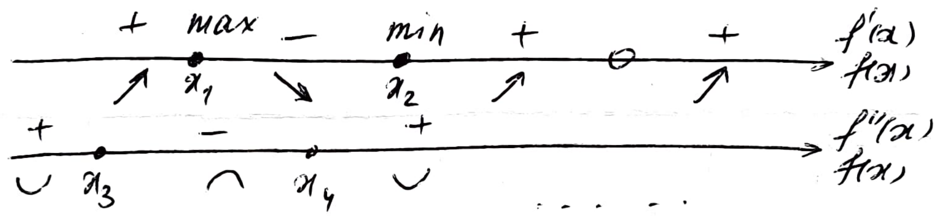
Построение графика функции с полным исследованием

1) D_f – область определения функции. Четность, нечетность, периодичность.

2) Точки пересечения с осями. Возможно, знак ф-ии на отдельных промежутках.

3) Если x_0 – точка разрыва, то:
$\lim\limits_{x\to x_0^+} f(x)$, $\lim\limits_{x\to x_0^-} f(x)$ (вертикальные асимптоты)

4) Возрастание-убывание, max, min, выпуклость, точки перегиба:
$f'(x)$, $f''(x)$



max: $\begin{cases} x = x_1 \\ y = f(x_1) \end{cases}$ min: $\begin{cases} x = x_2 \\ y = f(x_2) \end{cases}$ т-ки перегиба: $\begin{cases} x = x_3 \\ y = f(x_3) \end{cases}$, $\begin{cases} x = x_4 \\ y = f(x_4) \end{cases}$

5) Наклонные и горизонтальные асимптоты
$y_+ = k_+ x + b_+$ $\qquad$ $y_- = k_- x + b_-$

$k_+ = \lim\limits_{x\to+\infty} \dfrac{f(x)}{x}$, $\qquad b_+ = \lim\limits_{x\to+\infty} (f(x) - k_+ x)$

$k_- = \lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$, $\qquad b_- = \lim\limits_{x\to-\infty} (f(x) - k_- x)$

Только если полная пара конечных пределов $(k; b)$ существует.

6) Если хотя бы один конечный предел не существует, то полезно посчитать
$\lim\limits_{x\to a_\pm} f'(x)$.

Математический анализ

① $\lim\limits_{x\to0} \dfrac{\ln(1+\sin\pi x)}{\operatorname{arcctg}(2x)} = \left[\dfrac{0}{0}\right] = \lim\limits_{x\to0} \dfrac{\dfrac{1}{1+\sin\pi x}\cdot\pi\cos\pi x}{\dfrac{1}{1+(2x)^2}\cdot 2} = \lim\limits_{x\to0}\dfrac{\pi\cdot\cos\pi x}{1+\sin\pi x}\cdot\dfrac{1+(2x)^2}{2} =$

$$= \frac{\pi\cdot1\cdot(1+0)}{(1+0)\cdot2} = \frac{\pi}{2}$$

② $f(x,y) = x - 2y + \ln\sqrt{x^2+y^2} + 3\operatorname{arctg}\dfrac{y}{x}$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 1 + \dfrac{1}{\sqrt{x^2+y^2}}\cdot\dfrac{1}{2\sqrt{x^2+y^2}}\cdot 2x + \dfrac{-3}{1+\frac{y^2}{x^2}}\cdot\dfrac{y}{x^2} = 0 \\ \dfrac{\partial f}{\partial y} = -2 + \dfrac{1}{\sqrt{x^2+y^2}}\cdot\dfrac{1}{2\sqrt{x^2+y^2}}\cdot 2y + \dfrac{3}{1+\frac{y^2}{x^2}}\cdot\dfrac{1}{x} = 0 \end{cases}$$

$$\begin{cases} 1 + \dfrac{x}{x^2+y^2} - \dfrac{3y}{x^2+y^2} = 0 \\ -2 + \dfrac{y}{x^2+y^2} + \dfrac{3x}{x^2+y^2} = 0 \end{cases}$$

$$\begin{cases} 2x^2+2y^2+2x-6y=0 \\ -2x^2-2y^2+y+3x=0 \end{cases}$$

$5x - 5y = 0$

$x = y$

$-4x^2 + 4x = 0$

$x(x-1) = 0$

$x=0$	$x=1$	$\Delta_1 = 6/4$
$y=0$	$y=1$	$\Delta_2 = -72/4$
т. разрыва	$\longrightarrow$	седл. т.

$$\frac{\partial^2 f}{\partial x^2} = \frac{1\cdot(x^2+y^2)-(x-3y)\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1\,(x^2+y^2)-(y+3x)\,2y}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{3\,(x^2+y^2)-(3x+y)\cdot 2x}{(x^2+y^2)^2}$$

$$\frac{1}{(x^2+y^2)^2}\begin{pmatrix} -x^2+y^2+6xy & -3x^2+3y^2-6xy \\ -3x^2+3y^2-6xy & x^2-y^2-6xy \end{pmatrix}$$

$$\frac{1}{4}\begin{pmatrix} 6 & -6 \\ -6 & -6 \end{pmatrix} = \frac{1}{4}\cdot(-36-36) = -\frac{72}{4}$$

$$M_a = \frac{1}{1-MPC-MP1+MP2}$$

(3) $f(x,y,z) = x^2+y^2+2z^2+z(x^2-y^2)$

s.t. $x+y=2$

$\mathcal{L} = \dots$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 2x+2xz-\lambda=0 \\ \frac{\partial \mathcal{L}}{\partial y} = 2y-2yz-\lambda=0 \\ \frac{\partial \mathcal{L}}{\partial z} = 4z+x^2-y^2=0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = 2-x-y=0 \end{cases}$$

$\boxed{x=2-y}$

$$\begin{cases} 4-2y+4z-2zy-\lambda=0 \\ 2y-2yz-\lambda=0 \\ 4z+4-2y+\cancel{y^2}-\cancel{y^2}=0 \end{cases}$$

$z = \frac{1}{2}y-1$

$\boxed{y=2z+2}$

$$-\begin{cases} \cancel{4}-\cancel{4z}-4+\cancel{4z}-\cancel{4}z^2-4z-\cancel{\lambda}=0 \\ \cancel{4z}+\cancel{4}-\cancel{4z^2}-\cancel{4z}-\cancel{\lambda}=0 \end{cases}$$

$\Downarrow$

$4+4z=0 \Rightarrow \begin{cases} x=2 \\ y=0 \\ z=-1 \\ \lambda=0 \end{cases} \Rightarrow \max$

$\frac{\partial^2 \mathcal{L}}{\partial x^2} = 2+2z \qquad \frac{\partial \mathcal{L}}{\partial x \partial y} = 0$

$\frac{\partial^2 \mathcal{L}}{\partial y^2} = 2-2z \qquad \frac{\partial \mathcal{L}}{\partial y \partial z} = -2y$

$\frac{\partial^2 \mathcal{L}}{\partial z^2} = 4 \qquad \frac{\partial \mathcal{L}}{\partial x \partial z} = 2x$

xx yx zx
xy yy zy
xz yz zz

$d^2\mathcal{L} = (2+2z)(dx)^2 + (2-2z)(dy^2) + 4(dz)^2$

$- 4y\,dx\,dy + 4x\,dx\,dz$

при $M=(2,0,-1)$, $\lambda=0$ $\qquad d^2\mathcal{L} = 4(dy)^2+4(dz)^2 + 8\,dx\,dz$

>0

④ $\lim\limits_{x\to 0} \dfrac{1-\cos^2 4x}{\ln(1+tg\,3x)} = \left[\dfrac{1-1}{0}\right] = \lim\limits_{x\to 0} \dfrac{+8\cos 4x \cdot \sin 4x}{\dfrac{3\cdot \frac{1}{\cos^2 3x}}{1+tg\,3x}} =$

$= \lim\limits_{x\to 0} \dots = 0$

⑤ $f(x,y) = x^2 + xy + y^2 - 4\ln x - 10\ln y \quad \begin{cases} x>0 \\ y>0 \end{cases}$

$\begin{cases} \dfrac{\partial f}{\partial x} = 2x + y - \dfrac{4}{x} = 0 \\ \dfrac{\partial f}{\partial y} = x + 2y - \dfrac{10}{y} = 0 \end{cases} \qquad \begin{cases} xy + 2x^2 - 4 = 0 \\ xy + 2y^2 - 10 = 0 \end{cases}$

$y = \dfrac{4-2x^2}{x}$ $\qquad 2x^2 - 2y^2 = -6$

$y = \dfrac{4}{x} - 2x$ $\qquad y^2 - x^2 = 3$

$y^2 = 3 + x^2$

$x = \pm\sqrt{y^2-3}$ $\qquad y = \pm\sqrt{3+x^2}$

$-x\cdot\sqrt{3+x^2} + 2x^2 = 4$		$x\cdot\sqrt{3+x^2} + 2x^2 = 4$	
$x = -1$	$x = -\dfrac{4}{\sqrt3}$	$x = 1$	$x = \dfrac{4}{\sqrt3}$
$y = -2$	$y = \dfrac{5}{\sqrt3}$	$y = 2$	$y = -\dfrac{5}{\sqrt3}$

~~$-y\sqrt{y^2-3} + 2y^2 = 10$~~ ~~$y\sqrt{y^2-3} + 2y^2 = 10$~~

$$x^2 \cdot (3+x^2) = 16 - 16x^2 + 4x^4$$

$$3x^4 - 19x^2 + 16 = 0$$

$x_1^2 = 1$ или $x_2^2 = \frac{16}{3}$

$x_1 = 1 \quad x_2 = -1 \quad x_3 = \frac{4}{\sqrt3} \quad x_4 = -\frac{4}{\sqrt3}$

$y_1 = 2 \quad \underbrace{y_2 = -2 \quad y_3 = -\frac{5}{\sqrt3} \quad y_4 = \frac{5}{\sqrt3}}$

$x > 0$
$y > 0$

$$\frac{\partial^2 f}{\partial x^2} = 2 + \frac{4}{x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 2 + \frac{10}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 1$$

$$H = \begin{vmatrix} 6 & 1 \\ 1 & 7 \end{vmatrix} \qquad \Delta_1 > 0 \qquad \Delta_2 > 0$$

$\Rightarrow M(1.2)$ – т. минимума

⑥ $f(x,y,z) = x^2 + y^2 + z^2 + xyz$

s.t. $x+y+z=4$

$$\mathcal{L} = x^2 + y^2 + z^2 + xyz + \lambda(4 - x - y - z)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 2x + yz - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 2y + xz - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial z} = 2z + \cancel{x}y - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 4 - x - y - z = 0$$

$$\Downarrow$$

$$x = 4 - y - z$$

$$8 - 2y - \cancel{2z} + yz - \lambda = 0$$

$$y = (2z + \lambda - 8)/(\cancel{z} - 2)$$

$$\begin{cases} 2y + 4z - \cancel{z}y - z^2 - \cancel{\lambda} = 0 \\ 2z + 4y - \cancel{x}y^2 - \cancel{zy} - \cancel{\lambda} = 0 \end{cases}$$

$$\begin{cases} y = \dfrac{2z + \lambda - 8}{z - 2} \\ -2y + 2z - z^2 + y^2 = 0 \end{cases}$$

$$z^2 - 2z + 1 - y^2 + 2y - 1 = 0 \qquad (z-1)^2 = (y-1)^2$$

$y = x$ или $y = 2 - x$

$$\begin{cases} \cancel{y = x} \\ \cancel{2x + xz - \lambda = 0} \\ \cancel{2x + xz - \lambda = 0} \\ \cancel{2z + x^2 = 0} \end{cases}$$

$$\begin{cases} y = x \\ x = 2 - \frac{1}{2}z \end{cases}$$

$$\cancel{4} - 2 + \frac{1}{2}z = \frac{2z + \lambda - 8}{-(z - 2)}$$

$$\cancel{f(z-2)2+}$$

$$(z-2)^2 - 2(z-2) = \frac{(2z + \lambda - 8)}{(\cancel{z-2})} \cdot (-2)$$

$$z^2 - 4z + 4 - 2z - 2 + 4z + \lambda - 16 = 0$$

$\lambda = 2x + yz$

$$\begin{cases} x = 4 - y - z \\ 2y + xz = 2x + yz \quad \cancel{= 2\lambda z} \\ 2z + xy = 2x + yz \end{cases}$$

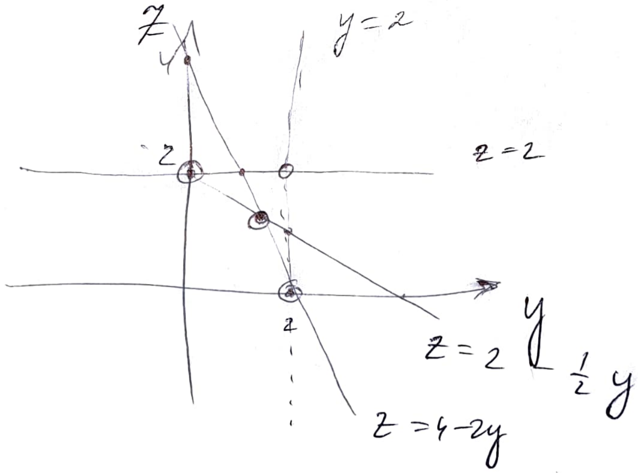
$$\begin{cases} 4y + 6z - 2yz - z^2 = 8 \cancel{- 2xy} \cancel{- 2z} \cancel{+ yz} \\ 4z + 6y - y^2 - 2zy = 8 \cancel{- 2y} \cancel{- 2z} \cancel{+ yz} \end{cases}$$

$-z(z + 2y - 4) + 2(z + 2y - 4) = 0$

$(z - 2)(z + 2y - 4) = 0$

$-y(y + 2z - 4) + 2(y + 2z - 4) = 0$

$(y - 2)(y + 2z - 4) = 0$



$x = 2$
$y = 0$
$z = 2$
$\lambda = 4$

$x = 2$
$y = 2$
$z = 0$
$\lambda = 4$

$x = 0$
$y = 2$
$z = 2$
$\lambda = 4$

$z = 4 - 2y$

$z = \frac{4 - y}{2} \Rightarrow 8 - 4y = 4y$

$y = 1$

$x =$
$y =$
$z =$
$\lambda =$

$$\int \frac{x e^{\operatorname{arctg} x}}{(1+x^2)^{3/2}} dx$$

$$\int u\,dv = uv - \int v\,du$$

$$\frac{\partial e^{\operatorname{arctg} x}}{\partial x} = \frac{e^{\operatorname{arctg} x}}{1+x^2}$$

$$v_1 = \int \frac{e^{\operatorname{arctg} x}}{1+x^2} dx = \int \partial \left(e^{\operatorname{arctg} x}\right) = e^{\operatorname{arctg} x}$$

$$u_1 = \frac{x}{\sqrt{1+x^2}} \Rightarrow \partial u = \frac{1}{(1+x^2)^{3/2}} \partial x$$

$$u_1 v_1 - \int v_1 du_1 = \frac{x e^{\operatorname{arctg} x}}{\sqrt{1+x^2}} - \underbrace{\int \frac{e^{\operatorname{arctg} x}}{(1+x^2)^{3/2}} \partial x}_{I^*}$$

$$I^* = \frac{e^{\operatorname{arctg} x}}{\sqrt{1+x^2}} + \int \frac{x\, e^{\operatorname{arctg} x}}{(1+x^2)^{3/2}} dx$$

$$v_2 = v_1$$

$$u_2 = \frac{1}{\sqrt{1+x^2}} \Rightarrow \partial u_2 = \frac{-x}{(1+x)^{3/2}} \partial x$$

$$\Downarrow$$

$$I^* = \frac{x e^{\operatorname{arctg} x}}{\sqrt{1+x^2}} - \frac{e^{\operatorname{arctg} x}}{\sqrt{1+x^2}} - I^*$$

$$I^* = \frac{x e^{\operatorname{arctg} x} - e^{\operatorname{arctg} x}}{2\sqrt{1+x^2}} = \frac{e^{\operatorname{arctg} x}(x-1)}{2\sqrt{1+x^2}}$$

$$y^2 = x^2 \cdot \frac{1+x}{1-x} \qquad y = \pm|x|\sqrt{\frac{1+x}{1-x}}$$

построить график, а) указать обл. опр.,

в) четн. ф-и, $x=0$, $x=-1$,

с) т. разрыва,

d) асимптоты,

e) экстремумы,

f) т. перегиба

а) • если $1+x>0$, то $1-x>0$ $(x>-1) \vee (x<1)$

• если $1+x<0$, то $1-x<0$ $(x<1) \vee (x>1)$

• $x \neq 1$

$$\boxed{x \in [-1;1)}$$

с) $\lim\limits_{x\to 1^-} \pm|x|\sqrt{\frac{1+x}{1-x}} = \pm\infty \Rightarrow$ верт. ас.

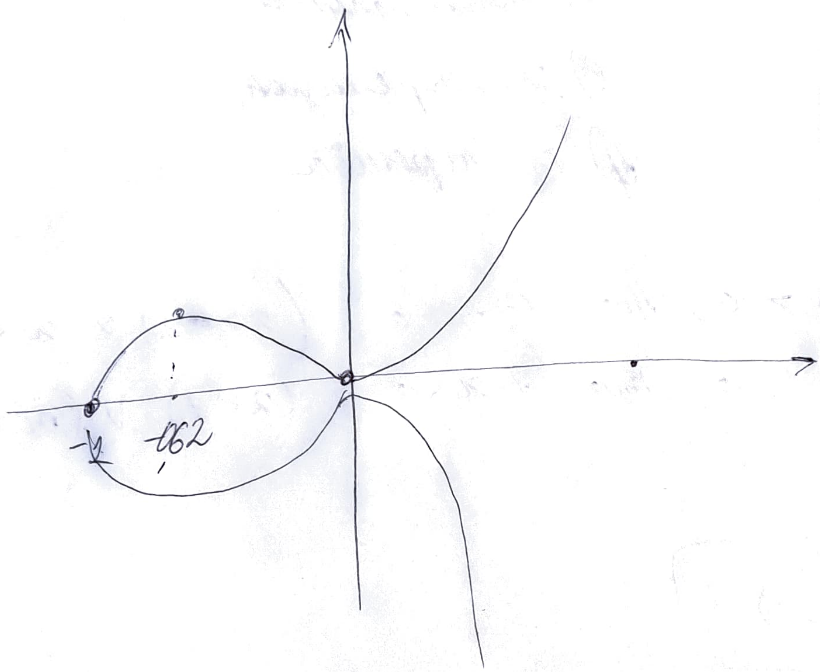
$$y_1 = |x|\sqrt{\frac{1+x}{1-x}}$$

$$\frac{\partial y_1}{\partial x} = \begin{cases} \dfrac{-(1+x-x^2)}{(\sqrt{1+x})(1-x)^{1/2}}, & x \in [-1,0) \\ \dfrac{1+x-x^2}{(\sqrt{1+x})(1-x)^{1/2}}, & x \in (0;1) \end{cases}$$

$1+x-x^2=0 \Rightarrow x_1^+ = -0{,}62$

$$\frac{\partial^2 y_1}{\partial x^2} = \begin{cases} \dfrac{-(2+x)}{(1+x)^{3/2}(1-x)^{5/2}}, & x \in (-1;0) \\ \dfrac{(2+x)}{\ldots}, & x \in (0;1) \end{cases}$$

т. $x=0$ – т. перегиба



$$z = x^2 + y^2 + 1 \longrightarrow \max/\min$$

$$\text{s.t. } (4x-8)^2 + 16(y+1)^2 = 80$$

glob min

$x_1 = 0$

$y_1 = 0$

$\lambda_1 = 0$

glob max

$x_2 = 4$

$y_2 = -2$

$\lambda_2 = -2$

Дифференциальные уравнения

(1) $y''' + 2y'' + y' = 9e^{2x}$

→ the general solution is the sum of complementary solution and particular solution

→ complementary solution:

$$\frac{d^3 y(x)}{dx^3} + 2\frac{d^2 y(x)}{dx^2} + \frac{d y(x)}{dx} = 0$$

assume solution is proportional to $e^{\lambda x}$ for some const. λ

$$\frac{d^3}{dx^3} e^{\lambda x} + 2\frac{d^2}{dx^2} e^{\lambda x} + \frac{d}{dx} e^{\lambda x} = 0$$

$$\lambda^3 e^{\lambda x} + 2\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} = 0$$

$$e^{\lambda x}(\lambda^3 + 2\lambda^2 + \lambda) = 0$$

$$\lambda^3 + 2\lambda^2 + \lambda = 0$$

$$\lambda(\lambda^2 + 2\lambda + 1) = 0$$

$\lambda_1 = 0$ $\quad \lambda_{2,3} = -1$

↳ multiplicity of root $\lambda = -1$ is 2

$\Downarrow$

$y_1(x) = C_1$ $\quad \Rightarrow \quad y_2(x) = C_2 e^{-x}$

$y_3(x) = C_3 e^{-x} \cdot x$

$y(x) = C_1 + C_2 e^{-x} + C_3 e^{-x} x$ ← compl. ~~general~~ solution

→ determine the particular solution to

$$\frac{d^3}{dx^3} y(x) + 2\frac{d^2}{dx^2} y(x) + \frac{d}{dx} y(x) = 9e^{2x}$$

by the method of undetermined coefs

the part. solution is of form $y_p(x) = a_1 e^{2x}$

$$8a_1 e^{2x} + 2\cdot 4\cdot a_1 e^{2x} + 2a_1 e^{2x} = 9e^{2x}$$

$18a_1 e^{2x} = 9e^{2x} \Rightarrow a_1 = \frac{1}{2} \Rightarrow$ Answer: general solution $y(x) = \frac{1}{2}\cdot e^{2x} + C_2 e^{-x} + C_3 e^{-x} x + C_1$

(2) $\begin{cases} \frac{dx}{dt} = 3x - 4y \\ \frac{dy}{dt} = 2x - y \end{cases}$

- $x = \frac{1}{2}\frac{dy}{dt} + \frac{1}{2}y$

$\frac{dx}{dt} = \frac{1}{2}\frac{d^2y}{dt^2} + \frac{1}{2}\frac{dy}{dt}$

- $\frac{1}{2}\frac{d^2y}{dt^2} + \frac{1}{2}\frac{dy}{dt} = \frac{3}{2}\frac{dy}{dt} + \frac{3}{2}y - 4y$

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$

$\lambda^2 - 2\lambda + 5 = 0$

$D = 4 - 20 = -16 = (4i)^2$

$\lambda_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$y(t) = e^t(C_1 \cos 2t + C_2 \cdot \sin 2t)$

- $\frac{dy}{dt} = e^t C_1 \cos 2t - 2e^t C_1 \sin 2t + e^t C_2 \sin 2t + 2e^t C_2 \cdot \cos 2t$

$x = \frac{1}{2}e^t((2C_1 + 2C_2)\cos 2t + (2C_2 - 2C_1)\sin 2t$

$\begin{cases} x = e^t((C_1 + C_2)\cos 2t + (C_2 - C_1)\sin 2t) \\ y = e^t(C_1 \cos 2t + C_2 \sin 2t) \end{cases}$, C_1, C_2 - const

(+) $\begin{cases} x = e^t(k_1 \cos 2t + (k_1 - 2k_2)\sin 2t) \\ y = e^t(k_2 \cos 2t + (k_1 - k_2)\sin 2t) \end{cases}$

$\begin{cases} k_2 = C_1 \\ k_1 - k_2 = C_2 \end{cases}$

$\begin{cases} k_1 = C_1 + C_2 \\ k_2 = C_1 \end{cases}$

$\begin{cases} C_1 = k_2 \\ C_2 = k_1 - k_2 \end{cases}$

Линейная алгебра

① $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} X^2 = \begin{pmatrix} 23 & 19 \\ 13 & 17 \end{pmatrix}$

$A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ $|A| = 6-2=4$

$A^{-1} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$

$\begin{pmatrix} 3/4 & -1/4 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} X^2 = \begin{pmatrix} 3/4 & -1/4 \\ -1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 23 & 19 \\ 13 & 17 \end{pmatrix}$

$X^2 = \begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}$

Найдем с.в. : $\begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{vmatrix} 14-\lambda & -5 \\ 10 & -1-\lambda \end{vmatrix} = 0 \iff \lambda^2 - 13\lambda + 36 = 0$

$\lambda_1 = 4$ $\lambda_2 = 9$

$\begin{cases} 10x - 5y = 0 \\ 10x - 5y = 0 \end{cases}$ $\begin{cases} 5x - 5y = 0 \\ 10x - 10y = 0 \end{cases}$

$2x = y$ $x = y$

$U = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ $|U| = 2-1=1$

$U^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

$Z = U^{-1} X^2 U = \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 14 & -5 \\ 10 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = -\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$

$K = \sqrt{-Z} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ $X = U K U^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}$ $\Delta_1 < 0$ $\Delta_2 > 0$

$\Rightarrow \boxed{X = \begin{pmatrix} -4 & 1 \\ -2 & -1 \end{pmatrix}}$

(2)

$$x_1 - 3x_2 + 4x_3 + 3x_5 = 2$$

$$3x_1 - 8x_2 + x_3 + 2x_4 = 5$$

$$2x_1 - 5x_2 - 3x_2 + 2x_4 - 3x_5 = 3$$

$$\left(\begin{array}{ccccc|c} 1 & -3 & 4 & 0 & 3 & 2 \\ 3 & -8 & 1 & 2 & 0 & 5 \\ 2 & -5 & -3 & 2 & -3 & 3 \end{array}\right) = \left(\begin{array}{ccccc|c} 1 & -3 & 4 & 0 & 3 & 2 \\ 0 & 1 & -11 & 2 & -9 & -1 \\ 0 & 1 & -11 & 2 & -9 & -1 \end{array}\right) =$$

$$= \left(\begin{array}{ccccc|c} 1 & 0 & -29 & 6 & -24 & -1 \\ 0 & 1 & -11 & 2 & -9 & -1 \end{array}\right)$$

$$x_1 = 29x_3 - 6x_4 + 24x_5 - 1$$

$$x_2 = 11x_3 - 2x_4 + 9x_5 - 1$$

(3)

$i = \begin{bmatrix} 2 \\ \vdots \\ 2 \end{bmatrix}$ в n-мерном пр-ве

Найти матрицу P – оператор орт. проектирования на лин. оболочку вектора i, найдите ранг P

Пусть v – вектор из n-мерного пр-ва

$Pv = \beta i$, где β – конст.

$(P_v)^T \cdot (v - P_v) = 0$ ← опр. орт. проекц. в 1 пр-во $\Rightarrow$ ранг $P = 1$

$(\beta i)^T \cdot (v - \beta i) = 0$

$\beta i^T \cdot v - \beta^2 \underbrace{i^T i}_{=4n} = 0 \quad | : \beta$, т.к. $\beta = 0$ трив.

$i^T v - \beta 4n = 0$

$\beta = \dfrac{i^T v}{4n}$

$Pv = \dfrac{\overset{\beta i}{\overbrace{i^T v\, i}}}{4n} = \dfrac{\overset{i\beta}{\overbrace{i\, i^T v}}}{4n} = \cancel{4\frac{E}{n} v} = \cancel{\frac{4E\,v}{4n}} = \cancel{\frac{E}{n} v}$

$= \dfrac{4 \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} v}{4n} = \begin{bmatrix} 1/n & \cdots & 1/n \\ \vdots & & \vdots \\ 1/n & \cdots & 1/n \end{bmatrix} v$

Выписать матрицу X, соотв. кв. форме

$$f(x,y) = 2x^2 - 2xy + 3y^2$$

а) $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$

б) $\left|AB^T BA^T\right|$, если $B = \begin{bmatrix} 1 & 0 \\ -2 & 3 \\ 4 & -1 \end{bmatrix}$

с) явл. ли A знакоопр.

$A^1 \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$A^2 \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$A^3 \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$A^4 \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 \\ 1+\sqrt5 & 1-\sqrt5 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1+\sqrt5 & 1-\sqrt5 \end{bmatrix} =$$

$$= \begin{bmatrix} 2 & 2 \\ 1+\sqrt5 & 1-\sqrt5 \end{bmatrix}^{-1}\begin{bmatrix} 1+\sqrt5 & 1-\sqrt5 \\ 3+\sqrt5 & 3-\sqrt5 \end{bmatrix} =$$

$\cancel{2}-2\sqrt5 - \cancel{2} - 2\sqrt5$

$= -4\sqrt5$

$[\ ]$

$\downarrow$

$\begin{bmatrix} 1-\sqrt5 & 1+\sqrt5 \\ 2 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1-\sqrt5 & -1+\sqrt5 \\ -2 & 2 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1-\sqrt5 & -2 \\ -1-\sqrt5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -\frac{1+\sqrt5}{4\sqrt5} & \frac{\cancel{1+1}}{\cancel{2}\sqrt5} \\ \frac{1+\sqrt5}{4\sqrt5} & \frac{-\cancel{1}}{\cancel{2}\sqrt5} \end{bmatrix}\begin{bmatrix} 1+\sqrt5 & 1-\sqrt5 \\ 3+\sqrt5 & 3-\sqrt5 \end{bmatrix} =$$

$$= \begin{bmatrix} -\frac{(1-5)}{4\sqrt5} + \frac{6+2\sqrt5}{4\sqrt5} & \frac{(1+\sqrt5)^2}{4\sqrt5} - 6 - 2\sqrt5 \\ & = 0 \\ \underbrace{-\frac{(1-\sqrt5)^2}{4\sqrt5} + \frac{6-2\sqrt5}{4\sqrt5}}_{=0} & \frac{1-5}{4\sqrt5} - \frac{6-2\sqrt5}{4\sqrt5} \end{bmatrix}$$

$-1 + 2\sqrt5 - 5 + 6 - 2\sqrt5$

$$= \begin{bmatrix} \frac{10+2\sqrt5}{4\sqrt5} & 0 \\ 0 & \frac{-10+2\sqrt5}{4\sqrt5} \end{bmatrix} = \begin{bmatrix} \frac{5+\sqrt5}{2\sqrt5} & 0 \\ 0 & -\frac{5-\sqrt5}{2\sqrt5} \end{bmatrix}$$

$$A^4 = \begin{bmatrix} \left(\frac{5+\sqrt5}{2\sqrt5}\right)^4 & 0 \\ 0 & \left(-\frac{5-\sqrt5}{2\sqrt5}\right)^4 \end{bmatrix} = \begin{bmatrix} \left(\frac{25+10\sqrt5+5}{20^2}\right)^2 & 0 \\ 0 & \left(\frac{25-10\sqrt5+5}{20^2}\right)^2 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{3\cancel{0}+1\cancel{0}\sqrt5}{40\cancel{0}}\right)^2 & 0 \\ 0 & \left(\frac{3\cancel{0}-1\cancel{0}\sqrt5}{40\cancel{0}}\right)^2 \end{bmatrix} = \begin{bmatrix} \left(\frac{3+\sqrt5}{4\cancel{0}}\right)^2 & 0 \\ 0 & \left(\frac{3-\sqrt5}{\cancel{40}}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{9+6\sqrt5+5}{4} & 0 \\ 0 & \frac{14-6\sqrt5}{4} \end{bmatrix}$$

Мат. статистика

① $P(T|B) = 0{,}9$

$P(T|\bar{B}) = 0{,}01$

$P(B) = 0{,}001$

$$P(\bar{B}|T) = \frac{P(T|\bar{B})\cdot P(\bar{B})}{P(T)} = \frac{0{,}01\cdot 0{,}999}{0{,}01\cdot 0{,}999 + 0{,}9\cdot 0{,}001} \approx 0{,}917.$$

② $X_1, \dots, X_n \sim Pois(\lambda)$; $P(X_i = x) = e^{-\lambda}\cdot \frac{\lambda^x}{x!}$

a) MLE for λ, $\exp(\lambda)$

$$\ell(\lambda) = \sum (x_i \log\lambda - \lambda - \log x_i!)$$

$$= \log\lambda \sum x_i - \lambda n - \sum \log x_i!$$

$$\frac{\partial \ell}{\partial \lambda} = 0 \iff \frac{\sum x_i}{\lambda} = n \iff \hat{\lambda}_{MLE} = \overline{x_i}$$

$$\exp(\hat{\lambda}_{MLE}) = e^{\overline{x_i}}$$

б) $P(X_i = 0) = e^{-\lambda}$ $\Rightarrow$ $X_i^* \sim \cancel{Bern(p(\lambda))}$

$P(X_i \neq 0) = 1 - e^{-\lambda}$

пусть $p = e^{-\lambda}$

~~k — число нулей~~

$$x_i = \begin{cases} 0, & X_i \neq 0 \\ 1, & X_i = 0 \end{cases}$$

$$\cancel{L(\lambda) = \prod \frac{n!}{x_i!(n-x_i)!} p^{x_i} (1-p)^{n-x_i}}, \quad p(\lambda) = e^{-\lambda}$$

$$\cancel{\ln L(\lambda) \Rightarrow \ell(\lambda) = \sum x_i \cdot \ln p + (n - \sum x_i)\ln(1-p)}$$

$$\cancel{\frac{\partial \ell(\lambda)}{\partial p}\frac{\partial p}{\partial \lambda} = \left(\frac{1}{p}\sum x_i + \frac{1}{1-p}(n - \sum x_i)\right)\cdot -\lambda\cdot e^{-\lambda} = 0}$$

$$\cancel{\left((1-p)\sum x_i + p\left(n-\sum x_i\right)\right)\frac{-\lambda e^{-\lambda}}{(1-p)\cdot p} = 0}$$

$$\cancel{l(\lambda) = \sum \ln \binom{\lambda}{x_i} + \ln p \cdot \sum x_i}$$

$$L(\lambda) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} = p^{\sum x_i} \cdot (1-p)^{n-\sum x_i}$$

$$l(\lambda) = \sum x_i \cdot p + \left(n-\sum x_i\right) \cdot (1-p)$$

$$\frac{\partial l}{\partial p} \cdot \frac{\partial p}{\partial \lambda} = \left(\frac{1}{p} \cdot \sum x_i - \frac{1}{1-p} \cdot \left(n - \sum x_i\right)\right)\left(-\lambda e^{-\lambda}\right) = 0$$

$$\frac{1}{p} \cdot \sum x_i = \frac{-\sum x_i + n}{1-p} \quad \text{или} \quad \lambda = 0 \quad \text{или} \quad \lambda = -\infty$$

$$\Updownarrow \qquad \qquad \Updownarrow$$

$$\sum x_i = 0 \qquad \qquad n = 0$$

$$\sum x_i - \cancel{\sum x_i p} = -\cancel{\sum x_i p} + np$$

$$\hat{p} = \frac{\sum x_i}{n}$$

$$\hat{\lambda}_{MLE} = \log \frac{n}{\sum x_i}$$

$$\exp\left(\hat{\lambda}_{MLE}\right) = \frac{n}{\sum x_i}$$

$$\boxed{\text{не сущ., если } \sum x_i = 0}$$

③ $\ln(Q) = \alpha + \beta \ln(P) + \varepsilon$, $n = 10$

$$\beta : \{ \hat{\beta} - t_{10-2,\,0{,}025} \cdot \hat{\sigma} \,;\, \hat{\beta} + t_{\sigma} \cdot \hat{\sigma} \}$$

а)

$$\hat{\beta} - \cancel{2{,}306} \cdot \hat{\sigma} = -2{,}435$$

$$\hat{\beta} + \cancel{2{,}306} \cdot \hat{\sigma} = -1{,}8802$$

$$\hat{\beta} = -2{,}1576 \qquad \hat{\sigma} = 0{,}12$$

б) $H_0 : \beta = -1$

$$t = \frac{\hat{\beta} - \beta}{se(\hat{\beta})} = \frac{-1 + 2{,}1576}{0{,}12} = 9{,}647$$

$$t_{8,\,0{,}005} = 3{,}355$$

$t_{obs} > t_{crit} \Rightarrow H_0$ отвергается а) <...>

4 $P(X_1 > 1 | H_0) = P\left(\frac{X_1 - 0}{\sqrt{4}} > \frac{1-0}{\sqrt{4}} \,\Big|\, H_0\right) =$ (если $\bar{X} > 1$?)

$= 1 - \left(\frac{X_1 - 0}{2} \le \frac{1}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right)$

$X_1 | H_0 \sim N(0, 4)$

в) при каком n мощность будет $\geq 98\%$

$1 - P(\text{ош. 2 рода}) = 0{,}98 \Rightarrow \beta = 0{,}02$

$\beta = P(\bar{X} \le 1 | H_1)$

$= P\left(\frac{\bar{X} - 2}{\frac{2}{\sqrt{n}}} \le \frac{1-2}{\frac{2}{\sqrt{n}}}\right)$

$= \Phi\left(\frac{1-2}{\frac{2}{\sqrt{n}}}\right) = 0{,}02$

$\Downarrow$

$(\ldots) = -2{,}05$

$\Downarrow$

$n \geq 17$

$\bar{X} \sim N\left(\mu, \frac{4}{n}\right)$

$E(\bar{X}) = E\left(\frac{1}{n}\sum x_i\right)$

$= \frac{1}{n}\sum E(x_i) =$

$= \frac{1}{n}\sum^{n} \mu = \mu$

$Var(\bar{X}) = \frac{1}{n^2}\sum Var(x_i) = \frac{4}{n}$

$\bar{X} | H_1 = N\left(2, \frac{4}{n}\right)$

(5)

$n_1 = 20$

$\varphi_1 : n_{11} = 5$

$\varphi_2 : n_{12} = 15$ $\xrightarrow[(1)]{1\varphi.}$ $n_2 = 9$, $n_{21} = 5$, $n_{22} = 4$ $\xrightarrow[(2)]{}$ $1\varphi.$

A : (1) → (2) φ_1

B : (2) → φ_1

a) $P(B) = P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A}) =$

$$= \frac{5+1}{9+1} \cdot \frac{5}{20} + \frac{5}{9+1} \cdot \frac{15}{20} = 0{,}525$$

б) $P(A|\bar{B}) = \dfrac{P(\bar{B}|A) \cdot P(A)}{P(\bar{B})} = \dfrac{(1 - P(B|A)) \cdot P(A)}{1 - P(B)}$

$$= \frac{\left(1 - \frac{5+1}{9+1}\right) \cdot \frac{5}{20}}{1 - 0{,}525}$$

$$Q_i = \alpha + \beta_1 P_i + \beta_2 \cdot D_i + \beta_3 P_i D_i + \varepsilon_i$$

$$D_i = \begin{cases} 1, & \text{зима/осень} \\ 0, & \text{лето/весна} \end{cases}$$

a) H о том, что спрос более чувств. при $D_i = 1$

$\hat{a} = 314,8$ $\quad \hat{Var}(\hat{a}) = 128,5$ $\quad n = 36$

$\hat{b}_1 = 13,5$ $\quad \hat{Var}(\hat{\beta}_1) = 3,2$

$\hat{b}_3 = 3,1$

$\hat{b}_2 = 211,3$ $\quad \hat{Var}(\hat{\beta}_2) = 15,1$

$\hat{Var}(\hat{b}_1 + \hat{b}_3) = 2,92$ $\quad 2,92 + 2 \cdot 1,18 - 3,2 = 2,09$

$\hat{Cov}(\hat{b}_1, \hat{b}_3) = -1,18$

$R^2 = 0,78$

$$Q_i = \alpha + (\beta_1 + \beta_3 D_i) P_i + \beta_2 D_i + \varepsilon_i$$

$H_0: \beta_3 = 0$

$H_a: \beta_3 > 0$

$$t = \frac{\hat{\beta}_3 - \beta_3}{se(\hat{\beta}_3)} = \frac{3,1}{\sqrt{2,09}} = 2,14 \quad \sim t_{n-k}$$

$t^{crit}_{36-4,\,0,95} \approx 2,042$

$t^{obs} > t^{crit} \Rightarrow \beta_3 > 0$

в) H_0 о значимости регрессии в целом

$H_0: \beta_1 = \beta_2 = \beta_3 = 0$

$$F = \frac{R^2}{1-R^2} \cdot \left(\frac{n-k}{k-1}\right) = \frac{0{,}78}{1-0{,}78} \cdot \left(\frac{36-4}{4-1}\right) = 37{,}8$$

$\sim F_{32,\,3,\,0{,}95}$

$F^{crit}_{32,\,3,\,0{,}95} \approx 2{,}84$

$F^{obs} > F^{crit}$

H_0 : отв.