} \sum \left(f(u_i) - \frac{\sum f(u)}{N}\right)^2$

M.C. acceptance - rejection

for sim. $E(h(X))$ for $X \sim \mu$ $h: \mathbb{R} \to \mathbb{R}$, s.t. $E[|h(x)|] < \infty$

let $(U_i, Y_i)_{n \geq 1}$ seq. of r.v. i.i.d.

$$\frac{\frac{1}{n} \sum h(Y_k) \mathbb{1}_{\left\{U_k \leq \frac{f_\mu(Y_k)}{c g_\nu(Y_k)}\right\}}}{\frac{1}{n} \cdot \sum \mathbb{1}_{\left\{U_k \leq \frac{f_\mu(Y_k)}{c g_\nu(Y_k)}\right\}}}$$

is emp. appox. of $E(h(X))$ weighted by the occurence of the event $\{...\}$

$$\to \frac{\to E[\dots]}{\to E[\dots]}$$

Importance sampling

$I = \int_0^1 \cos(\pi x/2)\, dx$

$g(x) = \cos(\pi x/2)$

$f(x) = 1$ $\qquad$ $\tilde{f}_+ = 1 - x \Rightarrow \tilde{f} = \frac{1-x}{1/2}$

$\tilde{f}$ близко к $g \times f$

$1 - \sqrt{u} \sim \tilde{f}$ по мет. обр. $\Rightarrow$ вместо $U \sim Unif$ берем $X \sim \tilde{f}$

$\Rightarrow Var(IS) < Var(MC)$

③ Markov Chains

M.C. : $\mathbb{P}(X_{n+1} = x \mid F_n) = P(X_{n+1} = x \mid X_n) \quad \forall x \in E$

i.e. $\{X_n\}_n$ – M.C. if $\forall n$ X_n has values in E is F – measurable.

- $\{F_n\}_n$ is filtration gen by $\{X_n\}_n$

  $F_0 \subset F_1 \dots \subset F_n$

- $p_{ij}(n, n+1) := P(X_{n+1} = j \mid X_n = i)$

  ↳ (one-step prob of transition

  $P(n, n+1) = (p_{ij}(n, n+1))_{i,j \in E}$ – one-step trans. matr

- $P(n, n+2) = P(n, n+1) \cdot P(n+1, n+2)$

- $A = (a_{ij})_{i,j \in E}$ is stochastic if

  $a_{ij} \geq 0$ and $\sum_{j \in E} a_{ij} = 1$ for $\forall i j$

- state distribution : $\pi^T(n) = \alpha^T P(0, n)$, $n \geq 0$

- M.C. is homogeneous if

  $\mathbb{P}(X_{n+1} = y \mid X_n = x)\ p_{x,y}$, $\forall n \geq 0$

- Chapman - Kolmogorov

  $p_{n+m}(x, y) = \sum_z p_m(z, y)\, p_n(x, z)$

- stationary measure of P

  $\sum_x \mu(x) \cdot P(x, y) = \mu(y)$

o $\{X_n\}_n$ admits a stat. measure

then $\frac{1}{n}\sum f(X_n) \xrightarrow[a.s.]{} \sum_x f(x)\cdot \pi(x)$

for $\forall f$ s.t. $\sum_x |f(x)|\pi(x) < \infty$

$\Rightarrow P(X_n = x) \to \pi(x)$

$p_n(x,y) \to \pi(y) \quad \forall x,y \in E$

<u>Stopping time</u>

$\{X_n\}_n$ - st. process $\to \{F_n\}_n$ - filtration

F_n - stop time is a r.v. $T: \Omega \to \mathbb{N}$ s.t. $\forall n$

the event $\{T=n\}$ is in F_n

<u>Lemma</u>

a) T - is F_n - st.t. $\Rightarrow \forall n \in \mathbb{N}$

$\{T \le n\}$ and $\{T > n\}$ are in F_n

b) $F_T = \{A \in F : \{T \le n\} \cap A \in F_n \ \forall n\}$

$\Rightarrow F_T$ - is a σ-algebra

<u>Markov prop.:</u>

$P(X_{T+n} = x \mid F_T) = P(X_{T+n} = x \mid X_T) \ \forall x \in E$

<u>Def.</u> sym. measure

$\pi(x)\, Q(x,y) = \pi(y)\, Q(y,x)$

<u>Thm.</u> ecum π - Q - sym. $\Rightarrow$ π - Q - inv.

$\triangleright \quad (\pi Q)(y) = \dots = \pi(y)$

④ MH

Th. $\{X_n\}_n$ is an irreducible and positive recurrent M.C. $\Rightarrow$ $\{X_n\}_n$ admits a unique st. prob. measure $\pi(x) = \dfrac{1}{E_x[T_x^1]}$

Thm.

Q – непр, апер. пер. матр. допуск. π_0

① $\Rightarrow \exists$ $\alpha \in [0,1)$, $M \in \mathbb{R}_+$: $\forall x, y$

$$|Q^n(x,y) - \pi_0(y)| \leq M\alpha^n$$

② $(X_n)_{n\geq 0}$ – ц.м., то $\forall f: E \to \mathbb{R}$.

$$\sqrt{n+1}\left(\frac{1}{n+1}\sum f(X_p) - \sum \pi_0(y) f(y)\right) \xrightarrow{d} (0, \sigma^2)$$

MH algorithm

π – закон по кот. хотим мод

или вычислить $\int_E f(x)\,\pi(dx)$

π – целевое распр.

Q – proposal kernel

хотим построить $(X_n)_{n\geq 0}$

- возьмем $X_0 = x_0$ s.t. $\pi(X_0) > 0$
- Если $X_n = x$ моделируем ind. Y_{n+1}, U_{n+1}

s.t. $Y_{n+1} \sim Q(x, \cdot)$, $U_{n+1} \sim U[0,1]$

↑ candidate for future sample

$$\alpha(x,y) = \min\left(1, \frac{\pi(y)\, Q(y,x)}{\pi(x)\, Q(x,y)}\right)$$

↑ acceptance coef.

Then

$$X_{n+1} = \begin{cases} Y_{n+1} & \text{if } U_{n+1} \le \alpha(X_n, Y_{n+1}) \\ X_n & \text{else} \end{cases}$$

Утв. $(X_n)_{n\geq 0}$ – ЦМ с пер матр. P,

где $\begin{cases} P(x,y) = Q(x,y)\cdot \alpha(x,y) \\ P(x,x) = 1 - \sum_{y \neq x} P(x,y) \end{cases}$

π P-инв.

⑤ Stoch. Processes

Right. cont filtration

$(\mathcal{F}_t, t \geq 0)$ if $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{n \geq 1} \mathcal{F}_{t+\frac{1}{n}}$ $\quad \forall t \geq 0$

$$\Leftrightarrow \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$$

Poisson process

- $N_0 = 0$
- $\forall \omega \in \Omega$ $\quad t \to N_t(\omega)$ is càdlàg

- $N_{t+s} - N_s$ independent to $\mathcal{F}_s$
- $N_{t+s} - N_s \sim \mathcal{P}(\lambda t)$

$$P(N_{t+s} - N_s = n) = \exp(-\lambda t)\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}$$

Construction:

- $T_1, \dots, T_n \sim \mathcal{E}(\lambda)$
- $S_n = \sum_{k=1}^{n} T_k \quad n \geq 1$
- $N_t = \sum_n \mathbb{1}_{\{S_n \leq t\}}$ is pois. process with int λ

S.B.M

- $W_0 = 0$
- $(W_t ; t \geq 0)$ has cont. paths a.s.
- $W_t - W_s$ is ind. to $\mathcal{F}_s$
- $W_t - W_s = W_{t-s} \sim N(0, t-s)$

Constr.: $\sum \xi_n$, where $\xi_n = \begin{cases} -1 \\ 1 \end{cases}$

(6) Martingales

Stop. time: $T : \Omega \to [0,\infty]$ is F_t - st. time, if $\{T \le t\}$ is F_t - measurable

Optional time —//— if $\{T < t\}$ is —//—

- every st. time is optional
- concepts coinside if $F_t = F_{t+}$ ⊕

Lemma $\{T_i\}_{i \ge 1}$ - seq. of opt. times

$\to$ $\sup T_i$, $\inf T_i$, $\overline{\lim}\, T_i$, $\underline{\lim}\, T_i$

are all opt times

Lemma if T, β - s.t.

$\Rightarrow$ $F_{T \wedge \beta} = F_T \cap F_\beta$ ⊛ $\{T < t\} = \bigcup_{n \ge 1} \{T \le t - \frac{1}{n}\}$

$\cap$ $\cap$

$\{T < \beta\}$, $\{\beta < T\}$, $\{T \le \beta\}$, $\{\beta \le T\}$, $\{\beta = T\}$ $F_{t - 1/n}$

$\cap$

(i) $E[Z | F_T] = E[Z | F_{T \wedge \beta}]$, $\cap$ $\Leftarrow$ F_t $\forall n$

F_t

$\mathbb{P}$ - a.s. on $\{T \le \beta\}$

(ii) $E[E[Z | F_T] | F_\beta] = E[Z | F_{T \wedge \beta}]$. $\mathbb{P}$-a.s.

Def. F_t - ad. process is said to be a F_t - mg iff $\forall\, 0 \le s < t$ $E[X_t | F_s] = X_s$

Doob max - ineq.

$$E\left[\max_{\tau\le t\le \beta} |X_t|^p\right] \le \frac{p}{p-1} E\left[|X_\beta|^p\right]$$

Doob's Optional Sampling thm:

$(X_t : t \ge 0)$ - r.c. submart.

Let τ, β - F_t st. times., s.t. $\tau \le \beta$

Then a.s. $E[X_\beta | F_\tau] \ge X_\tau$

If F_t - mart. => $\boxed{E[X_\beta | F_\tau] = X_\tau}$

$$\langle W_t \rangle_t = t$$

$$\langle N_t \rangle_t = \lambda t$$

$$\langle X, Y \rangle = \frac{1}{4}\left(\langle X+Y \rangle_t - \langle X-Y \rangle_t\right)$$

Ito's integral of $(b_t : 0 \le t \le T) \in L((0,T) \times \Omega)$

w.r.t. $(W_t : 0 \le t \le T)$

$$\int_0^t b_s \, dW_s$$

s.t. $\lim E\left[\max \left| \int_0^t b_s \, dW_s - \int_0^t b_s^{(n)} \, dW_s \right|^2\right] = 0$

Seq of simple processes conv. to b_t in L^2

Prop. of Ito's int:

Linearity

$$\int (b_s^1 + \lambda b_s^2)\, dW_s = \int_0^t b_s^1\, dW_s + \lambda \int_0^t b_s^2\, dW_s$$

Mart. prop.

$$E\left[\int_0^t b_r\, dW_r \mid F_s\right] = \int_0^s b_r\, dW_r \quad , \quad 0 \le s \le t \le T$$

Ito's isometry

$$E\left[\left(\int_0^T b_s\, dW_s\right)^2\right] = E\left[\int_0^T (b_s)^2\, ds\right]$$

Cross-variation

$$\langle \int_0 b_s^1\, dW_s, \int_0 b_s^2\, dW_s \rangle_t = \int_0^t b_s^1 b_s^2\, ds$$

$$E\left[\int_s^t b_p^1\, dW_p\right] 0$$

Ito's formula:

$$X_t = X_0 + W_t$$

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\, dX_s + \frac{1}{2}\int_0^t f''(X_s)\, d\langle X \rangle_s$$

$$(X_t)^2 = (X_0)^2 + 2\int_0^t X_s\, dX_s + \frac{2}{2}\int_0^t d\langle X \rangle_s$$

$$= (X_0)^2 + 2X_0 W_t + 2\int_0^t W_s\, dW_s + t$$

$$F(t,x) = e^{iux + 1/2 u^2 t}$$

$$\partial_t \cancel{(t,x)} = 1/2\, u\, F(t,x)$$

$$\partial_x F(\cdot) = iu\, F(t,x)$$

$$\partial_{xx} F(t,x) = -u^2 F(t,x)$$

$$dF = \partial_t F + \partial_x F + \frac{1}{2}\partial_{xx} F = iu F\, dM_t$$

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s\, dY_s + \int_0^t Y_s\, dX_s + \langle X, Y\rangle_t$$