Brownian Motion

$\exists$ prob. distr. over the set of contin. f $B: \mathbb{R}_{\geq 0} \to \mathbb{R}$

s.t. (i) $P(B(0)=0)=1$

(ii) $\forall\, 0 \leq s \leq t$ $B(t) - B(s) \sim N(0, t-s)$

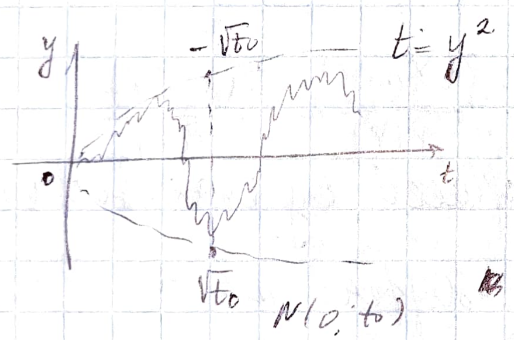
(iii) if $[s_i, t_i]$ aren't overlapping

$\Rightarrow$ $B(t_i) - B(s_i)$ are independent

Properties:

1) Crosses t-axis infin. often

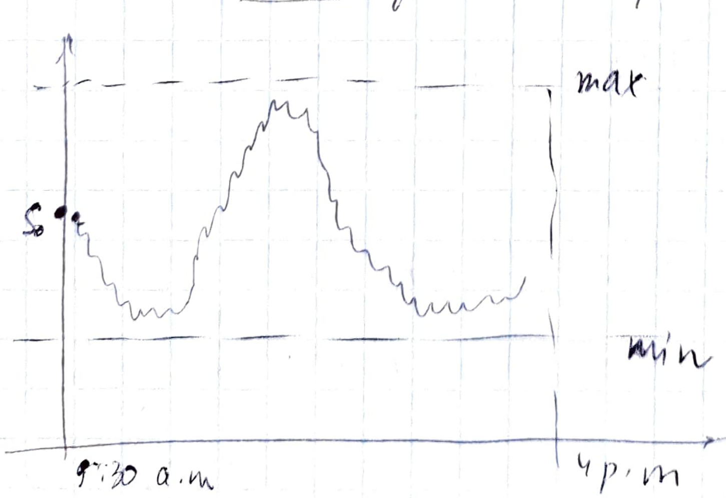
2) Doesn't deviate too much from $t = y^2$



3) Is nowhere differentiable $\Rightarrow$

~~classic~~ calculus $\Rightarrow$ Ito's calculus

Model for stock prices



$M(t) = \max_{0 \leq s \leq t} B(s)$

THM: $\forall t>0,\ a>0$

$$P(M(t)>a) = 2P(B(t)>a)$$

[Pf]: $t_a = \min\limits_{t} \{B(t)=a\}$

- $P(\{B(t)-B(t_a) \geq 0\} \cap \{t_a<t\}) = P(\{B(t)-B(t_a) \leq 0\} \cap \{t_a<t\})$
- $P(M(t)>a) = P(t_a<t) =$

$= P(\{B(t)-B(t_a)>0\} \cap \{t_a<t\}) + P(\{B(t)-B(t_a)<0\} \cap \{t_a<t\})$

$= 2P(\{B(t)-B(t_a)>0\} \cap \{t_a<t\}) =$

" a, as long as

$= 2P(\{B(t)>a\} \cap \{t_a<t\}) = 2P(\{B(t)>a\})$

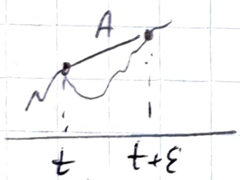
already inside

★ $P(X_1 \mid Y) = P(X_2 \mid Y) \Rightarrow \dfrac{P(X_1 \cap Y)}{P(Y)} = \dfrac{P(X_2 \cap Y)}{P(Y)}$

→ condition can be replaced by intersection

Prop: $\forall t \geq 0$ B.m. is not differ. at t with $p=1$

[Pf]



Suppose $\frac{dB}{dt}(t) = A$

then $|B(t+\varepsilon') - B(t)| \leq \varepsilon A \quad \forall \varepsilon' < \varepsilon$

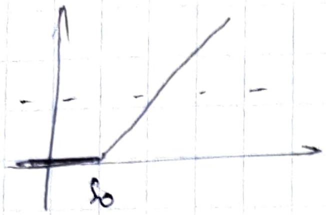
if dif. $\Rightarrow M(\varepsilon) \leq \varepsilon A$

but $P(M(\varepsilon) > \varepsilon A) = 2P(B(\varepsilon) > \varepsilon A) =$

$= 2P(N(0,\varepsilon) > \varepsilon A) = 2\underbrace{P(N(0,1) > \sqrt{\varepsilon}\underset{\downarrow 0}{A})}_{=1/2} = 1, \ \varepsilon \to 0$

Ito's calculus

f - smooth function



$f(B_t)$ - call option

function of the stock

if B_t was dif.

$$df = \cancel{\frac{dB_t}{dt} \cdot dt} = f'(B_t)\,dB_t$$

which is wrong, bcuz $(dB)^2 = dt$

$$f(t+x) = f(t) + f'(t)\cdot x + \underbrace{\frac{f''(t)}{2}\cdot x^2 + \frac{f'''(t)}{3!}\cdot x^3 + \dots}_{\text{ignorable}}$$

in classical calculus $\rightarrow$ ignorable

$$f(t+x) - f(t) \approx f'(t)\cdot x$$

$$f(B_{t+x}) - f(B_t) = f'(B_t)\cdot(\underbrace{B_{t+x} - B_t}_{dB_t}) + \underbrace{\frac{f''(B_t)}{2}\underbrace{(dB_t)^2}_{= dt}}_{\text{no longer ignor. bcuz of B.n}}$$

no longer ignor. bcuz of B.n

$$df = f'(B_t)\,dB_t + \frac{f''(B_t)}{2}\,dt$$

Ito's lemma

★ $(dB_t)^2 = dt$

$\sim N(0, t)$, bcuz $B_t^2 = t \Rightarrow (dB_t)^2 = dt$

Ito's lemma in general

$f(t,x)$; $f(t, B_t)$?

$$f(t+\Delta t, x+\Delta x) = f(t,x) + \frac{\partial f}{\partial t}(t,x)\Delta t + \frac{\partial f}{\partial x}(t,x)\Delta x$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}(t,x)(\Delta t)^2 + 2\frac{\partial^2 f}{\partial t \partial x}(t,x)\Delta t \Delta x + \frac{\partial^2 f}{\partial x^2}(t,x)(\Delta x)^2\right)$$
$$= f + \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{1}{2}\Big(\underbrace{\frac{\partial^2 f}{\partial t^2}(dt)^2 + 2\frac{\partial^2 f}{\partial t \partial x}dt\,dx}_{\text{insignif.}} + \frac{\partial^2 f}{\partial x^2}(dx)^2\Big) + \dots$$

$(dx)^2 = (dB_t)^2 = dt$

$$f(t+dt, B_t+dB_t) - f(t, B_t) =$$
$$= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt =$$
$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t$$

Ito's lemma $f(t, \underset{x}{\underset{\downarrow}{B_t}})$: $df = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x}dB_t$

Consider a It. process X_t : $dX_t = \underbrace{\mu dt}_{\text{drift term}} + \underbrace{\sigma dB_t}_{\text{B.m. term}}$

$X_t = \mu t + \sigma B_t$; μ, σ - const

THM (Ito's lemma) f - smooth function

X_t : $dX_t = \mu dt + \sigma dB_t$

$$df(t, X_t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dB_t$$

[Pf] $df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 =$

$$= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu dt + \sigma dB_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\underbrace{(\mu dt + \sigma dB_t)^2}_{= 0 + 0 + \sigma^2 dB_t^2}$$
$$= \frac{\partial f}{\partial t}dt + \mu\frac{\partial f}{\partial x}dt + \sigma\frac{\partial f}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt \cdot \sigma^2$$

Ito's lemma $f(t, X_t)$. $dx = \mu dt + \sigma dB_t$

$$df = \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + 1/2\, \sigma^2 \frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x} dB_t$$

[ex] $f(t,x) = x^2$ $\quad df(B_t) - ?$ $\quad \Rightarrow \mu = 0 \quad \sigma = 1$

$$df(B_t) = 0 + \mu \cdot 2x\, dt + 1/2\, \sigma^2 \cdot 2\, dt + \sigma \cdot 2x \cdot dB_t$$

$$= 2 \underset{\overset{\shortparallel}{x}}{B_t}\, dB_t + dt$$

[ex] $f(t,x) = e^{at + bx}$, $\mu = 0$, $\sigma = 1$

$$df(t, B_t) = a\, e^{at+bx} dt + 0 + 1/2 \cdot b^2 e^{at+bx} dt + b e^{at+bx} dB_t$$

$$= (a + 1/2\, b^2) f\, dt + b \cdot f\, dB_t$$

Modelling stock prices, using B_t - B.m. motion

$$\frac{dS_t}{S_t} = \sigma dB_t \quad - \text{\% dif. behave like B.m. with some Var } \sigma$$

$$dS_t = \underbrace{1/2\, \sigma^2 dt}_{\text{drift}} + \sigma dB_t$$

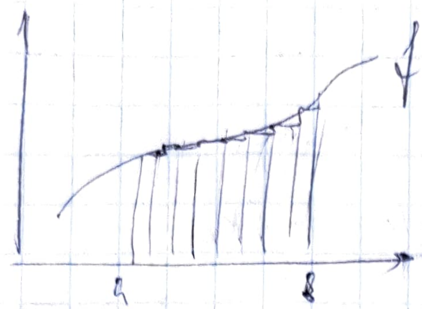
$\Rightarrow S_t \neq e^{\sigma B_t}$ instead $S_t = e^{-1/2 \sigma^2 t + \sigma B_t}$

– geom. B.m. without a drift

(after comp. dt term disappears)

<u>DEF</u> $F(t, B_t) - \int f(t, B_t)\, dB_t + \int g(t, B_t)\, dt$

if $dF = f\, dB_t + g\, dt$



Riemann sum

$$\int_0^8 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{1}^{t} f\left(\frac{t\cdot 8}{n}\right)$$

Ito's integral is the lim of Riem. sums,

when always take left most point of each int

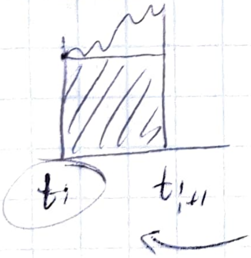
↳ equals inverse of Ito's differentiation

In fact, when rightmost point is taken

all the time we get equiv. theory of calculus

$\exists$ "equivalent" of Ito's calculus

s.t. $(dB_t)^2 = -dt$



decision is based on leftmost point, because we can't see future

DEF $\Delta(t)$ is adapted to X_t (st. process)

if $\forall t \geq 0$ $\Delta(t)$ depends only on $X_0 \sim X_t$

Intuition strategy based only on past values of your stock price

[ex] X_t is adapted to X_t

[ex] $Y_t = X_{t+1}$ is not adapted to X_t

[ex] $\Delta(t) = \min\{X_t, c\}$ is adapted to X_t

[ex] $T > 0$ $\Delta(t) = \max\limits_{t<s<T} X_s$ is not adapted to X_t

$B_t \sim N(0, t)$

$$X(t) = \sigma \cdot B(t) \quad \left(\sim N(0, \sigma^2 t) \right)$$
$$= \int \sigma \, dB_t$$

<u>THM</u> If $\Delta(t)$ is a process, that depends only on t
then $X(t) = \int \Delta(t) \, dB(t)$ has N distribution <u>at all time</u>

<u>THM</u> (Ito isometry)

$\Delta(t)$: adapted to B_t

$$\mathbb{E}\left[\left(\int_0^t \Delta(s) \, dB_s \right)^2 \right] = \mathbb{E}\left[\int_0^t \Delta^2(t) \, ds \right]$$

ito int. of ad. process

(var. => 2)

[ex] (<u>Quadr. variation</u>)

$\Delta(s) = 1 \implies E[(B_t)^2] = t$

When is Ito's int. a martingale?

$\mathbb{E}[X_s \mid \underbrace{\mathcal{F}_t}_{\overset{\shortparallel}{X_0 \sim X_t}}] = X_t$ for all $t < s$

→ conditional exp. based on whatever happened up to time t will be X_t

THM If $g(t, B_t)$ is adapted to B_t, then $\int g(t, B_t)\, dB_t$ is a martingale, as long as g is "reasonable" $\iint g^2(t, B_t)\, dt\, dB_t < \infty$

☆ $dX_t = \mu(t, B_t)\, dt + \sigma(t, B_t)\, dB_t$

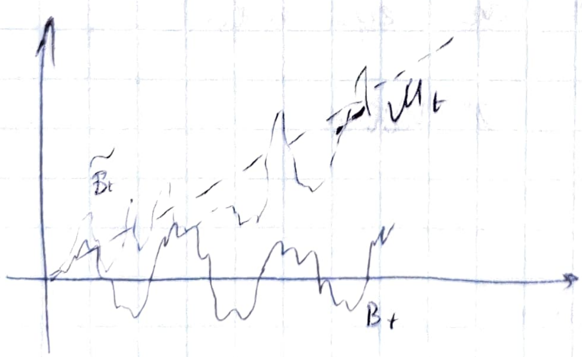
X_t - is a martingale iff $\mu = 0$, $\forall t$

(if $\mu \neq 0$ => drift => not a martingale)

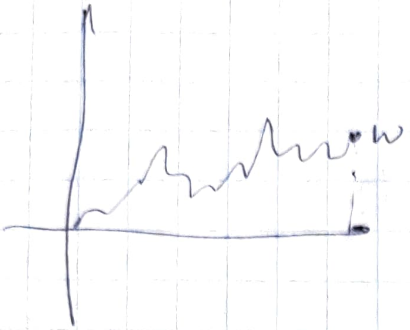
Change of measure (Girsanov thm)

B_t : B.m. w/o drift

$\tilde{B}_t$: B.m. w drift



Switch between B_t and $\tilde{B}_t$ by a change of measure!



$P(w)$ – p.d.f. given by B_t

$\tilde{P}(w)$ – // – $\tilde{B}_t$

$\exists\ Z = Z(w)$

s.t. $P(w) = Z(w) \cdot \tilde{P}(w)$?

$(\Omega, P); (\Omega, \tilde{P})$ – prob. distribution

__DEF__ P and $\tilde{P}$ are equivalent

if $P(A) > 0 \iff \tilde{P}(A) > 0 \quad \forall A \subseteq \Omega$

[ex] $\Omega = \{1, 2, 3\}$

P	$1/3$	$1/3$	$1/3$
$\tilde{P}$	$2/3$	$1/6$	$1/6$

$A = \{1, 2\}$ $\quad \underline{\underline{P(A) = 2/3}}_{>0} \neq \underline{\underline{\tilde{P}(A) = 5/6}}_{>0} \implies$ equivalent

$\tilde{P} = 2/3 \quad 1/3 \quad 0$ $\quad P(B) = 1/3 > 0 \quad \tilde{P}(B) = 0 \implies$ not eq.

$B = \{3\}$

__THM__ $\exists\ Z$ s.t. $P(w) = Z(w) \cdot \tilde{P}(w)$

iff P and $\tilde{P}$ are equivalent

Z – Radon-Nikodym derivative

__THM__ (Girsanov)

B_t and $\tilde{B}_t$ are equivalent.

You can change (switch) between by a multiplicative function

THM (Girsanov)

P : prob. distribution over $[0, T]^{\infty}$

defined by B.m. w/ drift μ

$\tilde{P}$: —//— w/o drift

Then P and $\tilde{P}$ are equivalent

and $Z(w) = \frac{d\tilde{P}}{dP}(w) = e^{-\mu \cdot w(T) - \mu^2 \cdot T/2}$

[ex] $\mathbb{E}[V_t] = \tilde{\mathbb{E}}[\tilde{Z}_t V_t]$

$\uparrow$ (Ω, P) prob. space $\quad \uparrow$ $(\Omega, \tilde{P})$

or $\tilde{\mathbb{E}}[V_t] = \mathbb{E}[Z_t V_t]$

$\hookrightarrow Z(w)$ from THM

Stochastic Dif. Equation

[DEF] SDE is a dif. eq. of the form

$$dX = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dB(t)$$

Goal is to find a st. process $X(t)$, satisfying the above

i.e. $X(t) = \int \mu(s, X(s))\,ds + \int \sigma(s, X(s))\,dB(s)$

* as long as μ and σ are reasonable functions there exist a unique solution (analogical to PDE)

THM SDE has a solution and given the initial point $X(0) = X_0$ then the solution is unique as long as μ, σ are "reasonable"

e.g. satisfies

$$|\mu(t,x) - \mu(t,y)|^2 + |\sigma(t,x) - \sigma(t,y)|^2 \le K \cdot |x-y|^2$$

and $|\mu(t,x)|^2 + |\sigma(t,x)|^2 \le K \cdot (1 + |x|^2)$

(ex) $dX(t) = \mu \cdot X(t) \cdot dt + \sigma \cdot X(t) \cdot dB(t)$, $X(0) = X_0$, $-\infty < \mu < \infty$, $\sigma > 0$

Guess $X(t) = f(t, B(t))$

then $dX(t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \right) dt + \frac{\partial f}{\partial x}\, dB_t$

$$\Rightarrow \quad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \mu \cancel{X(t)} \left[\mu f \right]$$

$$\frac{\partial f}{\partial x} = \sigma X(t) \left[= \sigma f \right] \quad \Rightarrow \quad f = e^{\sigma x + a g(t)}$$

$$\hookrightarrow \quad \frac{\partial f}{\partial t} = a g'(t) f \qquad \frac{\partial^2 f}{\partial x^2} = \sigma^2 f$$

$$\hookrightarrow \quad a g'(t) f + 1/2\, \sigma^2 f = \mu f$$

$$a g'(t) = \mu - 1/2\, \sigma^2$$

$$g(t) = (\mu - 1/2\, \sigma^2)\, c_1 + c_2$$

$$\hookrightarrow \quad f(t,x) = e^{\sigma x + (\mu - 1/2\, \sigma^2) \cdot t + c}$$

$\updownarrow$?

$X(0) = x_0$

$$f(0,0) = e^c = x_0 \quad \Rightarrow \quad \underline{\underline{f(t,x) = x_0 \cdot e^{\sigma x + (\mu - \frac{1}{2}\sigma^2) t}}}$$

[ex] $\quad dX(t) = -\alpha X(t)\, dt + \sigma\, dB(t) \qquad X(0) = x_0$