SOLUTIONS MANUAL FOR

Solving Applied Mathematical Problems with MATLAB

by

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Chapter 1

Computer Mathematics Languages — An Overview

Exercises and Solutions

1. Install MATLAB environment on your machine, and issue the command `demo`. From the dialog boxes and menu items of the demonstration program, experience the powerful functions provided in MATLAB.

SOLUTION Execute the setup.exe file on the MATLAB DVD, and follow the instructions, the MATLAB and the toolboxes licensed will be installed. Start MATLAB and run `demo` command under the prompt `>>`, you may appreciate the beauty of MATLAB.

2. Type the command `doc symbolic/diff`, and see whether it is possible, by reading the relevant help information, to solve the problem given in Example 1.1. If the solutions can be obtained, compare the solutions with the results in the example.

SOLUTION From the information provided in `doc`, one may specify the following commands

```
>> syms x; y=sin(x)/(x^2+4*x+3); dy=diff(y,x,4), pretty(dy)
```

3. The following Lyapunov equation is to be solved. Use the command `lookfor lyapunov` and see whether there are any function related with the keyword. If there is, say, the `lyap` function is found, type `doc lyap` and see whether there is a way to solve the Lyapunov equation. Check the accuracy of the solution.

$$\begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix} \boldsymbol{X} + \boldsymbol{X} \begin{bmatrix} 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$$

SOLUTION To solve the equation, one should know how a matrix can be expressed in MATLAB, which will be shown in Chapter 2. Comparing the help information obtained, it can immediately be found that $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ matrices can respectively be expressed by

$$\boldsymbol{A} = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$$

such that the Lyapunov equation $\boldsymbol{AX} + \boldsymbol{XB} = -\boldsymbol{C}$ can be established. The equation can be solved and checked with the following statements, where the norm of error matrix is then 9.5337×10^{-15}.

```
>> A=[8 1 6; 3 5 7; 4 9 2]; B=[16 4 1; 9 3 1; 4 2 1];
   C=-[1 2 3; 4 5 6; 7 8 0]; X=lyap(A,B,C)
   norm(A*X+X*B+C)
```

4. Write a simple subroutine which can be used to perform matrix multiplications in other languages such as C. Try to make the code bug free.

SOLUTION For an $n \times p$ matrix $\boldsymbol{A}$ and a $p \times m$ matrix $\boldsymbol{B}$, the multiplication of them can be obtained mathematically from

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \ i = 1, \cdots, n, j = 1, \cdots, m$$

Clearly, a triple-loop in C should be used to solve such a problem. From the above formula, the following C statements can be used as the main body

```
for (i=0: i<n; i++){ for (j=0; j<m; j++){
   c[i][j]=0; for (k=0; k<p; k++) c[i][j]+=a[i][k]*b[k][j];
}}
```

Unfortunately, there are bugs in the previous program, since the two matrices are not judged whether multiplicable or not. From the above equation it is easily found that the columns of $\boldsymbol{A}$ must equal to the rows of $\boldsymbol{B}$. Thus a statement in pseudo-code `if col(`$\boldsymbol{A}$`)==row(`$\boldsymbol{B}$`)` should be introduced. Even with such a patch, there are still bugs, since it excludes the case if one of the matrices is scalar. Also the case where $\boldsymbol{A}$ and/or $\boldsymbol{B}$ are complex is not considered.
From this example, it can be concluded that with C, even slight carelessness may cause serious problems in the results. High standard languages must be used to avoid the tedious and risky programming work.

5. Write a piece of code in C which is capable of generating the Fibonacci sequence, defined as $a_1 = a_2 = 1$, $a_{k+2} = a_k + a_{k+1}$, for $k = 1, 2, \cdots$. Generate the sequence with 50 terms. Observe whether the results are feasible. If there are serious problems, is there any possible solutions in C?

SOLUTION From the recursive formula, the following program can be written

```
main() { int a1,a2,a3,i;
  a1=1; a2=1; printf("%d  %d  ",a1,a2);
  for (i=3;i<=100;i++){a3=a1+a2; printf("%d  ",a3); a1=a2; a2=a3;
}}
```

It seems that the problem has been solved easily with the program. Once one executes the problem, it is found that after 24 terms, negative values of a_i suddenly appear. In the following terms, the values sometimes are positive, and sometimes are negative. Apparently the results are wrong. What is the reason for such kind of behavior? This is because the data type `int` is used, which only allows the terms within the interval $(-32767, 32767)$. If the results are beyond the range, wrong results may appear. Of course one may use the 64-bit `long` data type, however, even though with such a data type, wrong results still appear after 36 terms. A sophisticated mathematics language must be employed in solving reliably such a problem. For instance, when symbolic computation in MATLAB is used, it is easily found that $a_{100} = 354224848179261915075$.

Chapter 2

Fundamentals of MATLAB Programming

Exercises and Solutions

1. In MATLAB environment, the following statements can be given
`tic, A=rand(500); B=inv(A); norm(A*B-eye(500)), toc`
run the statements and observe results. If you are not sure with the commands, just use the on-line `help` facilities to display information on the related functions. Then explain in detail the statement and the results.

SOLUTION What the statements actually do is to calculate and verify the inverse matrix $\boldsymbol{B}$ of a 500×500 randomly generated matrix $\boldsymbol{A}$, and measure the total time consumes. It can be found that the precision reaches 10^{-12}-level, and the time required is around one second.

2. Suppose that a polynomial can be expressed by $f(x) = x^5 + 3x^4 + 4x^3 + 2x^2 + 3x + 6$. If one wants to substitute x by $\dfrac{s-1}{s+1}$, the function $f(x)$ can be changed into a function of s. Use the Symbolic Toolbox to do the substitution and get the simplest result.

SOLUTION One should declare the two variables s and x as symbolic variables, then the `subs()` function should be used to do variable substitution. Finally, simplification of the results should be performed

```
>> syms s x, f=x^5+3*x^4+4*x^3+2*x^2+3*x+6;
   F=subs(f,x,(s-1)/(s+1)), F=simple(F)
```

which leads to the result $F = \dfrac{3+23s+54s^2+70s^3+19s^5+23s^4}{(s+1)^5}$.

3. Input the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ into MATLAB workspace where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 4 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 1+\mathrm{j}4 & 2+\mathrm{j}3 & 3+\mathrm{j}2 & 4+\mathrm{j}1 \\ 4+\mathrm{j}1 & 3+\mathrm{j}2 & 2+\mathrm{j}3 & 1+\mathrm{j}4 \\ 2+\mathrm{j}3 & 3+\mathrm{j}2 & 4+\mathrm{j}1 & 1+\mathrm{j}4 \\ 3+\mathrm{j}2 & 2+\mathrm{j}3 & 4+\mathrm{j}1 & 1+\mathrm{j}4 \end{bmatrix}.$$

It is seen that $\boldsymbol{A}$ is a 4×4 matrix. If a command $\boldsymbol{A}(5,6) = 5$ is given, what will happen?

SOLUTION The two matrices can be specified easily with the following statements

```
>> A=[1 2 3 4; 4 3 2 1; 2 3 4 1; 3 2 4 1]
   B=[1+4i 2+3i 3+2i 4+1i;  4+1i 3+2i 2+3i 1+4i;
      2+3i 3+2i 4+1i 1+4i; 3+2i 2+3i 4+1i 1+4i];
```

If further the command

```
>> A(5,6)=5
```

is used, and columns and rows in the statements are all greater than the current size of $\boldsymbol{A}$, zero terms are introduced to the extended part of $\boldsymbol{A}$, then the (5,6)th term is assigned to 5.

4. For a matrix $\boldsymbol{A}$, if one wants to extract all the even rows to form matrix $\boldsymbol{B}$, what command should be used? Suppose that matrix $\boldsymbol{A}$ is defined by $\boldsymbol{A}$ =`magic(8)`, establish matrix $\boldsymbol{B}$ with suitable statements and see whether the results are correct.

SOLUTION Even row extraction of a matrix $\boldsymbol{A}$ can easily found by

```
>> A=magic(8), B=A(2:2:end,:)
```

5. Implement the following piecewise function where $\boldsymbol{x}$ can be given by scalar, vectors, matrices or even other multi-dimensional arrays, the returned argument $\boldsymbol{y}$ should be the same size as that of $\boldsymbol{x}$. The parameters h and D are scalars.

$$\boldsymbol{y} = f(\boldsymbol{x}) = \begin{cases} h, & \boldsymbol{x} > D \\ h/D\boldsymbol{x}, & |\, \boldsymbol{x} \,| \leqslant D \\ -h, & \boldsymbol{x} < -D \end{cases}$$

SOLUTION Two methods can be used, and the best one is with the use of the relationship expression in a clever way

```
>> y=h*(x>D) + h/D*x.*(abs(x)<=D) -h*(x<-D);
```

An alternative method is by the use of loops and condition structures

```
>> for i=1:length(x)
      if x(i)>D, y(i)=h;
      elseif abs(x(i))<=D, y(i)= h/D*x(i); else, y(i)=-h; end
   end
```

The structure of the latter statements are easy to understand, however the former is applicable not only for vector $\boldsymbol{x}$, but also to other data structures such as matrices or three-dimensional arrays.

6. Evaluate using numerical method the sum $S = 1+2+4+\cdots+2^{62}+2^{63} = \sum_{i=0}^{63} 2^i$, the use of vectorization form is suggested. Check whether accurate solutions can be found and explain why. Find the accurate sum using the symbolic computation methods.

SOLUTION The following statements can be used to evaluate numerically the sum, however due to the limitations of the 64-bit `double` data type, the result $s_1 = 1.844674407370955\times10^{19}$ is not accurate.

```
>> s1=sum(2.^[0:63])
```

To solve the problem accurately, the symbolic data type should be used instead. The new statement should be

```
>> s2=sum(sym(2).^[0:63])
```

with $s_2 = 18446744073709551615$. One may even replace the term 63 by 1000 to evaluate accurately $\sum_{i=0}^{1000} 2^i$, which is not possible by numerical data types. The accurate result is 2143017214372534641896850098120003621122809623411067214887500776740702102249872244986396757631391716255189345835106293650374290571384628087196915514939714960786913554964846197084214921012474228375590836430609294996716388253479753511833108789215412582914239295537308433532085966330524877367441133613875 1.

7. Write an M-function `mat_add()` with the syntax

   $\boldsymbol{A}$=`mat_add(`$\boldsymbol{A}_1$,$\boldsymbol{A}_2$,$\boldsymbol{A}_3$,$\cdots$`)`

   In the function, it is required that arbitrary number of input arguments $\boldsymbol{A}_i$ are allowed to be added up.

SOLUTION With the use of `varargin`, the function below can be designed.

```
function A=mat_add(varargin)
A=0; for i=1:length(varargin), A=A+varargin{i}; end
```

The `try-catch` structure can further be used to solve the above problem.

```
function A=mat_add(varargin)
try
   A=0; for i=1:length(varargin), A=A+varargin{i}; end
catch, error(lasterr); end
```

8. An MATLAB function can be written, whose syntax is

   $\boldsymbol{v}$=$[h_1, h_2, h_m, h_{m+1}, \cdots, h_{2m-1}]$ and $\boldsymbol{H}$=`myhankel(`$\boldsymbol{v}$`)`

   where the vector $\boldsymbol{v}$ is defined, and out of it, the output argument should be an $m \times m$ Hankel matrix.

SOLUTION Many methods can be used to solve the above problem:

(i) The most straightforward method is the use of double loop structure to implement $H_{i,j} = h_{i+j-1}$ such that

```
function H=myhankel(v)
m=(length(v)+1)/2;
for i=1:m, for j=1:m, H(i,j)=v(i+j-1); end, end
```

(ii) For a certain column (or row), $\boldsymbol{a}_i = [h_i, h_{i+1}, \cdots, h_{i+m-1}]$. Thus single loop structure can be used to generate the Hankel matrix

```
function H=myhankel(v)
m=(length(v)+1)/2; for i=1:m, H(i,:)=v(i:i+m-1); end
```

(iii) Based on the existing `hankel()` function, one can write

```
function H=myhankel(v)
m=(length(v)+1)/2; H=hankel(v(1:m),v(m:end));
```

9. From matrix theory, it is known that if a matrix $\boldsymbol{M}$ is expressed as $\boldsymbol{M} = \boldsymbol{A}+\boldsymbol{BCB}^{\mathrm{T}}$, where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are the matrices of relevant sizes, the inverse of $\boldsymbol{M}$ can be calculated by the following algorithm

$$\boldsymbol{M}^{-1} = \left(\boldsymbol{A}+\boldsymbol{BCB}^{\mathrm{T}}\right)^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}\left(\boldsymbol{C}^{-1}+\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{-1}$$

The matrix inversion can be carried out using the formula easily. Suppose that there is a 5×5 matrix $\boldsymbol{M}$, from which the three other matrices can be found.

$$\boldsymbol{M} = \begin{bmatrix} -1 & -1 & -1 & 1 & 0 \\ -2 & 0 & 0 & -1 & 0 \\ -6 & -4 & -1 & -1 & -2 \\ -1 & -1 & 0 & 2 & 0 \\ -4 & -3 & -3 & -1 & 3 \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

Write the statement to evaluate the inverse matrix. Check the accuracy of the inversion. Compare the accuracy of the inversion method and the direct inversion method with `inv()` function.

SOLUTION Based on the partition formula, the following function can be written

```
function Minv=part_inv(A,B,C)
Minv=inv(A)-inv(A)*B*inv(inv(C)+B'*inv(A)*B)*B'*inv(A);
```

For the given matrices, the following two methods can be used

```
>> M=[-1,-1,-1,1,0; -2,0,0,-1,0; -6,-4,-1,-1,-2;
     -1,-1,0,2,0;-4,-3,-3,-1,3];
   A=[1,0,0,0,0; 0,3,0,0,0; 0,0,4,0,0; 0,0,0,2,0; 0,0,0,0,4];
   B=[0,1,1,1,1; 0,2,1,0,1; 1,1,1,2,1; 0,1,0,0,1; 1,1,1,1,1];
   C=[1,-1,1,-1,-1; 1,-1,0,0,-1; 0,0,0,0,1; 1,0,-1,-1,0; 0,1,-1,0,1];
   M1=inv(M), % method 1, direct numerical solution
   M2=part_inv(A,B,C)   % method 2, with partition formula
   Ms=inv(sym(M)); e1=norm(double(Ms)-M1), e2=norm(double(Ms)-M2)
```

The appearance of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are the same, however, the precision might be different, since $e_1 = 1.5232\times10^{-16}, e_2 = 1.5271\times10^{-15}$. It can be concluded that normally when direct functions exist, it should be used, rather than using any other indirect methods, to avoid accumulative errors.

10. Consider the following iterative model

$$\begin{cases} x_{k+1} = 1 + y_k - 1.4x_k^2 \\ y_{k+1} = 0.3x_k \end{cases}$$

with initial conditions $x_0 = 0$, $y_0 = 0$. Write an M-function to evaluate the sequence x_i, y_i. 30000 points can be obtained by the function to construct the $\boldsymbol{x}$ and $\boldsymbol{y}$ vectors. The points can be expressed by a dot, rather than lines. In this case, the so-called Hénon attractor can be drawn.

SOLUTION Loop structure can be used to implement the recursive formula, and the Hénon attractor can be drawn as shown in Figure 2.1. Note that, the option `'.'` should be used to indicate the sequence.

```
>> n=30000; x=zeros(1,n); y=x;
for i=1:n-1, x(i+1)=1+y(i)-1.4*x(i)^2; y(i+1)=0.3*x(i); end
plot(x,y,'.')
```

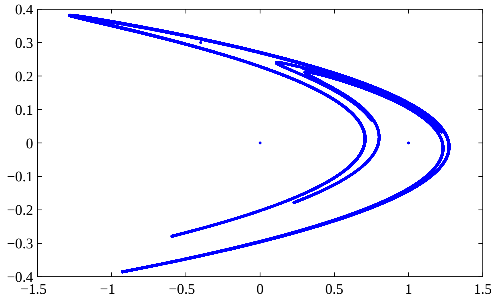


FIGURE 2.1: Hénon attractor

11. An equilateral triangle can be drawn by MATLAB statements easily. Use the loop structure to design an M-function that, in the same coordinates, a sequence of equilateral triangles can be drawn, each by rotating a certain angle from the previous one.

SOLUTION To rotate counter-clockwise an equilateral triangle by the angle θ, the new triangle can be illustrated as shown in Figure 2.2 (a). The critical points of the new triangle are respectively $(\cos\theta, \sin\theta)$, $(\cos(\theta+120^\circ), \sin(\theta+120^\circ))$ and $(\cos(\theta+240^\circ), \sin(\theta+240^\circ))$, then back to point $(\cos\theta, \sin\theta)$. The triangle can be drawn easily. Increment the angle θ continuously and draw in a loop a set of triangles, the resulted graphical display is shown in Figure 2.2 (b), with the command

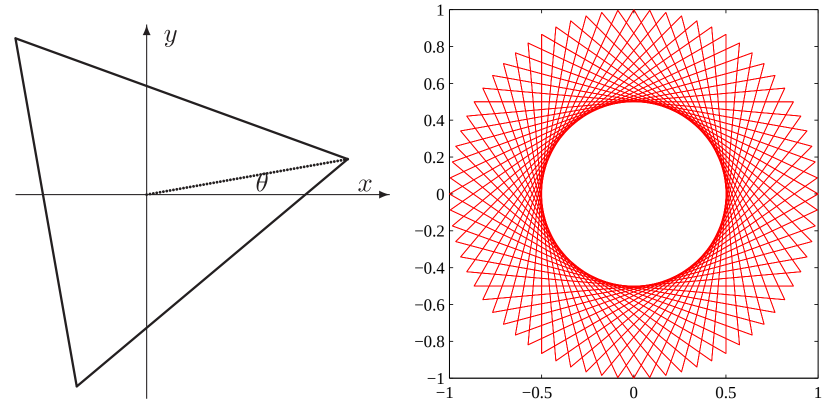
```
>> draw_triangles(5,'r') % 5 and 'r' for increment angle and color
```

and the M-function `draw_triangles()` is listed below

```
function draw_triangles(delta,col)
t=[0,120,240,0]*pi/180; xxx=[]; yyy=[];
for i=0:delta:360
   tt=i*pi/180; xxx=[xxx; cos(tt+t)]; yyy=[yyy; sin(tt+t)];
end
plot(xxx',yyy',col), axis('square')
```

Selecting the increment to other values such as $\Delta\theta = 2, 1, 0.1$, one may further observe the results.

12. Select suitable step-sizes and draw the function curve for $\sin(1/t)$, where $t \in (-1, 1)$.



(a) sketch of the rotation (b) final results

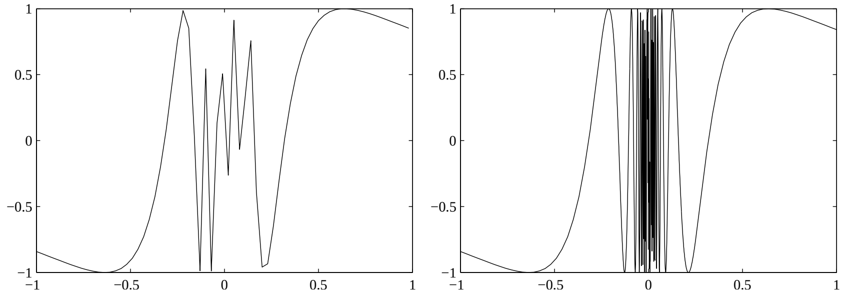
FIGURE 2.2: Graphical display of a set of triangles

SOLUTION In ordinary graphics mode, when a step-size of 0.03 is used, the curve of the function is shown in Figure 2.3 (a). However the curve is harsh.

```
>> t=-1:0.03:1; y=sin(1./t); plot(t,y)
```

If variable step-size is used, the curves are shown in Figure 2.3 (b).

```
>> t=[-1:0.03: -0.25, -0.248:0.001:0.248, 0.25:.03:1];
   y=sin(1./t); plot(t,y)
```



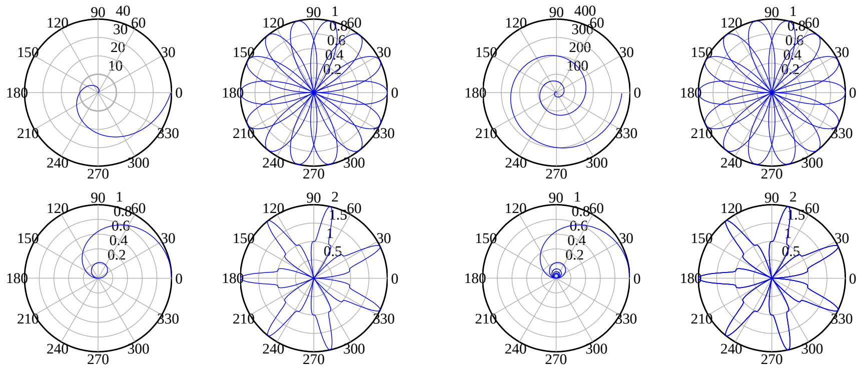
(a) fixed step-size (b) variable step-size

FIGURE 2.3: The curves of $\sin(1/t)$ under different step-sizes

It can be concluded from the example that, the curves obtained should be verified, before it can be put into practical used.

13. For suitably assigned ranges of θ, draw polar plots for the following parametric functions.
(i) $\rho = 1.0013\theta^2$, (ii) $\rho = \cos(7\theta/2)$,
(iii) $\rho = \sin\theta/\theta$, (iv) $\rho = 1 - \cos^3 7\theta$

SOLUTION It seems that drawing polar plot is an easy task. However one should be very careful in verifying the results by choosing different θ intervals. Also dot operation of vectors must be used. Comparing the plots for different θ ranges shown in Figures 2.4 (a) and (b).



(a) $\theta \in (0, 2\pi)$ (b) $\theta \in (0, 6\pi)$

FIGURE 2.4: Polar plots for different θ ranges

```
>> t=0:0.01:2*pi; subplot(221), polar(t,1.0013*t.^2),  % (i)
   subplot(222), t1=0:0.01:4*pi; polar(t1,cos(7*t1/2)) % (ii)
   subplot(223), polar(t,sin(t)./t)                    % (iii)
   subplot(224), polar(t,1-(cos(7*t)).^3)              % (iv)
   figure; t=0:0.01:2*pi; % repeat the previous commands, get figure (b)
```

14. Find the solutions to the following equations using graphical methods and verify the solutions.

$$\begin{cases} x^2 + y^2 = 3xy^2 \\ x^3 - x^2 = y^2 - y \end{cases}$$

SOLUTION The two equations can all be expressed by implicit function drawing command `ezplot()`, and the intersections are the solutions of the simultaneous equations, as shown in Figure 2.5. One may zoom the plots and find more accurate values of the intersections.

```
>> ezplot('x^2+y^2-3*x*y^2'); hold on, ezplot('x^3-x^2=y^2-y')
```

15. Draw the 3D surface plots for the functions xy and $\cos(xy)$ respectively. Also draw the contour lines of the functions. View the 3D surface plot from different angles.

SOLUTION The following commands can be used to draw the 3D surface and contour lines of the functions. The function `view()` can be used to change view points and also one can rotate the 3D surfaces manually.

```
>> [x,y]=meshgrid(-1:.1:1); z1=x.*y; z2=sin(z1);
   subplot(211), surf(x,y,z1), subplot(212), contour(x,y,z1,30)
```

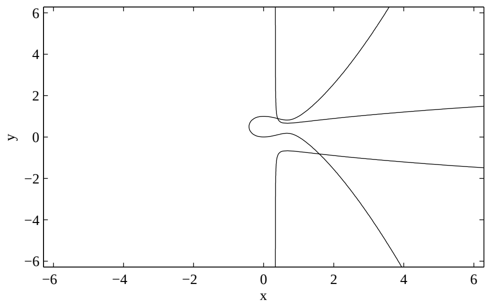


FIGURE 2.5: Graphical interpretations of the solutions

```
figure;
subplot(211), surf(x,y,z2), subplot(212), contour(x,y,z2,20)
```

16. In graphics command, there is a trick in hiding certain part of the plot. If the function values are assigned to NaN's, the point on the curve or the surface will not be shown. Draw first the surface plot of the function $z = \sin xy$. Then cut off the region satisfies $x^2 + y^2 \leqslant 0.5^2$.

SOLUTION The mesh grid data of a rectangular region can be generated first and the function values can be calculated. Then find all the points in the region satisfying $x^2 + y^2 \leqslant 0.5^2$, and set the values to NaN's. The 3D surface of the given function excluding the region, shown in Figure 2.6.

```
>> [x,y]=meshgrid(-1:.1:1); z=sin(x.*y);
   ii=find(x.^2+y.^2<=0.5^2); z(ii)=NaN; surf(x,y,z)
```

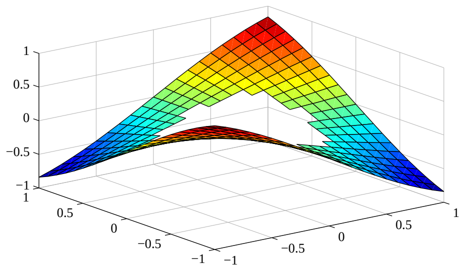


FIGURE 2.6: 3D surface with a region cut off

Chapter 3

Calculus Problems

Exercises and Solutions

1. Compute the limits.

(i) $\displaystyle\lim_{x\to\infty}(3^x+9^x)^{\frac{1}{x}}$, (ii) $\displaystyle\lim_{x\to\infty}\frac{(x+2)^{x+2}(x+3)^{x+3}}{(x+5)^{2x+5}}$

SOLUTION The limit problems can be solved with the following statements, with $L_1=9, L_2=\mathrm{e}^{-5}$.

```
>> syms x; f=(3^x+9^x)^(1/x); L1=limit(f,x,inf)
   f=(x+2)^(x+2)*(x+3)^(x+3)/(x+5)^(2*x+5); L2=limit(f,x,inf)
```

2. Compute the double limits.

(i) $\displaystyle\lim_{\substack{x\to-1\\y\to2}}\frac{x^2y+xy^3}{(x+y)^3}$, (ii) $\displaystyle\lim_{\substack{x\to0\\y\to0}}\frac{xy}{\sqrt{xy+1}-1}$, (iii) $\displaystyle\lim_{\substack{x\to0\\y\to0}}\frac{1-\cos\left(x^2+y^2\right)}{\left(x^2+y^2\right)\mathrm{e}^{x^2+y^2}}$

SOLUTION The double limit problems can be solved with the following statements, with $L_1=-6, L_2=2$, and $L_3=0$.

```
>> syms x y; fa=(x^2*y+x*y^3)/(x+y)^3; L1=limit(limit(fa,x,-1),y,2)
   fb=x*y/(sqrt(x*y+1)-1); L2=limit(limit(fb,x,0),y,0)
   fc=(1-cos(x^2+y^2))/(x^2+y^2)/exp(x^2+y^2);
   L3=limit(limit(fc,x,0),y,0)
```

3. Compute the derivatives of the functions.

(i) $y(x)=\sqrt{x\sin x\sqrt{1-\mathrm{e}^x}}$, (ii) $y=\dfrac{1-\sqrt{\cos ax}}{x\left(1-\cos\sqrt{ax}\right)}$

(iii) $\operatorname{atan}\dfrac{y}{x}=\ln(x^2+y^2)$, (iv) $y(x)=-\dfrac{1}{na}\ln\dfrac{x^n+a}{x^n}$, $n>0$

SOLUTION The derivatives of the functions can be found with the following statements and

```
>> syms a x; f=sqrt(x*sin(x)*sqrt(1-exp(x))); D1=simple(diff(f))
   y=(1-sqrt(cos(a*x)))/(x*(1-cos(sqrt(a*x)))); D2=simple(diff(y))
   syms y; f=atan(y/x)-log(x^2+y^2); D3=simple(-diff(f,x)/diff(f,y))
   syms n positive; f=-log((x^n+a)/x^n)/(n*a); D4=simple(diff(f,x))
```

with

$$D_1=\frac{1}{2\sqrt{x\sqrt{1-\mathrm{e}^x}\sin x}}\left(\sqrt{1-\mathrm{e}^x}\sin x+x\sqrt{1-\mathrm{e}^x}\cos x-\frac{x\mathrm{e}^x\sin x}{2\sqrt{1-\mathrm{e}^x}}\right)$$

$$D_2 = \frac{a\sin ax}{2x\sqrt{\cos ax}\,(1-\cos\sqrt{ax})} - \frac{1-\sqrt{\cos ax}}{x^2\,(1-\cos\sqrt{ax})} - \frac{a\,(1-\sqrt{\cos ax})\sin\sqrt{ax}}{2x\,(1-\cos\sqrt{ax})^2\,\sqrt{ax}}$$

$$D_3 = \frac{y+2x}{x-2y} \quad \text{and } D_4 = \frac{1}{x(x^n+a)}$$

4. Compute the fourth-order derivative of function $y(t) = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$.

SOLUTION One may find the fourth-order derivative of the given function with

```
>> syms x; f=sqrt((x-1)*(x-2)/(x-3)/(x-4)); D=simple(diff(f,x,4))
```

and the results can be expressed mathematically as

$$\frac{3\left(\begin{array}{c}16x^{11}-392x^{10}+4312x^9-28140x^8+121344x^7-364560x^6+\\ 783552x^5-1214604x^4+1342560x^3-1015348x^2+474596x-103741\end{array}\right)}{\left(\dfrac{(x-1)\,(x-2)}{(x-3)\,(x-4)}\right)^{7/2}(x-3)^8\,(x-4)^8}$$

5. In calculus courses, when the limit of a ratio is required, where both the numerator and the denominator tends to 0 or ∞, simultaneously, the L'Hôpital's law can be uses, i.e., to evaluate the limits of derivatives of numerator and denominator. Verify the $\lim\limits_{x\to 0}\dfrac{\ln(1+x)\ln(1-x)-\ln(1-x^2)}{x^4}$ by the consecutive use of the L'Hôpital's law, and compare with the results directly obtained.

SOLUTION From the denominator it can be seen that, to make sure the denominator is no longer zero, fourth-order derivative to both the numerator and denominator should be taken. Thus with the use of the L'Hôpital's law, the limit can be find as $L = 1/12$, which can also be verified with the direct method.

```
>> syms x; n=log(1+x)*log(1-x)-log(1-x^2); d=x^4;
   n4=diff(n,x,4); d4=diff(d,x,4); n4=subs(n4,x,0);
   L=n4/d4, L1=limit(n/d,x,0)
```

6. For parametric function $\begin{cases} x = \ln\cos t \\ y = \cos t - t\sin t \end{cases}$, compute $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=\pi/3}$.

SOLUTION With the `paradiff()` function given in the book, the derivatives required are obtained respectively as

$$D_1 = -\frac{(-2\sin t - t\cos t)\cos t}{\sin t},\ D_2 = \frac{5}{6} + \frac{7\sqrt{3}}{54}\pi.$$

```
>> syms t; x=log(cos(t)); y=cos(t)-t*sin(t);
   D1=paradiff(y,x,t,1),
   f=paradiff(y,x,t,2); D2=simple(subs(f,t,sym(pi)/3))
```

7. Assume that $u = \cos^{-1}\sqrt{\dfrac{x}{y}}$, verify $\dfrac{\partial^2 u}{\partial x\partial y} = \dfrac{\partial^2 u}{\partial y\partial x}$.

SOLUTION To show that the two sides are equal, one can evaluate the two side and simplify the difference between them to see whether it is zero or not. For this problem, it is easily shown that the difference is zero, hence the equation holds.

```
>> syms x y; u=acos(x/y);
   simple(diff(diff(u,x),y)-diff(diff(u,y),x))
```

8. For the given function $\begin{cases} xu + yv = 0 \\ yu + xv = 1 \end{cases}$, compute $\dfrac{\partial^2 u}{\partial x \partial y}$.

SOLUTION The solution to be problem can be found with the following statements and the result is $\dfrac{2x}{(x^2-y^2)^2} + \dfrac{8y^2 x}{(x^2-y^2)^3}$.

```
>> syms x y u v; [u,v]=solve('x*u+y*v=0','y*u+x*v=1','u,v');
   diff(diff(u,x),y)
```

9. Assume that $f(x,y) = \displaystyle\int_0^{xy} \mathrm{e}^{-t^2} \mathrm{d}t$, compute $\dfrac{x}{y}\dfrac{\partial^2 f}{\partial x^2} - 2\dfrac{\partial^2 f}{\partial x \partial y} + \dfrac{\partial^2 f}{\partial y^2}$.

SOLUTION The problem can easily be solved with the following statements, and $F = -2\mathrm{e}^{-x^2y^2}(-x^2y^2 + 1 + x^3 y)$.

```
>> syms x y t; f=int(exp(-t^2),t,0,x*y);
   F=x/y*diff(f,x,2)-2*diff(diff(f,x),y)+diff(f,y,2); F=simple(F)
```

10. Given the matrix $\boldsymbol{f}(x,y,z) = \begin{bmatrix} 3x + \mathrm{e}^y z \\ x^3 + y^2 \sin z \end{bmatrix}$, compute the Jacobian matrix.

SOLUTION The Jacobian matrix can be obtained as $\boldsymbol{J} = \begin{bmatrix} 3 & \mathrm{e}^y z & \mathrm{e}^y \\ 3x^2 & 2y \sin z & y^2 \cos z \end{bmatrix}$.

```
>> syms x y z; F=[3*x+exp(y)*z; x^3+y^2*sin(z)];
   J=jacobian(F,[x,y,z])
```

11. Compute the following indefinite integrals

(i) $I(x) = -\displaystyle\int \frac{3x^2 + a}{x^2 (x^2 + a)^2} \mathrm{d}x$, (ii) $I(x) = \displaystyle\int \frac{\sqrt{x(x+1)}}{\sqrt{x} + \sqrt{1+x}} \mathrm{d}x$

(iii) $I(x) = \displaystyle\int x\mathrm{e}^{ax} \cos bx \, \mathrm{d}x$, (iv) $I(t) = \displaystyle\int \mathrm{e}^{ax} \sin bx \sin cx \, \mathrm{d}x$

SOLUTION The indefinite integrals can be calculated from

```
>> syms x a; f=-(3*x^2+a)/(x^2+(x^2+a)^2); I1=int(f,x)
   f=sqrt(x*(x+1))/(sqrt(x)+sqrt(x+1)); I2=int(f,x)
   syms a b x; f=x*exp(a*x)*cos(b*x); I3=int(f,x)
   syms x a b c; f=exp(a*x)*sin(b*x)*sin(c*x); I4=int(f,x)
```

and the integral I_1 is too complicated and will not show here. The other three integrals are respectively

$$I_2 = \frac{2\sqrt{x(x+1)}x(3x+5)}{15\sqrt{x+1}} - 2\frac{\sqrt{x(x+1)}(x+1)(-2+3x)}{15\sqrt{x}}$$

$$I_3 = \left(\frac{ax}{a^2+b^2} - \frac{a^2-b^2}{(a^2+b^2)^2}\right)\mathrm{e}^{ax}\cos bx - \left(-\frac{bx}{a^2+b^2} + \frac{2ab}{(a^2+b^2)^2}\right)\mathrm{e}^{ax}\sin bx$$

$$I_4 = \frac{a\mathrm{e}^{ax}\cos((b-c)x)}{2[a^2+(b-c)^2]} - \frac{(-b+c)\mathrm{e}^{ax}\sin((b-c)x)}{2[a^2+(b-c)^2]} - \frac{a\mathrm{e}^{ax}\cos((b+c)x)}{2[a^2+(b+c)^2]} + \frac{(-b-c)\mathrm{e}^{ax}\sin((b+c)x)}{2[a^2+(b+c)^2]}$$

12. Compute the definite integrals and infinite integrals

$$\text{(i) } I = \int_0^\infty \frac{\cos x}{\sqrt{x}}\,\mathrm{d}x, \qquad \text{(ii) } I = \int_0^1 \frac{1+x^2}{1+x^4}\,\mathrm{d}x$$

SOLUTION The two integrals can be obtained as $I_1 = \sqrt{2\pi}/2$, $I_2 = \sqrt{2}\pi/4$.

```
>> syms x; I1=int(cos(x)/sqrt(x),x,0,inf),
   I2=int((1+x^2)/(1+x^4),x,0,1)
```

13. For the function $f(x) = \mathrm{e}^{-5x}\sin(3x+\pi/3)$, compute $\displaystyle\int_0^t f(x)f(t+x)\,\mathrm{d}x$.

SOLUTION Defining the function of x, `subs()` function can be used to establish the function of $t+x$. The following statements can be used to calculate R.

```
>> syms x t; f=exp(-5*x)*sin(3*x+sym(pi)/3);
   R=int(f*subs(f,x,t+x),x,0,t); simple(R)
```

and the result is

$$\frac{\begin{array}{c}15\sqrt{3}\mathrm{e}^{10t}\cos 3t - 68\cos 3t - 15\mathrm{e}^{10t}\sin 3t - 25\sqrt{3}\sin 9t + \\ 25\sqrt{3}\mathrm{e}^{10t}\sin 3t + 15\sin 9t - 25\cos 9t - 15\sqrt{3}\cos 9t + 93\mathrm{e}^{10t}\cos 3t\end{array}}{1360\mathrm{e}^{15t}}$$

14. For different values of a, compute the integral $I = \displaystyle\int_0^\infty \frac{\cos ax}{1+x^2}\,\mathrm{d}x$.

SOLUTION Loop structure can be used to calculate the infinite integral for different values of a, and the relationship is shown in Figure 3.1.

```
>> syms x a; f=cos(a*x)/(1+x^2); aa=[0:0.1:pi]; y=[];
   for n=aa, b=int(subs(f,a,n),x,0,inf); y=[y, double(b)]; end
   plot(aa,y)
```

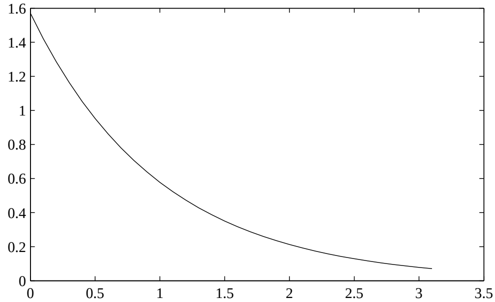


FIGURE 3.1: The integral for different values of a

15. Show that for any function $f(t)$, $\displaystyle\int_a^b f(t)\,\mathrm{d}t = -\int_b^a f(t)\,\mathrm{d}t$.

SOLUTION Function $f(t)$ can be defined with `f=sym('f(t)')`. Thus, the equation can be shown with the following statements, which yields zero difference.

```
>> syms a b t; f=sym('f(t)'); simple(int(f,t,a,b)+int(f,t,b,a))
```

16. Solve the multiple integral problems.

(i) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\sqrt{4-x^2-y^2}\,\mathrm{d}y\mathrm{d}x$, (ii) $\displaystyle\int_0^3\int_0^{3-x}\int_0^{3-x-y}xyz\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$

(iii) $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}z(x^2+y^2)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$

(iv) $\displaystyle\int_0^1\int_0^x\int_0^y\int_0^z xyzu\mathrm{e}^{6-x^2-y^2-z^2-u^2}\,\mathrm{d}u\mathrm{d}z\mathrm{d}y\mathrm{d}x$

SOLUTION The multiple integrals can easily be evaluated with

```
>> syms x y z u; f1=sqrt(4-x^2-y^2); f2=x*y*z; f3=z*(x^2+y^2);
   f4=x*y*z*u*exp(6-x^2-y^2-z^2-u^2);
   I1=int(int(f1,y,0,sqrt(4-x^2)),x,0,2)
   I2=int(int(int(f2,z,0,3-x-y),y,0,3-x),x,0,3)
   I3=int(int(int(f3,z,0,sqrt(4-x^2-y^2)),y,0,sqrt(4-x^2)),x,0,2)
   I4=int(int(int(int(f4,u,0,z),z,0,y),y,0,x),x,0,1)
```

where $I_1=\dfrac{4\pi}{3}$, $I_2=\dfrac{81}{80}$, $I_3=\dfrac{4\pi}{3}$, $I_4=\dfrac{1}{384}\mathrm{e}^6-\dfrac{1}{96}\mathrm{e}^5+\dfrac{1}{64}\mathrm{e}^4-\dfrac{1}{96}\mathrm{e}^3+\dfrac{1}{384}\mathrm{e}^2$.

17. Compute the Fourier series expansions to the following functions, and compare graphically the approximation results, if finite term series are used.

(i) $f(x)=(\pi-|x|)\sin x,\ -\pi\leqslant x<\pi$, (ii) $f(x)=\mathrm{e}^{|x|},\ -\pi\leqslant x<\pi$,

(iii) $f(x)=\begin{cases}2x/l, & 0<x<l/2\\ 2(l-x)/l, & l/2<x<l\end{cases}$, where $l=\pi$.

SOLUTION (i) and (ii) can easily be solved with

```
>> syms x; f=(sym(pi)-abs(x))*sin(x); [A,B,F1]=fseries(f,x,10); F1
   syms x; f=exp(abs(x)); [A,B,F2]=fseries(f,x,10); simple(F2)
```

which yield the results

$$F_1=\frac{1}{2}\pi\sin x+\frac{16}{9}\frac{\sin 2x}{\pi}+\frac{32}{225}\frac{\sin 4x}{\pi}+\frac{48}{1225}\frac{\sin 6x}{\pi}+\frac{64}{3969}\frac{\sin 8x}{\pi}+\frac{80}{9801}\frac{\sin 10x}{\pi}$$

$$\begin{aligned}F_2=&\frac{\mathrm{e}^\pi-1}{\pi}+\frac{(-\mathrm{e}^\pi-1)\cos x}{\pi}+2\frac{(\mathrm{e}^\pi-1)\cos 2x}{5\pi}-\frac{(\mathrm{e}^\pi+1)\cos 3x}{5\pi}\\&+\frac{2\,(\mathrm{e}^\pi-1)\cos 4x}{17\pi}-\frac{(\mathrm{e}^\pi+1)\cos 5x}{13\pi}+2\frac{(\mathrm{e}^\pi-1)\cos 6x}{37\pi}-\frac{(\mathrm{e}^\pi+1)\cos 7x}{25\pi}\\&+\frac{2\,(\mathrm{e}^\pi-1)\cos 8x}{65\pi}-\frac{(\mathrm{e}^\pi+1)\cos 9x}{41\pi}+\frac{2\,(\mathrm{e}^\pi-1)\cos 10x}{101\pi}\end{aligned}$$

The solutions to (iii) is a little bit difficult, the `heaviside()` function in the Symbolic Math Toolbox can be used to express the original function

$$f(x)=2\text{heaviside}\left(x-\frac{\pi}{2}\right)-\frac{2}{\pi}x\frac{|x-\pi/2|}{x-\pi/2}$$

Thus the following MATLAB statements can be used to find the Fourier series

```
>> syms x; pi1=sym(pi);
```

```
f=2*heaviside(x-pi1/2)-2/pi1*x*abs(x-pi1/2)/(x-pi1/2);
[a,b,F]=fseries(f,x,10,-pi,pi); F
```

which returns

$$-\frac{1}{4}+\frac{4}{\pi^2}\cos x+\frac{\left(4\pi^{-1}+2\right)}{\pi}\sin x-\frac{2}{\pi^2}\cos 2x-\frac{\sin 2x}{\pi}+4\frac{\cos 3x}{9\pi^2}$$
$$+\frac{\left(-\pi^{-1}+6\right)}{9\pi}\sin 3x-\frac{1}{2\pi}\sin 4x+\frac{4}{25\pi^2}\cos 5x+\frac{\left(4\pi^{-1}+10\right)}{25\pi}\sin 5x-\frac{2}{9\pi^2}\cos 6x$$
$$-\frac{1}{3\pi}\sin 6x+\frac{4}{49\pi^2}\cos 7x+\frac{\left(-4\pi^{-1}+14\right)}{49\pi}\sin 7x-\frac{1}{4\pi}\sin 8x+\frac{4}{81\pi^2}\cos 9x$$
$$+\frac{\left(4\pi^{-1}+18\right)}{81\pi}\sin 9x-\frac{2}{25\pi^2}\cos 10x-\frac{1}{5\pi}\sin 10x$$

18. Write the Taylor series expansions to the following functions, and compare graphically the approximation results, if finite term series are used.

(i) $\displaystyle\int_0^x \frac{\sin t}{t}\,\mathrm{d}t$, (ii) $\ln\left(\dfrac{1+x}{1-x}\right)$, (iii) $\ln\left(x+\sqrt{1+x^2}\right)$, (iv) $(1+4.2x^2)^{0.2}$,

(v) $\mathrm{e}^{-5x}\sin(3x+\pi/3)$ expansions about $x=0$ and $x=a$ points respectively.

SOLUTION Taylor series expansions for the first four functions can be obtained from the following statements

```
>> syms t x; f1=int(sin(t)/t,t,0,x); F1=taylor(f1,x,15)
   f2=log((1+x)/(1-x)),   F2=taylor(f2,x,15)
   f3=log(x+sqrt(1+x^2)); F3=taylor(f3,x,15)
   f4=(1+4.2*x^2)^0.2;    F4=taylor(f4,x,13)
   ezplot(f3,[-1,1]), hold on, ezplot(f4,[-1,1])
```

which yield respectively

(i) $F_1=x-\dfrac{x^3}{18}+\dfrac{x^5}{600}-\dfrac{x^7}{35280}+\dfrac{x^9}{3265920}-\dfrac{x^{11}}{439084800}+\dfrac{x^{13}}{80951270400}$

(ii) $F_2=2x+\dfrac{2}{3}x^3+\dfrac{2}{5}x^5+\dfrac{2}{7}x^7+\dfrac{2}{9}x^9+\dfrac{2}{11}x^{11}+\dfrac{2}{13}x^{13}$

(iii) $F_3=x-\dfrac{1}{6}x^3+\dfrac{3}{40}x^5-\dfrac{5}{112}x^7+\dfrac{35}{1152}x^9-\dfrac{63}{2816}x^{11}+\dfrac{231}{13312}x^{13}$

(iv) $F_4=1+\dfrac{21}{25}x^2-\dfrac{882}{625}x^4+\dfrac{55566}{15625}x^6-\dfrac{4084101}{390625}x^8+\dfrac{1629556299}{48828125}x^{10}-\dfrac{136882729116}{1220703125}x^{12}$

(v) The first four terms of Taylor series expansion can be obtained with

```
>> syms x a; f=exp(-5*x)*sin(3*x+sym(pi)/3); F5=taylor(f,x,4,a)
```

where

$$F_5=\mathrm{e}^{-5a}\sin\left(3a+\frac{\pi}{3}\right)+\left(3\mathrm{e}^{-5a}\cos\left(3a+\frac{\pi}{3}\right)-5\mathrm{e}^{-5a}\sin\left(3a+\frac{\pi}{3}\right)\right)(x-a)$$
$$+\left(8\mathrm{e}^{-5a}\sin\left(3a+\frac{\pi}{3}\right)-15\mathrm{e}^{-5a}\cos\left(3a+\frac{\pi}{3}\right)\right)(x-a)^2$$
$$+\left(33\mathrm{e}^{-5a}\cos\left(3a+\frac{\pi}{3}\right)+5/3\mathrm{e}^{-5a}\sin\left(3a+\frac{\pi}{3}\right)\right)(x-a)^3$$

19. Get the Taylor series expansion of the function $f(x,y)=\dfrac{1-\cos\left(x^2+y^2\right)}{\left(x^2+y^2\right)\mathrm{e}^{x^2+y^2}}$ about $x=1, y=0$ point.

SOLUTION Taylor series expansion of a two-dimensional function needs the facilities in Maple, with

```
>> syms x y; f=(1-cos(x^2+y^2))/((x^2+y^2)*exp(x^2+y^2));
   F=maple('mtaylor',f,'[x=1,y=0]',4)
```

where

$$\frac{1-\cos 1}{e^1}+\frac{(2\sin 1-4+4\cos 1)(x-1)}{e^1}+\frac{(-6\cos 1-7\sin 1+8)(x-1)^2}{e^1}$$
$$+\frac{(\sin 1-2+2\cos 1)y^2}{e^1}+\frac{\left(-\frac{34}{3}+\frac{32}{3}\sin 1+16/3\cos 1\right)(x-1)^3}{e^1}$$
$$+\frac{(-8\cos 1+10-8\sin 1)y^2(x-1)}{e^1}+\frac{\left(\frac{83}{6}-6\cos 1-\frac{34}{3}\sin 1\right)(x-1)^4}{e^1}$$
$$+\frac{(24\sin 1+16\cos 1-27)y^2(x-1)^2}{e^1}+\frac{(-2\cos 1+5/2-2\sin 1)y^4}{e^1}$$
$$+\frac{\left(\frac{134}{15}\cos 1+\frac{194}{15}\sin 1-\frac{244}{15}\right)(x-1)^5}{e^1}+\frac{\left(\frac{164}{3}-28\cos 1-\frac{140}{3}\sin 1\right)y^2(x-1)^3}{e^1}$$
$$+\frac{(14\sin 1-16+10\cos 1)y^4(x-1)}{e^1}$$

20. Compute the first n term finite sums and infinite sums.
   (i) $\dfrac{1}{1\times 6}+\dfrac{1}{6\times 11}+\cdots+\dfrac{1}{(5n-4)(5n+1)}+\cdots$
   (ii) $\left(\dfrac{1}{2}+\dfrac{1}{3}\right)+\left(\dfrac{1}{2^2}+\dfrac{1}{3^2}\right)+\cdots+\left(\dfrac{1}{2^n}+\dfrac{1}{3^n}\right)+\cdots$

SOLUTION The partial and infinite sums of the series are obtained with

```
>> syms n k; S1=simple(symsum(1/(5*k-4)/(5*k+1),k,1,n)),
   S2=symsum(1/(5*k-4)/(5*k+1),k,1,inf)
   S3=simple(symsum(1/2^k+1/3^k,k,1,n)),
   S4=symsum(1/2^k+1/3^k,k,1,inf)
```

with the results

$$S_1=\frac{n}{5n+1},\ S_2=\frac{1}{5},\ \ S_3=-2^{-n}-\frac{3^{-n}}{2}+\frac{3}{2}\ \text{and}\ S_4=\frac{3}{2}$$

21. Compute the following limits
   (i) $\displaystyle\lim_{n\to\infty}\left[\frac{1}{2^2-1}+\frac{1}{4^2-1}+\frac{1}{6^2-1}+\cdots+\frac{1}{(2n)^2-1}\right]$,
   (ii) $\displaystyle\lim_{n\to\infty} n\left(\frac{1}{n^2+\pi}+\frac{1}{n^2+2\pi}+\frac{1}{n^2+3\pi}+\cdots+\frac{1}{n^2+n\pi}\right)$

SOLUTION The limit of the series can be obtained as $L_1=1/2$, $L_2=1$.

```
>> syms k n; L1=limit( symsum(1/((2*k)^2-1),k,1,n),n,inf)
   L2=limit(n*symsum(1/(n^2+k*pi),k,1,n),n,inf)
```

22. Show that $\cos\theta+\cos 2\theta+\cdots+\cos n\theta=\dfrac{\sin(n\theta/2)\cos[(n+1)\theta/2]}{\sin\theta/2}$.

SOLUTION It is immediately found that the difference between the left-hand-side and the right-hand-side representations is zero, which proves the formula.

```
>> syms theta k n; S1=symsum(cos(k*theta),k,1,n);
   S2=sin(n*theta/2)*cos((n+1)*theta/2)/sin(theta/2); simple(S1-S2)
```

23. For the following tabulated measured data, evaluate numerically its derivatives and definite integral.

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1	1.2
y_i	0	2.2077	3.2058	3.4435	3.241	2.8164	2.311	1.8101	1.3602	0.9817	0.6791	0.4473	0.2768

SOLUTION The differentiation from the data set can be obtained as shown in Figure 3.2, and the integral is $I = 2.2642$.

```
>> x=[0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,1.1,1.2];
   y=[0,2.2077,3.2058,3.4435,3.241,2.8164,2.311,1.8101,...
      1.3602,0.9817,0.6791,0.4473,0.2768];
   [dy1,dx1]=diff_ctr(y,x(2)-x(1),1);
   [dy2,dx2]=diff_ctr(y,x(2)-x(1),2);
   [dy3,dx3]=diff_ctr(y,x(2)-x(1),3);
   [dy4,dx4]=diff_ctr(y,x(2)-x(1),4); I=trapz(x,y)
   plot(dx1+x(1),dy1,'-',dx2+x(1),dy2,'--',dx3+x(1),...
       dy3,':',dx4+x(1),dy4,'-.')
```

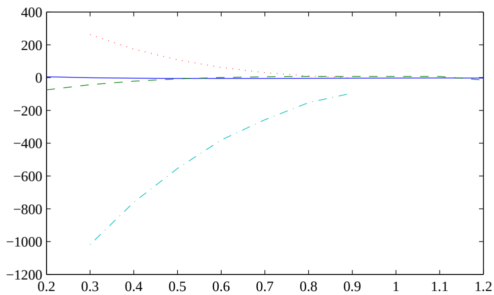


FIGURE 3.2: Numerical differentiation curve

24. Evaluate the definite integral $\displaystyle\int_0^{\pi}(\pi-t)^{\frac{1}{4}}f(t)\mathrm{d}t$, $f(t)=\mathrm{e}^{-t}\sin(3t+1)$ numerically.

Also evaluate the integration function $F(t) = \displaystyle\int_0^t (t-\tau)^{\frac{1}{4}} f(\tau)\,\mathrm{d}\tau$ numerically for different sample points of t, such that $t = 0.1, 0.2, \cdots, \pi$, and draw the $F(t)$ plot.

SOLUTION The definite integral can be obtained with $F_1 = 0.3415$.

```
>> f=@(t)exp(-t).*sin(3*t+1).*(pi-t).^(1/4); F1=quadl(f,0,pi)
```

For the second integral, select a vector $\boldsymbol{T} = [t_1, t_2, \cdots, t_{N+1}]$, $t_1 = 0$. The original integral can be expressed as the sum of the integrals on the sub interval

(t_i, t_{i+1}) such that $F(t) = \displaystyle\sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} (t-\tau)^{1/4} f(\tau) \mathrm{d}\tau$, where t is a discrete instance, with $t = t_{k+1}$. Thus the loop structure follows can be used to evaluate the integrals $F(\boldsymbol{t})$, and the curve of $F(t)$ is obtained as shown in Figure 3.3.

```
>> F0=0; T=[0:0.1:pi,pi]; F=0;
   for i=1:length(T)-1, xm=T(i); xM=T(i+1);
      f=@(tau)exp(-tau).*sin(3*tau+1).*(xM-tau).^(1/4);
      F0=F0+quadl(f,xm,xM); F=[F,F0];
   end
   plot(T,F)
```

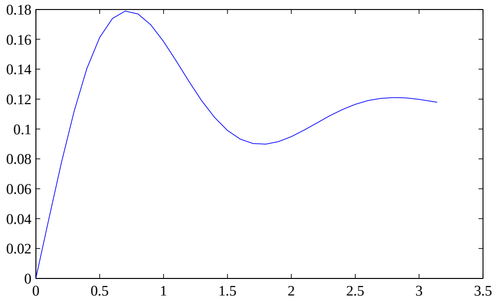


FIGURE 3.3: Integral curve

25. Evaluate numerically the multiple integral problems.

(i) $\displaystyle\int_0^2 \int_0^{\mathrm{e}^{-x^2/2}} \sqrt{4-x^2-y^2}\ \mathrm{e}^{-x^2-y^2}\,\mathrm{d}y\mathrm{d}x$

(ii) $\displaystyle\int_0^2 \int_0^2 \int_0^2 z(x^2+y^2)\,\mathrm{e}^{-x^2-y^2-z^2-xz}\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$

(iii) $\displaystyle\int_0^{7/10} \int_0^{4/5} \int_0^{9/10} \int_0^{1} \int_0^{11/10} \sqrt{6-x^2-y^2-z^2-w^2-u^2}\,\mathrm{d}w\mathrm{d}u\mathrm{d}z\mathrm{d}y\mathrm{d}x$

Solution

It should be noted that there are no analytical solutions to the problems, thus the obtained results should be verified.

(i) One may try first the theoretical method. Unfortunately, the analytical solution or numerical solution to the problem cannot be obtained with the `int()` function.

```
>> syms x y; f=sqrt(4-x^2-y^2)*exp(-x^2-y^2);
   int(int(f,y,0,exp(-x^2/2)),x,0,2)
```

Thus numerical solution methods should be used. For instance, the NIT toolbox can be used to solve (i), with $I_1 = 1.0633$.

```
>> f1=@(x,y)sqrt(4-x.^2-y.^2).*exp(-x.^2-y.^2);
   f1M=@(x)exp(-x.^2/2); f1m=@(x)0; I1=quad2dggen(f1,f1m,f1M,0,2)
```

(ii) Again it can be shown that the analytical solution to the problem cannot be obtained. Thus the following statements should be used to find the numerical solution, which is $I_2 = 0.2078$.

```
>> f=@(x,y,z)z.*(x.^2+y.^2).*exp(-x.^2-y.^2-z.^2-x.*z);
   I2=triplequad(f,0,2,0,2,0,2)
```

(iii) The NIT function `quadndg()` can be used in evaluating the n-dimensional integrals over hyper rectangular regions. There is a bug in the function since it does allow allow the integrand to be descried by anonymous function or inline function. The only way to use is the M-function.
For this example, one can denote that $x_1 = x, x_2 = y, x_3 = z, x_4 = u, x_5 = w$, thus the M-function for the integrand can be written as

```
function y=exc3fmi(x)
y=sqrt(6-x(1)^2-x(2)^2-x(3)^2-x(4)^2-x(5)^2);
```

The integral can be evaluated with $I_3 = 1.1888$.

```
>> I3=quadndg('exc3fmi',[0,0,0,0,0],[7/10,4/5,9/10,1,11/10])
```

Verification method hints: change error tolerance and see whether the same results can be obtained.

26. Compute the gradient of the measured data for a function of two variables. Assume that the data were generated by the function $f(x, y) = 4 - x^2 - y^2$. Generate the data and verify the results of gradient with theoretical results.

0	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0	4	3.96	3.84	3.64	3.36	3	2.56	2.04	1.44	0.76	0
0.2	3.96	3.92	3.8	3.6	3.32	2.96	2.52	2	1.4	0.72	−0.04
0.4	3.84	3.8	3.68	3.48	3.2	2.84	2.4	1.88	1.28	0.6	−0.16
0.6	3.64	3.6	3.48	3.28	3	2.64	2.2	1.68	1.08	0.4	−0.36
0.8	3.36	3.32	3.2	3	2.72	2.36	1.92	1.4	0.8	0.12	−0.64
1	3	2.96	2.84	2.64	2.36	2	1.56	1.04	0.44	−0.24	−1
1.2	2.56	2.52	2.4	2.2	1.92	1.56	1.12	0.6	0	−0.68	−1.44
1.4	2.04	2	1.88	1.68	1.4	1.04	0.6	0.08	−0.52	−1.2	−1.96
1.6	1.44	1.4	1.28	1.08	0.8	0.44	0	−0.52	−1.12	−1.8	−2.56
1.8	0.76	0.72	0.6	0.4	0.12	−0.24	−0.68	−1.2	−1.8	−2.48	−3.24
2	0	−0.04	−0.16	−0.36	−0.64	−1	−1.44	−1.96	−2.56	−3.24	−4

SOLUTION From the data, the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$ can easily found. Also from the known description of the surface function, the analytical representation of the partial derivatives can be obtained as $\partial z/\partial x = -2x$, and $\partial z/\partial y = -2y$, which are planes. The numerical partial derivatives and theoretical ones are shown in Figure 3.4, and it can be seen that the numerical solutions are quite accurate.

```
>> x=0:0.2:2; y=0:0.2:2; [x0,y0]=meshgrid(x,y);
   z0=[4,3.96,3.84,3.63,3.36,3,2.56,2.04,1.43,0.75,0;
      3.96,3.92,3.8,3.59,3.32,2.96,2.52,2,1.39,0.71,-0.04;
```

```
    3.84,3.8,3.68,3.47,3.2,2.84,2.4,1.88,1.27,0.59,-0.16;
    3.63,3.59,3.47,3.27,3,2.63,2.2,1.68,1.07,0.39,-0.36;
    3.36,3.32,3.2,2.99,2.72,2.36,1.92,1.4,0.79,0.11,-0.64;
    3,2.96,2.84,2.63,2.36,2,1.56,1.04,0.43,-0.24,-1;
    2.56,2.52,2.4,2.19,1.92,1.56,1.12,0.6,0,-0.68,-1.44;
    2.04,2,1.88,1.68,1.4,1.04,0.6,0.08,-0.52,-1.2,-1.95;
    1.43,1.39,1.27,1.07,0.79,0.43,0,-0.52,-1.12,-1.8,-2.56;
    0.75,0.71,0.59,0.39,0.11,-0.24,-0.68,-1.2,-1.8,-2.48,-3.24;
    0,-0.04,-0.16,-0.36,-0.64,-1,-1.44,-1.96,-2.56,-3.24,-4];
[fx,fy]=gradient(z0); fx=fx/0.2; fy=fy/0.2;
subplot(221), surf(x0,y0,fx), subplot(222), surf(x0,y0,fy)
syms x y z; z=4-x^2-y^2; zx=diff(z,x), zy=diff(z,y)
fx1=subs(zx,{x,y},{x0,y0}); fy1=subs(zy,{x,y},{x0,y0});
subplot(223), surf(x0,y0,double(fx1)),
subplot(224), surf(x0,y0,double(fy1))
```

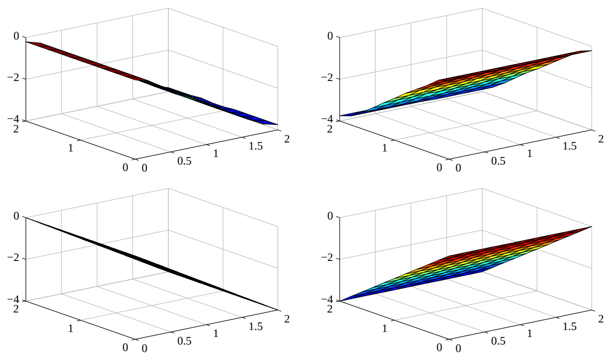


FIGURE 3.4: Partial derivatives

27. Evaluate the following path and line integrals

(i) $\displaystyle\int_l (x^2+y^2)\,\mathrm{d}s$, l: $x=a(\cos t+t\sin t), y=a(\sin t-t\cos t)$, for $0\leqslant t\leqslant 2\pi$

(ii) $\displaystyle\int_l (yx^3+\mathrm{e}^y)\,\mathrm{d}x+(xy^3+x\mathrm{e}^y-2y)\,\mathrm{d}y$, where l is given by the upper-semi-ellipsis of $a^2x^2+b^2y^2=c^2$.

(iii) $\displaystyle\int_l y\,\mathrm{d}x-x\,\mathrm{d}y+(x^2+y^2)\,\mathrm{d}z$, l: $x=\mathrm{e}^t, y=\mathrm{e}^{-t}, z=at$, $0\leqslant t\leqslant 1$, for $a>0$.

(iv) $\displaystyle\int_l (\mathrm{e}^x\sin y-my)\mathrm{d}x+(\mathrm{e}^x\cos y-m)\,\mathrm{d}y$, where l is defined as the closed path from $(a,0)$ to $(0,0)$, then with the upper-semi-circle $x^2+y^2=ax$.

SOLUTION The line and path integrals can easily be obtained with

```
>> syms a t; x=a*(cos(t)+t*sin(t)); y=a*(sin(t)-t*cos(t));
```

```
f=x^2+y^2; I1=int(f*sqrt(diff(x,t)^2+diff(y,t)^2),t,0,2*pi)
syms x y a b c t; x=c*cos(t)/a; y=c*sin(t)/b;
P=y*x^3+exp(y); Q=x*y^3+x*exp(y)-2*y;
ds=[diff(x,t);diff(y,t)]; I2=int([P Q]*ds,t,0,pi)
syms t; syms a positive; x=exp(t); y=exp(-t); z=a*t;
F=[y, -x, (x^2+y^2)];
ds=[diff(x,t);diff(y,t);diff(z,t)]; I3=int(F*ds,t,0,1)
syms t m; syms a positive; x1=t; y1=0;
F1=[exp(x1)*sin(y1)-m*y1, exp(x1)*cos(y1)-m];
x2=a/2+a/2*cos(t); y2=a/2*sin(t);
F2=[exp(x2)*sin(y2)-m*y2, exp(x2)*cos(y2)-m];
I4a=int(F1*[diff(x1,t);diff(y1,t)],t,0,a)
I4b=int(F2*[diff(x2,t);diff(y2,t)],t,0,pi); I4=I4a+I4b
```

and the results are respectively

$I_1 = 2a^3\pi^2 + 4a^3\pi^4$, $I_2 = \dfrac{2c(2c^4 - 15b^4)}{15ab^4}$, $I_3 = 2 + \dfrac{a\mathrm{e}^2}{2} - \dfrac{a\mathrm{e}^{-2}}{2}$, and $I_4 = \dfrac{a^2 m\pi}{8}$.

28. Compute the surface integrals, where S is the bottom side of the semi-sphere $z = \sqrt{R^2 - x^2 - y^2}$.

(i) $\displaystyle\int_S xyz^3\,\mathrm{d}s$, (ii) $\displaystyle\int_S (x + yz^3)\,\mathrm{d}x\mathrm{d}y$.

SOLUTION (i) The original integral can be converted into the double integral

$$I = -\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} xyz^3\sqrt{1 + z_x^2 + z_y^2}\,\mathrm{d}y\mathrm{d}x$$

Thus the integral can be evaluated with $I_1 = 0$.

```
>> syms x y z; syms R positive; F=x*y*z^3; z=sqrt(R^2-x^2-y^2);
   I1=-int(int(F*sqrt(1+diff(z,x)^2+diff(z,y)^2),y,...
          -sqrt(R^2-x^2),sqrt(R^2-x^2)),x,-R,R)
```

(ii) The parametric description of the semi-sphere can also be written as $x = R\sin u\sin v$, $y = R\sin u\cos v$, $z = R\cos u$, $\pi \leqslant u \leqslant 2\pi$, $0 \leqslant v \leqslant 2\pi$. The integral can be evaluated with the following statements and the result is $I_2 = 4\pi R^3/3$.

```
>> syms R positive; syms u v;
   x=R*sin(u)*sin(v); y=R*sin(u)*cos(v); z=R*cos(u); P=x+y*z^3;
   A=diff(y,u)*diff(z,v)-diff(z,u)*diff(y,v);
   I2=int(int(A*P,u,pi,2*pi),v,0,2*pi)
```

Chapter 4

Linear Algebra Problems

Exercises and Solutions

1. Jordanian matrix is a very practical matrix in matrix analysis courses. The general form the the matrix is described as

$$\boldsymbol{J}=\begin{bmatrix}-\alpha & 1 & 0 & \cdots & 0\\ 0 & -\alpha & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -\alpha\end{bmatrix},\quad \text{e.g.,}\quad \boldsymbol{J}_1=\begin{bmatrix}-5 & 1 & 0 & 0 & 0\\ 0 & -5 & 1 & 0 & 0\\ 0 & 0 & -5 & 1 & 0\\ 0 & 0 & 0 & -5 & 1\\ 0 & 0 & 0 & 0 & -5\end{bmatrix}.$$

Construct matrix $\boldsymbol{J}_1$ with the MATLAB function `diag()`.

SOLUTION With the sophisticated function `diag()`, the Jordanian matrix can easily be established

```
>> J1=diag([-5 -5 -5 -5 -5])+diag([1 1 1 1],1)
```

or even more systematically

```
>> n=5; J1=diag(-5*ones(1,n))+diag(ones(1,n-1),1))
```

2. Nilpotent matrix is a special matrix defined as $\boldsymbol{H}_n=\begin{bmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}$. Verify for any pre-specified n, $\boldsymbol{H}_n^i=0$ is satisfied for all $i\geqslant n$.

SOLUTION The `for` loop can be used to test whether there are exceptions to the above definitions. Actually there is no exception for $i<100$.

```
>> for i=1:100
      A=diag(ones(1,i),1); if norm(A^(1+i))>0, disp(i); end
   end
```

It is also interesting to observe the changes of $\boldsymbol{H}_n^i$ when i changes from 1 to n.

3. Can you recognize from the way of display, whether a matrix is a numeric matrix or a symbolic matrix. If $\boldsymbol{A}$ is a numeric matrix and $\boldsymbol{B}$ is a symbolic matrix, can you predict the product $\boldsymbol{C}$=$\boldsymbol{A}$*$\boldsymbol{B}$ is a numeric matrix or a symbolic matrix. Verify the judgement through a simple example.

SOLUTION $\boldsymbol{A}$ times $\boldsymbol{B}$ returns a symbolic matrix.

4. Compute the determinant of a Vandermonde matrix $\boldsymbol{A}=\begin{bmatrix} a^4 & a^3 & a^2 & a & 1 \\ b^4 & b^3 & b^2 & b & 1 \\ c^4 & c^3 & c^2 & c & 1 \\ d^4 & d^3 & d^2 & d & 1 \\ e^4 & e^3 & e^2 & e & 1 \end{bmatrix}$, and find the simplified results.

SOLUTION With the following statements, the Vandermonde matrix can be established first, then the determinant can be evaluated and converted.

```
>> syms a b c d e; A=vander([a b c d e]); simple(det(A))
```

the simplified result is

$$(a-d)(-a+c)(c-d)(b-a)(b-d)(b-c)(-a+e)(e-d)(e-c)(-b+e).$$

5. Input matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in MATLAB, and convert them into symbolic matrices.

$$\boldsymbol{A}=\begin{bmatrix} 5 & 7 & 6 & 5 & 1 & 6 & 5 \\ 2 & 3 & 1 & 0 & 0 & 1 & 4 \\ 6 & 4 & 2 & 0 & 6 & 4 & 4 \\ 3 & 9 & 6 & 3 & 6 & 6 & 2 \\ 10 & 7 & 6 & 0 & 0 & 7 & 7 \\ 7 & 2 & 4 & 4 & 0 & 7 & 7 \\ 4 & 8 & 6 & 7 & 2 & 1 & 7 \end{bmatrix}, \quad \boldsymbol{B}=\begin{bmatrix} 3 & 5 & 5 & 0 & 1 & 2 & 3 \\ 3 & 2 & 5 & 4 & 6 & 2 & 5 \\ 1 & 2 & 1 & 1 & 3 & 4 & 6 \\ 3 & 5 & 1 & 5 & 2 & 1 & 2 \\ 4 & 1 & 0 & 1 & 2 & 0 & 1 \\ -3 & -4 & -7 & 3 & 7 & 8 & 12 \\ 1 & -10 & 7 & -6 & 8 & 1 & 5 \end{bmatrix}.$$

SOLUTION The input and conversions of the matrices are easy and straightforward. Then the symbolic forms can be obtained.

```
>> A=[5,7,6,5,1,6,5; 2,3,1,0,0,1,4; 6,4,2,0,6,4,4; 3,9,6,3,6,6,2;
      10,7,6,0,0,7,7; 7,2,4,4,0,7,7; 4,8,6,7,2,1,7]; A=sym(A)
   B=[3,5,5,0,1,2,3; 3,2,5,4,6,2,5; 1,2,1,1,3,4,6; 3,5,1,5,2,1,2;
      4,1,0,1,2,0,1; -3,-4,-7,3,7,8,12; 1,-10,7,-6,8,1,5]; B=sym(B)
```

6. Check whether the matrices given in the above exercise are singular or not. Find the rank, determinant, trace and inverse matrices for them. Check whether the inverse matrices are correct or not.

SOLUTION The problem can be solved with the following statements

```
>> A=[5,7,6,5,1,6,5; 2,3,1,0,0,1,4; 6,4,2,0,6,4,4; 3,9,6,3,6,6,2;
      10,7,6,0,0,7,7; 7,2,4,4,0,7,7; 4,8,6,7,2,1,7]; A=sym(A);
   det(A), rank(A), trace(A), inv(A), inv(A)*A
```

and it is found that $\det(\boldsymbol{A})=-35432$, $\text{rank}(\boldsymbol{A})=7$, $\text{trace}(\boldsymbol{A})=27$, and the verified inverse matrix is obtained as

$$\boldsymbol{A}^{-1}=\begin{bmatrix} \frac{2297}{4429} & -\frac{1157}{4429} & \frac{2771}{8858} & -\frac{1709}{4429} & \frac{173}{4429} & -\frac{1278}{4429} & -\frac{178}{4429} \\ \frac{6465}{8858} & \frac{1537}{8858} & \frac{4151}{17716} & -\frac{1332}{4429} & -\frac{469}{4429} & -\frac{1758}{4429} & -\frac{1467}{8858} \\ -\frac{24047}{17716} & -\frac{4651}{17716} & -\frac{20641}{35432} & \frac{6091}{8858} & \frac{3079}{8858} & \frac{4827}{8858} & \frac{6439}{17716} \\ \frac{5515}{8858} & -\frac{977}{8858} & \frac{4185}{17716} & -\frac{1424}{4429} & -\frac{794}{4429} & -\frac{842}{4429} & -\frac{491}{8858} \\ -\frac{6163}{17716} & -\frac{887}{17716} & \frac{463}{35432} & \frac{1613}{8858} & -\frac{127}{8858} & \frac{1271}{8858} & \frac{1567}{17716} \\ \frac{3781}{17716} & \frac{2565}{17716} & \frac{219}{35432} & \frac{189}{8858} & -\frac{921}{8858} & \frac{429}{8858} & -\frac{3353}{17716} \\ -\frac{9225}{17716} & \frac{4959}{17716} & -\frac{6663}{35432} & \frac{2137}{8858} & -\frac{9}{8858} & \frac{2601}{8858} & \frac{1785}{17716} \end{bmatrix}$$

7. Find the characteristic polynomials, eigenvalues and eigenvectors for the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in Exercise 5. Validate Hamilton-Caylay Theorem and explain how the error, if any, may be eliminated.

SOLUTION Take matrix $\boldsymbol{A}$ as an example, the characteristic polynomial and the eigenvalues can be obtained with the following statements

```
>> A=[5,7,6,5,1,6,5; 2,3,1,0,0,1,4; 6,4,2,0,6,4,4; 3,9,6,3,6,6,2;
     10,7,6,0,0,7,7; 7,2,4,4,0,7,7; 4,8,6,7,2,1,7];
   e=eig(A), p=poly(sym(A)), p=sym2poly(p), polyvalm(p,A)
```

The characteristic polynomial is $x^7-27x^6-18x^5-1000x^4+3018x^3+24129x^2+2731x+35432$, and the Hamilton-Caylay Theorem is also validated.
If numerical method is used, the norm of error may be 1.494×10^{-5}. One should replace `poly()` by `poly1()`, to avoid the error.

```
>> p=poly(A), e=polyvalm(p,A), norm(e)
```

8. Perform singular value decompositions, LU decompositions and orthogonal decompositions to the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ in Exercise 5.

SOLUTION Still for matrix $\boldsymbol{A}$, the LU decomposition can be performed either in numerical or in symbolic method

```
>> A=[5,7,6,5,1,6,5; 2,3,1,0,0,1,4; 6,4,2,0,6,4,4; 3,9,6,3,6,6,2;
     10,7,6,0,0,7,7; 7,2,4,4,0,7,7; 4,8,6,7,2,1,7];
   [L,U]=lu(A), [L1,U1]=lu(sym(A))
```

where numerical solution are

$$\boldsymbol{L}=\begin{bmatrix} 0.5 & 0.50725 & 0.55556 & 0.10989 & 0.5 & 0.53756 & 1 \\ 0.2 & 0.23188 & -0.75 & 0.64286 & 0.57018 & 1 & 0 \\ 0.6 & -0.028986 & -0.94444 & 1 & 0 & 0 & 0 \\ 0.3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7 & -0.42029 & 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.75362 & 0.27778 & 0.64835 & 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{U}=\begin{bmatrix}10 & 7 & 6 & 0 & 0 & 7 & 7\\ 0 & 6.9 & 4.2 & 3 & 6 & 3.9 & -0.1\\ 0 & 0 & 1.5652 & 5.2609 & 2.5217 & 3.7391 & 2.058\\ 0 & 0 & 0 & 5.0556 & 8.5556 & 3.4444 & 1.7407\\ 0 & 0 & 0 & 0 & -8.7692 & -8.011 & 2.5751\\ 0 & 0 & 0 & 0 & 0 & 3.8534 & 1.5794\\ 0 & 0 & 0 & 0 & 0 & 0 & -1.9204\end{bmatrix}$$

and the analytical solutions are

$$\boldsymbol{L}_1=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ 2/5 & 1 & 0 & 0 & 0 & 0 & 0\\ 6/5 & -22 & 1 & 0 & 0 & 0 & 0\\ 3/5 & 24 & -1 & 1 & 0 & 0 & 0\\ 2 & -35 & 55/36 & 65/36 & 1 & 0 & 0\\ 7/5 & -39 & 59/36 & -17/36 & 3/17 & 1 & 0\\ 4/5 & 12 & -1/2 & -1 & -60/119 & -102/35 & 1\end{bmatrix}$$

$$\boldsymbol{U}_1=\begin{bmatrix}5 & 7 & 6 & 5 & 1 & 6 & 5\\ 0 & 1/5 & -7/5 & -2 & -2/5 & -7/5 & 2\\ 0 & 0 & -36 & -50 & -4 & -34 & 42\\ 0 & 0 & 0 & -2 & 11 & 2 & -7\\ 0 & 0 & 0 & 0 & -119/4 & -17/3 & 557/36\\ 0 & 0 & 0 & 0 & 0 & 5/3 & 479/153\\ 0 & 0 & 0 & 0 & 0 & 0 & 17716/1785\end{bmatrix}$$

9. For arbitrary matrices

$$\boldsymbol{A}_1=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},\ \boldsymbol{A}_2=\begin{bmatrix}a_{11} & a_{12} & a_{13} & a_{14}\\ a_{21} & a_{22} & a_{23} & a_{24}\\ a_{31} & a_{32} & a_{33} & a_{34}\\ a_{41} & a_{42} & a_{43} & a_{44}\end{bmatrix}$$

verify the Hamilton-Caylay Theorem.

SOLUTION The Hamilton-Caylay Theorem can be verified for the arbitrary 3×3 and 4×4 matrices with the following statements and it can be seen that both the resulted matrices are zero matrices.

```
>> syms a11 a12 a13 a21 a22 a23 a31 a32 a33 x;
   A=[a11 a12 a13; a21 a22 a23; a31 a32 a33]; p=poly(A)
   P=polycoef(p,x,3); simple(P(1)*A^3+P(2)*A^2+P(3)*A+P(4)*eye(3))
   syms a14 a24  a34 a41 a42 a43 a44;
   A=[A [a14; a24; a34]; a41 a42 a43 a44];
   p=poly(A); P=polycoef(p,x,4);
   simple(P(1)*A^4+P(2)*A^3+P(3)*A^2+P(4)*A+P(5)*eye(4))
```

10. Perform LU factorization and SVD decomposition to the following matrices

$$\boldsymbol{A}=\begin{bmatrix}8 & 0 & 1 & 1 & 6\\ 9 & 2 & 9 & 4 & 0\\ 1 & 5 & 9 & 9 & 8\\ 9 & 9 & 4 & 7 & 9\\ 6 & 9 & 8 & 9 & 6\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}1 & 2 & 2 & 2\\ 1 & 1 & 2 & 0\\ 1 & 1 & 1 & 0\\ 0 & 0 & 2 & 0\end{bmatrix}.$$

SOLUTION LU factorization and SVD decomposition of matrix $\boldsymbol{A}$ are found by

```
>> A=[8,0,1,1,6; 9,2,9,4,0; 1,5,9,9,8; 9,9,4,7,9; 6,9,8,9,6];
   [L,U]=lu(sym(A)), [L,A1,M]=svd(A)
```

and the matrices are

$$\boldsymbol{L}=\begin{bmatrix}1&0&0&0&0\\9/8&1&0&0&0\\1/8&5/2&1&0&0\\9/8&9/2&521/173&1&0\\3/4&9/2&451/173&1572/2101&1\end{bmatrix}$$

$$\boldsymbol{U}=\begin{bmatrix}8&0&1&1&6\\0&2&63/8&23/8&-27/4\\0&0&-173/16&27/16&193/8\\0&0&0&-2101/173&-6925/173\\0&0&0&0&-2242/2101\end{bmatrix}$$

$$\boldsymbol{N}=\begin{bmatrix}-0.23011&-0.6123&-0.32765&-0.57127&0.37207\\-0.35339&-0.44337&0.79124&0.014875&-0.22859\\-0.46818&0.59296&0.077909&-0.61995&-0.19699\\-0.54646&-0.17809&-0.50623&0.35009&-0.53929\\-0.55169&0.21259&0.065144&0.40808&0.69258\end{bmatrix}$$

$$\boldsymbol{A}_1=\mathrm{diag}([30.8316,9.8319,8.4099,4.8975,0.1796])$$

$$\boldsymbol{M}=\begin{bmatrix}-0.44492&-0.87705&0.049073&0.1109&0.13462\\-0.41941&0.24294&-0.23755&0.76644&-0.34817\\-0.46133&0.17518&0.71237&-0.27604&-0.41573\\-0.47509&0.36794&0.069106&0.006557&0.7963\\-0.43314&0.07553&-0.65492&-0.56924&-0.23179\end{bmatrix}$$

11. Compute the eigenvalues, eigenvectors and singular values of the following matrices.

$$\boldsymbol{A}=\begin{bmatrix}2&7&5&7&7\\7&4&9&3&3\\3&9&8&3&8\\5&9&6&3&6\\2&6&8&5&4\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}703&795&980&137&661\\547&957&271&12&284\\445&523&252&894&469\\695&880&876&199&65\\621&173&737&299&988\end{bmatrix}.$$

SOLUTION For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the problem can be solved by

```
>> A=[2,7,5,7,7; 7,4,9,3,3; 3,9,8,3,8; 5,9,6,3,6; 2,6,8,5,4];
   e=eig(A), [v,d]=eig(A), svd(A)
   B=[703,795,980,137,661; 547,957,271,12,284; 445,523,252,894,469;
      695,880,876,199,65; 621,173,737,299,988]
   e=eig(B), [v,d]=eig(B), svd(B)
```

and the eigenvalues of matrix $\boldsymbol{A}$ can be found as $27.8629, 2.6062, -2.2306 \pm \mathrm{j}1.8926, -5.0078$, and the eigenvalues of $\boldsymbol{B}$ are $2670, -580.9, -136.04, 572.99 \pm \mathrm{j}315.67$. More accurate eigenvalues can be obtained when one convert $\boldsymbol{A}$ and $\boldsymbol{B}$ into symbolic ones.

12. Please check whether the following matrices are positive-definite ones, if so, please find the Cholesky decomposited matrices.

$$\boldsymbol{A}=\begin{bmatrix}9&2&1&2&2\\2&4&3&3&3\\1&3&7&3&4\\2&3&3&5&4\\2&3&4&4&5\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}16&17&9&12&12\\17&12&12&2&18\\9&12&18&7&13\\12&2&7&18&12\\12&18&13&12&10\end{bmatrix}.$$

SOLUTION One may test the positive-definiteness of a symmetrical matrix via `chol()`. If a square matrix of the size of $\boldsymbol{A}$ is returned, it is a positive-definite matrix. So for $\boldsymbol{A}$, it is a positive-definite matrix.

```
>> A=[9,2,1,2,2; 2,4,3,3,3; 1,3,7,3,4; 2,3,3,5,4; 2,3,4,4,5]
   chol(A), L=chol(sym(A))
```

and the symbolic lower-triangular matrix

$$\boldsymbol{L}=\begin{bmatrix}3&0&0&0&0\\2/3&4\sqrt{2}/3&0&0&0\\1/3&25\sqrt{2}/24&\sqrt{302}/8&0&0\\2/3&23\sqrt{2}/24&25\sqrt{302}/1208&\sqrt{59041}/151&0\\2/3&\sqrt{2}/24&57\sqrt{302}/1208&215\sqrt{59041}/59041&\sqrt{193154}/391\end{bmatrix}$$

13. Perform the Jordanian transformation for $\boldsymbol{A}=\begin{bmatrix}-2&0.5&-0.5&0.5\\0&-1.5&0.5&-0.5\\2&0.5&-4.5&0.5\\2&1&-2&-2\end{bmatrix}$, and also find the corresponding transformation matrix.

SOLUTION The Jordanian transformation can be performed by

```
>> A=[-2,0.5,-0.5,0.5; 0,-1.5,0.5,-0.5; 2,0.5,-4.5,0.5; 2,1,-2,-2];
   [V,J]=jordan(sym(A))
```

with

$$\boldsymbol{V}=\begin{bmatrix}0&1/2&1/2&-1/4\\0&0&1/2&1\\1/4&1/2&1/2&-1/4\\1/4&1/2&1&-1/4\end{bmatrix},\quad \boldsymbol{J}=\begin{bmatrix}-4&0&0&0\\0&-2&1&0\\0&0&-2&1\\0&0&0&-2\end{bmatrix}$$

The above method also applies for a numerical matrix $\boldsymbol{A}$.

14. Find the basic set of solutions of the homogenous equations

$$\text{(i)}\ \begin{cases}6x_1+x_2+4x_3-7x_4-3x_5=0\\-2x_1-7x_2-8x_3+6x_4\quad=0\\-4x_1+5x_2+x_3-6x_4+8x_5=0\\-34x_1+36x_2+9x_3-21x_4+49x_5=0\\-26x_1-12x_2-27x_3+27x_4+17x_5=0,\end{cases}\qquad \text{(ii)}\ \boldsymbol{A}=\begin{bmatrix}-1&2&-2&1&0\\0&3&2&2&1\\3&1&3&2&-1\end{bmatrix}.$$

SOLUTION (i) The coefficient matrix of the homogenous equation can be established first and the basic set of solutions can be written as $\boldsymbol{e}_1^{\mathrm{T}}=[95/17,1,0,103/34,151/34]$ and $\boldsymbol{e}_2^{\mathrm{T}}=[191/34,0,1,109/34,173/34]$.

```
>> A1=[6,1,4,-7,-3; -2,-7,-8,6,0; -4,5,1,-6,8;
      -34,36,9,-21,49; -26,-12,-27,27,17]; A=sym(A);
   rank(A1), v=null(A1); e1=v(:,1), e2=v(:,2)
```

(ii) For the given matrix $\boldsymbol{A}$, the basic set of solutions can also be found such that $\boldsymbol{e}_3^{\mathrm{T}} = [-18, 1, 10, 0, -23]$, and $\boldsymbol{e}_4^{\mathrm{T}} = [-13, 0, 7, 1, -16]$.

```
>> A2=[-1,2,-2,1,0; 0,3,2,2,1; 3,1,3,2,-1]; A=sym(A);
   A2=sym(A2); rank(A2), v=null(A2); e3=v(:,1), e4=v(:,2)
```

Thus for any a_1 and a_2, the analytical equations are

```
>> syms a1 a2; x1=a1*e1+a2*e2, x2=a1*e3+a2*e4, A1*x1, A2*x2
```

with (i), $\boldsymbol{x}_1 = \begin{bmatrix} 191a_1/34 + 95a_2/17 \\ a_2 \\ a_1 \\ 109a_1/34 + 103a_2/34 \\ 173a_1/34 + 151a_2/34 \end{bmatrix}$, (ii) $\boldsymbol{x}_2 = \begin{bmatrix} -13a_1 - 18a_2 \\ a_2 \\ 7a_1 + 10a_2 \\ a_1 \\ -16a_1 - 23a_2 \end{bmatrix}$.

15. Find the numerical and analytical solutions to the following linear algebraic equations, and then validate the results.

$$\begin{bmatrix} 2 & -9 & 3 & -2 & -1 \\ 10 & -1 & 10 & 5 & 0 \\ 8 & -2 & -4 & -6 & 3 \\ -5 & -6 & -6 & -8 & -4 \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} -1 & -4 & 0 \\ -3 & -8 & -4 \\ 0 & 3 & 3 \\ 9 & -5 & 3 \end{bmatrix}.$$

SOLUTION One should check first the solvability problem

```
>> A=[2,-9,3,-2,-1; 10,-1,10,5,0; 8,-2,-4,-6,3; -5,-6,-6,-8,-4];
   B=[-1,-4,0; -3,-8,-4; 0,3,3; 9,-5,3];
   rank(A), rank([A B])
```

Since the two ranks are the same, there are infinite numbers of solutions with

```
>> syms a; v=null(sym(A))*a; x=[v v v]+sym(A)\B, A*x-B
```

The solutions are given for any a, that

$$\begin{bmatrix} a+967/1535 & a-943/1535 & a-159/1535 \\ -1535a/1524 & -1535a/1524 & -1535a/1524 \\ -3659a/1524-1807/1535 & -3659a/1524-257/1535 & -3659a/1524-141/1535 \\ 1321a/508+759/1535 & 1321a/508-56/1535 & 1321a/508-628/1535 \\ -170a/127-694/307 & -170a/127+719/307 & -170a/127+103/307 \end{bmatrix}.$$

16. Check whether the equation $\begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 7 \end{bmatrix}$ has a solution.

SOLUTION The rank of the two related matrices can be obtained and since they are not equal, there is no solution to the linear algebraic equation.

```
>> A=[16,2,3,13; 5,11,10,8; 9,7,6,12; 4,14,15,1];
   B=[1; 3; 4; 7]; [rank(A), rank([A B])]
```

17. Find the analytical solutions to the following linear algebraic equations, and then validate the results.

$$\begin{bmatrix} 2 & 9 & 4 & 12 & 5 & 8 & 6 \\ 12 & 2 & 8 & 7 & 3 & 3 & 7 \\ 3 & 0 & 3 & 5 & 7 & 5 & 10 \\ 3 & 11 & 6 & 6 & 9 & 9 & 1 \\ 11 & 2 & 1 & 4 & 6 & 8 & 7 \\ 5 & -18 & 1 & -9 & 11 & -1 & 18 \\ 26 & -27 & -1 & 0 & -15 & -13 & 18 \end{bmatrix} \boldsymbol{X} = \begin{bmatrix} 1 & 9 \\ 5 & 12 \\ 4 & 12 \\ 10 & 9 \\ 0 & 5 \\ 10 & 18 \\ -20 & 2 \end{bmatrix}.$$

SOLUTION We should check first the solvability of the equation

```
>> A=[2,9,4,12,5,8,6; 12,2,8,7,3,3,7; 3,0,3,5,7,5,10; 3,11,6,6,9,9,1;
     11,2,1,4,6,8,7; 5,-18,1,-9,11,-1,18; 26,-27,-1,0,-15,-13,18];
   B=[1,9; 5,12; 4,12; 10,9; 0,5; 10,18; -20,2];
   C=[A B]; rank(A), rank(C)
```

and it is found that rank($\boldsymbol{A}$) =rank($\boldsymbol{C}$) $= 5$, which means that there are infinite sets of equations. One may solve and verify the equation as follows

```
>> syms a1 a2; z=null(sym(A)); x0=sym(A)\B;
   x1=z*[a1; a2]; X=[x1 x1]+x0, A*X-B
```

The analytical solution to the equation can be found, and substituting it back to the original equation, zero error matrix can be returned.

$$\boldsymbol{X} = \begin{bmatrix} \frac{6386}{9453}a_1 - \frac{7118}{9453}a_2 - \frac{15139}{33710} & \frac{6386}{9453}a_1 - \frac{7118}{9453}a_2 - \frac{3599}{6742} \\ a_1 - \frac{3106}{16855} & a_1 - \frac{4302}{3371} \\ -\frac{14446}{9453}a_1 + \frac{15643}{9453}a_2 + \frac{60429}{33710} & -\frac{14446}{9453}a_1 + \frac{15643}{9453}a_2 + \frac{12527}{6742} \\ \frac{6437}{9453}a_1 - \frac{15716}{9453}a_2 - \frac{30043}{33710} & \frac{6437}{9453}a_1 - \frac{15716}{9453}a_2 + \frac{2153}{6742} \\ \frac{16855}{9453}a_1 - \frac{24190}{9453}a_2 & \frac{16855}{9453}a_1 - \frac{24190}{9453}a_2 \\ -\frac{25198}{9453}a_1 + \frac{25561}{9453}a_2 + \frac{29837}{33710} & -\frac{25198}{9453}a_1 + \frac{25561}{9453}a_2 + \frac{8671}{6742} \\ a_2 & a_2 \end{bmatrix}$$

18. For the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, calculate $\boldsymbol{A} \otimes \boldsymbol{B}$ and $\boldsymbol{B} \otimes \boldsymbol{A}$. Are they equal?

$$\boldsymbol{A} = \begin{bmatrix} -1 & 2 & 2 & 1 \\ -1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 3 & 0 & 3 \\ 3 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}.$$

SOLUTION The two Kronecker products can be obtained (display omitted). Obviously they are different.

```
>> A=[-1,2,2,1; -1,2,1,0; 2,1,1,0; 1,0,2,0];
   B=[3,0,3; 3,2,2; 3,1,1]; kron(A,B), kron(B,A)
```

19. Find the analytical and numerical solutions to the following Sylvester equation, and verify the results.

$$\begin{bmatrix} 3 & -6 & -4 & 0 & 5 \\ 1 & 4 & 2 & -2 & 4 \\ -6 & 3 & -6 & 7 & 3 \\ -13 & 10 & 0 & -11 & 0 \\ 0 & 4 & 0 & 3 & 4 \end{bmatrix} \boldsymbol{X} + \boldsymbol{X} \begin{bmatrix} 3 & -2 & 1 \\ -2 & -9 & 2 \\ -2 & -1 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -1 \\ 4 & 1 & 2 \\ 5 & -6 & 1 \\ 6 & -4 & -4 \\ -6 & 6 & -3 \end{bmatrix}.$$

SOLUTION The numerical and analytical solutions to the Sylvester equation can easily be found with the following statements

```
>> A=[3,-6,-4,0,5; 1,4,2,-2,4; -6,3,-6,7,3; -13,10,0,-11,0;
     0,4,0,3,4]; B=[3,-2,1; -2,-9,2; -2,-1,9];
   C=[-2,1,-1; 4,1,2; 5,-6,1; 6,-4,-4; -6,6,-3];
   X1=lyap(A,B,C), e1=norm(A*X1+X1*B+C), X2=lyap(sym(A),B,C)
```

and the results are

$$\boldsymbol{X}_1 = \begin{bmatrix} -4.0569 & -14.513 & 1.5653 \\ 0.035558 & 25.074 & -2.7408 \\ 9.4886 & 25.932 & -4.4177 \\ 2.6969 & 21.645 & -2.8851 \\ 7.7229 & 31.91 & -3.7634 \end{bmatrix}, \quad e_1 = 3.987\times 10^{-13},$$

$$\boldsymbol{X}_2 = \begin{bmatrix} \dfrac{-434641749950}{107136516451} & -\dfrac{4664546747350}{321409549353} & \dfrac{503105815912}{321409549353} \\ \dfrac{3809507498}{107136516451} & \dfrac{8059112319373}{321409549353} & -\dfrac{880921527508}{321409549353} \\ \dfrac{1016580400173}{107136516451} & \dfrac{8334897743767}{321409549353} & -\dfrac{1419901706449}{321409549353} \\ \dfrac{288938859984}{107136516451} & \dfrac{6956912657222}{321409549353} & -\dfrac{927293592476}{321409549353} \\ \dfrac{827401644798}{107136516451} & \dfrac{10256166034813}{321409549353} & -\dfrac{1209595497577}{321409549353} \end{bmatrix}$$

20. Find the analytical and numerical solution to the matrix equation given below and verify the results.

$$\begin{bmatrix} -2 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \boldsymbol{X} \begin{bmatrix} -2 & -1 & 2 \\ 1 & 3 & 0 \\ 3 & -2 & 2 \end{bmatrix} - \boldsymbol{X} + \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} = 0.$$

SOLUTION Denote the original equation by $\boldsymbol{AXB} - \boldsymbol{X} + \boldsymbol{C} = 0$, where

$$\boldsymbol{A} = \begin{bmatrix} -2 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -2 & -1 & 2 \\ 1 & 3 & 0 \\ 3 & -2 & 2 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

It is easily found that $\boldsymbol{AX} + \boldsymbol{X}(-\boldsymbol{B}^{-1}) = -\boldsymbol{CB}^{-1}$, and the solution to the equation can easily found and verified

```
>> A=[-2,2,1; -1,0,-1; 1,-1,2]; B=[-2,-1,2; 1,3,0; 3,-2,2];
   C=[0,-1,0; -1,1,0; 1,-1,-1];
   X=lyap(sym(A),-inv(B),C*inv(B)), A*X*B-X+C
```

It is found that $\boldsymbol{AXB} - \boldsymbol{X} + \boldsymbol{C} = 0$ with

$$\boldsymbol{X} = \begin{bmatrix} 4147/47149 & 3861/471490 & -40071/235745 \\ -2613/94298 & 2237/235745 & -43319/235745 \\ 20691/94298 & 66191/235745 & -10732/235745 \end{bmatrix}$$

21. Assume that a Riccati equation is given by $\boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{Q}=0$, where

$$\boldsymbol{A}=\begin{bmatrix}-27 & 6 & -3 & 9\\ 2 & -6 & -2 & -6\\ -5 & 0 & -5 & -2\\ 10 & 3 & 4 & -11\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0 & 3\\ 16 & 4\\ -7 & 4\\ 9 & 6\end{bmatrix},\ \boldsymbol{Q}=\begin{bmatrix}6 & 5 & 3 & 4\\ 5 & 6 & 3 & 4\\ 3 & 3 & 6 & 2\\ 4 & 4 & 2 & 6\end{bmatrix},\ \boldsymbol{R}=\begin{bmatrix}4 & 1\\ 1 & 5\end{bmatrix}.$$

Solve the equation and verify the result.

SOLUTION The numerical solution to the Riccati equation can be solved with an error-level of the 10^{-13}.

```
>> A=[-27,6,-3,9; 2,-6,-2,-6; -5,0,-5,-2; 10,3,4,-11];
   B=[0,3; 16,4; -7,4; 9,6];
   Q=[6,5,3,4; 5,6,3,4; 3,3,6,2; 4,4,2,6];  R=[4,1; 1,5];
   C=Q; B1=B*inv(R)*B'; P=are(A,B1,C),
   norm(P*A+A'*P-P*B*inv(R)*B'*P+Q)
```

with the solutions

$$\boldsymbol{P}=\begin{bmatrix}0.12264 & 0.10885 & 0.027266 & 0.11846\\ 0.10885 & 0.28127 & 0.16587 & 0.06371\\ 0.027266 & 0.16587 & 0.4023 & 0.013892\\ 0.11846 & 0.06371 & 0.013892 & 0.22087\end{bmatrix}$$

22. Certain functions can be expressed by polynomial functions, i.e., Taylor series expansions. In these functions, if x is substituted by matrix $\boldsymbol{A}$, the nonlinear function can also be expressed for matrices. Write M-functions for the matrix function evaluation problems and verify the results with `funm()`.

(i) $\cos\boldsymbol{A}=\boldsymbol{I}-\dfrac{1}{2!}\boldsymbol{A}^2+\dfrac{1}{4!}\boldsymbol{A}^4-\dfrac{1}{6!}\boldsymbol{A}^6+\cdots+\dfrac{(-1)^n}{(2n)!}\boldsymbol{A}^{2n}+\cdots$

(ii) $$\arcsin\boldsymbol{A}=\boldsymbol{A}+\frac{1}{2\cdot 3}\boldsymbol{A}^3+\frac{1\cdot 3}{2\cdot 4\cdot 5}\boldsymbol{A}^5+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\boldsymbol{A}^7 +\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}\boldsymbol{A}^9+\cdots+\frac{(2n)!}{2^{2n}(n!)^2(2n+1)}\boldsymbol{A}^{2n+1}+\cdots$$

(iii) $\ln\boldsymbol{A}=\boldsymbol{A}-\boldsymbol{I}-\dfrac{1}{2}(\boldsymbol{A}-\boldsymbol{I})^2+\dfrac{1}{3}(\boldsymbol{A}-\boldsymbol{I})^3-\dfrac{1}{4}(\boldsymbol{A}-\boldsymbol{I})^4+\cdots+\dfrac{(-1)^{n+1}}{n}(\boldsymbol{A}-\boldsymbol{I})^n+\cdots$.

SOLUTION (i) the $(n+1)$th term divided by the nth term is

$$\frac{\dfrac{(-1)^{n+1}}{(2n+2)!}\boldsymbol{A}^{2n+2}}{\dfrac{(-1)^n}{(2n)!}\boldsymbol{A}^{2n}}=-\frac{\boldsymbol{A}^2}{(2n+2)(2n+1)}$$

and the loop structure can be used to implement the sum of series in M-function

```
function A1=cosm1(A,eps0)
n=0; F=eye(size(A)); A1=F; if nargin==1, eps0=eps; end
while norm(F)>eps0, F=-A^2*F/(2*n+1)/(2*n+2); A1=A1+F; n=n+1; end
```

(ii) the $(n+1)$th term divided by the nth term is

$$\frac{\dfrac{(2n+2)!}{2^{2n+2}((n+1)!)^2(2n+3)}\boldsymbol{A}^{2n+3}}{\dfrac{(2n)!}{2^{2n}(n!)^2(2n+1)}\boldsymbol{A}^{2n+1}}=\frac{(2n+1)^2\boldsymbol{A}^2}{(2n+2)(2n+3)}$$

thus an M-function can be written

```
function A1=asinm1(A,eps0)
n=0; F=A; A1=F; if nargin==1, eps0=eps; end
while norm(F)>eps0,
   F=F*A^2*(2*n+1)^2/(2*n+2)/(2*n+3); A1=A1+F, n=n+1;
end
```

(iii) the $(n+1)$th term divided by the nth term is

$$\frac{\dfrac{(-1)^{n+2}}{n+1}(\boldsymbol{A}-\boldsymbol{I})^{n+1}}{\dfrac{(-1)^{n+1}}{n}(\boldsymbol{A}-\boldsymbol{I})^{n}}=-(\boldsymbol{A}-\boldsymbol{I})\frac{n}{n+1}$$

and an M-function can be written to implement the algorithm

```
function A1=logm1(A,eps0)
n=0; I=eye(size(A)); F=A-I; A1=F; if nargin==1, eps0=eps; end
while norm(F)>eps0, F=-F*(A-I)*n/(n+1); A1=A1+F; n=n+1; end
```

It should be noted that since the Taylor series for $\arcsin(x)$ and $\ln(x)$ are conditional, for instance, $\arcsin(x)$ for $|x|<1$ and $\ln(x)$ for $0<x\leqslant 2$, the use of the Taylor series for matrix function evaluation is restricted. One is suggested to use the `funm()` function designed in the book.

23. For an autonomous linear differential equation of the form $\dot{\boldsymbol{x}}(t)=\boldsymbol{A}\boldsymbol{x}(t)$, the analytical solution can be written as $\boldsymbol{x}(t)=\mathrm{e}^{\boldsymbol{A}t}\boldsymbol{x}(0)$. Find the analytical solution to the equation

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}-3&0&0&1\\-1&-1&1&-1\\1&0&-2&1\\0&0&0&-4\end{bmatrix}\boldsymbol{x}(t),\quad \boldsymbol{x}(0)=\begin{bmatrix}-1\\0\\3\\1\end{bmatrix}.$$

SOLUTION The analytical solution of the differential equation can be found as

$$\boldsymbol{x}^{\mathrm{T}}(t)=[-\mathrm{e}^{-4t},3\mathrm{e}^{-t}-3\mathrm{e}^{-2t},3\mathrm{e}^{-2t},\mathrm{e}^{-4*t}].$$

```
>> A=[-3,0,0,1; -1,-1,1,-1; 1,0,-2,1; 0,0,0,-4]; x0=[-1; 0; 3; 1];
   syms t; x=expm(A*t)*x0, e=diff(x)-A*x
```

24. If a block Jordanian matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_1&&\\&\boldsymbol{A}_2&\\&&\boldsymbol{A}_3\end{bmatrix},\quad\text{where } \boldsymbol{A}_1=\begin{bmatrix}-3&1&0\\0&-3&1\\0&0&-3\end{bmatrix},$$

$$\boldsymbol{A}_2=\begin{bmatrix}-5&1\\0&-5\end{bmatrix},\ \boldsymbol{A}_3=\begin{bmatrix}-1&1&0&0\\0&-1&1&0\\0&0&-1&1\\0&0&0&-1\end{bmatrix}$$

find the solutions to $\mathrm{e}^{\boldsymbol{A}t}$, $\sin\left(2\boldsymbol{A}t+\dfrac{\pi}{3}\right)$, $\mathrm{e}^{\boldsymbol{A}^2t}\boldsymbol{A}^2+\sin(\boldsymbol{A}^3t)\boldsymbol{A}t+\mathrm{e}^{\sin\boldsymbol{A}t}$.

SOLUTION The exponential of the block Jordanian matrix can be obtained

```
>> syms t x; A1=[-3 1 0; 0 -3 1; 0 0 -3]; A2=[-5 1; 0 -5];
   A3=[-1 1 0 0; 0 -1 1 0; 0 0 -1 1; 0 0 0 -1];
   A=diagm(A1,A2,A3); expm(A*t)
```

and it is found that

$$\mathrm{e}^{\boldsymbol{A}t}=\begin{bmatrix}\mathrm{e}^{-3t} & t\mathrm{e}^{-3t} & t^2\mathrm{e}^{-3t}/2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \mathrm{e}^{-3t} & t\mathrm{e}^{-3t} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \mathrm{e}^{-3t} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \mathrm{e}^{-5t} & t\mathrm{e}^{-5t} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \mathrm{e}^{-5t} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-t} & t\mathrm{e}^{-t} & t^2\mathrm{e}^{-t}/2 & t^3\mathrm{e}^{-t}/6\\ 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-t} & t\mathrm{e}^{-t} & t^2\mathrm{e}^{-t}/2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-t} & t\mathrm{e}^{-t}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{e}^{-t}\end{bmatrix}$$

The other matrix functions can be evaluated with the following statements

```
>> F1=funm(A,sin(2*x*t+pi/3),x),
   F2=funm(A,exp(x^2*t)*x^2+sin(x^3*t)*x*t+exp(sin(x*t)),x)
```

25. For the given matrix $\boldsymbol{A}$, find matrix functions $\mathrm{e}^{\boldsymbol{A}t}, \sin\boldsymbol{A}t, \mathrm{e}^{\boldsymbol{A}t}\sin\left(\boldsymbol{A}^2\mathrm{e}^{\boldsymbol{A}t}t\right)$.

$$\boldsymbol{A}=\begin{bmatrix}-4.5 & 0 & 0.5 & -1.5\\ -0.5 & -4 & 0.5 & -0.5\\ 1.5 & 1 & -2.5 & 1.5\\ 0 & -1 & -1 & -3\end{bmatrix}.$$

SOLUTION The exponential function of the matrix can be obtained

```
>> A=[-4.5,0,0.5,-1.5; -0.5,-4,0.5,-0.5; 1.5,1,-2.5,1.5; 0,-1,-1,-3];
   A=sym(A); syms t; expm(A*t)
```

and it is found that

$$\frac{1}{2}\begin{bmatrix}\mathrm{e}^{-3t}(1-t+t^2)+\mathrm{e}^{-5t} & \mathrm{e}^{-5t}+(-1+2t)\mathrm{e}^{-3t} & (t+t^2)\mathrm{e}^{-3t} & \mathrm{e}^{-5t}-(1+t-t^2)\mathrm{e}^{-3t}\\ (t-1)\mathrm{e}^{-3t}+\mathrm{e}^{-5t} & \mathrm{e}^{-3t}+\mathrm{e}^{-5t} & t\mathrm{e}^{-3t} & (t-1)\mathrm{e}^{-3t}+\mathrm{e}^{-5t}\\ (t+1)\mathrm{e}^{-3t}-\mathrm{e}^{-5t} & -\mathrm{e}^{-5t}+\mathrm{e}^{-3t} & (1+t)\mathrm{e}^{-3t} & (t+1)\mathrm{e}^{-3t}-\mathrm{e}^{-5t}\\ -t^2\mathrm{e}^{-3t} & -t\mathrm{e}^{-3t} & -(t+t^2)\mathrm{e}^{-3t} & (2-t^2)\mathrm{e}^{-3t}\end{bmatrix}.$$

The other matrix functions can also be obtained with

```
>> syms x t; funm(A,'sin(x*t)',x)
   syms x t; funm(A,'exp(x*t)*sin(x^2*exp(x*t)*t)',x)
```

and the results are tedious and are not displayed here.

Chapter 5

Integral Transforms and Complex Variable Functions

Exercises and Solutions

1. Perform Laplace transforms for the given functions

   (i) $f(t)=\dfrac{\sin\alpha t}{t}$, (ii) $f(t)=t^5\sin\alpha t$, (iii) $f(t)=t^8\cos\alpha t$,

   (iv) $f(t)=t^6\mathrm{e}^{\alpha t}$, (v) $f(t)=5\mathrm{e}^{-at}+t^4\mathrm{e}^{-at}+8\mathrm{e}^{-2t}$,

   (vi) $f(t)=\mathrm{e}^{\beta t}\sin(\alpha t+\theta)$, (vii) $f(t)=\mathrm{e}^{-12t}+6\mathrm{e}^{9t}$.

Solution The Laplace transforms of the functions can easily be obtained with the following statements

```
>> syms a t; f=sin(a*t)/t; F1=laplace(f)
   f=t^5*sin(a*t); F2=laplace(f)
   f=t^8*cos(a*t); F3=laplace(f)
   f=t^6*exp(a*t); F4=laplace(f)
   f=5*exp(-a*t)+t^4*exp(-a*t)+8*exp(-2*t); F5=laplace(f)
   syms a b c t; f=exp(b*t)*sin(a*t+c); F6=laplace(f)
   syms t; f=exp(-12*t)+6*exp(9*t); F7=laplace(f)
```

the solutions obtained are

$$F_1=\mathrm{atan}\left(\frac{\alpha}{s}\right),\ F_2=\frac{120\sin(6\mathrm{atan}(\alpha/s))}{(s^2+\alpha^2)^3},\ F_3=20160\,(s-\mathrm{j}\alpha)^9+20160\,(s+\mathrm{j}\alpha)^9,$$

$$F_4=\frac{720}{(s-a)^7},\quad F_5=\frac{5}{s+a}+\frac{24}{(s+a)^5}+\frac{8}{s+2},$$

$$F_6=\frac{\alpha\cos\theta+(s-\beta)\sin\theta}{s^2-2\beta s+\beta^2+\alpha^2},\quad F_7=7\,\frac{s+9}{(s+12)\,(s-9)}$$

2. Take inverse transforms for the problems solved above and see whether the original function can be restored.

Solution The inverse transforms can be obtain by, for instance,

```
>> ilaplace(F7)
```

and it can be seen that all the original functions can be restored.

3. The following properties are also given for Laplace transforms. Verify for different values of n, the following formula are satisfied.

(i) $\mathscr{L}[t^n f(t)] = (-1)^n \dfrac{\mathrm{d}^n \mathscr{L}[f(t)]}{\mathrm{d}s^n}$, (ii) $\mathscr{L}[t^{n-1/2}] = \Gamma(n+1/2)s^{-n-1/2}$

SOLUTION Arbitrarily choose an n by random number generator, then the two equations can be proved to be equal, if the differences are zeros.

```
>> n=fix(500*rand(1,1)); f=sym('f(t)');
   err1=simple(laplace(t^n*f)-(-1)^n*diff(laplace(f),s,n))
   err2=simple(laplace(t^(n-1/2))-gamma(sym(n)+1/2)*s^(-n-1/2))
```

4. Perform inverse Laplace transforms to the following $F(s)$.

(i) $F(s) = \dfrac{1}{s^2(s^2-a^2)(s+b)}$, (ii) $F(s) = \sqrt{s-a} - \sqrt{s-b}$,

(iii) $F(s) = \ln\dfrac{s-a}{s-b}$, (iv) $F(s) = \dfrac{1}{\sqrt{s}(s+a)}$, (v) $F(s) = \dfrac{3a^2}{s^3+a^3}$,

(vi) $F(s) = \dfrac{(s-1)^8}{s^7}$, (vii) $F(s) = \ln\dfrac{s^2+a^2}{s^2+b^2}$

(viii) $F(s) = \dfrac{s^2+3s+8}{\prod_{i=1}^{8}(s+i)}$, (ix) $F(s) = \dfrac{1}{2}\dfrac{s+\alpha}{s-\alpha}$

SOLUTION The inverse Laplace transforms can also be evaluated with

```
>> syms s a b; F=1/(s^2*(s^2-a^2)*(s+b)); f1=ilaplace(F)
   F=sqrt(s-a)-sqrt(s-b); f2=ilaplace(F)
   F=log((s-a)/(s-b)); f3=ilaplace(F)
   F=1/sqrt(s)/(s+a); f4=ilaplace(F)
   F=3*a^2/(s^3+a^3); f5=ilaplace(F)
   F=(s-1)^8/s^7; f6=ilaplace(F)
   F=log((s^2+a^2)/(s^2+b^2)); f7=ilaplace(F)
   F=s^2+3*s+8; for i=1:8, F=F/(s+i); end; f8=ilaplace(F)
   F=(s+a)/(s-a)/2; f9=ilaplace(F)
```

and the results are

$$f_1 = -\frac{t}{a^2 b} + \frac{\mathrm{e}^{at}}{2a^3(a+b)} + \frac{1}{b^2 a^2} - \frac{\mathrm{e}^{-bt}}{b^2(a^2-b^2)} + \frac{\mathrm{e}^{-at}}{2a^3(a-b)}$$

$$f_2 = \frac{\mathrm{e}^{bt}-\mathrm{e}^{at}}{2t^{3/2}\sqrt{\pi}},\quad f_3 = \frac{\mathrm{e}^{bt}-\mathrm{e}^{at}}{t},\quad f_4 = \frac{\mathrm{e}^{-at}\mathrm{erfi}\left(\sqrt{at}\right)}{\sqrt{a}}$$

$$f_5 = \mathrm{e}^{-at} + e^{at/2}\left(-\cos\left(\frac{\sqrt{3}}{2}at\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)$$

$$f_6 = \delta(1,t) - 8\delta(t) - 56\,t + 28 + \frac{1}{720}\,t^6 + 7/6\,t^4 + 35\,t^2 - \frac{28}{3}\,t^3 - 1/15\,t^5$$

$$f_7 = 2\frac{\cos bt - \cos at}{t},\ f_9 = \frac{1}{2}\delta(t) + a\mathrm{e}^{\alpha t}$$

$$f_8 = -\frac{1}{105}\mathrm{e}^{-8t} + \frac{1}{840}\mathrm{e}^{-t} - \frac{1}{12}\mathrm{e}^{-4t} + \frac{1}{30}\mathrm{e}^{-3t} - \frac{1}{120}\mathrm{e}^{-2t} - \frac{13}{120}\mathrm{e}^{-6t} + \frac{1}{20}\mathrm{e}^{-7t} + \frac{1}{8}\mathrm{e}^{-5t}$$

where $\mathrm{erfi}(x) = -\mathrm{j}\mathrm{erf}(\mathrm{j}x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t$, and $\delta(t)$ is the Dirac impulse function.

5. Show the Laplace transforms where the non-integer power of s is introduced, which is the fundamentals of fractional-order calculus.

(i) $\mathscr{L}[t^\gamma]=\dfrac{\Gamma(\gamma+1)}{s^{\gamma+1}}$, one should check different values of γ

(ii) $\mathscr{L}\left[\dfrac{1}{\sqrt{t}\,(1+at)}\right]=\dfrac{\pi}{\sqrt{a}}\,\mathrm{e}^{s/a}\mathrm{erfc}\left(\sqrt{s/a}\right)$ for $a>0$.

SOLUTION (i) One may try different fractional numbers, for instance, selecting $\gamma=1/3$, the Laplace transform of the function can be obtained and it is the same as the right-hand-side one.

```
>> syms t; g=sym(1/3); laplace(t^g)-gamma(g+1)/s^(g+1)
```

(ii) The Laplace transform can be evaluated directly to yield the right-hand-side representation.

```
>> syms a positive; syms t; laplace(1/sqrt(t)/(1+a*t))
```

6. One of the applications of Laplace transform is that it can be used in solving linear constant differential equations with zero initial conditions,using the property $\mathscr{L}\left[\dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right]=s^n\mathscr{L}[f(t)]$. Solve the differential equation $y''(t)+3y'(t)+2y(t)=\mathrm{e}^{-t}$, $y(0)=y'(0)=0$ using Laplace transforms.

SOLUTION Taking Laplace transform to both sides of the equation, such that $(s^2+3s+2)Y(s)=\mathscr{L}[\mathrm{e}^{-t}]$, the differential equation can equivalently be solved from $y(t)=\mathscr{L}^{-1}\left[\dfrac{\mathscr{L}[\mathrm{e}^{-t}]}{s^2+3s+2}\right]$, and the analytical solution can be obtained as $(-1+t)\mathrm{e}^{-t}+\mathrm{e}^{-2t}$. The solution is verified with also the following statements

```
>> syms s t; y=ilaplace(laplace(exp(-t))/(s^2+3*s+2)),
   subs(y,t,0), subs(diff(y),t,0), diff(y,2)+3*diff(y)+2*y-exp(-t)
```

7. Perform Fourier transforms to the following functions, and inverse Fourier transform should then be performed to see whether the original functions can be restored.

(i) $f(x)=x^2(3\pi-2|x|),\ 0\leqslant x\leqslant 2\pi$, (ii) $f(t)=t^2(t-2\pi)^2,\ 0\leqslant t\leqslant 2\pi$

(iii) $f(t)=\mathrm{e}^{-t^2},\ -l\leqslant t\leqslant l$, (iv) $f(t)=t\mathrm{e}^{-|t|},\ -\pi\leqslant t\leqslant\pi$

SOLUTION (i) The Fourier transform of the function can be written $F_1=-6(4+\pi^2\delta(2,\omega)\omega^4)/\omega^4$. Taking the inverse and finds that $f_1=x^2(-4x\mathrm{heaviside}(x)+3\pi+2x)$. Bear in mind the definition of the `heaviside()` function and it is immediately recognized that $(-4x\mathrm{heaviside}(x)+2x)$ is in fact $-2|x|$, thus the original function can be restored.

```
>> syms x; f=x^2*(3*sym(pi)-2*abs(x)); F1=fourier(f), f1=ifourier(F1)
```

(ii)-(iv) can be solved easily by

```
>> syms t; f=t^2*(t-2*sym(pi))^2; F2=fourier(f), f2=x^2*(-2*pi+x)^2
   f=exp(-t^2); F3=fourier(f), f3=ifourier(F)
   f=t*exp(-abs(t)); F4=fourier(f), f4=ifourier(F)
```

with

$$F_2=2\pi(4\mathrm{j}\pi\delta(3,\omega)-4\pi^2\delta(2,\omega)+\delta(4,\omega)),\ F_3=\pi^{1/2}\mathrm{e}^{\omega^2/4},\ F_4=-\frac{4\mathrm{j}\omega}{(1+\omega^2)^2}.$$

8. Perform Fourier sinusoidal and cosine transforms for the following functions and then perform inverse transformation and see whether the original functions can be restored.
   (i) $f(t) = \mathrm{e}^{-t}\ln t$, (ii) $f(x) = \dfrac{\cos x^2}{x}$, (iii) $f(x) = \ln\dfrac{1}{\sqrt{1+x^2}}$
   (iv) $f(x) = x(a^2 - x^2)$, $a > 0$, (iv) $f(x) = \cos kx$.

SOLUTION With the calling of corresponding Maple functions from MATLAB, the transforms can be obtained. Unfortunately the inverse transforms usually fail.

```
>> syms t w; f=exp(-t)*log(t);
   F1s=maple('fouriersin',f,t,w), F1c=maple('fouriercos',f,t,w)
   f1=maple('invfouriersin',Fs1,w,t),
   f2=maple('invfouriercos',F1c,w,t)
   syms x; f=cos(x^2)/x; F2s=maple('fouriersin',f,x,w)
   syms x w; f=sin(1+x^2); F3s=maple('fouriersin',f,x,w)
```

where the complicated F_{2s} and F_{3s} are not displayed, and

$$F_{1s} = \sqrt{\frac{2}{\pi}}\frac{\mathrm{atan}(\omega)-\gamma\omega-\omega\ln(1+\omega^2)/2}{1+\omega^2}, \quad F_{1c} = -\sqrt{\frac{2}{\pi}}\frac{\gamma+\ln(1+\omega^2)/2+\omega\mathrm{atan}(\omega)}{1+\omega^2}$$

where γ is the Euler Gamma constant.

9. Compute the discrete Fourier sinusoidal and cosine transforms for the functions (i) $f(x) = \mathrm{e}^{kx}$, and (ii) $f(x) = x^3$.

SOLUTION The discrete Fourier transforms can be obtained by direct integration methods such that

```
>> syms k x; syms a positive; f=exp(k*x);
   F1s=int(f*sin(k*sym(pi)*x/a),x,0,a),
   F1c=int(f*cos(k*sym(pi)*x/a),x,0,a)
   f=x^3; F2s=int(f*sin(k*sym(pi)*x/a),x,0,a),
   F2c=int(f*cos(k*sym(pi)*x/a),x,0,a)
```

where

$$F_{1s} = -\frac{a(\mathrm{e}^{ak}\pi\cos k\pi - \mathrm{e}^{ak}a\sin k\pi - \pi)}{k(a^2+\pi^2)}, F_{1c} = \frac{a\left(-a+\mathrm{e}^{ak}a\cos k\pi+\mathrm{e}^{ak}\pi\sin k\pi\right)}{k\left(a^2+\pi^2\right)}$$

$$F_{2s} = -\frac{a^4\left(-6k\pi\cos k\pi + 6\sin k\pi + k^3\pi^3\cos k\pi - 3k^2\pi^2\sin k\pi\right)}{k^4\pi^4}$$

$$F_{2c} = \frac{a^4\left(6 - 6k\pi\sin k\pi - 6\cos k\pi + k^3\pi^3\sin k\pi + 3k^2\pi^2\cos k\pi\right)}{k^4\pi^4}$$

It should be noted the computer generated results are not the simplest, since $\sin k\pi \equiv 0$ for integers k, and $\cos k\pi = (-1)^{k+1}$.

10. Write the Mellin transform for the function $f(x) = \begin{cases} \sin(a\ln x), & x \leqslant 1 \\ 0, & \text{othewise.} \end{cases}$

SOLUTION For the given piecewise function, the function `heaviside()` can be used. Then the Mellin transform can be obtained as $F = -a/(a^2+z^2)$. Again, unfortunately, the inverse transform of the function cannot be obtained.

```
>> syms x z a; f=sin(a*log(x))*heaviside(1-x);
   F=maple('mellin',f,x,z)
```

11. Perform Z transforms to the time sequences $f(kT)$, and verify the results.

    (i) $f(kT)=\cos(kaT)$, (ii) $f(kT)=(kT)^2\mathrm{e}^{-akT}$, (iii) $f(kT)=\dfrac{1}{a}(akT-1+\mathrm{e}^{-akT})$

    (iv) $f(kT)=\mathrm{e}^{-akT}-\mathrm{e}^{-bkT}$, (v) $f(kT)=1-\mathrm{e}^{-akT}(1+akT)$.

SOLUTION The Z transform and its inverse can easily be obtained with the `ztrans()` and `iztrans()` functions, respectively. Below are the transformation and inverse, and the inverse transformation restores the original functions.

```
>> syms k a T; f=cos(k*a*T); F1=ztrans(f), f1=iztrans(F1)
   f=(k*T)^2*exp(-a*k*T); F2=ztrans(f), f2=iztrans(F2)
   f=(a*k*T-1+exp(-a*k*T))/a; F3=ztrans(f), f3=iztrans(F3)
   syms b; f=exp(-a*k*T)-exp(-b*k*T); F4=ztrans(f), f4=iztrans(F4)
   f=1-exp(-a*k*T)*(1+a*k*T); F5=ztrans(f), f5=iztrans(F5)
```

and the results are

$$F_1=\frac{(z-\cos(aT))\,z}{z^2-2z\cos(aT)+1},\quad F_2=\frac{T^2z\mathrm{e}^{-aT}\left(z+\mathrm{e}^{-aT}\right)}{(z-\mathrm{e}^{-aT})^3}$$

$$F_3=\frac{1}{a}\left[\frac{aTz}{(z-1)^2}-\frac{z}{z-1}+z\mathrm{e}^{aT}\left(\frac{z}{\mathrm{e}^{-aT}}-1\right)^{-1}\right]$$

$$F_4=\mathrm{e}^{aT}\left(\frac{z}{\mathrm{e}^{-aT}}-1\right)^{-1}-z\mathrm{e}^{bT}\left(\frac{z}{\mathrm{e}^{-bT}}-1\right)^{-1}$$

$$F_5=\frac{z}{z-1}-z\mathrm{e}^{aT}\left(\frac{z}{\mathrm{e}^{-aT}}-1\right)^{-1}-aTz\mathrm{e}^{aT}\left(\frac{z}{\mathrm{e}^{-aT}}-1\right)^{-2}$$

12. Perform inverse Z transforms to the following functions.

    (i) $F(z)=\dfrac{10z}{(z-1)(z-2)}$, (ii) $F(z)=\dfrac{z^{-1}(1-\mathrm{e}^{-aT})}{(1-z^{-1})(1-z^{-1}\mathrm{e}^{-aT})}$

    (iii) $F(z)=\dfrac{z}{(z-a)(z-1)^2}$, (iv) $F(z)=\dfrac{Az[z\cos\beta-cos(\alpha T-\beta)]}{z^2-2z\cos(\alpha T)+1}$.

SOLUTION The inverse Z transform can be obtained directly

```
>> syms z; F=10*z/((z-1)*(z-2)); f1=iztrans(F)
   F=z^2/((z-0.8)*(z-0.1)); f2=iztrans(F)
   syms a T; F=z/((z-a)*(z-1)^2); f3=simple(iztrans(F))
   F=z^(-1)*(1-exp(-a*T))/((1-z^(-1))*(1-z^(-1)*exp(-a*T)));
   f4=simple(iztrans(F))
   syms b A; F=A*z*(z*cos(b)-cos(a*T-b))/(z^2-2*z*cos(a*T)+1);
   f5=simple(iztrans(F))
```

and the results are

$$f_1=10(2^n-1), f_2=\frac{8}{7}\left(\frac{4}{5}\right)^n-\frac{1}{7}\left(\frac{1}{10}\right)^n, f_3=-\frac{na-a^n+1-n}{(-1+a)^2},$$

$$f_4=-\mathrm{e}^{anT}+1, f_5=A\cos(b+aTn)$$

13. Take inverse Laplace transform to the following functions, then take Z transform and verify the results.

    (i) $G(s)=\dfrac{b}{s^2(s+a)}$, (ii) $G(s)=\dfrac{b}{s^2(s+a)^2}\dfrac{1-\mathrm{e}^{-Ts}}{s}$.

SOLUTION The idea for solving the problems is to take inverse Laplace transform to the given functions, and then take Z transform to the results

```
>> syms s a b; F=b/(s^2*(s+a)); f1=simple(ztrans(ilaplace(F)))
   syms T; F=b/s^2/(s+a)^2*(1-exp(-T*s))/s;
   f2=simple(ztrans(ilaplace(F)))
```

with

$$\mathscr{Z}\left[\mathscr{L}^{-1}[F(s)]\right]=b\frac{(aze^{a}+z-ze^{a}+e^{a}-a-1)\,zb}{(z^{2}e^{a}-z-ze^{a}+1)\,(z-1)\,a^{2}}$$

14. For $G(s)=\dfrac{1}{(s+1)^3}$, if one substitutes $s=\dfrac{2(z-1)}{T(z+1)}$ into $G(s)$, the function $H(z)$ can be obtained. This kind of transform is referred to as *bilinear transform*. For $T=1/2$, find $H(z)$. One may also assume that $z=\dfrac{1+Ts/2}{1-Ts/2}$, inverse bilinear transform can be performed. Check whether the original function can be restored.

SOLUTION Forward bilinear transform can be performed by

```
>> syms s z T; G=1/(s+3)^3; H=simple(subs(G,s,2*(z-1)/T/(z+1)));
   H=simple(H), H1=simple(subs(H,T,1/2))
   G1=subs(H,z,(s-T/2)/(s+T/2)); simple(G-G1)
```

where

$$H=\left(\frac{2\,z-2}{T\,(z+1)}+3\right)^{-3},\;H_1=\frac{(z+1)^3}{(7\,z-1)^3}$$

and the original function can be restored, if the inverse transform is taken.

15. Show that

$$\mathscr{Z}\left\{1-e^{-akT}\left[\cos(bkT)+\frac{a}{b}\sin(bkT)\right]\right\}=\frac{z(Az+B)}{(z-1)(z^2-2e^{-aT}\cos(bT)z+e^{-2aT})}$$

where

$$A=1-e^{-aT}\cos(bT)-\frac{a}{b}e^{-aT}\sin(bT)$$
$$B=e^{-2aT}+\frac{a}{b}e^{-aT}\sin(bT)-e^{-aT}\cos(bT).$$

SOLUTION To show an equation hold by computers, one may take and simplify the difference between the terms on both sides. If the difference is zero, the equation holds. Otherwise it dose not hold. The following statements can be given and for the problem, the difference is zero. Hence the equation holds.

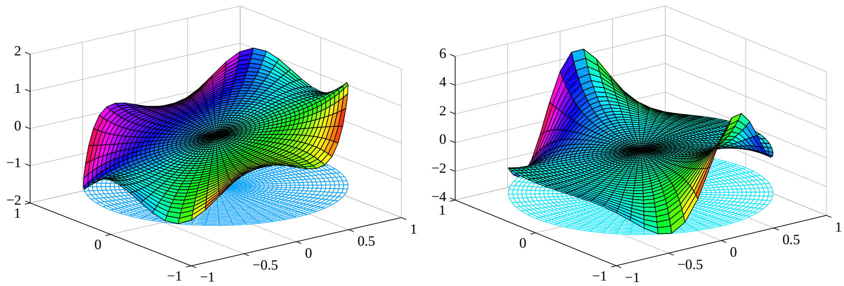
```
>> syms a b k T; f=1-exp(-a*k*T)*(cos(b*k*T)+a/b*sin(b*k*T));
   F=ztrans(f), A=1-exp(-a*T)*cos(b*T)-a/b*exp(-a*T)*sin(b*T);
   B=exp(-2*a*T)+a/b*exp(-a*T)*sin(b*T)-exp(-a*T)*cos(b*T);
   R=(z*(A*z+B))/((z-1)*(z^2-2*exp(-a*T)*cos(b*T)*z+exp(-2*a*T)));
   simple(F-R)
```

16. Draw the mapping surface of the following complex variable functions.

(i) $f(z)=z\cos z^2$, (ii) $f(z)=ze^{-z^2}(\cos z-\sin z)$.

SOLUTION The complex surfaces of the given functions can be drawn with the following statements, as shown in Figure 5.1.

```
>> z=cplxgrid(50); f1=z.*cos(z.^2); cplxmap(z,f1)
   figure; f2=z.*exp(-z.^2).*(cos(z)-sin(z))); cplxmap(z,f2)
```



(a) $f(z) = z\cos z^2$ (b) $f(z) = z\mathrm{e}^{-z^2}(\cos z - \sin z)$

FIGURE 5.1: Mapping surfaces of complex variable functions

17. For the function
$$f(x) = \frac{x^2+4x+3}{x^5+7x^4-2x^3-100x^2-232x-160}\mathrm{e}^{-5x}$$
find the poles and their multiplicities and compute the residues for each poles.

SOLUTION For polynomial denominators, the poles can be find by factorization technique,

```
>> syms x;
   f=(x^2+4*x+3)/poly2sym([1 7 -2 -100 -232 -160],x)*exp(-5*x);
   [n,D]=numden(f); D1=factor(D)
```

one finds that the factorized denominator is $D_1 = (x+5)(x-4)(x+2)^3$, and it is immediately seen that $x = -5, x = 4$ are single poles while $x = -2$ is a triple pole. The residues of the poles can then be solved as

```
>> r1=limit((x+5)*f,x,-5), r2=limit((x-4)*f,x,4)
   r3=limit(diff((x+2)^3*f,x,2)/2,x,-2)
```

The residues of the poles are respectively $r_1 = \dfrac{8\mathrm{e}^{25}}{243}, r_2 = \dfrac{35\mathrm{e}^{-20}}{1944}, r_3 = \dfrac{149\mathrm{e}^{10}}{216}$.

18. Judge whether the polynomials are coprime or not. If not, please find the terms which can simplify $B(s)/A(s)$.

(i) $B(x) = -3x^4 + x^5 - 11x^3 + 51x^2 - 62x + 24$
$A(x) = x^7 - 12x^6 + 26x^5 + 140x^4 - 471x^3 - 248x^2 + 1284x - 720$

(ii) $B(x) = 3x^6 - 36x^5 + 120x^4 + 90x^3 - 1203x^2 + 2106x - 1080$
$A(x) = x^9+15x^8+79x^7+127x^6-359x^5-1955x^4-3699x^3-3587x^2-1782x-360.$

SOLUTION To judge whether two polynomials are coprime or not, one can find the greatest common divisor of them and see whether there exists s term in it. The problem can then be solved with

```
>> syms s; B=poly2sym([-3 1 -11 51 -62 24],s);
   A=poly2sym([1 -12 26 140 -471 -248 1284 -720],s), d1=gcd(A,B)
   A=poly2sym([1 15 79 127 -359 -1955 -3699 -3587 -1782 -2360],s),
   B=poly2sym([3 -36 120 90 -1203 2106 -1080],s); d2=gcd(A,B)
```

where $d_1 = s - 1$, $d_2 = 1$, which means that (ii) is coprime while (i) is not.

19. Perform partial fraction expansions to the following functions

(i) $f(x) = \dfrac{3x^4 - 21x^3 + 45x^2 - 39x + 12}{x^7 + 15x^6 + 96x^5 + 340x^4 + 720x^3 + 912x^2 + 640x + 192}$

(ii) $f(s) = \dfrac{s+5}{s^8+21s^7+181s^6+839s^5+2330s^4+4108s^3+4620s^2+3100s+1000}$

(iii) $f(x) = \dfrac{3x^6 - 36x^5 + 120x^4 + 90x^3 - 1203x^2 + 2106x - 1080}{x^7 + 13x^6 + 52x^5 + 10x^4 - 431x^3 - 1103x^2 - 1062x - 360}$.

SOLUTION The partial fraction expressions of the functions can be obtained

```
>> syms s x;
   F=(3*x^4-21*x^3+45*x^2-39*x+12)/(x^7+15*x^6+96*x^5+...
     340*x^4+720*x^3+912*x^2+640*x+192); f1=residue(F,x)
   F=(s+5)/(s^8+21*s^7+181*s^6+839*s^5+2330*s^4...
     +4108*s^3+4620*s^2+3100*s+1000); f2=residue(F,s)
   F=(3*x^6-36*x^5+120*x^4+90*x^3-1203*x^2+2106*x-1080)/...
    (x^7+13*x^6+52*x^5+10*x^4-431*x^3-1103*x^2-1062*x-360);
   f3=residue(F,x)
```

with

$$f_1 = \frac{1344}{x+3} + \frac{486}{(x+2)^6} - \frac{1053}{(x+2)^5} + \frac{1296}{(x+2)^4} - \frac{1341}{(x+2)^3} + \frac{1344}{(x+2)^2} - \frac{1344}{x+2}$$

$$f_2 = \frac{-\dfrac{7}{2312} - \mathrm{j}\dfrac{23}{2312}}{(s+1+\mathrm{j})^2} - \frac{\dfrac{517}{39304} - \mathrm{j}\dfrac{143}{9826}}{s+1+\mathrm{j}} - \frac{\dfrac{7}{2312} - \mathrm{j}\dfrac{23}{2312}}{(s+1-\mathrm{j})^2} - \frac{\dfrac{517}{39304} + \mathrm{j}\dfrac{143}{9826}}{s+1-\mathrm{j}}$$
$$+ \frac{1}{36\,(s+2)} - \frac{1}{867(s+5)^2} - \frac{65}{44217(s+5)}$$

$$f_3 = \frac{945}{4(x+5)} - \frac{360}{x+4} + \frac{252}{x+2} + \frac{45}{(x+1)^2} - \frac{501}{4(x+1)}$$

20. Find the residues of the following functions at poles

(i) $f(z) = \dfrac{1 - \sin z\mathrm{e}^{-2z}}{z^7 \sin(z - \pi/3)}(z^4 + 10z^3 + 35z^2 + 50z + 24)$

(ii) $f(z) = \dfrac{(z-3)^4}{z^4 + 5z^3 + 9z^2 + 7z + 2}(\sin z - \mathrm{e}^{-3z})$

(iii) $f(z) = \dfrac{(1 - \cos 2z)(1 - \mathrm{e}^{-z^2})}{z^3 \sin z}$.

SOLUTION (i) poles at $z = 0$ (possible multiplicity of 7), $z = \pi/3 + k\pi$ for any integer k (single)

```
>> syms z; kk=[-3 -2 -1 0 1 2 3];
   f=(1-sin(z)*exp(-2*z))*poly2sym([1 10 35 50 24],z)/z^7/sin(z-pi/3);
   r1=limit(diff(z^7*f,z,6)/prod(1:6),z,0),
```

```
for i=1:length(kk),
   r(i)=limit((z-kk(i)*pi-pi/3)*f,z,kk(i)*pi+pi/3),
end
```

with $r_1 = -\dfrac{12973\sqrt{3}}{540} - \dfrac{11143}{270}$, and the rest of the residues can also be found for selected integers of k. The approximate values of $\boldsymbol{r}$ is $\boldsymbol{r} = [6262.7525, 10.7372, 0.0586, 82.4305, -0.0836, 0.0080, -0.0020]$.
(ii) The denominator can be expressed by $(z+1)^3(z+2)$, thus the pole -1 is a triple pole, and -2 is a single pole. The two residues can be evaluated with the following statements

```
>> syms z; D=z^4+5*z^3+9*z^2+7*z+2; factor(D)
   f=(z-3)^4/D*sin(z-exp(-3*z));
   r1=limit((z+2)*f,z,-2), r2=limit(diff((z+1)^3*f,z,2)/2,z,-1)
```

and the residues are respectively

$$r_1 = 625\sin 2\cos \mathrm{e}^6 + 625\cos 2\sin \mathrm{e}^6$$

$$\begin{aligned} r_2 = {} & 1152\mathrm{e}^6\sin 1\cos \mathrm{e}^3 + 1152\mathrm{e}^6\cos 1\sin \mathrm{e}^3 - 512\cos 1\cos \mathrm{e}^3 + 512\sin 1\sin \mathrm{e}^3 \\ & + 768\mathrm{e}^3\sin 1\cos \mathrm{e}^3 + 768\mathrm{e}^3\cos 1\sin \mathrm{e}^3 - 2688\mathrm{e}^3\cos 1\cos \mathrm{e}^3 + 2688\mathrm{e}^3\sin 1\sin \mathrm{e}^3 \\ & - 480\sin 1\cos \mathrm{e}^3 - 480\cos 1\sin \mathrm{e}^3 \end{aligned}$$

(iii) $z=0$ (possible multiplicity of 4), with a residue of $r_1 = 4$, and $z = k\pi$ for any integer k (single pole), and the residues are all zeros.

```
>> syms z; f=(1-cos(2*z))*(1-exp(-2*z))/z^3/sin(z);
   r1=limit(diff(z^4*f,z,3)/prod(1:3),z,0)
   kk=[-3 -2 -1 1 2 3];
   for i=1:length(kk), r(i)=limit((z-kk(i)*pi)*f,z,kk(i)*pi), end
```

21. Evaluate the closed-path integrals

(i) $\displaystyle\oint_\Gamma \frac{z^{15}}{(z^2-1)^2(z^4-2)^3}\,\mathrm{d}z$, where Γ is the positive circle $|z|=3$;

(ii) $\displaystyle\oint_\Gamma \frac{z^3}{1+z}\mathrm{e}^{1/z}\,\mathrm{d}z$, where Γ is the positive circle $|z|=2$.

(iii) $\displaystyle\oint_\Gamma \frac{\cos z(1-\mathrm{e}^{-z^2})\sin(3z+2)}{z\sin z}\,\mathrm{d}z$, where Γ is the positive circle $|z|=1$.

SOLUTION (i) The six poles, $z=\pm 1$, $z=\pm\sqrt[4]{2}$ and $z=\pm\mathrm{j}\sqrt[4]{2}$ are all encircled by the positive circle of $|z|=3$, thus one can evaluate the integral with

```
>> syms z; F=z^(15)/((z^2-1)^2*(z^4-2)^3); i=sqrt(-1);
   s=sym(2)^(1/4); r=0; pp=[-1,1,-s,s,-s*i,s*i];
   for j=1:6,
      r=r+limit(diff(F*(z-pp(j))^3,z,2)/2,z,pp(j));
   end
   R=simple(2*pi*i*r)
```

The value of the integral is $R = 2\mathrm{j}\pi$.
(ii) The single pole $z=-1$ is encircled by the closed-path, thus

```
>> syms z; F=z^3/(1+z)*exp(1/z); R=2*pi*i*limit(F*(z+1),z,-1)
```

and the integral is $R = -2\pi \mathrm{j}\mathrm{e}^{-1}$.
(iii) There are infinite number of poles in the function, at $z = 0$, and $z = k\pi$, for any integer k. However within the circle $|z| = 1$, there is only one double pole at $z = 0$. The closed-path integral can be evaluated with $R = 6\pi\mathrm{j}\cos 2$.

```
>> syms z; i=sqrt(-1); f=cos(1-exp(-z^2))*sin(3*z+2)/z/sin(z);
   R=2*pi*i*limit(diff(z^2*f,z),z,0)
```

Chapter 6

Nonlinear Equations and Optimization Problems

Exercises and Solutions

1. Find the solutions to the following equations, and verify the accuracy of the solutions.

(i) $\begin{cases} x_1^2 - x_2 - 1 = 0 \\ (x_1-2)^2 + (x_2-0.5)^2 - 1 = 0 \end{cases}$ (ii) $\begin{cases} x^2y^2 - zxy - 4x^2yz^2 = xz^2 \\ xy^3 - 2yz^2 = 3x^3z^2 + 4xzy^2 \\ y^2x - 7xy^2 + 3xz^2 = x^4zy \end{cases}$

SOLUTION (i) can be solved easily with the following statements, and the accuracy reaches to the 10^{-31} level.

```
>> [x1,x2]=solve('x1^2-x2-1=0','(x1-2)^2+(x2-0.5)^2-1=0','x1,x2')
   norm(double([x1.^2-x2-1 (x1-2).^2+(x2-0.5).^2-1]))
```

(ii) can be solved with the following statements, with 10^{-24} level.

```
>> [x,y,z]=solve('x^2*y^2-z*x*y-4*x^2*y*z^2=x*z^2',...
         'x*y^3-2*y*z^2=3*x^3*z^2+4*x*z*y^2',...
         'y^2*x-7*x*y^2+3*x*z^2=x^4*z*y','x,y,z')
   norm(double([x.^2.*y.^2-z.*x.*y-4*x.^2.*y.*z.^2-x.*z.^2,...
                x.*y.^3-2*y.*z.^2-3*x.^3.*z.^2-4*x.*z.*y.^2,...
                y.^2.*x-7*x.*y.^2+3*x.*z.^2-x.^4.*z.*y]))
```

It will be noticed once the solutions are displayed that there are 21 sets of solutions, and the first three are respectively $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$, which correspond to arbitrary combinations of x, y and z on the three axes. Thus there are infinite sets of solutions. The rest 18 sets are individuals.

2. Solve graphically the following equations, and verify the results.

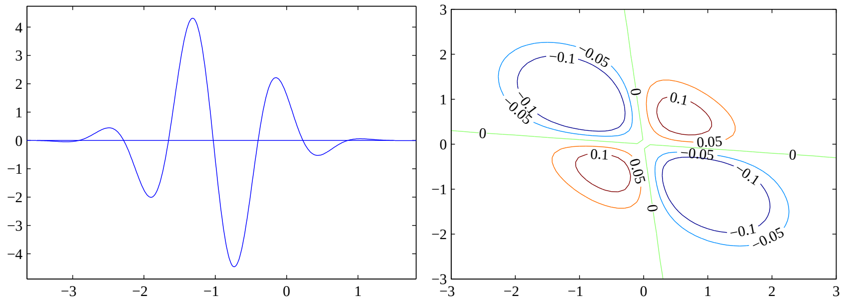
(i) $e^{-(x+1)^2+\pi/2}\sin(5x+2) = 0$ (ii) $(x^2+y^2+xy)e^{-x^2-y^2-xy} = 0$

SOLUTION (i) can be shown with the following statements, and the intersections with $y = 0$, i.e., x-axis, are the solutions of the equation. The solutions can be obtained from Figure 6.1 (a). One may zoom a particular intersection for a more accurate solution.

```
>> ezplot('exp(-(x+1)^2+pi/2)*sin(5*x+2)'), line([-3.5 1.5],[0,0])
```

(ii) The contour lines of the function can be can be drawn with the following statements, as shown in Figure d6.1 (b), an the contour line labeled 0 are the solutions of the equation.

```
>> [x,y]=meshgrid(-3:0.1:3);
   z=(0.1*x.^2+0.1*y.^2+x.*y).*exp(-x.^2-y.^2-x.*y);
   [C,h]=contour(x,y,z,[-0.1:0.05:0.1]); set(h,'ShowText','on')
```



(a) equation (i) (b) equation (ii)

FIGURE 6.1: Graphical solutions of equations

3. Find in numerical methods the solutions to the above problems, and verify the results.

SOLUTION (i) can be solved numerically, when one selects an initial point. For instance, if one selects $x_0 = 0$ as the initial point, the following statements can be used to find more accurate solution at $x = 0.2283$.

```
>> f=@(x)exp(-(x+1).^2+pi/2).*sin(5*x+2); x0=0; x=fsolve(f,x0)
```

(ii) Since the solutions to the equation are contour lines, it is not possible to find all the solutions on the continuous lines. However the analytical solutions of the equation can be obtained through symbolic computation such that $y = \left(-\frac{1}{2} \pm \frac{\mathrm{j}\sqrt{3}}{2}\right) x$. The last statement verifies the solutions.

```
>> syms x; y=solve('(x^2+y^2+x*y)*exp(-x^2-y^2-x*y)=0','y')
   simple(subs('(x^2+y^2+x*y)*exp(-x^2-y^2-x*y)','y',y1))
```

4. Find c such that the integral $\int_0^1 (\mathrm{e}^x - cx)^2 \, \mathrm{d}x$ is minimized.

SOLUTION The integral can be evaluated with the following statements

```
>> syms x c; y=int((exp(x)-c*x)^2,x,0,1)
```

and it is found that the integral $y = -\frac{1}{2} - 2c + \frac{1}{2}\mathrm{e}^2 + \frac{c^2}{3}$. The following statements can be used to find the minimum value of the original problem, where $c = 3$, $f = 0.1945$.

```
>> f=@(c)-1/2-2*c+1/2*exp(2)+c^2/3; c=fminsearch(f,0), y=f(c)
```

5. Find all the solutions to the modified Riccati equation, and verify the results.

$$\boldsymbol{AX}+\boldsymbol{XD}-\boldsymbol{XBX}+\boldsymbol{C}=0$$

where

$$\boldsymbol{A}=\begin{bmatrix}2&1&9\\9&7&9\\6&5&3\end{bmatrix},\ \boldsymbol{B}=\begin{bmatrix}0&3&6\\8&2&0\\8&2&8\end{bmatrix},\ \boldsymbol{C}=\begin{bmatrix}7&0&3\\5&6&4\\1&4&4\end{bmatrix},\ \boldsymbol{D}=\begin{bmatrix}3&9&5\\1&2&9\\3&3&0\end{bmatrix}.$$

SOLUTION One should write out the an M-file to describe the equation, with arguments $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{D}$ the additional ones.

```
function y=ex_ric(x,A,B,C,D)
X=reshape(x,size(A)); y1=A*X+X*D-X*B*X+C; y=y1(:);
```

and a Riccati equation solver can be written as

```
function X=ex_solve(A,B,C,D,x0)
if nargin==4, x0=rand(size(A)); end
x=fsolve('ex_ric',x0(:),[],A,B,C,D); X=reshape(x,size(A));
```

With the use of the functions, the solutions to the Riccati equation can all be found, by repeated calling of the function

```
>> A=[2,1,9; 9,7,9; 6,5,3]; B=[0,3,6; 8,2,0; 8,2,8];
   C=[7,0,3; 5,6,4; 1,4,4]; D=[3,9,5; 1,2,9; 3,3,0];
   x=ex_solve(A,B,C,D), norm(A*x+x*D-x*B*x+C)
```

Two solutions can be found for the equations such that

$$\boldsymbol{x}_1=\begin{bmatrix}0.78765&0.42984&-2.2827\\-0.60885&-0.26903&0.72972\\-0.65535&-0.34174&1.1487\end{bmatrix},\ \boldsymbol{x}_2=\begin{bmatrix}1.4567&1.3663&0.38581\\-0.28228&0.18807&2.0323\\-0.59091&-0.25153&1.4057\end{bmatrix}$$

6. Solve the unconstrained optimization problems $\min\limits_{\boldsymbol{x}} f(\boldsymbol{x})$, where

$$\begin{aligned}f(\boldsymbol{x})&=100(x_2-x_1^2)^2+(1-x_1)^2+90(x_4-x_3^2)+(1-x_3^2)^2+\\&\quad 10.1\left[(x_2-1)^2+(x_4-1)^2\right]+19.8(x_2-1)(x_4-1).\end{aligned}$$

SOLUTION One can declare the objective function first with anonymous function (or other types of function), then calling the solver to solve the problem.

```
>> f=@(x)100*(x(2)-x(1)^2)^2+(1-x(1))^2+90*(x(4)-x(3)^2)+...
     (1-x(3)^2)^2+10.1*((x(2)-1)^2+(x(4)-1)^2)+19.8*(x(2)-1)*(x(4)-1);
   x=fminunc(f,ones(7,1))
```

The optimal solution is $\boldsymbol{x}=[10.546,111.23,6.7823,-111.5,1,1,1]^{\mathrm{T}}$.

7. A set of challenging benchmark problems for evaluating optimization algorithms can be solved using MATLAB. Solve the following unconstrained optimization problems with MATLAB.

(i) De Jong's problems[1]

$$J=\min_{\boldsymbol{x}}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}=\min_{\boldsymbol{x}}(x_1^2+x_2^2+\cdots+x_p^2),\quad\text{where } x_i\in[-512,512]$$

where $i=1,\cdots,p$, with theoretic solution $x_1=\cdots=x_p=0$.

(ii) Griewank's benchmark problem

$$J = \min_{\boldsymbol{x}} \left(1 + \sum_{i=1}^{p} \frac{x_i^2}{4000} - \prod_{i=1}^{p} \cos\frac{x_i}{\sqrt{i}}\right), \text{ where } x_i \in [-600, 600].$$

(iii) Ackley's benchmark problem[2]

$$J = \min_{\boldsymbol{x}} \left[20 + 10^{-20} \exp\left(-0.2\sqrt{\frac{1}{p}\sum_{i=1}^{p} x_i^2}\right) - \exp\left(\frac{1}{p}\sum_{i=1}^{p}\cos 2\pi x_i\right)\right].$$

SOLUTION (i) De Jong's problem can be solved with the following statements, and a zero vector $\boldsymbol{x}$ can be found.

```
>> f=@(x)x.'*x; x=fminunc(f,ones(20,1)), norm(x)
```

(ii) One may use the following statements to solve the problem and the norm of the solution is 3×10^{-7}, which is close to a theoretical zero vector.

```
>> f=@(x)(1+x.'*x/4000-prod(cos(x./cos(sqrt(1:length(i))))));
   x=fminunc(f,ones(20,1)), norm(x)
```

(iii) The objective function can be established first, then the solutions to the problem can be found, such that the norm of the solution is 1.3×10^{-8}, with theoretical solution of 0.

```
>> f=@(x)20+1e-20*exp(-0.2*sqrt(sum(x.^2)/20))-...
          exp(sum(cos(2*pi*x))/20);
   x=fminunc(f,0.1*ones(20,1)), norm(x)
```

8. Consider the Rastrigin function[3]

$$f(x_1, x_2) = 20 + x_1^2 + x_2^2 - 10(\cos\pi x_1 + \cos\pi x_2).$$

3D surface plot for the objective function can be shown. An initial point can be selected from the plot, such that a good minimization to the problem. Understand the dependency of optimum point with respect to initial values.

SOLUTION For different initial vectors, the solutions may be completely different, of course, most of the solutions obtained are local minima. To solve it thoroughly, one should use global optimum solution algorithms, such as GA to be presented in Chapter 10.

```
>> y=@(x)20+x(1)^2+x(2)^2-10*(cos(pi*x(1))+cos(pi*x(2)));
   x1=fminunc(y,[1;0])
```

9. Solve the nonlinear programming problem with graphical methods and verify the results using numerical methods.

$$\min_{\boldsymbol{x} \text{ s.t.} \begin{cases} x_1-x_2+2\geqslant 0 \\ -x_1^2+x_2-1\geqslant 0 \\ x_1\geqslant 0, x_2\geqslant 0 \end{cases}} (x_1^3 + x_2^2 - 4x_1 + 4)$$

SOLUTION The following statements can be used to find graphically the solution of the problem, with all the points which do not satisfy the constraints set to `NaN`. The 3D surface is shown in Figure 6.2. It can be read from the surface that $x_1 = 0, x_2 = 1$.

```
>> [x1,x2]=meshgrid(0:0.02:1,1:0.02:2); z=x1.^3+x2.^2+4*x1+4;
   ii=find(x1-x2+2<0); z(ii)=NaN; ii=find(-x1.^2+x2-1<0); z(ii)=NaN;
   ii=find(x1<0); z(ii)=NaN; ii=find(x2<0); z(ii)=NaN; surf(x1,x2,z)
```

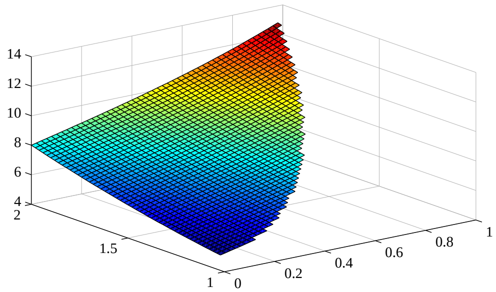


FIGURE 6.2: Graphical solutions to the problem

Set the nonlinear constraints

```
function [c,ce]=exc6f1(x), ce=[];  c=[x(1)^2-x(2)+1];
```

Numerical solution to the problem can also be found with the following statements that the solution vector is $\boldsymbol{x} = [0,1]^{\mathrm{T}}$, which agrees well with the graphical method.

```
>> f=@(x)x(1)^3+x(2)^2+4*x(1)+4;
   A=[-1 1]; B=2; Aeq=[]; Beq=[]; xm=[0;0];
   x=fmincon(f,[0;1],A,B,Aeq,Beq,xm,[],'exc6f1')
```

10. Try to solve the following linear programming problems.

(i) $$\min_{\boldsymbol{x}\ \text{s.t.}\begin{cases} 4x_1-x_2+2x_3-x_4=-2 \\ x_1+x_2-x_3+2x_4\leqslant 14 \\ 2x_1-3x_2-x_3-x_4\geqslant -2 \\ x_{1,2,3}\geqslant -1,\ x_4\ \text{unconstrained}\end{cases}} -3x_1+4x_2-2x_3+5x_4$$

(ii) $$\min_{\boldsymbol{x}\ \text{s.t.}\begin{cases} x_1+x_2+x_3+x_4=4 \\ -2x_1+x_2-x_3-x_6+x_7=1 \\ 3x_2+x_3+x_5+x_7=9 \\ x_{1,2,\cdots,7}\geqslant 0\end{cases}} x_6+x_7$$

Solution (i) The linear programming problem can be solved with the following statements, with the solution of $\boldsymbol{x} = [-1, 2.5, -1, -6.5]^{\mathrm{T}}$.

```
>> f=[-3 4 -2 5]; Aeq=[4 -1 2 -1]; Beq=-2;
   A=[1 1 -1 2; -2 3 1 1]; B=[14; 2]; xm=[-1;-1;-1;-inf];
   x=linprog(f,A,B,Aeq,Beq,xm)
```

(ii) The solution to the problem is found with

```
>> f=[0,0,0,0,0,1,1]; xm=[0;0;0;0;0;0;0]; A=[]; B=[];
   Aeq=[1 1 1 1 0 0 0; -2 1 -1 0 0 -1 1; 0 3 1 0 1 0 1];
   Beq=[4; 1; 9]; x=linprog(f,A,B,Aeq,Beq,xm)
```

and the solution is $\boldsymbol{x} = [0.3952, 2.3213, 0.5309, 0.7526, 1.5053, 0, 0]^{\mathrm{T}}$.

11. Solve the following quadratic programming problems and also illustrate the solutions using graphical methods.

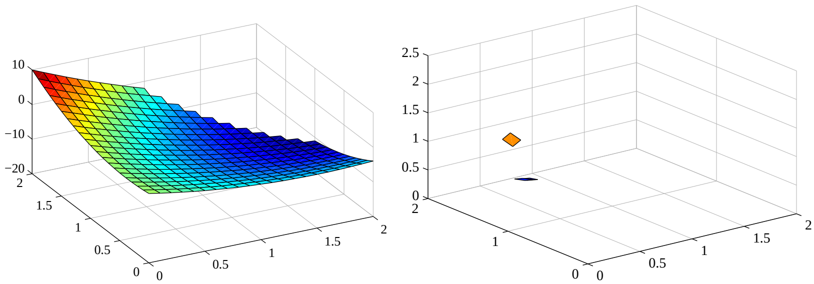
(i) $\min_{\boldsymbol{x} \text{ s.t.} \begin{cases} x_1+x_2\leqslant 3 \\ 4x_1+x_2\leqslant 9 \\ x_{1,2}\geqslant 0 \end{cases}} 2x_1^2-4x_1x_2+4x_2^2-6x_1-3x_2$ (ii) $\min_{\boldsymbol{x} \text{ s.t.} \begin{cases} -x_1+x_2=1 \\ x_1+x_2\leqslant 2 \\ x_{1,2}\geqslant 0 \end{cases}} (x_1-1)^2+(x_2-2)^2$

Solution (i) The quadratic programming matrices $\boldsymbol{H}$ and $\boldsymbol{f}$ can be written as

$$\boldsymbol{H} = \begin{bmatrix} 4 & -4 \\ -4 & 8 \end{bmatrix}, \quad \boldsymbol{f} = [-6, -3]$$

The problem can be solved and $\boldsymbol{x} = [1.95, 1.05]^{\mathrm{T}}$. The graphical method results are shown in Figure 6.3 (a), which also confirmed that the solutions obtained are correct.

```
>> H=[4 -4; -4 8]; f=[-6 -3]; Aeq=[]; Beq=[]; A=[1 1; 4 1];
   B=[3;9]; xm=[0;0]; x=quadprog(H,f,A,B,Aeq,Beq,xm)
   [x1,x2]=meshgrid(0:0.1:2); z=2*x1.^2-4*x1.*x2+4*x2.^2-6*x1-3*x2;
   ii=find(x1+x2>3 | 4*x1+x2>9); z(ii)=NaN; surf(x1,x2,z)
```



(a) problem (i) (b) problem (ii)

FIGURE 6.3: Graphical solutions of the QP problems

(ii) The objective function can be expanded and from which, the matrices

$$\boldsymbol{H} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \boldsymbol{f} = [-2, -4]$$

The problem can then be solved and $\boldsymbol{x} = [0.5, 1.5]^{\mathrm{T}}$. The graphical method shown in Figure 6.3 (b) verifies the results. Also since equation constraints are involved, it is the only feasible solution to the problem.

```
>> H=diag([2 2]); f=[-2 -4]; Aeq=[-1 1]; Beq=1;
   A=[1 1]; B=2; xm=[0;0]; x=quadprog(H,f,A,B,Aeq,Beq,xm)
   [x1,x2]=meshgrid(0:0.1:2); z=(x1-1).^2+(x2-2).^2;
   ii=find(abs(-x1+x2-1)>=0.1|x1+x2>3); z(ii)=NaN; surf(x1,x2,z)
```

12. Solve numerically the following nonlinear programming problems.

(i)
$$\min_{\boldsymbol{x} \text{ s.t.} \begin{cases} x_1+x_2\leqslant 0 \\ -x_1x_2+x_1+x_2\geqslant 1.5 \\ x_1x_2\geqslant -10 \\ -10\leqslant x_1,x_2\leqslant 10 \end{cases}} \mathrm{e}^{x_1}(4x_1^2+2x_2^2+4x_1x_2+2x_2+1)$$

(ii)
$$\min_{\boldsymbol{x} \text{ s.t.} \begin{cases} 0.003079x_1^3x_2^3x_5-\cos^3 x_6\geqslant 0 \\ 0.1017x_3^3x_4^3-x_5^2\cos^3 x_6\geqslant 0 \\ 0.09939(1+x_5)x_1^3x_2^2-\cos^2 x_6\geqslant 0 \\ 0.1076(31.5+x_5)x_3^3x_4^2-x_5^2\cos^2 x_6\geqslant 0 \\ x_3x_4(x_5+31.5)-x_5[2(x_1+5)\cos x_6+x_1x_2x_5]\geqslant 0 \\ 0.2\leqslant x_1\leqslant 0.5, 14<\leqslant x_2\leqslant 22, 0.35\leqslant x_3\leqslant 0.6, \\ 16\leqslant x_4\leqslant 22, 5.8\leqslant x_5\leqslant 6.5, 0.14\leqslant x_6\leqslant 0.2618 \end{cases}} \frac{1}{2\cos x_6}\left[x_1x_2(1+x_5)+x_3x_4\left(1+\frac{31.5}{x_5}\right)\right]$$

SOLUTION (i) The objective function can be expressed as

```
>> f=@(x)exp(x(1))*(4*x(1)^2+2*x(2)^2+4*x(1)*x(2)+2*x(2)+1);
```

the constraints function can also be designed

```
function [c,ce]=exc6fun6a(x)
ce=[]; c=[x(1)+x(2); x(1)*x(2)-x(1)-x(2)+1.5; -10-x(1)*x(2)];
```

The optimization problem solver can be used to solve the nonlinear programming problem.

```
>> A=[]; B=[]; Aeq=[]; Beq=[]; xm=[-10; -10]; xM=[10; 10];
   x0=(xm+xM)/2; ff=optimset; ff.TolX=1e-10; ff.TolFun=1e-20;
   x=fmincon(f,x0,A,B,Aeq,Beq,xm,xM,'exc6fun6a',ff)
```

However it is prompted after execution that "Maximum number of function evaluations exceeded; increase OPTIONS.MaxFunEvals", the solution obtained is not the expected one. One may use it as an initial point to further search for better ones with a loop structure, until a good solution is found at $\boldsymbol{x} = [1.1825, -1.7398]^{\mathrm{T}}$, after 5 loop executions.

```
>> i=0; x=x0;
   while (1)
      [x,a,b]=fmincon(f,x,A,B,Aeq,Beq,xm,xM,'exc6fun6a',ff); i=i+1;
      if b>0, break; end
   end
```

It should be noted that sometimes, due to the improperly assigned control parameters, such as the maximum allowed interation time, the solutions obtained may not be the one expected. One should further search for better solutions.
(ii) The constraints can be expressed as

```
function [c,ce]=exc6fun5a(x)
ce=[];
c=-[0.003079*x(1)^3*x(2)^3*x(5)-cos(x(6))^3;
   0.1017*x(3)^3*x(4)^3-x(5)^2*cos(x(6))^3;
   0.09939*(1+x(5))*x(1)^3*x(2)^2-cos(x(6))^2;
```

```
  0.1076*(31.5+x(5))*x(3)^3*x(4)^2-x(5)^2*cos(x(6))^2;
x(3)*x(4)*(x(5)+31.5)-x(5)*(2*(x(1)+5)*cos(x(6))+x(1)*x(2)*x(5))];
```

The objective function is expressed as

```
>> f=@(x)(x(1)*x(2)*(1+x(5))+x(3)*x(4)*(1+31.5/x(5)))/cos(x(6))/2;
```

Thus the solution found is $\boldsymbol{x} = [0.2012, 18.3629, 0.3596, 18.5780, 5.8, 0.2618]^{\mathrm{T}}$.

```
>> xm=[0.2; 14; 0.35; 16; 5.8; 0.14]; A=[]; B=[];
   xM=[0.5; 22; 0.6;  22; 6.5; 0.2618]; Aeq=[]; Beq=[];
   x0=(xm+xM)/2; ff=optimset; ff.TolX=1e-10; ff.TolFun=1e-20;
   x=fmincon(f,x0,A,B,Aeq,Beq,xm,xM,'exc6fun5a',ff)
```

13. Solve the constrained optimization problem $\boldsymbol{q}$ and k[4].

(i)
$$\min_{\boldsymbol{q},w,k \text{ s.t.}} k \quad \left\{\begin{array}{l} q_3+9.625q_1w+16q_2w+16w^2+12-4q_1-q_2-78w=0 \\ 16q_1w+44-19q_1-8q_2-q_3-24w=0 \\ 2.25-0.25k\leqslant q_1\leqslant 2.25+0.25k \\ 1.5-0.5k\leqslant q_2\leqslant 1.5+0.5k \\ 1.5-1.5k\leqslant q_3\leqslant 1.5+1.5k \end{array}\right.$$

(ii)
$$\min_{\boldsymbol{q},k \text{ s.t.}} k \quad \left\{\begin{array}{l} g(\boldsymbol{q})\leqslant 0 \\ 800-800k\leqslant q_1\leqslant 800+800k \\ 4-2k\leqslant q_2\leqslant 4+2k \\ 6-3k\leqslant q_3\leqslant 6+3k \end{array}\right.$$

where

$$\begin{aligned} g(\boldsymbol{q}) = {} & 10q_2^2q_3^3+10q_2^3q_3^2+200q_2^2q_3^2+100q_2^3q_3+q_1q_2q_3^2+q_1q_2^2q_3+1000q_2q_3^3 \\ & +8q_1q_3^2+1000q_2^2q_3+8q_1q_2^2+6q_1q_2q_3-q_1^2+60q_1q_3+60q_1q_2-200q_1 \end{aligned}$$

SOLUTION (i) Assume that $x_1=q_1, x_2=q_2, x_3=q_3, x_4=k, x_5=k$, the problem can be posed as

$$\min_{\boldsymbol{x} \text{ s.t.}} x_5 \quad \left\{\begin{array}{l} x_3+9.625x_1x_4+16x_2x_4+16x_4^2+12-4x_1-x_2-78x_4=0 \\ 16x_1x_4+44-19x_1-8x_2-x_3-24x_4=0 \\ -x_1-0.25x_5\leqslant -2.25 \\ x_1-0.25x_5\leqslant 2.25 \\ -x_2-0.5x_5\leqslant -1.5 \\ x_2-0.5x_5\leqslant 1.5 \\ -x_3-1.5x_5\leqslant -1.5 \\ x_3-1.5x_5\leqslant 1.5 \end{array}\right.$$

Thus one can write the nonlinear constraints as

```
function [c,ceq]=exc6c2(x)
c=[];
ceq=[x(3)+9.625*x(1)*x(4)+16*x(2)*x(4)+16*x(4)^2+...
          12-4*x(1)-x(2)-78*x(4)
     16*x(1)*x(4)+44-19*x(1)-8*x(2)-x(3)-24*x(4)];
```

With the following statements

```
>> Aeq=[]; Beq=[]; B=[-2.25; 2.25; -1.5; 1.5; -1.5; 1.5];
   A=[-1 0 0 0 -0.25; 1 0 0 0 -0.25; 0 -1 0 0 -0.5; 0 1 0 0 -0.5;
      0 0 -1 0 -1.5; 0 0 1 0 -1.5]; f=@(x)x(5); x0=ones(5,1);
   x=fmincon(f,x0,A,B,Aeq,Beq,[],[],'exc6c2')
```

and the solution is $\boldsymbol{x} = [2.0309, 1.9382, 2.8147, 1.5196, 0.8765]^{\mathrm{T}}$.
(ii) Assume that $x_1 = q_1$, $x_2 = q_2$, $x_3 = q_3$, $x_4 = k$, the original problem can be rewritten as

$$\min \quad x_4$$
$$\boldsymbol{x} \text{ s.t.} \begin{cases} g(\boldsymbol{x}) \leqslant 0 \\ -x_1 - 800x_4 \leqslant -800 \\ x_1 - 800x_4 \leqslant 800 \\ -x_2 - 2x_4 \leqslant -4 \\ x_2 - 2x_4 \leqslant 4 \\ -x_3 - 3x_4 \leqslant -6 \\ x_3 - 3x_4 \leqslant 6 \end{cases}$$

where

$$g(\boldsymbol{x}) = 10x_2^2x_3^3 + 10x_2^3x_3^2 + 200x_2^2x_3^2 + 100x_2^3x_3 + x_1x_2x_3^2 + x_1x_2^2x_3 + 1000x_2x_3^3 \\ + 8x_1x_3^2 + 1000x_2^2x_3 + 8x_1x_2^2 + 6x_1x_2x_3 - x_1^2 + 60x_1x_3 + 60x_1x_2 - 200x_1$$

The nonliear constraints can be modeled with

```
function [c,ceq]=exc6c3(x)
ceq=[];
c=10*x(2)^2*x(3)^3+10*x(2)^3*x(3)^2+200*x(2)^2*x(3)^2+...
   100*x(2)^3*x(3)+x(1)*x(2)*x(3)^2+x(1)*x(2)^2*x(3)+...
   1000*x(2)*x(3)^3+8*x(1)*x(3)^2+1000*x(2)^2*x(3)+...
   8*x(1)*x(2)^2+6*x(1)*x(2)*x(3)-x(1)^2+60*x(1)*x(3)+...
   60*x(1)*x(2)-200*x(1);
```

and the problem can be solved with

```
>> f=@(x)x(4); Aeq=[]; Beq=[]; B=[-800; 800; -4; 4; -6; 6];
   A=[1 0 0 -800; -1 0 0 -800; 0 -1 0 -2;
      0 1 0 -2; 0 0 -1 -3; 0 0 1 -3];
   x0=ones(4,1); x=fmincon(f,x0,A,B,Aeq,Beq,[],[],'exc6c3')
```

The solution obtained is $\boldsymbol{x} = [214.7013, 1.4632, 2.1949, 1.2684]^{\mathrm{T}}$.

14. Solve the following integer linear programming problems.

(i)
$$\max \quad (592x_1 + 381x_2 + 273x_3 + 55x_4 + 48x_5 + 37x_6 + 23x_7)$$
$$\boldsymbol{x} \text{ s.t.} \begin{cases} \boldsymbol{x} \geqslant 0 \\ 3534x_1 + 2356x_2 + 1767x_3 + 589x_4 + 528x_5 + 451x_6 + 304x_7 \leqslant 119567 \end{cases}$$

(ii)
$$\max \quad (120x_1 + 66x_2 + 72x_3 + 58x_4 + 132x_5 + 104x_6)$$
$$\boldsymbol{x} \text{ s.t.} \begin{cases} x_1 + x_2 + x_3 = 30 \\ x_4 + x_5 + x_6 = 18 \\ x_1 + x_4 = 10 \\ x_2 + x_5 \leqslant 18 \\ x_3 + x_6 \geqslant 30 \\ x_{1,\cdots,6} \geqslant 0 \end{cases}$$

SOLUTION (i) The maxization problem should first be converted into the standard minimization problem. The objective function can then be expressed by

```
function y=exc6o1(x)
y=-[592 381 273 55 48 37 23]*x;
```

and the integer linear programming problem can be solved with

```
>> intlist=ones(7,1); Aeq=[]; Beq=[]; xm=zeros(7,1);
   A=[3534 2356 1767 589 528 451 304]; B=119567;
   xM=20000*ones(7,1); x0=xM;
   [a b x]=bnb20('exc6o1',ones(7,1),intlist,xm,xM,A,B,Aeq,Beq);
   ix=(intlist==1); x(ix)=round(x(ix))
```

and the solutions is $\boldsymbol{x} = [32, 2, 1, 0, 0, 0, 0]^{\mathrm{T}}$.
(ii) The objective function can be modeled as

```
function y=exc6o2(x)
y=-[120 66 72 58 132 104]*x;
```

Thus one can issue the following MATLAB commands

```
>> Aeq=[1 1 1 0 0 0; 0 0 0 1 1 1; 1 0 0 1 0 0]; Beq=[30; 18; 10];
   A=[0 1 0 0 1 0; 0 0 -1 0 0 -1]; B=[18; -30];
   intlist=ones(6,1); xm=zeros(6,1); xM=20000*ones(6,1); x0=xm;
   [errmsg,f,x]=bnb20('exc6o2',x0,intlist,xm,xM,A,B,Aeq,Beq);
   if length(errmsg)==0, x=round(x), end
```

with the solution $\boldsymbol{x} = [10, 0, 20, 0, 8, 10]^{\mathrm{T}}$.

15. Solve the following binary linear programming problems and verify the results in problems (i) and (ii) using the enumerate methods.

(i) $$\min_{\boldsymbol{x} \text{ s.t. } \begin{cases} x_1-x_2+5x_3+x_4-4x_5 \geqslant 2 \\ -2x_1+6x_2-3x_3-2x_4+2x_5 \geqslant 0 \\ -2x_2+2x_3-x_4-x_5 \leqslant 1 \\ 0 \leqslant x_i \leqslant 1 \end{cases}} (5x_1 + 7x_2 + 10x_3 + 3x_4 + x_5)$$

(ii) $$\min_{\boldsymbol{x} \text{ s.t. } \begin{cases} x_1-x_6 \leqslant 0 \\ x_1-x_5 \leqslant 0 \\ x_2-x_4 \leqslant 0 \\ x_2-x_5 \leqslant 0 \\ x_3-x_4 \leqslant 0 \\ x_1+x_2+x_3 \leqslant 2 \\ 0 \leqslant x_i \leqslant 1 \end{cases}} (-3x_1 - 4x_2 - 5x_3 + 4x_4 + 4x_5 + 2x_6)$$

SOLUTION (i) For the binary linear programming problem, the following commands can be given, which returns $\boldsymbol{x} = [0, 1, 1, 0, 0]^{\mathrm{T}}$, with a minimum of 17.

```
>> f=[5 7 10 3 1]; B=[-2; 0; 1];
   A=[-1 1 -5 -1 4; 2 -6 3 2 -2; 0 -2 2 -1 -1];
   x=bintprog(f,A,B,[],[])
```

Enumerate method can also be used for this small-scale problem

```
>> [x1,x2,x3,x4,x5]=ndgrid([0,1]);
   i=find((x1-x2+5*x3+x4-4*x5>=2)& (-2*x1+6*x2-3*x3-2*x4+2*x5>=0)& ...
```

```
        (-2*x2+2*x3-x4-x5<=1));
   f=5*x1(i)+7*x2(i)+10*x3(i)+3*x4(i)+x5(i); [fmin,ii]=sort(f);
   idx=i(ii(1)); x=[x1(idx),x2(idx),x3(idx),x4(idx),x5(idx)]
```

(ii) Again `bintprog()` function can be used

```
>> A=[1 0 0 0 0 -1; 1 0 0 0 -1 0; 0 1 0 -1 0 0;
      0 1 0 0 -1 0; 0 0 1 -1 0 0; 1 1 1 0 0 0];
   B=[0; 0; 0; 0; 0; 2]; f=[-3 -4 -5 4 4 2];
   x=bintprog(f,A,B,[],[])
```

with $\boldsymbol{x} = [0,0,1,1,0,0]^{\mathrm{T}}$. Using the enumerate method, the following statements can be given

```
>> [x1,x2,x3,x4,x5,x6]=ndgrid([0,1]);
   i=find((x1-x6<=0)& (x1-x5<=0)& (x2-x4<=0)& (x2-x5<=0)&...
         (x3-x4<=0)& (x1+x2+x3<=2));
   ff=-3*x1(i)-4*x2(i)-5*x3(i)+4*x4(i)+4*x5(i)+2*x6(i);
   [fmin,ii]=sort(ff)
```

However by displaying the sorted values of objective function `fmin`, it is found that there are two solutions with the minimum objective function of -1. Thus the above result by `bintprog()` could be only one of them. Use the enumerate method, one can find the two solutions $\boldsymbol{x}_1 = [0,0,1,1,0,0]^{\mathrm{T}}$, and $\boldsymbol{x}_1 = [0,1,1,1,1,0]^{\mathrm{T}}$.

```
>> idx=i(ii(1:2));
   x=[x1(idx),x2(idx),x3(idx),x4(idx),x5(idx),x6(idx)]
```

16. Solve the binary linear programming problem.

$$\max_{\boldsymbol{x} \text{ s.t.} \begin{cases} [\boldsymbol{A}_1, \boldsymbol{A}_2]\boldsymbol{x} \leqslant \begin{bmatrix} 600 \\ 600 \end{bmatrix} \\ 0\leqslant x_i \leqslant 1 \end{cases}} -\boldsymbol{f}\boldsymbol{x}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} 45 & 0 & 85 & 150 & 65 & 95 & 30 & 0 & 170 & 0 & 40 & 25 & 20 & 0 \\ 30 & 20 & 125 & 5 & 80 & 25 & 35 & 73 & 12 & 15 & 15 & 40 & 5 & 10 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 0 & 25 & 0 & 0 & 25 & 0 & 165 & 0 & 85 & 0 & 0 & 0 & 0 & 100 \\ 10 & 12 & 10 & 9 & 0 & 20 & 60 & 40 & 50 & 36 & 49 & 40 & 19 & 150 \end{bmatrix}$$

$$\begin{aligned}\boldsymbol{f} = [&1898, 440, 22507, 270, 14148, 3100, 4650, 30800, 615, 4975, 1160, 4225,\\ &510, 11880, 479, 440, 490, 330, 110, 560, 24355, 2885, 11748, 4550,\\ &750, 3720, 1950, 10500]\end{aligned}$$

Solution The binary linear programming problem can easily be solved with

```
>> A1=[45,0,85,150,65,95,30,0,170,0,40,25,20,0;
       30,20,125,5,80,25,35,73,12,15,15,40,5,10,];
   A2=[0,25,0,0,25,0,165,0,85,0,0,0,0,100;
       10,12,10,9,0,20,60,40,50,36,49,40,19,150];
   A=[A1 A2];  B=[600; 600];
   f=-[1898,440,22507,270,14148,3100,4650,30800,615,4975,1160,...
         4225,510,11880,479,440,490,330,110,560,24355,2885,...
```

```
        11748,4550,750,3720,1950,10500];
x=bintprog(f,A,B)
```

and it is found that the solution is

$$\boldsymbol{x} = [0,0,1,0,1,1,1,1,0,1,0,1,1,1,0,0,0,0,1,0,1,0,1,1,0,1,0,0]^{\mathrm{T}}.$$

17. Solve the optimization problem with Robust Control Toolbox and YALIMP.

$$\min_{\boldsymbol{X} \text{ s.t.} \begin{cases} \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}}\boldsymbol{X}+\boldsymbol{X}\boldsymbol{A}+\boldsymbol{Q} & \boldsymbol{X}\boldsymbol{B} \\ \boldsymbol{B}^{\mathrm{T}}\boldsymbol{X} & -\boldsymbol{I} \end{bmatrix} < 0 \\ \boldsymbol{X} < 0 \end{cases}} \operatorname{tr}(\boldsymbol{X})$$

where $\boldsymbol{A} = \begin{bmatrix} -1 & -2 & 1 \\ 3 & 2 & 1 \\ 1 & -2 & -1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\boldsymbol{Q} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -3 & -12 \\ 0 & -12 & -36 \end{bmatrix}$

Solution (i) With Robust Control Toolbox

```
>> A=[-1 -2 1; 3 2 1; 1 -2 -1]; B=[1;0;1];
   Q=[1 -1 0; -1 -3 -12; 0 -12 -36];
   setlmis([]); X=lmivar(1,[3 1]);
   lmiterm([1 1 1 X],A',1,'s'); lmiterm([1 1 1 0],Q)
   lmiterm([1 1 2 X],1,B); lmiterm([1 2 2 0],-1)
   lmiterm([2,1,1,X],1,1); G=getlmis;
   [c,x]=mincx(G,trace(X))
```

(ii) With YALMIP, the following statements can be applied

```
>> A=[-1 -2 1; 3 2 1; 1 -2 -1]; B=[1;0;1];
   Q=[1 -1 0; -1 -3 -12; 0 -12 -36]; X=sdpvar(3);
   F=[[A'*X+X*A'+Q, X*B; B'*X -1]<0, X<0];
   sol=solvesdp(F,trace(X)); X=double(X)
```

with the results $\boldsymbol{X} = \begin{bmatrix} -0.1620 & -0.3488 & -0.4880 \\ -0.3488 & -1.3216 & -2.5964 \\ -0.4880 & -2.5964 & -5.6625 \end{bmatrix}$.

18. Solve the following linear matrix inequalities

$$\begin{cases} \boldsymbol{P}^{-1} > 0, \quad \text{or equavelently} \quad \boldsymbol{P} > 0 \\ \boldsymbol{A}_1\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_1^{\mathrm{T}} + \boldsymbol{B}_1\boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}_1^{\mathrm{T}} < 0 \\ \boldsymbol{A}_2\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}_2^{\mathrm{T}} + \boldsymbol{B}_2\boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{B}_2^{\mathrm{T}} < 0 \end{cases}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 & 2 & -2 \\ -1 & -2 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \ \boldsymbol{B}_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \ \boldsymbol{A}_2 = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

Solution This problem is also known as the simutaneous stabilization problem in control. With the Robust Control Toolbox, the problem can be solved with

```
>> A1=[-1,2,-2; -1,-2,1; -1,-1,0]; B1=[-2; 1; -1];
   A2=[0,2,2; 2,0,2; 2,0,1]; B2=[-1; -2; -1];
   setlmis([]); P=lmivar(1,[3,1]); Y=lmivar(2,[1,3]);
   lmiterm([1,1,1,P],-1,1);
```

```
   lmiterm([2,1,1,P],A1,1,'s'), lmiterm([2,1,1,Y],B1,1,'s')
   lmiterm([3,1,1,P],A2,1,'s'), lmiterm([3,1,1,Y],B2,1,'s')
   G=getlmis; [a,b]=feasp(G); P=dec2mat(G,b,P), Y=dec2mat(G,b,Y)
```

The solutions obtained are

$$\boldsymbol{P}=\begin{bmatrix}17.968 & -11.958 & -11.555\\ -11.958 & 21.607 & 10.857\\ -11.555 & 10.857 & 12.778\end{bmatrix},\quad \boldsymbol{Y}^{\mathrm{T}}=\begin{bmatrix}2.1076\\ 14.058\\ 13.585\end{bmatrix}$$

If YALMIP toolbox is used, the same problem can be solved with the following statements, and the results are also the same.

```
>> P=sdpvar(3); Y=sdpvar(1,3);
   F=[A1*P+P*A1'+B1*Y+Y'*B1'<0,A2*P+P*A2'+B2*Y+Y'*B2'<0,P>0];
   sol=solvesdp(F); P=double(P), Y=double(Y)
```

Chapter 7

Differential Equation Problems

Exercises and Solutions

1. Find the general solutions to the following linear differential equation.

$$\frac{\mathrm{d}^5y(t)}{\mathrm{d}t^5}+13\frac{\mathrm{d}^4y(t)}{\mathrm{d}t^4}+64\frac{\mathrm{d}^3y(t)}{\mathrm{d}t^3}+152\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}+176\frac{\mathrm{d}y(t)}{\mathrm{d}t}+80y(t)$$
$$=\mathrm{e}^{-2t}\left[\sin\left(2t+\frac{\pi}{3}\right)+\cos 3t\right]$$

Assume that the initial conditions are assumed to be $y(0)=1$, $y(1)=3$, $y(\pi)=2, \dot{y}(0)=1, \dot{y}(1)=2$, find the analytical solution to the problem. Verify the result.

Solution The analytical solutions of the equation can be found with the following statements. The final results may take many pages to display, and will not be given here. The solution may take some time.

```
>> syms t; u=exp(-2*t)*(sin(2*t+sym(pi)/3)+cos(3*t));
   y=dsolve(['D5y+13*D4y+64*D3y+152*D2y+176*Dy+80*y=',char(u)],...
        'y(0)=1','y(1)=3','y(pi)=2','Dy(0)=1','Dy(1)=2')
```

To verify the results, one may give the following statements

```
>> simple(diff(y,5)+13*diff(y,4)+64*diff(y,3)+152*diff(y,2)+...
          176*diff(y)+80*y-u)
```

2. Find the general analytical solution to the linear differential equations, and also the solution satisfying $x(0)=1, x(\pi)=2, y(0)=0$. Verify the results.

$$\begin{cases}\ddot{x}(t)+5\dot{x}(t)+4x(t)+3y(t)=\mathrm{e}^{-6t}\sin 4t\\ 2\dot{y}(t)+y(t)+4\dot{x}(t)+6x(t)=\mathrm{e}^{-6t}\cos 4t\end{cases}$$

Solution The general solutions of the equations can be obtained with the following statements

```
>> syms t; syms C1 C2 C3;
   [x,y]=dsolve('D2x+5*Dx+4*x+3*y=exp(-6*t)*sin(4*t)',...
          '2*Dy+y+4*Dx+6*x=exp(-6*t)*cos(4*t)')
   simple(2*diff(y)+y+4*diff(x)+6*x-exp(-6*t)*cos(4*t))
   simple(diff(x,2)+5*diff(x)+4*x+3*y-exp(-6*t)*sin(4*t))
```

with the verified results

$$x(t)=\frac{1013}{40820}\mathrm{e}^{-6t}\cos 4t-\frac{681}{40820}\mathrm{e}^{-6t}\sin 4t+C_1\mathrm{e}^{t}+C_2\mathrm{e}^{-\frac{13}{4}t-\frac{\sqrt{57}}{4}t}+C_3\mathrm{e}^{-\frac{13}{4}t+\frac{\sqrt{57}t}{4}}$$

$$y(t)=-\frac{433}{4082}\mathrm{e}^{-6t}\cos 4t+\frac{279}{4082}\mathrm{e}^{-6t}\sin 4t-\frac{10}{3}C_1\mathrm{e}^{t}-\frac{5}{8}C_2\mathrm{e}^{-\frac{13}{4}t-\frac{\sqrt{57}t}{4}}$$
$$-\frac{\sqrt{57}}{8}C_2\mathrm{e}^{-\frac{13}{4}t-\frac{\sqrt{57}t}{4}}-\frac{5}{8}C_3\mathrm{e}^{-\frac{13}{4}t+\frac{\sqrt{57}t}{4}}+\frac{\sqrt{57}}{8}C_3\mathrm{e}^{-\frac{13}{4}t+\frac{\sqrt{57}t}{4}}.$$

Substituting the known conditions into the solutions, the undetermined coefficients C_i can be uniquely found. However again they are very lengthy and will not be displayed here.

```
>> [x,y]=dsolve('D2x+5*Dx+4*x+3*y=exp(-6*t)*sin(4*t)',...
           '2*Dy+y+4*Dx+6*x=exp(-6*t)*cos(4*t)',...
           'x(0)=1','x(pi)=2','y(0)=0')
```

3. Write out the analytical solutions to the linear time-varying differential equations given below.

(i) Legendre equation: $(1-t^2)\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}-2t\dfrac{\mathrm{d}x}{\mathrm{d}t}+n(n+1)x=0$

(ii) Bessel equation: $t^2\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}+t\dfrac{\mathrm{d}x}{\mathrm{d}t}+(t^2-n^2)x=0$

SOLUTION The solutions of the equations are all based on special functions such Legendre's function and Bessel's function.

```
>> syms t n; x1=dsolve('(1-t^2)*D2x-2*t*Dx+n*(n+1)*x=0','x')
   x2=dsolve('t^2*D2x+t*Dx+(t^2-n^2)*x=0')
```

4. Find the analytical solution to the differential equation

$$\ddot{y}(x)-\left(2-\frac{1}{x}\right)\dot{y}(x)+\left(1-\frac{1}{x}\right)y(x)=x^2\mathrm{e}^{-5x}.$$

Also, if the boundaries are specified by $y(1)=\pi,\ y(\pi)=1$, find the analytical solution. Verify the results.

SOLUTION The general solutions to the differential equation can be obtained from

```
>> syms x; y=dsolve('D2y-(2-1/x)*Dy+(1-1/x)*y=x^2*exp(-5*x)','x')
```

with $\mathrm{Ei}(\cdot)$ the special function and

$$y=\mathrm{e}^{x}C_2+\mathrm{e}^{x}\ln(x)\,C_1+\frac{1}{216}\mathrm{Ei}\,(1,6x)\,\mathrm{e}^{x}+\frac{11}{1296}\mathrm{e}^{-5x}+\frac{5}{216}\mathrm{e}^{-5x}x+1/36x^2\mathrm{e}^{-5x}$$

If the boundary conditions are involved, the special solution to the equation can be obtained with the following statements

```
>> syms x; y=dsolve('D2y-(2-1/x)*Dy+(1-1/x)*y=x^2*exp(-5*x)',...
              'y(1)=sym(pi)','y(sym(pi))=1','x')
```

5. Solve the following differential equations using Laplace transforms.

$$\begin{cases}\ddot{x}(t)+\ddot{y}(t)+x(t)+y(t)=0,\ x(0)=2,y(0)=1\\ 2\ddot{x}(t)-\ddot{y}(t)-x(t)+y(t)=\sin t,\ \dot{x}(0)=\dot{y}(0)=-1\end{cases}$$

Compare the results with the ones by other methods.

SOLUTION Assume that the Laplace transform of $y(t)$ is $Y(s)$, then $\mathscr{L}[y''(t)]=s^2Y(s)-sy(0)-y'(0)$. Thus the equations can be transformed to

$$\begin{cases} s^2X(s) - sx(0) - x'(0) + s^2Y(s) - sy(0) - y'(0) + X(s) + Y(s) = 0 \\ 2s^2X(s) - 2sx(0) - 2x'(0) - s^2Y(s) + sy(0) + y'(0) - X(s) + Y(s) = \dfrac{1}{s^2+1} \end{cases}$$

The equations now become the algebraic equations of $X(s)$ and $Y(s)$, and can be solved with the function `solve()` discussed in Chapter 6. Inverse Laplace transform can be used to find the solutions of original differential equations.

```
>> syms s X Y;
   [X,Y]=solve('s^2*X-s*2-(-1)+s^2*Y-s*1-(-1)+X+Y=0',...
      '2*(s^2*X-s*2-(-1))-(s^2*Y-s*1-(-1))-X+Y=1/(s^2+1)','X,Y')
   x0=ilaplace(X), y0=ilaplace(Y)
```

The solutions are respectively

$$x = \frac{6}{5}\cos t - \sin t + \frac{4}{5}\cosh\frac{\sqrt{6}}{3}t, \text{ and } y = \frac{9}{5}\cos t - \sin t - \frac{4}{5}\cosh\frac{\sqrt{6}}{3}t.$$

Using the direct differential equation solution functions, the following statements can be given

```
>> syms t
   [x,y]=dsolve('D2x+D2y+x+y=0','2*D2x-D2y-x+y=sin(t)',...
       'x(0)=2','y(0)=1','Dx(0)=-1','Dy(0)=-1')
```

and the new solutions are

$$x = \frac{2}{5}\mathrm{e}^{\sqrt{6}t/3} + \frac{2}{5}\mathrm{e}^{-\sqrt{6}t/3} - \sin t + \frac{6}{5}\cos t,\ y = -\frac{2}{5}\mathrm{e}^{\sqrt{6}t/3} - \frac{2}{5}\mathrm{e}^{-\sqrt{6}t/3} - \sin t + \frac{9}{5}\cos t.$$

The appearances of the two sets of solutions are slightly different, however it can easily be shown that they are identical.

6. Find the general solutions to the following equations.

   (i) $\ddot{x}(t) + 2t\dot{x}(t) + t^2x(t) = t + 1$ (ii) $\dot{y}(x) + 2xy(x) = x\mathrm{e}^{-x^2}$

   (iii) $y^{(3)} + 3\ddot{y} + 3\dot{y} + y = \mathrm{e}^{-t}\sin t$

SOLUTION The general solutions of the equations can be solved with

```
>> syms t; y1=dsolve('D2x+2*t*Dx+t^2*x=t+1')
   syms x; y2=dsolve('Dy+2*x*y=x*exp(-x^2)','x')
   syms t; y3=dsolve('D3y+3*D2y+3*Dy+y=exp(-t)*sin(t)')
```

and the general solutions are

$$y_1 = C_2\mathrm{e}^{-t-t^2/2} + C_1\mathrm{e}^{t-t^2/2} - \frac{\sqrt{2\pi}\mathrm{j}}{2}\mathrm{erf}\left(\frac{\sqrt{2}\mathrm{j}}{2}(t-1)\right)\mathrm{e}^{-t^2/2+t-1/2}$$

$$y_2 = \frac{1}{2}\left(x^2 + 2C_1\right)\mathrm{e}^{-x^2},\ y_3 = \mathrm{e}^{-t}\left(\cos t + C_1 + C_2t^2 + C_3t\right).$$

It should be noted that in equation (ii), the independent variable is x rather than the default t, thus in the function call, one should declare x at the end, otherwise the solution obtained may be incorrect.

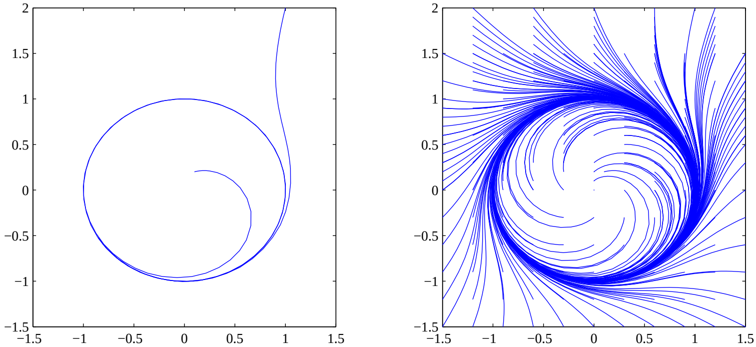
7. Limit cycle is a common phenomenon in nonlinear differential equations. For system nonlinear differential equations, no matter what the initial values are selected, the phase trajectory will settle down on the same closed path, which is referred to as the *limit cycle*. Solve the differential equation and draw the limit cycle.

$$\begin{cases} \dot{x} = y + x(1 - x^2 - y^2) \\ \dot{y} = -x + y(1 - x^2 - y^2) \end{cases}$$

Try different initial values, and check whether the phase plane plot converges to the limit cycle.

SOLUTION Since the equation is nonlinear, the above mentioned analytical solutions are not possible. We have to rely on numerical solutions. One should assign a set of state variables such as $x_1 = x, x_2 = y$, the following statements can be used to evaluate numerically the differential equations, for different initial values. The phase-plane trajectories are shown in Figure 7.1, and it can be seen that all the curves converge to a closed-path, referred to as limit cycles.

```
>> f=@(t,x)[x(2)+x(1)*(1-x(1)^2-x(2)^2);-x(1)+x(2)*(1-x(1)^2-x(2)^2)];
   [t,x]=ode45(f,[0,10],[1;2]); plot(x(:,1),x(:,2));
   [t,x]=ode45(f,[0,10],[0.1;0.2]); line(x(:,1),x(:,2));
   [x1,y1]=meshgrid(-1.5:0.3:1.5); x1=x1(:); y1=y1(:); figure
   for i=1:length(x1)
      [t,x]=ode45(f,[0,10],[x1(i); y1(i)]); line(x(:,1),x(:,2));
   end
```



(a) limit cycle (b) from different initial conditions

FIGURE 7.1: Demonstrations of limit cycles

8. Consider the ordinary differential equation

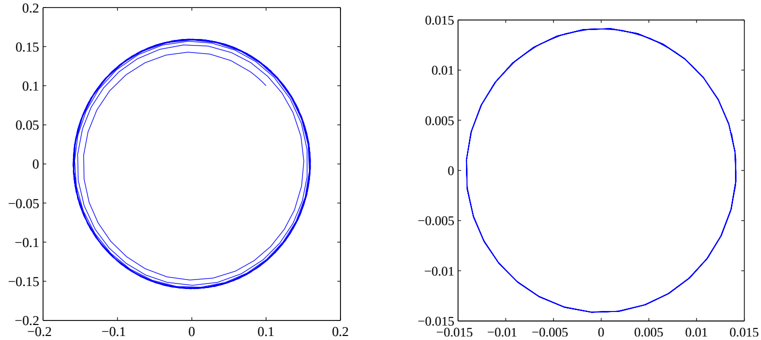
$$\begin{cases} \dot{x} = -y + xf\left(\sqrt{x^2 + y^2}\right) \\ \dot{y} = x + yf\left(\sqrt{x^2 + y^2}\right) \end{cases}$$

with function $f(r) = r^2 \sin(1/r)$. It is pointed out in [5] that there are multiple limit cycles for $r = 1/(n\pi), n = 1, 2, 3, \cdots$. Solve the different equations and observe the limit cycles for different initial points.

SOLUTION Again one should introduce the state variables $x_1 = x, x_2 = y$, the differential equation can be modeled by an anonymous function and the problem can be solved numerically. Under two different initial points, two different limit cycles are obtained as shown in Figures 7.2 (a) and (b) respectively.

```
>> f=@(t,x)[-x(2)+x(1)*(x(1)^2+x(2)^2)*sin(1/sqrt(x(1)^2+x(2)^2));
```

```
          x(1)+x(2)*(x(1)^2+x(2)^2)*sin(1/sqrt(x(1)^2+x(2)^2))];
x0=[0.1,0.1]; [t,x]=ode45(f,[0,100],x0); plot(x(:,1),x(:,2));
figure; [t,x]=ode45(f,[0,100],0.1*x0); plot(x(:,1),x(:,2));
```



(a) limit cycle (b) another limit cycles

FIGURE 7.2: Demonstrations of multiple limit cycles

9. Consider the well-known Rössler equation described as $\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + (x - c)z, \end{cases}$

where $a = b = 0.2$, $c = 5.7$, and $x_1(0) = x_2(0) = x_3(0)$. Draw the 3D phase trajectory and also its projection on the x-y plane. The parameters a, b, c are suggested to be used as additional parameters. If the parameters are changed to $a = 0.2, b = 0.5, c = 10$, solve the problem again.

SOLUTION Let $x_1 = x, x_2 = y, x_3 = z$, the numerical method can be used to solve the equation. The 3D phase-space trajectories are obtained as shown Figure 7.3 (a).

```
>> f=@(t,x,a,b,c)[-x(2)-x(3); x(1)+a*x(2); b+(x(1)-c)*x(3)];
   [t,x]=ode45(f,[0,100],[0;0;0],[],0.2,0.2,5.7);
   plot3(x(:,1),x(:,2),x(:,3)); grid
```

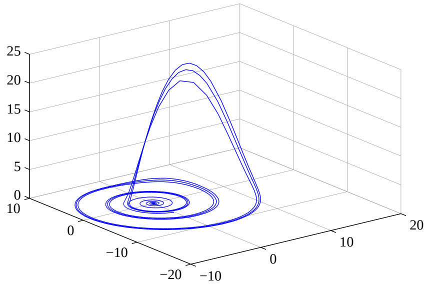
Change the parameters of a, b, c, a new phase-space trajectory can be drawn as shown in Figure 7.3 (b).

```
>> [t,x]=ode45(f,[0,100],[0;0;0],[],0.2,0.5,10);
   plot3(x(:,1),x(:,2),x(:,3)); grid
```

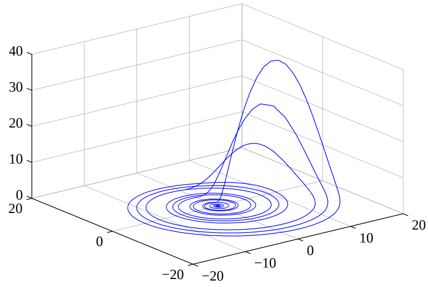
10. For the well-known Chua's circuit equation in chaotic studies that

$$\begin{cases} \dot{x} = \alpha[y - x - f(x)] \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y - \gamma z \end{cases}$$

where the nonlinear function $f(x)$ is described by



(a) $a = b = 0.2, c = 5.7$



(b) $a = 0.2, b = 0.5, c = 10$

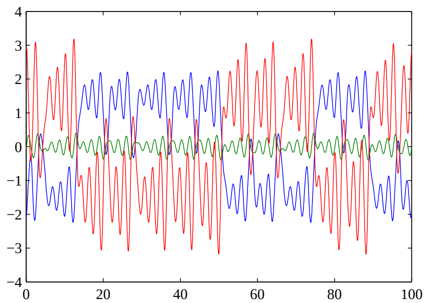
FIGURE 7.3: Phase-space trajectories of Rössler equations

$$f(x) = bx + \frac{1}{2}(a-b)(|x+1| - |x-1|), \text{ and } a < b < 0.$$

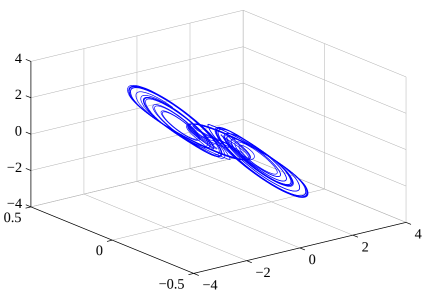
Write an M-function to describe the equation, and draw the phase trajectory for the parameters $\alpha = 9, \beta = 100/7, \gamma = 0, a = -8/7, b = -5/7$, and initial conditions $x(0) = -2.121304, y(0) = -0.066170, z(0) = 2.881090$.

Solution Select a set of state variables, $x_1 = x$, $x_2 = y$, $x_3 = z$, the Chua's circuit can be modeled by anonymous function, and the differential equations can be solved numerically. The time responses of the states, and the phase space trajectory are obtained as shown in Figures 7.4 (a) and (b). The chaotic behavior of the circuit can be observed from the results.

```
>> f=@(t,x,a,b,alpha,beta,gamma)...
       [alpha*(x(2)-x(1)-(b*x(1)+(a-b)*(abs(x(1)+1)-abs(x(1)-1))/2));
        x(1)-x(2)+x(3); -beta*x(2)-gamma*x(3)];
   a=-8/7; b=-5/7; alpha=9; beta=100/7; gamma=0;
   ff=odeset; ff.RelTol=1e-8;
   [t,x]=ode15s(f,[0,100],[-2.121304;-0.06617; 2.88109],ff,...
         a,b,alpha,beta,gamma);
   plot(t,x), figure, plot3(x(:,1),x(:,2),x(:,3))
```



(a) time responses



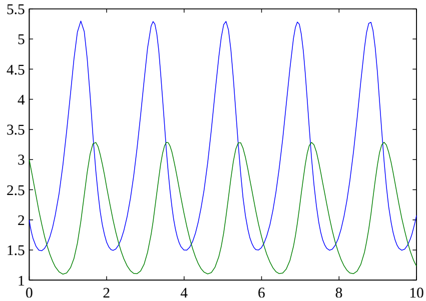
(b) phase space trajectory

FIGURE 7.4: Chaotic behavior of Chua's circuit

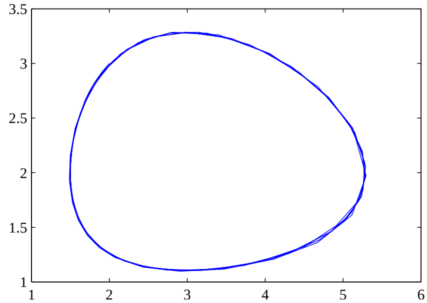
11. For the Lotka-Volterra's predator-prey equations $\begin{cases} \dot{x}(t) = 4x(t) - 2x(t)y(t) \\ \dot{y}(t) = x(t)y(t) - 3y(t) \end{cases}$ with initial conditions $x(0) = 2, y(0) = 3$, solve the time responses of $x(t)$ and $y(t)$, and also the phase plane trajectory.

SOLUTION One can describe the equation first, then the time response and the phase-plane trajectory of the solution can both be drawn as in Figures 7.5 (a) and (b), respectively.

```
>> f=@(t,x)[4*x(1)-2*x(1)*x(2); x(1)*x(2)-3*x(2)];
   [t,x]=ode45(f,[0,10],[2;3]); plot(t,x)
   figure; plot(x(:,1), x(:,2))
```



(a) time response



(b) phase-plane trajectory

FIGURE 7.5: Solutions of the Lotka-Volterra Predator-prey model

12. Select state variables to convert the following equations into first-order explicit ones, and solve the solutions for phase trajectories.

(i) $\begin{cases} \ddot{x} = -x - y - (3\dot{x})^2 + \dot{y}^3 + 6\ddot{y} + 2t \\ y^{(3)} = -\ddot{y} - \dot{x} - \mathrm{e}^{-x} - t \end{cases}$, with $\begin{cases} x(1) = 2, \dot{x}(1) = -4 \\ y(1) = -2, \dot{y}(1) = 7 \\ \ddot{y}(1) = 6 \end{cases}$

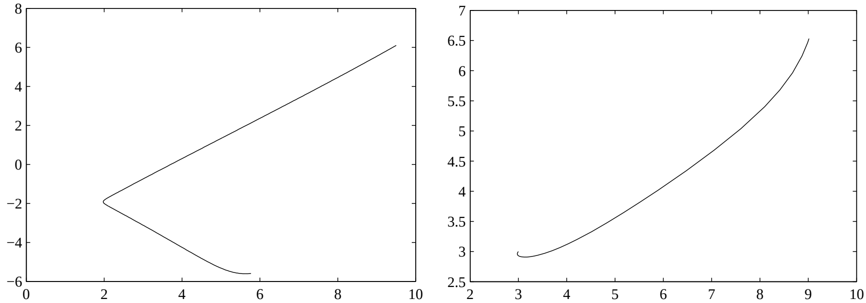
(ii) $\begin{cases} \ddot{x} - 2xz\dot{x} = 3x^2yt^2 \\ \ddot{y} - \mathrm{e}^y\dot{y} = 4xt^2z \\ \ddot{z} - 2t\dot{z} = 2t\mathrm{e}^{-xy} \end{cases}$, with $\begin{cases} \dot{z}(1) = \dot{x}(1) = \dot{y}(1) = 2 \\ \dot{z}(1) = x(1) = y(1) = 3 \end{cases}$

SOLUTION (i) Select $x_1 = x, x_2 = \dot{x}, x_3 = y, x_4 = \dot{y}, x_5 = \ddot{y}$, the explicit first-order state space equation can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_3 - (3x_2)^2 + x_4^3 + 6x_5 + 2t \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = x_5 \\ \dot{x}_5 = -x_5 - x_2 - \mathrm{e}^{-x_1} - t \end{cases}$$

with $\boldsymbol{x}(1) = [2, -4, -2, 7, 6]^T$. The other difficulties in the equation is that, since $\boldsymbol{x}(1)$ is given rather than $\boldsymbol{x}(0)$, $\boldsymbol{x}(0)$ should be find first and then the solutions over the () One can solve it with the following statements and the phase-plane trajectory is shown in Figure 7.6 (a).

```
>> f=@(t,x)[x(2); -x(1)-x(3)-(3*x(2))^2+(x(4))^3+6*x(5)+2*t; ...
          x(4); x(5); -x(5)-x(2)-exp(-x(1))-t];
   [t1,x1]=ode45(f,[1,0],[2, -4, -2, 7, 6]');
   [t2,x2]=ode45(f,[1,2],[2, -4, -2, 7, 6]');
   t=[t1(end:-1:1); t2]; x=[x1(end:-1:1,:); x2]; plot(x(:,1),x(:,3))
```



(a) equation (i) (b) equation (ii)

FIGURE 7.6: Phase-plane trajectories of the equations

(ii) Select $x_1 = x, x_2 = \dot{x}, x_3 = y, x_4 = \dot{y}, x_5 = z, x_6 = \dot{z}$, the new state space equation can be established.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 2x_1x_2x_5 + 3x_1^2x_3t^2 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \mathrm{e}^{-x_3}x_4 + 4x_1t^2x_5 \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = 2tx_6 + 2t\mathrm{e}^{-x_1x_3} \end{cases}$$

with $\boldsymbol{x}(1) = [3, 2, 3, 2, 2, 3]^T$. The problem can be solve numerically and the phase-space trajectory can be obtained as shown in Figure 7.6.

```
>> f=@(t,x)[x(2); 2*x(1)*x(2)*x(5)+3*x(1)^2*x(3)*t^2; x(4); ...
      exp(-x(3))*x(4)+4*x(1)*t^2*x(5); x(6); ...
      2*t*x(6)+2*t*exp(-x(1)*x(3))];
   [t,x]=ode15s(f,[1,0],[3,2,3,2,2,3]'); plot(x(:,1),x(:,3))
```

13. Solve the following ordinary differential equations

$$y^{(3)} + ty\ddot{y} + t^2\dot{y}y^2 = \mathrm{e}^{-ty},\ y(0) = 2,\ \dot{y}(0) = \ddot{y}(0) = 0$$

and draw the $y(t)$ curve. Select the fixed-step Runge-Kutta algorithm for solving the same problem. Compare in speed and accuracy with the MATLAB functions for this problem.

Solution Since there are nonlinear terms in the differential equation, there may not be analytical solutions. Thus one has to rely on numerical solutions.

Normally one should select a set of state variables as $x_1 = y, x_2 = \dot{y}, x_3 = \ddot{y}$. The first-order explicit differential equations can be established such that

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 & x_1(0) = 2,\ x_2(0) = x_3(0)0 \\ \dot{x}_3 = -tx_1x_3 - t^2x_2x_1^2 + \mathrm{e}^{-tx_1} \end{cases}$$

and MATLAB can be used for finding the numerical solutions. The time responses of the state variables are shown in Figure 7.7.

```
>> f=@(t,x)[x(2); x(3); -t^2*x(1)*x(3)-t^2*x(2)*x(1)^2+exp(-t*x(1))];
   [t,x]=ode45(f,[0,10],[2;0;0]); plot(t,x)
```

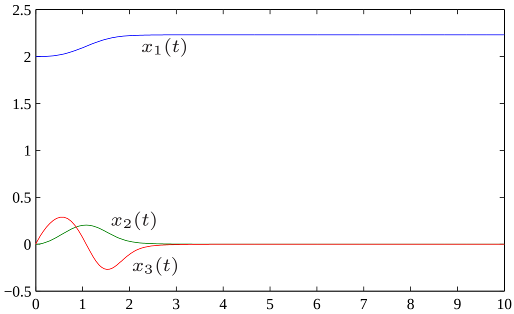


FIGURE 7.7: Time responses of the state variables

14. Find the analytical and numerical solutions to the following differential equations and verify the results.

$$\begin{cases} \ddot{x}(t) = -2x(t) - 3\dot{x}(t) + \mathrm{e}^{-5t}, & x(0) = 1, \dot{x}(0) = 2 \\ \ddot{y}(t) = 2x(t) - 3y(t) - 4\dot{x}(t) - 4\dot{y}(t) - \sin t, & y(0) = 3, \dot{y}(0) = 4 \end{cases}$$

SOLUTION The analytical solution of the equation can be obtained with

```
>> syms t; [x,y]=dsolve('D2x=-2*x-3*Dx+exp(-5*t)',...
      'D2y=2*x-3*y-4*Dx-4*Dy-sin(t)',...
      'x(0)=1','Dx(0)=2','y(0)=3','Dy(0)=4')
```

where

$$x(t) = \frac{1}{12}\mathrm{e}^{-5t} - \frac{10}{3}\mathrm{e}^{-2\,t} + \frac{17}{4}\mathrm{e}^{-t}$$

$$y(t) = -\frac{71}{5}\mathrm{e}^{-3t} - \frac{265}{16}\mathrm{e}^{-t} + \frac{11}{48}\mathrm{e}^{-5t} + \frac{100}{3}e^{-2t} + \frac{1}{5}\cos t - \frac{1}{10}\sin t + \frac{51}{4}t\mathrm{e}^{-t}.$$

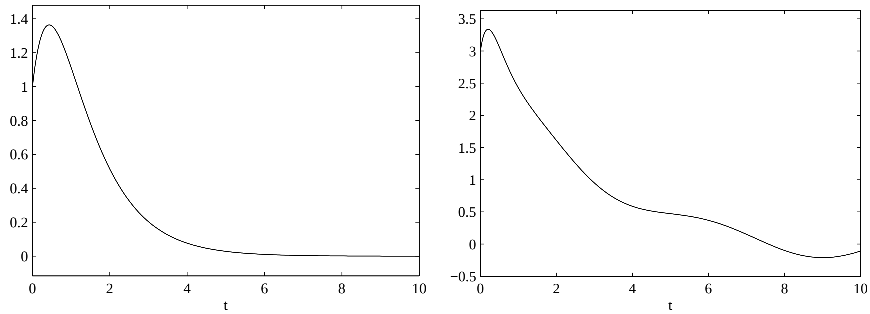
Also assume that $x_1 = x(t), x_2 = x'(t), x_3 = y(t), x_4 = y'(t)$, the state space equation can be written as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - 3x_2 + \mathrm{e}^{-5t} \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = 2x_1 - 3x_3 - 4x_2 - 4x_4 - \sin t, \quad \dot{x}^T(0) = [1, 2, 3, 4] \end{cases}$$

and the numerical solutions can be obtained with

```
>> f=@(t,x)[x(2); -2*x(1)-3*x(2)+exp(-5*t); x(4);...
            2*x(1)-3*x(3)-4*x(2)-4*x(4)-sin(t)];
   [t1,x1]=ode45(f,[0,10],[1;2;3;4]);
   ezplot(x,[0,10]), line(t1,x1(:,1))
   figure; ezplot(y,[0,10]), line(t1,x1(:,3))
```

The numerical and analytical solutions of $x(t)$ and $y(t)$ are obtained as shown in Figures 7.8 (a) and (b). It can be seen that they are almost undistinguishable from the plots.



(a) $x(t)$ (b) $y(t)$

FIGURE 7.8: Comparisons of numerical and analytical solutions

15. For the differential equations $\begin{cases} \ddot{u}(t)=-u(t)/r^3(t) \\ \ddot{v}(t)=-v(t)/r^3(t) \end{cases}$, where $r(t)=\sqrt{u^2(t)+v^2(t)}$, and $u(0) = 1, \dot{u}(0) = 2, \dot{v}(0) = 2, v(0) = 1$. Select a set of state variables and convert the equations to the form solvable by MATLAB. Draw and verify the curves of $u(t), v(t)$, and the phase plane trajectory.

SOLUTION This equation is quite similar to the Apollo model in the book. Select first a set of state variables such that $x_1 = u, x_2 = \dot{u}, x_3 = v, x_4 = \dot{v}$, the original equation can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t)/r^3(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = -x_3(t)/r^3(t) \end{cases}$$

where $r(t) = \sqrt{x_1^2(t) + x_3^2(t)}$, and $\boldsymbol{x}(0) = [1, 2, 1, 2]$. The differential equation can be expressed by an M-function

```
function dx=app_eq1(t,x)
r=sqrt(x(1)^2+x(3)^2); dx=[x(2); -x(1)/r^3; x(4); -x(3)/r^3];
```

The equation can be solved numerically with

```
>> x0=[1;2;1;2]; [t,x]=ode45(@app_eq1,[0,100],x0);
```

16. Consider the differential equation[6]

$$\begin{cases} \dot{u}_1 = u_3 \\ \dot{u}_2 = u_4 \\ 2\dot{u}_3 + \cos(u_1 - u_2)\dot{u}_4 = -\mathrm{g}\sin u_1 - \sin(u_1 - u_2)u_4^2 \\ \cos(u_1 - u_2)\dot{u}_3 + \dot{u}_4 = -\mathrm{g}\sin u_2 + \sin(u_1 - u_2)u_3^2 \end{cases}$$

where $u_1(0) = 45, u_2(0) = 30, u_3(0) = u_4(0) = 0$, and g$= 9.81$. Solve the equations and draw the time responses to the states.

SOLUTION From the given equation, it is easily recognized that it is a DAE, where the original equation can be expressed in a matrix manner such that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & \cos(u_1 - u_2) \\ 0 & 0 & \cos(u_1 - u_2) & 1 \end{bmatrix} \dot{\boldsymbol{u}} = \begin{bmatrix} u_3 \\ u_4 \\ -\mathrm{g}\sin u_1 - \sin(u_1 - u_2)u_4^2 \\ -\mathrm{g}\sin u_2 + \sin(u_1 - u_2)u_3^2 \end{bmatrix}$$

The following statements can be used to solve the equation, with the time responses of the state shown in Figure 7.9.

```
>> fM=@(t,u)[1,0,0,0; 0,1,0,0; 0,0,2,cos(u(1)-u(2));...
          0,0,cos(u(1)-u(2)),1]; g=9.81; x0=[45;30;0;0];
   f=@(t,u)[u(3); u(4); -g*sin(u(1))-sin(u(1)-u(2))*u(4)^2;...
       -g*sin(u(2))+sin(u(1)-u(2))*u(3)^2];
   ff=odeset; ff.Mass=fM; [t,x]=ode45(f,[0,1.2],x0,ff);
   plot(t,x)
```

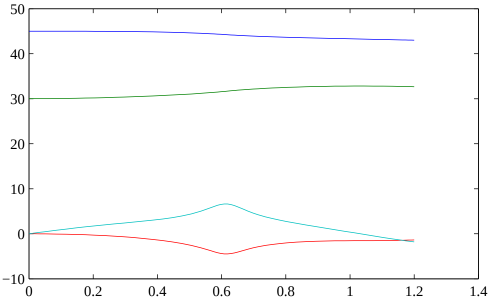


FIGURE 7.9: Differential algebraic equation solution

17. For the implicit differential equation

$$\begin{cases} \dot{x}_1\ddot{x}_2\sin(x_1x_2) + 5\ddot{x}_1\dot{x}_2\cos(x_1^2) + t^2x_1x_2^2 = \mathrm{e}^{-x_2^2} \\ \ddot{x}_1x_2 + \ddot{x}_2\dot{x}_1\sin(x_1^2) + \cos(\ddot{x}_2x_2) = \sin t \end{cases}$$

where $x_1(0) = 1, \dot{x}_1(0) = 1, x_2(0) = 2, \dot{x}_2(0) = 2$. Find the numerical solutions and draw the solution trajectory.

SOLUTION Defining $y_1 = x_1$, $y_2 = \dot{x}_1$, $y_3 = x_2$, $y_4 = \dot{x}_2$, the implicit differential equation can be rewritten as

$$\begin{cases} y_3\dot{y}_4\sin(y_1y_3)+5\dot{y}_2y_4\cos(y_1^2)+t^2y_1y_3^2-\mathrm{e}^{-y_3^2}=0 \\ \dot{y}_2y_3+\dot{y}_4y_2\sin(y_1^2)+\cos(\dot{y}_4y_3)-\sin t=0 \end{cases}$$

where $y_1(0)=1, y_2(0)=1, y_3(0)=2, y_4(0)=2$. Thus the formal method should be

```
>> f=@(t,y,yd)[y(3)*yd(4)*sin(y(1)*y(3))+5*yd(2)*y(4)*cos(y(1)^2)+...
            t^2*y(1)*y(3)^2-exp(-y(3)^2);
       yd(2)*y(3)+yd(4)*y(2)*sin(y(1)^2)+cos(yd(4)*y(3))-sin(t)];
   x0=[1;1;2;2]; ix=[1;1;1;1]; xF0=[1;1;2;2]; idx=[0;0;0;0];
   [x0,xd0]=decic(f,0,x0,ix,xF0,idx)
   res=ode15i(f,[0,20],x0,xd0); plot(res.x,res.y)
```

However it is not likely to find consistent initial values using `decic()` function, thus one is not able to solve the equation. It is suggested to use the combination of ODE solver, with the algebraic equation solver discussed in the book.
From the original equations, assume that $p_1=\dot{y}_2$, $p_2=\dot{y}_4$, then the following equation can be established

$$\begin{cases} y_3p_2\sin(y_1y_3)+5p_1y_4\cos(y_1^2)+t^2y_1y_3^2-\mathrm{e}^{-y_3^2}=0 \\ p_1y_3+p_2y_2\sin(y_1^2)+\cos(p_2y_3)-\sin t=0 \end{cases}$$

which can be regarded as the algebraic equation of p_1 and p_2. One can then write out the M-function to describe the explicit equation as

```
function dy=c7impode(t,y)
dx=@(p,y)[y(3)*p(2)*sin(y(1)*y(3))+5*p(1)*y(4)*cos(y(1)^2)+...
      t^2*y(1)*y(3)^2-exp(-y(3)^2);
      p(1)*y(3)+p(2)*y(2)*sin(y(1)^2)+cos(p(2)*y(3))-sin(t)];
ff=optimset; ff.Display='off'; dx1=fsolve(dx,y([1,3]),ff,y);
dy=[y(2); dx1(1); y(4); dx1(2)];
```

In this case, the implicit differential equation can be solved with the following statements, and the results are as shown in Figure 7.10.

```
>> [t,x]=ode45(@exc7ide,[0,1],x0); plot(t,x)
```

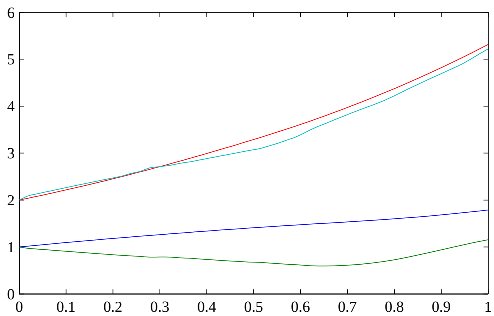


FIGURE 7.10: Solutions of implicit differential equations

18. The following equations are regarded as stiff equations in classical differential equation textbooks. Solve the problems using ordinary solver and stiff equation solver. Also solve the equations using analytical methods and verify the accuracy of the numerical results.

(i) $\begin{cases} \dot{y}_1 = 9y_1 + 24y_2 + 5\cos t - \dfrac{1}{3}\sin t, \ y_1(0) = \dfrac{1}{3} \\ \dot{y}_2 = -24y_1 - 51y_2 - 9\cos t + \dfrac{1}{3}\sin t, \ y_2(0) = \dfrac{2}{3} \end{cases}$

(ii) $\begin{cases} \dot{y}_1 = -0.1y_1 - 49.9y_2, \ y_1(0) = 1 \\ \dot{y}_2 = -50y_2, \ y_2(0) = 2 \\ \dot{y}_3 = 70y_2 - 120y_3, \ y_3(0) = 1 \end{cases}$

SOLUTION (i) The analytical solutions to the differential equation can be obtained

```
>> syms t;
   [y1,y2]=dsolve('Dy1=9*y1+24*y2+5*cos(t)-sin(t)/3',...
      'Dy2=-24*y1-51*y2-9*cos(t)+sin(t)/3','y1(0)=1/3','y2(0)=2/3')
```

with $y_1 = -\dfrac{2}{3}\mathrm{e}^{-39t} + \dfrac{2}{3}\mathrm{e}^{-3t} + \dfrac{1}{3}\cos t$, and $y_2 = \dfrac{4}{3}\mathrm{e}^{-39t} - \dfrac{1}{3}\mathrm{e}^{-3t} - \dfrac{1}{3}\cos t$.

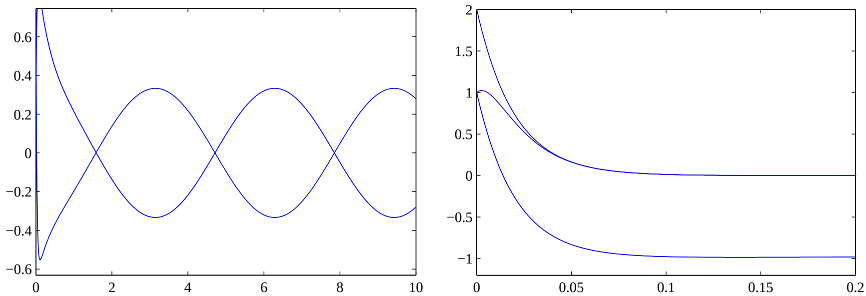
An anonymous function can be written to describe the differential equation

```
>> f=@(t,y)[9*y(1)+24*y(2)+5*cos(t)-sin(t)/3;
           -24*y(1)-51*y(2)-9*cos(t)+sin(t)/3];
```

Thus the equation can be solved numerically with an ordinary equation solver.

```
>> y0=[1/3; 2/3]; tf=10; [t,y]=ode45(f,[0,tf],y0); plot(t,y)
   hold on; ezplot(y1,[0,10]); ezplot(y2,[0,10])
```

The numerical solutions to the equation are shown in Figure 7.11 (a), superimposed by the analytical solutions. It can be seen that the numerical solutions are very accurate and there is no use to apply the stiff equation algorithms.



(a) solutions of (i) (b) solutions of (ii)

FIGURE 7.11: Comparisons of numerical and analytical solutions

(ii) Again, since the equation is linear, the analytical solutions of the equation can be obtained first with

```
>> [y1,y2,y3]=dsolve('Dy1=-0.1*y1-49.9*y2','Dy2=-50*y2',...
        'Dy3=70*y2-120*y3','y1(0)=1','y2(0)=2','y3(0)=1')
```

where $y_1 = 2\mathrm{e}^{-50t} - \mathrm{e}^{-t/10}, y_2 = 2\mathrm{e}^{-50t}, y_3 = 2\mathrm{e}^{-50t} - \mathrm{e}^{-120t}$.
The numerical solutions can also be obtained with

```
>> f=@(t,y)[-0.1*y(1)-49.9*y(2); -50*y(2); 70*y(2)-120*y(3)];
   y0=[1; 2; 1]; tf=0.2; [t,y]=ode45(f,[0,tf],y0); plot(t,y)
   hold on, ezplot(y1,[0,tf]); ezplot(y2,[0,tf]); ezplot(y3,[0,tf])
```

and the results are shown in Figure 7.11 (b), together with the analytical solutions. It can be seen again that ordinary solver is accurate enough for the solutions.

19. Consider the chemical reaction equation

$$\begin{cases} \dot{y}_1 = -0.04y_1 + 10^4 y_2 y_3 \\ \dot{y}_2 = 0.04y_1 - 10^4 y_2 y_3 - 3 \times 10^7 y_2^2 \\ \dot{y}_3 = 3 \times 10^7 y_2^2 \end{cases}$$

where the initial values are $y_1(0) = 1, y_2(0) = y_3(0) = 0$. This equation can be regarded as stiff equation. Solve the problem with `ode45()` and check whether it is correct or not. If not so, how can such a problem be solved.

SOLUTION Two solvers, `ode45()`, and `ode15s()`, are going to be used and compared in this example. The response curves are shown in Figure 7.12, and it can be seen that they are quite close.

```
>> f=@(t,y)[-0.04*y(1)+10^4*y(2)*y(3); ...
       0.04*y(1)-10^4*y(2)*y(3)-3e7*y(2)^2; 3e7*y(2)^2];
   [t1,y1]=ode45(f,[0,10],[1;0;0]); length(t1)
   [t2,y2]=ode15s(f,[0,10],[1;0;0]); length(t2),
   plot(t1,y1,t2,y2)
```

Further comparisons on the points calculated reveal that the ordinary solver takes 29109 steps, while the stiff equation solver only takes 41 steps. Thus for this example, the stiff equation solver is more effective in the solutions.

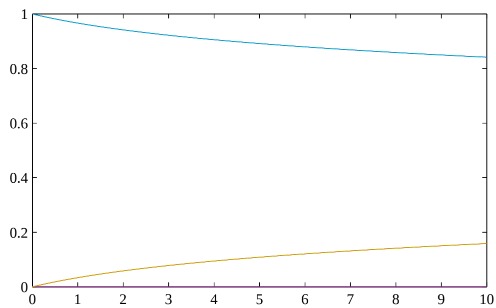


FIGURE 7.12: Solutions of differential equation

20. Solve the boundary value equation in Problem 4 using numerical methods and draw the solution $y(t)$. Compared the accuracy of the results with the analytical results obtained earlier.

SOLUTION The analytical solutions of the differential equation can be obtained

```
>> syms x
   y=dsolve('D2y-(2-1/x)*Dy+(1-1/x)*y=x^2*exp(-5*x)',...
               'y(1)=pi','y(pi)=1','x')
```

Numerical solutions of the problem can also be obtained with the following statements, and the curves are shown in Figure 7.13.

```
>> f=@(t,x)[x(2); t^2*exp(-5*t)+(2-1/t)*x(2)-(1-1/t)*x(1)];
   fb=@(xa,xb)[xa(1)-pi; xb(1)-1]; s=bvpinit(linspace(1,pi,5),[1;1]);
   v=bvpset; v.RelTol=1e-8; sol=bvp5c(f,fb,s,v); plot(sol.x,sol.y)
```

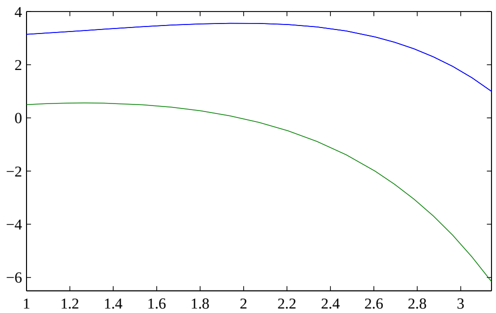


FIGURE 7.13: Solution of the boundary value problem

21. Solve the boundary value problem where $\ddot{x}+\dfrac{1}{t}\dot{x}+\left(1-\dfrac{1}{4t^2}\right)x=\sqrt{t}\cos t$, with $x(1)=1, x(6)=-0.5$.

SOLUTION For the linear time varying equation, the analytical solution exists, however the solution is too complicated to be displayed here.

```
>> x=dsolve('D2x+Dx/t+(1-1/(4*t^2))*x=sqrt(t)*cos(t)',...
            'x(1)=1','x(6)=-0.5')
```

The numerical solution can also be obtained with the `bvp5c()` function, with the results shown graphically in Figure 7.14.

```
>> f=@(t,x)[x(2); sqrt(t)*cos(t)-x(2)/t-(1-1/4/t^2)]*x(1);
   fb=@(xa,xb)[xa(1)-1; xb(1)+0.5]; s=bvpinit(linspace(1,6,5),[1;0]);
   v=bvpset; v.RelTol=1e-8; sol=bvp5c(f,fb,s,v); plot(sol.x,sol.y)
```

22. For the Van der Pol equation $\ddot{y}+\mu(y^2-1)\dot{y}+y=0$, if $\mu=1$, find the numerical solutions for boundary conditions $y(0)=1, y(5)=3$. If μ is a undetermined

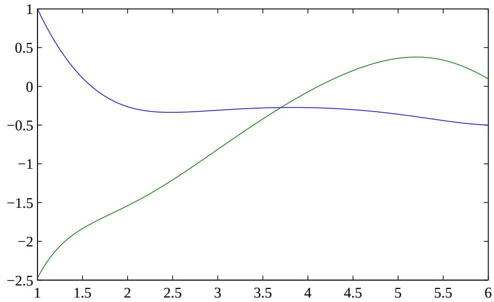


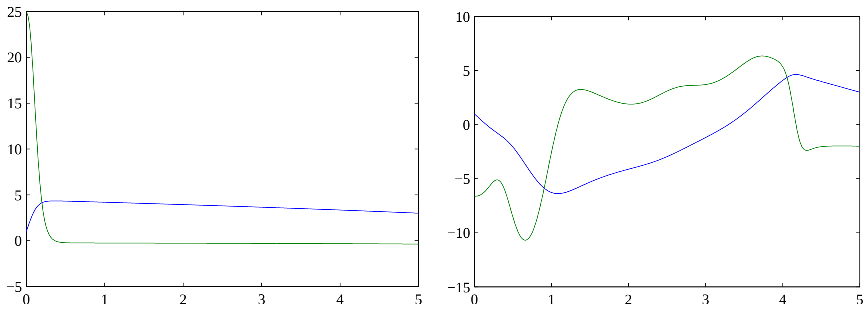
FIGURE 7.14: Solution of the boundary value problem

parameter, an extra condition $\dot{y}(5) = -2$ can be used. Solve the parameter μ as well as the equation. Draw the solution and verify the results.

SOLUTION If μ is known, the problem can be solved with

```
>> mu=1; f=@(t,x)[x(2); -mu*(x(1)^2-1)*x(2)-x(1)];
   fb=@(xa,xb)[xa(1)-1; xb(1)-3]; s=bvpinit(linspace(0,5,5),[0,1]);
   v=bvpset; v.RelTol=1e-8; sol=bvp5c(f,fb,s,v); plot(sol.x,sol.y)
```

and the time response can be obtained as shown in Figure 7.15 (a). It should be noted that some improperly chosen initial conditions may cause singularity problems in the Jacobian, while calculating the initial values.



(a) with known μ (b) without known μ

FIGURE 7.15: Study of boundary conditions in Van der Pol equations

If μ is not specified, it should be used as an additional parameter. The following statements can be used to solve the equations and also the value of μ.

```
>> f=@(t,x,mu)[x(2); -mu*(x(1)^2-1)*x(2)-x(1)];
   fb=@(xa,xb,mu)[xa(1)-1; xb(1)-3; xb(2)+2];
   s=bvpinit(linspace(0,5,5),[0,1],0); v=bvpset; v.RelTol=1e-8;
   sol=bvp5c(f,fb,s,v); plot(sol.x,sol.y), mu=sol.parameters
```

It is found that $\mu = 0.1694$, and the time responses of the equation can be obtained as shown in Figure 7.15 (b).

23. Solve numerically the partial differential equations below and draw the surface plot of the solution u.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{x=0,y>0} = 1, \quad u\big|_{y=0,\ x\geqslant 0} = 0 \\ x > 0, \quad y > 0 \end{cases}$$

SOLUTION It can be seen that the equation is an elliptic equation with $c = 1$ and $f = 0$. This equation is suitable for `pdetool` graphical user interface. The final results are shown in Figure 7.16.

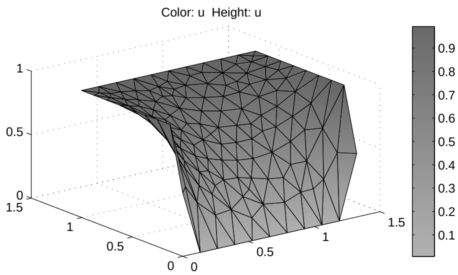


FIGURE 7.16: Solution surface of the partial differential equation

24. Consider the simple linear differential equation

$$y^{(4)} + 4y^{(3)} + 6\ddot{y} + 4\dot{y} + y = \mathrm{e}^{-3t} + \mathrm{e}^{-5t}\sin(4t + \pi/3)$$

with the conditions $y(0) = 1, \dot{y}(0) = \ddot{y}(0) = 1/2, y^{(3)}(0) = 0.2$. Construct the simulation model with Simulink, and find the simulation results.

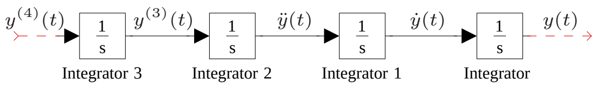
SOLUTION To represent a differential equation with Simulink, one has first to define the signals $y(t), \dot{y}(t), \ddot{y}(t), y^{(3)}(t)$ and $y^{(4)}(t)$ using integrators. As shown in Figure 7.17 (a).

Based on the previous considerations, the full simulation model can be established with Simulink as shown in Figure 7.17 (b). The numerical solution and simulation results are shown together in Figure 7.18, and they agree very well.

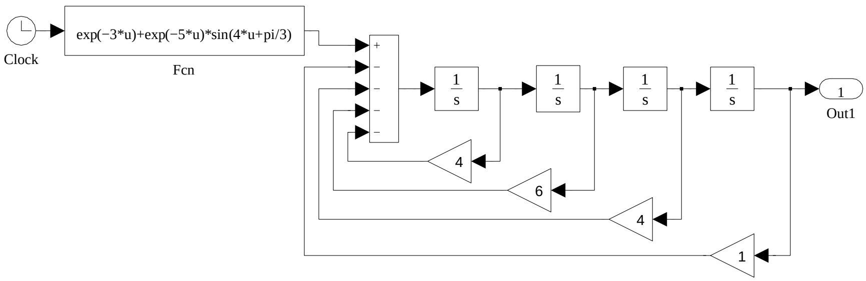
```
>> [t,x,y]=sim('exc7s2'); plot(t,y);
   eq='D4y+4*D3y+6*D2y+4*Dy+y=exp(-3*t)+exp(-5*t)*sin(4*t+pi/3)';
   y=dsolve(eq,'y(0)=0','Dy(0)=0','D2y(0)=1/2','D3y(0)=0.2')
   hold on; ezplot(y,[0,10])
```

the analytical solution can be written as

$$y = -\frac{1}{2048}\mathrm{e}^{-5t}\sin 4t - \frac{\sqrt{3}}{2048}\mathrm{e}^{-5t}\cos 4t + \frac{1}{16}\mathrm{e}^{-3t} + \left(-\frac{1}{16} + \frac{\sqrt{3}}{2048}\right)\mathrm{e}^{-t}$$



(a) defining key signals (file: exc7s1.mdl)



(b) modeling differential equation (file: exc7s2.mdl)

FIGURE 7.17: Simulink description of differential equations

$$+\left(\frac{65}{512}-\frac{\sqrt{3}}{512}\right)\mathrm{e}^{-t}t+\frac{15}{128}\mathrm{e}^{-t}t^2+\left(\frac{181}{480}+\frac{\sqrt{3}}{96}\right)\mathrm{e}^{-t}t^3$$

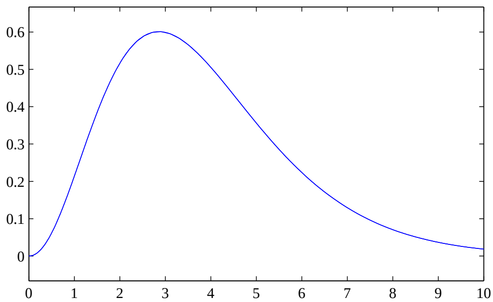


FIGURE 7.18: Solution of differential equation

25. Consider further a time-varying linear differential equations
$$y^{(4)}+4ty^{(3)}+6t^2\ddot{y}+4\dot{y}+y=\mathrm{e}^{-3t}+\mathrm{e}^{-5t}\sin(4t+\pi/3)$$
with the initial conditions $y(0)=1$, $\dot{y}(0)=\ddot{y}(0)=1/2, y^{(3)}(0)=0.2$. Construct a Simulink model to describe and solve the equation and draw the solutions.

SOLUTION It is not likely to get the analytical solutions to the revised time-varying differential equation. Simulink can still be used to model the equation. The Simulink model for the system is established as shown in Figure 7.19, and the solution to the equation is obtained via simulation method, and is shown in Figure 7.20.

26. Consider the delay differential equation
$$y^{(4)}(t)+4y^{(3)}(t-0.2)+6\ddot{y}(t-0.1)+6\ddot{y}(t)+4\dot{y}(t-0.2)+y(t-0.5)=\mathrm{e}^{-t^2}.$$

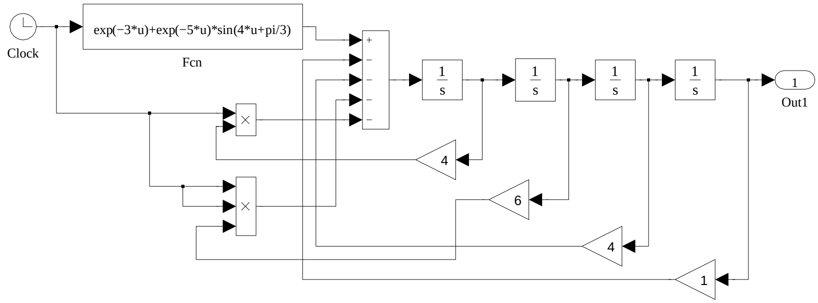


FIGURE 7.19: Time-varying equation (file: exc7s3.mdl)

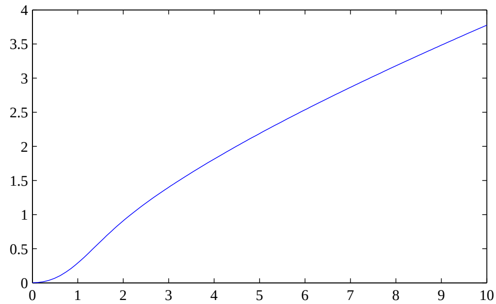


FIGURE 7.20: Time-varying system response

It is assumed that for $t \leqslant 0$, the equation has zero initial conditions. Construct a Simulink model and solve the solutions. Also the function `dde23()` can be used to solve the same problem. Compare the two methods and draw the solution $y(t)$.

Solution Delay systems can be modeled easily with Simulink. The delay blocks in Continuous group can be applied directly to generate delay signals. The Simulink diagram of the equation is shown in Figure 7.21.

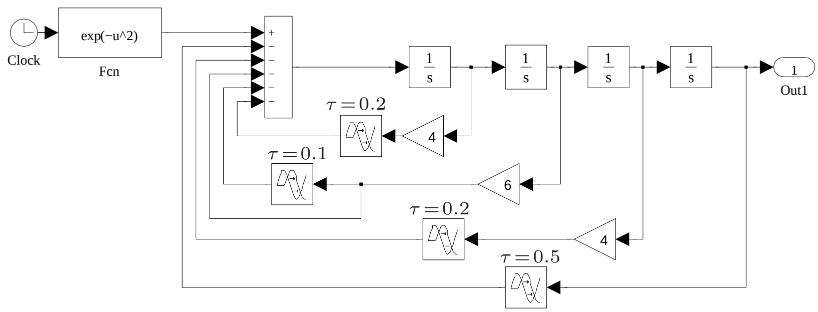


FIGURE 7.21: Delay equation (file: exc7s4.mdl)

Running the simulation model, the system response can be obtained as shown in Figure 7.22.

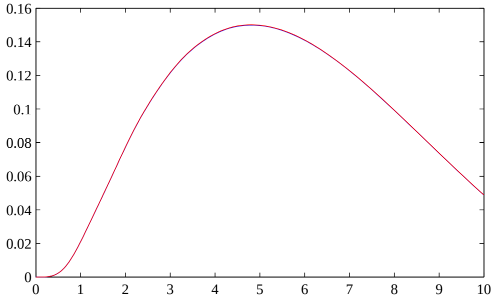


FIGURE 7.22: Delay system response

To solve the problem with MATLAB functions, one has to introduce state variables as $x_1 = y$, $x_2 = \dot{y}$, $x_3 = \ddot{y}$, and $x_4 = y^{(3)}$. Thus the delay differential equation can be rewritten as

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = x_4(t) \\ \dot{x}_4(t) = -x_1(t-0.5) - 4x_2(t-0.2) - 6x_3(t) - 6x_3(t-0.1) - 4x_4(t-0.2) + \mathrm{e}^{-t^2} \end{cases}$$

Assume that $\tau_1 = 0.5, \tau_2 = 0.2, \tau_3 = 0.1$. The M-function for such an equation can be written as

```
function dx=exc7md(t,x,z)
xd1=z(:,1); xd2=z(:,2); xd3=z(:,3);
dx=[x(2); x(3); x(4);
    -xd1(1)-4*xd2(2)-6*x(3)-6*xd3(3)-4*xd2(4)+exp(-t^2)];
```

The following MATLAB statements can be used to solve the problem by function calls, and the curve is also shown in Figure 7.22. It can be seen that the results by the two methods agree well.

```
>> del=[0.5,0.2,0.1]; tx=dde23('exc7md',del,zeros(4,1),[0,10]);
   hold on; plot(tx.x,tx.y(1,:))
```

It can also be seen that the block diagram method is far much simpler than the use of the `dde23()` function.

Chapter 8

Data Interpolation and Functional Approximation Problems

Exercises and Solutions

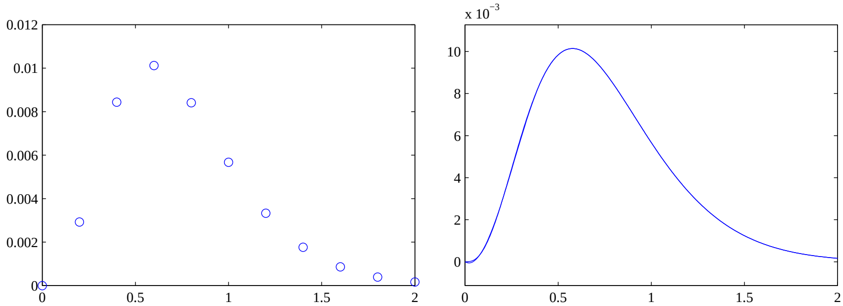
1. Generate a sparsely distributed data from the the following functions. Use one-dimensional interpolation method to fit the curves, with different fitting methods. Compare the interpolation results with the theoretical curves.

   (i) $y(t) = t^2 e^{-5t} \sin t$, where $t \in (0, 2)$,

   (ii) $y(t) = \sin(10t^2 + 3)$, for $t \in (0, 3)$.

SOLUTION (i) Sparsely distributed samples can be generated with the following statements and the samples are illustrated in Figure 8.1 (a).

```
>> t=0:0.2:2; y=t.^2.*exp(-5*t).*sin(t); plot(t,y,'o')
```



(a) sample distribution (b) fitting by spline interpolation

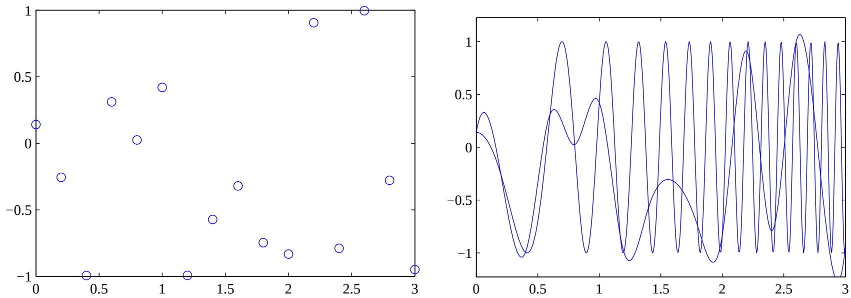
FIGURE 8.1: Spline fitting of the generated samples

From these samples, the interpolation are made using various of interpolation algorithms, and among these algorithms, the spline method gives the best fitting, as shown in Figure 8.1 (b), with other fittings omitted. It can be seen that the fitting quality is satisfactory.

```
>> ezplot('t.^2.*exp(-5*t).*sin(t)',[0,2]);
   x1=0:0.01:2; y1=interp1(t,y,x1,'spline'); line(x1,y1)
```

(ii) The sample data and interpolation results are shown respectively in Figures 8.2 (a) and (b), and it can be seen that the fitting quality is quite poor.

```
>> t=0:0.2:3; y=sin(10*t.^2+3); plot(t,y,'o')
   figure; ezplot('sin(10*t^2+3)',[0,3]); x1=0:0.001:3;
   y1=interp1(t,y,x1,'spline'); line(x1,y1)
```

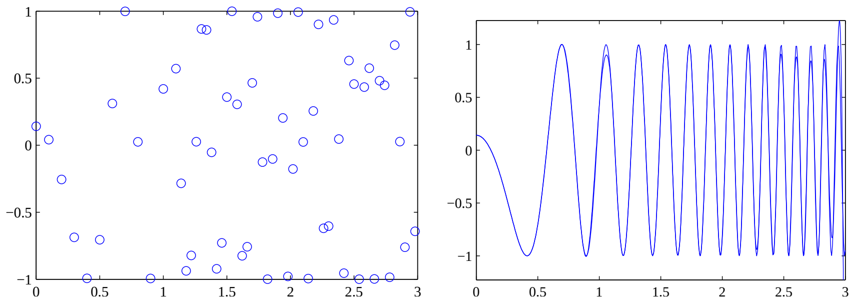


(a) samples generated (b) spline interpolation results

FIGURE 8.2: Curve fitting results

Due to the fast changing nature of the function, the fitting results cannot be made satisfactory, since the samples are not informative. In order to make sure good interpolation quality, samples must be increased. For this example, smaller step-size must be used in the fast changing region.

```
>> t=[0:0.1:1,1.1:0.04:3]; y=sin(10*t.^2+3); plot(t,y,'o')
   figure; ezplot('sin(10*t^2+3)',[0,3]);
   x1=0:0.001:3; y1=interp1(t,y,x1,'spline'); line(x1,y1)
```



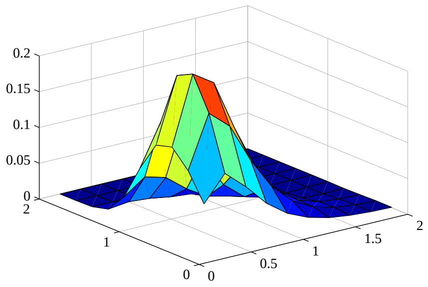
(a) improved sample distribution (b) improved fitting results

FIGURE 8.3: Fitting with more samples

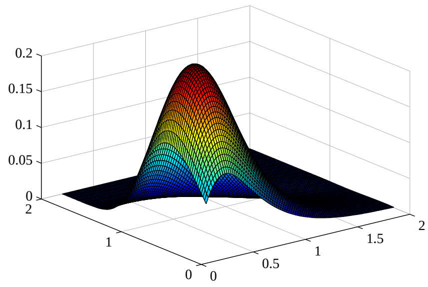
2. Generate a set of mesh grid data and randomly distributed data from the prototype function $f(x,y)=\dfrac{1}{3x^3+y}\mathrm{e}^{-x^2-y^4}\sin(xy^2+x^2y)$. Fit the original 3D surface with two-dimensional interpolation methods and compare the results with the theoretical ones.

Solution A set of mesh grid data can be generated and the surface plot can be obtained as shown in Figure 8.4 (a).

```
>> [x,y]=meshgrid(0.2:0.2:2);
   z=exp(-x.^2-y.^4).*sin(x.*y.^2+x.^2.*y)./(3*x.^3+y); surf(x,y,z)
```



(a) given sample surface



(b) interpolation results

FIGURE 8.4: Spline fitting of surface

Generate denser mesh grid and the surface can be calculated through two-dimensional interpolation and the new surface is as shown in Figure 8.4 (b). The error between the interpolated and theoretical surface are obtained as shown in Figure 8.5, and it can be seen that the interpolated surface is satisfactory.

```
>> [x1,y1]=meshgrid(0.2:0.02:2);
   z1=interp2(x,y,z,x1,y1,'spline'); surf(x1,y1,z1)
   z0=exp(-x1.^2-y1.^4).*sin(x1.*y1.^2+x1.^2.*y1)./(3*x1.^3+y1);
   figure; surf(x1,y1,abs(z1-z0))
```

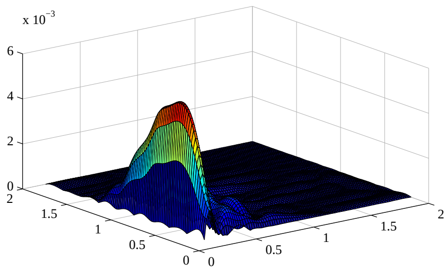
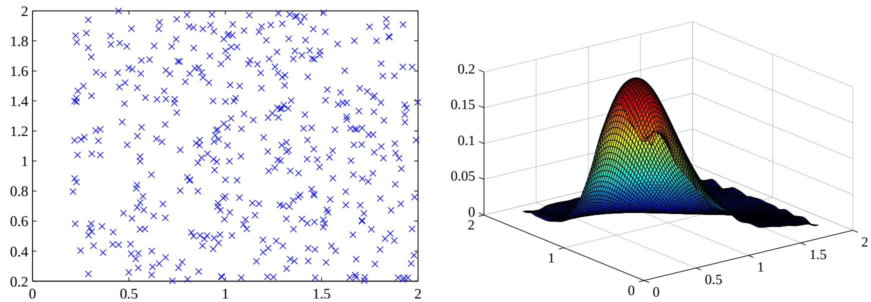


FIGURE 8.5: Absolute error surface of interpolation

If the samples are randomly distributed rather than in mesh grids, the following statements can be used to draw the distribution of the samples in Figure 8.6 (a) and the interpolation surface is shown in Figure 8.6 (b). It can be seen that the fitting is also satisfactory.

```
>> x=0.2+1.8*rand(400,1); y=0.2+1.8*rand(400,1);
   z=exp(-x.^2-y.^4).*sin(x.*y.^2+x.^2.*y)./(3*x.^3+y); plot(x,y,'x')
   [x1,y1]=meshgrid(0.3:0.02:1.9); z1=griddata(x,y,z,x1,y1,'v4');
   figure; surf(x1,y1,z1)
```



(a) sample distribution (b) interpolation surface

FIGURE 8.6: Interpolation from scatter distributed samples

3. Assume that a set of data is given as shown below. Fit the data into a smooth curve in the interval $x \in (-2, 4.9)$. Compare the advantages and disadvantages of the algorithms.

x_i	-2	-1.7	-1.4	-1.1	-0.8	-0.5	-0.2	0.1	0.4	0.7	1	1.3
y_i	0.1029	0.1174	0.1316	0.1448	0.1566	0.1662	0.1733	0.1775	0.1785	0.1764	0.1711	0.1630
x_i	1.6	1.9	2.2	2.5	2.8	3.1	3.4	3.7	4	4.3	4.6	4.9
y_i	0.1526	0.1402	0.1266	0.1122	0.0977	0.0835	0.0702	0.0579	0.0469	0.0373	0.0291	0.0224

SOLUTION For the given sample points, cubic and spline interpolation algorithms can be used and the interpolation results are obtained as shown in Figure 8.7. It can be seen that both the algorithms give satisfactory fitting effect.

```
>> x=[-2,-1.7,-1.4,-1.1,-0.8,-0.5,-0.2,0.1,0.4,0.7,1,1.3,...
      1.6,1.9,2.2,2.5,2.8,3.1,3.4,3.7,4,4.3,4.6,4.9];
   y=[0.10289,0.11741,0.13158,0.14483,0.15656,0.16622,0.17332,...
      0.1775,0.17853,0.17635,0.17109,0.16302,0.15255,0.1402,...
      0.12655,0.11219,0.09768,0.08353,0.07019,0.05786,0.04687,...
      0.03729,0.02914,0.02236];
   x0=-2:0.02:4.9; y1=interp1(x,y,x0,'cubic');
   y2=interp1(x,y,x0,'spline'); plot(x0,y1,':',x0,y2,x,y,'o')
```

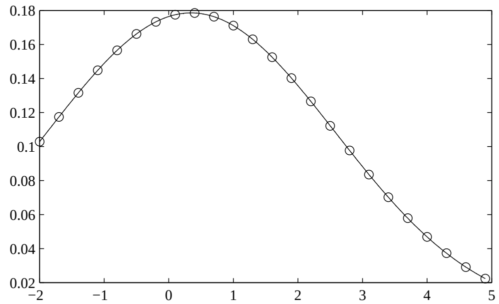


FIGURE 8.7: Fitting effect of two interpolation algorithms

4. Assume that a set of measured data is given in a file c8pdat.dat. Draw the 3D surface using interpolation methods.

SOLUTION The sample data in the scatter form, not the mesh grid form. Thus, the vectors $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ can be read from the data file, and interpolation range can be found from the data. Interpolation is shown in Figure 8.8.

```
>> load c8pdat.dat; x=c8pdat(:,1); y=c8pdat(:,2); z=c8pdat(:,3);
   v=[max(x), min(x) max(y), min(y)]  % find interpolation range
   [x1,y1]=meshgrid(0:0.02:1); z1=griddata(x,y,z,x1,y1,'v4');
   surf(x1,y1,z1)
```

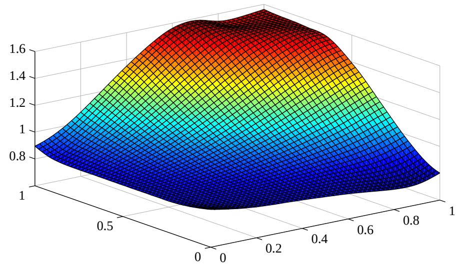


FIGURE 8.8: Interpolation surface

5. Assume that a set of measured data is given in a file c8pdat3.dat, whose 1∼3 columns are the coordinates of x, y, z, and the fourth column saves the measured function value $V(x, y, z)$. Perform three-dimensional interpolation from the data.

SOLUTION Four-dimensional display can be implemented using slice visualization approach, as shown in Figure 8.9.

```
>> load c8pdat3.dat
   x=c8pdat3(:,1); y=c8pdat3(:,2); z=c8pdat3(:,3); V=c8pdat3(:,4);
   [x1,y1,z1]=meshgrid(0.1:0.05:0.9); V1=griddata3(x,y,z,V,x1,y1,z1);
   xs=[0.1,0.6]; ys=[0.2 0.5]; zs=[0.2 0.6];
   slice(x1,y1,z1,V1,xs,ys,zs)
```

6. Generate a set of data from the function $f(x) = \dfrac{\sqrt{1+x}-\sqrt{x-1}}{\sqrt{2+x}+\sqrt{x-1}}$, for $x = 3:0.4:8$. The cubic splines and B-splines can be used to perform data interpolation tasks. From the fitted splines, take the second-order derivatives and compare the results with the theoretical curves.

SOLUTION Let's first consider piecewise cubic interpolation algorithm, the fitting results are shown in Figure 8.10 (a) and it can be seen that the fitting is satisfactory.

```
>> x=3:0.4:8; y=(sqrt(1+x)-sqrt(x-1))./(sqrt(2+x)+sqrt(x-1));
   S=csapi(x,y); S.coefs
   ezplot('(sqrt(1+x)-sqrt(x-1))./(sqrt(2+x)+sqrt(x-1))',[3,8]);
```

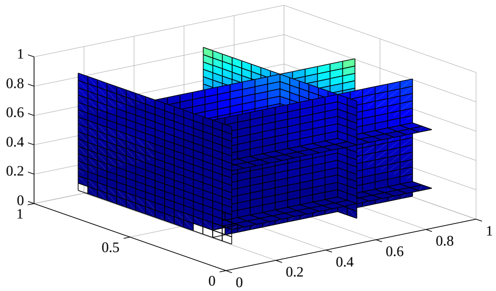
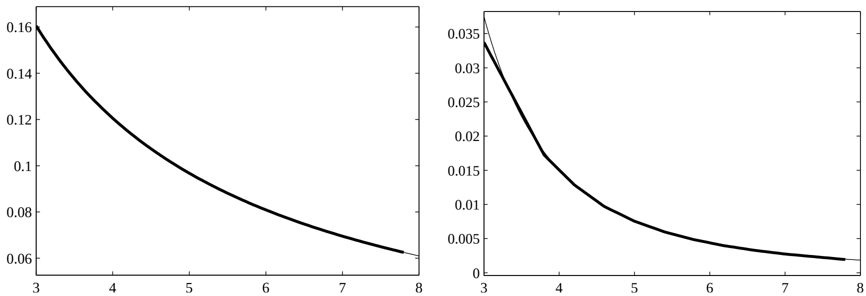


FIGURE 8.9: Interpolation of 3D functions and 4D slice view

```
hold on; fnplt(S)
```



(a) function fitting (b) second-order derivatives

FIGURE 8.10: Interpolation by piecewise cubic splines

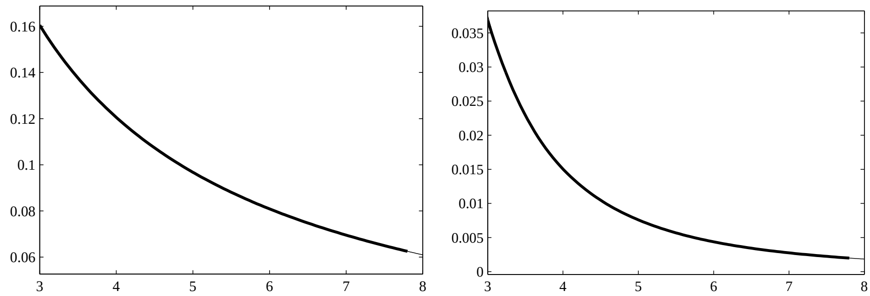
The second-order derivative of the function can be calculated from the spline object, as shown in Figure 8.10 (b). The calculated curve is compared with the theoretical curve, and it can be seen that when x is small, the fitting is not satisfactory.

```
>> syms x; y=(sqrt(1+x)-sqrt(x-1))/(sqrt(2+x)+sqrt(x-1));
   y2=diff(y,x,2); ezplot(y2,[3,8]); hold on
   S2=fnder(S,2); fnplt(S2)
```

Now let's try the B-spline. The curve fitting and derivative fitting are shown in Figures 8.10 (a) and (b), and it can be seen that the results are far much better than the piecewise cubic interpolations.

```
>> x=3:0.4:8; y=(sqrt(1+x)-sqrt(x-1))./(sqrt(2+x)+sqrt(x-1));
   S=spapi(6,x,y);
   ezplot('(sqrt(1+x)-sqrt(x-1))./(sqrt(2+x)+sqrt(x-1))',[3,8]);
   hold on; fnplt(S)
```

7. Assume that the measured data are given below. Draw the 3D surface plot for (x, y) within the rectangular intervals $(0.1, 0.1) \sim (1.1, 1.1)$.



(a) function fitting (b) second-order derivative

FIGURE 8.11: B-spline interpolation

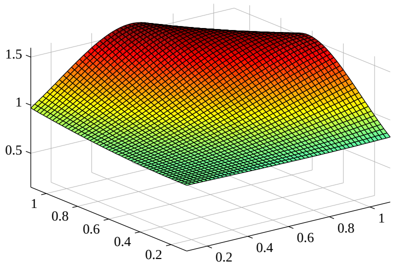
y_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
0.1	0.8304	0.8272	0.824	0.8209	0.8182	0.8161	0.8148	0.8146	0.8157	0.8185	0.823
0.2	0.8317	0.8324	0.8358	0.842	0.8512	0.8637	0.8797	0.8993	0.9226	0.9495	0.9801
0.3	0.8358	0.8434	0.8563	0.8746	0.8986	0.9284	0.9637	1.0045	1.0502	1.1	1.1529
0.4	0.8428	0.8601	0.8853	0.9186	0.9598	1.0086	1.0642	1.1253	1.1903	1.2569	1.3222
0.5	0.8526	0.8825	0.9228	0.9734	1.0336	1.1019	1.1763	1.254	1.3308	1.4017	1.4605
0.6	0.8653	0.9104	0.9684	1.0383	1.118	1.2045	1.2937	1.3793	1.4539	1.5086	1.5335
0.7	0.8807	0.9439	1.0217	1.1117	1.2102	1.311	1.4063	1.4859	1.5377	1.5484	1.5052
0.8	0.899	0.9827	1.082	1.1922	1.3061	1.4138	1.5021	1.5555	1.5572	1.4915	1.346
0.9	0.92	1.0266	1.1482	1.2768	1.4005	1.5034	1.5661	1.5678	1.4888	1.3156	1.0454
1	0.9438	1.0752	1.2191	1.3624	1.4866	1.5684	1.5821	1.5032	1.315	1.0155	0.6247
1.1	0.9702	1.1278	1.2929	1.4448	1.5564	1.5964	1.5341	1.3473	1.0321	0.6126	0.1476

SOLUTION Direct interpolation method can be used to solve the problem, and the interpolation surface can be obtained as shown in Figure 8.12 (a).

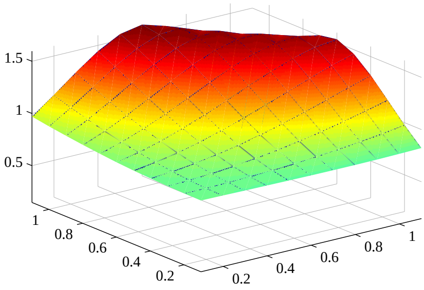
```
>> [x,y]=meshgrid(0.1:0.1:1.1);
   z=[0.8304,0.8272,0.824,0.8209,0.8182,0.8161,0.8148,0.8146,0.8157,0.8185,0.823;
      0.8317,0.8324,0.8358,0.842,0.8512,0.8637,0.8797,0.8993,0.9226,0.9495,0.9801;
      0.8358,0.8434,0.8563,0.8746,0.8986,0.9284,0.9637,1.0045,1.0502,1.1,1.1529;
      0.8428,0.8601,0.8853,0.9186,0.9598,1.0086,1.0642,1.1253,1.1903,1.2569,1.3222;
      0.8526,0.8825,0.9228,0.9734,1.0336,1.1019,1.1763,1.254,1.3308,1.4017,1.4605;
      0.8653,0.9104,0.9684,1.0383,1.118,1.2045,1.2937,1.3793,1.4539,1.5086,1.5335;
      0.8807,0.9439,1.0217,1.1117,1.2102,1.311,1.4063,1.4859,1.5377,1.5484,1.5052;
      0.899,0.9827,1.082,1.1922,1.3061,1.4138,1.5021,1.5555,1.5572,1.4915,1.346;
      0.92,1.0266,1.1482,1.2768,1.4005,1.5034,1.5661,1.5678,1.4888,1.3156,1.0454;
      0.9438,1.0752,1.2191,1.3624,1.4866,1.5684,1.5821,1.5032,1.315,1.0155,0.6247;
      0.9702,1.1278,1.2929,1.4448,1.5564,1.5964,1.5341,1.3473,1.0321,0.6126,0.1476];

   [x1,y1]=meshgrid(0.1:0.02:1.1);
   z1=interp2(x,y,z,x1,y1,'spline'); surf(x1,y1,z1)
   axis([0.1,1.1,0.1,1.1,min(z1(:)),max(z1(:))])
```

In fact, if the user is not interested in the interpolation data, it is better to perform interpolation using the `shading interp` command, with the effect



(a) interpolation results



(b) MATLAB internal interpolation effect

FIGURE 8.12: Interpolation surface

shown in Figure 8.12 (b). It can be seen that the method gives much smoother surface.

```
>> surf(x,y,z); shading interp
```

8. For the measured data samples (x_i, y_i) given below, piecewise cubic polynomial splines can be used and find the coefficients of each polynomial.

x_i	1	2	3	4	5	6	7	8	9	10
y_i	244.0	221.0	208.0	208.0	211.5	216.0	219.0	221.0	221.5	220.0

SOLUTION The cubic spline interpolation obtained is shown in Figure 8.13. It can be seen that the fitting is satisfactory.

```
>> x=1:10;
   y=[244.0,221.0,208.0,208.0,211.5,216.0,219.0,221.0,221.5,220.0];
   S=csapi(x,y); fnplt(S); hold on; plot(x,y,'o')
```

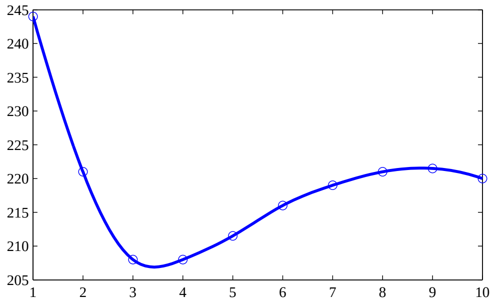
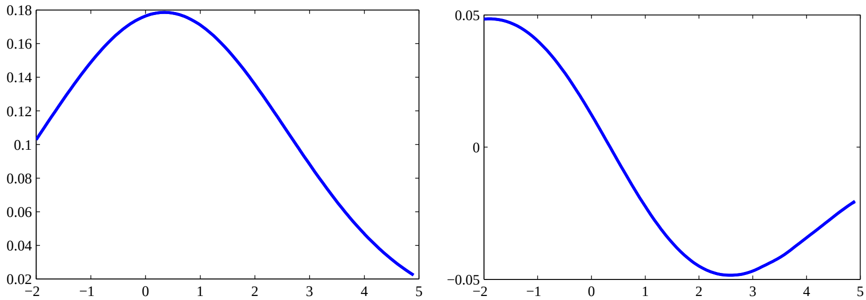


FIGURE 8.13: Cubic spline interpolation surface

9. The one-dimensional and two-dimensional data given in Exercises 3 and 7 can be used for cubic splines and B-splines interpolation. Find the derivatives of the related interpolated functions.

SOLUTION Consider first the problem in Exercise 3, the cubic and B-splines can be used to interpolate the function and its derivative, the interpolation results are shown respectively in Figures 8.14 (a) and (b).

```
>> x=[-2,-1.7,-1.4,-1.1,-0.8,-0.5,-0.2,0.1,0.4,0.7,1,1.3,...
      1.6,1.9,2.2,2.5,2.8,3.1,3.4,3.7,4,4.3,4.6,4.9];
   y=[0.10289,0.11741,0.13158,0.14483,0.15656,0.16622,0.17332,...
      0.1775,0.17853,0.17635,0.17109,0.16302,0.15255,0.1402,...
      0.12655,0.11219,0.09768,0.08353,0.07019,0.05786,0.04687,...
      0.03729,0.02914,0.02236];
   S=csapi(x,y); S1=spapi(6,x,y); fnplt(S); hold on; fnplt(S1)
   figure; Sd=fnder(S); S2=fnder(S1); fnplt(Sd), hold on; fnplt(S2)
```



(a) function fitting (b) first-order derivative

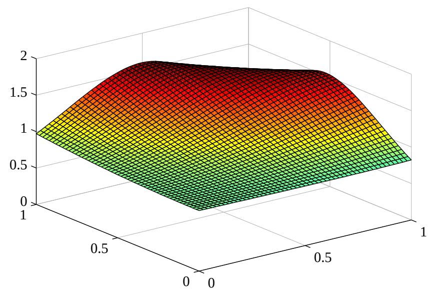
FIGURE 8.14: Interpolation comparisons with cubic and B-splines

Now consider the problem in Exercise 7. The original data format is slightly different from the necessary `ndgrid()` function. Here $\boldsymbol{x}$ and $\boldsymbol{y}$ should be given in vectors, and $\boldsymbol{z}$ should be assigned as the transpose of the original matrix $\boldsymbol{z}$. The following statements can be used to establish cubic and B-spline interpolation models, and the interpolation surfaces are respectively shown in Figures 8.15 (a) and (b). It can be seen that they are very close.

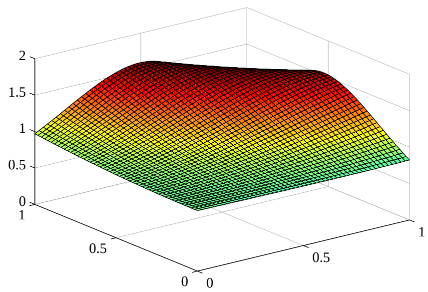
```
>> [x,y]=meshgrid(0.1:0.1:1.1);
   z=[0.8304,0.8272,0.824,0.8209,0.8182,0.8161,0.8148,0.8146,0.8157,0.8185,0.823;
      0.8317,0.8324,0.8358,0.842,0.8512,0.8637,0.8797,0.8993,0.9226,0.9495,0.9801;
      0.8358,0.8434,0.8563,0.8746,0.8986,0.9284,0.9637,1.0045,1.0502,1.1,1.1529;
      0.8428,0.8601,0.8853,0.9186,0.9598,1.0086,1.0642,1.1253,1.1903,1.2569,1.3222;
      0.8526,0.8825,0.9228,0.9734,1.0336,1.1019,1.1763,1.254,1.3308,1.4017,1.4605;
      0.8653,0.9104,0.9684,1.0383,1.118,1.2045,1.2937,1.3793,1.4539,1.5086,1.5335;
      0.8807,0.9439,1.0217,1.1117,1.2102,1.311,1.4063,1.4859,1.5377,1.5484,1.5052;
      0.899,0.9827,1.082,1.1922,1.3061,1.4138,1.5021,1.5555,1.5572,1.4915,1.346;
      0.92,1.0266,1.1482,1.2768,1.4005,1.5034,1.5661,1.5678,1.4888,1.3156,1.0454;
      0.9438,1.0752,1.2191,1.3624,1.4866,1.5684,1.5821,1.5032,1.315,1.0155,0.6247;
      0.9702,1.1278,1.2929,1.4448,1.5564,1.5964,1.5341,1.3473,1.0321,0.6126,0.1476];
   x0=[0.0:0.1:1]; y0=x0; z=z'; S=csapi({x0,y0},z); fnplt(S)
   figure; S1=spapi({5,5},{x0,y0},z); fnplt(S1)
```

The partial derivative surface can also be obtained with the B-spline, as shown respectively in Figures 8.16 (a) and (b).

```
>> S1x=fnder(S1,[0,1]); fnplt(S1x)
   figure; S1y=fnder(S1,[0,1]); fnplt(S1y)
```

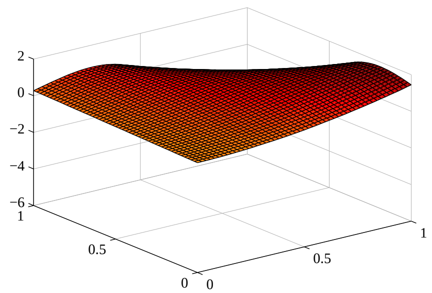


(a) cubic spline

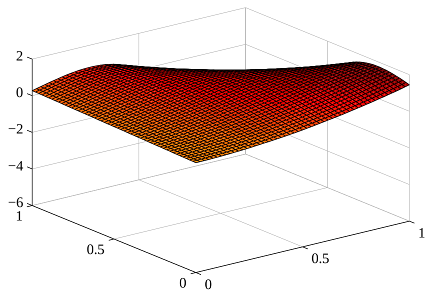


(b) B-spline

FIGURE 8.15: Fitting effect comparisons of the two algorithms



(a) $\partial z/\partial x$



(b) $\partial z/\partial y$

FIGURE 8.16: B-spline interpolation of partial derivatives

10. Consider again the data in Exercise 3. Polynomial fitting can be used to model the data. Select a suitable degree such that good approximation by polynomials can be achieved. Compare the results with interpolation methods.

SOLUTION Select different degrees such as 3,5,7,9, and 11, the polynomial fitting results are shown in Figure 8.17.

```
>> x=[-2,-1.7,-1.4,-1.1,-0.8,-0.5,-0.2,0.1,0.4,0.7,1,1.3,...
      1.6,1.9,2.2,2.5,2.8,3.1,3.4,3.7,4,4.3,4.6,4.9];
   y=[0.10289,0.11741,0.13158,0.14483,0.15656,0.16622,0.17332,...
      0.1775,0.17853,0.17635,0.17109,0.16302,0.15255,0.1402,...
      0.12655,0.11219,0.09768,0.08353,0.07019,0.05786,0.04687,...
      0.03729,0.02914,0.02236];
   x0=-2:0.02:4.9; p3=polyfit(x,y,3); y3=polyval(p3,x0);
   p5=polyfit(x,y,5); y5=polyval(p5,x0); p7=polyfit(x,y,7);
   y7=polyval(p7,x0); p9=polyfit(x,y,9); y9=polyval(p9,x0);
   p11=polyfit(x,y,11); y11=polyval(p11,x0);
   plot(x0,[y3; y5; y7; y9; y11])
```

It can be seen from the results that fifth-degree polynomial can fit the data satisfactorily.

11. Consider again the data in Exercise 3. Assumed that it is known the prototype

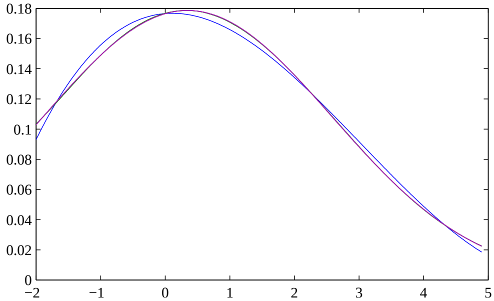


FIGURE 8.17: Polynomial fitting results

of the function for the data is $y(x) = \dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$. However the values of the parameters μ and σ are not known. Use least squares curve fitting methods to see whether suitable μ and σ can be identified. Observe the fitting results.

SOLUTION Let $a_1 = \mu, a_2 = \sigma$, the prototype function can be expressed by

$$y(\boldsymbol{a}, x) = \frac{1}{\sqrt{2\pi}a_2} e^{-(x-a_1)^2/2a_2^2}$$

which can be modeled by an anonymous function

```
>> f=@(a,x)exp(-(x-a(1)).^2/2/a(2)^2)/(sqrt(2*pi)*a(2));
```

where a_1, a_2 are undetermined parameters, with the values $a_1 = 0.3461, a_2 = 2.2340$. The fitting results are shown in Figure 8.18, and the fitting results are satisfactory.

```
>> x=[-2,-1.7,-1.4,-1.1,-0.8,-0.5,-0.2,0.1,0.4,0.7,1,1.3,...
      1.6,1.9,2.2,2.5,2.8,3.1,3.4,3.7,4,4.3,4.6,4.9];
   y=[0.10289,0.11741,0.13158,0.14483,0.15656,0.16622,0.17332,...
      0.1775,0.17853,0.17635,0.17109,0.16302,0.15255,0.1402,...
      0.12655,0.11219,0.09768,0.08353,0.07019,0.05786,0.04687,...
      0.03729,0.02914,0.02236];
   a=lsqcurvefit(f,[1,1],x,y), x0=-2:0.02:5; y0=f(a,x0);
   plot(x0,y0,x,y,'o')
```

12. Express the constant e in terms of continued fractions. Observe that how much of continued fraction stages are expected to get suitable approximations.

SOLUTION Continued fraction to a constant can be obtained with

```
>> maple('with(numtheory):'); f=maple(['cfe:=cfrac(exp(1),20)'])
   n1=maple('nthnumer','cfe',8); d1=maple('nthdenom','cfe',8);
   [vpa(n1),vpa(d1)], err1=abs(exp(1)-vpa(n1)/vpa(d1))
   n2=maple('nthnumer','cfe',20); d2=maple('nthdenom','cfe',20);
   [vpa(n2),vpa(d2)], err2=abs(exp(1)-vpa(n2)/vpa(d2))
```

Reserving the first 8 terms, an approximate rational 1264/465 can be used for the value of e, with an error level of 10^{-6}. With 20 terms, the rational

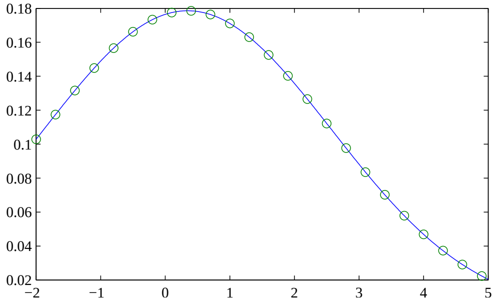


FIGURE 8.18: Data fitting by least squares method

approximation is 410105312/150869313, with error level of 10^{-15}. Continued fraction can be expressed as

$$e\approx 2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{8+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{10+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{12+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{14+}}}}}}}}}}}}}}}}}}}}$$

13. Find good approximations to the functions given below using continued fraction expansions and Padé approximations. Observe the fitting results obtained and find suitable degrees of the rational functions.

(i) $f(x)=\mathrm{e}^{-2x}\sin 5x$, (ii) $f(x)=\dfrac{x^3+7x^2+24x+24}{x^4+10x^3+35x^2+50x+24}\mathrm{e}^{-3x}$

SOLUTION (i) The 10-term continued fraction expression can be written as

```
>> syms x; fun='sin(5*x)*exp(-2*x)';
   maple('with(numtheory):'); f=maple(['cfe:=cfrac(' fun ',x,10)'])
```

the continued fraction is

$$f(x) \approx \cfrac{5x}{1+\cfrac{2x}{1-\cfrac{37x}{12+\cfrac{673x}{37+\cfrac{15983x}{673+\cfrac{76954709x}{79915-\cfrac{8067507978993x}{58235996+\cfrac{37499813958898841x}{\cdots}}}}}}}}.$$

The following statements can be used to extract rational approximation

```
>> n=collect(maple('nthnumer','cfe',4),x);
   d=collect(maple('nthdenom','cfe',4),x); [n,d]=numden(n/d); G1=n/d
   n=collect(maple('nthnumer','cfe',6),x);
   d=collect(maple('nthdenom','cfe',6),x); [n,d]=numden(n/d); G2=n/d
   n=collect(maple('nthnumer','cfe',8),x);
   d=collect(maple('nthdenom','cfe',8),x); [n,d]=numden(n/d); G3=n/d
   n=collect(maple('nthnumer','cfe',10),x);
   d=collect(maple('nthdenom','cfe',10),x); [n,d]=numden(n/d); G4=n/d
```

the first two rational approximations are

$$G_1 = -30\,\frac{x\,(58\,x-37)}{673\,x^2+96\,x+222}$$

$$G_2 = -5\,\frac{x\left(4048829\,x^2-484416\,x-958980\right)}{4159714\,x^3+2833713\,x^2+2402376\,x+958980}.$$

The quality of fitting is shown in Figure 8.19.

```
>> syms x;ezplot(sin(5*x)*exp(-2*x),[0,1]);hold on;ezplot(G1,[0,1]);
   ezplot(G2,[0,1]); ezplot(G3,[0,1]); ezplot(G4,[0,1]);
```

Padé approximation gives the same results.

```
>> f=taylor(sin(5*x)*exp(-2*x),x,12)
   syms x; ezplot(sin(5*x)*exp(-2*x),[0,1]); hold on; c=sym2poly(f);
   [n,d]=padefcn(c,5,6); G=poly2sym(n)/poly2sym(d); ezplot(G,[0,1])
   [n,d]=padefcn(c,3,4); G=poly2sym(n)/poly2sym(d); ezplot(G,[0,1])
```

14. Assume that the data in Exercise 7 satisfies a prototype function of $z(x,y) = a\sin(x^2y) + b\cos(y^2x) + cx^2 + dxy + e$. Identify the values of a,b,c,d,e with least squares method. Verify the identification results.

SOLUTION One may use least squares method to estimate the undetermined coefficients, where $\boldsymbol{a} = [-0.8922, 3.0939, -0.1220, 2.7085, -2.4253]$.

```
>> [x,y]=meshgrid(0.1:0.1:1.1);
   z=[0.8304,0.8272,0.824,0.8209,0.8182,0.8161,0.8148,0.8146,0.8157,0.8185,0.823;
```

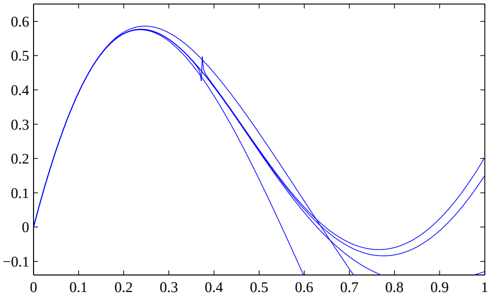


FIGURE 8.19: Continued fraction approximation results

```
    0.8317,0.8324,0.8358,0.842,0.8512,0.8637,0.8797,0.8993,0.9226,0.9495,0.9801;
    0.8358,0.8434,0.8563,0.8746,0.8986,0.9284,0.9637,1.0045,1.0502,1.1,1.1529;
    0.8428,0.8601,0.8853,0.9186,0.9598,1.0086,1.0642,1.1253,1.1903,1.2569,1.3222;
    0.8526,0.8825,0.9228,0.9734,1.0336,1.1019,1.1763,1.254,1.3308,1.4017,1.4605;
    0.8653,0.9104,0.9684,1.0383,1.118,1.2045,1.2937,1.3793,1.4539,1.5086,1.5335;
    0.8807,0.9439,1.0217,1.1117,1.2102,1.311,1.4063,1.4859,1.5377,1.5484,1.5052;
    0.899,0.9827,1.082,1.1922,1.3061,1.4138,1.5021,1.5555,1.5572,1.4915,1.346;
    0.92,1.0266,1.1482,1.2768,1.4005,1.5034,1.5661,1.5678,1.4888,1.3156,1.0454;
    0.9438,1.0752,1.2191,1.3624,1.4866,1.5684,1.5821,1.5032,1.315,1.0155,0.6247;
    0.9702,1.1278,1.2929,1.4448,1.5564,1.5964,1.5341,1.3473,1.0321,0.6126,0.1476];
x1=x(:); y1=y(:); z1=z(:);
A=[sin(x1.^2.*y1) cos(y1.^2.*x1) x1.^2 x1.*y1 ones(size(x1))];
theta=A\z1
```

The fitting quality can be obtained as shown in Figure 8.20.

```
>> [x,y]=meshgrid(0.1:0.02:1.1);
   z=theta(1)*sin(x.^2.*y)+theta(2)*cos(y.^2.*x)+theta(3)*x.^2+...
     theta(4)*x.*y+theta(5)
   surf(x,y,z)
```

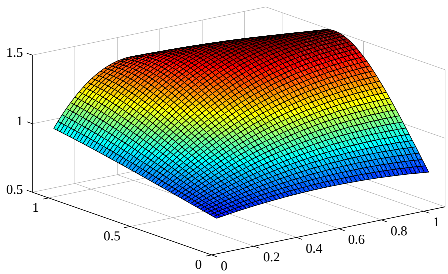


FIGURE 8.20: Surface obtained by least squares fitting

15. Assume that a function is given by $f(t) = \mathrm{e}^{-3t}\cos(2t+\pi/3)+\mathrm{e}^{-2t}\cos(t+\pi/4)$. Evaluate the formula of the auto-correlation function of the signal. Generate

a sequence of randomly distributed data and verify the results in a numerical way.

SOLUTION The auto-correlation function can be evaluated from

```
>> syms T t tau; f=exp(-3*t)*cos(2*t+pi/3)+exp(-2*t)*cos(t+pi/4);
   R=int(f*subs(f,t,t+tau),t,0,inf); R=simple(R)
```

with the result

$$-\frac{3\mathrm{e}^{-2\tau}}{68}\sin\left(\frac{7}{12}\pi+\tau\right)+\frac{\mathrm{e}^{-2\tau}}{52}\sin\left(-\frac{1}{12}\pi+\tau\right)+\frac{\mathrm{e}^{-2\tau}}{10}\cos\left(\frac{1}{2}\pi+\tau\right)$$
$$+\frac{\mathrm{e}^{-3\tau}}{12}\cos 2\tau-\frac{\mathrm{e}^{-3\tau}}{26}\sin\left(\frac{2}{3}\pi+2\tau\right)+\frac{3\mathrm{e}^{-3\tau}}{52}\cos\left(\frac{2}{3}\pi+2\tau\right)$$
$$-\frac{\mathrm{e}^{-3\tau}}{52}\sin\left(\frac{1}{12}\pi+2\tau\right)+\frac{5\mathrm{e}^{-3\tau}}{52}\cos\left(\frac{1}{12}\pi+2\tau\right)+\frac{\mathrm{e}^{-2\tau}}{8}\cos\tau$$
$$-\frac{3\mathrm{e}^{-3\tau}}{68}\sin\left(\frac{7}{12}\pi+2\tau\right)+\frac{5\mathrm{e}^{-3\tau}}{68}\cos\left(\frac{7}{12}\pi+2\tau\right)+\frac{5\mathrm{e}^{-2\tau}}{52}\cos\left(-\frac{1}{12}\pi+\tau\right)$$
$$+\frac{5\mathrm{e}^{-2\tau}}{68}\cos\left(\frac{7}{12}\pi+\tau\right)-\frac{\mathrm{e}^{-2\tau}}{20}\sin\left(\frac{1}{2}\pi+\tau\right)$$

16. Evaluate the auto-correlation function for the Gaussiandistribution function defined as $f(t)=\dfrac{1}{\sqrt{2\pi}3}\mathrm{e}^{-t^2/3^2}$. Generate a sequence of signals and compare them using Gaussian distributed data to check whether the results description is close to the theoretical results.

SOLUTION The following statements can be used to calculate the auto-correlation function

```
>> syms T t tau; f=exp(-t^2/2/3^2)/sqrt(2*pi)/3;
   R=int(f*subs(f,t,t+tau),t,0,inf); R=simple(R)
```

with

$$R_{tt}(\tau)=-\frac{2535301200456458802993406410752}{9557860351189671683947187847 6363}\sqrt{\pi}\mathrm{e}^{-\tau^2/36}\left(-1+\mathrm{erf}\left(\frac{1}{6}\tau\right)\right)$$

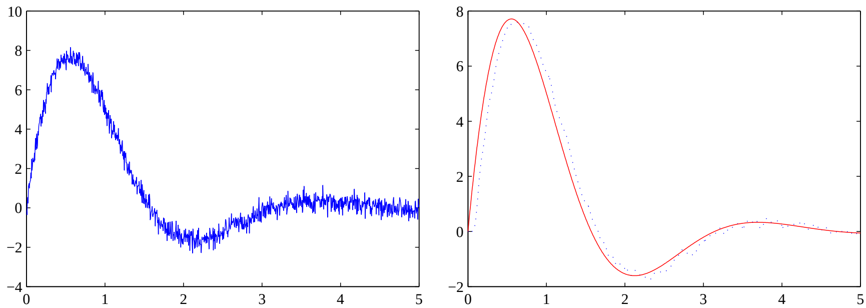
17. Assume that the noised signal can be generated with the following statements

```
>> t=0:0.005:5; y=15*exp(-t).*sin(2*t);
   r=0.3*randn(size(y)); y1=y+r;
```

find the Nyquist frequency of the signal. Based on such a frequency, design an eighth-order Butterworth filter which can be used to effectively filter out noises, while having a relatively small delay.

SOLUTION It can be found that the Nyquist frequency is 100000Hz. The corrupted signal is shown in Figure 8.21 (a).

```
>> h=0.005; t=0:0.005:5; y=15*exp(-t).*sin(2*t);
   r=0.3*randn(size(y)); y1=y+r; f=t/h;
   Nf=floor(length(f)/2)/h, plot(t,y1)
```



(a) corrupted signal (b) filtered signal

FIGURE 8.21: Signal filtering

With an eighth-order Butterworth filter, the filtered signal is shown in Figure 8.21 (b). It can be seen that the filtered signal is satisfactory, however inevitably, there exists delays in the filtered signal.

```
>> [b,a]=butter(8,0.1); y2=filter(b,a,y1); plot(t,y2,':',t,y,'r-')
```

18. High-pass filters can be used to filter out information with low-frequencies, and retain the high-frequency details. Design a high-pass filter for the data shown in Exercise 8.17, the noise information can be returned. Compare the statistical behavior of the noise signal obtained.

SOLUTION It can be seen from the filter that, the high-pass filter is in fact equivalent to 1 minus the low-pass filter. Thus the noise signal to be filtered out from the filter can be extracted by a high-pass filter, shown in Figure 8.22.

```
>> h=0.005; t=0:0.005:5; y=15*exp(-t).*sin(2*t);
   r=0.3*randn(size(y)); y1=y+r;
   [b,a]=butter(8,0.6); b1=b-a; y2=filter(b1,a,y1); plot(t,y2)
```

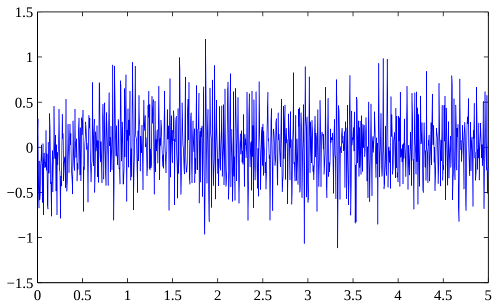


FIGURE 8.22: Extracted noise signal

Chapter 9

Probability and Mathematical Statistics Problems

Exercises and Solutions

1. The PDF of Rayleigh distribution is given by $p_{\rm r}(x) = \begin{cases} \dfrac{x}{b^2}\mathrm{e}^{-\frac{x^2}{2b^2}} & x \geqslant 0 \\ 0 & x < 0 \end{cases}$.

   Derive analytically the CDF, mean, variance, central moment and raw moments of the distribution. Generate a pseudo-random sequence satisfying Rayleigh distribution, verify numerically whether the calculation is correct.

SOLUTION Based on the relevant mathematical formulas, the distribution function, mean, variance and moments can be derived

```
>> syms x; syms b positive; p=x*exp(-x^2/2/b^2)/b^2*heaviside(x);
   syms tau; F=int(subs(p,x,tau),tau,-inf,x)
   m=simple(int(x*p,x,-inf,inf)),
   v=simple(int((x-m)^2*p,x,-inf,inf))
   Ev=int(x^r*p,x,0,inf), Evm=int((x-m)^r*p,x,0,inf)
```

and it is found that the distribution function is expressed by a piecewise function

$$F(x) = \begin{cases} 1 - \mathrm{e}^{x^2/(2b^2)}, & x > 0 \\ 0, & x \leqslant 0 \end{cases}$$

The mean and variance are $m = \sqrt{2\pi}b/2, v = -b^2(\pi - 4)/2$. The raw moment can be expressed by $2^{r/2}b^r\Gamma(r/2 + 1)$. Direct integration above cannot find the central moments for symbolic r. One may use loop structure to find the first five central moments

```
>> for r=1:5, mu=simple(int((x-m)^r*p,x,0,inf)), end
```

which can be written as $\mu_1 = 0$, $\mu_2 = -b^2(\pi - 4)/2$, $\mu_3 = b^3\sqrt{2\pi}(-3 + \pi)/2$, $\mu_4 = (8 - 3\pi^2)b^4/4$, and $\mu_5 = b^5\sqrt{2\pi}(\pi^2 - 25 + 5\pi)/2$.

2. Assume that between Locations A and B, there are six sets of traffic lights. The probability of red light at each set of traffic lights is the same, with $p = 1/3$. Suppose the number of red traffic lights on the road for Locations A to B satisfies a binomial distribution $B(6, p)$. Find the probability of one meets only once the red traffic light from A to B. Varying the value of p to draw the probability curve.

SOLUTION The probability density function of binomial distribution can be evaluated with `binopdf()`. Denote the time one meets red light by $\boldsymbol{x}$, which may contain $0, 1, 2, \cdots, 6$, the probability density function, in other words, the probability to meet red lights, can be evaluated from

```
>> x=0:6; y=binopdf(x,6,1/3)
```

It can be found that $\boldsymbol{y} = [0.0878, 0.2634, 0.3292, 0.2195, 0.0823, 0.0165, 0.0014]$. The probability to meet only once red traffic light is 0.2634. The probability to meet at least once the red light can be obtained by two ways. One way is to add up all the terms except the first one in vector $\boldsymbol{y}$. The other is to subtract the from term of $\boldsymbol{y}$ from 1. Thus the solution can be obtained as $P = 91.22\%$.

```
>> P=1-y(1)   % or P=sum(y(2:end))
```

One may change the value of p continuously to draw the probability curve with the following commabds, and the probability curve is shown in Figure 9.1.

```
>> p0=0.05:0.05:0.95; y=[];
   for p=p0, y=[y binopdf(1,6,p)]; end
   plot(p0,y)
```

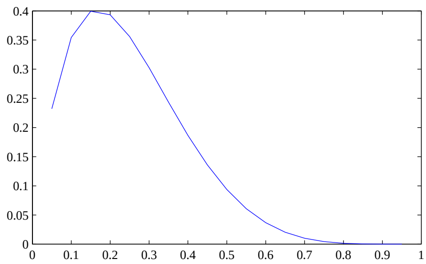


FIGURE 9.1: Probability of meet only once red traffic light

3. Assume that in a foreign language examination, the randomly selected samples indicate that the scores satisfies approximately the normal distribution, with a mean value of 72. The number of those whose scores are higher than 96 is 2.3% of all the number of students. Find the probability of a student whose score is between 60 and 80.

SOLUTION The standard deviation σ should be obtained first. The problem can mathematically be expressed as

$$P(x \geqslant 96) = P\left(\frac{X-72}{\sigma} \geqslant \frac{96-72}{\sigma}\right) = 1 - \Phi\left(\frac{24}{\sigma}\right) = 0.023$$

with $\Phi(24/\sigma) = 1 - 0.023$, which can be used to evaluate $\sigma = 12.0277$.

```
>> p1=norminv(1-0.023,0,1), sigma=24/p1
```

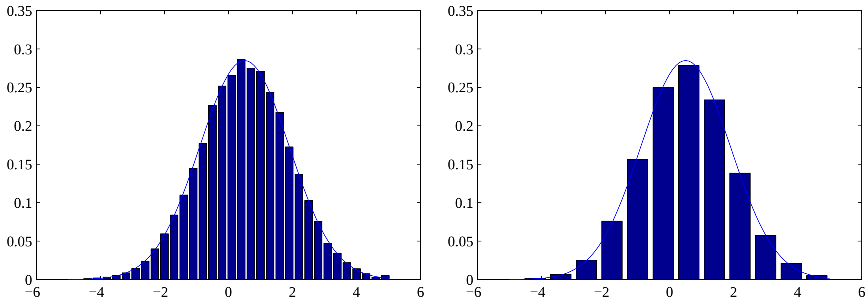
With the value of σ, the probability of the score between $(60, 80)$ can be obtained as $P = 58.78\%$.

```
>> P=normcdf(80,72,sigma)-normcdf(60,72,sigma)
```

4. Generate 30000 pseudo-random numbers satisfying normal distribution of $N(0.5, 1.4^2)$. Find the mean and standard deviation of the data. Observe the histogram of the data to see whether they agree with theoretical distribution. Change the width of the bins and see what may happen.

SOLUTION The random numbers can be generated and the mean and variance of the numbers can be calculated, with $m = 0.5040$, $s = 1.4045$. The PDF from the generated data can be obtained as shown in Figure 9.2 (a).

```
>> x=normrnd(0.5,1.4,30000,1); m=mean(x), s=std(x)
   xx=-5:0.3:5; yy=hist(x,xx); bar(xx,yy/length(x)/0.3);
   x0=-5:0.1:5; y0=normpdf(x0,0.5,1.4); line(x0,y0)
```



(a) probability density function (b) with wider bins

FIGURE 9.2: PDF fitting from the generated random numbers

If the width of the bins increase, the PDF is shown in Figure 9.2 (b).

```
>> xx=-5:0.8:5; yy=hist(x,xx); bar(xx,yy/length(x)/0.8); line(x0,y0)
```

5. Assume that a set of data was measured as shown below. Use MATLAB to perform the following hypothesis tests:
(i) Assume that the data satisfies normal distribution with a standard deviation of 1.5. Test whether the mean value of the data is 0.5.
(ii) If the standard deviation is not known, test whether the mean is still 0.5.
(iii) Test whether the distribution of the data is normal.

SOLUTION (i) Two hypotheses can be introduced first

$$\begin{cases} \mathscr{H}_0 : \mu = \mu_0 & \mu \text{ satisfies the requirement} \\ \mathscr{H}_1 : \text{reject the hypothesis } \mathscr{H}_0 & \end{cases}$$

Thus, using MATLAB statements, it is found that $u = -0.1886$.

```
>> x=[-1.7908,0.0903,3.9223,0.4135,3.2618,-1.0665,0.5169,-1.2615,...
      1.8206,-0.0652,1.5803,2.0033,0.3237,2.5006,5.6959,1.6804,...
      0.4734,2.5546,0.6258,-1.9909,1.5924,0.4887,-0.1214,3.372,...
      4.6927,-0.6757,0.7327,-1.3172,2.031,-4.8202,2.7277,0.9925,...
      1.0887,3.2303,-0.118,0.2004,-2.3586,-3.2431,-1.083,1.1319,...
     -0.7177,-2.5004,2.9135,-1.1022,0.4746,0.4981,-0.0612,1.3923,...
     -0.094,-3.244,-1.8152,1.047,-2.3273,-0.2811,-1.6181,-2.1427,...
```

−1.7908	0.3238	4.6927	−2.3586	−0.0940	2.8943	5.3067	3.1634	−3.2812	−3.4389
0.0903	2.5006	−0.6758	−3.2431	−3.2440	−0.0521	−0.0796	0.4653	−0.7905	2.0690
3.9223	5.6959	0.7327	−1.083	−1.8152	−2.9145	−2.6714	1.7065	0.0819	2.3258
0.4135	1.6804	−1.3172	1.132	1.047	0.5219	4.4827	−1.112	0.5201	1.9318
3.2618	0.4735	2.031	−0.7177	−2.3273	0.6606	1.2325	−0.9750	2.3831	3.4477
−1.0665	2.5546	−4.8203	−2.5004	−0.2812	1.2122	−2.0178	1.2073	−1.1251	1.236
0.5169	0.6259	2.7278	2.9135	−1.6181	1.6246	1.8958	0.7403	−1.1234	−1.0142
−1.2615	−1.9909	0.9925	−1.1022	−2.1428	3.3757	3.357	4.6585	0.04734	0.1640
1.8206	1.5924	1.0887	0.47461	−1.7976	−0.7326	−1.5161	−0.1190	0.4540	−5.0103
−0.0652	0.48874	3.2303	0.49816	−0.40375	1.0868	0.80414	5.4782	1.1275	1.5649
1.5803	−0.1215	−0.118	−0.0612	0.8908	0.4704	0.1872	3.8942	2.8812	0.7631
2.0033	3.372	0.2005	1.3923	0.23873	−0.80559	−2.1176	−3.8764	1.8988	−0.8300

```
    -1.7976,-0.4037,0.8907,0.2387,2.8943,-0.0521,-2.9145,0.5219,...
     0.6605,1.2122,1.6246,3.3757,-0.7325,1.0868,0.4703,-0.8055,...
     5.3067,-0.0796,-2.6714,4.4827,1.2325,-2.0177,1.8958,3.357,...
     -1.5161,0.8041,0.1871,-2.1176,3.1634,0.4652,1.7065,-1.112,...
     -0.975,1.2073,0.7403,4.6585,-0.1189,5.4782,3.8942,-3.8764,...
    -3.2812,-0.7904,0.0819,0.5201,2.3831,-1.1251,-1.1234,0.0473,...
     0.4539,1.1275,2.8812,1.8988,-3.4389,2.069,2.3258,1.9318,...
     3.4477,1.236,-1.0142,0.164,-5.0103,1.5649,0.7631,-0.8299];
  u=sqrt(length(x))*(mean(x)-0.5)/1.5
```

Since $|u| < 1.96$, the hypothesis $\mathscr{H}_0$ can be accepted.
(ii) Since the variance is not known, T test should be used

```
>> [H,p,ci]=ttest(x,0.5,0.02),
```

Since $H = 0$, the hypothesis $\mathscr{H}_0$ cannot be rejected. Also the confidential interval is $[0.014, 0.9343]$.
(iii) Jarque-Bera hypothesis test should be used and since $h = 0$, the distribution satisfies a normal distribution.

```
>> [h,s]=jbtest(x,0.05)
```

6. Suppose that tests have been made on a group of randomly selected fuse, and it is found that the burn-out current of the fuse are 10.4, 10.2, 12.0, 11.3, 10.7, 10.6, 10.9, 10.8, 10.2, 12.1 A. Suppose that these values satisfies normal distribution, find the mean value of burn-out current and its confidence interval under confidence level $\alpha \leqslant 0.05$.

Solution Method (i), `normfit()` function can be used to evaluate the mean and confidence interval, with the mean of 10.92 A, and the confidence interval of $(10.43, 11.41)$ A.

```
>> x=[10.4,10.2,12,11.3,10.7,10.6,10.9,10.8,10.2,12.1];
   [m1,s1,ma,sa]=normfit(x,0.05); m1, ma
```

Method (ii), to test whether the mean, `mean(x)`, can be accepted under T test. Meanwhile, the mean and confidence intervals can be obtained.

```
>> x=[10.4,10.2,12,11.3,10.7,10.6,10.9,10.8,10.2,12.1]; mean(x)
   [H,p,ci]=ttest(x,mean(x),0.05)
```

7. Assume that the boiling point under a certain atmospheric pressure are tested with the multiple measured data 113.53, 120.25, 106.02, 101.05, 116.46, 110.33, 103.95, 109.29, 93.93, 118.67°C. Check whether they satisfy normal distribution under the confidence level of $\alpha \leqslant 0.05$.

SOLUTION The data can be entered first, and since $H = 0$, the data satisfies a normal distribution. Further with the `normfit()` function, it can be found that the mean of the data is 109.3480, within the interval $[103.4030, 115.2930]$, and the standard deviation is 8.3105, within the interval $[5.7162, 15.1717]$.

```
>> x=[113.53,120.25,106.02,101.05,116.46,110.33,103.95,...
       109.29,93.93,118.67];
   [H,p,c,d]=jbtest(x,0.05)
   [m1,s1,ma,sa]=normfit(x,0.05)
```

8. Assume that 12 samples are measured for a random variable and they are 9.78, 9.17, 10.06, 10.14, 9.43, 10.60, 10.59, 9.98, 10.16, 10.09, 9.91, 10.36. Find the deviation of the data and its confidence interval.

SOLUTION Assume first the variable satisfy a normal distribution. Such a hypothesis can be tested with the following statements, and $H = 0$ indicates that the hypothesis can be accepted.

```
>> x=[9.78,9.17,10.06,10.14,9.43,10.6,10.59,9.98,...
          10.16,10.09,9.91,10.36];
   [H,p,c,d]=jbtest(x,0.05);
```

Since the normal distribution is confirmed, the `normfit()` function can be used to find the variance and its confidence interval, with $s_1 = 0.4220$, and $s_a = [0.2990, 0.7166]$.

```
>> [m1,s1,ma,sa]=normfit(x,0.05)
```

9. Twenty patients suffered from insomnia are divided randomly into groups A and B, with ten patients each. They were given medicines A, B respectively. The extended sleeping hours are measured as shown below. Judge whether there are significant differences in healing effect.

A	1.9	0.8	1.1	0.1	−0.1	4.4	5.5	1.6	4.6	3.4
B	0.7	−1.6	−0.2	−1.2	−0.1	3.4	3.7	0.8	0	2

SOLUTION The two sets of data can be entered into MATLAB first

```
>> x=[1.9,0.8,1.1,0.1,-0.1,4.4,5.5,1.6,4.6,3.4];
   y=[0.7,-1.6,-0.2,-1.2,-0.1,3.4,3.7,0.8,0,2];
```

There are two methods in solving the problem, one is to introduce $z = x - y$ and hypothesis $\mathscr{H}_0 : d = 0$, where d is the mean value of z. The alternative method is to test whether the hypothesis $\mathscr{H}_0 : \mu_1 = \mu_2$ is satisfied.
For the former method, the algorithm in the book can be used directly, where a statistics quantity $T = (\bar{z} - d)/\sigma_z^*$ can be introduced, who satisfies T distribution, with $T = 4.0621$, and $T_0 = 2.2622$

```
>> z=x-y; T=(mean(z)-0)/std(z)*sqrt(10), T0=abs(tinv(0.05/2,9))
```

Since $T > T_0$, the hypothesis should be rejected, i.e., there exists significant differences in the two groups.

For the second method, a statistics quantity $T = \dfrac{\bar{x}-\bar{y}}{\sqrt{s_1^{*2}+s_2^{*2}}}\sqrt{n}$ can be established, who satisfies T distribution, with the degree of freedom n_1+n_2-2. The following statements can be used to perform the hypothesis test, with $T = 1.8608$, $T_0 = 2.1009$.

```
>> T=(mean(x)-mean(y))/sqrt(cov(x)+cov(y))*sqrt(10)
   T0=abs(tinv(0.05/2,18))
```

Since $T < T_0$, the hypothesis cannot be rejected.

However for this problem, obvious the two conclusions are not the same. Normally the latter hypothesis test method should be adopted.

10. For a prototype function $y = a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5$, with five independent variables x_1, x_2, x_3, x_4, x_5 and one output y, the following data are obtained. Find the values a_i and their confidence intervals using linear regression method.

x_1	8.11	9.25	7.63	7.89	12.94	10.11	7.57	9.92	7.74	7.3	9.48	11.91
x_2	2.13	2.66	0.83	1.54	1.74	0.79	0.68	2.93	2.01	1.35	2.81	2.23
x_3	3.98	−0.68	1.42	−0.96	−0.28	3.37	4.58	2.15	2.66	3.69	1	−0.98
x_4	6.55	6.85	6.25	5.34	6.85	7.2	6.12	6.07	5.51	6.6	6.15	6.43
x_5	5.92	7.54	5.39	4.65	6.47	5.1	6.04	5.37	6.54	6.55	5.8	3.95
y	27.676	38.774	23.314	23.828	35.154	21.779	25.516	29.845	32.642	28.443	31.5	23.554

SOLUTION The undetermined coefficients can be obtained with $\boldsymbol{a} = [1, 2, -0.3, -2, 5]^{\mathrm{T}}$, and the maximum fitting error is 2.0606×10^{-13}.

```
>> X=[8.11,9.25,7.63,7.89,12.94,10.11,7.57,9.92,7.74,7.3,9.48,11.91;
      2.13,2.66,0.83,1.54,1.74,0.79,0.68,2.93,2.01,1.35,2.81,2.23;
      3.98,-0.68,1.42,-0.96,-0.28,3.37,4.58,2.15,2.66,3.69,1,-0.98;
      6.55,6.85,6.25,5.34,6.85,7.2,6.12,6.07,5.51,6.6,6.15,6.43;
      5.92,7.54,5.39,4.65,6.47,5.1,6.04,5.37,6.54,6.55,5.8,3.95];
   Y=[27.676,38.774,23.314,23.828,35.154,21.779,25.516,29.845,...
     32.642,28.443,31.5,23.554];
   a=X'\Y', norm(X'*a-Y')
```

Alternatively, `regress()` function can be used to get the regression parameters, which are the same as the ones obtained above.

```
>> [a b]=regress(Y',X')
```

11. Assume that a set of measured data x_i and y_i are given below, and the prototype function is $f(x) = a_1\mathrm{e}^{-a_2x}\cos(a_3x+\pi/3) + a_4\mathrm{e}^{-a_5x}\cos(a_6x+\pi/4)$. Estimate the values of a_i and their confidence intervals.

x	1.027	1.319	1.204	0.684	0.984	0.864	0.795	0.753	1.058	0.914	1.011	0.926
y	8.8797	5.9644	7.1057	8.6905	9.2509	9.9224	9.8899	9.6364	8.5883	9.7277	9.023	9.6605

SOLUTION Using least squares method, the coefficient vector can be obtained $\boldsymbol{c} = [23.8088, 0.9857, 1.8439, -26.1630, 3.7301, 6.1293]^{\rm T}$. It should be noted that the fitting vector is not unique, the fitting results using the coefficient vector is shown in Figure 9.3 and the quality of fitting is high.

```
>> x=[1.027,1.319,1.204,0.684,0.984,0.864,0.795,0.753,1.058,...
        0.914,1.011,0.926];
   y=[-8.8797,-5.9644,-7.1057,-8.6905,-9.2509,-9.9224,-9.8899,...
        -9.6364,-8.5883,-9.7277,-9.023,-9.6605];
   f=@(a,x)a(1)*exp(-a(2)*x).*cos(a(3)*x+pi/3)+...
          a(4)*exp(-a(5)*x).*cos(a(6)*x+pi/4);
   [c,ci]=nlinfit(x,y,f,[1;2;3;4;5;6])
   [x1,ii]=sort(x); y1=y(ii); y2=f(c,x1); plot(x1,y1,x1,y2)
```

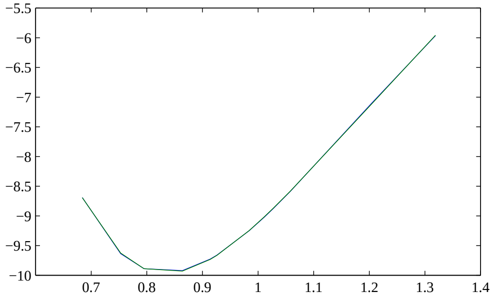


FIGURE 9.3: Fitting result by least squares method

12. For a prototype function $y = a_1 e^{-a_2 x_1}(a_3 x_2 + x_3) + a_4 x_4 (a_5 x_5 + 1)$, assume that the measured data are given below. Estimate the values of a_i and their confidence intervals.

x_1	8.11	9.25	7.63	7.89	12.94	10.11	7.57	9.92	7.74	7.3	9.48	11.91
x_2	2.13	2.66	0.83	1.54	1.74	0.79	0.68	2.93	2.01	1.35	2.81	2.23
x_3	3.98	−0.68	1.42	−0.96	−0.28	3.37	4.58	2.15	2.66	3.69	1	−0.98
x_4	6.55	6.85	6.25	5.34	6.85	7.2	6.12	6.07	5.51	6.6	6.15	6.43
x_5	5.92	7.54	5.39	4.65	6.47	5.1	6.04	5.37	6.54	6.55	5.8	3.95
y	8.19	7.68	5.42	3.98	5.99	6.19	7.09	6.83	7.23	8.06	6.74	4.24

SOLUTION The undetermined coefficients a_i can be obtained with the least squares method in Chapter 8

```
>> X=[8.11,9.25,7.63,7.89,12.94,10.11,7.57,9.92,7.74,7.3,9.48,11.91;
      2.13,2.66,0.83,1.54,1.74,0.79,0.68,2.93,2.01,1.35,2.81,2.23;
      3.98,-0.68,1.42,-0.96,-0.28,3.37,4.58,2.15,2.66,3.69,1,-0.98;
      6.55,6.85,6.25,5.34,6.85,7.2,6.12,6.07,5.51,6.6,6.15,6.43;
      5.92,7.54,5.39,4.65,6.47,5.1,6.04,5.37,6.54,6.55,5.8,3.95];
   Y=[8.19,7.68,5.42,3.98,5.99,6.19,7.09,6.83,7.23,8.06,6.74,4.24];
```

```
f=@(a,X)a(1)*exp(-a(2)*X(1,:)).*[a(3)*X(2,:)+X(3,:)]+...
    a(4)*X(4,:).*[a(5)*X(5,:)+1]; c0=zeros(5,1);
c=lsqcurvefit(f,c,X,Y)
```

and the vector $\boldsymbol{c} = [1.0015, 0.1002, 1.9987, 0.1010, 0.9875]^{\mathrm{T}}$. Of course, the function `nlinfit()` presented in Chapter 9 can also be used to identify the coefficients

```
>> [c,r,J]=nlinfit(X,Y,f,c0); c, ci=nlparci(c,r,J)
```

The estimated parameters are exactly the same as the ones obtained above. Moreover the estimated intervals for these parameters are $(0.9937, 1.0093)$, $(0.0991, 0.1013)$, $(1.9895, 2.0079)$, $(0.0986, 0.1034)$ $(0.9603, 1.0146)$.

13. Assume the measured data given below satisfies the following prototype function $y(t) = c_1\mathrm{e}^{-5t}\sin(c_2t) + (c_3t^2 + c_4t^3)\mathrm{e}^{-3t}$. Find from the data the parameters c_i's and their confidence interval.

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
y	0	0.1456	0.2266	0.2796	0.3187	0.3479	0.3677	0.3777	0.3782	0.37
t	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
y	0.3546	0.3335	0.3085	0.2812	0.253	0.225	0.198	0.1726	0.1492	0.1279
t	2	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
y	0.109	0.0922	0.0776	0.065	0.0541	0.0449	0.0371	0.0305	0.025	0.0204

SOLUTION The following statements can be used to find the estimated parameters can be obtained as $\boldsymbol{c} = [0.9858, 2.0268, 3.0096, 3.9926]^{\mathrm{T}}$. Under these parameters, the fitting results can be obtained as shown in Figure 9.4. It can be seen that the fitting results are satisfactory.

```
>> t=[0,0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8,0.9,1,1.1,1.2,1.3,1.4,...
     1.5,1.6,1.7,1.8,1.9,2,2.1,2.2,2.3,2.4,2.5,2.6,2.7,2.8,2.9];
   y=[0,0.1456,0.2266,0.2796,0.3187,0.3479,0.3677,0.3777,0.3782,...
     0.37,0.3546,0.3335,0.3085,0.2812,0.253,0.225,0.198,0.1726,...
     0.1492,0.1279,0.109,0.0922,0.0776,0.065,0.0541,0.0449,...
     0.0371,0.0305,0.025,0.0204];
   f=@(c,t)c(1)*exp(-5*t).*sin(c(2)*t)+(c(3)*t.^2+...
           c(4)*t.^3).*exp(-3*t);
   [chat,r,j]=nlinfit(t,y,f,[1;2;3;4]); chat
   y2=f(chat,t); plot(t,y,'-o',t,y2,'-')
```

14. Assume that 12 sample plants each are randomly selected from areas A and B. The iron element content in μg/g are measured as shown below. Assume that the iron element content in the plant satisfies a normal distribution and the variance of distribution is not affected by the area. Judge whether the distribution of the iron element content is the same.

area A	11.5	18.6	7.6	18.2	11.4	16.5	19.2	10.1	11.2	9	14	15.3
area B	16.2	15.2	12.3	9.7	10.2	19.5	17	12	18	9	19	10

SOLUTION The following statements can be used to calculate the value of $T = -0.2853$. Since $|T| < |T_0|$, the hypothesis can b e accepted, which means that there is no significant differences between them.

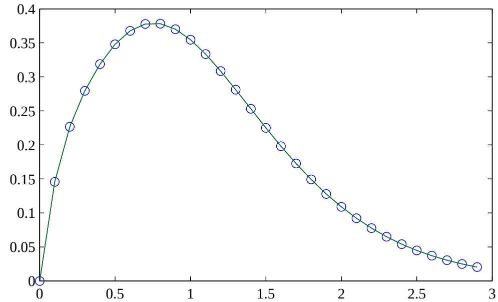


FIGURE 9.4: Fitting result by the estimated parameters

```
>> x=[11.5,18.6,7.6,18.2,11.4,16.5,19.2,10.1,11.2,9,14,15.3];
   y=[16.2,15.2,12.3,9.7,10.2,19.5,17,12,18,9,19,10];
   T=(mean(x)-mean(y))/sqrt(cov(x)+cov(y))*sqrt(length(x))
```

15. Assume that there are random variables A and B, whose samples are given below. Judge whether they have significant differences.

A	10.42	10.48	7.98	8.52	12.16	9.74	10.78	10.18	8.73	8.88	10.89	8.1
B	12.94	12.68	11.01	11.68	10.57	9.36	13.18	11.38	12.39	12.28	12.03	10.8

SOLUTION Similar to the previous examples, one can give the following statements

```
>> x=[10.42,10.48,7.98,8.52,12.16,9.74,10.78,10.18,8.73,...
       8.88,10.89,8.1];
   y=[12.94,12.68,11.01,11.68,10.57,9.36,13.18,11.38,...
       12.39,12.28,12.03,10.8];
   T=(mean(x)-mean(y))/sqrt(cov(x)+cov(y))*sqrt(length(x))
```

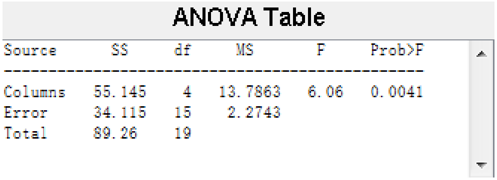
It is found that $T = -3.9518$. Since $|T| > |T_0|$, the hypothesis should be rejected, that is, there are significant differences between variables A and B.

16. Suppose that five different dyeing technics are tested for the same cloth. Different dyeing technics and different machines are tested randomly, and the percentage of washing shrinkage are given below. Judge whether the dyeing technics have significant effect on the washing shrinkage.

machine	dyeing technic					machine	dyeing technic				
number	1	2	3	4	5	number	1	2	3	4	5
1	4.3	6.1	6.5	9.3	9.5	2	7.8	7.3	8.3	8.7	8.8
3	3.2	4.2	8.6	7.2	11.4	4	6.5	4.2	8.2	10.1	7.8

SOLUTION The variance analysis to the problem can be performed with the following statements and the ANOVA table and boxed plot are obtained as shown in Figures 9.5 (a) and (b).

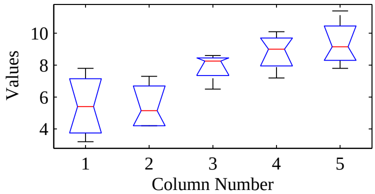
```
>> A=[4.3,6.1,6.5,9.3,9.5; 7.8,7.3,8.3,8.7,8.8;
      3.2,4.2,8.6,7.2,11.4; 6.5,4.2,8.2,10.1,7.8];
   [p,tbl,stats]=anova1(A)
```



ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	55.145	4	13.7863	6.06	0.0041
Error	34.115	15	2.2743		
Total	89.26	19			

(a) ANOVA table



(b) boxed plot

FIGURE 9.5: Variance analysis results of Exercise 16

Since the probability $p = 0.0041$ is too small, there exists significant differences among the means, i.e., the washing shrinkage. From the boxed plot, it is readily seen that the effect of dyeing tactics 1 and 5 have significant differences.

17. Assume that the heights of randomly selected Year 5 pupils in three schools are measured in the table given below. Check whether there are significant differences in the heights in the three school. ($\alpha = 0.05$)

school	measured height data					
1	128.1	134.1	133.1	138.9	140.8	127.4
2	150.3	147.9	136.8	126	150.7	155.8
3	140.6	143.1	144.5	143.7	148.5	146.4

SOLUTION Variance analysis to the problem can be performed with the following statements and the ANOVA table and boxed plot are obtained as shown in Figures 9.6 (a) and (b).

```
>> A=[ 128.1 134.1 133.1 138.9 140.8 127.4;
       150.3 147.9 136.8 126 150.7 155.8;
       140.6 143.1 144.5 143.7 148.5 146.4];
   [p,tbl,stats]=anova1(A)
```

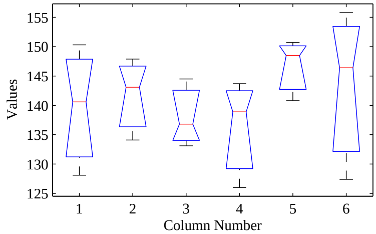
The probability obtained is $p = 0.7838$, which is not close to zero, thus the hypothesis, i.e., the mean of heights of the groups are the same, cannot be rejected.

18. The table below recorded the day output of three operators on four different machines. Check the following
(i) whether there are significant differences in the skill of the operators.
(ii) whether there are significant differences in the machines.
(iii) whether the interaction significant ($\alpha = 0.05$).

ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	211.34	5	42.2672	0.48	0.7838
Error	1053.8	12	87.8167		
Total	1265.14	17			

(a) ANOVA table



(b) boxed plot

FIGURE 9.6: Variance analysis results of Exercise 17

machine number	operator number 1			2			3			machine number	operator number 1			2			3		
M_1	15	15	17	19	19	16	16	18	21	M_3	15	17	16	18	17	16	18	18	18
M_2	17	17	17	15	15	15	19	22	22	M_4	18	20	22	15	16	17	17	17	17

Solution Two-way variance analysis can be performed with the following statements and the ANOVA table is obtained as shown in Figure 9.7.

```
>> A=[15,15,17,19,19,16,16,18,21; 17,17,17,15,15,15,19,22,22;
      15,17,16,18,17,16,18,18,18; 18,20,22,15,16,17,17,17,17];
   [p,tbl]=anova2(A)
```

ANOVA Table

Source	SS	df	MS	F	Prob>F
Columns	43	8	5.375	1.3	0.2887
Rows	2.75	3	0.91667	0.22	0.88
Error	99	24	4.125		
Total	144.75	35			

FIGURE 9.7: Two-way ANOVA table

Chapter 10

Non-traditional Solution Methods for Mathematical Problems

Exercises and Solutions

1. Consider a tipping problem in a restaurant[7]. Assume that the average rate for the tips is 15% of the consumption. The service level and food quality are used to calculate the tip. The service level can be written as “good”, “average” and “poor”, and the food quality can also be expressed as other fuzzy descriptions. Establish a fuzzy inference system for evaluating the tips.

SOLUTION In this problem, there are two inputs, u_1 for food quality, u_2 for service level, and one output, v for tip level. The universes of the inputs and output should be given first, for instance, u_1, u_2 for [0,100] interval, and v for [0,20], meaning 0% to 20%. One can then define the membership functions for these properties, an example of these is shown in Figure 10.1.

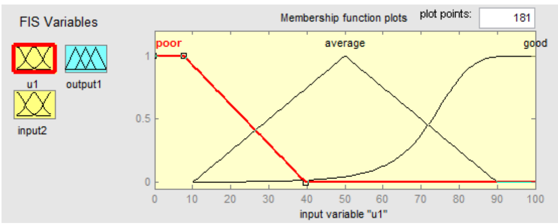


FIGURE 10.1: Fuzzy membership function dialog box

A fuzzy inference table can be established as shown in Table 10.1, and based on the table, the fuzzy inference system can be established as shown in Figure 10.2. It should be noted that the establishment of this fuzzy inference system depend heavily upon the opinion of the designer. One can save the system to file exc10fz.fis, from which the tip level can be evaluated directly from it.

2. Consider the sample data of (x_i, y_i) given below. Construct a neural network model, from which draw the curve in the interval $x \in (1, 10)$. Different neural network structures and training algorithms can be tested and compare the fitting results.

TABLE 10.1: Rule table

		u_2		
		poor	average	good
	poor	low	low	medium
u_1	average	average	average	high
	good	medium	high	high

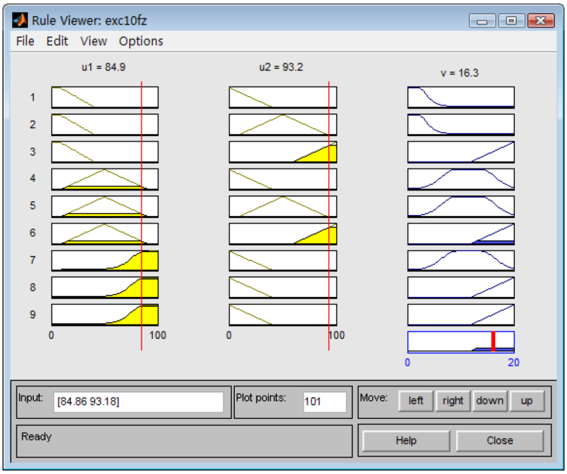


FIGURE 10.2: Fuzzy inference system

x_i	1	2	3	4	5	6	7	8	9	10
y_i	244.0	221.0	208.0	208.0	211.5	216.0	219.0	221.0	221.5	220.0

SOLUTION Six hidden layer nodes can be selected and also assume that the transfer functions of each layer are defined as `tansig()`. The following statements can be used to design and train the network.

```
>> x=1:10;
   y=[244.0,221.0,208.0,208.0,211.5,216.0,219.0,221.0,221.5,220.0];
   net=newff([1,10],[6,1],{'tansig','tansig'});
   net.trainParam.epochs=100; net=train(net,x,y);
   x1=1:0.1:10; y1=sim(net,x1); plot(x1,y1,x,y,'o')
```

It seems that the BP network may not work, another network structure, for instance, the RBF network can be used, and the fitting results in Figure 10.3. However even with RBF network, the fitting quality is not as good as the spline fitting method.

```
>> x1=1:0.1:10; net=newrbe(x,y); y1=sim(net,x1);
   y2=interp1(x,y,x1,'spline');
   plot(x,y,'o',x1,y1,x1,y2,':');
```

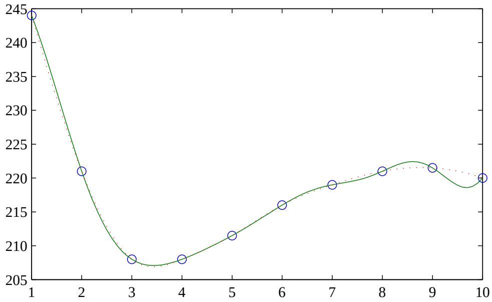


FIGURE 10.3: RBF network fitting

3. Assume that the actual measured data given below. Construct a neural network to fit the surface in the rectangular region $(0.1, 0.1) \sim (1.1, 1.1)$. Compare the results with data interpolation algorithms.

y_i	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
0.1	0.8304	0.8273	0.8241	0.8210	0.8182	0.8161	0.8148	0.8146	0.8158	0.8185	0.8230
0.2	0.8317	0.8325	0.8358	0.8420	0.8513	0.8638	0.8798	0.8994	0.9226	0.9496	0.9801
0.3	0.8359	0.8435	0.8563	0.8747	0.8987	0.9284	0.9638	1.0045	1.0502	1.1	1.1529
0.4	0.8429	0.8601	0.8854	0.9187	0.9599	1.0086	1.0642	1.1253	1.1904	1.257	1.3222
0.5	0.8527	0.8825	0.9229	0.9735	1.0336	1.1019	1.1764	1.254	1.3308	1.4017	1.4605
0.6	0.8653	0.9105	0.9685	1.0383	1.118	1.2046	1.2937	1.3793	1.4539	1.5086	1.5335
0.7	0.8808	0.9440	1.0217	1.1118	1.2102	1.311	1.4063	1.4859	1.5377	1.5484	1.5052
0.8	0.8990	0.9828	1.082	1.1922	1.3061	1.4138	1.5021	1.5555	1.5573	1.4915	1.346
0.9	0.9201	1.0266	1.1482	1.2768	1.4005	1.5034	1.5661	1.5678	1.4889	1.3156	1.0454
1	0.9438	1.0752	1.2191	1.3624	1.4866	1.5684	1.5821	1.5032	1.315	1.0155	0.6248
1.1	0.9702	1.1279	1.2929	1.4448	1.5564	1.5964	1.5341	1.3473	1.0321	0.6127	0.1476

SOLUTION The data can be entered with the following statements

```
>> A=[0.83,0.827,0.824,0.82,0.818,0.816,0.814,0.814,0.815,0.818,0.823;
      0.831,0.832,0.835,0.842,0.851,0.863,0.879,0.899,0.922,0.949,0.98;
      0.835,0.843,0.856,0.874,0.898,0.928,0.963,1.004,1.05,1.1,1.152;
      0.842,0.86,0.885,0.918,0.959,1.008,1.064,1.125,1.19,1.257,1.322;
      0.852,0.882,0.922,0.973,1.033,1.101,1.176,1.254,1.33,1.401,1.46;
      0.865,0.91,0.968,1.038,1.118,1.204,1.293,1.379,1.453,1.508,1.533;
      0.88,0.943,1.021,1.111,1.21,1.311,1.406,1.485,1.537,1.548,1.505;
      0.899,0.982,1.082,1.192,1.306,1.413,1.502,1.555,1.557,1.491,1.346;
      0.92,1.026,1.148,1.276,1.4,1.503,1.566,1.567,1.488,1.315,1.045;
      0.943,1.075,1.219,1.362,1.486,1.568,1.582,1.503,1.315,1.015,0.624;
      0.97,1.127,1.292,1.444,1.556,1.596,1.534,1.347,1.032,0.612,0.147];
```

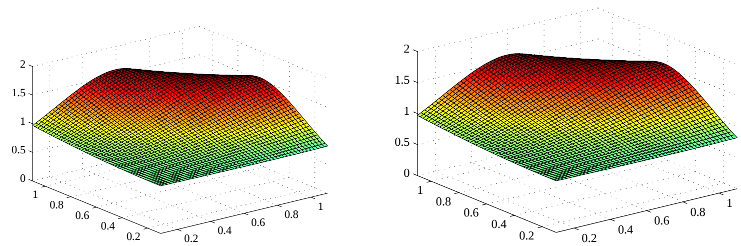
```
[X,Y]=meshgrid(0.1:0.1:1.1);
```

We can then set up a BP network with the following statements, and perform training, it can be seen that the fitting is again not satisfactory.

```
>> x=X(:); y=Y(:); z=A(:);
   [x1,y1]=meshgrid(0.1:0.02:1.1);
   x2=x1(:); y2=y1(:);
   net=newff([0.1,1.1; 0.1,1.1], ...
             [10,8,1],{'tansig','tansig','tansig'});
   net=train(net,[x'; y'],z');
```

The RBF network can be tested again, such that the fitting is should in Figure 10.4 (a).

```
>> net=newrbe([x'; y'],z');
   z1=sim(net,[x2'; y2']); z2=reshape(z1,size(x1));
   surf(x1,y1,z2)
```



(a) RBF network fitting (b) Interpolation fitting

FIGURE 10.4: Two-dimensional surface fitting

Using the interpolation method, the fitting results are obtained as shown in Figure 10.4 (b). It can be seen that the quality of two fitting methods is close for this example.

```
>> z3=interp2(X,Y,A,x1,y1,'spline'); surf(x1,y1,z3)
```

4. Solve the constrained optimization problem with genetic algorithms and PSO methods and compare the results with traditional algorithms.

$$\min_{\boldsymbol{x} \text{ s.t.} \begin{cases} 0.003079x_1^3x_2^3x_5-\cos^3 x_6\geqslant 0 \\ 0.1017x_3^3x_4^3-x_5^2\cos^3 x_6\geqslant 0 \\ 0.09939(1+x_5)x_1^3x_2^2-\cos^2 x_6\geqslant 0 \\ 0.1076(31.5+x_5)x_3^3x_4^2-x_5^2\cos^2 x_6\geqslant 0 \\ x_3x_4(x_5+31.5)-x_5[2(x_1+5)\cos x_6+x_1x_2x_5]\geqslant 0 \\ 0.2\leqslant x_1\leqslant 0.5, 14<\leqslant x_2\leqslant 22, 0.35\leqslant x_3\leqslant 0.6, \\ 16\leqslant x_4\leqslant 22, 5.8\leqslant x_5\leqslant 6.5, 0.14\leqslant x_6\leqslant 0.2618 \end{cases}} \frac{1}{2\cos x_6}\left[x_1x_2(1+x_5)+x_3x_4\left(1+\frac{31.5}{x_5}\right)\right]$$

SOLUTION (i) with GAOT, the objective function can be expressed by

```
function [sol,y]=exc10f2(sol,opt)
```

```
x=sol(1:6); cx6=cos(x(6));
c=[0.003079*x(1)^3*x(2)^3*x(5)-cx6^3;
   0.1017*x(3)^3*x(4)^3-x(5)^2*cx6^3;
   0.09939*(1+x(5))*x(1)^3*x(2)^2-cx6^2;
   0.1076*(31.5+x(5))*x(3)^3*x(4)^2-x(5)^2*cx6^2;
   x(3)*x(4)*(x(5)+31.5)-x(5)*(2*(x(1)+5)*cx6+x(1)*x(2)*x(5))];
if any(c<0), y=-100;
else, y=-(x(1)*x(2)*(1+x(5))+x(3)*x(4)*(1+31.5/x(5)))/2/cx6;
end
```

The following statements can be used to solve the problem of GA

```
>> xm=[0.2; 14; 0.35; 16; 5.8; 0.14];
   xM=[0.5; 22; 0.6; 22; 6.5; 0.2618]; xx=[xm xM];
   x=gaopt(xx,'exc10f2',[],[],[],'maxGenTerm',1500)
```

with $\boldsymbol{x} = [0.2216, 16.7496, 0.3596, 19.6057, 6.0943, 0.1661]^{\mathrm{T}}$, which is exactly the same as the one in Exercise 7 of Chapter 6.

5. Solve the benchmark problems in Exercise 7 of Chapter 6 using genetic algorithms and PSO methods.

SOLUTION For the benchmark problems, GA and PSO algorithms can be used. The objective functions for GAOT solver are written as

```
function [sol,f]=exc10o1(sol,x)
x=sol(1:end-1); x=x(:);  f=-x.'*x;
```

```
function [sol,f]=exc10o2(sol,x)
x=sol(1:end-1); x=x(:);
f=-(1+x.'*x/4000-prod(cos(x./cos(sqrt(1:length(i))))))
```

```
function [sol,f]=exc10o3(sol,x)
x=sol(1:end-1); x=x(:);
f=-20-1e-20*exp(-0.2*sqrt(sum(x.^2)/20))+exp(sum(cos(2*pi*x))/20);
```

```
>> xx=ones(20,1); xx=[-512*xx,512*xx];
   [a b c]=gaopt(xx,'exc10o1')
```

Unfortunately, the direct result may not be exact, or even acceptable, due to the fewer generations used. One should increase the number of generations to larger numbers. For instance, after 1500 generations, good solutions to the problem may be obtained.

```
>> [a b c]=gaopt(xx,'exc10o1',[],[],[],'maxGenTerm',1500);
   x1=a(1:end-1)
```

with $\boldsymbol{x}_1 = [-0.0008, -0.0002, -0.0000, -0.0004, -0.0014, -0.0001, -0.0004, 0.0006, 0.0001, -0.0001, -0.0001, 0.0004, -0.0004, 0.0000, -0.0005, 0.0008, 0.0003, -0.0000, 0.0015, -0.0031]^{\mathrm{T}}$.

The other two benchmark problems can be solved in a similar manner, and the results are all acceptable.

```
>> [a1 b c]=gaopt(xx,'exc10o2',[],[],[],'maxGenTerm',1500);
   [a2 b c]=gaopt(xx,'exc10o3',[],[],[],'maxGenTerm',1500);
   x2=a1(1:20), x3=a2(1:20)
```

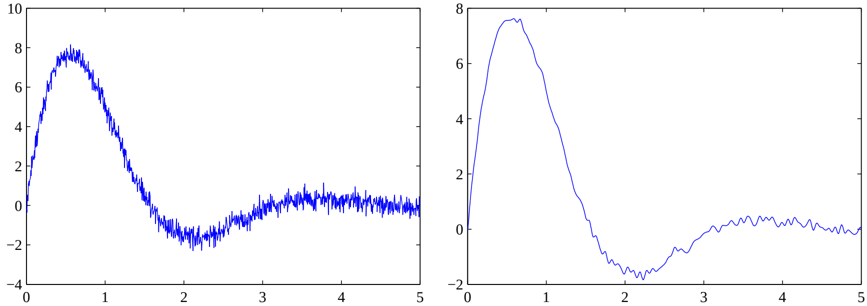
6. Assume that the corrupted signal is established from

```
>> t=0:0.005:5;y=15*exp(-t).*sin(2*t);r=0.3*randn(size(y));y1=y+r;
```

Perform de-noising tasks with wavelet transforms and compare the results with the filter techniques in Exercise 8.8.

SOLUTION For three-level wavelet decomposition, with `'coif4'` wavelet basis, the following statements can be used to perform de-noising tasks. The signal, before and after the process can be obtained as shown in Figures 10.5 (a) and (b) respectively. It can be seen that the de-noising effect is satisfactory.

```
>> t=0:0.005:5; y=15*exp(-t).*sin(2*t); r=0.3*randn(size(y));
   y1=y+r; plot(t,y1), figure
   [C,L]=wavedec(y1,3,coif4); A3a=wrcoef(a,C,L,coif4,3);
   plot(t,A3a)
```



(a) Original corrupted signal (b) de-noised signal

FIGURE 10.5: Corrupted signal de-noising

7. Assume that series of experimental data is given file c10rsdat.txt which is made up a 60×13 table. Each column corresponds to an attribute and the last column is the decision attribute. Use rough set reduction technique to check which of the attributes are important to the event in the decision.

SOLUTION Run `rsdav3` program, which gives the graphical user interface for rough set based data analysis program, shown in Figure 10.6. Click button Browse, one may load the data file into the interface. Then reduction can be done by clicking the redu(C,D,X) button. The reduced results are shown in the interface, which means that the conditional attributes 3,4,5,8,9,10 are important to the decision, and others can be neglected.

8. For the signal $f(t) = \mathrm{e}^{-3t}\sin(t+\pi/3)+t^2+3t+2$, find the 0.2th order derivative and 0.7th order integral. Draw the relevant curves.

SOLUTION The 0.2th-order differentiation and 0.7th-order integration, i.e. -0.7th-order differentiation can be obtained with the following statements and the

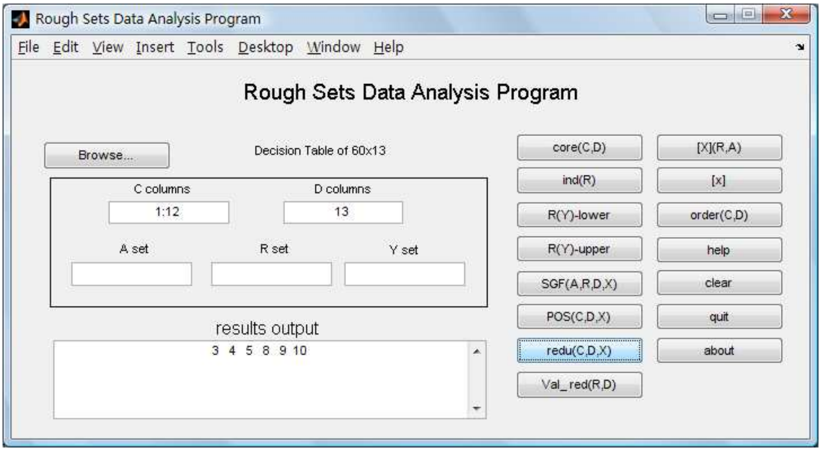


FIGURE 10.6: Graphical user interface for data analysis with rough sets

results are shown in Figure 10.7. It should be noted that, one can reduce the step-size to solve the problem again, which give the same results. This means that the results obtained are reliable and accurate.

```
>> t=0:0.01:10; y=exp(-3*t).*sin(t+pi/3)+t.^2+3*t+2;
   y1=glfdiff(y,t,0.2); y2=glfdiff(y,t,-0.7); plot(t,y1,t,y2,'--')
```

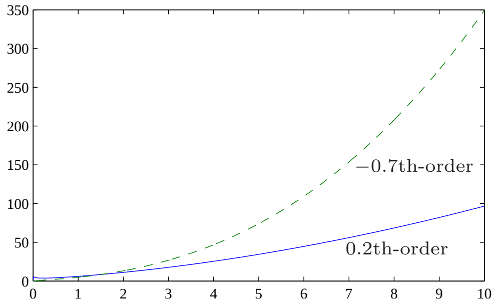


FIGURE 10.7: The 0.2th- and 0.7th-order derivative of $f(t)$

9. Design a filter for Exercise 8. The fractional-order derivatives and integrals can be obtained with the filter. Compare the results with the ones obtained with Grünwald-Letnikov method.

SOLUTION Select the interested frequency range of $(10^{-3}, 10^3)$, and $N = 3$, the Oustaloup filter and improved Oustaloup filter from

```
>> w1=1e-3; w2=1e3; N=3; n1=0.2; n2=-0.7;
   G11=zpk(ousta_fod(n1,N,w1,w2)), G12=zpk(new_fod(n1,N,w1,w2))
```

```
G21=zpk(ousta_fod(n2,N,w1,w2)), G22=zpk(new_fod(n2,N,w1,w2))
t=0:0.01:10; y=exp(-3*t).*sin(t+pi/3)+t.^2+3*t+2;
y11=lsim(G11,y,t); y12=lsim(G12,y,t);
y21=lsim(G21,y,t); y22=lsim(G22,y,t);
y1=glfdiff(y,t,0.2); y2=glfdiff(y,t,-0.7);
plot(t,y1,t,y2,t,y11,t,y12,t,y21,t,y22)
```

The filter by Oustaloup algorithm for the 0.2th order is

$$G_{11}=\frac{3.9811(s+306)(s+42.52)(s+5.91)(s+0.821)(s+0.1141)(s+0.016)(s+0.002)}{(s+454.1)(s+63.1)(s+8.767)(s+1.218)(s+0.1693)(s+0.02352)(s+0.003268)}$$

and other filters can also be obtained. The solutions of the problems are obtained and shown in Figure 10.8, together with the one evaluated from Grünwald-Letnikov definition. It can be seen that the Oustaloup filter is good for the 0.2th-order derivative, while for the −0.7th-order, the filter results are not satisfactory. The improved filter, however, gives accurate results.

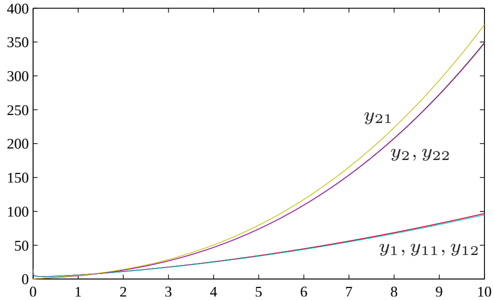


FIGURE 10.8: Comparisons of fractional-order derivatives

10. Consider a fractional-order linear differential equation is described by[8]

$$0.8\mathscr{D}_t^{2.2}y(t)+0.5\mathscr{D}_t^{0.9}y(t)+y(t)=1,\ y(0)=y'(0)=y''(0)=0.$$

Solve the solution using numerical methods. If one changes the orders of 2.2 and 0.9 respectively to 2 and 1, an approximate integer-order differential equation can be obtained. Compare the accuracy of the integer-order approximation.

SOLUTION If the fractional-orders are approximated by integer-orders, analytical solutions to the integer-order differential equation can be obtained with

```
>> y=dsolve('0.8*D2y+0.5*Dy+y=1','y(0)=0','Dy(0)=0')
```

and the solution is

$$y(t)=-\frac{\sqrt{295}}{59}\mathrm{e}^{-5t/16}\sin\frac{\sqrt{295}t}{16}-\mathrm{e}^{-5t/16}\cos\frac{\sqrt{295}t}{16}+1$$

The numerical solution of the fractional-order system can be obtained with the statements

```
>> t=0:0.01:10; u=ones(size(t));
   G=fotf([0.8 0.5 1],[2.2 0.9 0],1,0); y1=lsim(G,u,t);
```

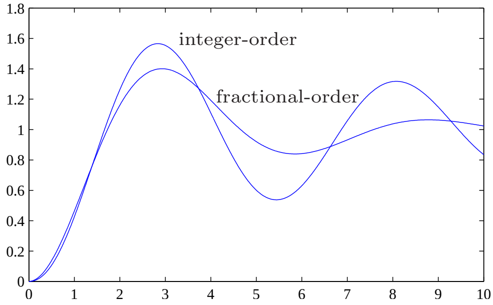


FIGURE 10.9: Solutions of the linear fractional-order equation

The exact and approximate solutions of the equations are obtained as shown in Figure 10.9. It can be seen that although with integer-order approximations the analytical solutions can be obtained, the accuracy is not very high for original fractional-order models. Fractional-order algorithms should be used.

11. Evaluate and draw the following Mittag-Leffler functions and verify (10.43).
(i) $\mathscr{E}_{1,1}(z)$, (ii) $\mathscr{E}_{2,1}(z)$, (iii) $\mathscr{E}_{1,2}(z)$, (iv) $\mathscr{E}_{2,2}(z)$

Solution With the `ml_fun()` function can be used to draw the requested Mittag-Leffler functions, as shown in Figure 10.10.

```
>> z=0:0.001:1;
   subplot(221), y1=ml_fun(1,1,z,0,1e-10); plot(z,y1)
   subplot(222), y2=ml_fun(2,1,z,0,1e-10); plot(z,y2)
   subplot(223), y3=ml_fun(1,2,z,0,1e-10); plot(z,y3)
   subplot(224), y4=ml_fun(2,2,z,0,1e-10); plot(z,y4)
```

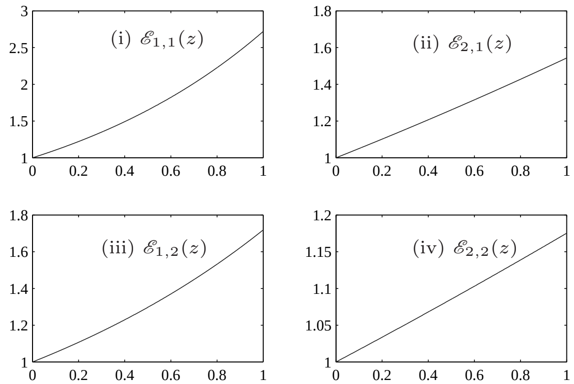


FIGURE 10.10: The Mittag-Leffler functions

12. Evaluate the following Mittag-Leffler functions and show graphically that the equations below are identical.

    (i) $\mathscr{E}_{\alpha,\beta}(x)+\mathscr{E}_{\alpha,\beta}(-x)=2\mathscr{E}_{\alpha,\beta}(x^2)$ (ii) $\mathscr{E}_{\alpha,\beta}(x)-\mathscr{E}_{\alpha,\beta}(-x)=2x\mathscr{E}_{\alpha,\alpha+\beta}(x^2)$

    (iii) $\mathscr{E}_{\alpha,\beta}(x)=\dfrac{1}{\Gamma(\beta)}+\mathscr{E}_{\alpha,\alpha+\beta}(x)$ (iv) $\mathscr{E}_{\alpha,\beta}(x)=\beta\mathscr{E}_{\alpha,\beta+1}(x)+\alpha x\dfrac{\mathrm{d}}{\mathrm{d}x}\mathscr{E}_{\alpha,\beta+1}(x)$

SOLUTION To show the two sides are equal, one can chose randomly the values of α and β, and assign a vector $\boldsymbol{x}$, then compare the curves of the two sides equal or not.

```
>> a=fix(100*rand(1))*0.01, b=fix(100*rand(1))*0.01
   x=0:0.0001:0.2; y11=ml_fun(a,b,x,0,1e-10)+ml_fun(a,b,-x,0,1e-10);
   y12=2*ml_fun(a,b,x.^2,0,1e-10); plot(x,y11,x,y12,'--')
   y21=ml_fun(a,b,x,0,1e-10)-ml_fun(a,b,-x,0,1e-10);
   y22=2*x.*ml_fun(a,a+b,x.^2,0,1e-10); plot(x,y21,x,y22,'--')
   y31=ml_fun(a,b,x,0,1e-10); y32=1/gamma(b)+ml_fun(a,a+b,x,0,1e-10);
   plot(x,y31,x,y32,'--') y41=y31;
   y42=b*ml_fun(a,b+1,x,0,1e-10)+a*x.*ml_fun(a,b+1,x,1,1e-10);
   plot(x,y41,x,y42,'--')
```

13. Find the closed-loop model from the typical feedback structure.

    (i) $G(s)=\dfrac{12\sqrt{s}+31}{(\sqrt{s}+20)(\sqrt{s}+100)(\sqrt{s}+1)},\ G_{\rm c}(s)=\dfrac{18s+20}{s(s+4)},\ H(s)=1$

    (ii) $G(s)=\dfrac{s^{0.4}+5}{s^{3.1}+2.8s^{2.2}+1.5s^{0.8}+4},\ G_{\rm c}(s)=3+2.5s^{-0.5}+1.4s^{0.8},\ H(s)=1$

SOLUTION (i) The plant model $G(s)$ can be entered first, then the closed-loop system model can be obtained with the following statements

```
>> G=fotf([1 20],[0.5 0],[12 31],[0.5 0])*fotf([1 100],[0.5,0],1,0)*...
     fotf([1 1],[0.5 0],1,0); Gc=fotf([1 4],[2 1],[18 20],[1 0]);
   H=fotf(1,0,1,0); Ga=feedback(G*Gc,H)
```

with the closed-loop model given by

$$G_a=\frac{216s^{1.5}+558s+240s^{0.5}+620}{s^{3.5}+121s^3+2124s^{2.5}+2484s^2+8696s^{1.5}+8558s+240s^{0.5}+620}.$$

(ii) The closed-loop model can be constructed with

```
>> G=fotf([1 2.8 1.5 4],[3.1 2.2 0.8 0],[1 5],[0.4 0]);
   Gc=fotf(1,0,3,0)+fotf(1,0.5,2.5,0)+fotf(1,0,1.4,0.8);
   H=fotf(1,0,1,0); Ga=feedback(G*Gc,H)
```

and the closed-loop model

$$G_a(s)=\frac{1.4s^{1.7}+7s^{1.3}+3s^{0.9}+15s^{0.5}+2.5s^{0.4}+12.5}{s^{3.6}+2.8s^{2.7}+1.4s^{1.7}+8.5s^{1.3}+3s^{0.9}+19s^{0.5}+2.5s^{0.4}+12.5}.$$

14. Consider the linear fractional-order differential equation given by

$$\mathscr{D}^{2\alpha}x(t)+\left(\frac{9}{1+2\lambda}\right)^{\alpha}\mathscr{D}^{\alpha}x(t)+x(t)=1,\ 0<\alpha<1$$

where $\lambda=0.5$, $\alpha=0.25$. Solve the equation numerically.

SOLUTION The solutions to the equation can be obtained with the `lsim()` function, and the solution is shown in Figure 10.11.

```
>> lam=0.5; alpha=0.25; a1=(9/(1+2*lam))^alpha;
   G=fotf([1 a1 1],[2*alpha alpha 0],1,0); t=0:0.002:1;
   u=ones(size(t)); x1=lsim(G,u,t); plot(t,x1)
```

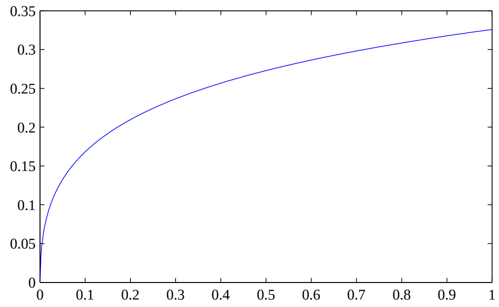
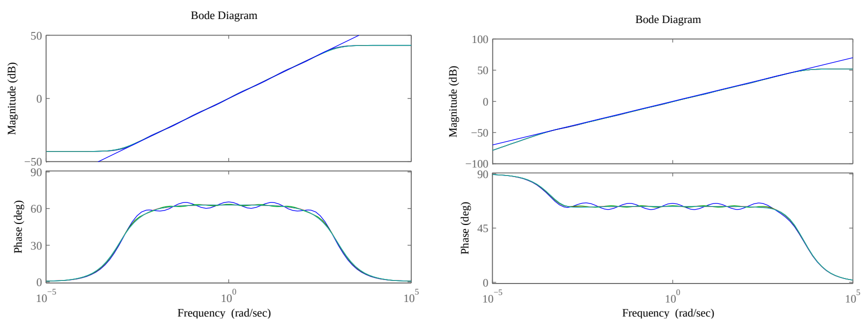


FIGURE 10.11: Solutions of the linear fractional-order equation

15. Find a good approximation for the modified Oustaloup's filter, to $s^{0.7}$ and see which N can best fit the fractional-order differentiator.

SOLUTION Suppose that the frequency range of interest is $(10^{-3}, 10^3)$, different values of N's can be tested for Oustaloup and improved Oustaloup filter, shown respectively in Figures 10.12 (a) and (b). It can be seen that $N = 2$ is not a good choice. If $N \geqslant 3$, the fitting quality is normally acceptable. Also for the same order N, the improved Oustaloup filter appears to be better than the original filter.

```
>> nn=[2:5]; gam=0.7; wm=1e-3; wM=1e3;
   for N=nn, G1=ousta_fod(gam,N,wm,wM); bode(G1); hold on; end
   figure; for N=nn, G1=new_fod(gam,N,wm,wM); bode(G1); hold on; end
```



(a) Oustaloup filter (b) improved Oustaloup filter

FIGURE 10.12: Filter approximation to differentiators

16. Solve the following nonlinear fractional-order differential equation with the block diagram based algorithm

$$\mathscr{D}^2 x(t) + 6\mathscr{D}^{1.455} x(t) + 13\left[\mathscr{D}^{0.555} x(t)\right]^2 + 12x^3(t) = \sin t.$$

SOLUTION The signal $\mathscr{D}^2 x(t)$ is in fact $\ddot{x}(t)$, thus the original equation can be converted to

$$x(t) = \sin t - 6\mathscr{D}^{1.455} x(t) - 13\left[\mathscr{D}^{0.555} x(t)\right]^2 - 12x^3(t).$$

The implementation of the equation can be modeled with Simulink, as shown in Figure 10.13.

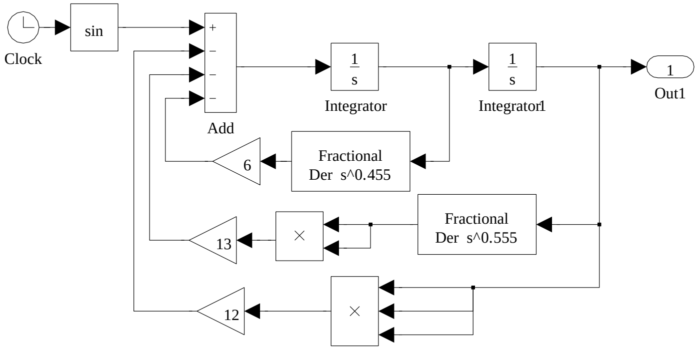


FIGURE 10.13: Simulink model (file: exc10m1.mdl)

Performing simulation to the original block diagram, the solutions of the fractional-order differential equation can be obtained as shown in Figure 10.14. Modifying the control parameters, the solutions are still the same, which means that the solutions are correct.

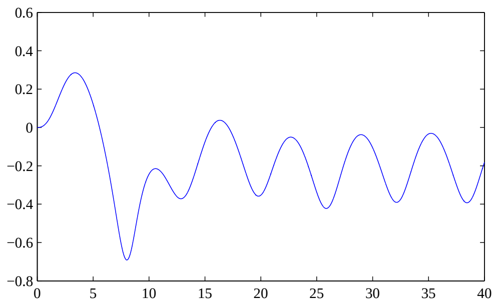


FIGURE 10.14: Solutions of the fractional-order equation

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