Problem 1

The Cobb-Douglas production function

$$f(z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2}$$

where for $l = 1, 2$, z_l is an input, and α_l is a parameter such that $\alpha_l > 0$.

1. The firm produces with decreasing returns to scale if:

$$\begin{aligned} f(\lambda z_1, \lambda z_2) &< \lambda f(z_1, z_2) \\ (\lambda z_1)^{\alpha_1} (\lambda z_2)^{\alpha_2} &< \lambda z_1^{\alpha_1} z_2^{\alpha_2} \\ \lambda^{\alpha_1+\alpha_2} z_1^{\alpha_1} z_2^{\alpha_2} &< \lambda z_1^{\alpha_1} z_2^{\alpha_2} \\ \lambda^{\alpha_1+\alpha_2} &< \lambda \\ \alpha_1 + \alpha_2 &< 1 \end{aligned}$$

   We assume that this condition holds.

2. To calculate the conditional factor demand (on the quantity produced q) we need to solve following optimization problem:

$$\begin{cases} \arg\min_{(z_1,z_2)\in\mathbb{R}^2_+} z_1 w_1 + z_2 w_2 \\ s.t. : q = z_1^{\alpha_1} z_2^{\alpha_2} \end{cases}$$

   where w_1, w_2 are the prices of factors.

   The Lagrangian for this problem:

$$\mathcal{L} = z_1 w_1 + z_2 w_2 + \lambda(q - z_1^{\alpha_1} z_2^{\alpha_2})$$

   FOC:

$$\begin{cases} \mathcal{L}'_{z_1} &= w_1 - \lambda \alpha_1 z_1^{\alpha_1 - 1} z_2^{\alpha_2} = 0 \\ \mathcal{L}'_{z_2} &= w_2 - \lambda z_1^{\alpha_1} \alpha_2 z_2^{\alpha_2 - 1} = 0 \\ \mathcal{L}'_{\lambda} &= q - z_1^{\alpha_1} z_2^{\alpha_2} = 0 \end{cases}$$

   We get:

$$\begin{aligned} \frac{w_1}{w_2} &= \frac{\alpha_1 \hat{z}_1^{\alpha_1 - 1} \hat{z}_2^{\alpha_2}}{\hat{z}_1^{\alpha_1} \alpha_2 \hat{z}_2^{\alpha_2 - 1}} = \frac{\alpha_1 \hat{z}_2}{\alpha_2 \hat{z}_1} \Rightarrow \hat{z}_2 = \frac{a_2 w_1}{a_1 w_2} \hat{z}_1 \\ q &= \hat{z}_1^{\alpha_1} \hat{z}_2^{\alpha_2} \end{aligned}$$

   That is:

$$q = \hat{z}_1^{\alpha_1} \left(\frac{a_2 w_1}{a_1 w_2} \hat{z}_1 \right)^{\alpha_2} = \hat{z}_1^{\alpha_1+\alpha_2} \left(\frac{a_2 w_1}{a_1 w_2} \right)^{\alpha_2} \quad q = \left(\frac{a_1 w_2}{a_2 w_1} \hat{z}_2 \right)^{\alpha_1} \hat{z}_2^{\alpha_2} = \hat{z}_2^{\alpha_1+\alpha_2} \left(\frac{a_1 w_2}{a_2 w_1} \right)^{\alpha_1}$$

$$\hat{z}_1 = q^{\frac{1}{a_1+a_2}} \left(\frac{a_1 w_2}{a_2 w_1} \right)^{\frac{a_2}{a_1+a_2}} \qquad \hat{z}_2 = q^{\frac{1}{a_1+a_2}} \left(\frac{a_2 w_1}{a_1 w_2} \right)^{\frac{a_1}{a_1+a_2}}$$

   where $\alpha_1 + \alpha_2 < 1$

The cost function:

$$
\begin{aligned}
C(w_1,w_2,q) &= w_1\hat{z}_1 + w_2\hat{z}_2 \\
&= w_1 q^{\frac{1}{a_1+a_2}}\left(\frac{a_1w_2}{a_2w_1}\right)^{\frac{a_2}{a_1+a_2}} + w_2 q^{\frac{1}{a_1+a_2}}\left(\frac{a_2w_1}{a_1w_2}\right)^{\frac{a_1}{a_1+a_2}} \\
&= w_1^{\frac{a_1}{a_1+a_2}} w_2^{\frac{a_2}{a_1+a_2}} q^{\frac{1}{a_1+a_2}}\left(\frac{a_1}{a_2}\right)^{\frac{a_2}{a_1+a_2}} + w_1^{\frac{a_1}{a_1+a_2}} w_2^{\frac{a_2}{a_1+a_2}} q^{\frac{1}{a_1+a_2}}\left(\frac{a_2}{a_1}\right)^{\frac{a_1}{a_1+a_2}} \\
&= w_1^{\frac{a_1}{a_1+a_2}} w_2^{\frac{a_2}{a_1+a_2}} q^{\frac{1}{a_1+a_2}}\left(\left(\frac{a_1}{a_2}\right)^{\frac{a_2}{a_1+a_2}} + \left(\frac{a_2}{a_1}\right)^{\frac{a_1}{a_1+a_2}}\right)
\end{aligned}
$$

where $\alpha_1+\alpha_2<1$

Profit function:

$$
\Pi(q) = pq - C(w_1,w_2,q) = pq - w_1^{\frac{a_1}{a_1+a_2}} w_2^{\frac{a_2}{a_1+a_2}} q^{\frac{1}{a_1+a_2}}\left(\left(\frac{a_1}{a_2}\right)^{\frac{a_2}{a_1+a_2}} + \left(\frac{a_2}{a_1}\right)^{\frac{a_1}{a_1+a_2}}\right)
$$

where $\alpha_1+\alpha_2<1$

Supply function:

$$
p(q) = MC(q) = C'_q = w_1^{\frac{a_1}{a_1+a_2}} w_2^{\frac{a_2}{a_1+a_2}} \frac{1}{a_1+a_2} q^{\frac{1-a_1-a_2}{a_1+a_2}}\left(\left(\frac{a_1}{a_2}\right)^{\frac{a_2}{a_1+a_2}} + \left(\frac{a_2}{a_1}\right)^{\frac{a_1}{a_1+a_2}}\right)
$$

where $\alpha_1+\alpha_2<1$

Problem 2

The production function

$$
f(z_1,z_2) = (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}}
$$

where $\rho \leqslant 1$

1. The producer profit maximization problem:

$$
\max \Pi(q) = pq - C(w_1,w_2,q)
$$

   where w_1, w_2 are prices of inputs respectively.

2. To compute the demand for inputs we need to solve following optimization problem:

$$
\begin{cases}
\arg\min_{(z_1,z_2)\in\mathbb{R}^2_+} z_1w_1 + z_2w_2 \\
s.t. : q = (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}}
\end{cases}
$$

   The Lagrangian for this problem:

$$
\mathcal{L} = z_1w_1 + z_2w_2 + \lambda(q - (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}})
$$

   FOC:

$$
\begin{cases}
\mathcal{L}'_{z_1} &= w_1 - \lambda z_1^{\rho-1}(z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} = 0 \\
\mathcal{L}'_{z_2} &= w_2 - \lambda z_2^{\rho-1}(z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} = 0 \\
\mathcal{L}'_{\lambda} &= q - (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}} = 0
\end{cases}
$$

We get:

$$\frac{w_1}{w_2} = \frac{\hat{z}_1^{\rho-1}(\hat{z}_1^\rho + \hat{z}_2^\rho)^{\frac{1}{\rho}-1}}{\hat{z}_2^{\rho-1}(\hat{z}_1^\rho + \hat{z}_2^\rho)^{\frac{1}{\rho}-1}} = \frac{\hat{z}_1^{\rho-1}}{\hat{z}_2^{\rho-1}} \Rightarrow \hat{z}_2 = \left(\frac{w_2}{w_1}\right)^{\frac{1}{\rho}-1} \hat{z}_1$$

$$q = (\hat{z}_1^\rho + \hat{z}_2^\rho)^{\frac{1}{\rho}}$$

That is:

$$q = \hat{z}_1^\rho \left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{\alpha}}\right)$$

$$\hat{z}_1 = q^{\frac{1}{\rho}} \left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{\alpha}}\right)^{-\frac{1}{\rho}}$$

$$\hat{z}_2 = q^{\frac{1}{\rho}} \left(1 + \left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\alpha}}\right)^{-\frac{1}{\rho}}$$

3. The cost function:

$$C(w_1, w_2, q) \;=\; w_1\hat{z}_1 \,+\, w_2\hat{z}_2 \;=\; w_1 q^{\frac{1}{\rho}} \left(1 + \left(\frac{w_2}{w_1}\right)^{\frac{\rho}{\alpha}}\right)^{-\frac{1}{\rho}} + \, w_2 q^{\frac{1}{\rho}} \left(1 + \left(\frac{w_1}{w_2}\right)^{\frac{\rho}{\alpha}}\right)^{-\frac{1}{\rho}}$$