b – уровень, после пот. ; $b = 1$

u – нач. капитал ; $u = 1/2$

θ – коэф. страх. нагрузки ; $\theta = 0{,}1$

λ – плот. пуас. пот. ; $\lambda = 0{,}01$

процесс нак. треб.

$S(t) \;-\; CP(\lambda, f_Y(x))$

$$S(t) = \sum_{j=1}^{N(t)} Y_j$$

$$N(t) = \#\{n \geq 1 : T_n \leq t\}$$

$$T_n = W_1 + \ldots + W_n$$

$$L(t) = S(t) - ct$$

процесс риск-резерва

$$U(t) = u - L(t)$$

capital

$f(x)$ – ПРВ разм. требований

$\hat{f}(x)$ – LT

$c = \lambda \cdot (1+\theta) \cdot E[Y]$ – премия

$E[Y] = \int_0^\infty x \cdot f(x) \cdot dx$ – средний размер требования

$$\hat{\varphi}(s) = \frac{c \cdot \left(\frac{\theta}{1+\theta}\right)}{c \cdot s - \lambda \cdot (1 - \hat{f}(s))} \qquad \hat{\varphi}(0) = \frac{\theta}{1+\theta}$$

$\varphi(x)$ – ILT – вер. неразорения (surv. prob.)

$\psi(x) = 1 - \varphi(x)$ – вероятность разорения (ruin prob.)

$$\chi(u, b) = \frac{1 - \psi(u)}{1 - \psi(b)}$$

$p(b) = \int_0^b \chi(b - x, b) \cdot f(x)\,dx$ – вероятность успеха

$E_u(D) = \chi(u, b) \cdot \dfrac{c}{\lambda(1 - p(b))}$ – (ср. величина дивидендов (общ. сумма див. выпл. до 1-ого разорения D))

$f(x)$	$e^{-2x} + \frac{1}{2} e^{-3/2\,x}$	$x \cdot e^{-x}$	$e^{-2x} + \frac{1}{3} e^{-\frac{2}{3}x}$
$\hat{f}(x)$	$\frac{1}{2(3/2 + x)} + \frac{1}{2+x}$	$\frac{1}{(1+x)^2}$	$\frac{1}{3(2/3 + x)} + \frac{1}{2+x}$
c	$\approx 0{,}005$	$= 0{,}022$	$= 0{,}011$
$p(b)$	$\approx 0{,}488$	$\approx 0{,}208$	$\approx 0{,}569$
$E_u(D)$	$\approx 0{,}593$	$\approx 2{,}287$	$\approx 2{,}028$

Больше риск => больше платеж:

$X \leq_{(1)} Y$

$$U(W) = E\,U(W + \pi(X) - X) \geq E\,U(W + \pi(X) - Y) \Rightarrow \pi(Y) \geq \pi(X)$$

$$U(W) = E\,U(W + \pi(Y) - Y)$$

Больше опасный риск => больше платеж:

$$W = u^{-1}(E\,u(W + \pi(X) - X)) \leq u^{-1}(E\,u(W + \pi(Y) - Y))$$

$\pi(X) > \pi(Y)$

$$W = u^{-1}(E(W + \pi(X) - X)) \geq u^{-1}(E(W + \pi(X) - Y)) \leq u^{-1}\,E\,u(W + \pi(X) + Y)$$

$$u^{-1}(E[u(W + \pi(X) - X)]) \geq u^{-1}(E[u(W + \pi(X) - Y)])$$

$\wedge$ $\quad\quad$ $\pi(Y)$

$$W = u^{-1}(E[W + \pi(X) - X])$$

$= W \Rightarrow \pi(X) > \pi(Y)$

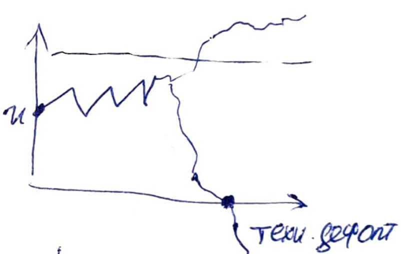
$\pi(u) > \pi(v)$

ⓐ **Shareholders:**

$U_0 = u$

$$U_t = U_{t-1} + \underbrace{C^* - iu}_{C} - S_t$$

$E S_t < C$



Ruin probability:

$$\psi(u) = P(\exists t : U(t) < 0)$$

Lundberg's inequality $\quad \psi(u) \leq e^{-Ru}$

$$\boxed{e^{Rc} = E\,e^{RS}}$$ ← премии, ← выплаты

$E S < C$

$$R = \frac{\ln \varepsilon}{u}$$ ур. над. комп. $\to 0$

$E S < C^{**} - iu < C$

R - coeff (Lundberg)

adjustment efficient

$$C = \frac{1}{R} \ln E\,e^{RS} =$$

$iu^* = \sigma(S)\left(\frac{\ln|\varepsilon|}{2i}\right)^{1/2}$ $\Leftarrow$ $\displaystyle = \frac{1}{R}\left(E[S]R + \frac{1}{2}\sigma^2(S)R^2 + \dots\right)$

$iu \to \infty$

$u \to 0$

$$C = E[S] + \frac{|\ln \varepsilon|}{2u}\sigma^2(S)$$

$$C^* = E[S] + |\ln \varepsilon|/2u \cdot \sigma^2(S) + iu$$

$\pi(S) = E[S] + \sigma(S)\sqrt{2i\,|\ln \varepsilon|}$

$\sum \pi(S) = E(S)$

$\pi(S_j) = E[S_j] + \sigma(S_j)\sqrt{2i\,|\ln \varepsilon|}$

$= R\,\sigma^2(S_j)$

$\pi S = E(S) + \frac{1}{2}\sigma^2 \frac{|\ln \varepsilon|}{4} =$

$= E(S) + \frac{1}{2}\sigma^2 \frac{|\ln \varepsilon| \cdot \sqrt{2i}}{\sigma\,|\ln \varepsilon|^{1/2}}$

$= E(S) + \frac{1}{2}\sigma\,|\ln \varepsilon|^{1/2} \cdot \sqrt{2i}$

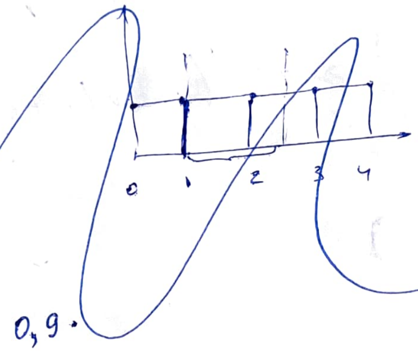
~~ex2~~ ~~$\boxed{3.1.1}$~~

~~$q_0 = 0,15$~~

~~$q_1 = 0,1$~~

~~$q_2 = 0,05$~~

~~$q_3 = 0,01$~~



Мат. методы 3

$$\langle fg \rangle \geq \langle f \rangle \langle g \rangle$$

$$\int (f(x) - f(y))(g(x) - g(y))\, \varphi(x)\, \varphi(y)\, dx\, dy \geq 0$$

$$\int f(x) g(x) \varphi(x) dx + \int f(y) g(y) \varphi(y) dy \geq \int f(x) g(y) \varphi(x) \varphi(y) dx dy$$
$$+ \int f(y) g(x) \varphi(x) \varphi(y) dx dy$$

correlation

$$2 \langle f, g \rangle \geq 2 \int f(y) \varphi(y) dy \cdot \int g(x) \varphi(x) dx$$

$$\int f(x) g(x) \varphi(x) dx$$

e.g. $\delta_0 A(s,t) = \delta_0 A(s,r) A(r,t) \Big|_{r=t}$

$\delta_0 A(s,t) = \delta_0 A(s,t) A(t,t)$

$\Downarrow$

$A(t,t) = 1$

$\mathrm{Cov}(f, g) = E\left[(f - \langle f \rangle)(g - \langle g \rangle)\right]$

≥ 0

$\Updownarrow$

$\langle f, g \rangle \geq \langle f \rangle \langle g \rangle$

$$\frac{A(t, t+\Delta) - 1}{\Delta} = \delta(t)$$

$$\frac{A(s,t)\ A(t, t+\Delta) - A(s,t)}{\Delta A(s,t)} = \delta(t)$$

$$\frac{A(s, t+\Delta t) - A(s,t)}{\Delta A(s,t)} = \left(\ln A(s,t)\right)' = \delta(t)$$

$\ln A(s,t) = \int_s^t \delta(u) du + c \qquad A(s,t) = e^{\int_s^t \delta(u) du + c}$

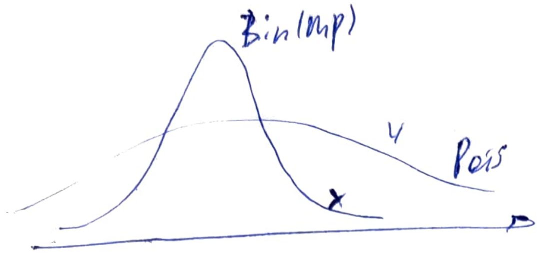
${}_t p_x = {}_s p_x \cdot {}_{t-s} p_{x+s}$

$A(x, x+t) = A(x, x+s) A(x+s, x+t)$

${}_t p_x = e^{-\int_x^t \mu(x+u) du} = e^{-\int_x^t \mu_x(u) du}$

для сил-ти мы знаем знач.

интенсивн. смерти



$X \leq_{(1)} Y$ Y хуже$_{(1)}$ X:

[$\star$ для $\forall$ вып. функц. $Ef(X) \leq Ef(Y)$

f convex (+non-decr. $EX \neq EY$)

$$E[(X-d)_+] \leq E[(Y-d)_+]$$

$\mathrm{Bin}(n,p) \leq_{(1)} \mathrm{Pois} \leq_{(1)} \mathrm{NegBin}$

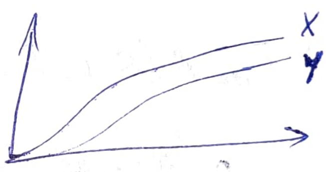
$$\int_d^\infty (x-d) f(x)\,dx = \int_d^\infty (1 - F_X(u))\,du$$

$-d(1-F)$

$f = -d\bar{F} = dF$

$$\Delta(d) = E[(Y-d)_+] - E[(X-d)_+] = \int_d^\infty (F_X(u) - F_Y(u))\,du$$

$$\Delta'(d) = F_Y(d) - F_X(d)$$



$X \leq_{(0)} Y$

в смысле (0)

ф-и распр. не пер. ($EX \neq EY$)

~~[illegible]~~

Ruin theory

W_i, Y_i — i.i.d. $i \geq 0$ ($W_i \sim \mathrm{Exp}(\lambda)$)

$T_n = \sum_{i=1}^n W_i$ $\qquad S(t) = \sum_{j=1}^{N(t)} Y_j$ ← сумма выплат по всем заявкам

$$N(t) = \#\{n : T_n \leq t\}$$

$L(t) = S(t) - ct$ (c – const) $\qquad U(t) = u - L(t)$

↑ risk-reserve $\qquad$ ↑ capital

Lundberg inequality $\psi(u) \leq e^{-Ru}$

$E[e^{rY}] = (1+\theta) \cdot E[Y] \cdot Y$ ← для Pois

$\Downarrow$

$$\boxed{e^{cr} = E[e^{rS(1)}]}$$ (всегда верно) ур. е Лангберга

$cr = \ln E[e^{rS(1)}]$ ← символ процесса Леви
производ. ф-я кумулянтов

для Pois.:

$\ln e^{\lambda(\varphi(t)-1)} = \lambda[E[e^{rY}] - 1] = cr$

$= \lambda E[Y](1+\theta)$

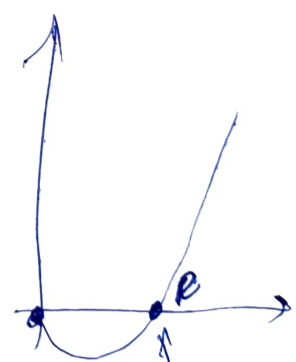
$(1+\theta)\cancel{\lambda} E[Y] = \cancel{\lambda}(E(e^{rY}) - 1)$

$f(x) = E[e^{rY}] - (1+\theta) E[Y] r$

$f(0) = 0$

$f'(0) = E[Y] - (1+\theta) E[Y] < 0$

$f''(r) = E[e^{rY} Y] > 0$



adjustment coefficient
коэф. Лангберга

$* \ \psi(u) \leq e^{-Ru} < \varepsilon \Rightarrow R = \frac{\ln 1/\varepsilon}{u}$

$$E[e^{RY}] = (1+\theta)\ E[Y]R$$

$$\cancel{1} + R\cancel{E[Y]} + \frac{1}{2} \underbrace{R}_{= \frac{\ln 1/\varepsilon}{u}} E[Y]^2 + \cancel{(-1)} = \cancel{E[Y]}R + \theta_i E[Y]$$

$\Downarrow$

$$\theta_\varepsilon \cdot u_\varepsilon \approx \frac{1}{2} \ln 1/\varepsilon \ \frac{E[Y^2]}{E[Y]}$$

$$c = \lambda E(Y)(1 + \theta_i)$$

Мат. методы (4)

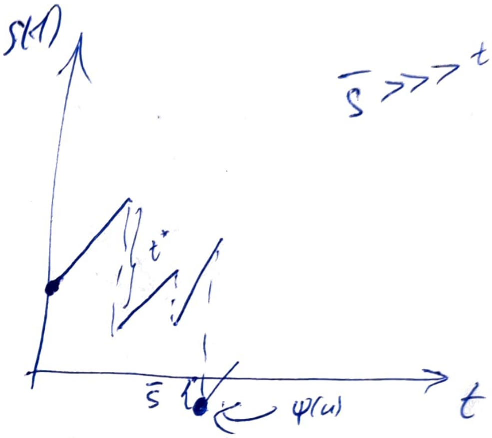
$Y_1, \dots, Y_n$ i.i.d.

$W_1, \dots, W_n$ i.i.d. $\sim Exp(\lambda)$

$T_n = \sum_{j=1}^{n} W_j$

$N(t) = \# \{ n : T_n \leq t \}$

$S(t) = \sum_{j=1}^{N(t)} Y_j$



$\xi >>> t^*$

Risk-reserve $L(t) = S(t) - ct$

$V(t) = u - S(t)$

$ES(t) = \lambda E(Y) < c$

$\psi(u) = P(\exists t : L(t) > u) = P(\exists t : V(t) < 0)$ P - разорение

P - выжить

$1 - \psi(u) = \phi(u)$

$\psi(u) < e^{-Ru}$

решение ур-я Лундберга

$$\psi(u) = \frac{1}{E[e^{R\xi} \mid \tau < \infty]} e^{-Ru} \left(< e^{-Ru} \right)$$

$\mathcal{K}(R) = cR$

ξ - overshoot

символ процесса Леви

произв. ф-я кумулянтов

$$E[e^{RY}] - 1 = R(1+\theta)E[Y]$$

$ES(t) = \lambda E(Y)(1+\theta) = c$

loading

$CP(\lambda, F_Y)$

$\phi(u), \psi(u)$?

$$\phi(u) = \int_0^\infty \lambda e^{-\lambda s} ds \cdot \int_0^{u+cs} \phi(u+cu-x) f_Y(x)\, dx =$$

$$\left\{ \begin{array}{l} v = u+cs \\ ds = \frac{1}{c} dv \end{array} \right\} = e^{-\lambda u/c} \cdot \frac{\lambda}{c} \int_u^\infty e^{-\lambda v/c} dv \int_0^v \phi(v-x) f_Y(x)\, dx$$

$$s = \frac{v-u}{c}$$

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u \phi(u-x) f(x)\, dx$$

$\int_0^\infty e^{-su}$ – лаплас $\Rightarrow$ Inverse Laplace [..., s, u]

Mathematica, Matlab

$$s\hat\phi(s) - \hat\phi(0) = \frac{\lambda}{c}\hat\phi(s) - \frac{\lambda}{c}\hat\phi(s)\hat f(s)$$

$$\hat\phi(s) = \frac{c\hat\phi(0)}{cs - \lambda[1-\hat f(s)]}$$

$$f(x) = \mu e^{-\mu x}$$

$$\hat f(s) = \frac{\mu}{\mu+s} \leftarrow \int_0^\infty e^{-sx} \mu e^{-\mu x} dx$$

$$\hat\phi(0) = \frac{\theta}{1+\theta};\ \hat\psi(0) = \frac{1}{1+\theta},\ c = \frac{\lambda}{\mu}(1+\theta)$$

$ES(t) = \lambda E[Y] = \lambda/\mu$

$c = \lambda(1+\theta)E[Y]$

$c = \lambda/\mu\,(1+\theta)$

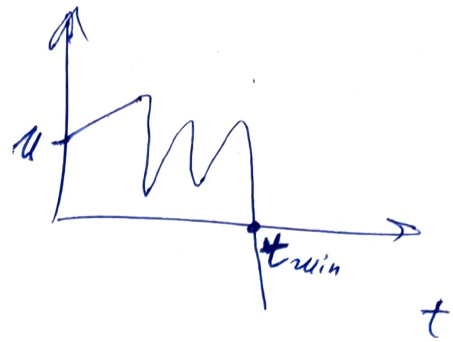
$$\hat\phi(s) = \frac{\lambda \frac{1}{\mu}(1+\theta)\cdot\frac{\theta}{1+\theta}}{\frac{1}{\mu}(1+\theta) - \lambda\cdot\frac{s}{s+\mu}} = \frac{(s+\mu)\cdot\theta}{(s\theta+\mu\theta+s)}$$

$$\hat\psi(s) = \frac{1}{s} - \hat\phi = \frac{1}{s}\left[1 - \frac{(s+\mu)\theta}{s\theta+\mu\theta+s}\right] = \underbrace{\frac{1}{1+\theta}}_{\text{const}} \cdot \frac{1}{s+\frac{\mu\theta}{1+\theta}}$$

$$\Rightarrow \psi(u) = \underbrace{\frac{1}{1+\theta}}_{\psi(0)} e^{-\overbrace{\frac{\mu\theta}{1+\theta}}^{R} u}$$

$\psi(u) = P(\sup_u L_u > u)$

ruin probability



$\varphi(u) = 1 - \psi(u)$

survival probability

$E\,S(t) = \lambda \cdot E[Y]$

страх. нагрузка

$c = \lambda E[Y] \cdot (1 + \theta)$

нез. число величин

$S(t) = \sum_{j=1}^{N(t)} Y_j$

нез. величины

$W_i \sim Exp(\lambda)$

$N(t) \sim Pois(\lambda)$

Moment gener. function (производящая ф-ция моментов)

$$E[e^{zS(t)}] = E\left[\underbrace{E[e^{zS(t)} \mid N(t)=k]}\right] =$$

$$= E[e^{z\sum_{j=1}^{k} Y_j}] = [\varphi(z)]^k$$

$$\varphi(z) = E[e^{zY}]$$

$$= \sum_{k=1}^{\infty} \varphi(z)^k \, P(N(t)=k) = \sum_{k=1}^{\infty} \frac{[\lambda\varphi(z)]^k \cdot e^{-\lambda}}{k} =$$

$$= e^{-\lambda} \cdot e^{\lambda\varphi(z)} = e^{\lambda(\varphi(z)-1)}$$

$$E[X^n] = \frac{d^n}{dt^n} M_X(t)\Big|_{t=0}$$

$$\left(e^{\lambda(\varphi(z)-1)}\right)'_z \Big|_{z=0} = \lambda\varphi'(z)\cdot e^{\lambda(\varphi(z)-1)} \Big|_{z=0} = \lambda\cdot\varphi'(0) = \lambda E[X]$$

$\cdot\ \varphi(z) = E(e^{z\cdot X})$

$\cdot\ \varphi'(z) = E[X\cdot \underset{1}{\underset{\shortparallel}{e^{zX}}}]\Big|_{z=0} = EX$

$$\lambda\varphi'' e^{\lambda(\varphi(z)-1)} + \lambda^2(\varphi')^2 e^{\lambda(\varphi(z)-1)}\Big|_{z=0} \ (=)$$

$\cdot\ \varphi''(z) = E[X^2 e^{zX}]\Big|_{z=0} = EX^2$

$$(=)\ \lambda EX^2 + \lambda^2(EX)^2 = (\lambda Ex)^2 = \lambda Ex^2$$

$$E[\varphi(z)^{N(t)}] = \sum_{j=1}^{\infty} \varphi(z)^j \frac{(\lambda t)^j}{j!} e^{-\lambda t} =$$

$$= e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t\ \varphi(t))^j}{j!} = e^{\lambda(\varphi(z)-1)t}$$

$$= \varphi(z)^{N(t)}$$

$$E[S(t)] = \lambda t E(Y)$$

$$Var[S(t)] = \lambda t E(Y^2)$$

Risk-reserve $\quad L(t) = S(t) - ct$

capital $\quad U(t) = u - L(t)$

$$\psi(u) = P(\sup_t L(t) > u) = P[\inf_t U(t) < 0]$$

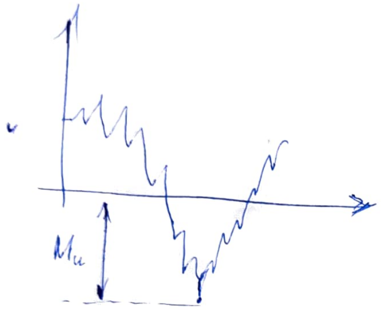
ϕ

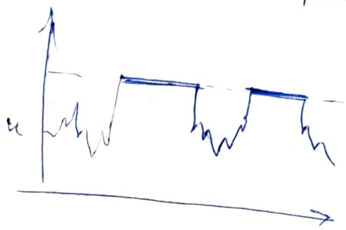
Мат. методы (5) $\quad \psi(u) = P_u(\tau_u < \infty)$

Let $M_u = \sup[\,|U(t)| : \tau_u \le t \le \tau_u'\,]$

$\tau_u = \inf[t>0 : U(t) < 0]$, $\tau_u' = \inf\, t > \tau_u : U(t) > 0]$

Lemma: $P(M_u < x \mid \tau_u < \infty) = \dfrac{\psi(u) - \psi(u+x)}{\psi(u)(1-\psi(x))}$





ψ – scale funct

$\psi_q(u) = E\left(e^{-q\tau_u} \mathbb{1}_{\{\tau_u < \infty\}}\right)$

$\lim\limits_{q \to u} \psi_q(u) = \psi(u)$

$S(t) = \sum\limits_{j=1}^{N(t)} Y_j$, $\quad L(t) = S(t) - ct$, $\quad U(t) = u - L(t)$

$\tau_u = \inf\{t : U(t) < 0\} = \inf\{t : L(t) > u\}$

$\tau_u' = \inf\{t > \tau_u : U(t) > 0\}$

E.g. $\psi(u) = \dfrac{1}{1+\theta} e^{-\frac{\theta\mu}{1+\theta} u}$

$M_u = \sup\{|U(t)| : \tau_u \le t \le \tau_u'\}$

Lemma: $P(\underset{J_u(x)}{M_u < x} \mid \tau_u < \infty) = \dfrac{\psi(u) - \psi(u+x)}{\psi(u)(1-\psi(x))} = 0{,}99$

$\dfrac{\psi(u+x)}{\psi(u)} = \psi(x) J_u(x) + (1 - J_u(x))$

1) $c\ 0{,}1$, $u = 10^6$ $\quad x - ?$

* $\ddot{a}_x^{(12)} = a_x - \dfrac{11}{24}$

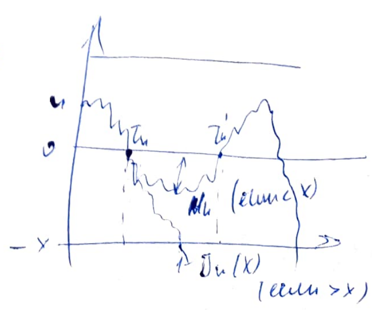
$i \le 5\%$

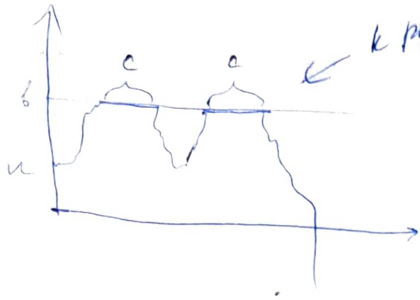
$\lim\limits_{p \to 0} \quad \ddot{a}_x^{(p)} = \underbrace{\dfrac{id}{i^{(p)} d^{(p)}}}_{\downarrow\, 1} \cdot \ddot{a}_x - \dfrac{i - i^{(p)}}{i^{(p)} d^{(p)}}$

$\downarrow$

$\dfrac{11}{12}$?

$\dfrac{p-1}{2p}$



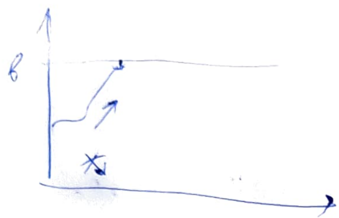


к раз $E_u(D) = \chi(u,b)\dfrac{c}{\lambda(1-p(b))}$

Злб

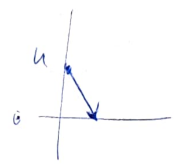
$\chi(u,b) = \varphi(u)/\varphi(b)$

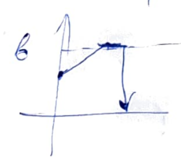
$p(b) = \int_0^b \chi(b-x, b)\, f_X(x)\, dx$



$\chi(u,b)$

$\psi(u) = (1-\chi(u,b)) + \chi(u,b)\cdot\psi(b)$





$$\chi(u,b) = \frac{1-\psi(u)}{1-\psi(b)} = \frac{\varphi(u)}{\varphi(b)}$$

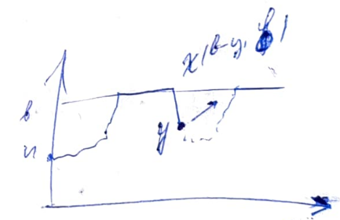
$$D = \sum_{j=1}^{N} \xi_j$$

т.к. по Exp

т.к. умн. на c

ξ_j – i.i.d. $\sim Exp(\lambda/c)$

$P(N=k) = p(b)^{k-1}(1-p(b))$



$p_b = \int_0^b \chi(b-y,b)\, f(y)\, dy$

↑ вер. успеха

$$E[e^{sD}] = E\left(E\left[e^{s\sum_{j=1}^{N}\xi_j} \mid N\right]\right) = E[\varphi(s)^N] = E\left[\left(\frac{\lambda}{\lambda-cs}\right)^N\right] =$$

$$\varphi(s) = \int_0^\infty e^{sx}\cdot \lambda/c \cdot e^{-\lambda/c\, x}\, dx = \frac{\lambda/c}{\lambda/c - s} = \frac{\lambda}{\lambda - cs}$$

$$= \sum_{k=1}^{\infty}(1-p(b))\,p(b)^{k-1}\left(\frac{\lambda}{\lambda-cs}\right)^k = \frac{(1-p(b))\,\lambda}{(1-p(b))\,\lambda - cs}$$

$$E_u(D) = \chi(u,\theta)\, E_\theta(D) = \chi(u,\theta) \cdot \left.\frac{(1-p(\theta))\lambda c}{((1-p(\theta))\lambda - cs)^2}\right|_{s=0}$$

$$= \chi(u,\theta) \frac{c}{\lambda(1-p(\theta))}$$

Wolfram / Matlab

5.1.13

5.1.14

$\theta = 1$

$u = 0{,}5$

$\Theta = 0{,}1$

$\lambda = 10^{-2}$

$E_u(D) = 2u$

≈ в 2 р больше вып. чем вер. в вып.

a) $f(x) = e^{-2x} + 1/2 \cdot e^{-3/2x}$ $\int_{-\infty}^{\infty} f(x) = 1$ (лаплас)

$\quad\quad\quad\quad \| \ \frac{1}{s+2} + \frac{1}{2}\cdot\frac{1}{s+3/2}$

b) $f(x) = x \cdot e^{-x}$

e.g. inverse. Laplace. transform $\left[\frac{1}{1+s}, s, u\right] = e^{-u}$

$\uparrow f \quad \uparrow$ пер. $\quad \uparrow$ куда ответ

$\Rightarrow \hat\psi, \psi \Rightarrow \chi(u,\theta) = \frac{1-\psi(u)}{1-\psi(\theta)}$

$\int_0^\theta \chi(u-x) f(x)\, dx = p(\theta)$

$\Downarrow$

$E_u(D) = \chi(u,\theta) \frac{c}{\lambda(1-p(\theta))}$

$$\int e^{-su} e^{-u}\, du = \frac{1}{1+s}$$

$$\hat\psi(s) = \frac{cs}{cs(1+\Theta) - \lambda(1-\hat f(s))}$$

4.11.

$$\begin{array}{c|cccc} & 0\% & 15\% & 30\% & 50\% \\ \hline 0\% & P(k\geq 1) & P(k=0) & 0 & 0 \\ 15\% & P(k\geq 1) & 0 & P(k=0) & 0 \\ 30\% & 0 & P(k\geq 1) & 0 & P(k=0) \\ 50\% & 0 & P(k\geq 1) & 0 & P(k=0) \end{array}$$

Cur. level	Claim			No claim			smallest for which will claim
0%	500	425	350	425	350	250	250
15%	500	425	350	350	250	250	425
30%	425	350	250	250	250	250	275
50%	425	350	250	250	250	250	275

$P(\text{cost} > 280) = e^{-280/1000} = 0{,}79$

$P(\text{cost} > 425) = e^{-425/1000} = 0{,}654$

$P(\text{cost} > 275) = e^{-275/1000} = 0{,}76$

У. М.
Скр. У. М.
+ задачи про Билли

1) Цепи марк.

2) Выч. дисп. для потерь

Байес vs.

3) Теор. игр. min max

4,5) 2 перестрах

6) пр. ф-и для Wolfram

$f(x)$ — П.Р. ~~приходит~~ того после приходит автобуса

$$h(x) = \frac{1-F(x)}{\mu_1} \qquad \int_0^\infty \frac{\bar F(x)}{\mu_1}\,dx = 1$$

П.Р. времени ожидания

$EX = \int_0^\infty \bar F(u)\,du$

$$E[\xi^k] = \frac{\mu_{k+1}}{(k+1)\mu_1}$$

$$E[L] = E[N]\,E[\xi]$$

$$E[N] = \frac{\psi(0)}{\phi(0)} = \frac{1}{\theta}$$

$$S = \sum_{j=1}^{N} X_j$$

$$E[\xi] = \int_0^\infty x \cdot \frac{1-F(x)}{\mu_1}\,dx = \frac{1}{\mu_1}\int_0^\infty x\bar F\,dx$$

$$= \underbrace{\frac{1}{\mu_1}(1-F(x))\frac{x^2}{2}\Big|_0^\infty}_{=0} + \frac{1}{\mu_1}\int_0^\infty \frac{x^2}{2} f(x)\,dx = \frac{\mu_2}{2\mu_1}$$

$$f_\xi(x) = \frac{1-F(x)}{\mu_1} \qquad \hat f_\xi(s) = \frac{1-\hat f(s)}{s\mu_1}$$

$$E[S] = E[X]\cdot E[N]$$

$$\mathrm{Var}[S] = E[N]\,\mathrm{Var}[X] + [E[X]]^2\,\mathrm{Var}[N]$$

$\star$ Уп. 4.1.1. + найти Var

Ц.М. (4×4) вып. матр. пер. и найти меру ⇓ найти ср. вып.

$$\hat h(s) = \frac{1}{\mu_1 s}(1-\hat f(s))$$

$$\hat h'(s) = \frac{1}{\mu_1}\left[\frac{-\hat f'(s)\cdot s + \hat f(s)}{s^2}\right]$$

$$\lim_{s\to 0}\hat h''(s) = \frac{1}{\mu_1}\lim_{s\to 0}\frac{\hat f''(s)\cdot s - \hat f''(s) + \hat f'(s)}{2s} = -\frac{\hat f''(s)}{2}$$

Мат методы (9)

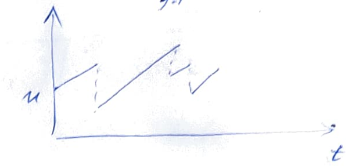
$S(t) = \sum_{j=1}^{N(t)} Y_j$

$N(t) = \#n\{T_n \le t\}$, $T_n = W_1 + \ldots + W_n$

$L(t) = S(t) - ct$ (risk-reserve)

$U(t) = u - L(t)$ (capital)

$U(t) = u + ct - \sum_{j=1}^{N(t)} Y_j$



$\psi(u) = P(\sup_t L(t) > u)$

$= P(\inf_t U(t) < 0)$

$1 - \psi(u) = \phi(u)$

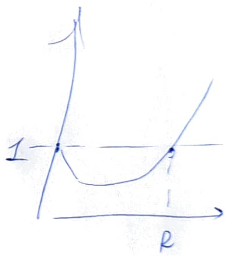
$L(T_n) \equiv L_n = Z_1 + \ldots + Z_n$, $Z_i = Y_i - cW_i$

$$E[Z_1] < 0$$

$$\Updownarrow$$

$$\frac{E[Y_1]}{E[W_1]} < c$$

Ур-е Ландберга $E[e^{RZ_1}] = E[e^{R(Y_1 - cW_1)}] = 1$



$\psi(u) \le e^{-Ru}$

к. Ланд. (насколько надежна компания)

Thm $\psi(u) = P(\sup_k L_k > u) < e^{-Ru}$

$\triangleright$ $\psi_n(u) = P(\sup_{k \le n} L_k > u) \le \psi(u)$

$\hookrightarrow \lim_{n\to\infty} \psi_n(u) = \psi(u)$

по инд. $\psi_1(u) = P(L_1 > u) < \underset{\text{н-во Маркова}}{e^{-Ru}} \underbrace{E[e^{RL_1}]}_{=1} = e^{-Ru}$

для $\psi_{n+1}(u) = P(\sup_{k \le n+1} L_k > u) = P(L_1 > u) + P(\{L_1 \le u\} \cap \{\max_{1 \le k \le n+1} L_k > u\})$

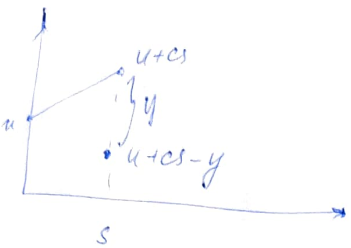
$\int_u^\infty e^{-R(x-u)}\, dF(x)$ $\psi_{n+1}(u) \le e^{-Ru} \overset{1}{\overset{\shortparallel}{\int_{-\infty}^{\infty} e^{Rx}\, dF_{L_1}(x)}}$ $\le \int_{-\infty}^{u} \underbrace{\psi_n(u-x)}_{e^{-R(u-x)}}\, dF_{L_1}(x)$

$$\phi(u) = \int_0^\infty \lambda e^{-\lambda s}\, ds \int_0^{u+cs} f_Y(y)\, \phi(u+cs-y)\, dy$$ ← для Пуас.

$ds = dv/c$

$u+cs = v$

$$= \frac{\lambda}{c} e^{\frac{\lambda u}{c}} \int_u^\infty e^{-\frac{\lambda v}{c}} \int_0^v f(y)\,\phi(v-y)\, dy\, dv$$



$$\int_0^\infty e^{-su} \frac{d\phi}{du}\, du = \int_0^\infty e^{-su} \frac{\lambda}{c}\phi(u)\, du - \frac{\lambda}{c} \cdot \cancel{e^{\frac{\lambda u}{c}}} \cdot \cancel{e^{-\frac{\lambda u}{c}}} \int_0^\infty e^{-su} du \int_0^u f(y)\,\phi(u-y)\, dy$$

Методом Лапласа решается.

$$\int_0^\infty e^{-su} \frac{d\phi}{du}\, du = s\hat{\phi}(s) - \phi(0)$$

$$s\hat{\phi}(s) = \int_0^\infty e^{-su} \phi(u)\, du$$

$$e^{-su}\phi \Big|_0^\infty - \int_0^\infty (-s) e^{-su} \phi(u)\, du =$$

$$= -\phi(0) + s\int_0^\infty e^{-su}\phi(u)\, du = -\phi(0) + s(\hat{\phi}(s))$$

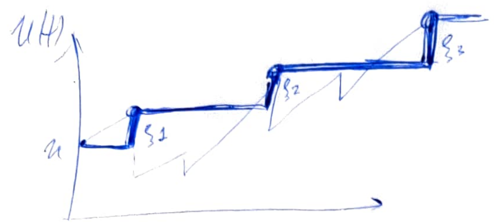
свертка → $$s\hat{\phi}(s) - \phi(u) = \frac{\lambda}{c}\hat{\phi}(s) - \frac{\lambda}{c}\hat{\phi}(s)\hat{f}(s)$$

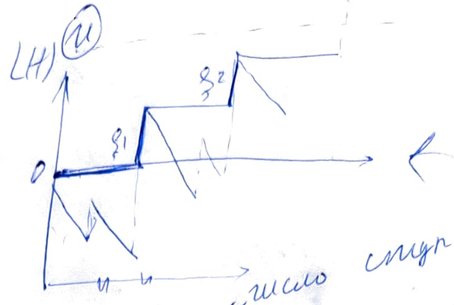
$$\hat{\phi}(s) = \frac{\phi(0)}{s - \frac{\lambda}{c}(1-\hat{f}(s))}$$, где $\phi(0) = \frac{\theta}{1+\theta} = \left(?\frac{\lambda\mu}{c}\right)$

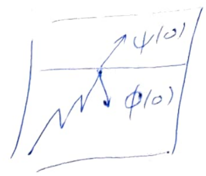
$$= \frac{c\,\phi(0)}{cs - \lambda(1-\hat{f}(s))}$$

комп. разорилась

Лестница Хинчина







число ступ.

$$\psi(u) = P(\sup_t L(t) > u) = P\left(\sum_{j=1}^{N} \xi_j > u\right)$$

к ступ. в лест.

больше не раз.

$$P(N=k) = \psi(0)^k \phi(0)$$

$$\psi(u) = P(L > u)$$

$$\phi(u) = P(L < u)$$

$$E[e^{-sL}] = \phi(0) + \int_0^\infty e^{-su} \frac{d\phi}{du} du$$

$$= \phi(0) + s\hat{\phi}(s) - \phi(0) = \frac{cs\phi(0)}{cs - \lambda(1 - \hat{f}(s))}$$

$$E\left[E\, e^{-s\sum_{j=1}^{N} \xi_j} \mid N=k\right] = E[\hat{h}(s)]^N = \sum_{k=0}^{\infty} \phi(0)\hat{\psi}(0)^k h(s)^k =$$

$$\boxed{= \frac{\phi(0)}{1 - \psi(0)\hat{h}(s)}}$$

?

$$\hat{h}(s) = E[e^{-s\xi}]$$

$$\frac{\phi(0)}{1 - \frac{\lambda}{cs}(1 - \hat{f}(s))} = \frac{\phi(0)}{1 - \frac{\lambda \mu_1}{c}\hat{h}(s)}$$

$$h(x) = \frac{1}{\mu_1}(1 - F(x)) \Longleftarrow \hat{h}(s) = \frac{1 - \hat{f}(s)}{s\mu_1}$$

(ex)

$$\int_0^\infty e^{-sx} \mu e^{-\mu x} dx = \frac{\mu}{s+\mu}$$

$$\psi(u) = 1 - \phi(u)$$

$$\hat{\psi}(s) = \frac{1}{s} - \hat{\phi}(s)$$

$$\frac{c\phi(0)}{cs - \lambda\underbrace{\left(1 - \frac{\mu}{\mu+s}\right)}_{\frac{s}{\mu+s}}} = \frac{1}{s} \frac{c\phi(0)}{c - \frac{\lambda}{\mu+s}}$$

$$\hat{\psi}(s) = \frac{1}{s} - \frac{1}{s} \frac{c\phi(0)}{c - \frac{\lambda}{\mu+s}} = \frac{1}{s}\left[1 - \frac{c\,\phi(0)\,(\mu+s)}{c(\mu+s) - \lambda}\right]$$

$$= \frac{1}{s}\left[1 - \frac{(1+\theta)\lambda/\mu\ \theta/(1+\theta)\cdot(\mu+s)}{(1+\theta)\lambda/\mu\cdot(\mu+s) - \lambda}\right]$$

$c = (1+\theta)\frac{\lambda}{\mu}$

$\phi(0) = \frac{\theta}{1+\theta}$

$$= \frac{1}{s}\left[1 - \frac{\lambda\theta(\mu+s)}{\cancel{\mu}(1+\theta)\cdot\lambda\cdot(\mu+s) - \lambda\mu}\right]$$

$$= \frac{1}{s}\left[\frac{(1+\theta)\cdot\lambda(\mu+s) - \lambda\mu - \lambda\theta(\mu+s)}{(1+\theta)\cdot\lambda\cdot(\mu+s) - \lambda\mu}\right]$$

$$= \frac{1}{s}\left[\frac{\lambda(\mu+s) - \lambda\mu}{(1+\theta)\cdot\lambda\cdot(\mu+s) - \lambda\mu}\right] =$$

$$= \frac{1}{s}\left[\frac{\cancel{\lambda}\ s}{(1+\theta)(\mu+s) - \mu}\right] =$$

$$= \cancel{\lambda}\ \frac{1}{1+\theta} \cdot \frac{\frac{\theta}{1+\theta}\mu}{s + \frac{\theta}{1+\theta}\mu}$$

$$\frac{1}{1+\theta} e^{-\frac{\theta}{1+\theta}\mu u}$$

$$\boxed{R = \frac{\theta}{1+\theta}\mu}$$

Перестраховка

X	2 див
	2 перестр. (EoL, проп)
Insurer $\quad 0 < h(x) < x$	1 авт. м.ч
	1 тайфуны
Reinsurer $\quad x - h(x)$	1 лет. Хинчина (пост. дисперсию)

Проп. перестрахование vs Excess of Loss

$x - h(x) =$ $\quad$ Ins. $\min(X, M)$

$(1-\alpha)x*$ $\quad$ Re-ins. $(X-M)_+$

Луч. (если фикс. $Var(Ins.)$, то $Var(Re\text{-}Ins)$ для проп страх.)

Теор. лучш. модель (наиболее устойчив.)

$R_e \geq R_h$ (для Ins.)

$C_{н.лау.}$ для EoL больше, чем для др.

$\triangleright E[e^{rY}] = 1 + r(1+\theta) \cdot EY$

$$\left\{ \begin{aligned} E[h(x)] = E\,[X \mathbb{1}_{\{X<M\}} \\ + M \mathbb{1}_{\{X \geq M\}}] \end{aligned} \right.$$

$\hookrightarrow M \Rightarrow R_e \geq R_h$

$$\int_0^\infty e^{rh(x)} f(x)\,dx \geq \int_0^\infty e^{r\varepsilon(x)} f(x)\,dx$$

$e^y > 1 + y$ $\qquad$ $e^{rh(x)} \geq e^{r\varepsilon(x)} + r e^{\varepsilon(x)}(h(x) - \varepsilon(x))$

e^r

$y = h(x) - \varepsilon(x)$

$$e^{\tau(h(x)-\varepsilon(x))} \geq 1+\tau(h(x)-\varepsilon(x)) \qquad (e^y \geq 1+y)$$

$$\int_0^M f(x)e^{\tau h(x)}dx \geq \int_0^M [e^{\tau\varepsilon(x)} + \tau e^{\varepsilon(x)}(h(x)-\varepsilon(x))] f(x)dx$$

$$(h(x)<X, \ \varepsilon(x)=X)$$

$$\int_0^M e^{\tau\varepsilon(x)}(h(x)-\varepsilon(x)) f(x)dx \leq e^{\tau M}\int_0^M (h(x)-\varepsilon(x)) f(x)dx$$

$$\int_0^\infty e^{\tau\varepsilon(x)}(h(x)-\varepsilon(x)) f(x)dx \leq e^{\tau M}\int_M^\infty (h(x)-\varepsilon(x)) f(x)dx$$

$$e^{\tau M}\int (h(x)-\varepsilon(x)) f(x)dx = 0$$

$$E\,\varepsilon(x) = E\,h(x)$$

3) $S=\sum_{j=1}^N X_j$ на капит. плат. $E_0 L$

найти ср. и дисперсии для Ins и Reins

$$S^{(1)} = \sum_{j=1}^N X_j \wedge M$$

$$S^{(2)} = \sum_{j=1}^{N \wedge N} (X_j - M)_+$$

4) мин. стр пропорц. и вычисл. R для перестр. и страх

найти опт. α для мот. спец. делить треб.

5) Цепи Маркова

6) Лемм. Хинчина

Пример (1.11.3)

1) J — собств. подмн-во конечного подпр-ва сост. I неприводимой ц.м. (X_n) с мат. перехода P

$$P = \begin{pmatrix} P^{JJ} & P^{J \setminus J} \\ P^{\setminus JJ} & P^{\setminus J \setminus J} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

част. наблюдаемые м.ц.

Если фикс. только пер. в сост. мн-ва J, то мн набл. ц.м. д.в. $(\tilde{X}_n)$ со знач. в J и её м-ца перехода имеет вид:

$$\tilde{P} = P^{JJ} + P^{J \setminus J} \sum_{n \geq 0} (P^{\setminus J \setminus J})^n P^{\setminus JJ} = P^{JJ} + P^{J \setminus J} (I_{\setminus J} - P^{\setminus J \setminus J})^{-1} \cdot P^{\setminus JJ}$$

ед. м-ца

▷ $\tilde{P} = A + B \cdot (1 - D)^{-1} \cdot C$

$$P(\tilde{X}_1 = j \mid \tilde{X}_0 = i) = p_{ij} + \sum_{n=2}^{\infty} P(X_n = j,\ X_r \in J\ \forall r = \overline{1, n-1} \mid X_0 = i)$$

$$= p_{ij} + \sum_{n=0}^{\infty} \sum_{k \notin J} \sum_{\ell \notin J} p_{ik} (D^n)_{k\ell} p_{ij}$$

2)

$$P = \begin{array}{c} \\ C \\ F \\ B \\ G \end{array} \begin{array}{c} \begin{array}{cccc} \text{П.О.} & \text{Кварт.} & \text{Бар} & \text{Подр.} \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \end{pmatrix} \end{array} \qquad P^W = \begin{array}{c} \\ C \\ F \\ B \end{array} \begin{array}{c} \begin{array}{ccc} C & F & B \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}$$

Знакомые видят Билла только, когда он нах. в $J = \{C, F, B\}$

$$\tilde{P} = A + B \cdot (1 - D)^{-1} \cdot C = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \\ 1/3 \end{pmatrix} \cdot \left(1 - 1/3\right)^{-1} \cdot \begin{pmatrix} 1/3 & 1/3 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1/6 & 1/6 & 0 \\ 1/6 & 1/6 & 0 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/2 & 1/3 \\ 1/2 & 1/6 & 1/3 \end{pmatrix}$$

— м-ца переходов движений Билла по мн. его друзей

Когда знакомые видят, что Билл переходит на новое место он звонит Жене.

м-ца переходов звонков Жене:

$$\begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/3 & 0 & 2/3 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

её инвариантное распределение $\pi = (\pi_C, \pi_F, \pi_B)$

удовл.
$$\pi_C = 1/3\, \pi_F + 3/4 \cdot \pi_B$$
$$\pi_F = 1/2\, \pi_C + 1/4 \cdot \pi_B$$
$$\pi_B = 1/2\, \pi_C + 2/3 \cdot \pi_F$$

$\Rightarrow$ $\pi = (4/11,\ 3/11,\ 4/11)$

Женя записывает отсюда посл. звонки и видит, что Билл приходит в квартиру недостаточно часто. Поэтому Билл выбирает новую матр переходов

$$P^{*} = \begin{pmatrix} 1/4 & 1/4 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 & 0 \\ 0 & 3/8 & 1/8 & 1/2 \\ 2/10 & 1/10 & 1/10 & 6/10 \end{pmatrix} \qquad P^{w} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$\tilde{P}^{*} = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \qquad \pi^{x} = (1/3,\ 1/3,\ 1/3)$$

Однако можно набл. разницу с P^w

а) звонки из B, следующие к звонкам из C происх. в 2р. чаще, чем зв. из B, след. к зв. из F

$$\begin{array}{cccc} & C & F & B \\ C & 0 & 1/3 & 2/3 \\ F & 2/3 & 0 & 1/3 \\ B & 1/3 & 2/3 & 0 \end{array}$$

$C \to B$ $2/3$

$F \to B$ $1/3$

б) ср число звонков $50/71 > \underbrace{2/3}_{\text{по } P^w}$

ср. число звонков
$$\frac{28}{71}\cdot\frac{3}{4} + \frac{34}{71}\cdot\frac{2}{4} + \frac{4}{71}\cdot\frac{3}{8} + \frac{5}{71}\cdot\frac{4}{10} = \frac{50}{71}$$

для инвар. расп. $\pi^* = (\pi_C^*, \pi_F^*, \pi_B^*, \pi_G^*)$ м-цы P^*

$\pi^* \cdot P^* = \pi^*$

$$\begin{aligned} \pi_C &= 1/4\, \pi_C + 1/2\, \pi_F + 1/5\, \pi_G \\ \pi_F &= 1/4\, \pi_C + 1/4\, \pi_F + 3/8\, \pi_B + 1/10\, \pi_G \\ \pi_B &= 1/2\, \pi_C + 1/4\, \pi_F + 1/8\, \pi_B + 1/10\, \pi_G \\ \pi_G &= 1/2\, \pi_B + 3/5\, \pi_G \end{aligned} \Rightarrow \begin{aligned} \pi_C &= 28/71 \\ \pi_F &= 34/71 \\ \pi_B &= 4/71 \\ \pi_G &= 5/71 \end{aligned}$$

Markov Chains

Markov property

$p_{ij} = P(X_{n+1} = j \mid X_n = i)$

$= P(X_{n+1} = j \mid X_n = i, X_{n-1}, \dots, X_0)$

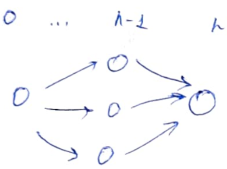
Model specification

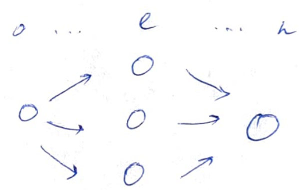
- identify possible states
- identify possible transitions
- identify transition probabilities

n - step transition probabilities given initial state i

$r_{ij}(n) = P(X_n = j \mid X_0 = i)$

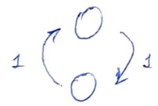
$r_{ij}(n) = \boxed{\sum_{k=1}^{m} r_{ik}(n-1)\, p_{kj}} = \sum_{\ell=1}^{m} r_{i\ell}(q)\, r_{\ell j}(n-q)$ key recursion



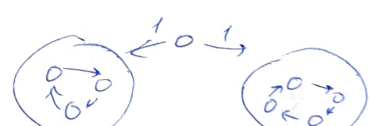


$P(X_n = j) = \sum_{i=1}^{m} P(X_0 = i)\, r_{ij}(n)$ with random in. state

Periodic states



Mult. recurrent classes



(not periodic

)

r_{ij} converge to π_j (independent of the initial state i)
(нач. распр. сход. к инвар. (стац.) распределению)
if recurrent states are all in a single class

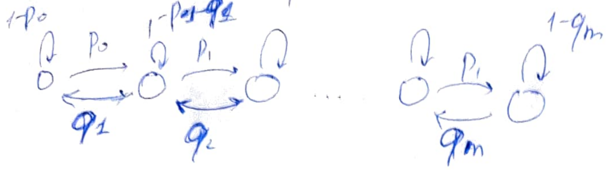
if single recurrent class is not periodic
возвр

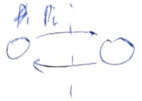
$r_{ij}(n) = \sum_k r_{ik}(n-1)\, p_{kj}$ $\qquad \lim_{n\to\infty} r_{ij}(n) = \pi_j$

$$\begin{cases} \pi_j = \sum_k \pi_k p_{kj}, \quad n \to \infty \quad \forall j \\ \sum_j \pi_j = 1 \end{cases}$$

(additional normalization condition)

Birth - death process





$$\pi_i p_i = \pi_{i+1} q_{i+1}$$

$$p_i = p \, , \; q_i = q \quad \forall i$$

$$\rho = p/q$$

$$\pi_{i+1} = \pi_i \, p/q = \pi_i \rho$$

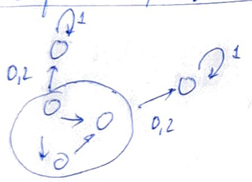
$$\pi_i = \pi_0 \cdot \rho^i \, , \quad i = \overline{0, m}$$

if $p < q$ $m \approx \infty$

$$\pi_0 = 1 - \rho$$

$$E[X_n] = \frac{\rho}{1-\rho} \quad \text{(in steady state)}$$

Absorption probabilities



$$a_i = \sum_j p_{ij} a_j \quad \forall i$$

prob. a_i ~~that~~ given in. state i process settles in abs state

Exp. time to absorption

$$\mu_i = 1 + \sum_j p_{ij} \mu_j$$

μ_i exp. number of transitions until reaching the abs. state

Hidden Markov Chains

Do not observe real state, but observe some other indicator

$p(y_t \mid x_t)$ obs. prob.

$p(x_t \mid x_{t-1})$ transition prob.

$$p(X, Y) = p(x_1) \prod_{t=1}^{T-1} p(x_{t+1} \mid x_t) \cdot \prod_{t'=1}^{T} p(y_{t'} \mid x_{t'})$$

$$\begin{tikzcd}
x_1 \arrow[r] \arrow[d] & x_2 \arrow[r] \arrow[d] & x_3 \arrow[r] \arrow[d] & x_4 \arrow[d] \\
y_1 & y_2 & y_3 & y_4
\end{tikzcd}$$

$\alpha_t(x_t) = p(x_t, y_1, \dots, y_t)$

$\beta_t(x_t) = P(y_{t+1}, \dots y_T \mid x_t)$

$$\alpha_t(x_t) \cdot \beta_T(x_t) = p(x_t, y_1, \dots, y_t) \cdot p(y_{t+1}, \dots, y_T \mid x_t) = p(x_t, Y)$$

$$\propto p(x_t \mid Y)$$

normalize to get cond. prob.

Forward inference

$\alpha_t(x_t) = p(x_t, y_1, \dots, y_t)$

$p(x_1, y_1) = p(x_1)\, p(y \mid x_1) = \alpha_1(x_1)$

$$p(x_2, y_1, y_2) = \Sigma_{x_1} p(x_1, y_1)\, p(x_2 \mid x_1)\, p(y_2 \mid x_2) = \alpha_2(x_2)$$

$$= \Sigma_{x_1} \alpha_1(x_1)\, p(x_2 \mid x_1) \cdot p(y_2 \mid x_2)$$

$$p(x_{t+1}, y_1, \dots, y_{t+1}) = \alpha_{t+1}(x_{t+1}) = \sum_{x_t} \alpha_t(x_t) \cdot p(x_{t+1} \mid x_t) \cdot p(y_{t+1} \mid x_{t+1})$$

Backward inference

$\beta_t(x_t) = p(y_{t+1}, \dots, y_T \mid x_t)$

$p(\{\} \mid x_T) = 1 = \beta_T(x_T)$

$$\beta_{t-1}(x_{t-1}) = p(y_t, \dots, y_T \mid x_{t-1}) = \Sigma_{x_t} p(x_t \mid x_{t-1})\, p(y_t, \dots, y_T \mid x_t)$$

$$= \sum_{x_t} p(x_t \mid x_{t-1}) \cdot p(y_t \mid x_t)\, p(y_{t+1}, \dots, y_T \mid x_t)$$

$$= \Sigma_{x_t} p(x_t \mid x_{t-1}) \cdot p(y_t \mid x_t) \cdot \beta_t(x_t)$$

$$p(x_t, x_{t+1} | Y) = \frac{p(x_t, x_{t+1}, Y)}{p(Y)} =$$

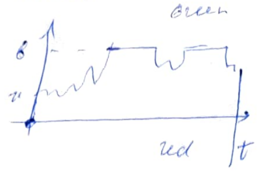
$$= \frac{p(x_t, y_1, \ldots, y_t) \cdot p(x_{t+1} | x_t) \cdot p(y_{t+2}, \ldots, y_T | x_{t+1}) \cdot p(y_{t+1} | x_{t+1})}{\sum_{x_t} p(x_t, Y)}$$

$$= \frac{\alpha_t(x_t) \cdot p(x_{t+1} | x_t) \cdot \beta_{t+1}(x_{t+1}) \cdot p(y_{t+1} | x_{t+1})}{\sum_{x_T} \alpha_T(x_T)}$$

Мат. методы (10)

1) Wolfram

2)



$D = \sum_{j=1}^{N} \xi_j$

$f_\eta(x)$ – ПРВ

(ξ_j) – i.i.d.

Pois (c/λ)

$S(t) = \sum_{j=1}^{N(t)} y_j - ct$

τ risk-reserve

$t = \inf_t \{S(t) > u\}$

$$\psi_\xi(s) = Ee^{s\xi} = \frac{\lambda}{\lambda - cs} \qquad \frac{\lambda/c}{\lambda/c - s} = \frac{\lambda}{\lambda - cs}$$

$$E\xi = c/\lambda$$

$$f(x) = \lambda e^{-\lambda x}$$

$$P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

$$X(u,b) = \phi(u)/\phi(b) = \frac{1-\psi(u)}{1-\psi(b)}$$

↑ p, что из u и коснулись b раньше, чем 0

$\xi(u,b)$ – наоборот

$$\psi(u) = \xi(u,b) + X(u,b)\cdot\psi(b)$$

$$\xi + X = 1$$

$$E_u D = X(u,b)\, E_b D$$

x потратим не больше чем b

$$P(M=k) = p(b)^{k-1}(1-p(b)) \quad \text{где} \quad p(b) = \int_0^{b} f_\eta(x)\, X(b-x, b)\, dx$$

↑ сколько периодов мы выпл. дивиденды

$$E\left[E_b\left(s^{\sum_{j=1}^{M}\xi_j} \mid M=k\right)\right] = E\left[\varphi(s)^k\right] = \sum_{k=1}^{\infty} (1-p(b))\, p(b)^{k-1} \varphi(s)^k =$$

$$= (1-p(b)) \cdot \varphi(s) \sum_{k=0}^{\infty} p(b)^k \varphi(s)^k = \ldots =$$

$$= \frac{(1-p(b))\,\varphi(s)}{1 - p(b)\varphi(s)} \qquad \varphi(s) \leftarrow \frac{\lambda}{\lambda - cs}$$

$$\Rightarrow E_a(D) = X(u,b) \cdot \frac{c}{\lambda(1-p(b))}$$

$$\hat{\phi}(s) = \frac{c\,\phi(0)}{cs - \lambda(1-\hat{f}(s))}$$

$$\psi(u) = 1 - \phi(u)$$

$$\hat{\psi}(s) = 1/s - \hat{\phi}(s)$$

$$\phi(0) = \frac{\theta}{1+\theta}$$

$$\psi(0) = \frac{1}{1+\theta}$$

2) все выплаты 9/11 25,5 млрд \$ (4)

катрина 76 млрд \$ (1)

Выбрать вид страх. от ест. катастрофы

D_0, D_1, D_2

без перестр. — перестр. на 1 кат. — перестр. на 2 соб.

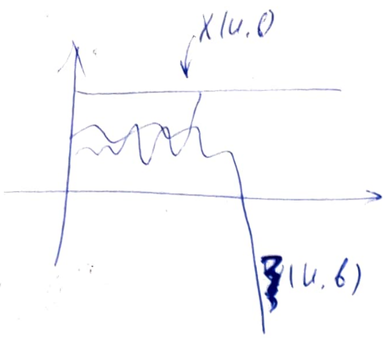
min max vs Bayes

$\Rightarrow D_0$ $\Rightarrow D_0$

Большие ресурсы (∞ денег) - можно иг. не нуж.

и наоборот если огр., то 2 стр лучше.

Мат. мет. (12)



$$X(u,b) + \xi(u,b) = 1$$

$$\psi(u) = \xi(u,b) + X(u,b)\cdot\psi(b)$$

$$X(u,b) = \frac{\phi(u)}{\phi(b)} = \frac{(1-\psi(u))}{(1-\psi(b))}$$

$$\xi(u,b) = \frac{\psi(u)-\psi(b)}{1-\psi(b)}$$

$$\phi'(u) = \frac{\lambda}{c}\phi(u) - \frac{\lambda}{c}\int_0^u f(x)\,\phi(u-x)\,dx$$

$$\hat{\phi}(s) = \frac{c\,\phi(0)}{cs - \lambda(1-\hat{f}(s))} \qquad \psi'(u) = \frac{\lambda}{c}\psi(u) - \frac{\lambda}{c}\int_0^u \psi(u-x)f(x)dx$$

$$-\int_u^\infty f(x)dx$$

$$\psi(0) = \frac{1}{1+\theta}$$

$$\psi(u) = 1-\phi(u)$$

$$\underbrace{\psi(\infty)}_{=0} - \psi(0) = \lambda/c \overbrace{\int_0^\infty \psi(u)\,du}^{=I_1} - \frac{\lambda}{c}\overbrace{\int_0^\infty du \int_0^u f(x)\psi(u-x)\,dx}^{=I_2}$$

$$-\frac{\lambda}{c}\int_0^\infty (1-F(u))\,du = I_3$$

$$I_1 = I_2$$

$$\psi(0) = \frac{\lambda}{c}EX = \frac{\lambda EX}{\lambda E(1+\theta)} = \frac{1}{1+\theta}$$

$$\int_0^\infty dx \underbrace{\left(\int_0^\infty dv f(v)\right)}_{=1}\psi(u) - \int_0^\infty \psi(u)du$$

* Дивиденды

$E_u D = X(u,b)\, E_b D$; $D = \sum_{j=1}^N \xi_j$; $\xi \sim Exp(c/\lambda)$ $P(N=k) = p(b)^{k-1}(1-p(b))$; $p(b) = \int_0^b f(x)X(b-x,b)dx$

	p_0	p_1	p_2	
	Θ_0	Θ_1	Θ_2	$\Theta_3 \geq 3$ катастр.
D_0	0	$2\cdot 10^6$	$4\cdot 10^6$	6
D_1	500000	$\boxed{E(X-3\cdot 10^6)_+}$ $2\cdot 10^6 - a + 1/2$	$4-a+\frac{1}{2}$	$6-a+1/2$
D_2	1000000	$2\cdot 10^6 - a + 1$	$4-2a+1$	$6-2a+1$

Katrina 16,254

Japan 35,735

Andrew 26,180

9/11 24,349

1500000 // 3000000 порог

M, 6

minimax ищу max потери в Θ_3

max должен быть мин

↳ D_2

перестрах → D_0 ничего

→ D_1 на 1 свыше порога плат. ЦБ

→ D_2 на 2 свыше пор. плат. ЦБ

Bayes

усредняем по весам

⇒ самое мал. D_0

ср. див. пол.

выпл. деньгу до 1 max. раз.

Див. ** $\sum_{i=1}^{N} \xi_i$ Tower Property

$$E\, E[e^{sD} \mid N=k] = \sum_k P(N=k)\cdot E[e^{s\sum \xi_i} \mid N=k]$$
$$= \sum P(N=k)(E e^{s\xi})^k$$

$$E[(\varphi(s))^N]$$

$\varphi(s) = E e^{s\xi}$

$$E[S^N] = \sum_{k=1}^{\infty} p(\theta)^{k-1}(1-p(\theta)) s^k$$

$$= \frac{(1-p(\theta))\frac{\lambda}{\lambda-cs}}{1-p(\theta)\cdot\frac{\lambda}{\lambda-cs}} = \frac{(1-p(\theta))s}{1-p(\theta)s} = \frac{\lambda}{c}\cdot\frac{1}{\frac{\lambda}{c}-s}$$

$$E[e^{s\xi}] = \int_0^{\infty} \frac{\lambda}{c} e^{sx} e^{-\lambda/c\, x} = \frac{\lambda}{\lambda - cs}$$

$$E_\theta(D) = \frac{c\lambda(1-p(\theta))}{\lambda^2(1-p(\theta))^2} = \frac{c}{\lambda(1-p(\theta))}; \quad E_u(D) = \chi(u,\theta)\cdot\frac{c}{\lambda(1-p(\theta))}$$

Задача 1

$E(X-3\cdot 10^6)_+$

(5–6 стр.)

Зад. 2

Wolf. (див.)

Зад. 4

θ, ψ, ϕ

Зад. 3

Мет. Хинчина пол. Var

Зад. 5

Мар. цен.

Зад. 6

+ перестрах.

E_{0L}

Зад. 7

перестрах

пропорц.

$$\psi(0) = \frac{1}{1+\theta}$$

$$\hat{\phi}(s) = \frac{c\phi(0)}{cs - \lambda(1-\hat{f}(s))}$$

$X(u, b)$

$X(u - x, b)$

$p(b)$

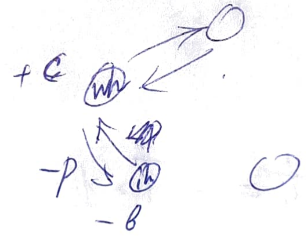
$$L = \sup_t \left[\sum_{j=1}^{N(t)} y_j - ct \right]$$

$$Var\left(\sum_{j=1}^{N} X_j\right) = (EN)^2 \, VarX + VarN \, EX$$

Tiele equations

$H = \{w, i, r, d\}$

$W = \{i, h, d\}$



$X(t) \in \{1, \dots, 12\}$

ЦМ н. в.

$$A = \begin{pmatrix} <0 & & >0 \\ & <0 & \\ & & \ddots \\ >0 & & & <0 \end{pmatrix}_{12 \times 12} \qquad \forall_j \ \sum \lambda_{ij} = 0$$

полугруппа операторов

$P(t) \cdot P(s) = P(t+s)$

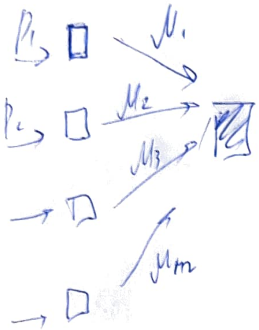
$\det e^{At} = e^{tr A \cdot t}$

process generator (матриц. величина)

$$P(t) = e^{At} = E + A + \frac{1}{2!}A^2 + \dots + \frac{1}{n!}A^n + \dots$$

Распр. фаз. типа (PH) 5.3.

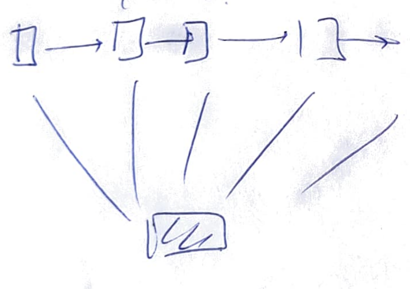
$F = 1 - e^{-\lambda x}$ $\qquad$ $\square \xrightarrow{\lambda} \blacksquare$ $\qquad$ $p(T < x) = 1 - e^{-\mu x}$



$$f(x) = \sum_{i=1}^{m} p_i \mu_i e^{-\mu_i x}$$

Erlang

$$\frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}$$



Cox

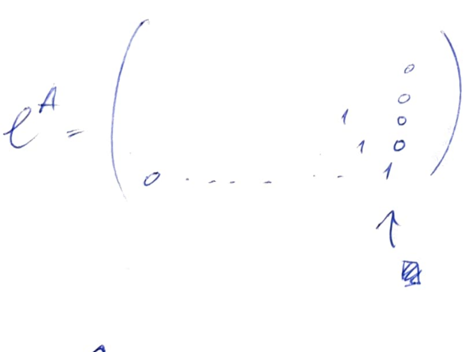
ф (в н.р.)

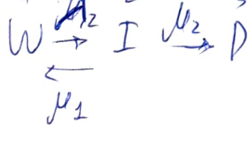
Ломакс, $x > 0$

$$f(x) = \frac{\lambda^\alpha}{(\lambda + x)^\alpha}$$

Евр. парет

$$A = \left(\begin{array}{c|c} T & t \\ \hline 0 & 0 \end{array}\right) \qquad t = -T \cdot \mathbb{1}$$





$$\begin{array}{c|c} & \begin{array}{ccc} T & & t \end{array} \\ \begin{array}{c} W \\ I \\ D \end{array} & \left(\begin{array}{cc|c} -\lambda & \lambda & 0 \\ \mu_1 & (-\mu_1-\mu_2) & \mu_2 \\ 0 & 0 & 0 \end{array}\right) \end{array}$$

$E[\xi^k] = (-1)^k k! \alpha \cdot T^{-k} \cdot 1$

$\uparrow$

время жизни

$\alpha = (1, 0, \cancel{0})$

$w, \; I) \; \cancel{0}$

$E[e^{t\xi}] = \int_0^\infty \alpha e^{(zI+T)x} t \, dx = \alpha(-zI - T)^{-1} t$

$E[\xi] = -\alpha T^{-1} \cdot 1 \qquad E[\xi^2] = 2\alpha T^{-2} 1$

$(-1)^k \cdot (1, 0) \cdot \begin{pmatrix} -\lambda & \lambda \\ \mu_1 & -\mu_1 - \mu_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} =$

$= 1 (1, 0) \cdot \begin{pmatrix} \lambda & -\lambda \\ -\mu_1 & +\mu_1 + \mu_2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E[\xi] = \dfrac{\lambda + \mu_1 + \mu_2}{\lambda \mu_2} \qquad E[\xi] = \dfrac{\mu + \lambda}{\lambda \mu_2}$

$F(x) = 1 - \alpha \cdot T^x \mathbb{1}$

$\bar{F}(x) = P(\xi > x) = \sum \alpha_i p_{ij}^t = \alpha \cdot e^{Tx} \cdot 1$

$f(x) = -\alpha e^{Tx} \cdot \underbrace{T \mathbb{1}}_{-T\mathbb{1} = t} = \alpha e^{Tx} \cdot t$

$\int_0^\infty e^{-zx} \alpha \cdot e^{Tx} t = \alpha \int_0^\infty e^{-(zI - T)} t \, dx = \alpha (zI - T)^{-1} t$

$(-1)\alpha(-T)^{-2} I \cdot 1 = (-1) \cdot 1! \, T^{-1} \mathbb{1}$

$f_\xi = -\alpha(zI - T)^{-2} t \Big|_{T=0} = (-1) \cdot \alpha \cdot (-T)^{-2} t$