Ornstein - Uhlenbeck processes. Novikov martingale method.

Kasianova, Shulyak

Higher School of Economics

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Some results on AR(1)

We shall assume that we have some underlying probability space $(\Omega, \mathcal{F}, \mathrm{P})$ in our discussions below and $\mathbf{E}$ will be expectations with respect to this probability.

We shall study some aspects of the simplest autoregressive time series : AR(1). It is given as a solution of

$$X_t = \alpha X_{t-1} + \varepsilon_t$$

for $t \in \mathrm{T}$ where $\mathrm{T} = \{0, 1, 2, 3, \ldots\}$ or $\mathrm{T} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$.

We are mainly interested in the following problem: given that $X_0 = x > 0$ and $\tau(x) = \min\{n \geq 0 \, : \, X_n \leq 0\}$, what is the distribution of τ? In other words we see values for $\mathrm{P}\{\tau(x) = n\}$ for each integer $n \geq 0$. The difficulty with this problem is the fact that we do not have $X_{\tau(x)} = 0$.

We shall assume that the process $\{X_t\}$ is real valued, but there are generalizations of this in the literature.

Definition 1

Let $\{X_t : t \in \mathrm{T}\}$ be a process with $V(X_t) < \infty$. Then the **autocorrelation** γ of it is defined by

$$\gamma(r,s) = \mathrm{cov}(X_r, X_s) = \mathbf{E}[(X_r - \mathbf{E}[X_r])(X_s - \mathbf{E}[X_s])]$$

for $r, s \in \mathrm{T}$.

Definition 2

Let $\mathrm{T} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$. We say that $\{X_t\}$ is stationary if

(i) $\mathbf{E}[|X_t|^2] < \infty$ for all $t \in \mathrm{T}$

(ii) $\mathbf{E}[X_t] = m$ for all $t \in \mathrm{T}$ and some real m

(iii) $\gamma(r, s) = \gamma(r + t, s + t)$ for all $s, t, r \in \mathrm{T}$

We note that (iii) implies $\gamma(r, s) = \gamma(r - s, 0)$ if we put $t = -s$.

We then write $\gamma(h)$ for $\gamma(h, 0)$.

Alternatively, such processes are also called weak stationary, covariance stationary, stationary in wide sense, and second order stationary.

Definition 3

We say that $\{X_t : t \in \mathrm{T}\}$ is strictly stationary if

$$\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\} \stackrel{d}{=} \{X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_n+h}\}$$

for all positive integers n and $t_1, t_2, ..., t_n, h \in \mathrm{T}$.

Some results on AR(1)

In the case of AR(1) we have

$$X_t = \alpha X_{t-1} + \varepsilon_t \tag{1}$$

with $\mathbf{E}[\varepsilon_t] = 0$ and $\mathbf{E}[\varepsilon_t^2] = \sigma^2$ for all t, and the correlation

$$\gamma(h) = \operatorname{cov}(\varepsilon_t, \varepsilon_{t+h}) = \begin{cases} 0 & \text{if} \quad h \neq 0 \\ \sigma^2 & \text{if} \quad h = 0 \end{cases}$$

If this is the case, the noises are uncorrelated and $\{\varepsilon_t\}$ is said to be $\mathrm{WN}(0, \sigma^2)$, a white noise process.

If $\{\varepsilon_t\}$ are iid, then $\{\varepsilon_t\}$ is said to be an $\mathrm{IID}(0, \sigma^2)$ process.

In the classic case we have $\{\varepsilon_t\} \sim \mathcal{N}(0, \sigma^2)$ and uncorrelated or we have $\{\varepsilon_t\} \sim \mathcal{N}(0, \sigma^2)$ and independent.

In the case that $\{\varepsilon_t\}$ are iid and $\mathbf{E}[\varepsilon_t^2] < \infty$, then it turns out that the two types of stationarity are the same.

Time series people are interested in **stationary solutions** of equation (1), and estimation of such time series are based on this assumption.

Ornstein - Uhlenbeck processes

Recall

- Arithmetic BM: $dX_t = \mu dt + \sigma dB_t$.
- Geometric BM: $dX_t = \mu X_t dt + \sigma X_t dB_t$.
- OU process: $dX_t = \theta(\mu - X_t)dt + \sigma dB_t$.

OU process is a mean-reverting process, i.e. over time, the process tends to drift towards its mean function.

Direction and the magnitude of the drift is not constant but changes depending on the difference between th value of the process and its long term mean.

When X_t is below μ the positive drift pulls it back up towards μ, whereas the opposite occurs when X_t is above μ. Therefore, μ may be interpreted as the "long run mean" of the process ($E[X_t] \to \mu \; as \; t \to \infty$) and the mean-reversion coefficient θ is the speed of adjustment of X_t towards its long run. Stochastic term σdB_t captures the instantaneous volatility, i.e. constant instantaneous variance σ^2, which ensures that the process erratically and continuously fluctuates around the level μ.

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t \tag{1}$$

Figure 1 shows three sample paths of equation (1) for $\sigma = 0.05$, $\mu = 1$, and for increasing values of θ. It can be seen that the smaller θ, the more X_t drifts away from μ. For $\theta = 2$, X_t displays only small and short-lived deviations from μ.

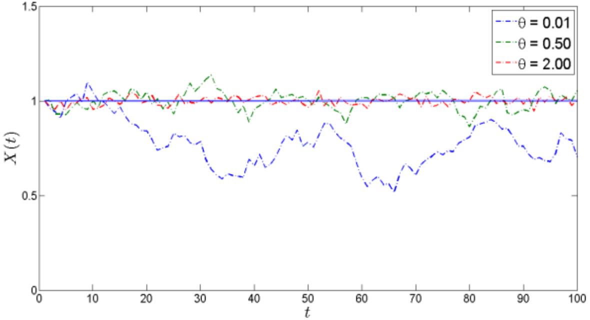


Figure 1: Three sample paths of the mean-reverting Ornstein-Uhlenbeck process.

To solve SDE

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t \tag{2}$$

let's first consider the deterministic version of this equation

$$dX_t = \theta(\mu - X_t)dt \tag{3}$$

$$dX_t + \theta X_t dt = \theta\mu dt \tag{4}$$

$$\frac{dX_t}{dt} + \theta X_t = \theta\mu \tag{5}$$

Can be solved via the integrating factor method.

The integrating factor is $e^{\int_0^t \theta ds} = e^{\theta t}$

$$e^{\theta t}\frac{dX_t}{dt} + e^{\theta t}\theta X_t = e^{\kappa t}\theta\mu \tag{6}$$

The left hand side is an exact differential of the product.

$$\frac{d}{dt}(e^{\theta t}X_t) = e^{\theta t}\theta\mu \tag{7}$$

Recall, that for OU process by Ito's lemma $dX_t^2 = \sigma^2 dt$

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t \tag{8}$$

We are looking for solution in a form

$$d(e^{\theta t}X_t) = e^{\theta t}\theta\mu dt \tag{9}$$

$$dX_t + \theta X_t dt = \theta\mu dt + \sigma dB_t \tag{10}$$

Using the same integrating factor

$$e^{\theta t}dX_t + \theta e^{\theta t}X_t dt = \theta\mu e^{\theta t}dt + \sigma e^{\theta t}dB_t \tag{11}$$

By Ito's product rule $d(e^{\theta t}X_t) = d(e^{\theta t})X_t + e^{\theta t}dX_t + d(e^{\theta t})dX_t$, $d(e^{\theta t})dX_t = 0$ as a covariance between a deterministic and stochastic function is zero

$$d(e^{\theta t}X_t) = \theta\mu e^{\theta t}dt + \sigma e^{\theta t}dB_t \tag{12}$$

$$\int_0^T d(e^{\theta t}X_t) = \int_0^T \theta\mu e^{\theta t}dt + \int_0^T \sigma e^{\theta t}dB_t \tag{13}$$

$$e^{\theta T}X_T - e^0 X_0 = \theta\mu\frac{e^{\theta T} - e^0}{\theta} + \int_0^T \sigma e^{\theta t}dB_t \tag{14}$$

$$X_T = e^{-\theta T}X_0 + \mu(1 - e^{-\theta T}) + \sigma e^{-\theta T}\int_0^T e^{\theta t}dB_t \tag{15}$$

SDE:

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t \tag{16}$$

Solution:

$$X_T = e^{-\theta T}X_0 + \mu(1 - e^{-\theta T}) + \sigma \int_0^T e^{-\theta(T-t)}dB_t \tag{17}$$

We now characterize its probability distribution. It is easy to see that X is normally distributed, as the integral of a deterministic function w.r.t. Brownian is Gaussian.
The expected value of a deterministic function w.r.t. Brownian is zero. Therefore,

$$E[X_T] = E[e^{-\theta T}X_0 + \mu(1 - e^{-\theta T}) + \sigma \int_0^T e^{-\theta(T-t)}dB_t = e^{-\theta T}X_0 + \mu(1 - e^{-\theta T}) \tag{18}$$

Using Ito's isometry rule

$$E[(\int_0^T X_t dW_t)^2] = E[\int_0^T X_t^2 dt] \tag{19}$$

gives us a deterministic integral that we can easily evaluate

$$Var[X_T] = E[(X_T - E[X_T])^2] = E[(\sigma \int_0^T e^{-\theta(T-t)}dB_t)^2] = \sigma^2 E[\int_0^T e^{-2\theta(T-t)}dt] = \sigma^2 \frac{1 - e^{-2\theta T}}{2\theta} \tag{20}$$

Up to this point we have assumed the start point to be 0 and the end point to be time T. The distribution of the value of the process at time T given the information at time 0.

$$X_T \sim N(e^{-\theta T}X_0 + \mu(1 - e^{-\theta T}), \sigma^2 \frac{1 - e^{-2\theta T}}{2\theta}) \tag{21}$$

We can generalize this to an arbitrary starting point X_t

$$X_{t+\Delta t} = e^{-\theta \Delta t}X_t + \mu(1 - e^{-\theta \Delta t}) + \sigma\sqrt{\frac{1 - e^{-2\theta \Delta t}}{2\theta}}N(0, 1) \tag{22}$$

It is easy to see that the process is similar to AR(1)

$$y_{t+1} = \beta y_t + \alpha + e_{t+1} \tag{23}$$

Ornstein - Uhlenbeck processes

In this section $\{X_t\}$ is an Ornstein-Uhlenbeck process.

Suppose that $X_0 = x$ and

$$\tau(x) = \inf\{t \geq 0 \, : \, X_t \leq 0\}$$

then we wish to find $\mathrm{P}(\tau(x) \leq t)$.

Clearly we need only study the case where $x > 0$.

Ornstein - Uhlenbeck processes

Yi, Chuang. "On the first passage time distribution of an Ornstein–Uhlenbeck process." Quantitative Finance 10, no. 9 (2010): 957-960.

Let

$$dX_t = -\alpha X_t dt + \sigma\sqrt{2\alpha} dB_t \tag{24}$$

and let $X_0 = x > 0$.

Then for each $t > 0$ the solution X_t is Gaussian and

$$X_t \sim \mathcal{N}(x\, e^{-\alpha t},\, \sigma^2(1 - e^{-2\alpha t}))$$

In fact

$$d(e^{\alpha t} X_t) = e^{\alpha t}(\alpha X_t dt + dX_t) = e^{\alpha t}\sigma\sqrt{2\alpha}\, dB_t$$

implies that

$$e^{\alpha t} X_t = X_0 + \int_0^t e^{\alpha s}\sigma\sqrt{2\alpha}\, dB_s$$

and hence

$$X_t = e^{-\alpha t}\, x + \sigma\sqrt{2\alpha}\int_0^t e^{-\alpha(t-s)}\, dB_s$$

Thus

$$X_t \sim \mathcal{N}(m(t),\, n(t)^2)$$

where

$$m(t) = \mathbf{E}[X(t)] = e^{-\alpha t}\, x$$

and

$$n(t)^2 = \mathbf{E}\left[\left(\sigma\sqrt{2\alpha}\int_0^t e^{-\alpha(t-s)}\, dB_s\right)^2\right] = \sigma^2 2\alpha \int_0^t e^{-2\alpha(t-s)}\, ds = \sigma^2(1-e^{-2\alpha t})$$

Theorem 1

We have

$$\mathrm{P}(\tau(x) \le t) = 2\,\mathrm{P}(X_t \le 0)$$

Corollary

From the theorem

$$
\begin{aligned}
\mathrm{P}(\tau(x) \le t) &= 2\,\mathrm{P}\left(\frac{X_t - m(t)}{n(t)} \le -\frac{m(t)}{n(t)}\right) \\
&= 2\,\Phi\left(-\frac{m(t)}{n(t)}\right) \\
&= 2\,\Phi\left(\frac{-x e^{-\alpha t}}{\sigma\sqrt{1 - e^{-2\alpha t}}}\right) \\
&= \int_0^t f(u)\,du
\end{aligned}
$$

where Φ stands for cumulative distribution function on $\mathcal{N}(0,1)$ and

$$
f(u) = \frac{x}{2\sigma\sqrt{\pi\alpha}} \exp\left\{-\frac{x^2\,e^{-\alpha u}}{4\,\sigma^2\,\sinh(\alpha u)} + \frac{\alpha u}{2}\right\}\left(\frac{\alpha}{\sinh(\alpha\,u)}\right)^{\frac{3}{2}}
$$

for $u > 0$ will be a consequence.

Proof of the Theorem

We note at

$$\begin{aligned} d(-X_t) &= -\lambda(-X_t)\,dt - \sigma dB_t \\ &= -\lambda(-X_t)\,dt + \sigma d\tilde{B}_t \end{aligned}$$

where $\tilde{B}_t = -B_t$.

Define a new process by

$$\tilde{X}_t = \begin{cases} X_t & \text{if} \quad t \leq \tau \\ -X_t & \text{if} \quad t > \tau \end{cases}$$

then $\tilde{X}_0 = X_0 = x$.

We now claim that $\tilde{X}_t \stackrel{d}{=} X_t$.

To show this, let g be an arbitrary bounded Borel function. Then

$$\begin{aligned}
\mathbf{E}[g(\tilde{X}_t)] &= \mathbf{E}[g(\tilde{X}_t)\mathrm{I}(t \le \tau)] + \mathbf{E}[g(\tilde{X}_t)\mathrm{I}(t > \tau)] \\
&= \mathbf{E}[g(X_t)\mathrm{I}(t \le \tau)] + \mathbf{E}[g(-X_t)\mathrm{I}(t > \tau)] \\
&= \mathbf{E}[g(X_t)] + \mathbf{E}[g(-X_t)\mathrm{I}(t > \tau)] - \mathbf{E}[g(X_t)\mathrm{I}(t > \tau)] \\
&= \mathbf{E}[g(X_t)]
\end{aligned}$$

using the fact that $X_\tau = -X_\tau = 0$.

To be very precise, the cancellation follows from the strong Markov property for Itô processes. See Øksendal[2] Theorem 7.2.4.

$$\mathrm{P}(\tau(x) \leq t) = \mathrm{P}(\{\tau(x) \leq t\} \cap \{X_t \leq 0\}) + \mathrm{P}(\{\tau(x) \leq t\} \cap \{X_t < 0\})$$

$$= \mathrm{P}(X_t \leq 0) + \mathrm{P}(\tilde{X}_t \leq 0)$$

To see this, we note that $X_t \leq 0$ implies that $\tau \leq t$ and hence

$$\{t \leq \tau\} \cap \{X_t \leq 0\} = \{X_t \leq 0\}$$

Also

$$\begin{aligned}
\{\tau \leq t\} \cap \{X_t > 0\} &= \{\tau \leq t\} \cap \{-X_t < 0\} \\
&= \{\tau < t\} \cap \{-X_t < 0\} \\
&= \{\tau < t\} \cap \{\tilde{X}_t < 0\} \\
&= \{\tau \leq t\} \cap \{\tilde{X}_t < 0\} \\
&\overset{\mathrm{P}}{=} \{\tau \leq t\} \cap \{\tilde{X}_t \leq 0\} \\
&= \{\tilde{X}_t \leq 0\}
\end{aligned}$$

where we wrote $\overset{\mathrm{P}}{=}$ to indicate equality up to a set of probability zero.

Therefore

$$\mathrm{P}(\tau(x) \leq t) = \mathrm{P}(X_t \leq 0) + \mathrm{P}(\tilde{X}_t \leq 0) = 2\,\mathrm{P}(X_t \leq 0)$$

Stationary Solutions

$$X_t = \alpha X_{t-1} + \varepsilon_t \quad (1)$$

We can iterate the AR(1) equation to obtain

$$X_t = \varepsilon_t + \alpha\varepsilon_{t-1} + \alpha^2\varepsilon_{t-2} + \ldots + \alpha^{k+1}X_{t-k-1} \tag{2}$$

If $|\alpha| < 1$ we shall show that

$$X_t = \sum_{j=0}^{\infty} \alpha^j \varepsilon_{t-j} \tag{3}$$

converges and provides the unique stationary solution to equation (1).

Let us first observe that stationarity of $\{X_t\}$ implies that

$$\mathbf{E}[X_t^2] = \gamma(0) + \mathbf{E}[X_t]^2 = \gamma(0) + m^2 = \mathbf{E}[X_0^2]$$

for all $t \in \mathrm{T}$.

The sum in (3) converges in mean square as (using (2))

$$\mathbf{E}\left[X_t - \sum_{j=0}^{k} \alpha^j \varepsilon_{t-j}\right]^2 = \mathbf{E}\left[\alpha^{2(k+1)} X_{t-k-1}^2\right] = \alpha^{2(k+1)}\,\mathbf{E}[X_0^2] \to 0$$

as $k \to \infty$.

From Chebyshev inequality

This implies convergence is probability, because

$$\mathrm{P}\left(\left|X_t - \sum_{j=0}^{k} \alpha^j \varepsilon_{t-j}\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^2}\, \alpha^{2(k+1)}\, \mathbf{E}[X_0^2] \to 0$$

as $k \to \infty$.

In fact (3) converges almost surely. In the case when $\{\varepsilon_t\}$ are independent, then we can use Lecture 55, Lemma 4, page 816 as

$$\sum_{j=0}^{\infty} \mathbf{E}[\alpha^{2j}\varepsilon_{t-j}^2] = \sigma^2 \sum_{j=0}^{\infty} \alpha^{2j} = \frac{\sigma^2}{1-\alpha^2} < \infty,$$

Stationary Solutions

Lemma 1

If X is a random variable with $\mathbf{E}[|X|] < \infty$, then X is finite almost surely.

Proof

Let

$$A = \{\omega \in \Omega \,:\, |X(\omega)| = \infty\}$$

It suffices to show that $A \in \mathcal{F}$ and $\mathrm{P}(A) = 0$.

In fact, for positive integers n, let

$$A_n = \{\omega \in \Omega \,:\, |X(\omega)| > n\}$$

Then $A_n \in \mathcal{F}$ and so

$$A = \bigcap_{n=1}^{\infty} A_n \;\in\; \mathcal{F}$$

and for each n

$$\infty > \mathbf{E}[|X|] \geq \mathbf{E}[|X|\,\mathrm{I}(A)] \geq n\,\mathbf{E}[\mathrm{I}(A)] = n\,\mathrm{P}(A)$$

implies

$$\mathrm{P}(A) \leq \frac{\mathbf{E}[|X|]}{n} \to 0$$

as $n \to \infty$. ∎

Stationary Solutions

We note, using monotone convergence, that

$$\mathbf{E}\left[\sum_{j=0}^{\infty}|\alpha|^j\,|\varepsilon_{t-j}|\right]=\lim_{n\to\infty}\mathbf{E}\left[\sum_{j=0}^{n}|\alpha|^j\,|\varepsilon_{t-j}|\right]$$

$$=\lim_{n\to\infty}\sum_{j=0}^{n}|\alpha|^j\,\mathbf{E}\left[|\varepsilon_{t-j}|\right]$$

$$\leq\lim_{n\to\infty}\sum_{j=0}^{n}|\alpha|^j\sqrt{\mathbf{E}\left[\varepsilon_{t-j}^2\right]}=\frac{\sigma}{1-|\alpha|}<\infty$$

So there exists $\Omega_0\subset\Omega$ with $\mathrm{P}(\Omega_0)=0$ and when $\omega\notin\Omega_0$ we have

$$\sum_{j=0}^{\infty}|\alpha|^j\,|\varepsilon_{t-j}(\omega)|<\infty$$

For such $\omega\notin\Omega_0$ the expression in (3) converges because if we set

$$S_n(\omega)=\sum_{j=0}^{n}\alpha^j\varepsilon_{t-j}(\omega)$$

then

$$|S_n(\omega)-S_m(\omega)|\equiv\left|\sum_{j=m+1}^{n}\alpha^j\varepsilon_{t-j}(\omega)\right|\leq\sum_{j=m+1}^{n}|\alpha|^j\,|\varepsilon_{t-j}(\omega)|\;\to 0$$

as $m,n\to\infty$.

Remarks

1. If $|\alpha| > 1$, there is also a stationary solution

$$X_t = -\sum_{j=1}^{\infty} \alpha^{-j} \, \varepsilon_{t+j}$$

as can easily be checked. However it is not what is called **causal** as X_t depend on the future values $\{\varepsilon_s \; : \; s > t\}$. Such a solution is regarded as unnatural.

2. When $|\alpha| = 1$ there is no stationary solution. For example, let $\alpha = 1$. If there were a stationary solution, then

$$X_t - X_0 = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t$$

implies

$$\mathbf{E}[(X_t - X_0)^2] = \mathbf{E}[(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_t)^2]$$

But now the right hand side is $t\,\sigma^2 \to \infty$ as $t \to \infty$ while the left hand side remains bounded as

$$\mathbf{E}[(X_t - X_0)^2] \leq \mathbf{E}[2\,X_t^2 + 2\,X_0^2] = 4\,\mathbf{E}[X_0^2] < \infty$$

for all $t \in \mathrm{T}$.

When $\alpha = -1$ we use

$$\mathbf{E}[(X_{2t} - X_0)^2] = \mathbf{E}[(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \cdots + \varepsilon_{2t})^2]$$

and come to a similar contradiction.

Remarks

3. When $|\alpha| < 1$, the formula for γ is obtained from (3). In fact, (3) implies that $\mathbf{E}[X_t] = 0$ for all $t \in \mathrm{T}$ and for $r \geq s$,

$$\gamma(r,s) = \mathbf{E}[X_r\,X_s] = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \alpha^{j+k}\,\mathbf{E}[\varepsilon_{r-j}\,\varepsilon_{s-k}] = \sigma^2 \sum_{k=0}^{\infty} \alpha^{2k+r-s} = \frac{\alpha^{r-s}}{1-\alpha^2}\,\sigma^2$$

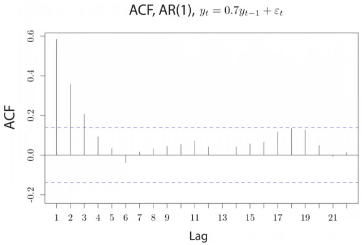
and if $r < s$

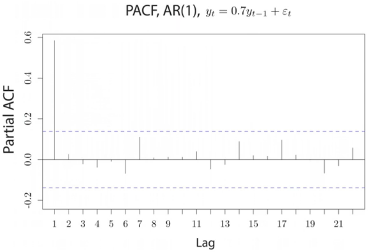
$$\gamma(r,s) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \alpha^{j+k}\,\mathbf{E}[\varepsilon_{r-j}\,\varepsilon_{s-k}] = \sigma^2 \sum_{j=0}^{\infty} \alpha^{2j+s-r} = \frac{\alpha^{s-r}}{1-\alpha^2}\,\sigma^2$$

so

$$\gamma(h) = \frac{\alpha^{|h|}}{1-\alpha^2}\,\sigma^2$$

for $h \in \mathrm{T}$.





Crossing-time problem

Let $x \in \mathbb{R}$ and define

$$\tau(x) = \min\{n \,:\, X_n \leq 0,\ X_0 = x\}^3$$

and

$$\psi_k(x) = \mathrm{P}(\tau(x) = k) \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

We would like to compute the values of $\psi_k(x)$, but this is an unsolved problem. So we need to find a method to finds its values or develop approximate methods.

There are two methods to study this question.

(i) Use martingales.

(ii) Use integral equations.

[3] We use the convention : if $X_n > 0$ for all n, then $\tau(x) = \infty$

Integral Equation Methods

We suppose that the $\{\varepsilon_t\}$ is iid with common cumulative distribution function F and density f.

We have

$$\psi_0(x) = \begin{cases} 0 & \text{if} \quad x > 0 \\ 1 & \text{if} \quad x \le 0 \end{cases}$$

and for $x > 0$

$$\psi_1(x) = \mathrm{P}(X_1 \le 0) = \mathrm{P}(\varepsilon_1 \le -\alpha\, x) = F(-\alpha\, x)\ .$$

We obtain the recurrence. For $x > 0$

$$\begin{aligned}
\psi_{k+1}(x) &= \mathbf{E}[\psi_k(X_1)\,\mathrm{I}(X_1 > 0)] \\
&= \mathbf{E}[\psi_k(\alpha x + \varepsilon_1)\mathrm{I}(\alpha x + \varepsilon_1 > 0)] \\
&= \int_{-\alpha x}^{\infty} \psi_k(\alpha x + u) f(u)\, du \\
&= \int_0^{\infty} \psi_k(v) f(v - \alpha x)\, dv \qquad\qquad (\text{putting } v = \alpha x + u)
\end{aligned}$$

Hence, the integral equation recurrence is

$$\psi_{k+1}(x) = \int_0^{\infty} \psi_k(v) f(v - \alpha x)\, dv$$

Integral Equation Methods

We can introduce a generating function

$$\psi(x, z) = \sum_{j=1}^{\infty} \psi_k(x)\, z^k$$

for $|z| \leq 1$. Then using $\psi_0(v) = 0$ when $v > 0$,

$$\sum_{k=0}^{\infty} z^{k+1}\psi_{k+1}(x) = \int_0^{\infty} \sum_{k=0}^{\infty} z^{k+1}\psi_k(v) f(v - \alpha x)\, dv$$

$$= z \int_0^{\infty} \sum_{k=1}^{\infty} z^k \psi_k(v) f(v - \alpha x)\, dv$$

and so

$$\psi(x, z) = z \int_0^{\infty} \psi(v, z) f(v - \alpha x)\, dv$$

The general solution of this equation is unknown, but perhaps for some good choices of f an explicit solution may be possible. This is clearly a problem for exploration. The example in Exercise 1 could be explored.

Alexander Novikov (University of Technology, Sydney) has several studies using martingale methods in the study of AR(1) processes. We will present some of these ideas, and provide some extra details not provided in these papers. While these methods will not lead to formulas for the characteristic function or the moment generating function of the crossing time, it can lead to some estimates and does lead under some technical conditions to the result that

$$\mathrm{P}(\tau(x) < \infty) = 1$$

Applied to our study, Novikov looks at a reflection of what we are interested in. In order to be able to read the Novikov papers we revert to his setup.

Abstract

Using the martingale approach we find sufficient conditions for exponential boundedness of first passage times over a level for ergodic first order autoregressive sequences. Further, we prove a martingale identity to be used in obtaining explicit bounds for the expectation of first passage times.

In applications, the distribution and expectation of such passage times are usually approximated via Monte-Carlo simulation or using Markov chain approximations (e.g. [16]). However, analytical bounds are also of interest (e.g. to control an accuracy of simulation algorithms).

Let $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ be iid random variables, and for $0 < \alpha < 1$ and $n = 1, 2, 3, \ldots$

$$X_n = \alpha\, X_{n-1} + \varepsilon_n$$

and $X_0 = x < 0$.

We let

$$\tau = \min\{n > 0 \,:\, X_n \geq 0\}$$

The study of this crossing time is equivalent to the one we defined earlier.

As martingales will be presented, these will be with respect to the natural filtration defined by

$$\mathcal{F}_n = \sigma\,\{X_0, X_1, \ldots, X_n\}$$

We shall assume that

$$M(u) = \mathbf{E}[\exp(u\,\varepsilon)] < \infty \tag{5}$$

for all $u \geq 0$.

Let us note that this implies that

$$\mathbf{E}[\varepsilon^+] = \mathbf{E}[\max(0, \varepsilon)] < \infty$$

This is because

$$\begin{aligned}\mathbf{E}[\varepsilon^+] &\le \mathbf{E}[\exp(\varepsilon^+)] \\ &= \mathbf{E}[\mathrm{I}(\varepsilon \ge 0)\exp(\varepsilon)] \\ &\le \mathbf{E}[\mathrm{I}(\varepsilon \ge 0)\exp(\varepsilon)] + \mathbf{E}[\mathrm{I}(\varepsilon < 0)\exp(\varepsilon)] = M(1) < \infty\end{aligned}$$

But (5) does not imply

$$\mathbf{E}[\varepsilon^-] = \mathbf{E}[\max(0, -\varepsilon)] < \infty$$

as the following example shows.

By the way, we recall that $\varepsilon = \varepsilon^+ - \varepsilon^-$ and $|\varepsilon| = \varepsilon^+ + \varepsilon^-$.

First, consider a martingale M_n of the form

$$M_n = \lambda^{\nu n} q_\nu(X_n), \tag{5}$$

where a deterministic function $q_\nu(y)$ depends on a parameter ν, the variable y takes values from the domain D of X_n . Note that, typically, $D = (-\infty, \infty)$ but if, for example, condition (3) holds and $H(1-\lambda) \geq x$ then $D \subset (-\infty, (\mathrm{H}/(1-\lambda)))$.

Under the assumption that M_n has a finite expectation, by definition of martingales

$$E[M_n | \mathcal{F}_{n-1}] = M_{n-1} \quad \text{a.s.}$$

which with (1) is equivalent to the equation

$$\lambda^{\nu n} E[q_\nu(\lambda X_{n-1} + \eta_n) | \mathcal{F}_{n-1}] = \lambda^{\nu(n-1)} q_\nu(X_{n-1}) \quad \text{a.s.}$$

Here, η_n is independent of $\mathcal{F}_{n-1}$ and X_{n-1} may take any value from the domain D of X_n. Therefore, if the function $q_\nu(y)$ is a solution of the equation

$$E q_\nu(\lambda y + \eta_1) = \lambda^{-\nu} q_\nu(y), \quad y \in D, \tag{6}$$

and the expectation $E q_\nu(\lambda y + \eta_n)$ is finite then $\lambda^{\nu n} q_\nu(X_n)$ is a martingale.

We present a result for AR(1) series which was presented in greater generality by Vervaat.[8]

Theorem (Vervaat)

Let $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ be iid random variables and suppose that

$$\mathbf{E}[\log^+ |\varepsilon|] < \infty \tag{1}$$

Let us suppose that $0 < |\alpha| < 1$ (for $\alpha = 0$, the result is trivial) and that

$$X_k = \alpha\, X_{k-1} + \varepsilon_k$$

for $k = 1, 2, \ldots$ and suppose $X_0 = x$.

Then there is a random variable η so that

$$X_n \xrightarrow{d} \eta \tag{2}$$

as $n \to \infty$. Further η and ε are independent and

$$\eta \stackrel{d}{=} \varepsilon + \alpha\, \eta\,. \tag{3}$$

We note that

$$\log^+ |\varepsilon| \equiv \max\{0, \log|\varepsilon|\} \leq \max\{0, \log(1+|\varepsilon|)\} = \log(1+|\varepsilon|) \leq |\varepsilon|$$

This says that condition (1) will hold if $\mathbf{E}[\log(1+|\varepsilon|)] < \infty$ or $\mathbf{E}[|\varepsilon|] < \infty$.

We also recall that (2) means that

$$\mathbf{E}[g(X_n)] \to \mathbf{E}[g(\eta)]$$

as $n \to \infty$ for any continuous bounded function g.

Lemma 1

Assuming the set up of the previous theorem, condition (2) implies (3).

Lemma 2

Assume that (1) holds and $a > 1$. There exists $\Omega_0 \subset \Omega$ with $P(\Omega_0) = 0$ so that for any $\omega \notin \Omega_0$ there is a positive integer $n_0 = n_0(\omega)$ so that for all $k \geq n_0(\omega)$,

$$|\varepsilon_k(\omega)| \leq a^k \tag{4}$$

Remark

In the next section we will saw something about the moment generating function of η.

The condition $\mathbf{E}[\log^+|\varepsilon|] < \infty$ often arises in the study of ARMA time series and is often required for the existence of stationary solutions. Brockwell refers to an early paper by Yohai and Maronna (1977) which indicated that for stationarity it is hard to relax this condition further.[9]

We now give some further observations that follow from Vervaat's Theorem.

Let us note that since

$$\mathbf{E}[\log^+|\varepsilon|] < \infty$$

we also have

$$\mathbf{E}[\log^+|\varepsilon^{\pm}|] < \infty \tag{1'}$$

where $\varepsilon^+ = \max(0, \varepsilon)$ and $\varepsilon^- = \max(0, -\varepsilon)$ so that $\varepsilon = \varepsilon^+ - \varepsilon^-$ and $|\varepsilon| = \varepsilon^+ + \varepsilon^-$ and so $0 \le \varepsilon^+, \varepsilon^- \le |\varepsilon|$ from which (1′) follows.

We can then define two sequences

$$\begin{aligned} \eta_n^{\curlywedge} &= \varepsilon_1^+ + \alpha\,\varepsilon_2^+ + \alpha^2\,\varepsilon_3^+ + \cdots + \alpha^{n-1}\,\varepsilon_n^+ \\ \eta_n^{\curlyvee} &= \varepsilon_1^- + \alpha\,\varepsilon_2^- + \alpha^2\,\varepsilon_3^- + \cdots + \alpha^{n-1}\,\varepsilon_n^- \end{aligned}$$

which each converge almost surely as $n \to \infty$ by Vervaat's Theorem since each sequence $\{\varepsilon_n^+\}$ and $\{\varepsilon_n^-\}$ are iid. Let the limits be $\eta^{\curlywedge}$ and $\eta^{\curlyvee}$.

Let us note that $\eta_n^{\curlywedge} \uparrow \eta^{\curlywedge}$ and $\eta_n^{\curlyvee} \downarrow \eta^{\curlyvee}$ as $n \to \infty$, that $\eta^{\curlyvee} \le \eta_n \le \eta^{\curlywedge}$ for each n and $\eta = \eta^{\curlywedge} - \eta^{\curlyvee}$.

We will use these constructions in some of the arguments that follow in the next section.

Lemma A

If (5) holds for $u \geq 0$ and $\mathbf{E}[|\varepsilon|] < \infty$, then the right hand derivative of $M(u)$ at $u = 0$ exists and is given by

$$M'(0+) = \mathbf{E}[\,\varepsilon\,]$$

Remark

We note that $\mathbf{E}[\varepsilon^+] \leq M_+(u) \leq M(u) < \infty$. We also note that $\mathbf{E}[\varepsilon^-] = \infty$ if and only if $M'(0+) = -\infty$.

Contributions by Novikov

The **cumulant function** of ε is defined by

$$\psi(u) = \log \mathbf{E}\,[\,\exp\,(\,u\,\varepsilon\,)\,]$$

Remarks

1. The cumulant function is convex. Suppose that $\psi(s)$ and $\psi(t)$ are finite and $0 < \lambda < 1$, then

$$\begin{aligned}\lambda\psi(s) + (1-\lambda)\psi(t) &= \log\left[\mathbf{E}[\exp(s\,\varepsilon)]^{\lambda}\,\mathbf{E}[\exp(t\,\varepsilon)]^{1-\lambda}\right] \\ &\geq \log\left[\mathbf{E}[\exp((\lambda\,s + (1-\lambda)\,t)\varepsilon)\right] \\ &= \psi(\lambda\,s + (1-\lambda)\,t)\end{aligned}$$

In the process we used Hölder's inequality. In fact

$$\begin{aligned}\mathbf{E}\,[\exp((\lambda\,s + (1-\lambda)\,t)\varepsilon)] &= \mathbf{E}[\exp(\lambda\,s\,\varepsilon)\,\exp((1-\lambda)\,t\,\varepsilon)] \\ &\leq \mathbf{E}[\exp(p\,\lambda\,s\,\varepsilon)]^{\frac{1}{p}}\,\mathbf{E}[\exp(q\,(1-\lambda)\,t\,\varepsilon)]^{\frac{1}{q}}\end{aligned}$$

Where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Choose $p = 1/\lambda$ and $q = 1/(1-\lambda)$.

2. It then follows that for $0 < u < 1$ that

$$\psi(u) = \psi(u \cdot 1 + (1-u)\cdot 0) \leq u\,\psi(1)$$

as $\psi(0) = 0$.

3. If $\mathbf{E}[\,|\varepsilon|\,] < \infty$, then from Lemma A,

$$\psi'(0+) = \mathbf{E}[\,\varepsilon\,]$$

Lemma B

If $\mathbf{E}[\,|\varepsilon|\,] < \infty$, then

$$\phi(u) = \sum_{k=0}^{\infty} \psi(\alpha^k u) \tag{8}$$

is convergent. It is differentiable for $u > 0$. It satisfies

$$\phi(u) = \log \mathbf{E}[\exp(u\,\eta)] \tag{9}$$

where η is the same random variable as in the previous section, and

$$\phi(u) = \phi(\alpha\, u) + \psi(u) \tag{10}$$

Remark

Novikov[10] shows that (8) converges under the weaker condition

$$\mathbf{E}[\log^+ |\varepsilon|] < \infty \tag{11}$$

Remark

Under the weaker condition

$$\mathbf{E}[\log^{+}|\varepsilon|] < \infty$$

there are two cases (1) $\mathbf{E}[\varepsilon^{-}] < \infty$ and (2) $\mathbf{E}[\varepsilon^{-}] = \infty$. In the first case we have $\mathbf{E}[|\varepsilon|] < \infty$ and we have the assumptions of Lemma B. In the second case we have $\psi'(0+) = -\infty$. This implies that for some $u_0 > 0$, $\psi(u) < 0$ for $0 < u < u_0$. This implies that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \psi(\alpha^k u)$$

is finite or $-\infty$. But the latter is not possible as

$$\begin{aligned}
\liminf_{n\to\infty} \sum_{k=0}^{n-1} \psi(\alpha^k u) &= \liminf_{n\to\infty} \log \mathbf{E}[\exp(u\,\eta_n)] \\
&\geq \liminf_{n\to\infty} \log \mathbf{E}[\exp(-u\,\eta_n^{\curlyvee})] \\
&= \log \mathbf{E}[\exp(-u\,\eta^{\curlyvee})] > -\infty
\end{aligned}$$

Remark

This follows because $\mathbf{E}[\log^+|\varepsilon^-|] < \infty$ implies that $0 \le \eta^\Upsilon < \infty$ with probability 1, and if $\log \mathbf{E}[\exp(-u\,\eta^\Upsilon)] = -\infty$ for $u > 0$ we would have a contradiction. This implies that under the condition (11) that $\phi(u)$ defined in (8) exists.

The proof of the Lemma B under (11) now follows as we have already described.

It is a further corollary to these discussions that

$$\phi(u) \ge \log \mathbf{E}[\exp(-u\,\eta^\Upsilon)] \ge -K = \log \mathbf{E}[\exp(-\,\eta^\Upsilon)] > -\infty$$

for all $0 \le u \le 1$.