Мат. анал. (н) - 1

$z = x^2 - 2y + y^2 + 9x - 6y + 20$

(I) $\frac{\partial z}{\partial x} = 2x - y + 9$, $\frac{\partial z}{\partial y} = -x + 2y - 6$

$$\begin{cases} 2x - y + 9 = 0 \\ -x + 2y - 6 = 0 \end{cases} \quad x = -4, \; y = 1$$

$M(-4;1)$ - крит. точка

(II) $\frac{\partial^2 z}{\partial x^2} = 2 = A$

$\frac{\partial^2 z}{\partial xy} = -1 = B$

$\frac{\partial^2 z}{\partial y^2} = 2 = C$

$\Delta = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 > 0$

$\Rightarrow$ есть extr

$\Rightarrow M(-4;1)$ - точка лок. мин.

лок. мин. $z_{min} = z(M)$

$z = x^3 + 8y^3 - 6xy + 1$

(I) $z'_x = 3x^2 - 6y$, $z'_y = 24y^2 - 6x$

$$\begin{cases} x^2 - 2y = 0 \\ 4y^2 - x = 0 \end{cases} \quad \begin{matrix} x = 0 \\ y = 0 \end{matrix} \text{ или } \begin{matrix} x = 1 \\ y = \frac{1}{2} \end{matrix}$$

$M_1(0;0)$, $M_2(1;\frac{1}{2})$ - крит. точ.

(II.) $z''_{xx} = 6x$

$z''_{xy} = -6$

$z''_{yy} = 48y$

$M_1(0;0)$

$z''_{xx}|_{M_1} = 0 = A_1$

$z''_{xy}|_{M_1} = 6 = B_1$

$z''_{yy}|_{M_1} = 0 = C_1$

$A_1 = 0$

$$\Delta_1 = \begin{vmatrix} A_1 & B_1 \\ B_1 & C_1 \end{vmatrix} = \begin{vmatrix} 0 & -6 \\ -6 & 0 \end{vmatrix} = -36 < 0$$

$\Rightarrow M_1(0;0)$ – нет extr

$M_2(1;\frac{1}{2})$

$z''_{xx}|_{M_1} = 6 = A_2$

$z'_{xy}|_{M_1} = -6 = B_2$

$z''_{yy}|_{M_1} = 24 = C_2$

$$\Delta_2 = \begin{vmatrix} 6 & -6 \\ -6 & 24 \end{vmatrix} = 6 \cdot 24 - 6^2 > 0$$

$\Rightarrow M_2(1;\frac{1}{2})$ – точ. лок. мин.

$z = e^{\frac{x}{2}}(x+y^2)$

(I) $\begin{cases} z'_x = \frac{1}{2}e^{\frac{x}{2}}(x+y^2) + e^{\frac{x}{2}} \cdot 1 = e^{\frac{x}{2}}(\frac{1}{2}x + \frac{1}{2}y^2 + 1) \\ z'_y = e^{\frac{x}{2}} 2y \end{cases}$

$\begin{cases} e^{x/2}(\frac{1}{2}x + 1/2\, y^2 + 1) = 0 \\ e^{x/2} \cdot 2y = 0 \end{cases}$ $\begin{cases} x = -2 \\ y = 0 \end{cases}$ $M(-2;0)$ – крит. точк.

(II) $z''_{xx} = \frac{1}{2}e^{x/2}(\frac{1}{2}x + \frac{1}{2}y^2 + 1) + e^{\frac{x}{2}} \cdot \frac{1}{2} = e^{\frac{x}{2}}(\frac{1}{4}x + \frac{1}{4}y^2 + 1)$

$z''_{xy} = e^{x/2} \cdot y$

$z''_{yy} = 2e^{x/2}$

$z''_{xx}|_{M_1} = e^{-1}\cdot\frac{1}{2} = A > 0$ $\qquad \Delta = \begin{vmatrix} \frac{1}{2}e^{-1} & 0 \\ 0 & 2e^{-1} \end{vmatrix} = e^{-2} > 0$

$z''_{xy}|_{M_1} = \frac{1}{2} = B$

$z''_{yy}|_{M_1} = 2e^{-1} = C$

– есть extr

$A > 0 \Rightarrow M(-2;0)$

– точка лок. мин.

$z = \sin x + \sin y + \sin(x+y)$ при $0 < x; y < \frac{\pi}{2}$

Ⅰ $z'_x = \cos x + \cos(x+y)$ $\qquad \cos x + \cos(x+y) = 0$

$z'_y = \cos y + \cos(x+y)$ $\qquad \cos y + \cos(x+y) = 0$

$\begin{cases} \cos x + \cos(x+y) = 0 \\ \cos x = \cos y \end{cases}$ $\qquad \begin{cases} \cos x + \cos(x+y) = 0 \\ x = y + 2\pi n \quad \vee \quad x = -y + 2\pi n \end{cases}$

$\begin{cases} x = y + 2\pi k \\ \cos(y+2\pi k) + \cos(2y+2\pi k) = 0 \end{cases} \vee \begin{cases} x = -y + 2\pi k \\ \cos(y+2\pi k) + \cos(2y+2\pi k) = 0 \end{cases}$

$\begin{cases} x = y + 2\pi k \\ \cos y + \cos 2y = 0 \end{cases} \vee \begin{cases} x = -y + 2\pi k \\ \cos 2y = -\cos y \end{cases}$

$\begin{cases} x = y + 2\pi k \\ \cos 2y = \cos(y+\pi) \end{cases} \vee \begin{cases} x = -y + 2\pi k \\ \cos 2y = \cos(y+\pi) \end{cases}$

$\begin{cases} x = y + 2\pi k, \quad k \in \mathbb{Z} \\ 2y - y - \pi = 2\pi l, \ l \in \mathbb{Z} \ \vee \ 2y + y + \pi = 2\pi m, \quad m \in \mathbb{Z} \end{cases}$

$\begin{cases} y = 2\pi l + \pi \\ x = \pi + 2\pi l + 2\pi k \quad l, k \in \mathbb{Z} \end{cases} \vee \begin{cases} y = -\frac{\pi}{3} + \frac{2\pi m}{3} \\ x = -\frac{\pi}{3} + \frac{2\pi m}{3} + 2\pi k \quad m, k \in \mathbb{Z} \end{cases}$

$0 < y < \frac{\pi}{2}$

$0 < \pi + 2\pi l < \frac{\pi}{2}$, $l \in Z$

$-\frac{1}{2} < l < -\frac{1}{4}$

$l \in \varnothing \Rightarrow y \in \varnothing \Rightarrow (x,y) \in \varnothing$

$0 < y < \frac{\pi}{2}$

$0 < -\frac{\pi}{3} + \frac{2\pi m}{3} < \frac{\pi}{2}$

$0 < -\frac{1}{3} + \frac{2m}{3} < \frac{1}{2}$

$\frac{1}{3} < \frac{2}{3} m < \frac{5}{6}$

$m = 1 \quad y = \frac{\pi}{3}$

$\frac{1}{2} < m < \frac{5}{4}$

$$\begin{cases} y = \frac{\pi}{3} \\ x = y + 2\pi k, \ k \in Z \end{cases}$$

$$\begin{cases} y = \frac{\pi}{3} \\ x = \frac{\pi}{3} + 2\pi k, \ k \in Z \end{cases}$$

$\Rightarrow \quad 0 < x < \frac{\pi}{2}$

$0 < \frac{\pi}{3} + 2\pi k < \frac{\pi}{2}$

$\Rightarrow k = 0 \Rightarrow x = \frac{\pi}{3}$

Аналогично:

$$\begin{cases} x = -y + 2\pi k \\ \cos x = -1 \end{cases} \qquad \begin{cases} x + y = 2\pi k \\ \cos x = -1 \\ 0 < x \, y < \frac{\pi}{2} \end{cases} \Rightarrow x \in \varnothing$$

$x, y \in \varnothing$

Вывод: $M\left(\frac{\pi}{3}; \frac{\pi}{3}\right)$ – крит. точка

④ $z''_{xx} = -\sin x - \sin(x+y)$ $\quad z''_{xx}(M) = -\frac{\sqrt3}{2} - \frac{\sqrt3}{2} = -\sqrt3 \quad A<0$

$z''_{xy} = -\sin(x+y)$ $\quad z''_{xy}(M) = -\frac{\sqrt3}{2} \quad B$

$z''_{yy} = -\sin y - \sin(x+y)$ $\quad z''_{yy}(M) = -\sqrt3 \quad C$

$$\Delta = \begin{vmatrix} -\sqrt3 & -\sqrt3/2 \\ -\sqrt3/2 & -\sqrt3 \end{vmatrix} = 3 - \frac{3}{4} > 0$$

есть лок. экстр.

$A < 0$

$\Rightarrow M\left(\frac{\pi}{3}; \frac{\pi}{3}\right)$ – т. лок. макс.

$F(x, y, z) = 0$

локально $z = z(x, y)$ – неявно задано

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$\overbrace{x^2 + y^2 + z^2 - 6x}^{F(x,y,z)} = 0$ $\quad \frac{\partial z}{\partial x}$ – ? $\quad \frac{\partial z}{\partial y}$ – ?

$\frac{\partial F}{\partial x} = 2x - 6$ $\quad \frac{\partial z}{\partial x} - \frac{2x-6}{2z} = \frac{3-x}{z} = \varphi(x, y, z)$

$\frac{\partial F}{\partial y} = 2y$ $\quad \frac{\partial z}{\partial x} = -\frac{2y}{2z} = -\frac{y}{z} = \psi(x, y, z)$

$\frac{\partial F}{\partial y} = 2z$

$$\underbrace{2\sin(x+2y-3z)-(x+2y-3z)}_{F(x,y,z)}=0 \qquad \text{Док.: } \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$$

$$\frac{\partial F}{\partial x}=2\cos(x+2y-3z)-1$$

$$\frac{\partial F}{\partial y}=4\cos(x+2y-3z)-2$$

$$\frac{\partial F}{\partial z}=-6\cos(x+2y-3z)+3$$

$$+\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{2\cos u-1}{-6\cos u+3}=\dfrac{1}{3}\\[2ex] \dfrac{\partial z}{\partial y}=\dfrac{4\cos u-2}{-6\cos u+3}=\dfrac{2}{3}\end{cases}$$

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{1}{3}+\frac{2}{3}=1$$

Мат.анал (п) 2

Первообразная на пром-ке

1) $F(x) + C, \quad C \in \mathbb{R}$
2) других нет

$\int f(x)dx = F(x) + C, \; C \in R$

1) $\int d f(x) = \int f'(x)\, dx = f(x) + C \quad C \in R$

2) $\int a f(x) dx = a \int f(x) dx$

3) $\int (f(x) + \varphi(x)) dx = \int f(x) dx + \int \varphi(x) dx$

4) $(\int f(x) dx)' = f(x)$

$\int (x^2 - 2x) dx = \int x^2 dx - \int 2x dx = \frac{1}{3}x^3 - x^2 + C, C \in R$

$\int (3x - 5x^3) dx = \frac{3}{2}x^2 - \frac{5}{4}x^4 + C, \; C \in R$

$\int (x^2 + 3\sin x) dx = \frac{1}{3}x^3 - 3\cos x + C, \; C \in R$

$\int (e^x + 5\cos x) dx = e^x + 5\sin x + C, \; C \in R$

$\int (\frac{1}{x} + x^5) dx = \ln|x| + \frac{1}{6}x^6 + C, \; C \in R$

$\int (\sqrt{x} + \frac{1}{x^2} + 3^x) dx = \frac{2}{3}x^{\frac{3}{2}} - x^{-1} + \frac{3^x}{\ln 3} + C = \frac{2}{3}x\sqrt{x} - \frac{1}{x} + \frac{3^x}{\ln 3}, C \in R$

$\int \frac{dx}{2\sqrt{x}} = \sqrt{x} + C, \; C \in R$

$\int (\sqrt{x} + 1)(x - \sqrt{x} + 1) dx = \int (x^{\frac{3}{2}} \cancel{- x} \cancel{+ x^{\frac{1}{2}}} \cancel{+ x} \cancel{- x^{\frac{1}{2}}} + 1) dx = \frac{2}{5}x^{\frac{5}{2}} + x + C =$

$= \frac{2}{5}x^2\sqrt{x} + x + C, \; C \in R$

$$\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}\,dx = \int \left(x^{-2,5} - e^x + x^{-1}\right)dx = -\frac{1}{1,5}x^{-1,5} - e^x + \ln(x) + C =$$

$$-\frac{2}{3}\cdot\frac{1}{x\sqrt{x}} - e^x + \ln(x) + C, \quad C \in R$$

$$\int \frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx = \int \left(\frac{1 + 3\sqrt{x} + 3x + (\sqrt{x})^3}{x^{\frac{1}{3}}}\right) = \int \left(x^{-\frac{1}{3}} + 3\cdot x^{\frac{1}{6}} + 3\cdot x^{\frac{2}{3}} + x^{\frac{7}{6}}\right)dx$$

$$= \frac{3}{2}x^{\frac{2}{3}} + \frac{18}{7}x^{\frac{7}{6}}$$

$$\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\,dx = \int \frac{x^{2/3} - x^{1/4}}{x^{1/2}}\,dx = \int \left(x^{\frac{1}{6}} - x^{-\frac{1}{4}}\right)dx =$$

$$=$$

$$\int \frac{3\cdot 2^x - 2x^3}{2^x}\,dx = \int \left(3 - \frac{2x^3}{2^x}\right)dx = 3x - 2\cdot\frac{(1,5)^x}{\ln(1,5)} + C, \; C \in R$$

$$\int \frac{dx}{\sqrt{3 - 3x^2}} = \frac{1}{\sqrt{3}}\arcsin x + C, \; C \in R$$

$$\int \frac{dx}{\sqrt{4 - x^2}} = \arcsin\frac{x}{2} + C, \; C \in R$$

$$\int \frac{dx}{\sqrt{5 - x^2}} = \arcsin\frac{x}{\sqrt{5}} + C, \; C \in R$$

$$\int \frac{dx}{4 + x^2} = \frac{1}{2}\operatorname{arctg}\frac{x}{2} + C, \; C \in R$$

$$\int \frac{dx}{5 + x^2} = \frac{1}{\sqrt{5}}\operatorname{arctg}\frac{x}{\sqrt{5}} + C, \; C \in R$$

$$\int 6\cos^2\frac{x}{2}\,dx = 3x + 3\sin x + C, \; C \in R$$

$$\int \operatorname{ctg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1 - \sin^2 x}{\sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} - 1\right)dx = -\operatorname{ctg}x - x + C, C \in R$$

$$\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x}\,dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right)dx = -\operatorname{ctg}(x) - \operatorname{tg}x + C, C \in R$$

$$\int \frac{1}{\cos^2 x \sin^2 x}\,dx = \int \frac{\sin^2 x + \cos^2 x}{\cos^2 x \sin^2 x}\,dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx = \operatorname{tg}x - \operatorname{ctg}x + C, \; C \in R$$

$$\int \cos(3x)\,dx = \frac{1}{3}\int \cos(3x)\,d(3x) = \frac{1}{3}\cdot\sin 3x + C, \; C \in R$$

$$\int \cos(2x+5)\,dx = \frac{1}{2}\int \cos(2x+5)\,d(2x+5) = \frac{1}{2}\sin(2x+5) + C, \; C \in R$$

$$\int \sin\left(\frac{5}{2}x - 3\right)dx = -\frac{2}{5}\cos\left(\frac{5}{2}x - 3\right) + C, \; C \in R$$

1) $\int x^{\alpha}dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C \quad C\in R,\ \alpha \neq -1$

$\int \frac{dx}{2\sqrt{x}} = \sqrt{x} + C,\ C\in R$

$\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C,\ C\in R$

2) $\int \cos x\,dx = \sin x + C,\ C\in R$

$2\cos^2\frac{x}{2} = 1+\cos x$

3) $\int \sin x\,dx = -\cos x + C,\ C\in R$

$2\sin\frac{x}{2} = \frac{\sin x}{\cos\frac{x}{2}}$

4) $\int e^x dx = e^x + C,\ C\in R$

$\int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + C,\ C\in R$

$\int a^x dx = \dfrac{a^x}{\ln a} + C,\ C\in R$

5) $\int \frac{1}{x}dx = \ln|x| + C,\ C\in R$

$= \ln|kx| + C,\ C\in R,\ k\neq 0$

$= \underbrace{\ln|k|} + \ln|x| + \underbrace{C},\ C\in R,\ k\neq 0$

6) $\int \dfrac{1}{\sqrt{1-x^2}}dx = \arcsin + C,\ C\in R$

$\int \dfrac{1}{\sqrt{1-x^2}}dx = -\arccos + C,\ C\in R$

$\int \dfrac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C,\ C\in R$

7) $\int \dfrac{dx}{1+x^2} = \operatorname{arctg} x + C,\ C\in R$

$\int \dfrac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a} + C,\ C\in R$

8) $\int \dfrac{dx}{\cos^2 x} = \operatorname{tg} x + C,\ C\in R$

$\int \dfrac{dx}{\sin^2 x} = -\operatorname{ctg} x + C,\ C\in R$

Подведение под знак дифференциала

$$\int f(x)\,dx = F(x) + C, \quad C \in R$$

$$\int f(u)\,du = F(u) + C, \quad C \in R$$

$$\int f(\varphi(x))\,d(\varphi(x)) = F(\varphi(x)) + C, \quad C \in R$$

$$d(ax+b) = (ax+b)' \cdot dx = a \cdot dx, \quad a \neq 0$$

$$\boxed{dx = \frac{1}{a}\,d(ax+b)}$$

$$\int e^{7x+1}\,dx = \frac{1}{7}\int e^{7x+1}\,d(7x+1) = \frac{1}{7}e^{7x+1} + C, \quad C \in R$$

$$\int e^{-5x+2}\,dx = -\frac{1}{5}e^{-5x+2} + C, \quad C \in R$$

$$\int (2x-9)^5\,dx = \frac{1}{2}\int (2x-9)^5\,d(2x-9) = \frac{1}{2}\cdot\frac{1}{6}(2x-9)^6 + C =$$

$$\int \sin\left(\frac{5x}{2}-3\right)dx = -\frac{2}{5}\cos\left(\frac{5x}{2}-3\right) + C, \quad C \in R$$

$$\int (3x-1)^{3/2}\,dx = \frac{1}{3}\cdot\frac{2}{5}\cdot(3x-1)^{5/2} + C, \quad C \in R$$

$$\int \frac{1}{\sqrt{4x+2}}\,dx = \frac{1}{4}\cdot 2\sqrt{4x+2} + C, \quad C \in R$$

$$\int \frac{dx}{5x-2} = \frac{1}{5}\ln|5x-2| + C, \quad C \in R$$

$$\int \frac{1}{(1-3x)}\,dx = -\frac{1}{3}\ln|1-3x| + C, \quad C \in R$$

$$\int \frac{1}{(1-3x)^3}\,dx = +\frac{1}{3}\cdot\frac{1}{2}\,\frac{1}{(1-3x)^2} + C, \quad C \in R$$

$$\int \frac{1}{\cos^2 5x}\,dx = \frac{1}{5}\int \frac{1}{\cos^2 5x}\,d\,5x = \frac{1}{5}\,tg\,5x + C, \quad C \in R$$

$$\int \frac{1}{\sin^2(2x-3)}\,dx = -\frac{1}{2}\,ctg\,2x-3 + C, \quad C \in R$$

№1031–1047

Мат. анал. (3)

$$df(x) = f'(x)\,dx \qquad df(x) \leftarrow \varphi(x)\,dx$$

1) $\int x e^{x^2} dx = \frac{1}{2}\int e^{(x^2)} d(x^2) = \frac{1}{2} e^{x^2} + c, \ c \in R$

$x\,dx = \frac{1}{2}\underbrace{dx^2}_{2x\,dx}$

2) $\int x \cdot \cos x^2 dx = \frac{1}{2}\int \cos x^2 dx^2 = \frac{1}{2}\sin x^2 + c, \ c \in R$

3) $\int x^2 \sin x^3 dx = \frac{1}{3}\int \sin(x^3)\, d(x^3) = -\frac{1}{3}\cos x^3 + c, \ c \in R$

4) $\int (x^4-3)^5 x^3 dx = \frac{1}{4}\int (x^4-3)^5 dx^4 = \frac{1}{4}\int (x^4-3)^5 d(x^4-3) =$

$= \frac{1}{24}(x^4-3)^6 + c, \ c \in R$

$(ax+b)'dx = a\,dx$

$f'(x)\,dx = df(x) = \frac{1}{a} d(af(x)+b) = a f'(x)\,dx$

5) $\int e^{5x^3-4} x^2 dx = \frac{1}{3}\int e^{5x^3-4} dx^3 = \frac{1}{15}\int e^{5x^3-4} d(5x^3-4) =$

$= \frac{1}{15} e^{5x^3-4} + c, \ c \in R$

6) $\int \frac{e^x dx}{e^x+2} = \int \frac{de^x}{e^x+2} = \int \frac{1}{e^x+2} d(e^x+2) = \ln|e^x+2| + c, \ c \in R$

7) $\int \frac{e^{5x} dx}{e^{5x}-3} = \frac{1}{5}\int \frac{de^{5x}}{e^{5x}-3} = \frac{1}{5}\ln|e^{5x}-3| + c, \ c \in R$

8) $\int ctg\, x\, dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d\sin x = \ln|\sin x| + c, \ c \in R$

9) $\int \frac{\cos x\, dx}{2\sin x+1} = \int \frac{d\sin x}{2\sin x + 1} = \frac{1}{2}\ln|2\sin x+1| + c, \ c \in R$

10) $\int \frac{\sin x\, dx}{1+3\cos x} = -\int \frac{d\cos x}{1+3\cos x} = -\frac{1}{3}\ln|3\cos x+1| + c, \ c \in R$

11) $\int \frac{\sin x\,dx}{\cos^3 x} = -\int \frac{d\cos x}{\cos^3 x} = -\frac{1}{-2}\cos^{-2} x + C,\ C\in\mathbb{R} = \frac{1}{2\cos^2 x},\ C\in\mathbb{R}$

12) $\int \frac{e^{\sqrt{x}}\,dx}{\sqrt{x}} = 2\int e^{\sqrt{x}}\,d\sqrt{x} = 2e^{\sqrt{x}} + C,\ C\in\mathbb{R}$

13) $\int \frac{\ln^2 x}{x}\,dx = \int \ln^2 x\cdot\frac{1}{x}\,dx = \int \ln^2 x\,d\ln|x| = \frac{1}{3}\ln^3 x + C,\ C\in\mathbb{R}$

14) $\int e^{-3x^2+1}x\,dx = \frac{1}{2}\int e^{-3x^2+1}\,dx^2 = -\frac{1}{6}e^{-3x^2+1} + C,\ C\in\mathbb{R}$

$x\,dx = \frac{1}{2}dx^2$ $\qquad$ $e^x dx = de^x$

$x^2dx = \frac{1}{3}dx^3$ $\qquad$ $\cos x\,dx = d\sin x$

$x^n dx = \frac{1}{n+1}dx^{n+1},\ n\neq -1$ $\qquad$ $\sin x\,dx = -d\cos x$

15) $\int\sqrt{5x^3+2}\,x^2\,dx = \frac{1}{3}\int\sqrt{5x^3+2}\,dx^3 = \frac{1}{15}\int\sqrt{5x^3+2}\,d(5x^3+2) =$

$= \frac{2}{45}(5x^3+2)^{3/2} + C,\ C\in\mathbb{R}$

16) $\int\frac{\sin x\,dx}{\sqrt{1+2\cos x}} = -\int\frac{d\cos x}{\sqrt{1+2\cos x}} = -\frac{1}{2}\int\frac{d(2\cos x+1)}{\sqrt{2\cos x+1}} = -\sqrt{2\cos x+1} + C,\ C\in\mathbb{R}$

17) $\int\frac{\ln^3 x}{x}\,dx = \int \ln^3 x\,d\ln x = \frac{1}{4}\ln^4 x + C,\ C\in\mathbb{R}$

18) $\int\sqrt{1+4\sin x}\,\cos x\,dx = \int\sqrt{1+4\sin x}\,d\sin x = \frac{1}{4}\int\sqrt{1+4\sin x}\,d(4\sin x+1) =$

$= \frac{2}{12}(1+4\sin x)^{3/2} + C,\ C\in\mathbb{R}$

19) $\int\frac{\sqrt{1-4\ln x}\,dx}{x} = -\frac{1}{4}\int\sqrt{1-4\ln x}\,d(-4\ln x+1) = -\frac{1}{6}(1-4\ln x)^{3/2} + C,\ C\in\mathbb{R}$

20) $\int\frac{e^{2x}\,dx}{1-3e^{2x}} = \frac{1}{2}\int\frac{de^{2x}}{1-3e^{2x}} = -\frac{1}{6}\int\frac{d(1-3e^{2x})}{1-3e^{2x}} = -\frac{1}{6}\ln|1-3e^{2x}| + C,\ C\in\mathbb{R}$

21) $\int\frac{\cos 2x\,dx}{\sin x\cos x} = 2\int\frac{d\sin 2x}{\frac{1}{2}\sin 2x} = \ln|\sin 2x| + C,\ C\in\mathbb{R}$

22) $\int\frac{\sin(\ln x)\,dx}{x} = \int\sin(\ln x)\,d(\ln x) = -\cos(\ln x) + C,\ C\in\mathbb{R}$

22) $\int \frac{e^{1/x}}{x^2}dx = \int e^{1/x} \cdot x^{-2}dx = -\int e^{1/x} d\frac{1}{x} = -e^{1/x} + c, \; c \in \mathbb{R}$

23) $\int \frac{1-2\sin x}{\cos^2 x}dx = \int \frac{1}{\cos^2 x}dx - 2\int \frac{\sin x}{\cos^2 x}dx = tg\,x + c + 2\int \frac{d\cos x}{\cos^2 x} =$

$= tg\,x - 2\frac{1}{\cos x} + c, \; c \in \mathbb{R}$

24) $\int \frac{1+\sin 2x}{\sin^2 x}dx = \int \frac{1}{\sin^2 x}dx + \int \frac{\sin 2x}{\sin^2 x}dx = -ctg\,x + c + \int \frac{2\sin x\cos x}{\sin^2 x}dx$

$= -ctg\,x + c + 2\int \frac{\cos x}{\sin x}dx = -ctg\,x + c + 2\int \frac{d\sin x}{\sin x} = -ctg\,x + 2\ln|\sin x| + c$

25) $\int \frac{1}{x^2-1}dx \ominus$

$\frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{Ax - A + Bx + B}{(x+1)(x-1)} = \frac{(A+B)x + (-A+B)}{x^2-1}$

$\begin{cases} A+B=0 \\ -A+B=1 \end{cases} \quad \begin{matrix} B = 1/2 \\ A = -1/2 \end{matrix}$

$\ominus -\frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx = -\frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + c =$

$= \frac{1}{2}(\ln|x-1| - \ln|x+1|) + c = \frac{1}{2}\left(\ln\left|\frac{x-1}{x+1}\right|\right) + c, \; c \in \mathbb{R}$

26) $\int \frac{dx}{x^2+x-6} \ominus$

$\frac{1}{x^2+x-6} = \frac{A}{x+3} + \frac{B}{x-2} = \frac{Ax - 2A + 3B + Bx}{x^2+x-6} = \frac{(A+B)x + (-2A+3B)}{x^2+x-6}$

$\begin{cases} A+B=0 \\ -2A+3B=1 \end{cases} \quad \begin{matrix} A = -1/5 \\ B = 1/5 \end{matrix}$

$\ominus -\frac{1}{5}\int \frac{1}{x+3}dx + \frac{1}{5}\int \frac{1}{x-2}dx = \frac{1}{5}\ln|x-2| - \frac{1}{5}\ln|x+3| + c =$

$= \frac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + c, \; c \in \mathbb{R}$

Мат. анал. (4)

1) $\int \underbrace{\ln x}_{u(x)} \underbrace{dx}_{v(x)} = \ln x \cdot x - \int x \, d\ln(x) = x\ln x - \int x \cdot \frac{1}{x} dx =$

$= x \cdot \ln x - x + c, \ c \in R$

2) $\int \underbrace{\operatorname{arctg} x}_{u(x)} \underbrace{dx}_{v'(x)} = \operatorname{arctg} x \cdot x - \int x \, d\operatorname{arctg}(x) = x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} dx$

$= x \cdot \operatorname{arctg} x - \underbrace{\frac{1}{2}\int \frac{dx^2}{x^2+1}}_{= \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}} = x \operatorname{arctg} x - \frac{1}{2} \ln(x^2+1) + C, \ C \in R$

3) $\int \arcsin x \, dx = x \cdot \arcsin x - \int x \cdot d\arcsin x = x \cdot \arcsin x -$

$- \int x \cdot \frac{1}{\sqrt{1-x^2}} dx = x \arcsin x - \frac{1}{2}\int \frac{dx^2}{\sqrt{1-x^2}} =$

$= x \arcsin x + \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x \arcsin x + \frac{1}{2} \cdot 2\sqrt{1-x^2} + c$

$= x \arcsin x + \sqrt{1-x^2} + c, \ c \in R$

4) $\int x \cdot \sin x \cdot dx = -\int x \, d\cos x = -x\cos x + \int \cos x \, dx =$

$= -x\cos x + \sin x + c, \ c \in R$

5) $\int (x^3 + 3x) \cos x \, dx = \int (x^3+3x) \, d\sin x = (x^3+3x) \cdot \sin x - \int \sin x \cdot d(x^3+3x)$

$= (x^3+3x) \cdot \sin x - \int \sin x \cdot (3x^2+3) dx = (x^3+3x)\sin x - 3\int (x^2+1) \, d(\cos x) =$

$= (x^3+3x)\sin x - \cancel{\int} \ 3(x^2+1)\cos x - 3\int \cos x \, d(x^2+1) =$

$= (x^3+3x)\sin x - 3(x^2+1)\cos x - 3\int \cos x \cdot 2x \, dx =$

$= (x^3+3x)\sin x - 3(x^2+1)\cos x - 6\int x \, d\sin x =$

$= (x^3+3x)\sin x - 3(x^2+1)\cos x - 6x\sin x - 6\cos x + c, \ c \in R$

$= (x^3-3x)\sin x + 3(x^2-1)\cos x + c, \ c \in R$

6) $\int xe^{3x}dx = \frac{1}{3}\int x\,d(e^{3x}) = \frac{1}{3}x\cdot e^{3x} - \frac{1}{3}\int e^{3x}dx =$

$= \frac{1}{3}xe^{3x} - \frac{1}{3}\cdot\frac{1}{3}\int e^{3x}d3x = \frac{1}{3}xe^{3x} - \frac{1}{9}\cdot e^{3x} + c$

$= \frac{1}{9}e^{3x}(3x-1) + c,\ c\in R$

7) $\int x\ln(x-1)\,dx = \frac{1}{2}\int \ln(x-1)\,dx^2 = \frac{1}{2}\int x^2 d\ln(x-1) + \frac{1}{2}x^2\ln(x-1) =$

$= \frac{1}{2}x^2\ln(x-1) + \frac{1}{2}\int x^2\cdot\frac{1}{x-1}dx = \frac{1}{2}x^2\ln(x-1) + \frac{1}{2}\int\frac{\overset{x+1}{\cancel{x^2-1}}+1}{\cancel{x-1}}dx =$

$= \frac{1}{2}x^2\ln(x-1) + \frac{1}{2}\int\left((x+1)+\frac{1}{x-1}\right)dx = \frac{1}{2}x^2\ln(x-1) - \frac{1}{2}\int(x+1)d(x+1)$

$- \frac{1}{2}\int\frac{1}{x-1}d(x-1) = \frac{1}{2}x^2\ln(x-1) - \frac{1}{4}(x+1)^2 - \frac{1}{2}\ln(x-1) + c,\ c\in R$

8) $\int x\,\mathrm{arctg}x\,dx = \frac{1}{2}\int\mathrm{arctg}x\,dx^2 = \frac{1}{2}x^2\mathrm{arctg}x - \frac{1}{2}\int x^2\,d\,\mathrm{arctg}x =$

$= \frac{1}{2}x^2\mathrm{arctg}x - \frac{1}{2}\int x^2\cdot\frac{1}{1+x^2}dx = \frac{1}{2}x^2\mathrm{arctg}x - \frac{1}{2}\int\left(1-\frac{1}{1+x^2}\right)dx =$

$= \frac{1}{2}x^2\mathrm{arctg}x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{1}{1+x^2}d(x) = \frac{1}{2}x^2\mathrm{arctg}x - \frac{1}{2} + \frac{1}{2}\mathrm{arctg}x + c,\ c\in R$

9) $\underline{\int e^x\sin x\,dx} = \int\sin x\,de^x = e^x\sin x - \int e^x d\sin x =$

$= e^x\sin x - \int e^x\cos x\,dx = e^x\sin x - \int\cos x\,de^x =$

$= e^x\sin x - e^x\cos x + \int e^x d\cos x = \underline{e^x(\sin x - \cos x)} - \underline{\int e^x\sin x\,dx}$

$I = \varphi(x) - I$ — ур-ие отн. интеграла $I = \frac{1}{2}\varphi(x)$

$\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c,\ c\in R$

10) $\int e^x\cos 3x\,dx = \int\cos 3x\,de^x = e^x\cos 3x + \int e^x d\cos 3x =$

$= e^x\cos 3x + 3\int e^x\sin 3x\,dx = e^x\cos 3x + 3\int\sin 3x\,de^x =$

$= e^x\cos 3x + 3e^x\sin 3x - 3\int e^x d\sin 3x =$

$= e^x(\cos 3x + 3\sin 3x) - 9\int e^x\cdot\cos 3x\,dx \longrightarrow$

$10\int e^x\cos 3x\,dx = e^x(\cos 3x + 3\sin 3x) + c,\ c\in R \longrightarrow$

$\int e^x\cos 3x\,dx = \frac{1}{10}e^x(\cos 3x + 3\sin 3x) + c,\ c\in R$

11) $\int \cos(\ln x)\,dx = x\cos(\ln x) - \int x\,d\cos(\ln x) = x\cos(\ln x) + \int x\sin(\ln x)\frac{1}{x}\,dx$

$= x\cos(\ln x) + \int \sin(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \int x\,d(\sin(\ln x))$

$= x\cos(\ln x) + x\sin(\ln x) - \int x\cos(\ln x)\cdot\frac{1}{x}\cdot dx =$

$= x(\cos(\ln x) + \sin(\ln x)) - \int \cos(\ln x)\,dx \longrightarrow$

$\int \cos(\ln x)\,dx = \frac{1}{2}\left(x(\cos(\ln x) + \sin(\ln x))\right) + c,\quad c\in\mathbb{R}$

12) $\int \sqrt{x}\ln x\,dx = \frac{2}{3}\int \ln x\,d(x^{3/2}) = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{3/2}\,d(\ln x) =$

$= \frac{2}{3}x^{\frac{3}{2}}\ln x - \frac{2}{3}\int x^{\frac{3}{2}}\cdot\frac{1}{x}\cdot dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{\frac{1}{2}}\,dx =$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\cdot\frac{2}{3}\cdot x^{\frac{3}{2}} + c = \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{\frac{3}{2}} + c =$

$= \frac{2}{9}x\sqrt{x}(3\ln x - 2) + c,\quad c\in\mathbb{R}$

$\int \sqrt{a^2+x^2}\,dx \qquad \int \sqrt{a^2-x^2}\,dx \qquad \int \sqrt{x^2-a^2}\,dx$

13) $\int \sqrt{x^2-a^2}\,dx = x\sqrt{x^2-a} - \int x\,d\sqrt{x^2-a^2} = x\sqrt{x^2-a} - \int x\frac{1}{2\sqrt{x^2-a^2}}2x\,dx$

$= x\sqrt{x^2-a^2} - \int \frac{x^2-a+a}{\sqrt{x^2-a^2}}\,dx = x\sqrt{x^2-a^2} - \int\left(\sqrt{x^2-a^2} + \frac{a^2}{\sqrt{x^2-a^2}}\right)dx$

$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}\,dx =$

$= x\sqrt{x^2-a^2} - \int\sqrt{x^2-a^2}\,dx - a^2\ln|x+\sqrt{x^2-a^2}| \longrightarrow$

$\int\sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}|$

$\int\frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c \qquad \int\frac{dx}{\sqrt{x^2+b}} = \ln|x+\sqrt{x^2+b}| + c$

1) $$\int \frac{x\,dx}{\sqrt{x+1}} = \left| \begin{array}{l} t=\sqrt{x+1} \\ x=t^2-1 \\ dx=2t\,dt \end{array}\right. = \int \frac{(t^2-1)2t\,dt}{t} = 2\int (t^2-1)\,dt =$$
$$= 2\int t^2\, dt - 2\int dt = \frac{2}{3}t^3 - 2t + C = \frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1} + C,\ C\in R$$

2) $$\int \frac{1+x}{1+\sqrt{x}}\,dx = \left| \begin{array}{l} t=\sqrt{x} \\ x=t^2 \\ dx=2t\,dt \end{array}\right. = \int \frac{1+t^2}{1+t}\,2t\,dt = 2\int \frac{t+t^3}{1+t}\,dt =$$
$$=2\int \frac{t^3+t^2-t^2-t+t+t+2-2}{1+t}\,dt = 2\int \left(t^2 - t + 2 - \frac{2}{t+1}\right)dt =$$
$$= \frac{2}{3} t^3 - t^2 + 4t - 4\ln|t+1| + C,\ C\in R =$$
$$= \frac{2}{3} x\sqrt{x} - x + 4\sqrt{x} - 4\ln|\sqrt{x}+1| + C,\ C\in R$$

3) $$\int \frac{dx}{e^x+1} = \left| \begin{array}{l} x=\ln t \\ dx=-\frac{1}{t}\,dt \\ e^x=\frac{1}{t} \end{array}\right. = \int -\frac{1}{t}\left(\frac{1}{\frac{1}{t}+1}\right)dt = -\int\frac{1}{1+t}\,dt$$
$$= -\ln|t+1| + C,\ C\in R = -\ln|e^x+1| + C,\ C\in R$$

4) $$\int \frac{x+\sqrt[3]{x+1}}{\sqrt{x+1}}\,dx = \left| \begin{array}{l} t=\sqrt[6]{x+1} \\ x=t^6-1 \\ dx=6t^5\,dt \end{array}\right. = \int \frac{t^6-1+t^2}{t^3}\,6t^5\,dt = 6\int(t^6+t^2-1)\,t^2\,dt =$$
$$= 6\int(t^8 - t^4 - t^2)\,dt = \frac{2}{3} t^9 - \frac{6}{5} t^5 - 2t^3 + C =$$
$$= \frac{2}{3} (x+1)\sqrt{x+1} - \frac{6}{5} (x+1)^{5/6} - 2\sqrt{x+1} + C,\ C\in R$$

5) $$\frac{3x-1}{(x-1)^3(x+2)(x^2+x+9)^2} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{x+2} + \frac{px+q}{x^2+x+9} + \frac{mx+n}{(x^2+x+9)^2}$$
$$\int\frac{dx}{x^2+4x+7} + \int \frac{dx}{(x^2+4x+4)+3} + \int\frac{dx}{(x-1)^3} + \int\frac{d(x+2)}{(x-2)^2+3} + \frac{1}{\sqrt{3}}\,\text{arctg}\,\frac{x+2}{\sqrt{3}} + C,\ C\in R$$

6) $$\int\frac{dx}{2x^2-5x+7} = \frac{1}{2}\int\frac{dx}{x^2-5/2x+7/2} = \frac{1}{2}\int\frac{dx}{(x-\frac{5}{4})^2+\frac{31}{16}} = \frac{1}{2}\int\frac{d(x-\frac{5}{4})}{(x-\frac{5}{4})^2+\frac{31}{16}} =$$
$$= \frac{1}{2}\cdot\frac{4}{\sqrt{31}}\,\text{arctg}\,\frac{(x-\frac{5}{4})\cdot 4}{\sqrt{31}} = \frac{2}{\sqrt{31}}\,\text{arctg}\,\frac{4x-5}{\sqrt{31}} + C,\ C\in R$$

7) $$\int\frac{dx}{3x^2-x+1} = \frac{1}{3}\int\frac{dx}{x^2-\frac{1}{3}x+\frac{1}{3}} = \frac{1}{3}\int\frac{d(x-\frac{1}{6})}{(x-\frac{1}{6})^2+\frac{11}{36}} = \frac{1}{3}\cdot\frac{6}{\sqrt{11}}\,\text{arctg}\,\frac{(x-\frac{1}{6})\cdot 6}{\sqrt{11}} + C =$$
$$= \frac{2}{\sqrt{11}}\cdot\text{arctg}\,\frac{6x-1}{\sqrt{11}} + C,\ C\in R$$

8) $\int \frac{x\,dx}{x^2-7x+13} = \int \frac{x\,dx}{(x-\frac{7}{2})^2+\frac{3}{4}} = \int \frac{(x-\frac72+\frac72)\,d(x-\frac72)}{(x-\frac72)^2+\frac34} = \left| \begin{array}{l} t = x-\frac72 \\ x = t+\frac72 \end{array}\right| =$

$= \int \frac{t+\frac72}{t^2+\frac34}dt = \int\frac{t\,dt}{t^2+\frac34} + \frac72\int\frac{dt}{t^2+\frac34} = \frac12\int\frac{d(t^2+\frac34)}{t^2+\frac34} + \frac72\cdot\frac{2}{\sqrt3}\operatorname{arctg}\frac{t\cdot 2}{\sqrt3}$

$= \frac12\ln|t^2+\frac34| + \frac{7}{\sqrt3}\operatorname{arctg}\frac{2\sqrt3}{3}t = \frac12\ln(x^2-7x+13)+\frac{7}{\sqrt3}\operatorname{arctg}\frac{2x-7}{\sqrt3}+C, C\in R$

9) $\int\frac{3x-2}{2x^2+4x+5}dx = \frac32\int\frac{x-\frac23}{x^2+2x+\frac52}dx = \frac32\int\frac{(x-\frac23)dx}{(x+1)^2+\frac32} = \frac32\int\frac{(x+1)\,d(x+1)}{(x+1)^2+\frac32}+$

$+\frac32\int\frac{-5/3\,d(x+1)}{(x+1)^2+\frac32} = |x+1=t| = \frac32\int\frac{t\,dt}{t^2+\frac32}+\frac32\int\frac{-5/3\,dt}{t^2+\frac32}=$

$=\frac34\int\frac{d(t^2+\frac32)}{t^2+\frac32}-\frac52\int\frac{dt}{t^2+\frac32} = \frac34\ln|t^2+\frac32|-\frac52\cdot\frac{\sqrt2}{\sqrt3}\operatorname{arctg}t\frac{\sqrt2}{\sqrt3}+c=$

$=\frac34\ln|x^2+2x+\frac52|-\frac{5}{\sqrt6}\operatorname{arctg}(x+1)\sqrt{\frac23}+c,\ c\in R$

10) $\int\frac{dx}{\sqrt{x^2+2x+3}} = \int\frac{dx}{\sqrt{(x+1)^2+2}} = |t=x+1| = \int\frac{dt}{\sqrt{t^2+2}} = \ln|t+\sqrt{t^2+2}| =$

$=\ln|x+1+\sqrt{x^2+2x+3}|+c,\ c\in R$

11) $\int\frac{dx}{\sqrt{4x^2+4x-3}} = \frac12\int\frac{dx}{\sqrt{x^2+1-\frac34}} = \frac12\int\frac{dx}{\sqrt{(x+\frac12)^2+\frac14}} = \frac12\int\frac{dt}{\sqrt{t^2+\frac14}} = \frac12\ln|t+\sqrt{t^2+\frac14}|=$

$=\frac12\ln|x+\frac12+\sqrt{x^2+x+\frac34}|+c,\ c\in R$

12) $\int\frac{dx}{\sqrt{3x^2-2x+1}} = \frac{1}{\sqrt3}\int\frac{dx}{\sqrt{x^2-\frac23x+\frac13}} = \frac{1}{\sqrt3}\int\frac{d(x-\frac13)}{\sqrt{(x-\frac13)^2+\frac23}} = \frac{1}{\sqrt3}\ln(x-\frac13+\sqrt{x^2+\frac23x+\frac13})+c$

13) $\int\frac{dx}{\sqrt{1-2x-x^2}} = \int\frac{dx}{\sqrt{-(x^2+2x-1)}} = \int\frac{dx}{\sqrt{-(x^2+2x-1)-1+1}} = \int\frac{dx}{\sqrt{-(x+1)^2+1}} = \int\frac{d(x+1)}{\sqrt{-(x+1)^2+1}}$

$= -\arccos(x+1)+c$

14) $\int\frac{(8x-11)\,dx}{\sqrt{10-4x-2x^2}} = 8\int\frac{(x-\frac{11}{8})dx}{\sqrt{-(x^2-2x-5)}} = \frac{8}{\sqrt2}\int\frac{(x-\frac{11}{8})dx}{\sqrt{-(x-1)^2-6}} = \frac{8}{\sqrt2}\int\frac{(x-1+1-\frac{11}{8})dx}{\sqrt{-(x-1)^2-6}}$

$=\frac{8}{\sqrt2}\int\frac{(x-1)\,d(x-1)}{\sqrt{-(x-1)^2-6}} - \frac{8}{\sqrt2}\cdot\frac38\int\frac{d(x-1)}{\sqrt{-(x-1)^2-6}} = \frac{8}{\sqrt2}\int\frac{t\,dt}{\sqrt{6-t^2}}-\frac{3}{\sqrt2}\int\frac{dt}{\sqrt{6-t^2}}=$

$=-\frac{4}{\sqrt2}\int\frac{d(t^2+6)}{\sqrt{6+t^2}} - \frac{3}{\sqrt2}\arcsin\frac{t}{\sqrt6}+c = -\frac{4}{\sqrt2}2\sqrt{t}-\frac{3}{\sqrt2}\arcsin\frac{t}{\sqrt6}+c=$

$=-\frac{8}{\sqrt2}\sqrt{x-1}-\frac32\arcsin\frac{x-1}{\sqrt6}+c,\ c\in R$

Мат. анал. 6

1) $\int \frac{7x+4}{(x-3)(x+2)}\,dx \overset{\circ}{=}$

$\frac{7x+4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2} = \frac{Ax+2A+Bx-3B}{(x-3)(x+2)} =$

$= \frac{(A+B)x+(2A-3B)}{(x-3)(x+2)}$ $\quad \left.\begin{array}{l} A+B=7 \\ 2A-3B=4 \end{array}\right\}$ $\quad \begin{array}{l} 5A=25 \\ A=5 \quad B=2 \end{array}$

$\overset{\circ}{=} 5\int \frac{1}{x-3}\,d(x-3) + 2\int \frac{1}{x+2}\,d(x+2) = 5\ln|x-3| + 2\ln|x+2| + C, \quad C \in R$

2) $\int \frac{x\,dx}{x^2-4x-5} = \int \frac{x\,dx}{(x+1)(x-5)} \overset{\circ}{=}$

$\frac{x}{(x+1)(x-5)} = \frac{A}{(x+1)} + \frac{B}{(x-5)} = \frac{Ax-5A+Bx+B}{(x+1)(x-5)} = \frac{(A+B)x+(-5A+B)}{(x+1)(x-5)}$

$\begin{cases} A+B=1 \\ -5A+B=0 \end{cases} \Rightarrow \quad \begin{array}{l} 6B=5 \\ B=5/6 \quad A=1/6 \end{array}$

$\overset{\circ}{=} \frac{5}{6}\int \frac{1}{x+1}\,d(x+1) + \frac{1}{6}\int \frac{1}{x-5}\,d(x-5) = \frac{5}{6}\ln|x+1| + \frac{1}{6}\ln|x-5| + C, \quad C \in R$

3) $\int \frac{dx}{x^3-1} \overset{\circ}{=}$

$\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} = \frac{Ax^2+Ax+A+Bx^2-Bx+Cx-C}{(x-1)(x^2+x+1)}$

$= \frac{(A+B)x^2+(A-B+C)x+(A-C)}{(x-1)(x^2+x+1)}$

$\begin{cases} A+B = 0 \\ A-B+C = 0 \\ A-C = 1 \end{cases} \quad \begin{array}{l} A = 1/3 \\ B = -1/3 \\ C = -2/3 \end{array}$

$\overset{\circ}{=} \frac{1}{3}\int \frac{1}{x-1}\,d(x-1) + \frac{1}{3}\int \frac{-x-2}{x^2+x+1}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+\frac{1}{2}-\frac{1}{2}+2}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx =$

$\frac{1}{3}\ln|x-1| - \frac{1}{3}\cdot\frac{1}{2}\int \frac{d((x+\frac{1}{2})^2+3/4)}{(x+1/2)^2+3/4} + \frac{1}{3}\cdot\left(-\frac{3}{2}\right)\int \frac{d(x+1/2)}{(x+1/2)^2+3/4} = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$

4) $\int \frac{dx}{(x+1)^2(x^2+1)}$

$$\frac{1}{(x+1)^2(x^2+1)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{(x^2+1)} = \frac{A(x+1)(x^2+1) + Bx^2+B + (Cx+D)(x+1)^2}{Z} =$$

$$= \frac{(A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + (A+B+D)}{Z}$$

$$\begin{cases} A + C = 0 \\ A+B+2C+D = 0 \\ A + C + 2D = 0 \\ A+B+D = 1 \end{cases} \qquad \begin{aligned} C &= -1/2 \\ A &= 1/2 \\ D &= 0 \\ B &= +1/2 \end{aligned}$$

$$= \frac{1}{2}\int \frac{d(x+1)}{x+1} + \frac{1}{2}\int \frac{dx}{(x+1)^2} - \frac{1}{2}\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln|x+1| - \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{2}\cdot\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} =$$

$$= \frac{1}{2}\ln|x+1| - \frac{1}{2(x+1)} - \frac{1}{4}\ln|x^2+1| + C, \quad C \in \mathbb{R}$$

5) $2\int \cos x \sin 3x\,dx = \int (\sin(x-3x) + \sin(x+3x))\,dx = \int(-\sin 2x + \sin 4x)\,dx =$

$= \frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x + C; \quad C \in \mathbb{R}$

6) $\int \sin 2x \sin 5x\,dx = \frac{1}{2}\int(\cos(2x-5x) - \cos(2x+5x))\,dx = \frac{1}{2}\int \cos 3x\,dx - \frac{1}{2}\int \cos 7x\,dx =$

$= \frac{1}{6}\sin 3x - \frac{1}{4}\sin 7x\,dx + C, \quad C \in \mathbb{R}$

7) $\int \cos x \cos 2x \cos 3x\,dx = \frac{1}{2}\int(\cos(x+2x) + \cos(x-2x))\cos 3x\,dx =$

$= \frac{1}{2}\int \cos 3x \cos 3x\,dx + \frac{1}{2}\int \cos x \cos 3x\,dx = \frac{1}{4}\int(1+\cos 6x)\,dx + \frac{1}{4}\int(\cos(x+3x) + \cos(x-3x))\,dx$

$= \frac{1}{4} + \frac{1}{6}\cdot\frac{1}{4}\int \cos 6x\,d(6x) + \frac{1}{4}\int \cos 4x\,dx + \frac{1}{4}\int \cos 2x\,dx = \frac{1}{4} + \frac{1}{24}\sin 6x + \frac{1}{16}\sin 4x + \frac{1}{8}\sin 2x + C$

8) $\int \cos(ax+b)\cos(ax-b)\,dx = \frac{1}{2}\int(\cos(ax+b+ax-b) + \cos(ax+b-ax+b))\,dx =$

$= \frac{1}{2}\int(\cos 2ax + \cos 2b)\,dx =$ 1) $\frac{1}{4a}\sin 2ax + \frac{1}{2}\cos(2b)\cdot x + C, \; C \in \mathbb{R}$

2) $\frac{1}{2}x + \frac{1}{2}\cos(2b)\cdot x + C, \; C \in \mathbb{R}$

9) $\int \sin^3 x\,dx = \int \sin^2 x \cdot \sin x\,dx = -\int \sin^2 x \cdot d(\cos x) = -\int(1-\cos^2 x)\,d\cos x =$

$= -\cos x + \frac{1}{3}\cos^3 x + C, \quad C \in \mathbb{R}$

10) $\int \cos^5 x\, dx = \int \cos x \cdot \cos^4 x\, dx = \int \cos^4 x\, d(\sin x) = \int (\cos^2 x)^2\, d \sin x =$

$= \int (1-\sin^2 x)^2\, d\sin x = \int (1 - 2\sin^2 x + \sin^4 x)\, d\sin x = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

11) $\int \sin^2 x \cdot \cos^3 x\, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos x\, dx = \int \sin^2 x \cdot (1-\sin^2 x)\, d\sin x =$

$= \int (\sin^2 x - \sin^4 x)\, d\sin x = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C,\ C \in \mathbb{R}$

12) $\int \cos^5 x \sin^9 x\, dx = \int \cos^4 x \cdot \cos x \cdot \sin^9 x\, dx = \int \cos^4 x \sin^9 x\, d\sin x =$

$= \int (1-\sin^2 x)^2 \sin^9 x\, d\sin x = \int (1 - 2\sin^2 x + \sin^4 x)\sin^9 x\, d\sin x =$

$= \int (\sin^9 x - 2\sin^{11} x + \sin^{13} x)\, d\sin x = \frac{1}{10}\sin^{10} x - \frac{1}{6}\sin^{12} x + \frac{1}{14}\sin^{14} x + C,\ C \in \mathbb{R}$

13) $\int \frac{\sin 2x}{\cos^7 x}\, dx = \int \frac{2\sin x \cos x}{\cos^7 x}\, dx = 2\int \frac{\sin x}{\cos^6 x}\, dx = -2\int \frac{d\cos x}{\cos^6 x} =$

$= 2 \cdot \frac{1}{5}\, \frac{1}{\cos} + C = \frac{2}{5\cos^5 x} + C,\ C \in \mathbb{R}$

14) $\int \frac{\sin^5 x}{\cos^3 x}\, dx = \int \frac{\sin^4 x \cdot \sin x\, dx}{\cos^3 x} = -\int \frac{\sin^4 x}{\cos^3 x}\, d\cos x = -\int \frac{(1-\cos^2 x)^2}{\cos^3 x}\, d\cos x =$

$= -\int \frac{1 - 2\cos^2 x + \cos^4 x}{\cos^3 x}\, d\cos x = -\int \left(\frac{1}{\cos^3 x} - \frac{2}{\cos x} + \cos x\right) d\cos x =$

$= +\frac{1}{2} \cdot \frac{1}{\cos^2 x} + 2\ln|\cos x| - \frac{1}{2}\cos^2 x + C,\ C \in \mathbb{R}$

15) $\int \sin^6 x\, dx = \int (\sin^2 x)^3\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^3 dx = \frac{1}{8}\int dx - \frac{3}{8}\int \cos 2x\, dx + \frac{3}{8}\int \cos^2 2x\, dx$

$- \frac{1}{8}\int \cos^3 2x\, dx = \frac{1}{8}x - \frac{3}{16}\sin 2x + \frac{3}{8}\int \frac{1+\cos 4x}{2}\, dx - \frac{1}{8}\int \cos^2 2x \cdot \cos 2x\, dx =$

$= \frac{1}{8}x - \frac{3}{16}\sin 2x + \frac{3}{16}\int (1+\cos 4x)\, dx - \frac{1}{8}\cdot\frac{1}{2}\int \cos^2 2x \cdot d\sin 2x =$

$= \frac{1}{8}x - \frac{3}{16}\sin 2x + \frac{3}{16}x + \frac{3}{64}\sin 4x - \frac{1}{16}\int (1-\sin^2 x)\, d\sin 2x =$

$= \frac{5}{16}x - \frac{3}{16}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{16}\sin 2x + \frac{1}{48}\sin^3 2x + C =$

$= \frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C,\ C \in \mathbb{R}$

16) $\int \cos^2 x \sin^4 x \, dx = \int (\sin^2 x \cos^2 x) \sin^2 x \, dx = \frac{1}{4} \int (2 \sin x \cos x)^2 \sin^2 x \, dx$

$= \frac{1}{4} \int \sin^2 2x \cdot \frac{1 - \cos 2x}{2} \, dx = \frac{1}{8} \int \sin^2 2x \, dx - \frac{1}{8} \int \sin^2 2x \cos 2x \, dx =$

$= \frac{1}{8} \int \frac{1 - \cos 4x}{2} \, dx - \frac{1}{2} \cdot \frac{1}{8} \int \sin^2 2x \, d \sin 2x = \frac{1}{16} \int 1 \cdot dx - \frac{1}{16} \int \cos 4x \, dx$

$- \frac{1}{16} \cdot \frac{1}{3} \sin^3 x = \frac{1}{16} x - \frac{1}{16} \cdot \frac{1}{4} \cdot \frac{1}{2}$

Мат. анал. (7)

$\cos^2 x + \sin^2 x = 1$

$1 + tg^2 x = \frac{1}{\cos^2 x}$

$1 + \frac{1}{ctg^2 x} = \frac{1}{\cos^2 x} = \frac{1 + ctg^2 x}{ctg^2 x}$

$ctg^2 x + 1 = \frac{1}{\sin^2 x}$

$\frac{1}{tg^2 x} + 1 = \frac{1}{\sin^2 x} = \frac{1 + tg^2 x}{tg^2 x}$

$\sin \arcsin x = x, \ -1 \le x \le x$

$\arcsin \sin x = x$

$arctg\, tg\, x = x + C_0$

1) $\int \frac{1}{\cos^6 x} dx = \int \frac{1}{\cos^4 x} \cdot \frac{1}{\cos^2 x} dx = \int \frac{1}{\cos^4 x} d(tgx) = \int \left(\frac{1}{\cos^2 x}\right)^2 d(tgx) =$

$\int (1 + tg^2 x)^2 d\, tgx = \int (1 + 2tg^2 x + tg^4 x)\, d\, tgx = tgx + \frac{2}{3} tg^3 x + \frac{1}{5} tg^5 x + C, C \in$

2) $\int \frac{dx}{\sin^4 7x} = \left| \begin{matrix} u = 7x \\ x = u/7 \\ dx = \frac{1}{7} du \end{matrix} \right| = \frac{1}{7} \int \frac{du}{\sin^4 u} = -\frac{1}{7} \int (1 + ctg^2 u)\, d\, ctgu =$

$= -\frac{1}{7} ctg u - \frac{1}{7} \cdot \frac{1}{3} ctg^3 u + C = -\frac{1}{7} ctg 7x - \frac{1}{21} ctg^3 7x + C, \ C \in R$

3) $\int tg^6 x dx = \int tg^4 x \frac{\sin^2 x}{\cos^2 x} dx = \int tg^4 x \cdot \sin^2 x\, d(tgx) = \int tg^4 x \frac{tg^2 x}{1 + tg^2 x} d(tgx) =$

$\{tgx = u\} = \int \frac{u^6}{u^2+1} du = \int \frac{u^6 + u^4 - u^4 + u^2 - u^2 + 1 - 1}{u^2+1} du =$

$\int \frac{u^4(u^2+1) - u^2(u^2+1) + (u^2+1) - 1}{u^2+1} du = \int \left(u^4 - u^2 + 1 - \frac{1}{u^2+1}\right) du =$

$= \frac{1}{5} u^5 - \frac{1}{3} u^3 + u - arctg u + C = \frac{1}{5} tg^5 x - \frac{1}{3} tg^3 x + tgx - \underbrace{arctg(tgx)}_{= x + C_0} + C_1$

$= \frac{1}{5} tg^5 x - \frac{1}{3} tg^3 x + tgx - x + C, \ C \in R$

4) $\int ctg^8 x\, dx = \int ctg^6 x \frac{\cos^2 x}{\sin^2 x} dx = -\int ctg^6 x \cdot \cos^2 x \cdot d\, ctgx = -\int ctg^6 x \cdot \frac{ctg^2 x}{1 + ctg^2 x} d\, ctgx$

$= \{ctgx = u\} = \int \frac{u^8}{u^2+1} du = \int \frac{u^8 + u^6 - u^6 + u^4 - u^4 + u^2 - u^2 - 1 + 1}{u^2+1} du =$

$= \int (u^6 - u^4 + u^2 - 1) du - \int \frac{1}{1+u^2} du = -\frac{1}{7} u^7 + \frac{1}{5} u^5 - \frac{1}{3} u^3 + u + arctg u + C$

$$= -\frac{1}{7}ctg^7x + \frac{1}{5}ctg^5x - \frac{1}{3}ctg^3x + ctg x + \overset{x+C_0}{arcctg(ctg x)} + C =$$

$$= -\frac{1}{7}ctg^7x + \frac{1}{5}ctg^5x - \frac{1}{3}ctg^3x + ctg x + \cancel{arcctg}\ x + C$$

5) $$\int \frac{dx}{\sin^4x\cos^4x} = 16\int \frac{dx}{16\sin^4x\cos^4x} = 16\int \frac{dx}{\sin^4 2x} = \left\{ \begin{matrix} 2x = u \\ x = u/2 \\ dx = 1/2\,du \end{matrix} \right\} = 8\int \frac{du}{\sin^4 u} =$$

$$= -8\int \frac{du}{\sin^2 u}\, d(ctg\, u) = -8\int (1 + ctg^2 u)\, d\, ctg\, u = -8\, ctg u - \frac{8}{3} ctg^3 u + C =$$

$$= -8\, ctg\, 2x - 8/3\, ctg^3 2x + C$$

$$\sin x = \frac{2tg\, x/2}{1+tg^2 x/2} \qquad tg\frac{x}{2} = t \qquad \frac{x}{2} = arctg\, t + C_0$$

$$\cos x = \frac{1-tg^2 x/2}{1+tg^2 x/2} \qquad \sin x = \frac{2t}{1+t^2} \qquad x = 2\, arctg\, t + C_0$$

$$tg\, x = \frac{2tg\, x/2}{1-tg^2 x/2} \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2\,dt}{1+t^2}$$

$$tg\, x = \frac{2t}{1-t^2}$$

6) $$\int \frac{dx}{3+5\cos x} = \int \frac{\frac{2dt}{1+t^2}}{3+5\cdot\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{3(1+t^2)+5(1-t^2)} = \int \frac{2dt}{8-2t^2} = \int \frac{dt}{4-t^2} =$$

$$\frac{1}{4}\ln\left|\frac{2+t}{2-t}\right| + C = \frac{1}{4}\ln\left|\frac{2+tg\, x/2}{2-tg\, x/2}\right| + C$$

7) $$\int \frac{\cos x\, dx}{1+\cos x} = \int \frac{\frac{1-t^2}{1+t^2}\cdot\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} = \int \frac{(1-t^2)\cdot 2dt}{(1+t^2)^2+(1-t^2)(1+t^2)} = 2\int \frac{(1-t^2)\,dt}{1+2t^2+t^4+1-t^4} =$$

$$= \frac{2}{2}\int \frac{(1-t^2)\,dt}{t^2+1} = -\int \frac{t^2-1}{t^2+1}\,dt = -\int \left(1 - \frac{2}{t^2+1}\right) dt = -\int dt + 2\int \frac{1}{t^2+1}\,dt =$$

$$= -t + 2\, arctg\, t + C = -tg\frac{x}{2} + 2\, arctg\left(tg\frac{x}{2}\right) + C = -tg\frac{x}{2} + x + C$$

8) $$\int \frac{dx}{1+3\cos^2 x} = \int \frac{dx}{\left(\frac{1}{\cos^2 x}+3\right)\cos^2 x} = \int \frac{d\, tg x}{\left(\frac{1}{\cos^2 x}+3\right)} = \langle \dots \rangle$$

$$\int \frac{2\,dx}{2(1+3\cos^2 x)} = 2\int \frac{dx}{2+3(1+\cos 2x)} = 2\int \frac{dx}{5+\cos 2x} = \{2x = y\} = \int \frac{dy}{5+3\cos y} =$$

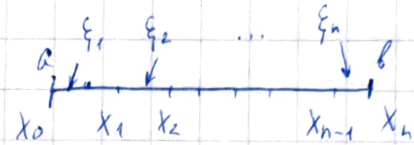
$$= \int \frac{\frac{2dt}{t^2+1}}{5+3\cdot\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{5(1+t^2)+3(1-t^2)} = \int \frac{2dt}{8+2t^2} = \int \frac{dt}{4+t^2} =$$

$$= \frac{1}{2}\operatorname{arctg}\frac{t}{2} + C = \frac{1}{2}\operatorname{arctg}\left(\frac{1}{2}\operatorname{tg}\frac{x}{2}\right) + C = \frac{1}{2}\operatorname{arctg}\left(\frac{1}{2}\operatorname{tg} x\right) + C$$

9) $$\int \frac{\cos x\,dx}{\sin^2 x - 6\sin x + 5} = \int \frac{d\sin x}{\sin^2 x - 6\sin x + 5} = \{\sin x = t\} = \int \frac{dt}{t^2-6t+5} =$$

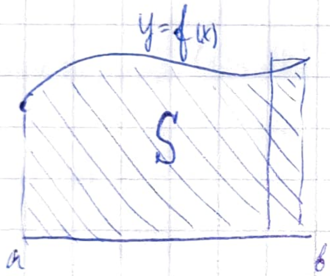
$$\int \frac{1}{t-1}\cdot\frac{1}{t-5}\,dt = -\frac{1}{4}\int \frac{1}{(t-1)} - \frac{1}{(t-5)}\,dt = \langle\ldots\rangle$$

Опр. интеграл:



$$\lim_{d\to 0} \sum_{k=1}^{n} f(\xi_k)\cdot\Delta x_k$$

$\exists$ кон. lim $\Rightarrow$ $\exists \int_a^b f(x)\,dx$



$f(\xi_k)\,\Delta x_k$

Мат. анализ (8)

$dt =$

Определённые интегралы

$f(x)$ – четная на $[-a; a]$, то

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$f(x)$ – нечетная на $[-a; a]$, то

$$\int_{-a}^{a} f(x)\,dx = 0$$

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

1) $\int_1^3 (x^2 - 2x + 3)\,dx = \left(\frac{x^3}{3} - x^2 + 3x\right)\Big|_1^3 = (9 - 9 + 9) - \left(\frac{1}{3} - 1 + 3\right) = 6\frac{2}{3}$

2) $\int_0^{\pi} (2x + \sin 2x)\,dx = \int_0^{\pi} 2x\,dx + \int_0^{\pi} \sin 2x\,dx = x^2\Big|_0^{\pi} + \frac{1}{2}\int_0^{\pi} \sin 2x\,d2x =$

$= x^2\Big|_0^{\pi} - \frac{1}{2}(\cos 2x)\Big|_0^{\pi} = (\pi^2 - 0) - \left(\frac{1}{2}\underbrace{(\cos 2\pi - \cos 0)}_{0}\right) = \pi^2$

$\int_0^{2\pi} \sin x\,dx = 0 \qquad \int_0^{\frac{2\pi}{k}\cdot n} \sin kx\,dx = 0$

3) $\int_{-5}^{-2} \frac{dx}{2x-3} = \frac{1}{2}\int_{-5}^{-2} \frac{d(2x-3)}{2x-3} = \frac{1}{2}\ln|2x-3|\Big|_{-5}^{-2} =$

$= \frac{1}{2}\left(\ln|2\cdot(-2) - 3| - \ln|2\cdot(-5) - 3|\right) = \frac{1}{2}\ln\frac{7}{13}$

4) $\int_1^e \frac{x + \sqrt{x}}{x\sqrt{x}}\,dx = \int_1^e \frac{x}{x\sqrt{x}}\,dx + \int_1^e \frac{\sqrt{x}}{x\sqrt{x}}\,dx = \int_1^e \frac{1}{\sqrt{x}}\,dx + \int_1^e \frac{1}{x}\,dx =$

$= 2\sqrt{x}\Big|_1^e + \ln|x|\Big|_1^e = 2\sqrt{e} - 2 + \underbrace{\ln|e|}_{=1} - \underbrace{\ln|1|}_{=0} = 2\sqrt{e} - 1$

5) $\int_1^5 \frac{x\,dx}{1+x^2} = \frac{1}{2}\int_1^5 \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln|x^2+1|\Big|_1^5 = \frac{1}{2}\left|\ln\frac{26}{2}\right| = \frac{1}{2}\ln 13$

6) $2\int_0^{\frac{\pi}{6}} \sin 2x \cos 8x\,dx = \int_0^{\pi/6} (\sin 10x + \sin(-6x))\,dx = \int_0^{\pi/6} \sin 10x\,dx - \int_0^{\pi/6} \sin 6x\,dx$

$= \frac{1}{10}\int_0^{\pi/6} \sin 10x\,d10x - \frac{1}{6}\int_0^{\pi/6} \sin 6x\,d6x = -\frac{1}{10}(\cos 10x)\Big|_0^{\frac{\pi}{6}} + \frac{1}{6}(\cos 6x)\Big|_0^{\frac{\pi}{6}} =$

$= -\frac{1}{10}\left(\cos\frac{5\pi}{3} - 1\right) + \frac{1}{6}(\cos\pi - 1) = -\frac{1}{10}\cdot\left(-\frac{1}{2}\right) + \frac{1}{6}(-2) = -\frac{17}{60}$

7) $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\overset{\text{неч}}{\sin x + \sin 5x}}{\underset{\text{чет}}{\cos 2x + \cos 4x + 4}}\, dx = 0$$

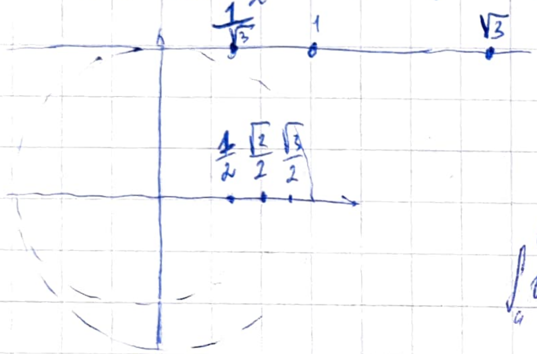
8) $$\int_1^3 \frac{dx}{x^2+6x+10} = \int_1^3 \frac{d(x+3)}{(x+3)^2+1} = \operatorname{arctg}(x+3)\Big|_1^3 = \operatorname{arctg} 6 - \operatorname{arctg} 4$$

9) $$\int_{-2/5}^{0} (2+5x)^4\, dx = \left\{ \begin{array}{l} 2+5x=t \\ dx=\frac{1}{5}dt \end{array} \right\} = \frac{1}{5}\int_0^2 t^4\, dt = \frac{1}{5}\cdot\frac{t^5}{5}\Big|_0^2 = \frac{1}{25}(32-0) = \frac{32}{25}$$

10) $$\int_{\pi/6}^{\pi/4} \frac{dx}{1-\cos 6x} = \left\{ \begin{array}{l} 6x=t \\ dx=\frac{1}{6}dt \end{array} \right\} = \frac{1}{6}\int_{\pi}^{3\pi/2} \frac{dt}{1-\cos t} = \frac{1}{6}\int_{\pi}^{3\pi/2} \frac{dt}{2\sin^2\frac{t}{2}} =$$

$t_a = 6\cdot\frac{\pi}{6} = \pi$
$t_b = 6\cdot\frac{\pi}{4} = \frac{3\pi}{2}$

$$= \frac{1}{6}\int_{\pi}^{3\pi/2} \frac{dt/2}{\sin^2\frac{t}{2}} = -\frac{1}{6}\operatorname{ctg}\frac{t}{2}\Big|_{\pi}^{3\pi/2} = -\frac{1}{6}\left(\operatorname{ctg}\frac{3\pi}{4} - \operatorname{ctg}\frac{\pi}{2}\right) = -\frac{1}{6}(-1-0) = \frac{1}{6}$$



$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

11) $$\int_0^1 \operatorname{arctg} x\, dx = x\cdot\operatorname{arctg} x\Big|_0^1 - \int_0^1 x\, d\operatorname{arctg} x = \operatorname{arctg} 1 - \int_0^1 x\frac{1}{1+x^2}\,dx =$$
$$\frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{d(x^2+1)}{x^2+1} = \frac{\pi}{4} - \frac{1}{2}\ln|x^2+1|\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1) = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

12) $$\int_1^e \ln x\, dx = x\cdot\ln x\Big|_1^e - \int_1^e x\, d(\ln x) = e\ln e - \ln 1 - \int_1^e x\cdot\frac{1}{x}\,dx =$$
$$= e - x\Big|_1^e = 1$$

13) $$\int_0^{\frac{\pi}{6}} x\sin x\, dx = -\int_0^{\pi/6} x\, d\cos x = -x\cdot\cos x\Big|_0^{\pi/6} + \int_0^{\pi/6}\cos x\, dx =$$
$$= -\left(\frac{\pi}{6}\cos\frac{\pi}{6} - 0\right) + \sin\Big|_0^{\pi/6} = -\frac{\pi\sqrt{3}}{12} + \frac{1}{2}$$

14) $$\int_0^{\pi/2} e^{2x}\cos x\, dx = \int_0^{\pi/2} e^{x}\, d\sin x = e^{2x}\cdot\sin x\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin x\, d(e^{2x}) =$$
$$e^{\pi}\cdot\sin\frac{\pi}{2} - e^0\cdot\sin 0 - 2\int_0^{\pi/2}\sin x\, d\, e^{2x}\,dx = e^{\pi} + 2\int_0^{\pi/2} e\, d\cos x =$$
$$= e^{\pi} + 2e^{2x}\cdot\cos x\Big|_0^{\pi/2} - 2\int_0^{\pi/2}\cos x\, d(e^{2x}) = e^{\pi} + \cancel{2e^{\pi}\cos\frac{\pi}{2}}^{=0} - 2 - 4\int_0^{\pi/2}\cos x\, e^{2x}\,dx$$

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx = \frac{1}{5}\left(e^{\pi} - 2\right) = \frac{e^{\pi}}{5} - \frac{2}{5}$$

15) $$\int_0^a \sqrt{a^2 - x^2}\, dx = \left\{ \begin{array}{l} x = a \sin t \\ dx = a \cos t \, dt \\ t \in [0; \frac{\pi}{2}] \end{array} \right\} = \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 t} \; a \cos t \, dt =$$

$$= a^2 \int_0^{\pi/2} \sqrt{1 - \sin^2 t} \cos t \, dt = a^2 \int_0^{\pi/2} \sqrt{\cos^2 t} \cos t \, dt =$$

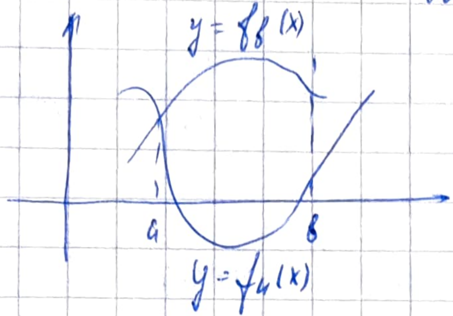
$$= a^2 \int_0^{\pi/2} |\cos t| \cos t \, dt = a^2 \int_0^{\pi/2} \cos^2 t \, dt =$$

$$= a^2 \cdot \frac{1}{2} \int_0^{\pi/2} (1 + \cos 2t)\, dt = \frac{a^2}{2} \cdot t \Big|_0^{\pi/2} + \frac{a^2}{2} \cdot \frac{1}{2} \cdot \sin 2t \Big|_0^{\pi/2} =$$

$$= \frac{a^2 \pi}{4}$$

Мат. анализ (9)

$$\int_a^b (f_1(x) - f_4(x))$$

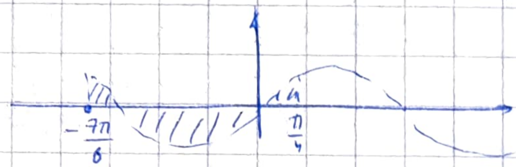


$x \in [a; b]$

1) S = $y = \sin x$

$y = 0$

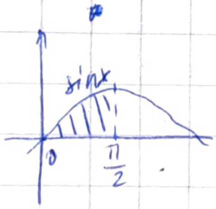
$x = -\frac{7\pi}{6}$ $\quad x = \frac{\pi}{4}$



$$S = \int_{-\frac{7\pi}{6}}^{\pi} (\sin x - 0)dx + \int_0^{\pi} (0 - \sin x) + \int_0^{\frac{\pi}{4}} (\sin x - 0)dx = \langle \ldots \rangle$$

$$= 4 - \frac{\sqrt{2}+}{2}$$

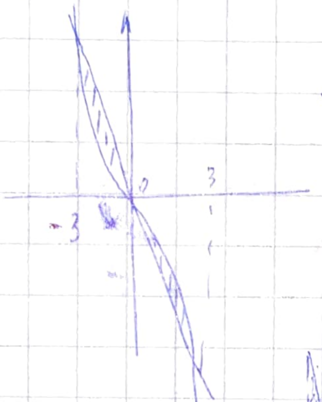
2)



$$S = \int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x \Big|_0^{\frac{\pi}{2}} = 1$$

3) $y = -x^3$

$y = -9x$



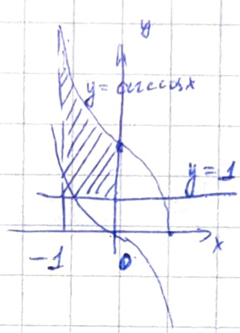
$$S = 2\int_0^3 (-x^3 + 9x)\,dx$$

$$= 2\left(-\frac{x^4}{4} + \frac{9x^2}{2}\right)\Big|_0^3 = \frac{81}{2}$$

4) S : $y = \arccos x$

$y = 1$

$x = -1 \quad x = 0$



$$S = \int_{-1}^{0} (\arccos x - 1)\,dx = \int_{-1}^{0} \arccos x\,dx - \int_{-1}^{0} dx =$$

$$= x \arccos x \Big|_{-1}^{0} - \int_{-1}^{0} x \, d\arccos x - (0 - (-1)) =$$

$$= 0 - (-1)\cdot\pi - \int_{-1}^{0} x\left(-\frac{dx}{\sqrt{1-x^2}}\right) - 1 =$$

$$= \pi - 1 + \int_{-1}^{0} \frac{x\,dx}{\sqrt{1-x^2}} = \pi - 1 + \frac{1}{2}\int_{-1}^{0} \frac{d(x^2)}{\sqrt{1-x^2}} = \pi - 1 - \frac{1}{2}\int_{-1}^{0} \frac{d(1-x^2)}{\sqrt{1-x^2}}$$

$$= \pi - 1 - \sqrt{1-x^2}\Big|_{-1}^{0} = \pi - 2$$

$\int_a^c f(x)\,dx = 0$

$\left|\int_a^{a+\varepsilon} f(x)\,dx\right| \le \int_a^{a+\varepsilon} |f(x)|\,dx < C \cdot \int_a^{a+\varepsilon} dx = C\cdot\varepsilon$

Опр. Несобственный интеграл I рода

Пусть $f(x) \in R[a;b]$ $\forall b > a$

$\int_a^{+\infty} f(x)\,dx = \lim\limits_{b\to+\infty} \int_a^b f(x)\,dx =$ $\left[\begin{array}{l} \to \exists \text{ конечный} - \text{сход.} \\ \exists \text{ бесконечный} \\ \text{не сущ.} \end{array}\right\}$ – расх.

1) $\int_1^{+\infty} \frac{1}{x}\,dx = \lim\limits_{b\to+\infty} \int_1^b \frac{1}{x}\,dx = \lim\limits_{b\to+\infty} \ln|x|\Big|_1^b = \lim\limits_{b\to+\infty} (\ln|b| - 0) = +\infty$ (расход.)

2) $\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim\limits_{b\to+\infty} \int_1^b \frac{dx}{\sqrt{x}} = \lim\limits_{b\to+\infty} 2\sqrt{x}\Big|_1^b = \lim\limits_{b\to+\infty} (\underbrace{2\sqrt{b}}_{\to+\infty} - 2) = +\infty$ – расход.

3) $\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim\limits_{b\to+\infty} \int_1^b \frac{dx}{x^2} = \lim\limits_{b\to+\infty} -\frac{1}{x}\Big|_1^b = \lim\limits_{b\to+\infty} (\underbrace{-\frac{1}{b}}_{\to 0} + 1) = 1$ – сход.

$\int_a^{+\infty} \frac{1}{x^p}\,dx$ $\begin{array}{l} p > 1 \text{ сход.} \\ p \le 1 \text{ расход.} \end{array}$

4) $\int_0^{+\infty} e^{-4x}\,dx = \lim\limits_{b\to+\infty} \int_0^b \frac{1}{e^{4x}}\,dx = \lim\limits_{b\to+\infty} -\frac{1}{4}\int_0^b e^{-4x}\,d(-4x) =$

$\lim\limits_{b\to+\infty} -\frac{1}{4} e^{-4x}\Big|_0^b = \lim\limits_{b\to+\infty} (\underbrace{-\frac{1}{4}e^{-4b}}_{\to 0} + \frac{1}{4}) = \frac{1}{4}$ – сход.

5) $\int_{13}^{+\infty} \frac{dx}{x\ln x} = \lim\limits_{b\to+\infty} \int_{13}^b \frac{1}{\ln x}\,d\ln x = \lim\limits_{b\to+\infty} \ln|\ln(x)|\Big|_{13}^b =$

$= \lim\limits_{b\to+\infty} (\underbrace{\ln|\ln b|}_{\to\infty} - \ln|\ln 13|) = \infty$ – расход

Мат. анализ (10)

1) $\int_{13}^{+\infty} \frac{dx}{x \ln^2 x} = \lim_{b \to +\infty} \int_{13}^{b} \frac{1}{\ln^2 x} d\ln x = \lim_{b \to +\infty} \left[-\frac{1}{\ln x} \right] \Big|_{13}^{b} =$

$= \lim_{b \to +\infty} \left(-\underbrace{\frac{1}{\ln b}}_{\to 0} + \frac{1}{\ln 13} \right) = \frac{1}{\ln 13}$ – сход.

2) $\int_{0}^{+\infty} x \cos x \, dx = \lim_{b \to +\infty} \int_{0}^{b} x \cos x \, dx = \langle \ldots \rangle = \lim_{b \to +\infty} (x \sin x + \cos x) \Big|_{0}^{b} =$

$= \lim_{b \to +\infty} (b \sin b + \cos b - 1)$ – не сущ. => расход

при $b = 2\pi n$, $n \in \mathbb{N}$ $\quad b \sin b + \cos b = 1$

$b = \pi + 2\pi n$, $n \in \mathbb{N}$ $\quad b \sin b + \cos b = -1$

3) $\int_{-\infty}^{+\infty} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{+\infty} f(x) \, dx =$

$= \underbrace{\lim_{b \to +\infty} \int_{-b}^{a} f(x) \, dx}_{\to \text{ сход}} + \underbrace{\lim_{c \to +\infty} \int_{a}^{c} f(x) \, dx}_{\to \text{ сход}}$ $\Big\} \Rightarrow$ сход.

$\int_{-\infty}^{+\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{+\infty} \frac{1}{1+x^2} dx$

$= \lim_{a \to +\infty} \int_{-a}^{0} \frac{1}{1+x^2} dx + \lim_{c \to +\infty} \int_{0}^{c} \frac{1}{x^2+1} dx =$

$= \lim_{a \to +\infty} \operatorname{arctg} x \Big|_{-a}^{0} + \lim_{c \to +\infty} \operatorname{arctg} x \Big|_{0}^{c} =$

$= \lim_{a \to +\infty} (\operatorname{arctg} 0 - \operatorname{arctg}(-a)) + \lim_{c \to +\infty} (\operatorname{arctg} c - \operatorname{arctg} 0) =$

$= \lim_{a \to +\infty} \operatorname{arctg} a + \lim_{c \to +\infty} \operatorname{arctg} c = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

Интегрирование по Риману $\int_0^1 \frac{1}{x}\,dx$

Опр. Несобственный интеграл II рода

$f(x)$ – не огр. в $\forall U_\delta(a)$, $f(x) \in R : [a+\varepsilon, b]$, $\forall \varepsilon > 0$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0_+} \int_{a+\varepsilon}^b f(x)\,dx$$

1) $$\int_a^b \frac{1}{x-a}\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{1}{x-a}\,dx-a =$$

$$\lim_{\varepsilon \to 0} \ln|x-a| \Big|_{a+\varepsilon}^b = \lim_{\varepsilon \to 0} (\ln|b-a| - \ln|\varepsilon|) = \lim_{\varepsilon \to 0} \underbrace{(\ln|b-a|)}_{\downarrow\ \ln|b-a|} - \lim_{\varepsilon \to 0} \underbrace{\ln|\varepsilon|}_{\downarrow\ -\infty}$$

расход

2) $$\int_a^b \frac{1}{(x-a)^2}\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{1}{(x-a)^2}\,d(x-a) = \lim_{\varepsilon \to 0} \left(-\frac{1}{x-a} \Big|_{a+\varepsilon}^b \right) =$$

$$= \lim_{\varepsilon \to 0} \Big(\underbrace{-\frac{1}{b-a}}_{const} + \underbrace{\frac{1}{\varepsilon}}_{\to\infty} \Big) = \infty$$ – расход.

3) $$\int_a^b \frac{1}{\sqrt{x-a}}\,dx = \lim_{\varepsilon \to 0} \int_{a+\varepsilon}^b \frac{1}{\sqrt{x-a}}\,d(x-a) = \lim_{\varepsilon \to 0} 2\sqrt{x-a} \Big|_{a+\varepsilon}^b =$$

$$= \lim_{\varepsilon \to 0} \Big(\underbrace{2\sqrt{b-a}}_{\to const} - \underbrace{2\sqrt{\varepsilon}}_{\to 0} \Big) = 2\sqrt{b-a}$$

$\int_0^1 \frac{1}{x^p}$ $\quad p \geq 1$ расх, $\quad p < 1$ сход

$\int_1^{+\infty} \frac{1}{x^p}$ $\quad p \geq 1$ расх, $\quad p < 1$ сход

4) $$\int_0^1 \ln x\,dx = \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \ln x\,dx = \lim_{\varepsilon \to 0} \left(x \ln x \Big|_{0+\varepsilon}^1 - \int_{0+\varepsilon}^1 x\,d\ln x \right) =$$

$$= \lim_{\varepsilon \to 0} \left(x\ln x \Big|_\varepsilon^1 - \int_\varepsilon^1 x \cdot \frac{1}{x}\,dx \right) = \lim_{\varepsilon \to 0} \Big(\underbrace{1 \cdot \ln 1}_{=0} - \underbrace{\varepsilon \ln \varepsilon}_{\to 0} - 1 + \underbrace{\varepsilon}_{\to 0} \Big) = -1$$

• $$\lim_{\varepsilon \to 0} (\varepsilon \cdot \ln \varepsilon) = [0 \cdot \infty] = \lim_{\varepsilon \to 0} \frac{\ln \varepsilon}{\frac{1}{\varepsilon}} = \left[\frac{\infty}{\infty} \right] = \lim_{\varepsilon \to 0} \frac{\frac{1}{\varepsilon}}{\underbrace{-\frac{1}{\varepsilon^2}}_{= -\varepsilon}} = 0$$

5) $\int_0^{\frac{\pi}{4}} \frac{dx}{1-\cos 2x} = \lim\limits_{\varepsilon\to 0} \int_\varepsilon^{\frac{\pi}{4}} \frac{dx}{1-\cos 2x} = \lim\limits_{\varepsilon\to 0} \int_0^{\frac{\pi}{4}} \frac{dx}{2\sin^2 x} = \lim\limits_{\varepsilon\to 0} -\frac{1}{2} \cdot ctg\, x \Big|_\varepsilon^{\frac{\pi}{4}} =$

$= -\frac{1}{2} \lim\limits_{\varepsilon\to 0} (\underbrace{ctg \frac{\pi}{4}}_{=1} - \underbrace{ctg\, \varepsilon}_{\to\infty}) = \infty \to$ расх.

6) $\int_0^1 x \ln x\, dx = \lim\limits_{\varepsilon\to 0} \int_\varepsilon^1 x \ln x\, dx = \lim\limits_{\varepsilon\to 0} \frac{1}{2} \int_\varepsilon^1 \ln x\, dx^2 =$

$= \frac{1}{2} \lim\limits_{\varepsilon\to 0} \left(x^2 \ln x \Big|_\varepsilon^1 - \int_\varepsilon^1 x^2\, d\ln x \right) = \frac{1}{2} \lim\limits_{\varepsilon\to 0} \left(0 - \varepsilon^2 \ln\varepsilon - \int_\varepsilon^1 \frac{x^2}{x} dx \right) =$

$= \frac{1}{2} \lim\limits_{\varepsilon\to 0} \left(\underbrace{-\varepsilon^2 \ln\varepsilon}_{\to 0} - \frac{1}{2} x^2 \Big|_\varepsilon^1 \right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ – сход.

7) $\int_1^4 \frac{dx}{x-2} = \int_1^2 \frac{dx}{x-2} + \int_2^4 \frac{dx}{x-2} = \lim\limits_{\varepsilon\to 0} \int_1^{2-\varepsilon} \frac{dx}{x-2} + \lim\limits_{\delta\to 0} \int_{2+\delta}^4 \frac{dx}{x-2} =$

$\lim\limits_{\varepsilon\to 0} \ln|x-2| \Big|_1^{2-\varepsilon} + \lim\limits_{\delta\to 0} \ln|x-2| \Big|_{2+\delta}^4 =$

$\lim\limits_{\varepsilon\to 0} (\underbrace{\ln\varepsilon}_{-\infty} - \underbrace{\ln 1}_{=0}) + \ldots \to$ расх

8) $\int_{-1}^1 \frac{1}{x^3} dx = \int_{-1}^0 \frac{1}{x^3} dx + \int_0^1 \frac{1}{x^3} dx = \lim\limits_{\varepsilon\to 0} \int_{-1}^{\varepsilon} \frac{1}{x^3} dx + \lim\limits_{\delta\to 0} \int_\delta^1 \frac{1}{x^3} dx =$

$= \lim\limits_{\varepsilon\to 0} \left(-\frac{1}{2x^2} \Big|_{-1}^{\varepsilon} \right) + \lim\limits_{\delta\to 0} \left(-\frac{1}{2x^2} \Big|_\delta^1 \right) = -\frac{1}{2} \lim\limits_{\varepsilon\to 0} \left(\underbrace{-\frac{1}{2\varepsilon^2}}_{\to\infty} - \frac{1}{2} \right) + \ldots \to$ расх.

9) $\int_a^b f(x)\, dx = \lim\limits_{\varepsilon\to 0} \left(\int_a^{c+\varepsilon} f(x) dx + \int_{\varepsilon+c}^b f(x)\, dx \right)$ — инт. в смысле гл. знач.

$c \in [a; b]$ f – не огр. $\forall U_\delta(c)$

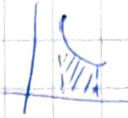
$\int_{-1}^1 \frac{1}{x^3} dx = \lim\limits_{\varepsilon\to 0} \left(\int_1^{-\varepsilon} \frac{1}{x^3} dx + \int_\varepsilon^1 \frac{1}{x^3} dx \right) =$

$= -\frac{1}{2} \lim\limits_{\varepsilon\to 0} \left(\frac{1}{\varepsilon^2} - 1 + 1 - \frac{1}{\varepsilon^2} \right) = 0 \to$ сход

Объемы:

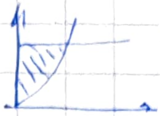
$V_{Ox} = \pi \int_a^b (f(x))^2 dx \qquad V_{Ox} = \pi \int_a^b (f^2(x) - g^2(x))\, dx$

1) $xy = 6 \quad x = 1 \quad x = 4 \quad y = 0$



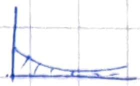
$$V_{Ox} = \pi \int_1^4 \left(\frac{6}{x}\right)^2 dx = 36\pi \int_1^4 \frac{1}{x^2}\,dx = 36\pi\left(\frac{1}{x}\Big|_1^4\right) = 27\pi$$

2) $y = x^3 \quad x = 0 \quad y = 8$



$$V_{Ox} = \pi \int_0^2 \left(8^2 - (x^3)^2\right) dx = \pi \left(64 - \frac{x^7}{7}\Big|_0^2\right) = \frac{\pi \cdot 2^8 \cdot 3}{7}$$

3) $y = e^{-x} \quad x = 0 \quad y = 0$



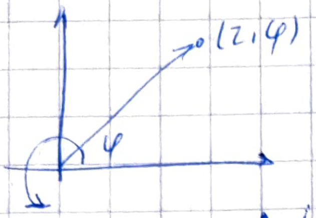
$$V_{Ox} = \pi \int_0^{+\infty} (e^{-x})^2 dx = \pi \lim_{b\to+\infty} \int_0^b e^{-2x}\,dx = -\frac{\pi}{2} \lim_{b\to+\infty} \left(e^{-2x}\Big|_0^b\right) =$$

$$= -\frac{\pi}{2} \lim_{b\to+\infty} \left(\underbrace{\frac{1}{e^{2b}}}_{\to 0} - \frac{1}{e^0}\right) = -\frac{\pi}{2}\cdot(-1) = \frac{\pi}{2}$$

4) $$\int_0^3 \frac{2dx}{x^2-1} = \int_0^3 \left(\frac{1}{x-1} + \frac{1}{x+1}\right) dx = \int_0^3 \frac{dx}{x-1} + \int_0^3 \frac{dx}{x+1} =$$

$$= \lim_{\varepsilon\to0} \int_0^{1-\varepsilon} \frac{dx}{x-1} + \lim_{\delta\to0} \int_{1+\delta}^3 \frac{dx}{x-1} + \ln|x+1|\Big|_0^3 \to \text{расх.}$$

Полярные координаты

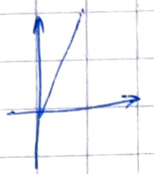


$$\begin{cases} x = r \cdot \cos\varphi \\ y = r \cdot \sin\varphi \end{cases}$$

$\varphi \in (-\infty; +\infty)$

$r \geq 0$

$\varphi = \frac{\pi}{3}$



$r = 5$

1) $r = \sin 2\varphi$ 1) $r(\varphi) \geq 0$ $\sin 2\varphi \geq 0$

$0 + 2\pi n \leq 2\varphi \leq \pi + 2\pi n$, $n \in \mathbb{Z}$

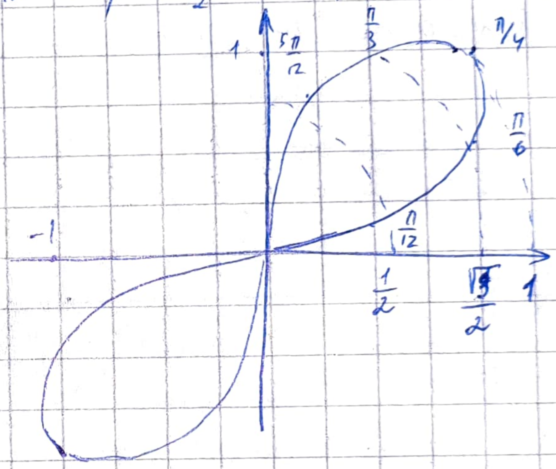
$\pi n \leq \varphi \leq \frac{\pi}{2} + \pi n$, $n \in \mathbb{Z}$

2)

φ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
r	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{3}$	0

период $T = \pi$

двух лепестковая роза



2) $r = \cos 3\varphi$ 1) $r(\varphi) \geq 0$ $\cos 3\varphi \geq 0$

$-\frac{\pi}{2} + 2\pi n \leq 3\varphi \leq \frac{\pi}{2} + 2\pi n$, $n \in \mathbb{Z}$

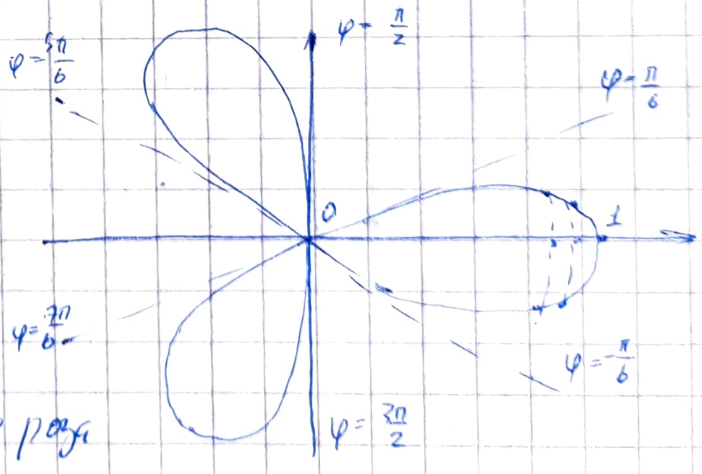
$-\frac{\pi}{6} + \frac{2\pi n}{3} \leq \varphi \leq \frac{\pi}{6} + \frac{2\pi n}{3}$, $n \in \mathbb{Z}$

2)

φ	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	$-\frac{\pi}{18}$	0	$\frac{\pi}{18}$	$\frac{\pi}{12}$	$\frac{\pi}{6}$
r	0	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	0

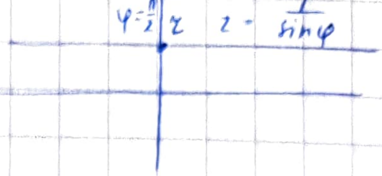
период $T = \frac{2\pi}{3}$

3 лепестковая роза



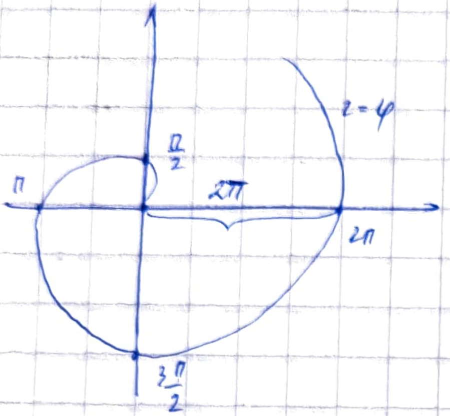
$\cos k\varphi$ – k-лепестковая роза

$r = \frac{1}{\sin\varphi}$ $\quad \sin\varphi > 0$ $\quad \varphi \in (0; \pi)$

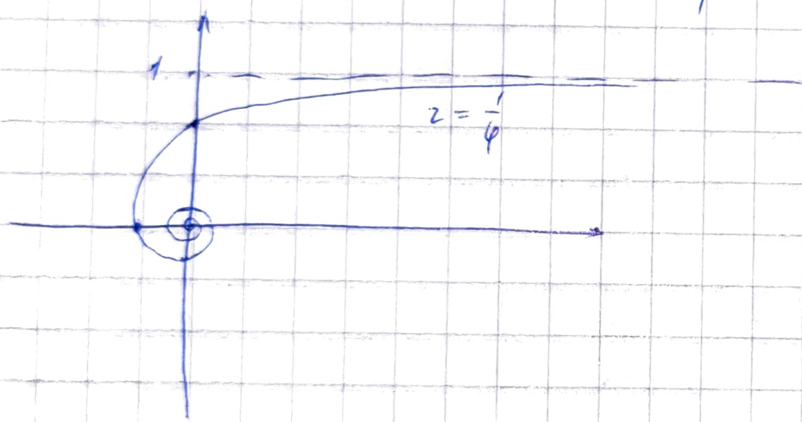


$y = r\sin\varphi = 1$

Спираль Архимеда $\quad r = \varphi \quad r(\varphi) \geq 0 \quad \varphi \geq 0$

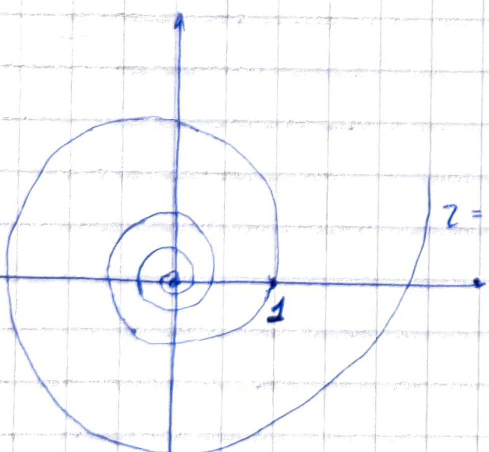


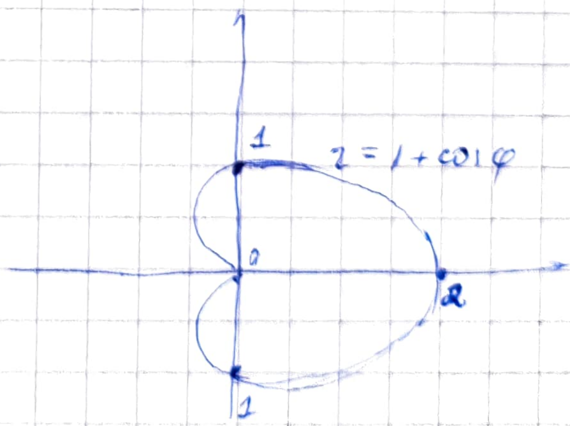
Гиперболическая спираль $\quad r = \frac{1}{\varphi} \quad r(\varphi) \geq 0 \quad \varphi > 0$



$\varphi \to 0 : r\left(\frac{1}{\varphi}\right) \to r\left(\frac{1}{\sin\varphi}\right)$

Логарифмическая спираль $\quad r = e^{\varphi} \quad r(\varphi) \geq 0 \quad \varphi \in [-\infty; \infty]$





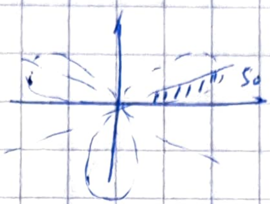
Кардиоида $\quad r = 1 + \cos\varphi \quad r(\varphi)$ – четн. по φ с $T = 2\pi$

φ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
r	2	$1+\frac{\sqrt{2}}{2}$	1	$1-\frac{\sqrt{2}}{2}$	0

Площадь сектора (пол. коорд.)

$$S = \frac{\pi r^2 \varphi}{2\pi} = \frac{1}{2} r^2 \varphi - \frac{1}{2}\int_{\varphi_1}^{\varphi_2} (r(\varphi))^2 \, d\varphi$$

1) $r = a \cdot \sin 3\varphi$



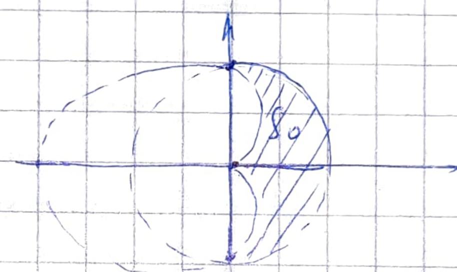
$S = 6 S_0$ –

$$S_0 = \frac{1}{2}\int_0^{\frac{\pi}{6}} a^2 \sin^2 3\varphi \, d\varphi = \frac{a^2}{2} \cdot \frac{1}{2}\int_0^{\frac{\pi}{6}} (1-\cos 6\varphi)\, d\varphi =$$

$$= \frac{a^2}{4}\int_0^{\frac{\pi}{6}} d\varphi - \frac{a^4}{4\cdot 6}\int \cos 6\varphi \, d6\varphi = \langle \ldots \rangle = \frac{\pi a^4}{24}$$

$S = \frac{\pi a^2}{24} \cdot 6 = \frac{\pi a^2}{4}$ – площадь к леп. розы

2) $S:$ $r = a(1-\cos\varphi)$
$r = a$



$$S = 2 S_0 = 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{2}} (a^2 - a^2(1-\cos\varphi)^2)\, d\varphi =$$

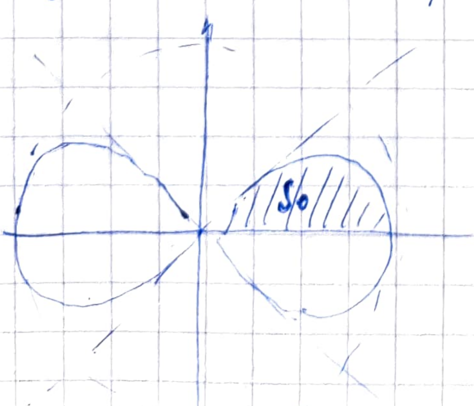
$$= a^2 \int_0^{\frac{\pi}{2}} (1 - 1 + 2\cos\varphi - \cos^2\varphi)\, d\varphi =$$

$$= a^2 \int_0^{\frac{\pi}{2}} 2\cos\varphi \, d\varphi - a^2/2 \int_0^{\frac{\pi}{2}} (1+\cos 2\varphi)\, d\varphi =$$

$$= 2a^2 - \frac{a^2}{2}\int_0^{\frac{\pi}{2}} d\varphi - \frac{1}{2}\cdot\frac{a^2}{2}\int_0^{\frac{\pi}{2}} \cos 2\varphi \, d2\varphi =$$

$$= 2a^2 - \frac{a^2\pi}{4} - \frac{a^2}{4}(\sin\pi - \sin 0) = 2a^2 - \frac{a^2\pi}{4}$$

3) Лемниската Бернулли $S:$ $r^2 = a^2\cos 2\varphi$
$\cos 2\varphi \geq 0$
$-\frac{\pi}{4} + \pi k \leq \varphi \leq \frac{\pi}{4} + \pi k$



$$S = 4 \cdot \frac{1}{2}\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} a^2 \cos 2\varphi \, d2\varphi =$$

$$= 2\frac{a^2}{2} \cdot \sin 2\varphi \Big|_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2} - \sin 0\right) = a^2$$

Мат. анализ (12)

Кратный интеграл

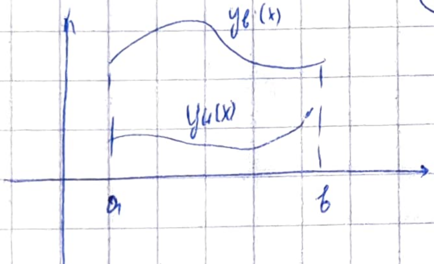
$L: \; y = f(x) \qquad x \in [a; b] \qquad S(L) = 0$

1) $M = \bigcup_{k=1}^{n} M_k$

2) $S(M_k \cap M_j) = 0$

$$\iint\limits_M f(x,y)\,dx\,dy = \int_{a}^{b} dx \int_{y_н(x)}^{y_в(x)} f(x,y)\,dy$$

const!



1) $\int_0^2 dy \int_0^1 (x^2+2y)\,dx = \int_0^2 dy \left(\frac{x^3}{3} + 2yx\right)\Big|_0^1 =$

$= \int_0^2 dy \left(\frac{1}{3} + 2y\right) = \left(\frac{1}{3}y + y^2\right)\Big|_0^2 = 4\frac{2}{3}$

2) $\int_{-3}^{3} dy \int_{y^2-4}^{5} (x+2y)\,dx = \int_{-3}^{3} dy \; \frac{x^2}{2} + 2yx \Big|_{y^2-4}^{5} =$

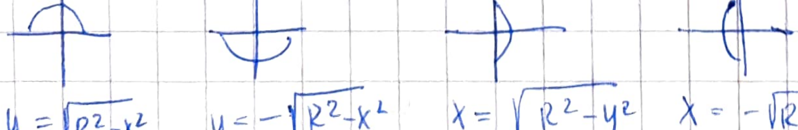
$= 2\int_0^3 dy \; (25 - (y^2-4))^2 = 2\left(-\frac{y^5}{5} + \frac{8y^3}{3} + 9\right)\Big|_0^3 = \frac{252}{5}$

3) $\int_0^{2\pi} d\varphi \int_{a\sin\varphi}^{a} r\,dr = \frac{1}{2}\int_0^{2\pi} d\varphi \; r^2\Big|_{a\sin\varphi}^{a} = \frac{a^2}{4}\int_0^{2\pi}(1+\cos 2\varphi)\,d\varphi =$

$= \frac{a^2}{4}(2\pi - 0) = \frac{\pi a^2}{2}$

4) $\int_0^1 dy \int_y^{\sqrt{y}} f(x,y)\,dx = \int_0^1 dx \int_{x^2}^{x} f(x,y)\,dy$

$\int_{-1}^{1} dx \int_0^{\sqrt{1-x^2}} f(x,y)\,dy = \int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}$



$y = \sqrt{R^2 - x^2} \qquad y = -\sqrt{R^2 - x^2} \qquad x = \sqrt{R^2 - y^2} \qquad x = -\sqrt{R^2 - y^2}$

4) $\int_{-2}^{2} dx \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} f(x,y)\, dy = \int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-\sqrt{4-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx$

5) $\int_{1}^{2} dx \int_{x}^{2y} f(x,y)\, dy = \int_{1}^{4} dy \int_{1}^{y} f\, dx + \int_{2}^{4} dy \int_{\frac{y}{2}}^{2} f\, dx$

6) $\iint\limits_{D} \cos(x+y)\, dx\, dy = \int_{0}^{\pi} dx \int_{x}^{\pi} \cos(x+y)\, dy = -2$

$$\int_a^b f(\varphi(x))\, d\varphi(x) = \int_a^b f(\varphi(x))\, \varphi'(x)\, dx.$$

$M_1 \longrightarrow M$

$(u, v) \qquad (x, y)$

$x = x(u, v)$

$y = y(u, v)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\iint_{M(x,y)} f(x, y)\, dx\, dy = \iint f(x(u,v), y(u,v)) \cdot |J| \cdot du\, dv$$

1) $x = r\cos\varphi$

$y = r\sin\varphi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -r\sin\varphi & r\cos\varphi \end{vmatrix} = r \begin{vmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{vmatrix} =$$

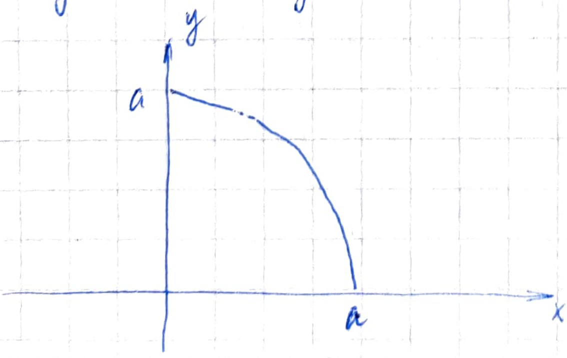
$|J| = r$

$dx\,dy \to r\, d\varphi\, dr$

2) $$\int_0^a dx \int_0^{\sqrt{a^2-x^2}} e^{x^2+y^2}\, dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r\, dr \cdot e^{r^2} =$$

x: от 0 до a

y: от 0 до $\sqrt{a^2-x^2}$



$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi \int_0^a e^{r^2}\, d(r^2) =$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} d\varphi\; e^{r^2}\Big|_0^a =$$

$$= 1/2 \int_0^{\pi/2} d\varphi\,(e^{a^2} - 1) =$$

$$= 1/2\left(\frac{\pi}{2} - 0\right)(e^{a^2} - 1) = \frac{\pi}{4}(e^{a^2} - 1)$$

3) $\iint\limits_D \sqrt{x^2+y^2-9}\,dxdy = \int_0^{2\pi} d\varphi \int_3^5 r\,dr\,\sqrt{r^2-9} =$

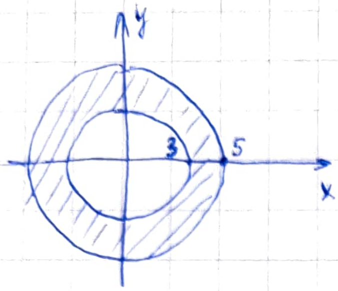
D: кольцо: $x^2+y^2 \geq 9$, $x^2+y^2 \leq 25$

$= 1/2 \int_0^{2\pi} d\varphi \int_3^5 \sqrt{r^2-9}\; d(r^2-9) =$

$= 2/3 \cdot 1/2 \int_0^{2\pi} d\varphi \sqrt{(r^2-9)^3}\Big|_3^5 =$

$= \frac{1}{3} \int_0^{2\pi} d\varphi \left(\sqrt{(25-9)^3} - \sqrt{(9-9)^3}\right) =$

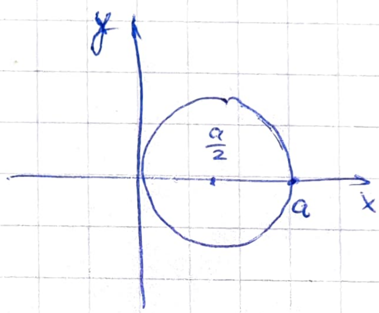
$= \frac{1}{3} \int_0^{2\pi} d\varphi\,(4^3) = \frac{4^3}{3}\, 2\pi = \frac{128}{3}\pi$



4) $\iint\limits_D (x^2+y^2)\,dxdy$

D: $x^2+y^2 = ax$

$a > 0$



I: D – ?

$x^2 - ax + y^2 = 0$

$x^2 - 2\cdot x \cdot \frac{a}{2} + \left(\frac{a}{2}\right)^2 - \left(\frac{a^2}{2}\right)^2 - y^2 = 0$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$

II. Пред. интегр. по r, φ

1) В ур-е кривой, огр. компакт в дек. коорд вместо x, y подст. их выражение через r, φ. Получить ур-ие отношения r, φ

2) Решить ур-ие отн. r. Корни – пределы интегр. по r

3) Из усл. $r(\varphi) \geq 0$ найти пределы инт. по φ

1) $x^2+y^2=ax \qquad a>0$

$r^2 = a \cdot r \cdot \cos\varphi$

2) $r=0 \qquad r=a\cos\varphi$

$r(\varphi)=0 \qquad\qquad r(\varphi)=a\cos\varphi$

$\Rightarrow r(\varphi)\geq 0 \qquad r(\varphi)\geq 0$
$\forall\varphi$

$a\cos\varphi \geq 0$

$$\boxed{-\frac{\pi}{2}\leq\varphi\leq\frac{\pi}{2}}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} r\,dr\,r^2 = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} r^2 d(r^2) =$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\, r^4\Big|_0^{a\cos\varphi} = \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\left((a\cos\varphi)^4-0\right) =$$

$$= \frac{a^4}{4}\int_{-\pi/2}^{\pi/2} d\varphi\, \underbrace{\cos^4\varphi}_{=(\cos^2\varphi)^2} = \frac{a^4}{4}\left(\frac{1}{2}\right)^2\int_{-\pi/2}^{\pi/2} d\varphi\,(1+\cos 2\varphi)^2 =$$

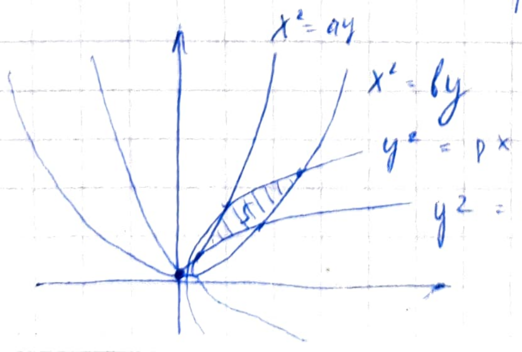
$$= \frac{a^4}{16}\int_{-\pi/2}^{\pi/2} d\varphi\,(1+\underbrace{2\cos 2\varphi}_{\to 0}+\cos^2 2\varphi) =$$

$$= \frac{a^4}{16}\int_{-\pi/2}^{\pi/2}\left(1+\frac{1}{2}+\underbrace{\frac{1}{2}\cos 4\varphi}_{\to 0}\right)d\varphi$$

$$= \frac{a^4}{16}\cdot\frac{3}{2}\cdot\pi = \frac{3a^2\pi}{32}$$

3) $D: x^2=ay \qquad y^2=px$
$x^2=by \qquad y^2=qx$

$0<a<b \qquad 0<p<q$



$$\begin{cases} x^2/y = u & u \in [a; b] \\ y^2/x = v & v \in [p; q] \end{cases}$$

$$\Leftrightarrow \begin{cases} x = u^{2/3} v^{1/3} \\ y = u^{1/3} v^{2/3} \end{cases}$$

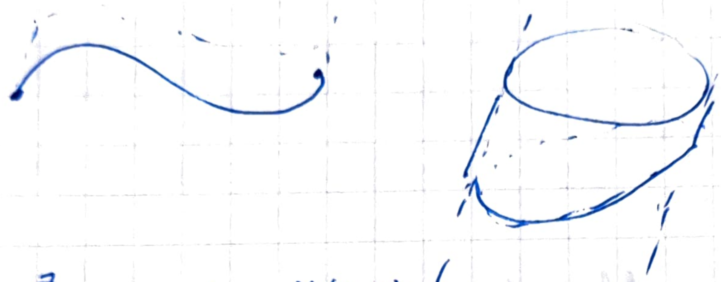
$$|J| = \begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \frac{1}{3}$$

$$\iint_D 1 \cdot dx\,dy = \int_a^b du \int_p^q dv \cdot \frac{1}{3} = \frac{1}{3} \iint_{\substack{a \le u \le b \\ p \le v \le q}} du\,dv$$

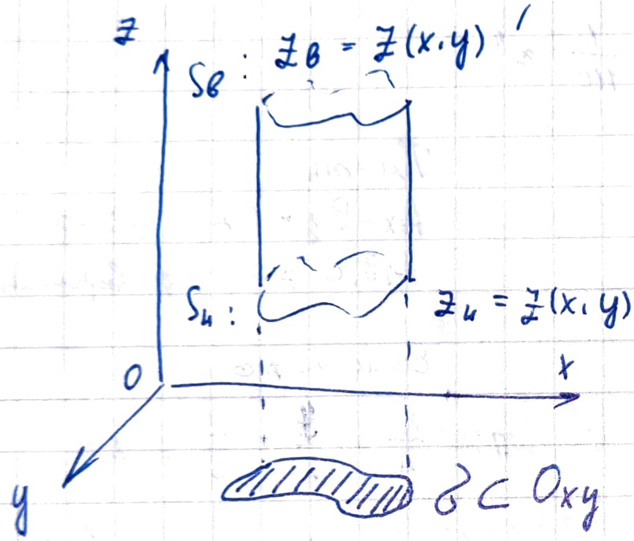
Цилиндр

Пусть l – нек. кривая, кот. лежит в п-ти π

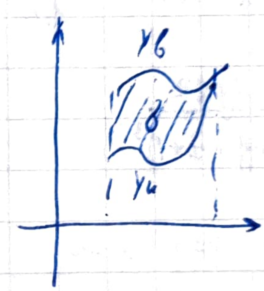
Пусть l – прямая и $l \cap \pi$ в 1 точке <...>



часть цилиндра – цилиндроид



проекция – компакт



$$\iiint\limits_M f(x,y,z)\,dx\,dy\,dz =$$

$$\iint\limits_\sigma dx\,dy \int\limits_{z_н(x,y)}^{z_в(x,y)} f(x,y,z)\,dz =$$

$$\int_a^b dx \int_{y_н(x)}^{y_в(x)} dy \int_{z_н(x,y)}^{z_в(x,y)} f(x,y,z)\,dz$$

1) $\int_0^a dx \int_0^x dy \int_0^{xy} x^3 y^2 z \, dz =$

$= \int_0^a x^3 dx \int_0^x y^2 dy \int_0^{xy} z \, dz =$

$= \frac{1}{2} \int_0^a x^3 dx \int_0^x y^2 dy \; z^2 \Big|_0^{xy} =$

$= \frac{1}{2} \int_0^a x^3 dx \int_0^x y^2 dy \, (x^2 y^2 - 0) =$

$= \frac{1}{2} \int_0^a x^5 dx \int_0^x y^4 dy =$

$= \frac{1}{10} \int_0^a x^5 dx \; y^5 \Big|_0^x =$

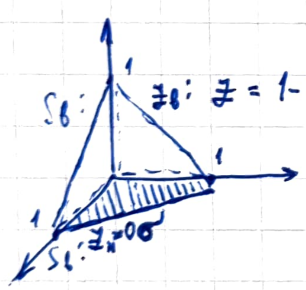
$= \frac{1}{10} \int_0^a x^5 dx \, (\cancel{y} x^5 - 0) =$

$= \frac{1}{10} \int_0^a x^{10} dx = \frac{1}{110} x^{11} \Big|_0^a = \frac{1}{110} a^{11}$

2) $\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$

$\Omega: \; x=0 \quad y=0 \quad z=0$

$x+y+z=1$



Пл-ть

$Ax + By + Cz + D = 0$

$A, B, C \neq 0$ – одновременно

Если $D \neq 0$

$\Pi: \frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$

– ур-е пл-ти в отрезках

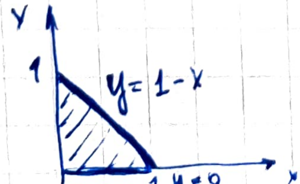
$M_1(a,0,0) \in \Pi$

$M_2(0,b,0) \in \Pi$

$M_3(0,0,c) \in \Pi$

$\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \iint\limits_{\sigma} dx\,dy \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz$

$\begin{cases} x+y+z=1 \\ z=0 \end{cases} \Leftrightarrow \begin{cases} x+y=1 \\ z=0 \end{cases}$



$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{1}{(x+y+z+1)^3} dz =$$

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (x+y+z+1)^{-3} d(x+y+z+1) =$$

$$\frac{1}{2}\int_0^1 dx \int_0^{1-x} dy \, (x+y+z+1)^{-2}\Big|_0^{1-x-y} =$$

$$-\frac{1}{2}\int_0^1 dx \int_0^{1-x} dy\left((\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2} - (x+y+1)^{-2}\right) =$$

$$-\frac{1}{2}\int_0^1 dx \int_0^{1-x} dy \left(\frac{1}{4} - \frac{1}{(x+y+1)^2}\right) =$$

$$-\frac{1}{8}\int_0^1 dx \int_0^{1-x} dy + \frac{1}{2}\int_0^1 dx \int_0^{1-x} \frac{1}{(x+y+1)^2} dy =$$

$$-\frac{1}{8}\iint_{G} 1\cdot dx\,dy + \frac{1}{2}\int_0^1 dx \int_0^{1-x} (x+y+1)^{-2} d(x+y+1) =$$

$$= -\frac{1}{8}\cdot\frac{1}{2} + \frac{-1}{2}\int_0^1 dx \,(x+y+1)^{-1}\Big|_0^{1-x} =$$

$$= -\frac{1}{16} - \frac{1}{2}\int_0^1 dx\left((\cancel{x}+1-\cancel{x}+1)^{-1} - (x+1)^{-1}\right) =$$

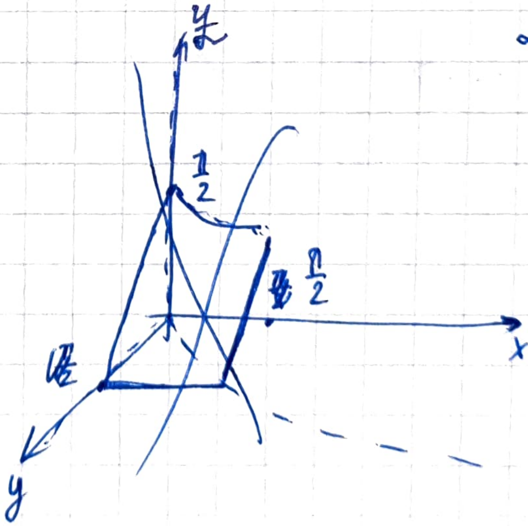
$$= -\frac{1}{16} - \frac{1}{2}\int_0^1 dx \left(\frac{1}{2} - \frac{1}{x+1}\right) =$$

$$= -\frac{1}{16} - \frac{1}{2}\cdot\frac{1}{2}\int_0^1 dx + \frac{1}{2}\int_0^1 \frac{1}{x+1} dx =$$

$$= -\frac{1}{16} - \frac{1}{4} + \frac{1}{2}\ln|x+1|\Big|_0^1 = -\frac{5}{16} + \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1 = \frac{1}{2}\ln 2 - \frac{5}{16}$$

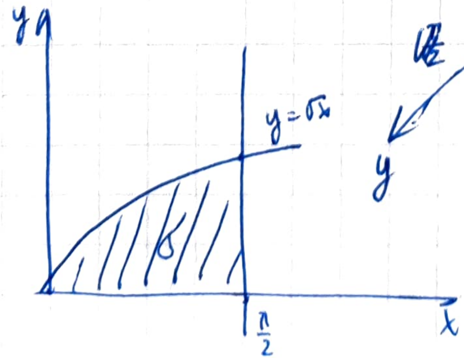
3) $$\iiint_{\Omega} y\cos(x+z)\,dx\,dy\,dz$$

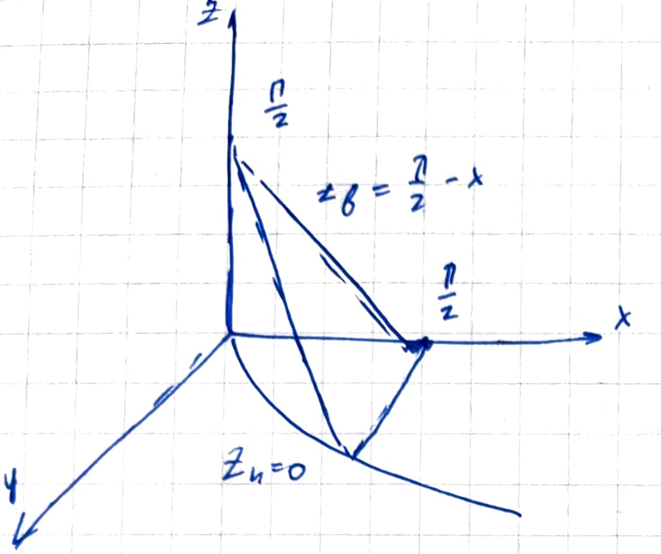
Ω: $y=\sqrt{x}$
$y=0$
$z=0$
$x+z=\frac{\pi}{2}$



$\alpha \cap Ozy$ $\begin{cases} x+z=\frac{\pi}{2} \\ x=0 \\ y=0 \end{cases} \Leftrightarrow \begin{cases} z=\frac{\pi}{2} \\ x=0 \\ y=0 \end{cases}$

$\alpha \cap Oxy$ $\begin{cases} x+z=\frac{\pi}{2} \\ z=0 \end{cases} \Leftrightarrow \begin{cases} x=\frac{\pi}{2} \\ z=0 \end{cases}$





$$\iiint\limits_{\Omega} y\cos(x+z)\,dx\,dy\,dz = \iint\limits_{\sigma} dx\,dy \int_0^{\frac{\pi}{2}-x} y\cos(x+y)\,dz =$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} y\,dy \int_0^{\frac{\pi}{2}-x} \cos(x+\cancel{y})\,d(z+x) =$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} y\,dy \ \sin(x+\cancel{y})\Big|_0^{\frac{\pi}{2}-x} =$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} y\,dy \ \left(\sin\left(\frac{\pi}{2}\right) - \sin x\right) =$$

$$= \int_0^{\frac{\pi}{2}} dx \int_0^{\sqrt{x}} y\,dy \ - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cancel{dx}\,\sin x\,dx \ y^2\Big|_0^{\sqrt{x}} =$$

$$\cancel{= \frac{1}{2}\int_0^{\frac{\pi}{2}} dx\ y^2\Big|_0^{\sqrt{x}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\,dx\ y^2\Big|_0^{\sqrt{x}} =}$$

$$\cancel{= \frac{1}{2}\int_0^{\frac{\pi}{2}} dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin x\ y^2\Big|_0^{\sqrt{x}}}$$

$$\cancel{= \frac{1}{2}\left(\frac{\pi}{2} - \right.}$$

< . . . >

Цилиндр. система координат

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad |J| = r$$

$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int\limits_{\pi/4}^{\pi/3} d\varphi \int\limits_0^2 r\,dr \int\limits_0^1 dz\, f(r\cos\varphi, r\sin\varphi, z)$$

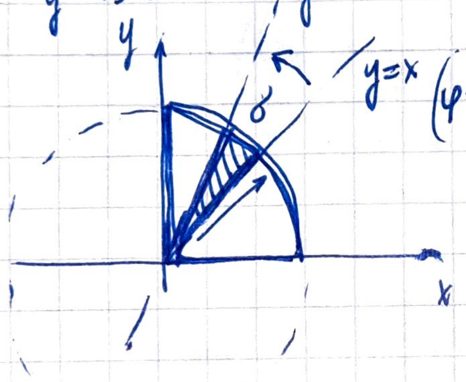
Ω : часть 1 окт.

$x^2+y^2=2^2$

$z=0,\ z=1$

$y=x$

$y=\sqrt{3}x$ $\left(\varphi = \operatorname{arctg}\sqrt{3} = \frac{\pi}{3}\right)$



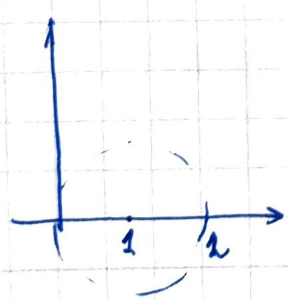
$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{2\cos\varphi} r\,dr \int\limits_0^{r^2} dz\, f(r\cos\varphi, r\sin\varphi, z)$$

Ω : $x^2+y^2=2x$

$z=0$

$z=x^2+y^2$

1) $(x-1)^2+y^2=1$



2) $r^2 = 2r\cos\varphi$

$r=0 \ \vee\ r=2\cos\varphi$

$\varphi \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

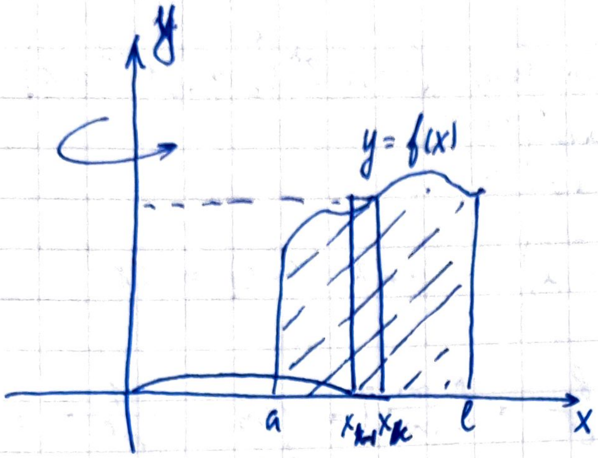
Контрольная работа №2 в 14:00 6 июня

1 прим. опр. интеграла

2 несобств. интегралы

1. двойной инт.

1 тройной инт.



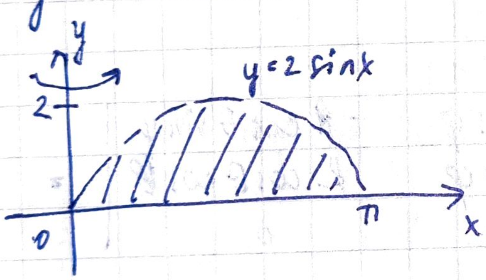
$$\pi x_k^2 \cdot f(\xi_k) - \pi x_{k-1}^2 f(\xi_k) =$$

$$= \pi f(\xi_k) \underbrace{(x_{k-1} + x_k)}\underbrace{(-x_{k-1} + x_k)}$$

$$\Delta x_k \to 0 : \quad 2x_k - \underbrace{\Delta x_k}_{\to 0} \quad \Delta x_k \longrightarrow \pi \int_a^b f(x) 2x\, dx$$

$$= 2\pi \int_a^b x\, y(x)\, dx$$

1) $y = 2\sin x \qquad 0 \le x \le \pi$

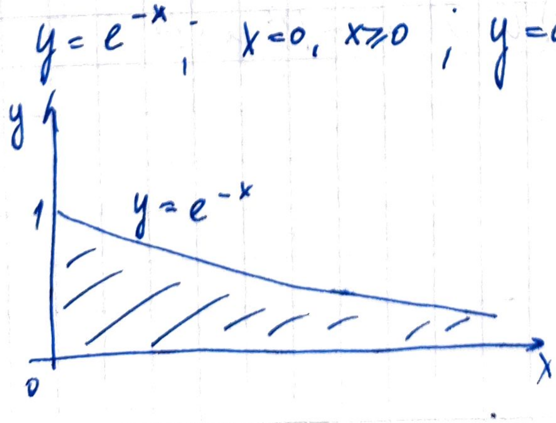


$$V_y = 2\pi \int_0^\pi x \cdot 2\sin x\, dx = -4\pi \int_0^\pi x\, d\cos x =$$

$$= -4\pi \cdot x\cos x \Big|_0^\pi + 4\pi \int_0^\pi \cos x\, dx =$$

$$= -4\pi \cdot (-\pi) + 4\pi \underbrace{\sin x \Big|_0^\pi}_{=0} = 4\pi^2$$

2) $y = e^{-x}; \quad x = 0, \ x \ge 0; \quad y = 0$



$$V_y = 2\pi \int_0^{+\infty} x \cdot e^{-x} dx = 2\pi \lim_{b \to +\infty} \left(\int_0^b x \cdot e^{-x} dx \right) =$$

$$= 2\pi \lim_{b \to +\infty} \left(-\int_0^b x \, d(e^{-x}) \right) = 2\pi \lim_{b \to +\infty} \left(-x \cdot e^{-x} \Big|_0^b + \int_0^b e^{-x} dx \right) =$$

$$= 2\pi \lim_{b \to +\infty} \left(-b \cdot e^{-b} - e^{-x} \Big|_0^b \right) =$$

$$= -2\pi \lim_{b \to +\infty} \left(\underset{\infty}{b} \cdot \underset{0}{e^{-b}} + \underset{0}{e^{-b}} - 1 \right) = -2\pi \cdot (-1) = 2\pi$$

$$\lim_{b \to +\infty} (b \cdot e^{-b}) = [\infty \cdot 0] = \lim_{b \to +\infty} \frac{b}{e^b} = \left[\frac{\infty}{\infty} \right] = \lim_{b \to +\infty} \frac{1}{e^b} = 0$$

Q – "широта" $\quad \varphi$ – долгота

max $\quad \theta \in \left[-\frac{\pi}{2} ; \frac{\pi}{2} \right]$

$\varphi \in [0 ; 2\pi]$

$$\begin{cases} x = R \cos\theta \cos\varphi \\ y = R \cos\theta \sin\varphi \\ z = R \sin\theta \end{cases}$$

$$|J| = \begin{vmatrix} \cos\theta \cdot \cos\varphi & -R \sin\theta \cos\varphi & -R \cos\theta \sin\varphi \\ \cos\theta \cdot \sin\varphi & -R \sin\theta \sin\varphi & R \cos\theta \cos\varphi \\ \sin\theta & R\cos\theta & 0 \end{vmatrix} =$$

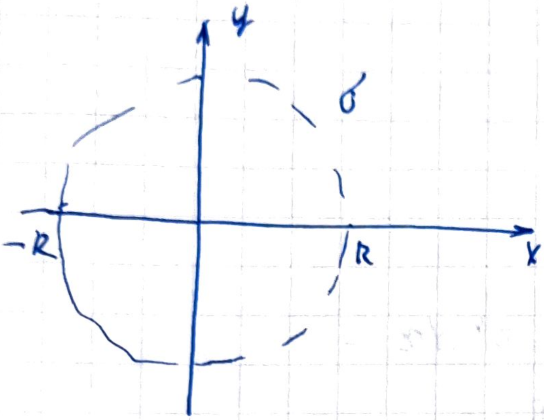
$$= R^2 (-\cos^3\theta \sin^2\varphi - \cos\theta \sin^2\varphi \cos^2\varphi) - (\sin^2\varphi \cos\theta \sin^2\varphi + \cos^3\theta \cos^2\varphi) =$$

$$= -R^2 \left(\cos^3\theta (\sin^2\varphi + \cos^2\varphi) + \cos\theta \sin^2\varphi (\sin^2\varphi + \cos^2\varphi) \right) =$$

$$= -R^2 (\cos\theta + \sin^2\varphi) \cos\theta = -R^2 \cos\theta$$

$$\begin{cases} |J| = R^2 \cos\theta \\ x^2 + y^2 = R^2 \cos^2\theta \\ x^2 + y^2 + z = R^2 \end{cases}$$

3) $\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\, dz =$



z : от $z=0$ до $z=\sqrt{R^2-x^2-y^2}$ $\rightarrow$ сфера

M - верхнее полушарие:

φ от 0 до 2π

θ от 0 до $\frac{\pi}{2}$

r от 0 до R

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^R r^2 \cos\theta \, dr \cdot (r^2 \cos^2\theta)$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta \int_0^R r^4 \, dr =$$

$$= \frac{1}{5} \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta \cdot r^5 \Big|_0^R =$$

$$= \frac{1}{5} \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta \cdot R^5 =$$

$$= \frac{1}{5} R^5 \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \underbrace{\cos^2\theta}_{=(1-\sin^2\theta)} \, d \sin\theta =$$

$$= \frac{1}{5} R^5 \int_0^{2\pi} d\varphi \left(\sin\theta - \frac{1}{3}\sin^3\theta\right) \Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{5} R^5 \int_0^{2\pi} d\varphi \left(1 - \frac{1}{3}\right)$$

$$= \frac{2}{15} R^5 \cdot 2\pi = \frac{4\pi}{15} R^5$$

$$\int_a^b dx \int_c^d dy \int_p^q f_1(x) f_2(y) f_3(z) \, dz$$

$$= \underbrace{\int_a^b f_1(x)\, dx}_{C_3} \underbrace{\int_c^d f_2(y)\, dy}_{C_2} \underbrace{\int_p^q f_3(z)\, dz}_{C_1} = \underbrace{C_1 \cdot C_2 \cdot C_3}_{const}$$

4) $\iiint\limits_{\Omega} (x^2+y^2)\,dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_{\rho}^{R} r^2\cos^2\theta \, dr \, (r^2\cos^2\theta) \;(=)$

$\Omega: \; z \geq 0$

$\rho^2 \leq x^2+y^2+z^2 \leq R^2$

$(=) \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\theta \cos^3\theta \int_{\rho}^{R} r^4 \, dr =$

$= \int_0^{2\pi} d\varphi \cdot \int_0^{\frac{\pi}{2}} \cos^3\theta \, d\theta \cdot \int_{\rho}^{R} r^4 \, dr =$

$= 2\pi \cdot \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)\, d\sin\theta \cdot \frac{1}{5} r^5 \Big|_{\rho}^{R} =$

$= \frac{2}{5}\pi \left(\sin\theta - \frac{1}{3}\sin^3\theta\right)\Big|_0^{\frac{\pi}{2}} (R^5-\rho^5) =$

$= \frac{2}{5}\pi \cdot \frac{2}{3} \cdot (R^5-\rho^5) = \frac{4\pi}{15}(R^5-\rho^5)$

5) $\iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\, dx\,dy\,dz \;(=)$

Ω огр : $x^2+y^2+z^2 = x$

(I) $\Omega: \; x^2+y^2+z^2 \leq x \qquad x^2 - 2\cdot\frac{1}{2}x + \left(\frac{1}{2}\right)^2 + y^2+z^2 = 0$

$\left(\begin{array}{l} x^2+y^2+z^2 = R^2 \\ x^2+y^2+z^2 < R^2 \end{array} \right) \qquad \left(x-\frac{1}{2}\right)^2 + y^2 + z^2 = \left(\frac{1}{2}\right)^2$

(II) $\sqrt{x^2+y^2+z^2} = x \iff r^2 = r\cos\theta\cos\varphi$

$\underset{(1)}{r=0} \;\vee\; \underset{(2)}{r=\cos\theta\cos\varphi}$, $\cos\theta \geq 0$, $\cos\varphi \geq 0$

$\forall \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$\forall \varphi \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$(=) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{\cos\theta\cos\varphi} r^2\cos\theta \, dr \cdot r =$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \cos\theta \int_0 r^3 \, dr =$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \, \frac{1}{4} r^4 \Big|_0^{\cos\theta\cos\varphi} =$

$$= \frac{1}{4}\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi \, d\varphi}_{I_1} \cdot \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\theta \, d\theta}_{I_2} = \frac{1}{4} \cdot \frac{\cancel{3}\pi}{\cancel{8}} \cdot \frac{\overset{2}{\cancel{16}}}{\underset{5}{\cancel{15}}} = \frac{\pi}{10}$$

$$\cancel{= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi \, d\varphi}$$

$$I_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi \, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^2\varphi)^2 \, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\varphi}{2}\right)^2 d\varphi =$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + 2\cos 2\varphi + \cos^2 2\varphi) \, d\varphi =$$

$$= \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big(1 + \underbrace{2\cos 2\varphi}_{\Downarrow \; 0 \; (T=\pi)} + \frac{1}{2} + \frac{1}{2}\underbrace{\cos 4\varphi}_{\Downarrow \; 0 \; (T=\frac{\pi}{2})}\Big) \, d\varphi = \frac{1}{4} \cdot \frac{3\pi}{8}$$

$$I_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\theta \, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta \, d\sin\theta =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2\theta)^2 \, d\sin\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta + \sin^4\theta) \, d\sin\theta$$

$$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} =$$

$$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) \cdot 2 = \frac{8 \cdot 2}{15} = \frac{16}{15}$$

$r = \sin k\varphi$ $\quad$ k - леп. роза $\quad S = \frac{\pi a^2}{4}$

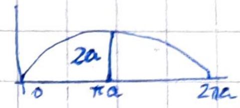
$r = \frac{k}{\sin\varphi}$ $\quad y = k$

$r = \varphi \quad r = \frac{1}{\varphi} \quad r = e^{\varphi}$ - спирали

$r = 1 + \cos\varphi$ - кардиоида

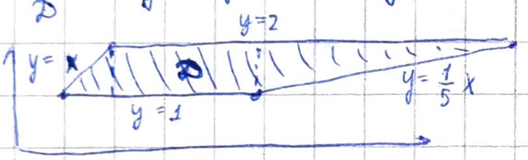
$r^2 = a^2\cos 2\varphi$ - лемниската $\quad S = a^2$

$\begin{cases} x = a(\varphi - \sin\varphi) \\ y = a(1-\cos\varphi) \end{cases}$ циклоида $\quad S_{арки} = 3\pi a^2$, $\quad l_{арки} = 8a$



$S_{0x} = \pi\int_a^b (f(x))^2\,dx \qquad \boxed{S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2}(r(\varphi))^2\,d\varphi}$

3) $\iint\limits_D \sqrt{xy - y^2}\,dxdy = \int_1^2 dx\int_1^x f(x)\,dy + \int_2^5 dx\int_1^2 f(x)\,dy + \int_5^{10} dx\int_{\frac{x}{5}}^2 f(x)\,dy$



$= \int_1^2 dy\int_y^{5y} f(x)\,dx =$

$= \int_1^2 dy\int_y^{5y}\sqrt{xy-y^2}\,dx =$

$= \int_1^2 dy\,\sqrt{y}\int_y^{5y}\sqrt{x-y}\,d(x-y) =$

$= \int_1^2 \sqrt{y}\,dy\,\frac{2}{3}(x-y)^{\frac{3}{2}}\Big|_y^{5y} = \frac{2}{3}\int_1^2\sqrt{y}\,dy\,\left((2\sqrt{y})^3 - 0\right) = \frac{2}{3}\int_1^2 2^3\cdot y^2\,dy =$

$= \frac{16}{3}\,y^3\Big|_1^2 = \frac{16}{9}(8-1) = \frac{112}{9}$

3) $\iint\limits_S f(x,y)\,dxdy = \int_0^{\frac{\pi}{2}} d\varphi\int_0^{\sqrt{\sin 2\varphi}} r\cdot f\cdot dr$

$S: (x^2+y^2)^2 \le 2xy \qquad 0 \le r^4 \le \underline{2r^2\sin\varphi\cos\varphi = r^2\sin 2\varphi}$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $\qquad r^2 = 0 \;\vee\; \sin 2\varphi = r^2$

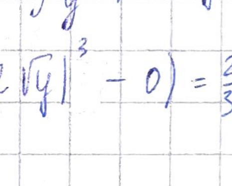
$\Downarrow \qquad r = \pm\sqrt{\sin 2\varphi}, \; r \ge 0 \Rightarrow r = \sqrt{\sin 2\varphi}$

$r = 0$

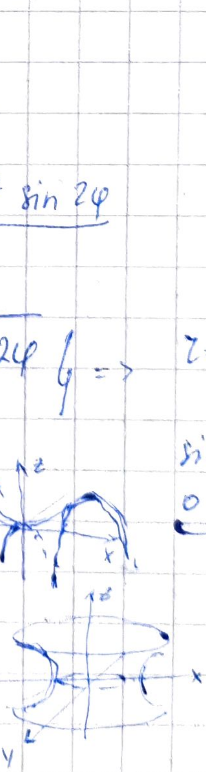
$\sin 2\varphi \ge 0$

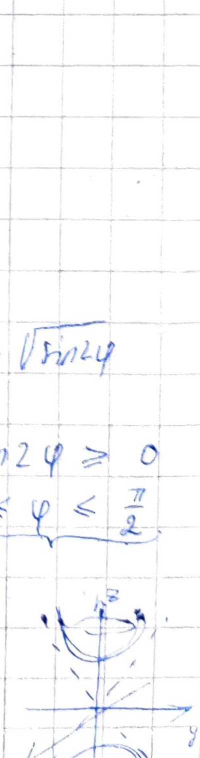
$0 \le \varphi \le \frac{\pi}{2}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ гиперб. параболоид (седло)



$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ однопол. гиперб.





$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$ двупол. гиперб.

4) $\iiint_\Omega xyz^2\,dx\,dy\,dz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^1 r^2\cos\theta \cdot f \, dr$

$x^2+y^2+z^2=1$

$x \geq 0$ $\qquad \cos\varphi\cos\theta \geq 0 \qquad 0 \leq \varphi \leq \frac{\pi}{2} \Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$

$y \geq 0$ $\qquad \sin\varphi\cos\theta \geq 0$

$z \geq 0$ $\qquad \sin\theta \geq 0 \qquad (0 \leq \theta \leq \pi)$?

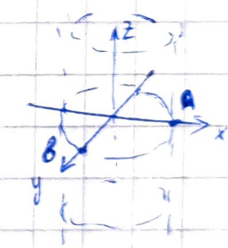
$$\begin{cases} x = r\cos\varphi\cos\theta \\ y = r\sin\varphi\cos\theta \\ z = r \cdot \sin\theta \end{cases} \qquad |J| = r^2\cos\theta$$

$r^2\cos^2\varphi\cos^2\theta + r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\theta = 1$

$r^2(\cos^2\theta+\sin^2\theta) = 1 \qquad \left.\begin{matrix} r \geq 0 \\ r = \pm 1\end{matrix}\right\} \Rightarrow r = 1$

$$\begin{cases} Ax+By+Cz+D=0 & \text{— ур-е плоскости } F(x,y,z)=0 \\ z = -\frac{A}{C}x - \frac{B}{C}y - \frac{D}{C} \quad (C \neq 0) & \text{— } z = f(x,y) \end{cases}$$

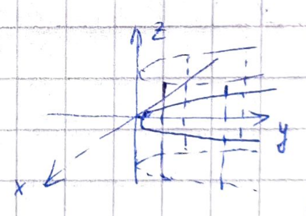
$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ $(a,b,c \neq 0)$ — ур-е плоскости в отрезках

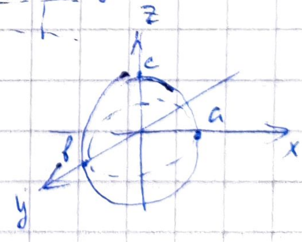


$\frac{x^2}{a^2}+\frac{y^2}{b^2}+0\cdot z = 1$ — эллиптический цилиндр

$\leq$ (труба)

$0\cdot z + y = x^2$ — парабола

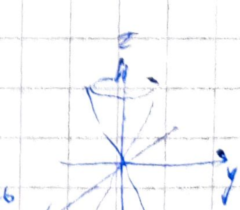




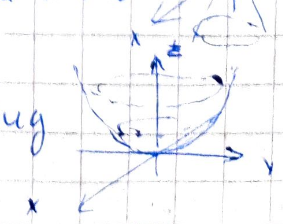
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ — эллипсоид (п-ть)

$\leq$ — эллипсоид (тело)

$$\begin{cases} x^2+y^2+z^2=4 \\ z^2 = 4-x^2-y^2 \\ z = \sqrt{4-x^2-y^2} & \text{верх. полусф.} \\ z = -\sqrt{4-x^2-y^2} & \text{нижн. полусф.} \end{cases}$$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2} = 0$ $(a,b,c>0)$ — коническая п-ть



$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 2z$ $(a,b>0)$ — эллипт. параболоид