Stochastic Solow Model w/BM

$F(K_t, L_t) = A K_t^{\alpha} L_t^{1-\alpha}$

$f(k_t) = F(k_t, 1) = y$, where $f'(k_t) > 0$, $f''(k_t) < 0$

s.t. $k_t = \frac{K_t}{L_t}$ $\qquad f(k_t) > k_t \cdot f'(k_t)$

+ Inada conditions:

$f(0) = f'(\infty) = 0$

$f'(0) = f(\infty) = \infty$

$\dot{K}_t = s Y_t - \delta K_t$, $\dot{K}_t = \frac{dK_t}{dt}$

$\dot{L}_t = n L_t$, $\dot{L}_t = \frac{dL_t}{dt}$

$\dot{k}_t = s f(k_t) - (n+\delta) k_t$

$$\dot{k}_t = \frac{dk_t}{dt} = \frac{d(K_t/L_t)}{dt} = \frac{(sY_t - \delta K_t)L_t dt - nL_t \cdot K_t dt}{L_t^2 \cdot dt} =$$

$$= s\frac{Y_t}{L_t} - \delta \frac{K_t}{L_t} - n\frac{K_t}{L_t} = s y_t - (\delta + n) k_t$$

$dK_t = (sY_t - \delta K_t) dt + K_t dB_t^K$

$dL_t = n L_t dt + L_t dB_t^L$

, where B_t^K, B_t^L – BM

$dB_t^k \sim N(0; \sigma_k^2 dt)$

$dB_t^L \sim N(0; \sigma_L^2 dt)$

$cov(dB_t^k; dB_t^L) = \sigma_{KL} dt$

$dk_t = \left(s f(k_t) - \mu k_t\right) dt + k_t dB_t$ (by Ito's Lemma)

where $\mu = n + \delta + \sigma_{KL} - \sigma_L^2$

$dB_t = dB_t^k - dB_t^L$

$$\boxed{\begin{aligned} &2F''_{xy}\, dx\, dy = \dots \\ &dX = A^x du + B^x dw^x \\ &dY = A^y du + B^y dw^y \\ &dx dy = B^x B^y \underbrace{dw^x dw^y}_{\rho \cdot dt} \end{aligned}}$$

$$dk_t = F'_t dt + F'_K dK_t + F'_L dL_t +$$

$$+ \frac{1}{2}\Big(F''_{tt}\underbrace{(dt)^2}_{\to 0} + F''_{KK}\underbrace{(dK)^2}_{\to dt \cdot K_t^2 \cdot \sigma_K^2} + F''_{LL}\underbrace{(dL)^2}_{\to dt \cdot L_t^2 \cdot \sigma_L^2} + 2F''_{tK}\underbrace{dt\, dK}_{\to 0} + 2F''_{tL}\underbrace{dt\, dL}_{\to 0} + 2F''_{KL}\underbrace{dK dL}_{\to \sigma_{KL} dt \cdot K_t \cdot L_t}\Big)$$

$$= F'_t dt + F'_K dK_t + F'_L dL_t + \frac{1}{2}F''_{KK}\sigma_K^2 K_t^2 dt + \frac{1}{2}F''_{LL}\sigma_L^2 L_t^2 dt + F''_{KL} K_t \cdot L_t\, \sigma_{KL} dt$$

$$= 0 + \frac{1}{L_t}\left((sY_t - \delta K_t)dt + K_t dB_t^K\right) + \left(-\frac{K_t}{L_t^2}\right)\cdot\left(nL_t dt + L_t dB_t^L\right)$$

$$+ 0 + \frac{1}{2}2\cdot\frac{K_t}{L_t^3}\sigma_L^2 L_t^2\, dt + \left(-\frac{1}{L_t^2}\right)\cdot K_t \cdot L_t \cdot \sigma_{KL}\, dt$$

$\displaystyle \sim\sim = sf(k_t)\,dt - \delta k_t\,dt + \underbrace{k_t\,dB_t^k} - n \cdot k_t\,dt - \underbrace{k_t\,dB_t^L}$
$+ \sigma_L^2 k_t\,dt - \sigma_{kL} k_t\,dt = \left(sf(k_t) - (\delta + n + \sigma_{kL} - \sigma_L^2) k_t\right) dt$
$+ k_t (dB_t^k - dB_t^L)$

① $dk_t = (sf(k_t) - \mu k_t)\,dt + k_t\,dB_t$

(k_t) – homogenous diffusion process

$sf(k_t) - \mu k_t$ – drift coef.

$k_t^2(\sigma_k^2 - 2\sigma_{kL} + \sigma_L^2)$ – diffusion coef.

② $s > t \quad \forall (a; b)$

$\mathbb{P}(a < k_s < b \mid k_t = k) = \int_a^b p(s-t, k, y)\,dy$

③ $\psi_k := E\left(\frac{dk_t}{k}\right) = \frac{sf(k_t)}{k} - \mu = \frac{sf(k_t)}{k} - (n + \delta + \sigma_{kL} - \sigma_L^2)$

$g_k = \frac{\dot{k}_t}{k} = \frac{sf(k_t)}{k} - (n + \delta)$ ← determenistic growth rate (W/O st. disturbance)

$\psi_k - g_k = \sigma_{kL} - \sigma_L^2$

$\sigma_{kL} > \sigma_L^2 \Rightarrow$ st. disturb. can raise the growth

$\sigma_{kL} < \sigma_L^2 \Leftrightarrow \sigma_{kL} < \sigma_k \sigma_L < \sigma_L \Leftrightarrow$ magn of L_t is bigger than k_t

$\Rightarrow \psi_k < g_k$

$\sigma_L^2 > 0, \sigma_k^2 = 0$ (*) L_t is disturbed, k_t - is not $\Rightarrow$ growth rate ↓

$\sigma_L^2 = 0, \sigma_k^2 > 0$ (*) growth rate will remain the same

$\Rightarrow$ L_t has bigger impact than k_t

Stoch. Solow w/ BM. Stability analysis

Steady state: various quantities grow at constant rate

(Det. Solow) $\dot{k}_t = 0$ $\Leftrightarrow$ $sf(k_t^*) = (n+\delta)k_t^*$, $t \to 0$, $k_t \to k^*$

$k_t \equiv k_t^*$ - asymp stable in $(0; \infty)$ globally

A "Jump" in the Stochasticity of The Solow-Swan Growth Model
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1 Introduction

актуальность

Solow-Swan (1956) economic growth model has been extensively celebrated in economic growth literature, the properties and extension of which have been intuitively investigated in, for instance, Barro and Sala-i-Martin (2004). A number of recent research (e.g., Binder and Pesaran, 1999; Stachurski, 2002; Barossi-Filho et al., 2005) have also focused on modelling this growth model in a stochastic environment. The main motivation of the need for a stochastic version of Solow-Swan model arises due to the necessity of reproducing the statistical characteristics of business cycle fluctuations in actual economies. Indeed, this approach to growth modelling in economics has had a tremendous impact on the way we think about the analysis of effects of the different exogenous shocks in an economy. It is possible, by using the modelling approach to analyze which among the possible shocks is more likely to produce a given statistical characteristic of the solution, or which one is more useful in order for a model to replicate a given statistical regularity observed in actual time series data. In addition, one can characterize the role of economic policy in determining the dynamics of relevant variables, as well as the co-movements between them.

Following on the leads of the recent literature, in this note we aim to expand the existing work by introducing a Jump process in the growth of the main determinants of income. Our broad idea is to investigate the stability of the economic growth system under stochastic disturbances when these disturbance bring sudden changes in the growth process. As is well-known the Solow-Swan model is an exogenous growth model, an economic model of long-run economic growth set within the framework of neoclassical economics (see [3], pp26). It attempts to explain long-run economic growth by looking at capital accumulation, labor or population growth, and increases in productivity, commonly referred to as technological progress. The key assumption of the neoclassical growth model is that capital is subject to diminishing returns in a closed economy. More specifically, it incudes two aspects: The total output $Y(t)$ is determined by the overall input; The factor accumulation does not depend on decisions of economic entities. The main difference with the deterministic model is: we assume that the factor accumulation is affected by some stochastic disturbance such that the Ordinary differential equation (ODE) can be transformed into the Stochastic differential equation (SDE). Then we can do the stability analysis for the stochastic system.

Moreno et al. (2011) discuss the usefulness of jump process to reflect how economic variables respond to the arrival of sudden information. While analyzing the dynamics of the model, the authors find that the degree of serial autocorrelation is related to the occurrence and magnitude of abnormal information. In addition, the authors provide some useful approximations in a particular case that considers exponential-type decay. Empirically, the authors propose a GMM approach to estimate the parameters of the model and present an empirical application for the stocks included in the Dow Jones Averaged Index. Our findings seem to confirm the presence of shot-noise effects in 73% of the stocks and a strong relationship between the shot-noise process and the autocorrelation pattern embedded in data.

Jumps were initially analyzed by Merton (1976) for modeling the arrival of uncommon information at financial markets, introducing discontinuities in the stock charts. Jumps account adequately situations such as, for example, the sudden reaction of stock prices to unexpected news about a company, the consequences of extreme fluctuations in supply and demand in electricity markets, or the failure and thus abandonment of a R&D firms investment project, see Pennings and Sereno (2011). Not surprisingly, the empirical evidence about jumps is vast, see Andersen et al. (2002), Eraker et al. (2003), and Escribano et al. (2011).

The rest of the paper is planned as follows. In Section 2, we characterize Solow-Swan model by Brownian Motion. Section 3 introduces Jump process in Solow-Swan framework, whereby stability analysis and properties of the determinants of growth are discussed. Section 4 investigates stationary features of capital stock. Finally, Section 5 concludes with main findings.

2 Stochastic Solow Model with the Brownian Motion

2.1 Construct

We initially consider the neoclassical production function. A neoclassical production function $F(K_t, L_t)$ satisfies in the general the following properties:

1. The function $F(\cdot)$ is a linear homogenous function;
2. Let $f(k_t) = F(k_t, 1) = y$, where $k_t = \frac{K_t}{L_t}$, then $f(k_t)$ satisfies
$$f'(k_t) > 0, f''(k_t) < 0, f(k_t) > k_t f'(k_t); \tag{2.1}$$
3. Inada condition:
$$f(0) = f'(\infty) = 0, \quad f'(0) = f(\infty) = \infty, \tag{2.2}$$

where Y_t is the flow of output produced at time t, K_t is the physical capital, such as machines, building, pencils, etc.; L_t represents the labor, including the number of workers and the amount of time they work, as well as their physical strength, skills, and health; k_t is the per capital capital; y_t is output per worker.

Let $F(\cdot)$ be the Cobb-Douglas function, then

$$F(K_t, L_t) = AK_t^{\alpha} L_t^{\beta},$$

where $A > 0$ is the level of technology, $\alpha \in (0, 1)$, $\beta = 1 - \alpha$.

In the deterministic model, the net increase in the stock of physical capital at a point in time equals gross investment less depreciation:

$$\dot{K}_t = sY_t - \delta K_t, \quad L_t = nL_t \tag{2.3}$$