

American Derivatives

General American Derivatives

American Call Option

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We work within the context of N -period binomial model with up factor u , down factor d , and interest rate r satisfying the no-arbitrage condition $0 < d < 1 + r < u$. We define S_n to be the set of all stopping times τ that take values in set $\{n, n+1, \dots, N\}$. In particular, S_0 contains every stopping time and S_N only N .

Definition 4.4.1

For each n , $n = 0, 1, \dots, N$ let G_n be a random variable depending on the first n coin tosses. An American derivative security with intrinsic value process G_n is a contract that can be exercised at any time prior to and including time N or not exercised at all, and, if exercised at time n , pays G_n . We define the price process V_n for this contract by the American risk-neutral pricing formula:

$$V_n = \max_{\tau \in S_n} \tilde{E}_n[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau]. \quad (1)$$

One of the immediate consequences of this definition is that $V_N = \max(G_N, 0)$.

To see that, we take $N = n$ and get

$$V_N = \sup_{\tau \in S_N} \mathbb{I}_{(\tau \leq N)} \frac{1}{(1+r)^{\tau-N}} G_\tau. \quad (2)$$

A stopping time in S_n takes only values N and and for such stopping time

$$\mathbb{I}_{(\tau \leq N)} \frac{1}{(1+r)^{\tau-N}} G_\tau = \mathbb{I}_{(\tau=N)} G_N \quad (3)$$

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$Strike = 5, r = 0.25, p = q = 0.5.$

American put option: if owner exercises at time n , he receives $5 - S_n$

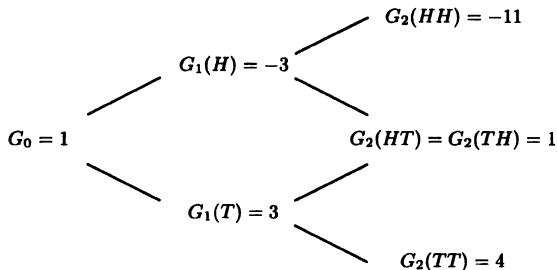


Figure: Intrinsic value

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We apply

$$V_n = \max_{\tau \in S_n} \tilde{E}_n[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau] \quad (4)$$

with $n = 1$, first considering the case of H on the first toss. Then

$$V_1(H) = \max_{\tau \in S_1} \tilde{\mathbb{E}}_1 \left[\mathbb{I}_{(\tau \leq 2)} \left(\frac{4}{5} \right)^{\tau-1} G_\tau \right] (H). \quad (5)$$

To make conditional expectation as much as possible, we should take $\tau(HH) = \infty$ (do not exercise) and take $\tau(HT) = 2$ (exercise at time one in case of HT).

With this exercise policy we get:

$$V_1(H) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \left(\frac{4}{5} \right)^{2-1} G_2(HT) = 0.4 \quad (6)$$

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We apply

$$V_n = \max_{\tau \in S_n} \tilde{E}_n[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau] \quad (7)$$

with $n = 1$, first considering the case of T on the first toss. Then

$$V_1(H) = \max_{\tau \in S_1} \tilde{\mathbb{E}}_1 \left[\mathbb{I}_{(\tau \leq 2)} \left(\frac{4}{5} \right)^{\tau-1} G_\tau \right] (T). \quad (8)$$

To make this conditional expectation as large as possible, knowing that the first toss results in tail, we must consider two possibilities: exercise at time one or at time two. If we take $\tau(TH) = \tau(TT) = 1$, then

$$\tilde{\mathbb{E}}_1 \left[\mathbb{I}_{(\tau \leq 2)} \left(\frac{4}{5} \right)^{\tau-1} G_\tau \right] (T) = G_1(T) = 3. \quad (9)$$

If we take $\tau(TH) = \tau(TT) = 2$, then

$$\tilde{\mathbb{E}}_1 \left[\mathbb{I}_{(\tau \leq 2)} \left(\frac{4}{5} \right)^{\tau-1} G_\tau \right] (T) = \frac{4}{5} \left(\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 4 \right) = 2. \quad (10)$$

Hence, $V_1(T) = 3$.

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Finally, when $n=0$, we have

$$V_0 = \max_{\tau \in S_0} \tilde{\mathbb{E}} \left[\mathbb{I}_{(\tau \leq 2)} \left(\frac{4}{5} \right)^\tau G_\tau \right]. \quad (11)$$

Stopping time that makes (11) as large as possible is

$\tau(HH) = \infty, \tau(TH) = \tau(TT) = 1, \tau(HT) = 2$. With this stopping time it becomes:

$$V_0 = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \left(\frac{4}{5} \right)^2 \cdot G_2(HT) + \frac{1}{2} \cdot \frac{4}{5} \cdot G_1(T) = \frac{1}{4} \cdot \frac{16}{25} \cdot 1 + \frac{1}{2} \cdot \frac{4}{5} \cdot 3 = 1.36 \quad (12)$$

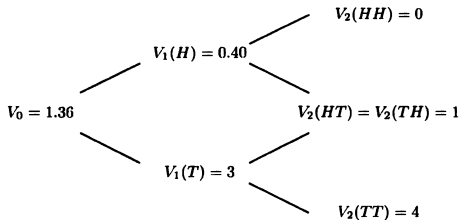


Figure: American put prices

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Theorem 4.4.2

The American derivative security price process given by Definition 4.4.1 has the following properties:

- (i) $V_n \geq \max\{G_n, 0\}$ for all n ;
- (ii) the discounted process $\frac{1}{(1+r)^n} V_n$ is a supermartingale;
- (iii) if Y_n is another process satisfying $Y_n \geq \max\{G_n, 0\}$ for all n and for which $\frac{1}{(1+r)^n} Y_n$ is a supermartingale. then $Y_n \geq V_n$ for all n .

We summarize property (iii) by saying that V_n is the smallest process satisfying (i) and (ii).

We shall see that property (ii) in Theorem 4.4.2 guarantees that an agent beginning with initial capital V_0 can construct a hedging portfolio whose value at each time n is V_n . Property (i) guarantees that if an agent does this, he has hedged a short position in the derivative security; no matter when it is exercised, the agent's hedging portfolio value is sufficient to pay off the derivative security. Thus, (i) and (ii) guarantee that the derivative security price is acceptable to the seller. Condition (iii) says that the price is no higher than necessary in order to be acceptable to the seller. This condition ensures that the price is fair for the buyer.

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We first establish (i). Let n be given, and consider the stopping time $\hat{\tau}$ in \mathcal{S}_n that takes the value n , regardless of the coin tossing. Then

$$\tilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\hat{\tau} \leq N\}} \frac{1}{(1+r)^{\hat{\tau}-n}} G_{\hat{\tau}} \right] = G_n. \quad (13)$$

Since V_n is the largest possible value we can obtain for $\tilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_{\tau} \right]$ when we consider all stopping times $\tau \in \mathcal{S}_n$, we must have $V_n \geq G_n$. On the other hand, if we take $\bar{\tau}$ to be the stopping time in \mathcal{S}_n that takes the value ∞ , regardless of the coin tossing, then

$$\tilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\bar{\tau} \leq N\}} \frac{1}{(1+r)^{\bar{\tau}-n}} G_{\bar{\tau}} \right] = 0. \quad (14)$$

Again, V_n is the maximum of expressions of this type, and hence $V_n \geq 0$. We conclude that (i) holds.

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We next prove (ii). Let n be given, and suppose τ^* attains the maximum in the definition of V_{n+1} ; i.e., $\tau^* \in \mathcal{S}_n$ and

$$V_{n+1} = \tilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n-1}} G_{\tau^*} \right]. \quad (4.4.8)$$

But $\tau^* \in \mathcal{S}_n$ also, which together with iterated conditioning implies

$$\begin{aligned} V_n &\geq \tilde{\mathbb{E}}_n \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*} \right] = \tilde{\mathbb{E}}_n \left[\tilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*} \right] \right] \\ &= \tilde{\mathbb{E}}_n \left[\frac{1}{1+r} \tilde{\mathbb{E}}_{n+1} \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n-1}} G_{\tau^*} \right] \right] = \tilde{\mathbb{E}}_n \left[\frac{1}{1+r} V_{n+1} \right]. \end{aligned} \quad (4.4.9)$$

Dividing both sides by $(1+r)^n$, we obtain the supermartingale property for the discounted price process:

$$\frac{1}{(1+r)^n} V_n \geq \tilde{\mathbb{E}}_n \left[\frac{1}{(1+r)^{n+1}} V_{n+1} \right]. \quad (15)$$

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Finally, we prove (iii). Let Y_n be another process satisfying conditions (i) and (ii). Let $n \leq N$ be given and let τ be a stopping time in \mathcal{S}_n . Because $Y_k \geq \max\{G_k, 0\}$ for all k , we have

$$\begin{aligned}\mathbb{I}_{\{\tau \leq N\}} G_\tau &\leq \mathbb{I}_{\{\tau \leq N\}} \max\{G_\tau, 0\} \leq \mathbb{I}_{\{\tau \leq N\}} \max\{G_{N \wedge \tau}, 0\} + \mathbb{I}_{\{\tau = \infty\}} \max\{G_{N \wedge \tau}, 0\} \\ &= \max\{G_{N \wedge \tau}, 0\} \leq Y_{N \wedge \tau}.\end{aligned}\quad (16)$$

We next use the Optional Sampling Theorem 4.3.2 and the supermartingale property for $\frac{1}{(1+r)^k} Y_k$ to write

$$\begin{aligned}\tilde{\mathbb{E}}_n \left[\mathbb{I}_{\tau \leq N} \frac{1}{(1+r)^\tau} G_\tau \right] &= \tilde{\mathbb{E}}_n \left[\mathbb{I}_{\tau \leq N} \frac{1}{(1+r)^{N \wedge \tau}} G_\tau \right] \leq \tilde{\mathbb{E}}_n \left[\frac{1}{(1+r)^{N \wedge \tau}} Y_{N \wedge \tau} \right] \\ &\leq \frac{1}{(1+r)^{n \wedge \tau}} Y_{n \wedge \tau} = \frac{1}{(1+r)^n} Y_n\end{aligned}\quad (17)$$

the equality at the end being a consequence of the fact that $\tau \in \mathcal{S}_n$ is greater than or equal to n on every path. Multiplying by $(1+r)^n$, we obtain

$$\tilde{\mathbb{E}}_n \left[\mathbb{I}_{\tau \leq N} \frac{1}{(1+r)^{\tau-n}} G_\tau \right] \leq Y_n. \quad (18)$$

Since V_n is the maximum value we can obtain for $\tilde{\mathbb{E}}_n \left[\mathbb{I}_{\tau \leq N} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$ as τ ranges over \mathcal{S}_n , and all these values are less than or equal to Y_n , we must have $V_n \leq Y_n$.

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Theorem 4.4.3

We have the following American pricing algorithm for the path-dependent derivative security price process given by Definition 4.4.1:

$$V_N(\omega_1 \dots \omega_N) = \max\{G_N(\omega_1 \dots \omega_N), 0\} \quad (4.4.10)$$

$$V_n(\omega_1 \dots \omega_n) = \max\{G_n(\omega_1 \dots \omega_n), \frac{1}{1+r} [\tilde{p}V_{n+1}(\omega_1 \dots \omega_n H) + \tilde{q}V_{n+1}(\omega_1 \dots \omega_n T)]\} \quad (4.4.11)$$

for $n = N - 1, \dots, 0$.

We shall prove that V_n defined recursively by (4.4.10) and (4.4.11) satisfies properties (i) and (ii) of Theorem 4.4.2 and is the smallest process with these properties. According to Theorem 4.4.2, the process V_n given by (4.4.1) is the smallest process with these properties, and hence the algorithm (4.4.10), (4.4.11) must generate the same process as formula (4.4.1).

We first establish property (i) of Theorem 4.4.2. It is clear that V_N defined by (4.4.10) satisfies property (i) with $n = N$. We proceed by induction backward in time. Suppose that for some n between 0 and $N - 1$ we have $V_{n+1} \geq \max\{G_{n+1}, 0\}$. Then, from (4.4.11), we see that

$$V_n(\omega_1 \dots \omega_n) \geq \max\{G_n(\omega_1 \dots \omega_n), 0\}. \quad (19)$$

This completes the induction step and shows that V_n defined recursively by (4.4.10), (4.4.11) satisfies property (i) of Theorem 4.4.2.

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We next verify that $\frac{1}{(1+r)^n} V_n$ is a supermartingale. From (4.4.11), we see immediately that

$$V_n(\omega_1 \dots \omega_n) \geq \frac{1}{1+r} [\tilde{p} V_{n+1}(\omega_1 \dots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \dots \omega_n T)] = \tilde{\mathbb{E}}_n \left[\frac{1}{1+r} V_{n+1} \right] (\omega_1 \dots \omega_n) \quad (20)$$

Multiplying both sides by $\frac{1}{(1+r)^n}$ we obtain the desired supermartingale property.

Finally, we must show that V_n defined by (4.4.10), (4.4.11) and satisfying (i) and (ii) of Theorem 4.4.2 is the smallest process satisfying (i) and (ii). It is immediately clear from (4.4.10) that V_N is the smallest random variable satisfying $V_N \geq \max\{G_N, 0\}$. We proceed by induction backward in time. Suppose that, for some n between 0 and $N-1$, V_{n+1} is as small as possible.

The supermartingale property (ii) implies that V_n must satisfy (4.4.12). In order to satisfy property (i), V_n must be greater than or equal to G_n . Therefore, properties (i) and (ii) of Theorem 4.4.2 imply

$$V_n(\omega_1 \dots \omega_n) = \max\{G_n(\omega_1 \dots \omega_n), \frac{1}{1+r} [\tilde{p} V_{n+1}(\omega_1 \dots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \dots \omega_n T)]\} \quad (4.4.13)$$

for $n = N-1, \dots, 0$. But (4.4.11) defines $V_n(\omega_1 \dots \omega_n)$ to be equal to the right-hand side of (4.4.13), which means $V_n(\omega_1 \dots \omega_n)$ is as small as possible.

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Consider an N -period binomial asset-pricing model with no-arbitrage rule $0 < d < 1 + r < u$ and with risk-neutral probabilities

$$\tilde{p} = \frac{1 + r - d}{u - d}, \tilde{q} = \frac{u - 1 - r}{u - d}. \quad (21)$$

Let G_n be a random variable depending on the first n coin tosses. An American derivative security with intrinsic value process G_n is a contract that can be exercised at any time prior to and including time N and, if exercised at time n , pays off G_n with the price process V_n :

$$V_n = \max_{\tau \in S_n} \tilde{E}_n[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1 + r)^{\tau - n}} G_\tau]. \quad (22)$$

In order to justify the definition for American derivative security prices, we must show that a short¹ position can be hedged using these prices.

This requires a generalization of Theorem 4.2.2 (Replication of path-independent derivative) to the path-dependent case.

¹ A short, or a short position, is created when a trader sells a security first with the intention of repurchasing it or covering it later at a lower price.

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Let

$$\Delta_n(w_1 \dots w_n) = \frac{V_{n+1}(w_1 \dots w_n H) - V_{n+1}(w_1 \dots w_n T)}{S_{n+1}(w_1 \dots w_n H) - S_{n+1}(w_1 \dots w_n T)} \quad (23)$$

$$C_n(w_1 \dots w_n) = V_n(w_1 \dots w_n) - \frac{1}{1+r} [\tilde{p} V_{n+1}(w_1 \dots w_n H) + \tilde{q} V_{n+1}(w_1 \dots w_n T)] \quad (24)$$

where S_n – the stock price at time n , $w_n \in \{H, T\}$ – result of a coin toss at time n .

Theorem 4.4.4

- 1 We have $C_n > 0, \forall n$.
- 2 If we set $X_0 = V_0$ and define recursively forward in time the portfolio values X_1, X_2, \dots, X_N by

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n) \quad (25)$$

then we have

$$X_n(w_1 \dots w_n) = V_n(w_1 \dots w_n), \forall n, \forall w_1 \dots w_n. \quad (26)$$

- 3 In particular $X_n > G_n, \forall n$.

Proof 4.4.4

- 1 The nonnegativity of C_n is a consequence of the fact that the discounted process

$\frac{1}{(1+r)^n} V_n$ is a supermartingale.

The induction hypothesis is that $X_n(w_1 \dots w_n) = V_n(w_1 \dots w_n)$ for some $n \in \{0, \dots, N-1\}$, $\forall w_1 \dots w_n$. We fix $w_1 \dots w_n$ for the rest of the proof, so by definition:

$$V_n - C_n = \frac{1}{1+r} [\tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T)] \quad (27)$$

2

$$\begin{aligned} X_{n+1}(H) &= \Delta_n S_{n+1}(H) + (1+r)(X_n - C_n - \Delta_n S_n) = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} \\ (S_{n+1}(H) - (1+r)S_n) + (1+r)(V_n - C_n) &= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n} (uS_n - (1+r)S_n) + \\ &+ \tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T) = (V_{n+1}(H) - V_{n+1}(T)) \frac{u-1-r}{u-d} + \tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T) = \\ &= (V_{n+1}(H) - V_{n+1}(T)) \tilde{q} + \tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T) = (\tilde{q} + \tilde{p}) V_{n+1}(H) = V_{n+1}(H) \end{aligned} \quad (28)$$

The proof of $X_n(w_1 \dots w_n T) = V_n(w_1 \dots w_n T)$ is analogous.

- 3 The fact that $X_n > G_n, \forall n$ follows from $X_n(w_1 \dots w_n) = V_n(w_1 \dots w_n), \forall n, \forall w_1 \dots w_n$ and $V_n \geq \max G_n, 0, \forall n$

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Theorem 4.4.4 shows that the American derivative security price given by (33) is acceptable to the seller because he can construct a hedge for the short position. We next argue that it is also acceptable to the buyer.

Let us fix n , imagine we have gotten to time n without the derivative security being exercised, and denote by $\tau^* \in S_n$ the stopping time that attains the maximum price:

$$V_n = \tilde{E}_n[\mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau}]. \quad (29)$$

If the owner of the derivative security exercises it according to the stopping time τ^* , then he will receive the cash flows C_n, C_{n+1}, \dots, C_N , where

$$C_k = \mathbb{I}_{\{\tau^*=k\}} G_k, \quad k = n, n+1, \dots, N \quad (30)$$

Actually, at most one of these C_k values is non-zero. If the option is exercised at or before the expiration time N , then the C_k corresponding to the exercise time is the only nonzero payment among them. However, on different paths, this payment comes at different times. In any case, (31) becomes the net present value at time n of the cash flows:

$$V_n = \tilde{E}_n \left[\sum_{k=n}^N \mathbb{I}_{\{\tau^*=k\}} \frac{1}{(1+r)^{k-n}} G_k \right] = \tilde{E}_n \left[\sum_{k=n}^N \frac{C_k}{(1+r)^{k-n}} \right]. \quad (31)$$

Once the option holder decides on the exercise strategy τ^* , this is exactly the contract she holds. Thus, the American derivative security price V_n is acceptable to him.

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It remains to provide a method for the American derivative security owner to choose an optimal exercise time. We shall consider this problem with $n = 0$ (i.e., seek a stopping time $\tau^* \in S_0$ that achieves the maximum price when $n = 0$).

Theorem 4.4.5

The stopping time $\tau^* = \min\{n; V_n = G_n\}$ maximizes the right-hand side of

$$V_n = \max_{\tau \in S_n} \tilde{E}_n[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau]. \quad (32)$$

when $n = 0$; i.e.,

$$V_0 = \tilde{E}[\mathbb{I}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}]. \quad (33)$$

The value of an American derivative security is always greater than or equal to its intrinsic value. The stopping time τ^* is the first time these two are equal.

It may be that they are never equal. For example, the value of an American put is always greater than or equal to zero, but the put can always be out of the money (i.e., with negative intrinsic value). In this case, the minimum is taken over the empty set (the set of integers n for which $V_n = G_n$ is the empty set), and we follow the mathematical convention that the minimum over the empty set is ∞ .

For us, $\tau^* = \infty$ is synonymous with the derivative security expiring unexercised.

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Proof 4.4.5

We first observe that the stopped process

$$\frac{1}{(1+r)^{n \wedge \tau^*}} V_{n \wedge \tau^*} \quad (34)$$

is a martingale under the risk-neutral probability measure. This is a consequence of theorem about algorithm for the path dependent derivative security price process.

Indeed, if the first n coin tosses result in $w_1 \dots w_n$ (fixed from here) and along this path $\tau^* \geq n+1$, then we know that $V_n > G_n$ and by Theorem 4.4.3:

$$V_{n \wedge \tau^*} = V_n = \frac{1}{1+r} [\tilde{p} V_{n+1}(H) + \tilde{q} V_{n+1}(T)] = \frac{1}{1+r} [\tilde{p} V_{(n+1) \wedge \tau^*}(H) + \tilde{q} V_{(n+1) \wedge \tau^*}(T)]. \quad (35)$$

This is the martingale property for the process (34).

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Proof 4.4.5 (continued)

On the other hand, if along the path $w_1 \dots w_n$ we have $\tau^* < n$, then

$$V_{n \wedge \tau^*} = V_{\tau^*}^* = \tilde{p} V_{\tau^*} + \tilde{q} V_{\tau^*} = \tilde{p} V_{(n+1) \wedge \tau^*}(H) + \tilde{q} V_{(n+1) \wedge \tau^*}(T). \quad (36)$$

Again we have the martingale property.

Since the stopped process (34) is a martingale, we have

$$V_0 = \tilde{E} \left[\frac{1}{(1+r)^{N \wedge \tau^*}} V_{N \wedge \tau^*} \right] = \tilde{E} \left[\mathbb{I}_{\{\tau^* < N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*} \right] + \tilde{E} \left[\mathbb{I}_{\{\tau^* = \infty\}} \frac{1}{(1+r)^N} V_N \right]. \quad (37)$$

But on those paths for which $\tau^* = \infty$, we must have $V_n > G_n, \forall n$ and, in particular, $V_N > G_N$.

Knowing the fact that

$$V_N(\omega_1 \dots \omega_N) = \max\{G_N(\omega_1 \dots \omega_N), 0\} \quad (38)$$

this can only happen if $G_N < 0$ and $V_N = 0$. Therefore, $\mathbb{I}_{\{\tau^* = \infty\}} V_N = 0$ and

$$V_0 = \tilde{E} \left[\mathbb{I}_{\{\tau^* < N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*} \right]. \quad (39)$$

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Call option, often simply labeled a "call", is a contract, between the buyer and the seller of the call option, to exchange a security at a set price.

We can depict the payoff (to the buyer) of a call option upon expiration:

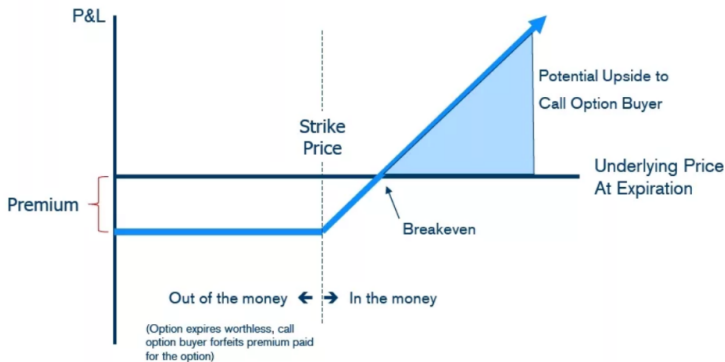


Figure: Call option payoff diagram

Financial risk is limited by the amount of the premium paid.

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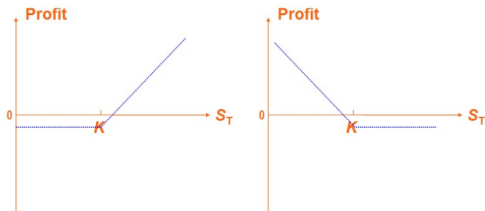


Figure: Call option vs Put option

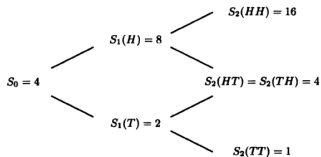


Figure: A two period model

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Let $g: [0, \infty) \rightarrow \mathbb{R}$

$$g(\lambda s_1 + (1 - \lambda)s_2) \leq \lambda g(s_1) + (1 - \lambda)g(s_2), \quad (40)$$

where $s_1 \geq 0, s_2 \geq 0$ and $0 \leq \lambda \leq 1$.

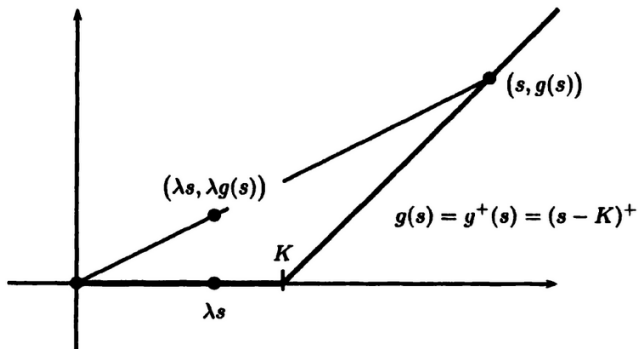


Figure: Convex function with $g(0) = 0$

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Theorem 4.5.1

Consider an American derivative security with convex payoff function $g(s)$ satisfying $g(0) = 0$. The value of this derivative security at time zero, which is:

$$V_0^A = \max_{\tau \in S_0} \tilde{E}_n[\mathbb{I}_{\{\tau \leq N\}} \frac{1}{(1+r)^\tau} g(S_\tau)] \quad (41)$$

is the same as the value of the European derivative security with payoff $g(S_N)$ at expiration N , which is

$$V_0^E = \tilde{E}_N \left[\frac{1}{(1+r)^\tau} \max\{g(S_N, 0)\} \right]. \quad (42)$$

Theorem 4.5.1 shows that the early exercise feature of the American call contributes nothing to its value, because the discounted intrinsic value of the call is a submartingale under the risk-neutral probabilities.

The discounted intrinsic value of an American put is not a submartingale. Because the owner of the put receives upon exercise, he may exercise early in order to prevent the value of this payment from being discounted away. For low stock prices early exercise becomes optimal.

For the call, the owner pays and prefers that the value of this payment be discounted away before exercise. This reinforces the convexity effect and makes early exercise undesirable.