Prob. Theory 1

ФСБ ≤ ФУБ

complementation:

$A^c = \{ x \in \Omega : x \notin A \}, \quad A \subseteq \Omega$

T - index set $\qquad \mathcal{T} = \{ A_t \subset \Omega : t \in T \}$

sequence of subset

e.g.: $\Omega = \mathbb{R}$

$A_t = [0 : 1/(t+1))$

$t \in T = \mathbb{N}$

$t \in T = \mathbb{Z}$

$$\bigcap_{t \in T} A_t = \{ \omega \in \Omega : \omega \in A_t \ \forall t \in T \}$$

(and)

disjoint sets $A \cap B = \phi$

mutually exclusive events $P(A \cap B) = 0$

(C_k):

$C_0 \supseteq C_1 \supseteq \ldots \supseteq C_{k+1}$

nested seq.

$\rightarrow$ by Cauchy-Cantor

$\left(\bigcap_k C_k \right) \neq \phi$

$\{A_t\}_{t \in T}$ is pairwise disjoint collection (mutually) of sets of Ω if $\forall t \in T$.

$A_t \cap A_{t'} = \phi, \quad \forall t' \in T$ s.t. $t' \neq t$

Union over arbitrary index sets

$\{A_t\}_{t \in T}$: $\bigcup_{t \in T} A_t = \{\omega \in \Omega : \omega \in A_t$ for some $t \in T\}$

(or)

$\subset$

Rem. $\{A_t\}_{t \in T}$ $\quad \bigcap_{t \in T} A_t \subseteq \bigcup_{t \in T} A_t$

Set Difference: $A \setminus B = \{\omega \in \Omega ; \omega \in A, \omega \notin B\}$

Symmetric diff: $A \triangle B = (A \setminus B) \cup (B \setminus A)$

HW: i) $\bigcup_{n=1}^{\infty} [0, \frac{n}{n+1}) = [0, 1)$

$\lim_{n \to \infty} \frac{n}{n+1} = 1$

ii) $\bigcap_{k=1}^{\infty} (0, \frac{1}{k}) = \emptyset$

$\lim_{n \to \infty} \frac{1}{n} = 0 \Rightarrow |A_\infty| \to 0$

if $[0, a] \in \bigcap_{k=1}^{\infty} (0, \frac{1}{k}) \Rightarrow |a| > 0$

Properties of sets:

1) $A \subset A$
2) $A \subset B$ and $B \subset C$
3) $A \subset B$ and $B \subset C \Rightarrow A \cup B \subset C$
4) $C \subset A$ and $C \subset B \Rightarrow C \subset A \cap B$
5) $A \subset B$ iff $B^c \subset A^c$

Indicator functions $A \subseteq \Omega$

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{---}\,{''}\,\text{---} \end{cases}$$

1) $\mathbb{1}_A \leq \mathbb{1}_B$ iff $A \subseteq B$

2) $\mathbb{1}_{A^c} = 1 - \mathbb{1}_A$

Limits of sets $\{A_n\}_{n \in \mathbb{N}}$

$$\inf_{k \geq n} A_k := \bigcap_{k=n}^{\infty} A_k$$

$$\sup_{k \geq n} A_k := \bigcup_{k=n}^{\infty} A_k$$

$$\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

$$\limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

ex $A_n := [0; \frac{n}{n+1})$

$$\liminf_{n \to \infty} A_n = [0, 1)$$

$$\bigcup_{n=1}^{\infty} [0; \tfrac{n}{n+1})$$

$$\limsup_{n \to \infty} A_n \overset{?}{=} [0, 1)$$

$$\bigcap_{n=1}^{\infty} [0; 1)$$

Def.

$$\liminf_{n \to \infty} A_n = \limsup_{n \to \infty} A_n = A$$

$\Rightarrow$ convergence

$\{A_n\}_{n \in \mathbb{N}} \xrightarrow{\text{converges}}$ to A

$$\lim_{n \to \infty} A_n = A$$

$$A_n \to A$$

- $\lim_{n \to \infty} \left(\underset{=\,A_n}{\inf_{k \geq n} B_k} \right) = \liminf_{n \to \infty} B_n$

$\Longleftrightarrow$ by def. $\begin{cases} \liminf_{n \to \infty} A_n = \liminf_{n \to \infty} B_n \\ \limsup_{n \to \infty} A_n = \limsup_{n \to \infty} B_n \end{cases}$

a) $\limsup_{n \to \infty} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k = \ldots$

b) $\liminf_{n \to \infty} B_n = \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} B_k = \bigcup_{n=1}^{\infty} A_n$

$$\liminf_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \bigcap_{j \geq k}^{\infty} B_j$$

6) complementation

$(A^c)^c = A$, $\phi^c = \Omega$, $\Omega^c = \phi$

7) commutativity

$A \cup B = B \cup A$, $A \cap B = B \cap A$

8) Associativity

$(A \cup B) \cup C = A \cup (B \cup C)$

9) $(A \cap B) \cap C = A \cap (B \cap C)$

10) De Morgans law

11) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

$\{A_t\}_{t \in T}$: $\left(\bigcup_{t \in T} A_t\right)^c = \bigcap_{t \in T} A_t^c$

$\left(\bigcap_{t \in T} A_t\right)^c = \bigcup_{t \in T} A_t^c$

<u>UW</u> . <u>Distributivity</u>:

$\{A_t\}_{t \in T}$, B : $B \cap \left(\bigcup_{t \in T} A_t\right) = \bigcup_{t \in T} (B \cap A_t)$

$B \cup \left(\bigcap_t A_t\right) = \bigcap_t (B \cup A_t)$

Indicator functions

Prob. Theory (2)

I. Set Theory

Lemma $\{A_n\}_{n\geq 1}$ seq. of subsets of Ω

$\limsup A_n = \{\omega \in \Omega \quad \sum_{n\geq 1} \mathbb{1}_{A_n}(\omega) = \infty\}$

$= \{\omega \in \Omega : \omega \in A_{n_k},\ k=1,2\dots\}$

for some subseq. n_k depend. on ω

We denote that $\limsup_{n\to\infty} A_n = [A_n, \text{ i.o.}]$

i.o. — ~~infinitely~~ often

$\exists n \quad \forall k\geq n \quad \omega \in A_k$

b) $\liminf A_n = \{\omega \in \Omega : A_n, \forall n$ except
a finite number

$\omega : \Omega \quad \sum_n \mathbb{1}_n(\omega)\}$

$= \{\omega : \omega \in A_n \quad \forall n \geq n_0(\omega)\}$

Proof: a) $\omega' \in \limsup A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k \iff$

$\forall n \geq 1 \quad \exists k \geq n, \quad \omega' \in A_k \iff \forall n \geq 1 : k_n \geq n \quad \omega' \in A_{k_n}$

$\sum_{n\geq 1} \mathbb{1}_{A_n}(\omega') \geq \sum_{n\geq 1} \mathbb{1}_{A_{k_n}}(\omega') = \infty$

$\Rightarrow \omega' \in \{\omega : \sum_{n\geq 1} \mathbb{1}_{A_n}(\omega) = \infty\}$

$\Rightarrow \limsup A_n \subseteq \{\omega : \sum_{n\geq 1} \mathbb{1}_{A_n}(\omega) = \infty\}$

$\omega' \in \{\omega : \sum \mathbb{1}_{A_n}(\omega) = \infty\}$

$\Rightarrow \exists\, n_k \uparrow \infty,\ \omega \in A_{n_k} \quad \forall n_k$

$\forall n \geq 1 \quad \exists\, n_k' \geq n \qquad n_k \geq n \qquad \forall n_k \geq n_k'$

$$\omega' \in \bigcup_{n_k \geq n_k'} A_{n_k} \subset \bigcup_{k \geq n} A_k$$

$$\forall n \geq 1 \qquad \omega' \in \bigcup_{k \geq n} A_n$$

$$\omega' \in \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k$$

$$= \limsup A_n \qquad \Rightarrow$$

$\limsup A_n = \{\omega : \omega \in A_{n_k},\ \forall k \geq 1\}$
for some seq. n_k that depend on ω $\square$

For the lemma above we can verify:

$\liminf\limits_{n \to \infty} A_n = \{\omega : \omega \in A_n,\ \forall n > n_0(\omega)\} \subset \{\omega : \omega \in A_n \text{ i.o.}\}$
$= \{\omega : \omega \in A_{n_k},\ \forall k \geq 1\}$
for some subseq n_k

From here we see $\liminf A_n \subset \limsup A_n$

<u>Corollary</u> $(\liminf A_n)^c = \limsup A_n^c$

<u>Prop.</u>: (i) $\{A_n\}$ is an increasing seq.

$A_n \subset A_{n+1}, \quad \forall n \geq 1$

$$\lim A_n = \bigcup_{n \geq 1}^{\infty} A_n$$

(ii) $\{A_n\}$ is decreasing

$$A_n \supset A_{n+1} \quad , \forall n \geq 1$$

$$\lim_{n\to\infty} A_n = \bigcap_{n\geq 1} A_n$$

Proof

(i) $\liminf A_n = \bigcup_{n\geq 1} \underbrace{\bigcap_{k\geq n} A_k}_{B_n = A_n, \; (A_n \subset A_k \; , \forall k \geq n) \; \forall n \geq 1} = \bigcup_{n\geq 1} B_n = \bigcup_{n\geq 1} A_n$

$$\limsup A_n = \bigcap_{n\geq 1} \bigcup_{k\geq n} A_k = \bigcup_{k\geq 1} A_k$$

$$\subset \bigcup_{k\geq 1} A_k = \liminf A_n \subset \limsup A_n$$

(ii) -//- ■

- $\limsup_{n\to\infty} A_n = \lim_{n\to\infty} \sup_{k\geq n} A_n$

  $\liminf_{n\to\infty} A_n = \lim_{n\to\infty} \inf_{k\geq n} A_n$

- $\{A_n\}$ any seq of subsets

  1) $\mathbb{1}_{\inf_{k\geq n} A_n} = \inf_{k\geq n} \mathbb{1}_{A_n}$

  $\{X_n\}$ is a seq of real numbers

  $\inf_{k\geq n} \{X_k\} = l \; : \; l \leq X_k \; , \forall k \geq n$

  2) $\mathbb{1}_{\sup_{k\geq n} A_k} = \sup_{k\geq n} \mathbb{1}_{A_k}$

$$\mathbb{1}_{\sup_{k\geq n} A_k} = \sup_{k\geq n} \mathbb{1}_{A_k}$$

$$\mathbb{1}_{\bigcup_{n\geq 1} A_n} \leq \sum_{n\geq 1} \mathbb{1}_{A_n}$$

$$\mathbb{1}_{\limsup A_n} = \limsup \mathbb{1}_{A_n}$$

$$\mathbb{1}_{\liminf A_n} = \limsup \mathbb{1}_{A_n}$$

II. Algebras

Let $\mathcal{A}$ collection of subsets of Ω

$A \in \mathcal{A} \Rightarrow A \subset \Omega$.

i) $\mathcal{A}$ is an algebra if

1) $\Omega \in \mathcal{A}$
2) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$
3) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

ii) $\mathcal{A}$ is a σ-algebra if.

1) $\Omega \in \mathcal{A}$
2) $A \in \mathcal{A} \to A^c \in \mathcal{A}$
3) $\{A_n\}_{n\geq 1}$ is a seq. of elements in $\mathcal{A}$
$\Rightarrow \bigcup_{n\geq 1} A_n \in \mathcal{A}$

ex. 1) $\mathcal{P}(\Omega)$ is σ-algebra and algebra
- the power set

2) $\mathcal{A} = \{\phi, \Omega\}$

3) The countable / co-countable σ-algebra

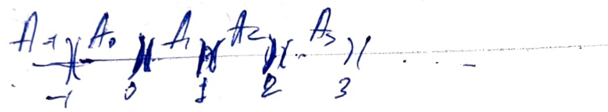
$\Omega = \mathbb{R}$

$\mathcal{A} = \{A \subset \mathbb{R} : A \text{ is countable}\}$

$\cup \{A \subset \mathbb{R} : A^c \text{ is countable}\}$

(ex) $\mathbb{Q} \in \mathcal{A}$ $\quad \mathbb{I} \in \mathcal{A}$ $\quad \mathbb{I}^c = \mathbb{Q}$

$A = \bigcup_{n \in \mathbb{Z}} A_n \in \mathcal{A}$ $\quad A^c = \mathbb{Z}$



1) $\Omega \in \mathcal{A}$ due to $\Omega^c = \emptyset$ is countable

2) $A \in \mathcal{A} \overset{?}{\Rightarrow} A^c \in \mathcal{A}$

$A \in \mathcal{A}$ if A^c is countable

if $A \in \mathcal{A}$ $\quad A$ is countable $\quad A^c \in \mathcal{A}$ (since $(A^c)^c = A$)

3) $\{A_n\}$ a seq of elements in $\mathcal{A}$

$\bigcup A_n \in \mathcal{A}$

if A_n is countable $\forall n \geq 1 \Rightarrow \bigcup A_n$ is counable

$\Rightarrow \bigcup A_n \in \mathcal{A}$

Assume that $A_{n'}$ is not countable, but $A_{n'}^c$ is countable

A_n is countable, $\forall n$, $n \neq n'$

$\bigcup A_n$ is not countable

$(\bigcup A_n)^c = \bigcap A_n^c \subset$ but it is countable $\therefore \bigcup A_n^c \in \mathcal{A}$

$\Rightarrow \mathcal{A}$ is a σ-algebra

$\mathcal{A}$ is a σ-algebra

$\mathcal{A} \subsetneq \mathcal{P}(\mathbb{R})$

$(-\infty, t] \in \mathcal{A}$ $\quad (-\infty, t]^c = (t, \infty)$

$\mathbb{R}$ uncountable

...

$\mathcal{A}$ consists of the $\emptyset$ + all finite unions of disjoint intervals of form $(a, a']$ $\quad 0 < a < a' \leq 1$

$A \in \mathcal{A}$

$$A = \bigcup_{k=1}^{n} (a_k, a_k']$$

The σ-algebra generated by given class $\mathcal{G}$

Lemma

$\bigcap$ of σ algebras is σ algebra

$\mathcal{N} = \{\mathcal{B} : \mathcal{B}$ σ-algebra$\}$

$$\mathcal{B}_0 = \bigcap_{\mathcal{B} \in \mathcal{N}} \mathcal{B}$$

$$\mathcal{B}_1 = \bigcup_{\mathcal{B} \in \mathcal{N}} \mathcal{B}$$

$\Omega \in \mathcal{B}$

$A \in \mathcal{B}_1 \Rightarrow \exists \mathcal{B} \in \mathcal{N}$

$A \in \mathcal{B} \Rightarrow A^c \in \mathcal{B} \Rightarrow A^c \in \mathcal{B}_1$

$\{A_n\}$ is a sequence of elements in $\mathcal{B}_1$

$A_n \in \mathcal{B}_1 \quad \forall n$

$\exists \mathcal{B} \in \mathcal{N},\ A_n \in \mathcal{B}$

$\Rightarrow \bigcup \mathcal{B}$ is not a σ-algebra

Def: Let G be a collection of subsets of Ω

The σ-algebra generated by G

$\sigma(G)$ satisfy then:

i) $G \subset \sigma(G)$

ii) If $\mathcal{B}$ is a σ-algebra containing G

the $\sigma(G) \subset \mathcal{B}$

where $\sigma(G)$ is the minimal σ-algebra over G

Prop: given a class G of subsets of Ω

there is a unique min σ-alg. cont. G

Proof: $\mathcal{N} = \{\mathcal{B} : \mathcal{B}\ \sigma\text{-alg cont. } G\}$

$\mathcal{B} = \bigcap_{\mathcal{B} \in \mathcal{N}} \mathcal{B}$ is a σ alg.

$G \subset \mathcal{B},\ \forall \mathcal{B} \in \mathcal{N} \Rightarrow G \subset \mathcal{B}_0$

Let $\mathcal{B}'$ a σ-algebra s.t. $G \subset \mathcal{B}' \Rightarrow \mathcal{B}' \in \mathcal{N}$

$\Rightarrow \mathcal{B}_0 \subset \mathcal{B}' \Rightarrow \mathcal{B}$

ex Proof?

1) If $\mathcal{A} = \{A\}$ a single set $\Rightarrow \sigma(\mathcal{A}) = \{\emptyset, \Omega, A, A^c\}$

2) If $\mathcal{A}$ is σ-algebra $\Rightarrow \sigma(\mathcal{A}) = \mathcal{A}$

<u>Borel set</u> on the real line $\Omega = \mathbb{R}$

$\mathcal{G}^{(]} = \{ (a, b] : -\infty \leq a \leq b < \infty \}$

$\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G}^{(]})$

$\mathcal{B}(\mathbb{R})$ is the Borel sets of $\mathbb{R}$

Borel σ-algebra

$$\mathcal{B}(\mathbb{R}) = \sigma(\overbrace{\{(a,b) : -\infty \leq a < b \leq \infty\}}^{\mathcal{C}^{()}})$$

$$= \sigma(\{[a,b) : -\infty < a \leq b \leq \infty\})$$

$$= \sigma(\{[a,b] : -\infty < a \leq b < \infty\})$$

$$= \sigma(\{\text{open subsets of } \mathbb{R}\})$$

$\mathbb{R} = (-\infty, \infty)$

$(a,b) \in \mathcal{C}^{()}$

$$(a,b) = \bigcup_{n \geq 1} (a, b - 1/n]$$

$(a, b - 1/n] \in \sigma(\mathcal{G}^{(]})$, $\forall n$

$$\bigcup_{n \geq 1} (a, b - 1/n] \in \sigma(\mathcal{C}^{()})$$

$(a,b) \in \sigma(\mathcal{C}^{()})$

$\mathcal{C}^{()} \subset \sigma(\mathcal{C}^{(]})$

$\sigma(G^{(1)}) \subset \sigma(G^{(3)})$

$(a, b] = \bigcap_{n \geq 1} (a, b + 1/n) \in \sigma(G^{(1)})$

$C^{(3)} \subset \sigma(G^{(1)})$

$\sigma(G^{(3)}) \subset \sigma(C^{(1)})$.

$\mathcal{B}(\mathbb{R}) = \sigma(C^{(1)})$

Remark:

If $\mathbb{E}$ is a metric space

$\mathcal{B}(\mathbb{E})$ - σ-algebra generated by collection of open subsets of $\mathbb{E}$

$\mathcal{B}(\mathbb{R}^d)$

$\|x\|^2 = \sum_{i=1}^{d} x_i^2 \Rightarrow \|x\| = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$

III. Probability Space

$(\Omega, \mathcal{F}, \mathbb{P})$

where 1) Ω is the sample space

2) $\mathcal{F}$ is a σ-algebra

3) $\mathbb{P}$ is a prob. measure $\mathbb{P} : \mathcal{F} \longrightarrow [0;1]$

$A \longrightarrow \mathbb{P}(A)$

1) $\mathbb{P}(A) \geq 0, \forall A \in \mathcal{F}$

2) $\mathbb{P}$ is σ-additive

If $\{A_n\}$ seq. of events (disjoint) $\Rightarrow \mathbb{P}(\bigcup_{n \geq 1} A_n) = \sum_{n \geq 1} \mathbb{P}(A_n)$

3) $\mathbb{P}(\Omega) = 1$

Properties :

1) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

$\triangleright$ $\left.\begin{array}{l} A \cap A^c = \phi \\ \Omega = A \cup A^c \end{array}\right\} \Rightarrow \mathbb{P}(A) + \mathbb{P}(A^c) = 1$ $\square$

2) $\mathbb{P}(\phi) = 0$

$\triangleright$ $\Omega^c = \phi \Rightarrow \mathbb{P}(\phi) = \mathbb{P}(\Omega^c) = 1 - \mathbb{P}(\Omega) = 1-1$ $\blacksquare$

3) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$\triangleright$ $A = (A \setminus B) \cup (A \cap B)$ $\qquad \mathbb{P}(A) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)$

$B = (B \setminus A) \cup (A \cap B)$ $\qquad \mathbb{P}(B) = \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ $\qquad \mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)$

$\Rightarrow \ldots$ $\square$

Prob. Theory (3)

Def. Independence

$\{A_n\}_{n=1}^{m}$ are independent iff $P(\bigcap_{k=1}^{j} A_{i_k}) = \prod_{k=1}^{j} P(A_{i_k})$ $\quad \forall i \in \{1,\dots,m\}$, $k \in \{1,\dots,j\}$, $i_1 < \dots < i_k$

- $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$, $i,j \in \{1,\dots,m\}$
- To check if $\{A_k\}_{k=1}^{n}$ are independent, we need $C_2^n + C_3^n + \dots + C_n^n$

$n = 3 \Rightarrow C_2^3$
$n = 4 \Rightarrow C_2^4$
...

ex.
Ω - people with B.D.
A - women with B.D. $\quad P(A) = 1/2$
B - people with B.D. in business $\quad P(B) = 1/5$

$P(A \cap B) = 0{,}1 \Rightarrow A, B$ are indep. since $P(A \cap B) = P(A) \cdot P(B) = 0{,}1$

Prop. If A and B are independent $\Rightarrow$ so $\{A, B^c\}; \{A^c, B\}; \{A^c, B^c\}$

$\triangleright$ $P(A \cap B) = P(A) \cdot P(B)$?

$P(A) \cdot P(B^c) = P(A) \cdot (1 - P(B)) = P(A) - P(A) \cdot P(B) = P(A) - P(A \cap B)$

$P(A \cap B^c) = 1 - P((A \cap B^c))^c) = 1 - P(A^c \cup B) =$

$= 1 - [P(A^c) + P(B) - P(A^c \cap B)] = P(A) - [P(B) - P(A^c \cap B)]$

$B = B \setminus (A \cap B) \cup A \cap B \longrightarrow B \setminus (A \cap B) = B \cap (A \cap B)^c$
$= B \cap (A^c \cup B^c) = B \cap A^c$

$P(B) = P(B \setminus A \cap B) + P(A \cap B) = P(B \cap A^c) + P(A \cap B) \Rightarrow$

$P(A \cap B^c) = P(A) - P(A \cap B)$

$\Rightarrow P(A \cap B^c) = P(A) \cdot P(B^c)$

Def. Pairwise ind.

$\{A_k\}_{k=1}^{n}$ are pairwise ind. iff $P(A_i \cap A_j) = P(A_i) \cdot P(A_j)$, $i < j$

$\{A_k\}_{k=1}^{n}$ are ind. $\underset{\not\Leftarrow}{\Rightarrow}$ $\{A_n\}_{n=1}^{n}$ are pairwise ind.

ex. $\Omega = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$
$P(B_i) = 1/4$

A_k - k^{th} equal to 1 $\quad A_1 = \{B_1, B_4\}$ $\quad A_2 = \{B_2, B_4\}$
$A_3 = \{B_3, B_4\}$

$P(A_k) = 1/2$

$P(A_k \cap A_j) = P(B_4) = 1/4 \qquad P(A_k) \cdot P(A_j) = 1/4$

Def. Cond. probability

Let A, B events : $P(A) > 0$

$\Rightarrow P(B|A) = \dfrac{P(A \cap B)}{P(A)}$

Remark:

$(A, \mathcal{F} \cap A, P(\cdot\, | A))$

$\mathcal{F} \cap A := \{B \cap A : B \in \mathcal{F}\}$ - σ-Algebra

1) $P(A|A) = \dfrac{P(A)}{P(A)} = 1$

$P(B|A) \geq 0 \quad \forall B \in \mathcal{F} \cap A$

$\{B_i\}_{i \geq 1}$ are disjoint $\Rightarrow P(\bigcup B_i | A) = \sum_{i \geq 1} P(B_i | A)$

2) $P(A|B) = P(A)$ iff A, B are independent

Def. A part. of Ω is a collection of disjoint sets, s.t. the union of which is eq. to Ω:

$\{H_k\}_{k \geq 1}^{m}$, $H_i \cap H_j = \emptyset$, $\forall i, j \in \{1, \dots, m\}$

$\bigcup_{k=1}^{m} H_k = \Omega$

Prop. The law of tot. prob.:

Let $\{H_n\}_{n \geq 1}^{m}$ be a partition of Ω

$\Rightarrow \forall$ event A : $P(A) = \sum_{k=1}^{n} P(A|H_k) \cdot P(H_k)$

▷ $\Omega = \bigcup_{k=1}^{m} H_k$

$A = A \cap \Omega = A \cap (\bigcup_{k=1}^{m} H_k) = \bigcup_{k=1}^{m} (A \cap H_k)$

$\{A \cap H_k\}_{k=1}^{n}$ are disjoint

$P(A) = P(\bigcup_{k=1}^{m} (A \cap H_k)) = \sum_{k=1}^{m} \dfrac{P(A \cap H_k)}{P(H_k)} \cdot P(H_k) = \sum_{k=1}^{n} P(A| H_k) \cdot P(H_k)$ □

Prop. Bayes formula

$\{H_k\}_{k=1}^{n}$ a part. of Ω $\Rightarrow$ $P(H_k \mid A) = \dfrac{P(A \mid H_k)\, P(H_k)}{P(A)} = \dfrac{P(A \mid H_k)\, P(H_k)}{\sum_{j=1}^{n} P(A \mid H_j)\, P(H_j)}$

ex. a) is the last N of your soc. security N odd?

b) have you ever lied on an employment application?

If coin $\Rightarrow$ H respond to (a)

T respond to (b)

$p(\text{yes}) = 0{,}37$

$p(\text{yes} \mid (b)) = \dfrac{P((b) \mid \text{yes})\, P(\text{yes})}{P((b))} = \dfrac{p' \cdot 0{,}37}{1/2} = p$

- $p((b) \mid \text{yes}) = p'$

$p((b)) = P((b) \mid \text{yes})\, P(\text{yes}) + P((b) \mid \text{No})\, P(\text{No}) = 1/2$

$p(\text{yes} \mid (a)) = 1/2$

- $p(\text{yes}) = P(\text{yes} \mid (a))\, P((a)) + P(\text{yes} \mid (b)) \cdot P((b))$

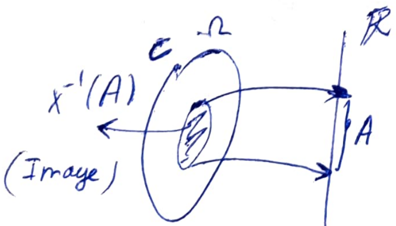
$0{,}37 = 1/2 \cdot 1/2 + p \cdot 1/2$

$P(\text{yes} \mid (b)) = \dfrac{0{,}37 - 0{,}25}{1/2} = 0{,}24$

Random variable $(\Omega, \mathcal{F}, \mathbb{P})$ a prob. space

Def. A r.v. X is a measurable function from the sample set Ω to $\mathbb{R}$: $\forall A \in \mathcal{B}(\mathbb{R})$,

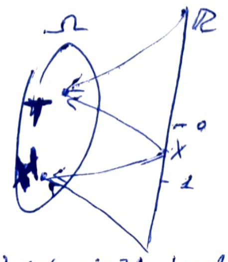
$X^{-1}(A) = \{w \in \Omega : X(w) \in A\} \in \mathcal{F}$



$\Omega = \{H, T\}$

$X(w) = \begin{cases} 1 & \text{if } w = H \\ 0 & \text{if } w = T \end{cases}$

$A = (-\infty, x) \ni 0$



$X^{-1}((-\infty; x]) = \{T\} \in \Omega$

$X^{-1}((x; \infty)) = \{H\} \in \Omega$

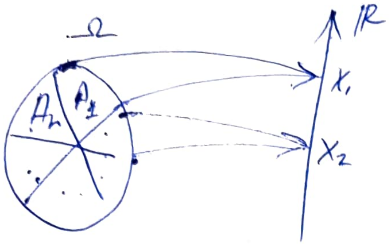
$X^{-1}((-\infty; 2]) = \{w : X(w) \in (-\infty; 2]\} = \{H, T\}$

$X^{-1}((-\infty; 0.5]) = \{w : X(w) \in (-\infty; 0.5]\} = \{T\} \in \Omega$

X is simple if $\exists n$ $X = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$

where $\{x_k\}_{k=1}^{n}$ are real numbers and $\{A_i\}_{i=1}^{k}$

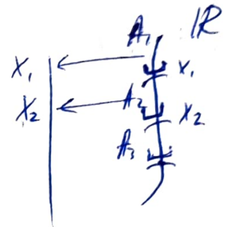
is a finite partition of Ω



$\omega_1 \in A_1$

$X(\omega_1) = x_1$

X is elementary if $X = \sum_{k \geq 1} x_n \mathbb{1}_{A_n}$



$\cap [\ ] = \mathbb{R}$

$\{x_k\}_{k \geq 1}$ are real numbers

$\{A_k\}_{k \geq 1}$ is a finite part. of Ω

$X : \Omega \to [-\infty ; \infty] = \mathbb{R} \cup \{-\infty, \infty\}$

$\Rightarrow X$ is an extended r.v.

__Def.__ null set

a set B is a null set if $\exists C$ s.t. $C \supset B$ and $P(C) = 0$

__Def.__ r.vs equivalent

r.v. which only differ on a null set are equivalent

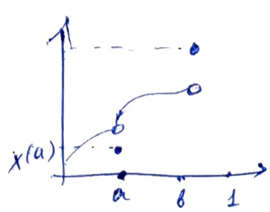
$\Omega = (0,1)$

$\mathcal{B}(\Omega)$

$P((a,b)) = b - a$, $(a,b) \in \mathcal{B}(\Omega)$

$C = \{a_1, \dots, a_n\}$ $P(C) = 0$

$$Y(\omega) = \begin{cases} X(\omega) & \omega \in (0,a) \cup (a,b) \cup (b,1) \\ 0 & \omega \in (a,b) \end{cases}$$



The eq. class of r.v. X is the collection that differ from X ~~on~~ a null set $\Rightarrow X, Y$ are equiv.

$X \sim Y$

$X : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$

$\forall$ Borel set $A : \mathbb{R}$

$X^{-1}(A) \in \mathcal{F}$

$\mathbb{P}(A) := P(X^{-1}(A))$

<u>Thm</u>: the ind. space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P})$ with $\mathbb{P}$ def. in (1), is a p. measure

Prob. Theory (5)

Def: Let X be a simple r.v. $X = \sum_{i=1}^{n} x_i \mathbb{1}_{A}$, where $\{x_i\}_{i=1}^{n}$ are real numbers and $\{A_i\}_{i=1}^{n}$ is a finite partition of Ω

$E[X] := \sum_{i=1}^{n} x_i P(A_i)$

If $\{A_k\}_{k=1}^{n}$ and $\{B_i\}_{i=1}^{m}$ are partition of Ω s.t.

$X = \sum_{k=1}^{n} x_k \mathbb{1}_{A_k}$

$X = \sum_{i=1}^{m} y_i \mathbb{1}_{B_i}$

then
$$\sum_{k=1}^{n} x_k P(A_k) = \sum_{i=1}^{m} y_i P(B_i)$$

Proof
$$P(A_k) = \sum_{i=1}^{m} P(A_k \cap B_i) \quad \left\{ \begin{array}{l} P(A_k) = P(A_k \cap \Omega) = \\ = P(A_k \cap (\bigcup_{i=1}^{m} B_i)) = \sum P(A_k \cap B_i) \end{array} \right.$$

$$P(B_i) = \sum_{k=1}^{n} P(A_k \cap B_i)$$

Then
$$\sum_{k=1}^{n} x_k P(A_k) = \sum^{n} \sum^{m} x_i P(A_k \cap B_i)$$

$$\sum_{i=1}^{m} y_i P(B_i = \sum^{n} \sum^{m} y_i P(A_k \cap B_i)$$

~~since $(A_k \cap B_i)$~~ ...

Properties: $X, Y \geq 0$ r.v.

1) If $X=0$ a.s. $\Rightarrow E[X]=0$

▷ If $X=0$ a.s. $P(X=0)=1$

$P(X=x_k) \leq P(X>0)=0$ for $x_k>0$

Then $E[X] = 0 \cdot P(X=0) + \sum_{k=1}^{n} x_k \mathbb{1}_{\{x_k>0\}} P(X=x_k) = 0$

2) $E[X] \geq 0$

3) If $E[X]=0 \Rightarrow X=0$ a.s.

▷ by law of contraposition $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$

Assume $P(X=0)<1 \Rightarrow 0<P(X>0)<1$

$$\boxed{\begin{array}{l} \overset{A}{E(X)=0} \Rightarrow \overset{B}{P(X=0)} = 1 \\ \overset{\neg B}{P(X=0)<1} \Rightarrow \overset{\neg A}{E(X)>0} \end{array}}$$

Then there $\exists\ x_k>0$ s.t. $0<P(X=x_k)<1$

$E[X] = \sum_{j=1}^{n} x_k P(A_k) \geq x_k P(X=x_k) > 0$

4) If $E[X]>0 \Rightarrow P(X>0)>0$

5) Linearity $E[aX+bY] = aE[X] + bE[Y]$ $\forall a,b \in \mathbb{R}^+$

▷ Let $X = \sum_{k=1}^{n} x_k P(A_k)$ $\Rightarrow aX+bY = \sum_{k=1}^{n}\sum_{i=1}^{m} (ax_k+by_i) \mathbb{1}_{A_k \cap B_i}$

$Y = \sum_{i=1}^{m} y_i P(B_i)$

$\Rightarrow E[aX+bY] = \sum_{k=1}^{n}\sum_{i=1}^{m} (ax_k+by_i) P(A_k \cap B_i) =$

$= a\sum_{k=1}^{n} x_k \sum_{i=1}^{m} P(A_k \cap B_i) + b\sum_{i=1}^{m} y_i \sum_{k=1}^{n} P(A_k \cap B_i) =$

$= aE[X] + bE[Y]$

6) $E[X \mathbb{1}_{\{X>0\}}] = E[X]$

$\triangleright$ $E[X] = E[X \mathbb{1}_{\{X>0\}} + X \mathbb{1}_{\{X=0\}}] = E[X \mathbb{1}_{\{X>0\}}] + 0$

7) Equivalence: If $X = Y$ a.s. $\Rightarrow$ $E[Y] = E[X]$

$\triangleright$ $X = Y$ a.s. $\Rightarrow$ $X - Y = 0$ a.s.

Then $E[X-Y] = 0$ $\Rightarrow$ $E[X] = E[(X-Y) + Y] = E[X-Y] + E[Y] = E[Y]$

8) Domination: If $Y \leq X$ a.s. $\Rightarrow$ $E[Y] \leq E[X]$

$\triangleright$ $Y \leq X$ a.s. $\Rightarrow$ $X - Y \geq 0$ a.s.

Then $E[X-Y] \geq 0$ $\Rightarrow$ $E[X] = E[(X-Y) + Y] = E[X-Y] + E[Y] \geq E[Y]$

---

if $X \geq 0$ r.v. $\Rightarrow$ seq. of non-neg. r.v.

$$X_n = \begin{cases} \frac{k-1}{2^n} & \text{for } \frac{k-1}{2^n} \leq X < \frac{k}{2^n}, \quad k = 1, 2, \ldots, n2^n \\ n & \text{for } X \geq n \end{cases}$$

converges monotonically to X, $n \to \infty$

Def $X \geq 0$ r.v.

$$E[X] = \lim_{n \to \infty} \sum_{k=1}^{n2^n} \frac{k-1}{2^n} P\left(\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\right)$$

Thm Consistence

$X \geq 0$, r.v., $\{Y_n\}_{n \geq 1}$ $\{Z_n\}_{n \geq 1}$ - seq. of simple r.v. s.t.

$Y_n \nearrow X$ $Z_n \nearrow X$ as $n \to \infty$

Then $\lim_{n \to \infty} E[Y_n] = \lim_{m \to \infty} E[Z_m]$ $(= E[X])$

$\triangleright$? $\lim_{n \to \infty} E[Y_n] \geq E[Z_m]$

$$\lim_{m\to\infty} \lim_{n\to\infty} [Y_n] \geq \lim E[Z_m]$$

$$\lim_{n\to\infty} E[Y_n] \geq \lim_{m\to\infty} E[Z_m] \geq E[Y_n]$$

$$\lim_{m\to\infty} E[Z_m] \geq \lim_{n\to\infty} E[Y_n]$$

$$X = \sum_{k=1} Y_k \cdot \mathbb{1}_{\{B_i\}}$$

Let n be an int. number (large enough)

$$E[Y_n] = E[Y_n \mathbb{1}_{\{Y_n \geq Z_m\}} + Y_n \mathbb{1}_{\{Y_n < Z_m\}}] \geq E[Y_n \mathbb{1}_{\{Y_n \geq Z_m\}}] \geq E[Z_m \mathbb{1}_{\{Y_n \geq Z_m\}}]$$

$$= \sum_{k=1}^{(m)} x_k^{(m)} P(A_k^{(m)} \cap \{Y_n \geq Z_m\})$$

$\{Y_n \geq Z_m\} \uparrow \Omega$ since $Y_n \uparrow X$ $n \to \infty$, $X \geq Z$

$Y_n \leq Y_{n+1} \leq X$

$Y_n \uparrow X$, $n \to \infty$

$\{Y_n \geq Z_m\} \subseteq \{Y_{n+1} \geq Z_m\}$

$\{Y_n \geq Z_n\} \uparrow \{X \geq Z_m\} = \Omega$

if $\alpha_n \to \alpha$, $\beta_n \to \beta$

$\Rightarrow \lim(\alpha_n + \beta_n) = \alpha + \beta$

Then $P(A_k^{(m)} \cap \{Y_n \geq Z_m\}) \uparrow P(A_k^{(m)})$, $n \to \infty$

$$\Rightarrow \lim_{n\to\infty} E[Y_n] \geq E[Z_m]$$

Def. $E[X] = E[X^+] - E[X^-]$

$X^+ := \max\{0, X\}$

$X^- := -\min\{0, X\}$

If $E[X^+]$, $E[X^-]$ are finite $\Rightarrow E[|X|] < \infty \Rightarrow X$ is integrable

$E X = E(X^+) + E(X^-)$

$E[aX + bY] = aE[X] + bE[Y]$

$E[|aX + bY|] < \infty$ $\qquad$ $E[|X|] = E[X^+] + E[X^-] < \infty$

$X = X^+ - X^-$ $\qquad$ $E[|Y|] = E[Y^+] + E[Y^-] < \infty$

$|X| = X^+ - X^-$

$Y = Y^+ - Y^-$

$[aX + bY] = [aX + bY]^+ - [aX + bY]^- \ldots$

<u>Thm</u> $\{X_n\}_{n \geq 1} \geq 0$ r.v.

If $X_n \uparrow X$, $n \to \infty$ $\Rightarrow$ $E[X_n] \uparrow E[X]$, $n \to \infty$

$\triangleright$ $\forall X_n$ $\exists \{Y_{k,n}\}_{k \geq 1}$ s.t. $Y_{k,n} \uparrow X_n$ as $k \to \infty$

def. $\{Z_n\}_{k \geq 1}$ as $Z_k = \max_{1 \leq n \leq k} \{Y_{k,n}\}$

and $\{Z_k\}$ is a seq. of simple r.v. s.t. $Z_n \uparrow X$ as $k \to \infty$

then $E[X] = \lim_{k \to \infty} E[Z_k]$

$Y_{k,n} \leq Z_n \leq X_k \Rightarrow X_n \leq \lim_{k \to \infty} Z_k \leq X \Rightarrow X \leq \lim_{k \to \infty} Z_k \leq X$

$\Rightarrow \lim_{k \to \infty} E[Z_n] = E[X] = E[\lim_{k \to \infty} Z_k]$

$E[Y_{k,n}] \leq E[Z_k] \leq E[X_k] \Rightarrow E[X_n] \leq \lim_{k \to \infty} E[Z_n] \leq \lim_{k \to \infty} E[X_k]$

$\Rightarrow \lim_{n \to \infty} E[X_n] \leq \lim_{k \to \infty} E[Z_k] \leq \lim_{k \to \infty} E[X_k]$

$\Rightarrow \lim_{k \to \infty} E[X_k] = \lim_{k \to \infty} E[Z_k] = E[\lim_{k \to \infty} Z_k] = E[X]$

Thm Fatou's Lemma

If $\{X_n\}_{n\geq 1} \geq 0$ r.v. $\Rightarrow E[\liminf_{n\to\infty} X_n] \leq \liminf_{n\to\infty} E[X_n]$

also If Y and Z are integrable r.v. s.t.

1) 5 : $p = 0{,}4$

a) $P(1 \text{ at most}) = 0{,}4 \cdot 0{,}6^4 \cdot C_5^1 = 0{,}2592$

b) $P(2 \text{ to } 4 \text{ sales}) = 0{,}4^2 \cdot 0{,}6^3 \cdot C_5^2 + 0{,}4^3 \cdot 0{,}6^2 \cdot C_5^3 + 0{,}4^4 \cdot 0{,}6 \cdot C_5^4$

3) 3 failures during 100 days $\qquad m = 0{,}03$

a) $P(0 \text{ failures}) = P(X=0) = \dfrac{0{,}03^0 \, e^{-0{,}03}}{0!} = e^{-0{,}03} = 0{,}97$

b) $P(X \geq 1) = 1 - 0{,}97 = 0{,}03$

c) $P(X \geq 2 \text{ for three days}) = 1 - P(X=0 \text{ or } X=1 \text{ for three days})$

$= 1 - \left(P(X=0)^3 + P(X=1) \cdot P(X=0)^2 \cdot C_3^1\right) =$

$= 1 - \left(0{,}97^3 + 0{,}97^2 \cdot \dfrac{0{,}03^1 \cdot e^{-0{,}03}}{1!} \cdot 3\right) =$

$= 1 - (0{,}94) = 0{,}06$

5) $E(X) = 10^6 \qquad Var(X) = 3 \cdot 10^4$

$E(X) = 1 \qquad Var(X) = 0{,}03$

$P(0{,}97 < X < 1{,}06) = P\left(\dfrac{0{,}97 - 1}{\sqrt{0{,}03}} < X < \dfrac{1{,}06 - 1}{\sqrt{0{,}03}}\right)$

$= P\left(\dfrac{-0{,}03}{\sqrt{0{,}03}} < X < \dfrac{0{,}06}{\sqrt{0{,}03}}\right) = P\left(-\sqrt{0{,}03} < X < 2\sqrt{0{,}03}\right)$

$= P(-0{,}17 < X < 0{,}34) = \Phi(0{,}17) + \Phi(0{,}34) =$

$= 0{,}06749 + 0{,}13307 = 0{,}2$

7) exp, $1/\theta = 0{,}05$

$P(X \geq 20) = \int_{20}^{\infty} 0{,}05 \, e^{-x \cdot 0{,}05} \, dx = -e^{-x \cdot 0{,}05} \Big|_{20}^{\infty} =$

$= +e^{-20 \cdot 0{,}05} = 0{,}3678$

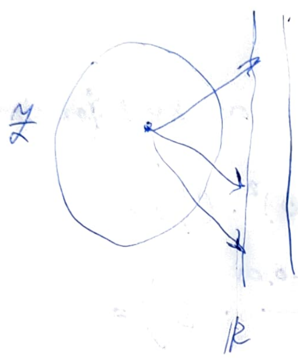
9) 8 ← {8M, 8W} $N=16$ $k_1=4$ $n-k=4$ $S=8$ $N-S=8$

$n=8$

$$P(4W)=\frac{C_8^4\cdot C_8^4}{C_{16}^8}=\frac{8!\,8!\,8!\,8!}{4!\,4!\,4!\,4!\,16!}=0{,}38$$

11) X, Y - r.v. $A\in F$

$Z = X\cdot 1\!\!1_A + Y\cdot 1\!\!1_A$ is a r.v.



$$Z=(X+Y)\cdot 1\!\!1_A=\begin{cases}X+Y & \text{if } A\\ 0 & \text{if } \bar{A}\end{cases}$$

$X+Y$ – r.v.

13) $F(x)$ - contin. distr. fun. $\in\mathbb{R}$

F is unif. cont.

$f: X\to Y$ is unif cont

- if $\forall\,\varepsilon>0$ $\exists\,\delta>0$ s.t. $\forall x,y\in X$: $d_1(x,y)<\delta$
  $d_2(f(x),f(y))<\varepsilon$

- $\forall\,\varepsilon>0$ $\exists\,\delta>0$ s.t. $\forall x,y\in X$, $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$F_X'(x)>0$

$\lim\limits_{x\to-\infty} F_X(x)=0$ $\lim\limits_{x\to\infty} F_X(x)=1$

F_X should be right-cont. function + cont. $\Big\} \Rightarrow$ unif cont.

Lecture 1

$(\Omega, \mathcal{F}, \mathbb{P})$ p.s. $\quad \Omega$ – collection of outcomes

$\mathcal{F}$ – collection of events

$\mathbb{P}$ – measure sat. Kol. axioms.

Rem. 1 $\bigcap_{t\in T} A_t \subseteq \bigcup_{t\in T} A_t$, $\{A_t\}_{t\in T}$ – seq. of subsets of Ω

Def. 1 $\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$

$\Rightarrow \mathbb{1}_A \leq \mathbb{1}_B$ iff $A \subseteq B$

$\Rightarrow \mathbb{1}_{A^c} = 1 - \mathbb{1}_A$

Def. 2 $\inf_{k\geq n} A_k := \bigcap_{k=n}^{\infty} A_k \qquad \sup_{k\geq n} \bigcup_{k=n}^{\infty} A_k$

$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \qquad \limsup_{n\to\infty} \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$

Def. 3 $\lim_{n\to\infty} B_n = B = \limsup_{n\to\infty} B_n = \liminf_{n\to\infty} B_n \qquad (B_n \to B)$

$\Rightarrow \liminf_{n\to\infty} B_n = \lim_{n\to\infty} (\inf_{k\geq n} B_k)$

$\Rightarrow \limsup_{n\to\infty} B_n = \lim_{n\to\infty} (\sup_{k\geq n} B_k)$

Lemma 1 $\{A_n\}$ – seq of subsets Ω

$\limsup_{n\to\infty} A_n = \{\omega \in \Omega : \sum_{n=1}^{\infty} \mathbb{1}_{A_n}(\omega) = \infty\} = \{\omega \in \Omega : \omega \in A_{n_k}, k = 1, 2 \dots\}$

$= [A_n \text{ i.o.}]$

$\liminf_{n\to\infty} A_n = \{\omega : \omega \in A_n \ \forall n \text{ except a finite number}\} =$

$= \{\omega : \sum_n \mathbb{1}_{A_n^c}(\omega) < \infty\} = \{\omega : \omega \in A_n \ \forall n \geq n_0(\omega)\}$

$\Rightarrow \liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n$

$\Rightarrow (\liminf_{n\to\infty} A_n)^c = \limsup_{n\to\infty} A_n^c$

Mon. seq $A_n \uparrow \Rightarrow \lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$

$A_n \downarrow \Rightarrow \lim_{n\to\infty} A_n = \bigcap_{n\geq 1}^{\infty} A_n$