Begin to write your assignment here

Problem 1

To understand if David has rational preferences we need to define them first. We assume that David is rational if he aims to maximize his satisfaction (utility) with given prices of the goods and budget constraints. In order to do that his preferences should satisfy following axioms: they should be complete (David can compare any two bundles of paint) and transitive (if x $\succ$ y and y $\succ$ z, then x $\succ$ z)

From the text follows that David's preferences are:

1. Canary Yellow $\succ$ Bumblebee Yellow
2. Canary Yellow $\succ$ Lime Yellow
3. Canary Yellow $\succ$ Crayola Yellow
4. Sunrise Yellow $\succ$ Canary Yellow
5. School Bus Yellow $\succ$ Canary Yellow
6. Sunrise Yellow $\succ$ School Bus Yellow

That means, that for David U(School Bus Yellow) $>$ U(Lime Yellow) since he ranked the colors that way (assuming transitivity of his preferences): Sunrise Yellow $\succ$ School Bus Yellow $\succ$ Canary Yellow $\succ$ Lime Yellow.

First, David acts rationally choosing School Bus Yellow, since he the store is out of Sunrise Yellow. When David decides to repaint his house with Lime Yellow he choses the color that gives him smaller utility, also spending twice the money on paint. Assuming that his preferences should stay the same with time we can say either he's not maximizing his utility with given constraints or his preferences aren't transitive. That means the statement " David has rational preferences" is **False**.

Problem 2

A consumer's utility function:

$$U(x_1, x_2) = \sqrt{x_1} + x_2$$

1. The marginal rate of substitution of the consumer at an arbitrary point (x_1^*, x_2^*)

$$MRS = \frac{U'_{x_1}}{U'_{x_2}}\Big|_{(x_1^*, x_2^*)} = \frac{\frac{1}{2\sqrt{x_1^*}}}{1} = \frac{1}{2\sqrt{x_1^*}}$$

   where $x_1^* > 0, x_2^* > 0$

2. Suppose $p_1 = p, p > 0$ - price of the first good, $p_2 = 1$ - price of the second good, $w > 0$ - consumer's income.

To obtain the optimal consumption bundle of the consumer we solve consumer's optimization problem by maximizing his utility with given budget constraints:

$$Max\ U(x_1, x_2)$$
$$s.t. : px_1 + x_2 = w$$

The Lagrangian for this problem:

$$\mathcal{L} = \sqrt{x_1} + x_2 + \lambda(w - px_1 - x_2)$$

FOC:

$$\begin{cases} \mathcal{L}'_{x_1} = \dfrac{1}{2\sqrt{x_1^*}} - \lambda p = 0 \\ \mathcal{L}'_{x_2} = 1 - \lambda = 0 \\ \mathcal{L}'_{\lambda} = w - px_1 - x_2 = 0 \end{cases}$$

From which we derive:

$$\lambda = 1$$
$$\frac{1}{2\sqrt{x_1^*}} = p \Rightarrow x_1^* = \frac{1}{4p^2}$$
$$x_2^* = w - px_1 = w - p\frac{1}{4p^2} = w - \frac{1}{4p}$$

There are two different cases depending on the amount of income:

- In case w is relatively low: $w \leq \frac{1}{4p} \Rightarrow x_2^* = 0, x_1^* = \frac{w}{p}$
- In case w is relatively high: $w \geq \frac{1}{4p} \Rightarrow x_1^* = \frac{1}{4p^2}, x_2^* = w - \frac{1}{4p})$

So the optimal consumption bundle of the consumer is:

$$(x_1^*, x_2^*) = \begin{cases} (\dfrac{1}{4p^2}; w - \dfrac{1}{4p}) & if\ w \geq \dfrac{1}{4p} \\ (\dfrac{w}{p}; 0) & if\ w \leq \dfrac{1}{4p} \end{cases}$$

Problem 3

Consider the three-good setting in which consumer has utility function:

$$u(x_1, x_2, x_3) = (x_1 - b_1)^{\alpha_1}(x_2 - b_2)^{\alpha_2}(x_3 - b_3)^{\alpha_3}, \alpha_k \geq 0, k = 1, 2, 3.$$

1. A monotonic transformation of a utility function is a utility function that represents the same preferences as the original utility function. The following utility functions represents the same preferences because of monotonic transformations:

   - $\tilde{u}(x_1, x_2, x_3) = \ln(u(x_1, x_2, x_3))$, where $\ln(x)$ is a strictly increasing function
   - $\hat{u}(x_1, x_2, x_3) = f(u(x_1, x_2, x_3)) = u(x_1, x_2, x_3)^{\overline{(\alpha_1 + \alpha_2 + \alpha_3)}} = (x_1 - b_1)^{\beta_1}(x_2 - b_2)^{\beta_2}(x_3 - b_3)^{\beta_3}, \beta_k = \dfrac{\alpha_k}{(\alpha_1 + \alpha_2 + \alpha_3)}, k =$ 1,2,3, where $f(x)$ is a strictly increasing function

2. Suppose w is the agent's available income. Then the agent's budget constraint is:

$$p_1x_1 + p_2x_2 + p_3x_3 = w$$

   where p_k - price of the k^{th} good

3. The agent's utility maximization problem is:

$$\begin{gathered} Max\ u(x_1, x_2, x_3) \\ s.t.: p_1x_1 + p_2x_2 + p_3x_3 = w \end{gathered}$$

4. To compute the agent's demand functions for x_1, x_2, x_3 more easily let's use variable substitution $\tilde{x}_k = x_k - b_k, k =$ 1,2,3, then $\tilde{w} = w - b_1p_1 - b_2p_2 - b_3p_3$. Also we should use monotonic transformation on agent's utility function: $\tilde{u}(x_1, x_2, x_3) = \ln(u(x_1, x_2, x_3)) = \alpha_1 \ln(x_1 - b_1) + \alpha_2 \ln(x_2 - b_2) + \alpha_3 \ln(x_3 - b_3) = \alpha_1 \ln \tilde{x}_1 + \ln \tilde{x}_2 + \ln \tilde{x}_3$. So we have same utility maximization problem written in new terms:

$$\begin{gathered} Max\ \tilde{u}(x_1, x_2, x_3) \\ s.t.: p_1\tilde{x}_1 + p_2\tilde{x}_2 + p_3\tilde{x}_3 = \tilde{w} \end{gathered}$$

   The Lagrangian for this problem:

$$\mathcal{L} = \alpha_1 \ln \tilde{x}_1 + \ln \tilde{x}_2 + \ln \tilde{x}_3 + \lambda(\tilde{w} - p_1\tilde{x}_1 - p_2\tilde{x}_2 - p_3\tilde{x}_3)$$

   FOC:

$$\begin{cases} \mathcal{L}'_{x_1} = \dfrac{\alpha_1}{\tilde{x}_1} - \lambda p_1 = 0 \\ \mathcal{L}'_{x_2} = \dfrac{\alpha_2}{\tilde{x}_2} - \lambda p_2 = 0 \\ \mathcal{L}'_{x_3} = \dfrac{\alpha_3}{\tilde{x}_3} - \lambda p_3 = 0 \\ \mathcal{L}'_{\lambda} = \tilde{w} - p_1\tilde{x}_1 - p_2\tilde{x}_2 - p_3\tilde{x}_3 = 0 \end{cases}$$

   From which we derive that

$$\tilde{x}_k^* = \frac{\alpha_k \tilde{w}}{p_k} \Rightarrow x_k - b_k = \frac{\alpha_k(w - b_1p_1 - b_2p_2 - b_3p_3)}{p_k}, k = 1, 2, 3$$

   therefore demand functions for x_1, x_2, x_3 are:

$$x_k = b_k + \frac{\alpha_k(w - \hat{w})}{p_k}, k = 1, 2, 3$$

   where $\hat{w} = b_1p_1 + b_2p_2 + b_3p_3$