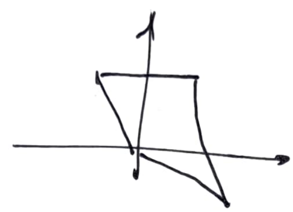
Dynamic Games

Perfect monitoring   rate of impatience r.

$= $ rate of frequency of actions Δt

u, u	$-B$, $u+\Delta u$
$u+\Delta u$, $-B$	0, 0



$u > \Delta u - B$

$(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_t$

$\delta = e^{-r\Delta t} < 1$

$\uparrow$ patience

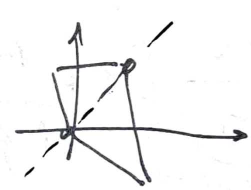
	c	D
c	$p_2\, q_2$	$p_1\, q_1$
D	$p_1\, q_1$	$p_0\, q_0$

$p_2 > p_1 > p_0$

$\uparrow$ 2 cooperators

$\Omega = \xi \pm i\xi$

if we have strongly symmetric strategy



public randomization device (α)

$CC \xrightarrow{+1} R\&R$ --- repeat ··· good news

$CC \xrightarrow{-1}$ ··· bad news

$\xrightarrow{1-\alpha} R\&R$ --- repeat

$\xrightarrow{\alpha} DD^{\infty}$ --- never repeat

$V = (1-\delta)u + \delta[p_2V + q_2((1-\alpha)V + \alpha 0))$

(1) $V = (1-\delta)u + \delta[V - q_2\alpha V]$

$\uparrow$ value recursion eq.

(2) $0 \leq \alpha \leq 1$

(3) $V \geq (1-\delta)(u+\Delta u) + \delta[V - q_1\alpha V]$

(so no one will deviate)

$\max_{\alpha} V$ s.t. (1), (2), (3)

$(1) \Rightarrow V = u - \frac{\delta}{1-\delta} q_2 \alpha v$

$\ell = q_1 / q_2 > 1$

likelihood ratio

$(1-\delta)\Delta u \leq \delta(q_1 - q_2)\alpha v$

$\Delta u \underset{=}{\leq} \frac{\delta}{1-\delta}(\ell - 1) q_2 \alpha v$

$\frac{\delta}{1-\delta} q_2 \alpha v = \frac{\Delta u}{\ell - 1}$

$V^* = u - \frac{\Delta u}{\ell - 1}$

deviating

getting caught

with (2):

$\alpha v = \frac{\Delta u}{\ell - 1} \cdot \frac{1-\delta}{\delta q_2} \leq v = u - \frac{\Delta u}{\ell - 1}$

$u \geq \frac{\Delta u}{\ell - 1}\left[1 + \frac{1-\delta}{\delta \cdot q_2}\right], \quad \delta = e^{-r\Delta t} < 1$

$\downarrow \Delta t \to 0$

∞

$r \uparrow \quad \delta \to 1 \Rightarrow$ constraint is easier to satisfy

$\Delta t \to 0 \Rightarrow$ no cooperation

$\ell(\Delta t) \quad \ell - 1 = \frac{q_1 - q_2}{q_2} \approx \frac{(x_2 - x_1)\sqrt{\Delta t}}{1/2} \to 0$

$q_2(\Delta t) \to 1/2$

Br. Motion

$X_0 = 0$

$X_{t+\Delta t} = X_t + \begin{cases} \sqrt{\Delta t} & \text{w.p. } p_i = 1/2\,[1 + x\sqrt{\Delta t}\,] \\ -\sqrt{\Delta t} & \text{w.p. } q_i = 1 - p \end{cases}$

Poiss. process

$$= \Pr(+1 \mid CD(DC))$$

$$\Pr(+1 \mid CC) = \lambda_2 \Delta t > \lambda_1 \Delta t$$ ← good news - poisson event

$$\Pr(-1 \mid CC) = 1 - \lambda_2 \Delta t$$

$$\ell - 1 = \frac{q_1 - q_2}{q_2} = \frac{(\lambda_2 - \lambda_1)\Delta t}{1 - \lambda_2 \Delta t} \longrightarrow 0, \quad \Delta t \to 0$$

⇒ again no cooperation

$$\Pr(+1 \mid CC) = 1 - \mu_2 \Delta t$$ $\mu_1 > \mu_2$

(accident happen more freq. if less people cooperates)

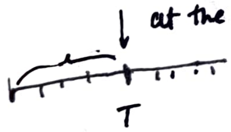
$$\Pr(-1 \mid CC) = \mu_2 \Delta t$$ ← bad news - pois. proc.

$$\ell - 1 = \frac{q_1 - q_2}{q_2} = \frac{\mu_1 - \mu_2}{\mu_2}$$ ⇒ V doesn't explode

we can sustain cooperation

at some cost

all news come at the end



⊛ with perfect monitoring delaying information doesn't help

⊛ with imperfect monitoring helps a lot

$$\begin{tikzcd} CC^T \arrow[r, "1-(1-\delta)^T"] \arrow[d, "(1-\delta)^T"'] & R\&R \\ 1 \arrow[r, "1-\alpha"] \arrow[d, "\alpha"'] & R\&R \\ DD^\infty & \end{tikzcd}$$

$$V = (1-\delta^T)u + \delta^T[(1-q_2^T)V + q_2^T(1-\alpha)V]$$

$$V = u - \frac{\delta^T}{1-\delta^T} q_2^T \alpha V \qquad , 0 < \alpha \leq 1$$

1 deviation

$$(1-\delta)\Delta u \leq \delta^T \alpha V (q_1 q_2^{T-1} - q_2^T)$$

$$= \underbrace{\delta^T q_2^T \alpha v (\ell - 1)}_{(1)} \qquad \ell = q_1/q_2$$

1 deviation => every deviation

t deviations

$$(1-\delta)t\Delta u \leq \delta^T \alpha V (q_1 q_2^{T-t} - q_2^T)$$

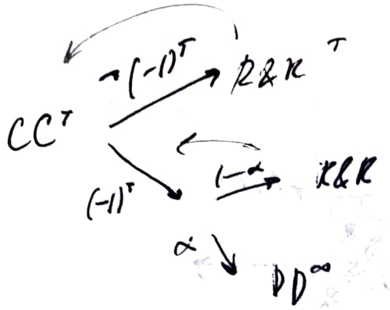
t upper bound

$$= \delta^T q_2^T \alpha v \underbrace{(\ell^t - 1)}_{\geq\, t(\ell-1)} \quad (2)$$

$$t(1-\delta)\Delta u \leq t \cdot (1) \leq (2)$$

$$\frac{(1-\delta)\Delta u}{\delta^T(\ell-1)} = q^T \alpha v$$

$$V^* = u - \frac{1-\delta}{1-\delta^T} \cdot \frac{\Delta u}{\ell - 1} \xrightarrow{L'H} u - \frac{1}{T}\frac{\Delta u}{\ell-1} \to u \qquad \begin{matrix}\delta \to 1 \\ T \to \infty\end{matrix}$$



	$u,\ u$	$-B,\ u+\Delta u$
	$u+\Delta u,\ -B$	$0,\ 0$

$$v = u - \frac{1-\delta}{1-\delta^T} \cdot \frac{\Delta u}{\ell - 1}$$

$$(1-\delta)\,\Delta u \leq \delta^T \alpha v\, q_2^T (\ell - 1)$$

$$=$$

$$\alpha v \geq \frac{1-\delta}{\delta^T q_2^T} \cdot \frac{\Delta u}{\ell - 1} \leq \ldots \qquad v \to u \text{ as } \delta \to 1,\ T \to \infty$$

$\alpha \leq 1$ (for feasability)

$$\leq \quad v = u - \frac{1-\delta}{1-\delta^T}\,\frac{\Delta u}{\ell - 1}$$

$$\Rightarrow \quad u \geq \frac{\Delta u}{\ell - 1} \underbrace{\left[\frac{1-\delta}{1-\delta^T} + \frac{1-\delta}{\delta^T q_2^T}\right]}_{\downarrow}$$

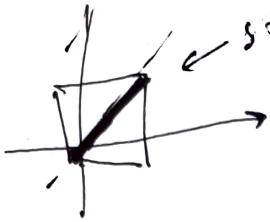
Δt - fixed

$\delta \to 1$

$$\frac{\Delta u}{\ell - 1}\left[\frac{1}{T} + 0\right]$$

Now we can obtain efficiency

(though we could not without T)



strongly symmetric strategies

$(1,1)$

$\delta \to 1$ compact set

$\Omega = \{\pm 1\}$

		$1-\mu_j$	μ_j
$1-\mu_i$	c	$1, 1$	$-1, 2$
μ_i	d	$2, -1$	$0, 0$

p_2	p_1
p_1	p_0

$X_0 = 0 \quad X_{t+\Delta t} = X_t + \begin{cases} \sqrt{\Delta t} & \text{w.p. } p_k \\ -\sqrt{\Delta t} & \text{w.p. } q_k \end{cases}$

$p_k = \frac{1}{2}\left[1 + x_k\sqrt{\Delta t}\right]$

$\Delta t \to 0 \quad dX_t = x_k\, dt + dW_t$

Prop. Let $\gamma = \max\{V_1 + V_2 \mid (V_1, V_2) \in E(\delta)\}$

$\exists\, \underline{\Delta} > 0$ s.t. $\forall \Delta t < \underline{\Delta} \quad \forall \delta > 0$

$\gamma \leq 1$

suppose not

$$c: (1-\delta)(1-2\mu_j) + \delta\left[\left((1-\mu_j)\cancel{p_2} + \mu_j p_1\right)W_i^+ + \left((1-\mu_j)q_2 + \mu_j q_1\right)W_i^-\right]$$

$$\geq (1-\delta)2(1-\mu_j) + \delta\left[\left((1-\mu_j)\cdot p_1 + \mu_j p_0\right)W_i + (1-\mu_j)q_1 + \mu_j q_0\right)W_i^-\right]$$

$$\frac{1-\delta}{\delta} \leq \left[\underbrace{(1-\mu_j)(p_2 - p_1) + \mu_j(p_1 - p_0)}_{\Delta p(\mu_j)}\right]\left(\underline{W_i^+ - W_i^-}\right)$$

diff def. vs coop

$$w_i^- \leq w_i^+ - \frac{1-\delta}{\delta \Delta p(\mu_j)}$$

$\delta = e^{-r\Delta t}$

$1-e^{-r\Delta t} \approx r\Delta t$

$(1-\delta)/\delta \Delta p \approx C\sqrt{\Delta t}$

$$V_i = (1-\delta)u_i + \delta[p w_i^+ + q w_i^-] \qquad w_i^+ - w_i^-$$

$$+ \quad V_j = \dots$$

$$\gamma = (1-\delta)(u_i+u_j) + \delta[p(\underbrace{w_j^+ + w_i^+}_{\leq \gamma}) + q\underbrace{(w_j^- + w_i^-)}_{\leq \gamma - \frac{1-\delta}{\delta}\left[\frac{1}{\Delta p(\mu_i)} + \frac{1}{\Delta p(\mu_j)}\right]}]$$

from principle of optimality

$$\gamma \leq \underbrace{u_i + u_j}_{=2} - \underbrace{\left[\frac{1}{\Delta p(\mu_i)} + \frac{1}{\Delta p(\mu_j)}\right]}_{\Delta t \to 0,\ \to \infty}$$

$\Rightarrow$ contradiction

for BM: (we have a dif. eq for equilibrium frontier has curvature)

F

		W	S
$\sigma > 0$	M	0, 0	0, 1
$1-\sigma$	R	1, $*$0	, 1
		μ	$1-\mu$

R

↑

pr of monitoring

	W	S
M	1, 0	0, 1
R	$\frac{1}{2}$, $\frac{1}{2}$	$\frac{1}{2}$, $\frac{1}{2}$

← for F same exp. reward ⇒ F is better "S"

signal (news)

	R, F
g	2, 1/8
b	0, 0

F: $\cancel{0 + \sigma \tfrac{1}{6} + (1-\sigma) \cdot \tfrac{1}{2} \cdot \tfrac{1}{6} \geq}$ ✓

$\cancel{1 + \sigma \cdot 0 + (1-\sigma) \tfrac{1}{2} \tfrac{1}{6}}$

if only R sees news, but he can lie

⇓

"costly subjective evaluation"

R: $0 + 2 = 1 + \tfrac{1}{2} \cdot 2$ ✓

$\delta \to 0$

	μ (M, W)	$1-\mu$ (M, S)	(R, W)	(R, S)
g	$\frac{1}{\mu}$, $\frac{1}{8}$	0, 0	0, 0	0, 0
b	0, 0	$\frac{1}{1-\mu}$, 0	0, 0	0, 0

IF R ~~tells~~ tells T / tells F

he gets $1/\mu$ / pays $\frac{1}{1-\mu}$