Chapter 1

1) Toss a coin, $\{H, T\}$, $\frac{1}{2}$, $\frac{1}{2}$

2) Draw a card, {ace, king, queen, jack, 10, 9, ..., 2 - clubs, same for hearts, diamonds and spades}, $1/52$, $1/52$, ..., $1/52$

3) Toss a die, $\{1, 2, 3, 4, 5, 6\}$, $\frac{1}{6}$, $\frac{1}{6}$, ..., $\frac{1}{6}$

4) RI : max error in 1921 $\approx$ 7 in (about 17 in or 7 in above average)
AZ : max error in 1921 $\approx$ 4 in (about 8.4 in or 4 in above average)
1921 was very rainy year!

5) a) c b) d c) d d) c e) d

6) $1/10$, $1/1000$, $1/1000000$ Prob $\to 0$

7) Yes since $\sum_{k=1}^{\infty} (\frac{1}{2})^k = \frac{1/2}{1-1/2} = 1$

Yes but number of outcomes must be countable and probabilities can't be equal

8) {HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT, TTTT, TTTH, TTHT, TTHH, THTT, THTH, THHT, THHH}

4 outcomes with 3 heads

$p[4] = 4/16 = 1/4$

From (1.1) $p[4] = \binom{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1$

$$= \frac{4!}{1!\,3!}\,\frac{1}{16} = \frac{4}{16} = \frac{1}{4}$$

9) $p[k] = \binom{4}{0}\frac{1}{16} = \frac{1}{16}$ $\quad k=0$

$\binom{4}{1}\frac{1}{16} = \frac{4}{16}$ $\quad k=1$

$\binom{4}{2}\frac{1}{16} = \frac{6}{16}$ $\quad k=2$ ← most probable

$\binom{4}{3}\frac{1}{16} = \frac{4}{16}$ $\quad k=3$

$\binom{4}{4}\frac{1}{16} = \frac{1}{16}$ $\quad k=4$

2 heads is most probable

10) $p[k=9] = \binom{12}{9}\left(\frac{1}{2}\right)^{12} = \frac{12\cdot 11\cdot 10}{3\cdot 2\cdot 1}\left(\frac{1}{2}\right)^{12}$

$= 0.0537$

Probably not a fair coin - note that $p[k=6] = 0.2255$ is maximum over k

11) For $N=1 \Rightarrow a+b = \sum_{k=0}^{1}\binom{1}{k}a^k b^{1-k}$

$$= \binom{1}{0}a^0 b^1 + \binom{1}{1}a^1 b^0 = b+a$$

Now assume true for N and consider $N+1$.

$$\sum_{k=0}^{N+1}\binom{N+1}{k}a^k b^{N+1-k} = \sum_{k=0}^{N}\binom{N+1}{k}a^k b^{N+1-k} + \binom{N+1}{N+1}a^{N+1}$$

$$= \sum_{k=0}^{N} \left[\binom{N}{k} + \binom{N}{k-1} \right] a^k b^{N+1-k} + a^{N+1}$$

$$= b \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k} + \underbrace{\sum_{k=0}^{N} \binom{N}{k-1} a^k b^{N+1-k}} + a^{N+1}$$

$$= \sum_{k=1}^{N} \binom{N}{k-1} a^k b^{N+1-k}$$

let $l = k-1$

$$= \sum_{l=0}^{N-1} \binom{N}{l} a^{l+1} b^{N-l}$$

$$= b \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k} + a \left(\underbrace{\sum_{l=0}^{N-1} \binom{N}{l} a^l b^{N-l} + a^N} \right)$$

$$= \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k}$$

$$= b(a+b)^N + a(a+b)^N = (a+b)^{N+1}.$$

Now $\sum_{k=0}^{N} \binom{N}{k} p^k (1-p)^{N-k} = [p + (1-p)]^N = 1.$

12) Must be 1 since it represents the probability that T is in the interval $(-\infty, \infty)$ or all possible outcomes.

13) $$\int_7^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-7)^2} dt = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

↑ let $u = t-7$

But
$$1 = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \int_0^\infty + \int_{-\infty}^0$$

and by symmetry each integral is the same and must equal $1/2$ $\Rightarrow$

$$\int_7^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(t-7)^2} dt = 1/2.$$

14)

Δ	Integral
0.1	0.9977
0.01	0.9973
0.001	0.9973

```
% probprobl_14.m
%
clear all
Delta=[0.1 0.01 0.001]';
for i=1:3
   n=[round(-3/Delta(i)):round(3/Delta(i))]';
   arg=(1/sqrt(2*pi))*exp(-0.5*(n*Delta(i)).^2)*Delta(i);
   I(i,1)=sum(arg);
end
[Delta I]
```

15) $p[3] = \binom{4}{3}\left(\frac{1}{2}\right)^4 = 0.25$

From MATLAB simulation

$\hat{p}[3] = 0.2490$

```
% probprobl_15.m
%
clear all
rand('state',0)
for k=1:1000
   for i=1:4
      if rand(1,1)<0.5
         x(i,1)=1;
      else
         x(i,1)=0;
      end
   end
   heads(k,1)=sum(x);
end
count=0;
for k=1:1000
   if heads(k)==3
      count=count+1;
   end
end
probheads=count/1000
```

16) $P(3) = \binom{4}{3}(0.75)^3(0.25)^1 = 0.4219$

$\hat{P}(3) = 0.4460$

```
% probprobl_16.m
%
clear all
rand('state',0)
for k=1:1000
   for i=1:4
      if rand(1,1)<0.75 % change this
         x(i,1)=1;
      else
         x(i,1)=0;
      end
   end
   heads(k,1)=sum(x);
end
count=0;
for k=1:1000
   if heads(k)==3
      count=count+1;
   end
end
probheads=count/1000
```

Chapter 2

1) $\hat{P}[Y=0] = 0.7490$

$\hat{P}[Y=1] = 0.2510$

$Y=0$ if we have HT, TH, TT

$Y=1$ if we have HH

$\Rightarrow P[Y=0] = 3/4, \quad P[Y=1] = 1/4$

```
% probprob2_1.m
%
clear all
rand('state',0)
for i=1:1000
   if rand(1,1)<0.5
      x1=0;
   else
      x1=1;
   end
if rand(1,1)<0.5
      x2=0;
   else
      x2=1;
   end
   y(i,1)=x1*x2;
end
p1=sum(y)/1000
p0=1-p1
```

2) $\hat{P} = 0.0290$

True value is $1/36 = 0.0278$

```
% probprob2_2.m
%
clear all
rand('state',0)
count=0;
for i=1:1000
   u1=rand(1,1);
   if u1<=1/6
      x1=1;
   elseif u1>1/6&u1<=2/6
      x1=2;
   elseif u1>2/6&u1<=3/6
      x1=3;
   elseif u1>3/6&u1<=4/6
      x1=4;
   elseif u1>4/6&u1<=5/6
      x1=5;
   else
      x1=6;
   end
u2=rand(1,1);
   if u2<=1/6
      x2=1;
   elseif u2>1/6&u2<=2/6
      x2=2;
   elseif u2>2/6&u2<=3/6
      x2=3;
   elseif u2>3/6&u2<=4/6
      x2=4;
   elseif u2>4/6&u2<=5/6
      x2=5;
   else
      x2=6;
   end
   if x1==1&x2==1
      count=count+1;
   end
end
prob=count/1000
```

3) $\hat{P}[-1 \le x \le 1]$

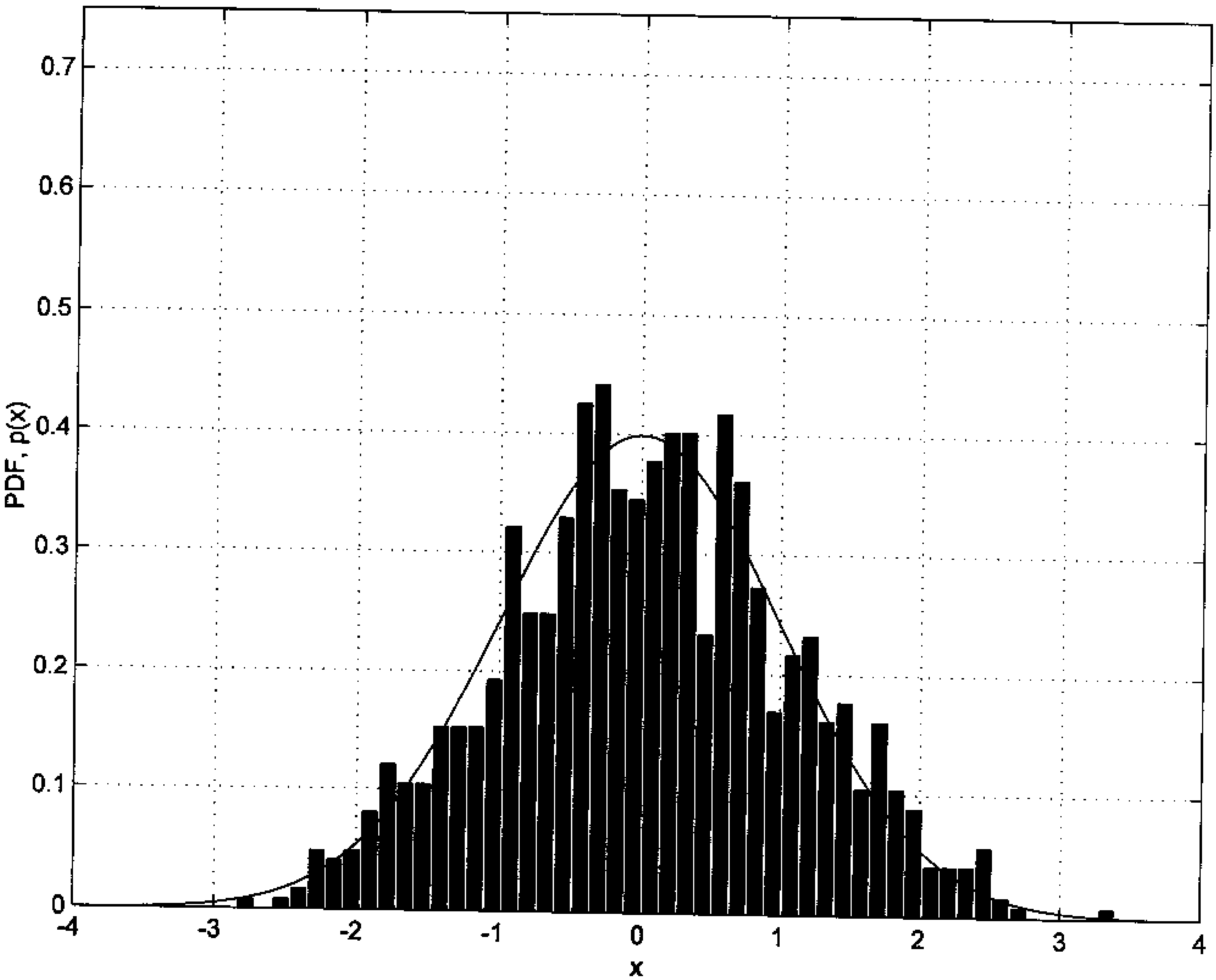
$= 0.6863$

Value from numerical integration

$= 0.6851$

```
% probprob2_3.m
%
clear all
randn('state',0)
count=0;
for i=1:10000
   x=randn(1,1);
   if x>=-1 & x<=1
      count=count+1;
   end
end
probest=count/10000
x=[-1:0.01:1]';
px=(1/sqrt(2*pi))*exp(-0.5*x.^2);
probtrue=sum(px)*0.01
```

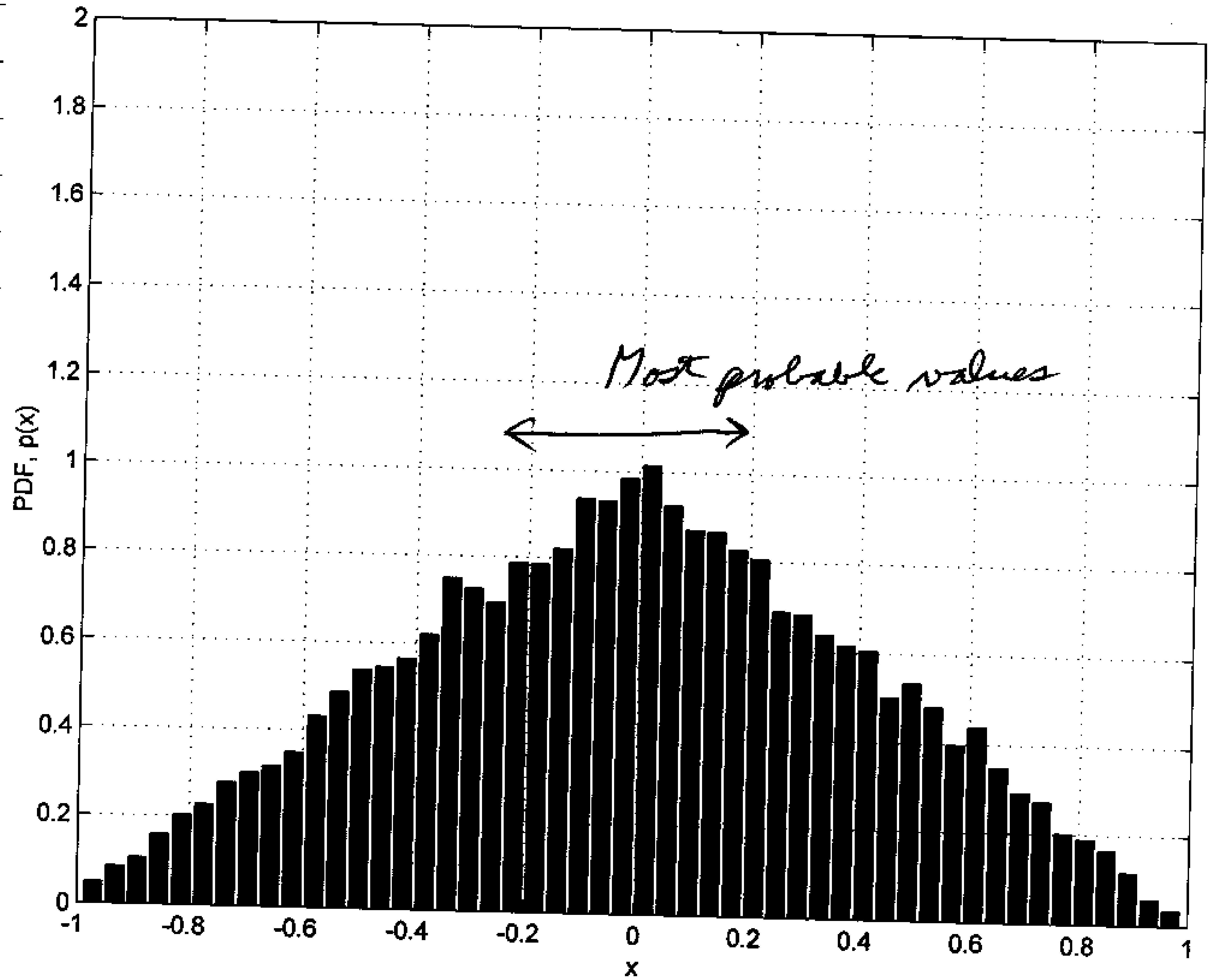
4)




```
% probprob2_4.m
%
clear all
rand('state',0)
x=zeros(1000,1);
for i=1:1000
   for j=1:12
      x(i,1)=x(i,1)+(rand(1,1)-0.5);
   end
end
pdf(x,1000,50,-4,4,0.75)
hold on
xaxis=[-4:0.01:4]';
px=(1/sqrt(2*pi))*exp(-0.5*xaxis.^2);
plot(xaxis,px)
hold off
```

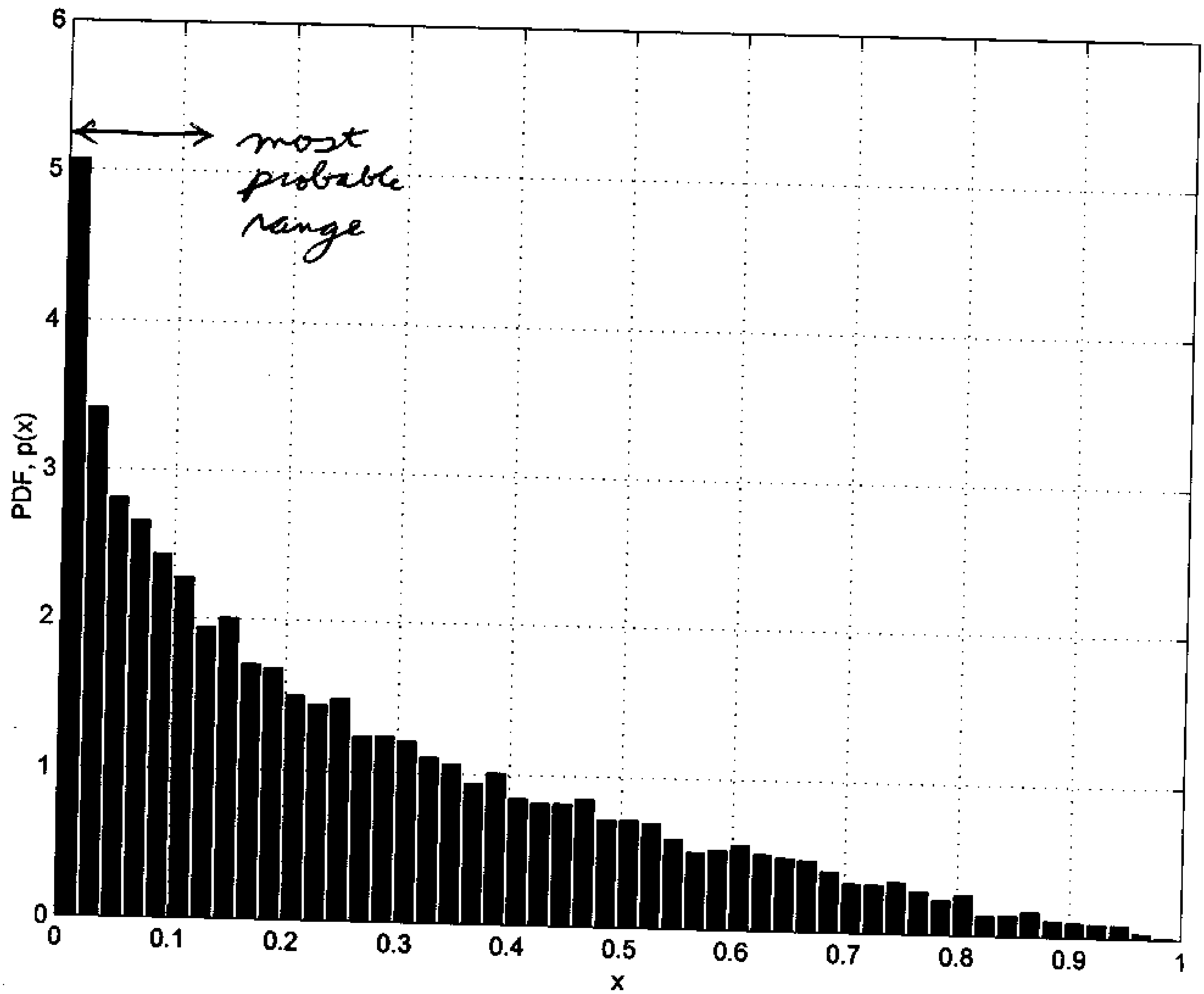
```
% pdf.m
%
%  This function subprogram computes and plots the
%  PDF of a set of data.
%
%  Input parameters:
%
%    x    - Nx1 data array
%    N    - number of data points
%    nbins - number of bins (<N/10)
%    xmin,xmax,ymax - axis scaling
%
 function pdf(x,N,nbins,xmin,xmax,ymax)
 [y,xx]=hist(x(1:N),nbins);
 delx=xx(2)-xx(1);
 bar(xx,y/(N*delx))
 grid
 axis([xmin xmax 0 ymax]);
 xlabel('x')
 ylabel('PDF, p(x)')
```

5)




```
% probprob2_5.m
%
clear all
rand('state',0)
x=zeros(10000,1);
   for i=1:10000
      x(i,1)=rand(1,1)-rand(1,1);
   end
pdf(x,10000,50,-1,1,2)
```

6)




```
% probprob2_6.m
%
clear all
rand('state',0)
x=zeros(10000,1);
   for i=1:10000
      x(i,1)=rand(1,1)*rand(1,1);
   end
pdf(x,10000,50,0,1,6)
```

7) $\hat{p}_1 = 0.1000$

$\hat{p}_2 = 0.1990$

$\hat{p}_3 = 0.7010$

```
% probprob2_7.m
%
clear all
rand('state',0)
x=zeros(1000,1);
for i=1:1000
   u=rand(1,1);
   if u<=0.1
      x(i,1)=1;
   elseif u>0.1 & u<=0.3
      x(i,1)=2;
   else
      x(i,1)=3;
   end
end
count1=0;count2=0;count3=0;
for i=1:1000
   if x(i,1)==1
      count1=count1+1;
   elseif x(i,1)==2
      count2=count2+1;
   else
      count3=count3+1;
   end
end
p1est=count1/1000
p2est=count2/1000
p3est=count3/1000
```

8) $\widehat{\text{Mean}} = 0.5021$

True value = 1/2

```
% probprob2_8.m
%
clear all
rand('state',0)
meanest=0;
for i=1:1000
   meanest=meanest+(1/1000)*rand(1,1);
end
meanest
```

9) $\widehat{\text{Mean}} = 0.9569$

True value = 1

```
% probprob2_9.m
%
clear all
randn('state',0)
meanest=0;
for i=1:1000
   y(i,1)=1+randn(1,1);
   meanest=meanest+(1/1000)*y(i);
end
meanest
```

10) $\hat{Mean} = 1.0022$

True value = 1

```
% probprob2_10.m
%
clear all
randn('state',0)
meanest=0;
for i=1:10000
   y(i,1)=randn(1,1)^2;
   meanest=meanest+(1/10000)*y(i);
end
meanest
```

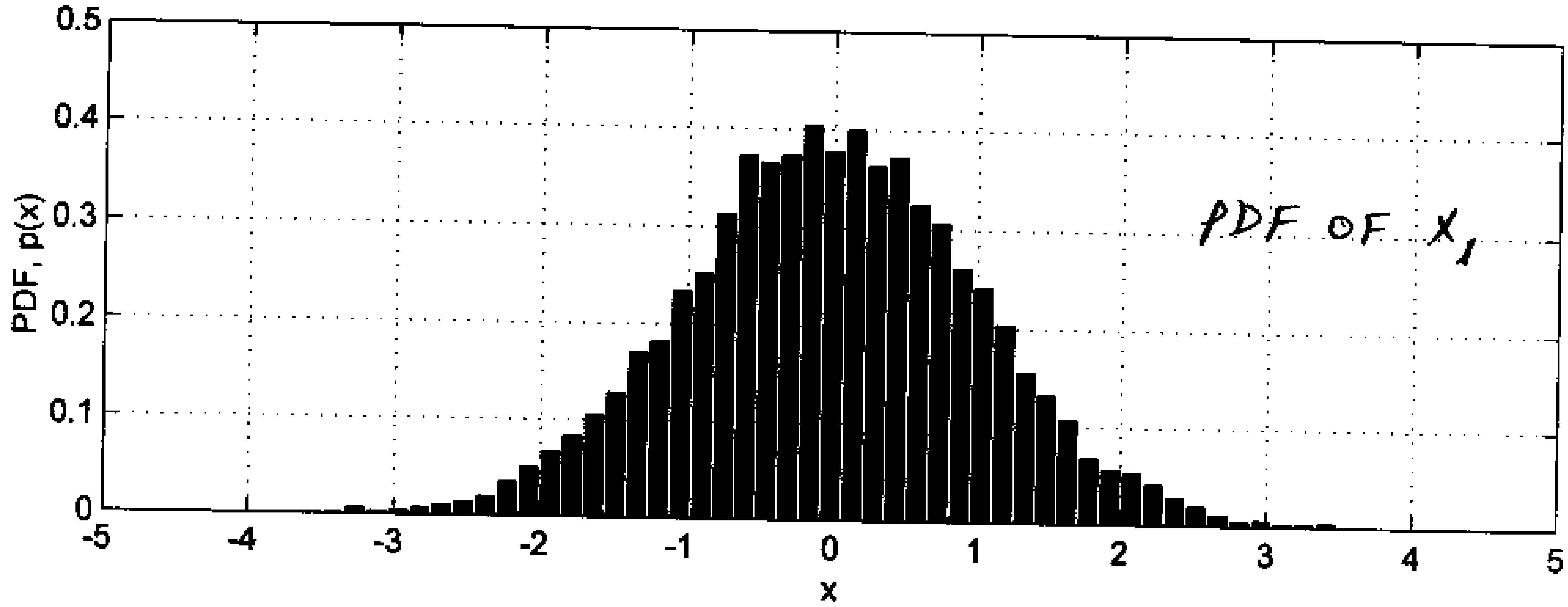
11) $\hat{Mean} = 1.0042$

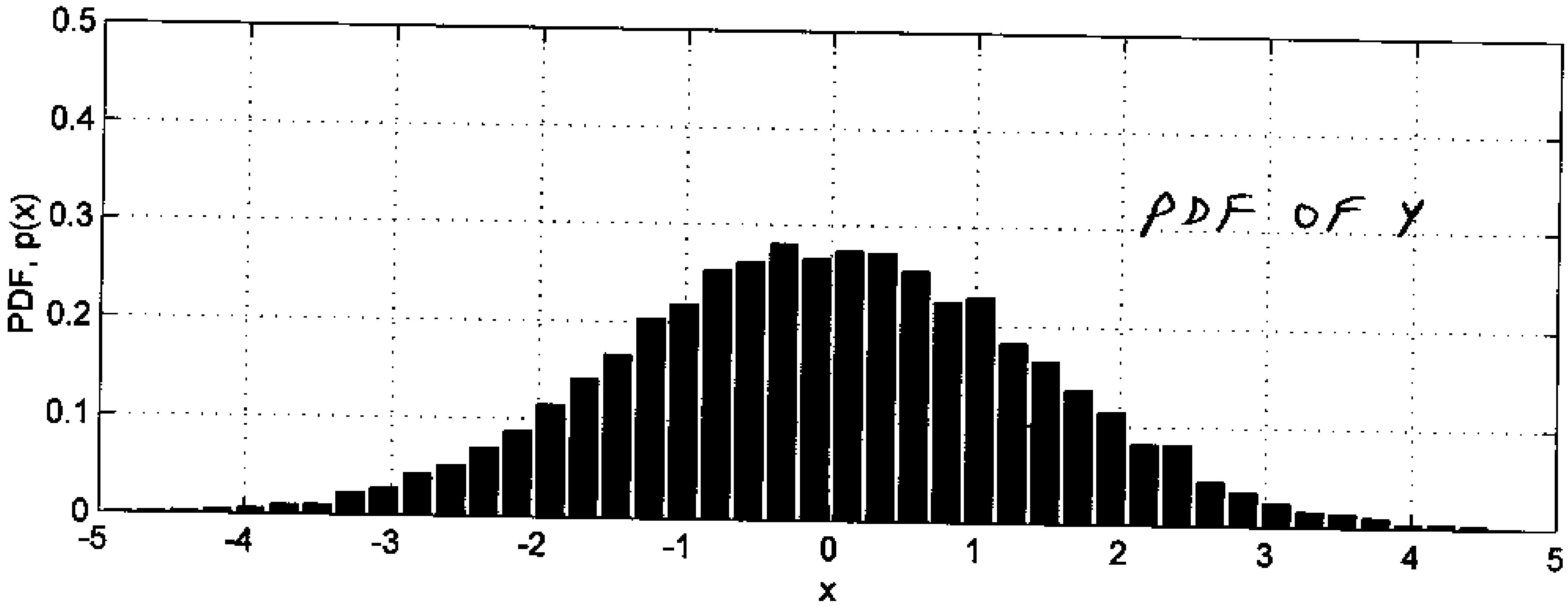
True value = 1

```
% probprob2_11.m
%
clear all
rand('state',0)
meanest=0;
for i=1:1000
   y(i,1)=2*rand(1,1);
   meanest=meanest+(1/1000)*y(i);
end
meanest
```

12) No, values become more spread out
See estimated PDFs on next page.

```
% probprob2_12.m
%
clear all
randn('state',0)
for i=1:10000
   x1(i,1)=randn(1,1);
   x2(i,1)=randn(1,1)-randn(1,1);
end
subplot(2,1,1)
pdf(x1,10000,50,-5,5,0.5)
subplot(2,1,2)
pdf(x2,10000,50,-5,5,0.5)
```





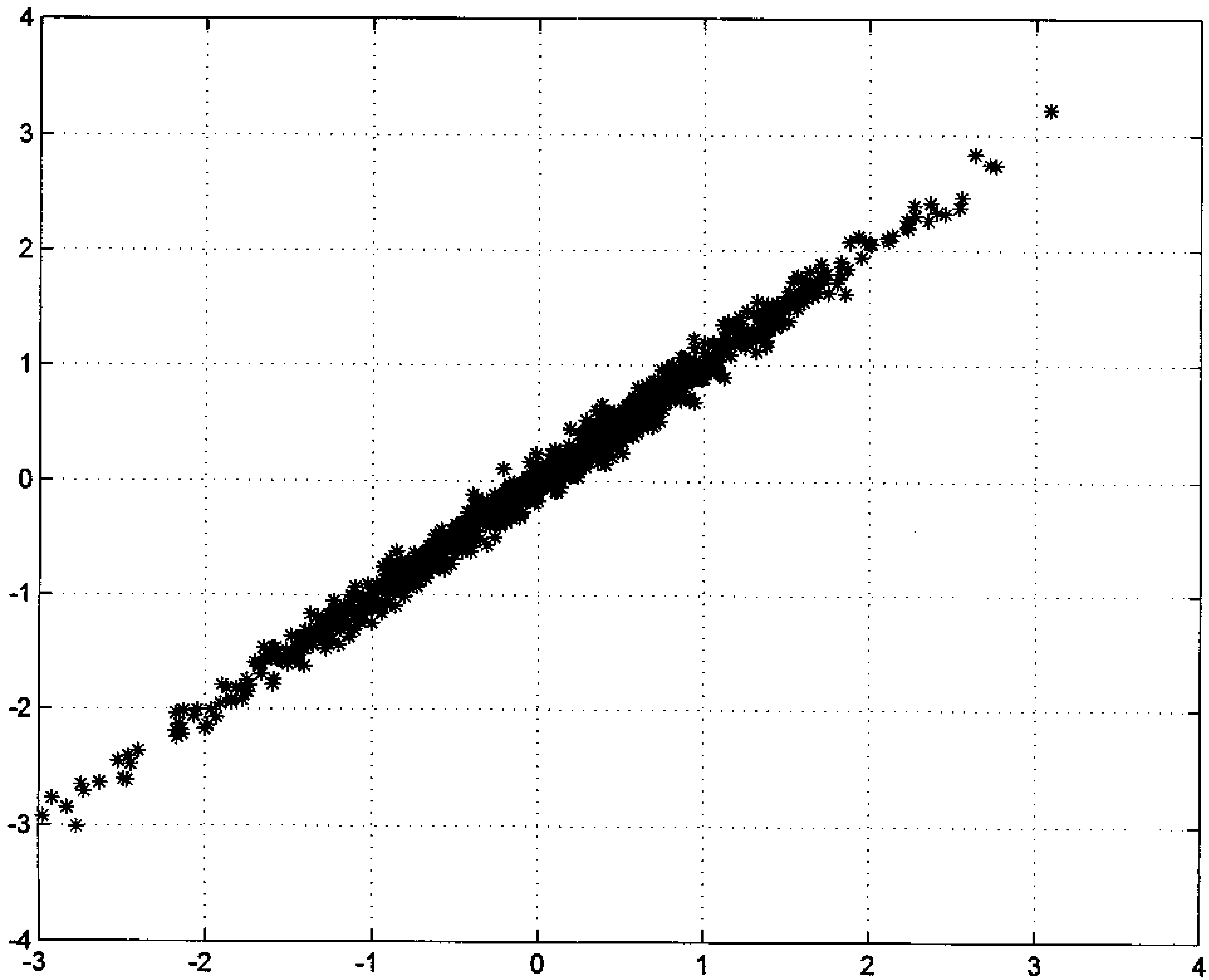
13) Average distance = 1.2381
True value is mean of Rayleigh random variable = $\sqrt{\pi/2}$ = 1.2533

```
% probprob2_13.m
%
clear all
randn('state',0)
meanest=0;
for i=1:1000
   R(i,1)=sqrt(randn(1,1)^2+randn(1,1)^2);
   meanest=meanest+(1/1000)*R(i);
end
  meanest
```

14) No, mean of $U = \frac{1}{2}$ (program gives 0.5077)
$\Rightarrow \sqrt{\text{mean of } U} = \sqrt{1/2} = 0.7071$
(program gives $\sqrt{0.5077} = 0.7125$)
From program, mean of $\sqrt{U} = 0.6589$

```
% probprob2_14.m
%
clear all
rand('state',0)
meanuest=0;meansqrtuest=0;
for i=1:1000
   u(i,1)=rand(1,1);
   sqrtu(i,1)=sqrt(rand(1,1));
   meanuest=meanuest+(1/1000)*u(i);
   meansqrtuest=meansqrtuest+(1/1000)*sqrtu(i);
end
meanuest
meansqrtuest
```

15)

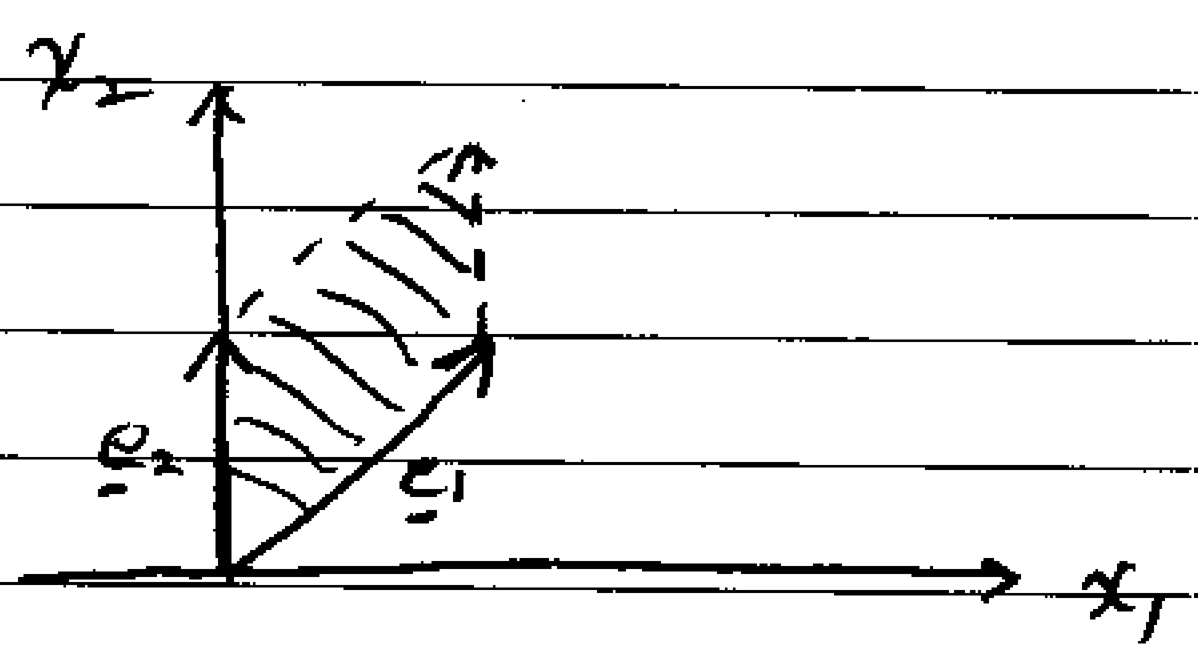


Yes, if $Y_1 = 1$, we could expect $Y_2 = 1$.

```
% probprob2_15.m
%
clear all
randn('state',0)
for i=1:1000
   x1=randn(1,1);x2=randn(1,1);
   y1(i,1)=x1+0.1*x2;
   y2(i,1)=x1+0.2*x2;
   end
   plot(y1,y2,'*')
   grid
```

16) $$\underline{X} = \begin{bmatrix} U_1 \\ U_1 + U_2 \end{bmatrix}$$

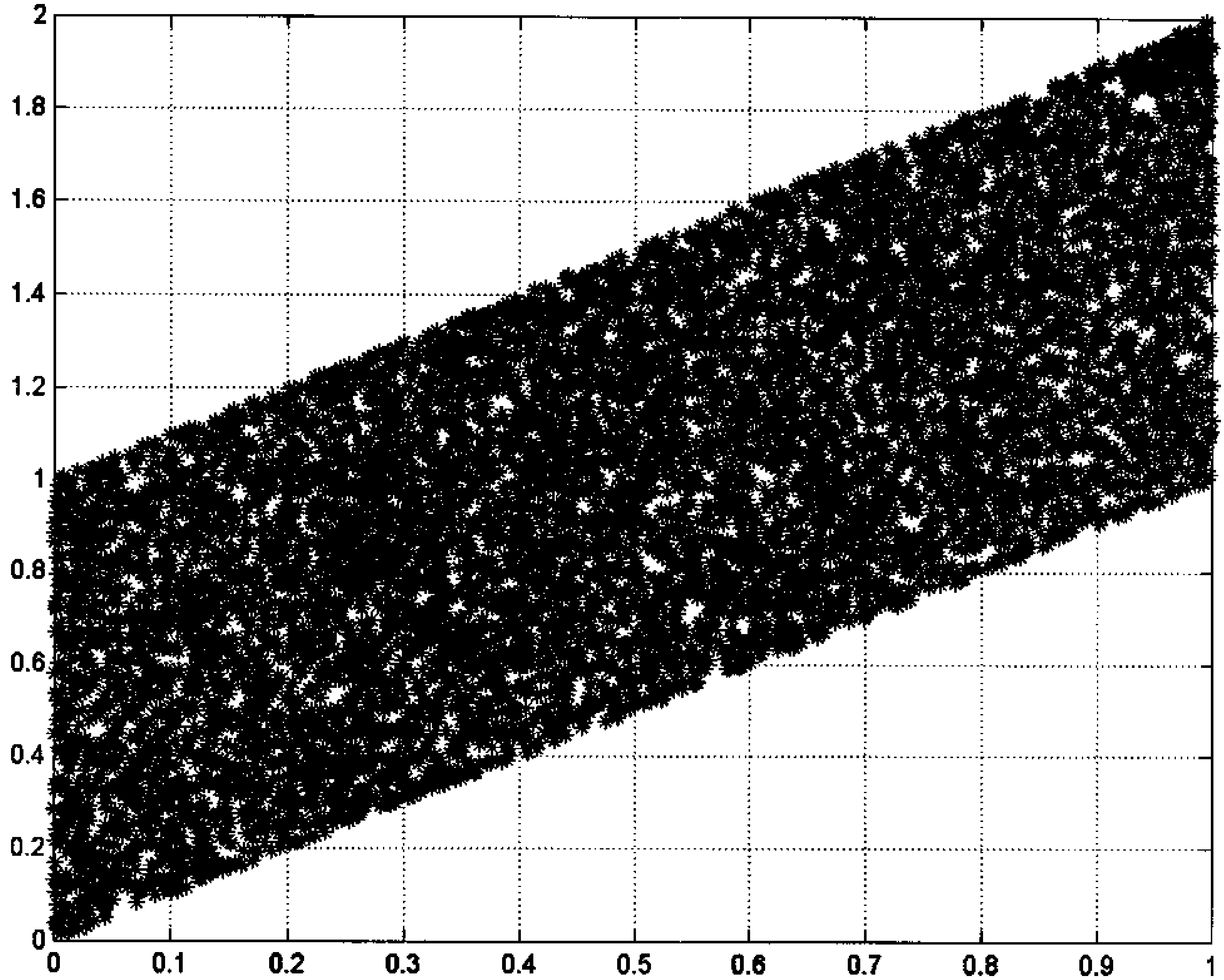
$$= U_1 \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\underline{e}_1} + U_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\underline{e}_2} \qquad \begin{array}{l} 0 < U_1 < 1 \\ 0 < U_2 < 1 \end{array}$$



See next page

All points shown as cross-hatched can be obtained by choosing U_1, U_2 within $(0,1)$.

See next page




```
% probprob2_16.m
%
clear all
rand('state',0)
for i=1:10000
   u1=rand(1,1);u2=rand(1,1);
   x1(i,1)=u1;
   x2(i,1)=u1+u2;
   end
   plot(x1,x2,'*')
   grid
```

Chapter 3

1) a) $A^c = \{x : x \le 1\}$ $B^c = \{x : x > 2\}$

b) $A \cup B = \{x : -\infty < x < \infty\} = S$ $A \cap B = \{x : 1 < x \le 2\}$

c) $A - B = \{x : x > 2\}$ $B - A = \{x : x \le 1\}$

2) a) $A^c = \{x : 0 \le x \le 1\}$ $B^c = \{x : x > 2\}$

b) $A \cup B = \{x : x \ge 0\} = S$ $A \cap B = \{x : 1 < x \le 2\}$

c) $A - B = \{x : x > 2\}$ $B - A = \{x : 0 \le x \le 1\}$

3) a) $A = \{$ John, Susan, Phillip, Fred $\}$

b) $B = \{$ Lisa Ashley Brad $\}$

c) $C = \{$ John Phillip Fred $\}$

d) $D = \{$ Lisa Ashley $\}$

e) $E = \{$ Lisa John Ashley, Phillip, Fred, Brad $\}$

f) $F = \{$ Susan $\}$

A and B

4) $\bigcup_{i=1}^{N} A_i = \{x : 0 \le x \le N\} = A_N$

$\bigcap_{i=1}^{N} A_i = \{x : 0 < x \le 1\} = A_1$

No, since $A_1 \subset A_2 \subset \ldots \subset A_N$

5) If $x \in A \Rightarrow x \ge -1 \Rightarrow 2x + 2 \ge 0 \Rightarrow x \in B$

If $x \in B \Rightarrow 2x + 2 \ge 0 \Rightarrow x \ge -1 \Rightarrow x \in A$

$\Rightarrow A = B$

6) If $x \in A \cap B^c \Rightarrow x \in A$ and $x \in B^c$ since set is the intersection. But if $x \in B^c$ $\Rightarrow x \notin B$. Thus $x \in A$ and $x \notin B$. By definition $x \in A - B$. Thus, $A \cap B^c \subset A - B$

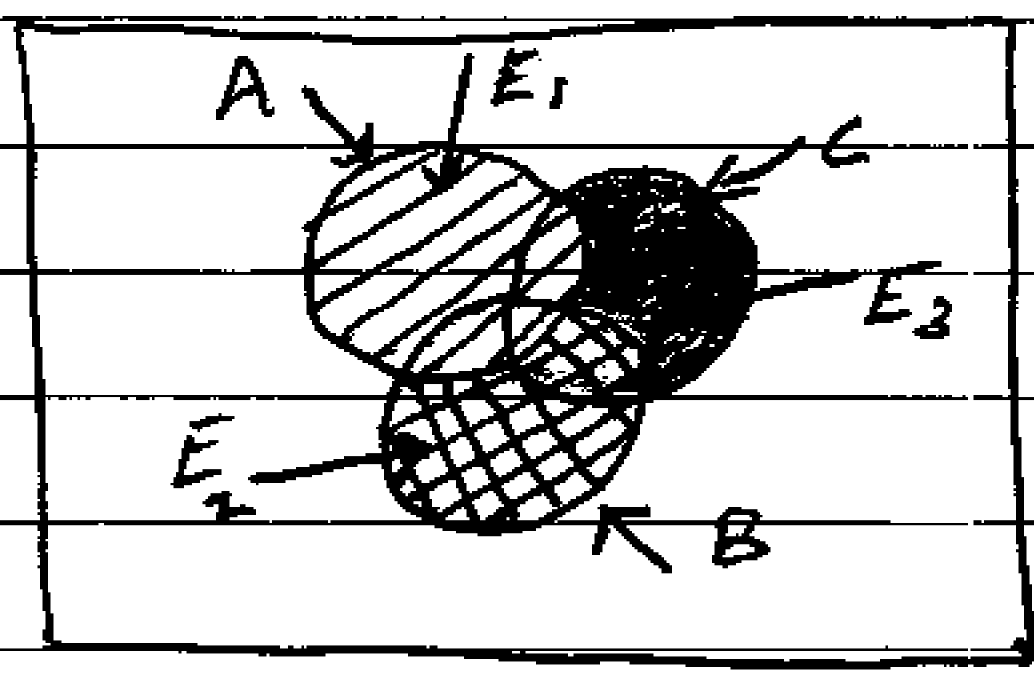
7) $A = \{1, 2, 3\}$ $B = \{4, 5\}$ disjoint
$C = \{1, 2, 3\}$ $D = \{4, 5, 6\}$ partition

$A \cap B = \phi$, $C \cap D = \phi$ and $C \cup D = S$

8) $A = \{(x, y) : 0 \le x \le \frac{1}{2}, \ 0 \le y \le 1\}$
$B = \{(x, y) : \frac{3}{4} \le x \le 1, \ 0 \le y \le 1\}$

$C = \{(x, y) : 0 \le x \le \frac{1}{2}, \ 0 \le y \le 1\}$
$D = \{(x, y) : \frac{1}{2} < x \le 1, \ 0 \le y \le 1\}$

9)

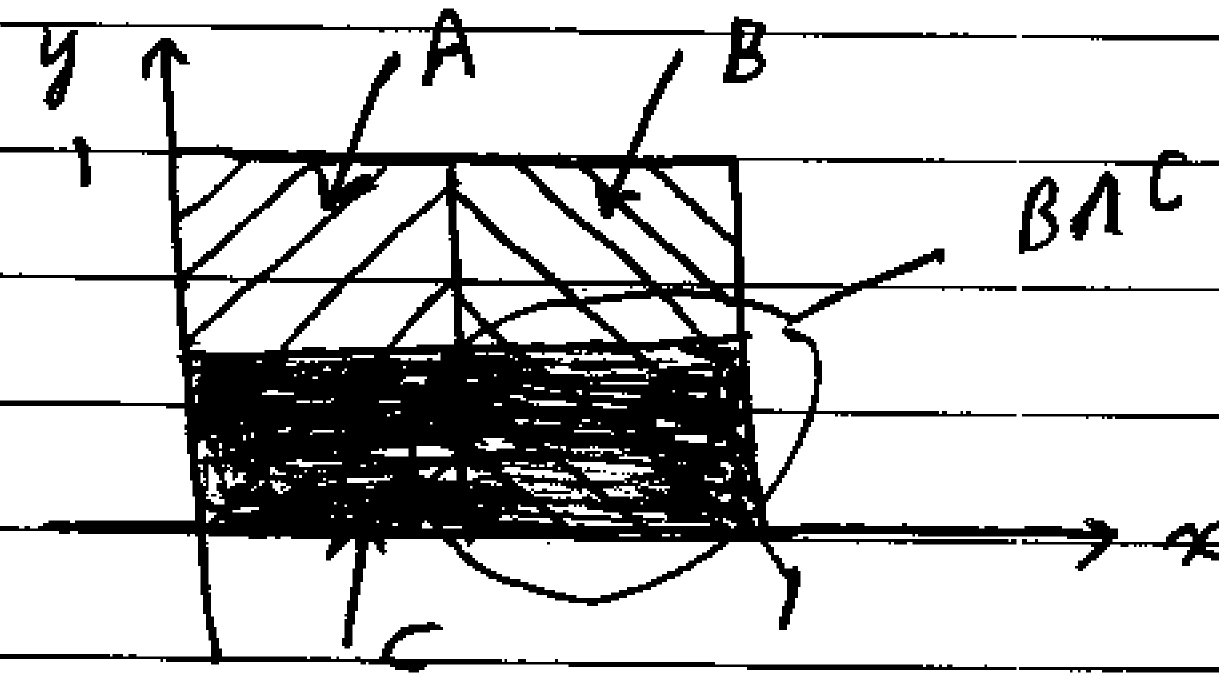


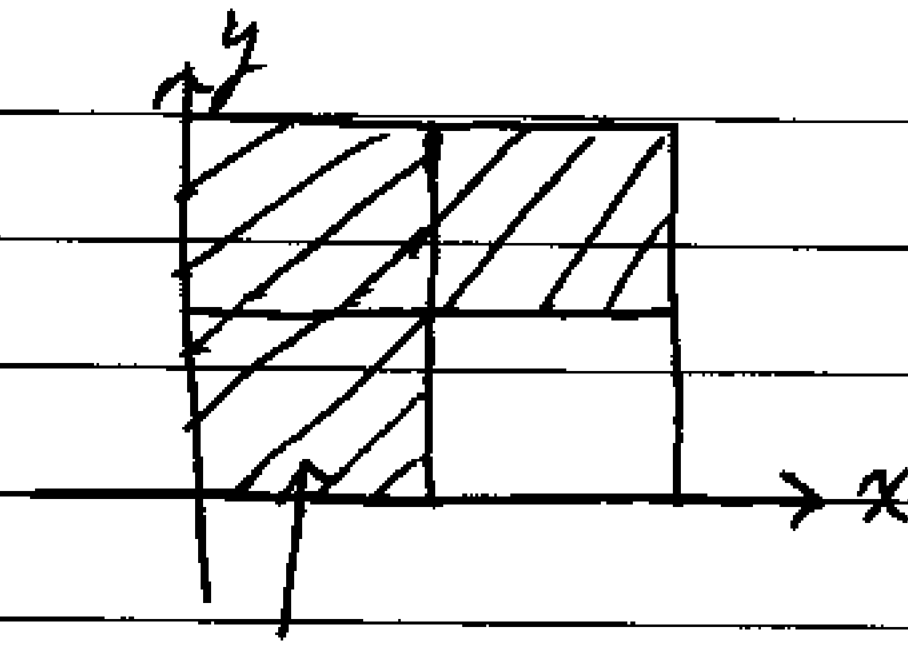
$E_1 = A$
$E_2 = B - E_1$
$E_3 = C - (E_1 \cup E_2)$

$E_1 = A_1$
$$E_k = A_k - \bigcup_{i=1}^{k-1} E_i \qquad k = 2, 3, \ldots, N$$

10) $A \cup B \cup C = (A^c \cap B^c \cap C^c)^c$

$A \cap B \cap C = (A^c \cup B^c \cup C^c)^c$

11) 



By De Morgan's law

$$A \cup (B \cap C)^c = A \cup (B^c \cup C^c)$$

12) a) 10^7, discrete b) 1, discrete
c) ∞ (uncountable), continuous
d) ∞ (uncountable), continuous
e) 2, discrete
f) ∞ (countable), discrete

13) a) $S = \{2, 3, \ldots, 12\}$
b) outcomes are 2, 3, ..., 12
c) impossible event is $\{\ \}$ or ϕ (the set of outcomes having no outcome)
d) $A = \{3\}$, $B = \{3, 6, 7\}$, $C = \{8, 9, 10\}$
A is simple event
e) $A = \{1, 2, 3\}$ $B = \{4, 5, 6\}$

14) a) $S = \{t : 30 \le t \le 100\}$
b) outcomes are all t in the range $30 \le t \le 100$
c) impossible event is $\{\ \}$ or ϕ (the set of outcomes having no outcome)
d) $A = \{t : 40 \le t \le 60\}$ $B = \{t : 40 \le t \le 50$ or $60 \le t \le 70\}$

$c = \{100\}$ simple event

e) $A = \{t : 40 \le t \le 50\}$ $B = \{t : 60 \le t \le 70\}$

15) 0 outcomes $\Rightarrow \binom{N}{0}$
1 outcome $\Rightarrow \binom{N}{1}$
$\vdots$
N outcomes $\Rightarrow \binom{N}{N}$

$$\text{Total} = \sum_{k=0}^{N} \binom{N}{k} = \sum_{k=0}^{N} \binom{N}{k} 1^k 1^{N-k}$$

$$= (1+1)^N = 2^N$$

16) $A = \{$ 2 black balls or 3 black balls $\}$
An element of A is the balls numbered 3 4 5 whose sum ≥ 10 or an element of $\bar{A}$ belongs to B $\Rightarrow$ not mutually exclusive

17) Assume $P[E_1 \cup E_2] = P(E_1) + P(E_2)$ is true or true for $N=2$ (just axiom 3). Next assume axiom 3' true for N (some fixed integer > 2). Then,

$$P\left[\bigcup_{i=1}^{N+1} E_i\right] = P\left[\left(\bigcup_{i=1}^{N} E_i\right) \cup E_{N+1}\right] \quad \text{(associative)}$$

$$= P\left[\bigcup_{i=1}^{N} E_i\right] + P(E_{N+1}) \text{ since}$$

if all E_i's are disjoint, then $\bigcup_{i}^{N} E_i$ and E_{N+1} must be disjoint

$$= \sum_{i=1}^{N} P[E_i] + P(E_{N+1})$$

since we assumed axiom 3' true for N.

$$= \sum_{i=1}^{N+1} P(E_i)$$

$\Rightarrow$ axiom 3' true for $N+1$

Thus, axiom 3' true for $N=2$ since this is axiom 3. But then it must be true for $N+1=3$ (just shown), $\Rightarrow$ true for $3+1=4$, etc $\Rightarrow$ true for all N.

18) a) $P(\text{red}) = \frac{1}{2}$, $P(\text{black}) = \frac{1}{2}$ from (3.10)
where $s_i = i$ $i = 1, 2, \ldots, 36$ and each $P(\{s_i\}) = 1/36$ and
$E_{red} = \{1, 3, 5, \ldots, 35\}$
$E_{black} = \{2, 4, 6, \ldots, 36\}$

b) $E = \{26, 28, 30, 32, 34, 36\} \Rightarrow P(E) = 6/36$

c) $E = \{2, 4, 6, \ldots, 24, 25, 26, 27, \ldots, 36\}$
$P(E) = 24/36$

19) $P[\text{even}] = 1/2$
$\hat{P}(\text{even}) = 0.5080$

```
% probprob3_19.m
%
clear all
rand('state',0)
M=1000;count=0;
for i=1:M
   outcome=floor(6*rand(1,1))+1; % floor rounds down to nearest integer
   if outcome==2 | outcome ==4 | outcome ==6
      count=count+1;
   end
end
probest=count/M
```

20) $P(\text{even}) = P(2 \text{ or } 4 \text{ or } 6)$

$= P(2) + P(4) + P(6) = 3/6 = \frac{1}{2}$

$P(\text{odd}) = 1 - P(\text{even}) = 3/6 = \frac{1}{2}$

$P(\text{even or odd}) = P(S) = 1$

$P(\text{even and odd}) = P(\phi) = 0$

21) Let $P(\text{even}) = 2p$

$P(\text{odd}) = p$

But $P(\text{even}) + P(\text{odd}) = 1 \Rightarrow p = 1/3$

$\Rightarrow P(\text{even}) = 2/3$

$P(\text{odd}) = 1/3$

$P(\text{even or odd}) = P(\text{even}) + P(\text{odd}) = 1$

$P(\text{even and odd}) = P(\phi) = 0$

22) $S = \{A, B, C\}$ $\quad 2^3 = 8$ events

Event	Prob.
ϕ	0
$\{A\}$	1/3
$\{B\}$	1/3
$\{C\}$	1/3
$\{A, B\}$	2/3
$\{A, C\}$	2/3
$\{B, C\}$	2/3
$S = \{A, B, C\}$	1

Probs. do not sum to one since these events do not partition S.

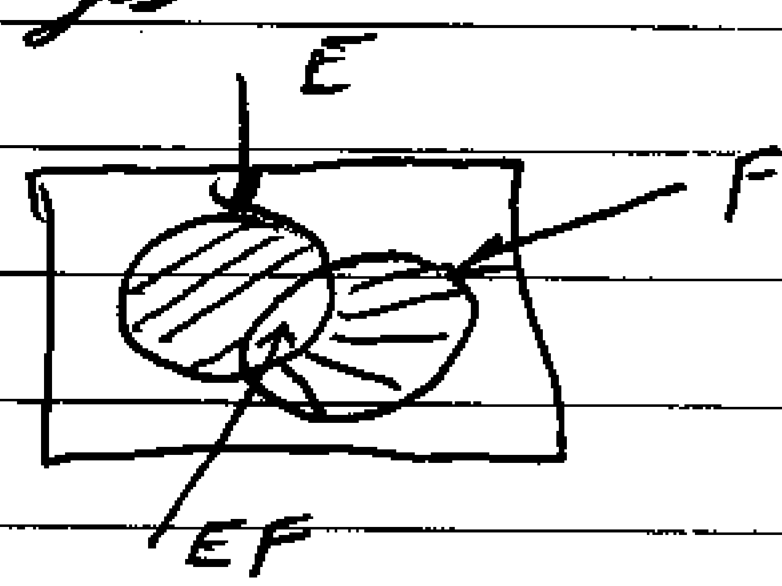
23) $p(i \geq 4) = 1 - p(i<4)$
$= 1 - (p(1) + p(2) + p(3))$
$= 1 - (4/7 + 2/7 + \frac{1}{8})$
$= 1/7 - 1/8 = 1/56$

24) $\sum_{i=0}^{\infty} e^{-2} \frac{2^i}{i!} = e^{-2} \sum_{i=0}^{\infty} \frac{2^i}{i!}$

But $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \sum_{i=0}^{\infty} \frac{2^i}{i!} = e^2$

$\Rightarrow \sum_{i=0}^{\infty} p(i) = 1$ and $p(i) \geq 0$ Yes

25) $E = \{$ die 1 comes up 6 $\}$
$F = \{$ die 2 comes up 6 $\}$
$EF = \{$ both dice come up 6 $\}$



$p($ only one die comes up a 6 $) = p(E) + p(F) - 2p(EF)$
$= 1/6 + 1/6 - 2(1/36) = 10/36 = 5/18$
or just use counting method

26) $p($ circuit closed $) = 1 - p[$ circuit open $]$
$= 1 - p($ all switches fail $)$
$= 1 - p[(f, f, \ldots, f)]$
All simple events equally likely $\Rightarrow$
2^N simple events $\Rightarrow p((f, f, \ldots, f)) = 1/2^N$

$p($ circuit closed $) = 1 - 2^{-N}$

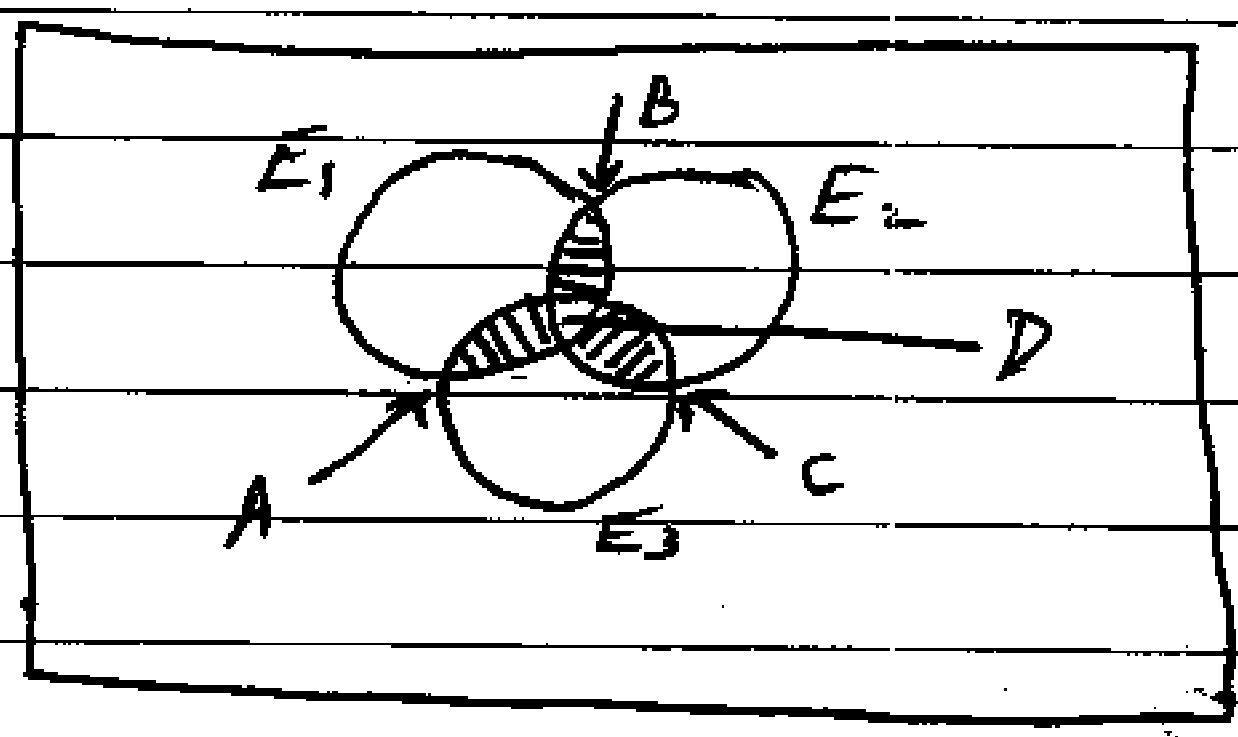
27) $p[$ series circuit closed $] = p^2$
$p[$ parallel circuit closed $] = p[$ switch 1 or switch 2 or both close $] =$

p(switch 1 closes) + p(switch 2 closes)
 − p(both switches close) = $p + p - p^2$
 $= 2p - p^2$ since this is

$p(A \cup B) = p[A] + p[B] - p[AB]$

Is $p^2 > 2p - p^2$?
 $\Rightarrow p^2 > p \Rightarrow p(p-1) > 0$ No
Both perform the same if $p = 0$ or $p = 1$, however.

28)



$p[E_1] + p(E_2) + p(E_3) \Rightarrow$ count A twice
count B twice
count C twice
count D 3 times

$\Rightarrow$ subtract out $p(E_2E_3) + p(E_1E_2) + p(E_1E_3)$
This also subtracts out D 3 times, $\Rightarrow$
add $p(E_1E_2E_3)$ back in.

29) $p(E_1E_2) + p(E_1E_3) + p(E_2E_3)$
 $\geq p(E_1E_2)$ since $p(\ ) \geq 0$
 $\geq p(E_1E_2E_3)$ monotonicity

30) $P[8:10\,AM] = 0$

since $S = \{t : 8:00 \le t \le 8:10\}$
which is continuous sample space

From (3.23)
$P[E] = \frac{5}{20} = \frac{1}{4}$

31) simple events are $\{x\}$ where x satisfies $0 < x < 1$. No, number of simple events is uncountably infinite.

32) Using enumeration of outcomes
$(i,i) \Rightarrow 6 \qquad P[(i,i)] = 6/36$
$(i,j) \quad i \ne j \Rightarrow 30 = 6 \cdot 5 \qquad P[(i,j), i \ne j] = \frac{30}{36}$

33) The event of interest is (i,i,j) for $i \ne j$ and can have in any order

$P[E] = N_E / N_S = N_E / 6^3$
To count outcomes
1) choose value for $i \Rightarrow 6$
2) choose value for $j \Rightarrow 5 \quad (j \ne i)$
3) choose place for $j \Rightarrow 3$

$N_E = 6 \cdot 5 \cdot 3 = 90 \qquad P[E] = 90/216$
You can enumerate all 3-tuples using MATLAB to check or try
$P[(i,i,j)] = 1 - P[(i,i,i)] - P[(i,j,k)]$

34) Want $P[(R,B) \text{ or } (B,R)]$

R is numbered 1, 2, 3, 4
B " " 5, 6

$N_S = 6 \cdot 5$ Sampling without replacement

For $(R,B) \Rightarrow 4 \cdot 2$
For $(B,R) \Rightarrow 2 \cdot 4$

$P(\text{one red and one black}) = \frac{8+8}{30} = 16/30$

```
% probprob3_34.m
%
clear all
count=0;
E=[];
for i=1:6
   for j=1:6
      if (i<=4 &j>=5)|(i>=5&j<=4)
         E=[E; [i j]];
         count=count+1;
      end
   end
end
count
```

```
» E

E =
```

(R,B)	1	5
	1	6
	2	5
	2	6
	3	5
	3	6
	4	5
	4	6
(B,R)	5	1
	5	2
	5	3
	5	4
	6	1
	6	2
	6	3
	6	4

35) $N_S = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$
$= 676{,}000$

36) $N_S = 26^4$

A =

N	$\ln N!$	$\ln \hat{N!}$
10.0000	15.1044	15.0961
20.0000	42.3356	42.3315
30.0000	74.6582	74.6555
40.0000	110.3206	110.3186
50.0000	148.4778	148.4761
60.0000	188.6282	188.6268
70.0000	230.4390	230.4379
80.0000	273.6731	273.6721
90.0000	318.1526	318.1517
100.0000	363.7394	363.7385

```
% probprob3_37.m
%
clear all
A=[];
for N=1:100
   xi(N,1)=(sqrt(2*pi)*(N^(N+0.5)*exp(-N)));
   if mod(N,10)==0
      A=[A;N log(prod(1:N)) log(xi(N))];
      end
end
A
```

Can't calculate 200! using $\prod_{i=1}^{200} i$ on a computer but could use

$x = \sum_{i=1}^{200} \ln i$ and represent as e^x.

38) $P[2 \text{ or more}] = 1 - P[0] - P[1]$

$$= 1 - \frac{364^{23}}{365^{23}} - \frac{23 \cdot 364^{22}}{365^{23}} = 0.00183$$

since $P[1] = P[1$ birthday on Jan 1$]$ found by choosing which student has Jan 1 birthday and then others have 364 possible birthdays

39) $\hat{P} = 0.0020$

```
% probprob3_39.m
%
clear all
rand('state',0)
BD=[0:365]';
event=zeros(100000,1);% initialize to no successful events
for ntrial=1:100000
for i=1:23
   x(i,1)=ceil(365*rand(1,1));% chooses birthdays at random
                              % (ceil rounds up to nearest integer)
 end
 y=sort(x);
 if y(1)==1 & y(2)==1
    event(ntrial)=1;% event occurs if two or more birthdays on Jan. 1
 end
end
prob=sum(event)/100000
```

40) Total number

$\binom{4}{0}$ no toppings

$\binom{4}{1}$ 1 "

$\binom{4}{2}$ 2 "

$\binom{4}{3}$ 3 "

$\binom{4}{4}$ 4 "

$$\sum_{k=0}^{4}\binom{4}{k} = 2^4 = 16$$

Could also solve by considering each pizza designated as $(1, 0, 0, 1) \Rightarrow$ Toppings 1 and 4

topping 1 topping 4

only. Thus, $2^4 = 16$ possible 4-tuples.

For 2 toppings $\binom{4}{2} = 6$

41) $\binom{5}{3} = \frac{5!}{2!3!} = \frac{5 \cdot 4}{2} = 10$

42) $\binom{4}{2} = 6$ Combinations

$p, n = 6$
$p, d = 11$
$p, q = 26$
$n, d = 15$
$n, q = 30$
$d, q = 35$

43) $(a+b)^3 = (a+b)(a^2+2ab+b^2)$

$= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$

$= b^3 + 3ab^2 + 3a^2b + a^3$

$= \binom{3}{0}b^3 + \binom{3}{1}ab^2 + \binom{3}{2}a^2b + \binom{3}{3}a^3$

$(a+b)^4 = (a+b)(a^3+3a^2b+3ab^2+b^3)$

$= a^4 + 3a^3b + 3a^2b^2 + ab^3$
$+ a^3b + 3a^2b^2 + 3ab^3 + b^4$

$= b^4 + 4ab^3 + 6a^2b^2 + 4$

$= \binom{4}{0}b^4 + \binom{4}{1}ab^3 + \binom{4}{2}a^2b^2$
$+ \binom{4}{3}a^3b + \binom{4}{4}a^4$

44) $P(4 \text{ of kind}) = \dfrac{13 \cdot 48}{\binom{52}{5}} = 2.4 \times 10^{-4}$

choose value

choose remaining card

$P(\text{flush}) = \dfrac{4 \cdot \binom{13}{5}}{\binom{52}{5}}$

choose suit

choose cards

$= 2.0 \times 10^{-3}$

45) Use hypergeometric law with $N = 30$, $M = 5$

$N_R \to N_S$ sophomores

$N_B \to N_F$ freshmen

$$P(k=5) = \frac{\binom{10}{5}\binom{20}{0}}{\binom{30}{5}} = \frac{\frac{10!}{(5!)^2}}{\frac{30!}{25!\,5!}}$$

$$= \frac{25!\,10!}{30!\,5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{30\cdot 29\cdot 28\cdot 27\cdot 26} = 0.0018$$

46) Can use binomial law since $N \to \infty$.

$M = 2$, $k = 1$

$$P(k=1) = \binom{2}{1} p^1(1-p)^1 = 2p(1-p)$$

$$\frac{dP}{dp} = 2 - 4p = 0 \Rightarrow p = \tfrac{1}{2}$$

47) Each draw produces a head or tail with $P(\text{head}) = P(\text{tail}) = \frac{1}{2}$ since the proportions are equal. This is the same as if the urn contained an infinite number of red balls (heads) and black balls (tails). Each outcome is an M-tuple of R's and B's or of H's and T's. If each outcome of M successive coin tosses is equally likely, then this is equivalent to choosing M balls at random.

48) $p_B[k] = \binom{100}{k}(0.94)^k (0.06)^{100-k}$

$k = 95, \ldots, 100$

$$p_{HG}[k] = \frac{\binom{N_R}{k}\binom{N_B}{M-k}}{\binom{N}{M}}$$

where $p = N_R/N \Rightarrow N_R = Np = 940$

$N_B = N - N_R = 60$

$M = 100$

$$p_{HG}[k] = \frac{\binom{940}{k}\binom{60}{100-k}}{\binom{1000}{100}}$$

$$= \frac{\dfrac{940!}{(940-k)!\,k!}\;\dfrac{60!}{(k-40)!\,(100-k)!}}{\dfrac{1000!}{900!\,100!}}$$

$$= \frac{940!\,60!\,900!\,100!}{(940-k)!\,k!\,(k-40)!\,(100-k)!\,1000!}$$

Use ln $p_{HG}[k]$ and convert back

```
» probprob3_48

ans =
```

k	BINOMIAL	HYPERGEO.
95	0.1639	0.1696
96	0.1338	0.1333
97	0.0864	0.0814
98	0.0414	0.0362
99	0.0131	0.0104
100	0.0021	0.0015

```
% probprob3_48.m
%
clear all
PB=zeros(100,1);
for k=95:100
   c=prod(k+1:100)/prod(1:100-k);
   PB(k,1)=c*(0.94^k)*(0.06^(100-k));
   x=lnfact(940)+lnfact(60)+lnfact(900)+lnfact(100)-lnfact(940-k)-...
      lnfact(k)-lnfact(k-40)-lnfact(100-k)-lnfact(1000);
   PHG(k,1)=exp(x);
end
  [PB(95:100) PHG(95:100)]
```

```
% lnfact.m
%
function lnf=lnfact(N) % N must be zero or positive integer
x=[1:N]';
if N>0
   lnf=sum(log(x));
elseif N==0
   lnf=0;
end
```

Note: assumptions for binomial probability are only partially satisfied since

$N >> M \qquad 1000 >> 100$

$N_R >> M \qquad 940 >> 100$

$N_B >> M \qquad 60 >> 100$ No good

but we are only computing $\binom{N_B}{M-k}$ for $k = 95, \ldots, 100 \Rightarrow$ actually require

$N_B >> M-k$

$60 >> 100-k = 0, 1, 2, 3, 4, 5$

and not $N_R >> M$ unless we want $k=0, 1, \ldots, M$

49) For sampling with replacement the binomial law is exact.

$$P[k \geq 95] = \sum_{k=95}^{100} \binom{100}{k} (0.94)^k (0.06)^{100-k}$$

$$= 0.4407$$

$$\hat{P}[k \geq 95] = 0.4430$$

```
% probprob3_49.m
%
clear all
rand('state',0)
ntrials=1000;
for n=1:ntrials
   good=0;
   for i=1:100
      k=ceil(1000*rand(1,1)); %choose chip at random, good chips are
      % numbered 1,2,...,940 and bad chips are numbered 941,...,1000
      if k<=940
         good=good+1;
      end
   end
   if good>=95
      accept(n,1)=1;
   end
   end
   probaccept=mean(accept)
   PBtrue=0;
   for k=95:100
   c=prod(k+1:100)/prod(1:100-k);
   PBtrue=PBtrue+c*(0.94^k)*(0.06^(100-k));
end
PBtrue
```

50) Probability of rejecting a <u>good</u> batch =
1 - Prob of accepting a <u>good</u> batch

$= 1 - P[k \geq 95]$ strategy 1

$1 - P[k \geq 98]$ strategy 2

Both computations use $p = 0.95$

$\hat{P}$[reject good batch] = 0.3940 strategy 1
0.8830 strategy 2

To accept a good batch we require 95 out of 100 successes for strategy 1 but 98 out of 100 successes for strategy 2. Therefore it is much more likely that strategy 2 will produce a rejection.

Figures 3.10 3.11 give acceptance probs vs p. For good batch $p = 0.95$

$\Rightarrow p$[accept] = 0.61 from Fig. 3.10
= 0.12 from Fig 3.11

$\Rightarrow p$[reject] = 1 − p[accept]
= 0.39 strategy 1
0.88 strategy 2

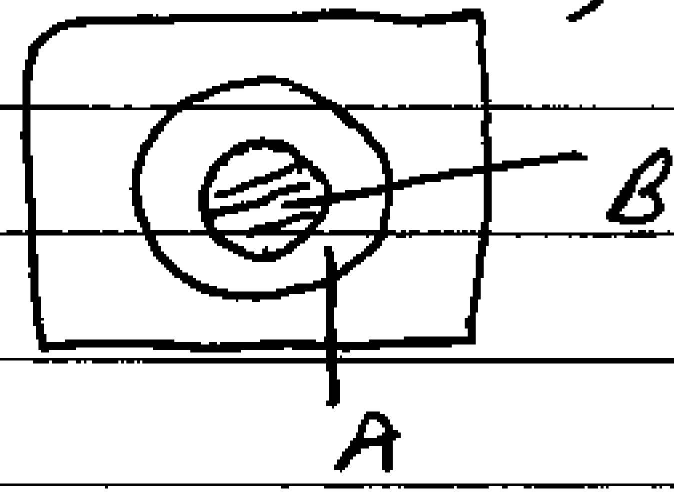
same as above from simulation

```
% probprob3_50.m
%
clear all
rand('state',0)
ntrials=1000;
threshold=95;
%threshold=98;
for n=1:ntrials
   good=0;
   for i=1:100
      k=ceil(1000*rand(1,1)); %choose chip at random, good chips are
      % numbered 1,2,...,950 and bad chips are numbered 951,...,1000
      if k<=950
         good=good+1;
      end
   end
   if good>=threshold
      accept(n,1)=1;
   end
   end
probreject=1-mean(accept)
```

Chapter 4

1) $P[A \mid B] = P(AB)/P(B) = P(B)/P(B) = 1$

If B occurs, then since $B \subset A$, A must also occur.



2) $$P[X \geq 7/8 \mid X \geq 1/2] = \frac{P[X \geq 7/8 \cap X \geq \frac{1}{2}]}{P[X \geq \frac{1}{2}]}$$

$$= P[X \geq 7/8]/P[X \geq \tfrac{1}{2}] = \frac{1/8}{1/2} = \frac{1}{4}$$

3) 1) Reduced sample space

$= S' = \{(H,H,T), (H,T,H), (T,H,H)\}$

$P[(H,T,H)] = 1/3$

2) $$P[(H,T,H) \mid 2 \text{ heads}] = \frac{P[(H,T,H) \cap 2 \text{ heads}]}{P[2 \text{ heads}]}$$

$$= P[(H,T,H)]/P[\{(H,H,T),(H,T,H),(T,H,H)\}]$$

$$= \frac{1/8}{3/8} = 1/3$$

4) $P[\text{die } 1 = \text{die } 2 \mid \text{sum is even}] =$

$P[(1,1) \cup (2,2) \cup \ldots \cup (6,6) \mid \text{sum is even}]$

$= P[(1,1) \cup \ldots \cup (6,6)]/P[\text{sum is even}]$

$$= \frac{6/36}{1/2} = \frac{1}{3}$$

To verify $P[\text{sum is even}] = \frac{1}{2}$ just list out all 2-tuples and count.

5) $$P[B_2 | B_1] = \frac{P[(B,B)]}{P[\underbrace{(B,B) \cup (B,R)}_{\text{mutually exclusive}}]}$$

$$= \frac{\frac{2\cdot 1}{5\cdot 4}}{\frac{2\cdot 1}{5\cdot 4} + \frac{2\cdot 3}{5\cdot 4}} = \frac{2}{8} = \frac{1}{4}$$

6) $P[11 \text{ heads} \,|\, \text{first } 10 \text{ heads}] =$

$$\frac{P[\text{first } 10 \text{ heads} \cap 11 \text{ heads}]}{P[\text{first } 10 \text{ heads}]}$$

$$= \frac{P[11 \text{ heads}]}{P[\underbrace{(H,\ldots,H,H) \cup (H,H,\ldots,T)}_{\text{mutually exclusive}}]}$$

$$= \frac{1/2^{11}}{1/2^{11} + 1/2^{11}} = \frac{1}{2}$$

7) $P[W > 190 \,|\, H > 5'8''] =$

$$\frac{P[W > 190, H > 5'8'']}{P[H > 5'8'']}$$

$$= \frac{\sum_{i=3}^{5}\sum_{j=4}^{5} P((H_i, W_j))}{\sum_{i=3}^{5}\sum_{j=1}^{5} P((H_i, W_j))} \leftarrow = \sum_{i=3}^{5} P(H_i)$$

$$= \frac{(0.06+0)+(0.10+0.04)+(0.08+0.04)}{0.26+0.22+0.12}$$

$$P[W > 190] = \sum_{i=1}^{5}\sum_{j=4}^{5} P\{(H_i, W_j)\} = 0.34$$

$$= \frac{0.32}{0.60} = \frac{8}{15} = 0.53$$

8) $P[W < 160 / H > 5'4''] =$

$$\frac{P[H > 5'4'', W < 160]}{P[H > 5'4'']} = \frac{\sum_{i=2}^{5}\sum_{j=1}^{2} P[(H_i, W_j)]}{\sum_{i=2}^{5}\sum_{j=1}^{5} P[(H_i, W_j)]}$$

$$= \frac{(0.06+0.12)+(0+0.06)+(0+0.02)+(0+0)}{0.26+0.26+0.22+0.12}$$

$$= \frac{0.26}{0.86} = \frac{13}{43}$$

$$P[W < 160 / H < 5'4''] = \frac{\sum_{i=1}^{1}\sum_{j=1}^{2} P((H_i, W_j))}{\sum_{i=1}^{1}\sum_{j=1}^{5} P((H_i, W_j))}$$

$$= \frac{0.08+0.04}{0.14} = \frac{0.12}{0.14} = 6/7$$

Not related - see next problem

9) From Fig 4.2a

$P(A|B) = 2/3$

$P(A|B^c) = 0$ since $A \subset B$

$\Rightarrow P(A|B) + P(A|B^c) = 2/3$

10) $P[A \cup B | C] = \dfrac{P((A \cup B) \cap C)}{P(C)}$

$= \dfrac{P[(A \cap C) \cup (B \cap C)]}{P(C)}$ distributive

$= \dfrac{P(AC \cup BC)}{P(B)} = \dfrac{P(AC) + P(BC) - P(AC \cap BC)}{P(C)}$

But $AC \cap BC = (A \cap C) \cap (B \cap C)$

$= A \cap C \cap B \cap C = A \cap B \cap C = ABC$

(associative)

$= \dfrac{P(AC) + P(BC) - P(ABC)}{P(C)}$

$= P(A|C) + P(B|C) - P(AB|C)$

11) $P(\text{cure}) = P(\text{cure} | \text{control}) P(\text{control})$
$+ P(\text{cure} | \text{drug}) P(\text{drug})$

$= (0.2)(\tfrac{1}{2}) + (0.8)(\tfrac{1}{2}) = 0.5$

12) $P\{\text{on time}\} = P[\text{on time} \mid \text{new bus}) P(\text{new bus})$
$\quad + P(\text{on time} \mid \text{old bus}) P(\text{old bus})$
$= (2/3)(4/7) + (\tfrac{1}{3})(3/7) = 11/21$

13) $$Pe_1 = \sum_{i=-1}^{1} P\{\text{error} \mid i) P(i)$$

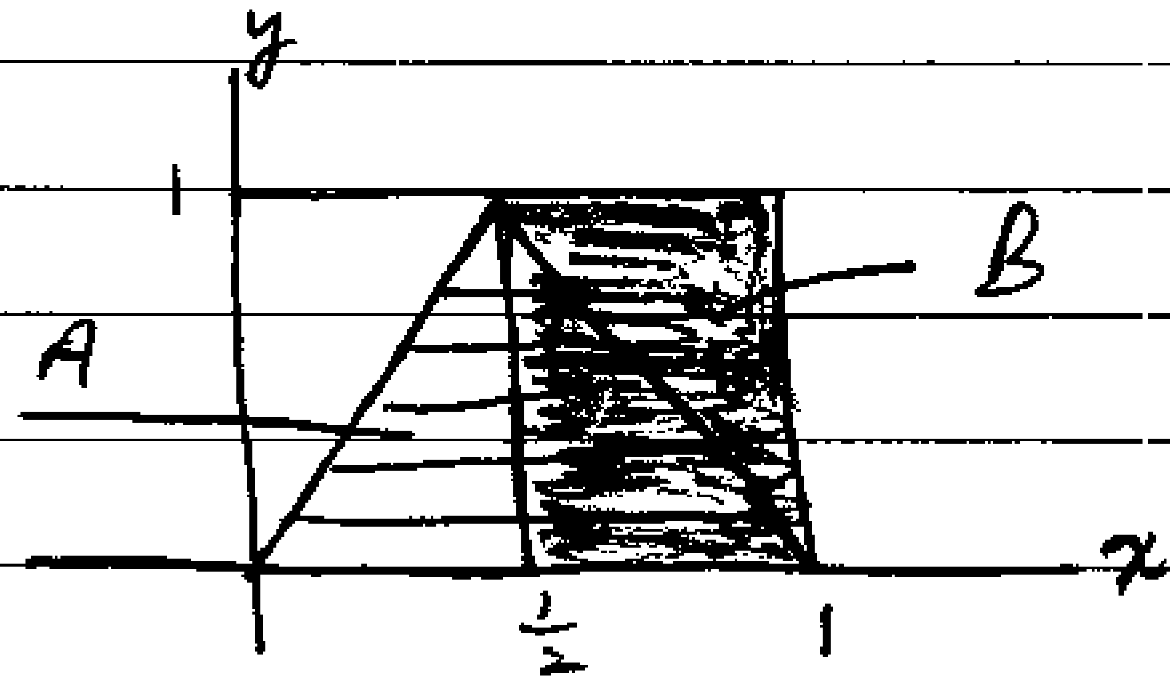
$$= (\tfrac{1}{8})(\tfrac{1}{4}) + (\tfrac{3}{4})(\tfrac{1}{2}) + (\tfrac{1}{8})(\tfrac{1}{4})$$

$$= \frac{7}{16}$$

$$Pe_2 = (\tfrac{1}{8})(\tfrac{1}{3}) + (\tfrac{3}{4})(\tfrac{1}{3}) + (\tfrac{1}{8})(\tfrac{1}{3})$$
$$= \frac{1}{3} < Pe_1$$

Error prob decreases since prob of transmitting a 0, which has a high error rate, is decreased.

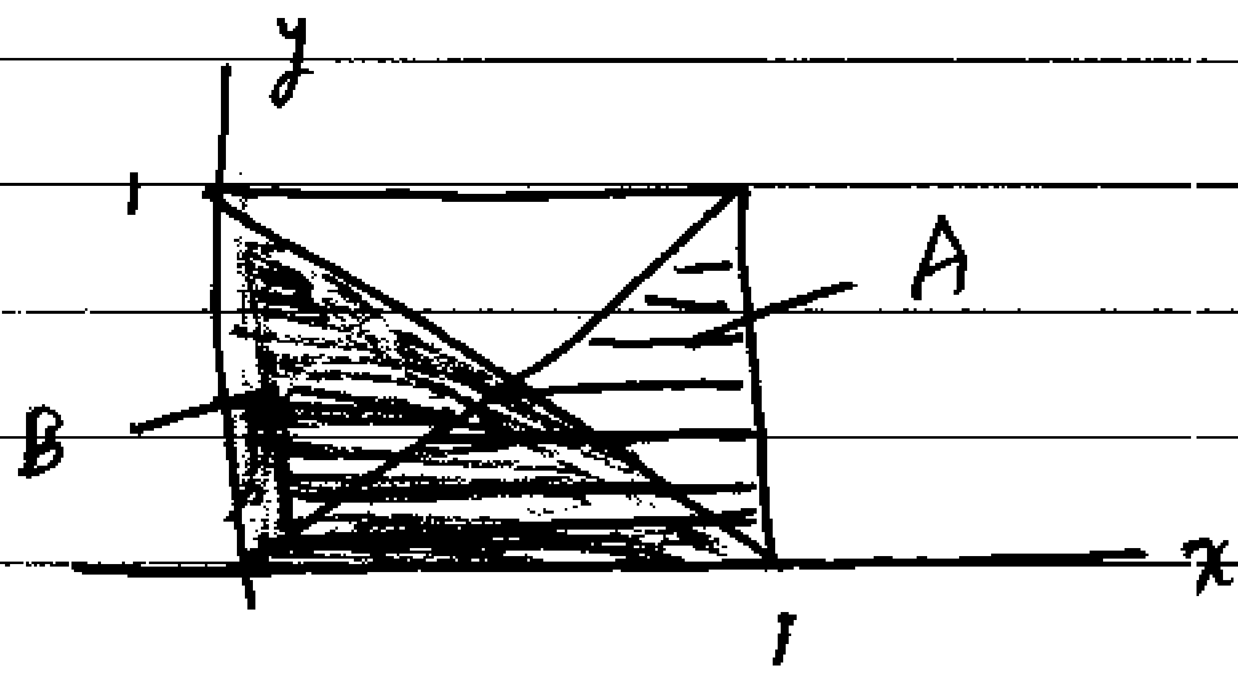
14)



$$P(A/B) = \frac{P(AB)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

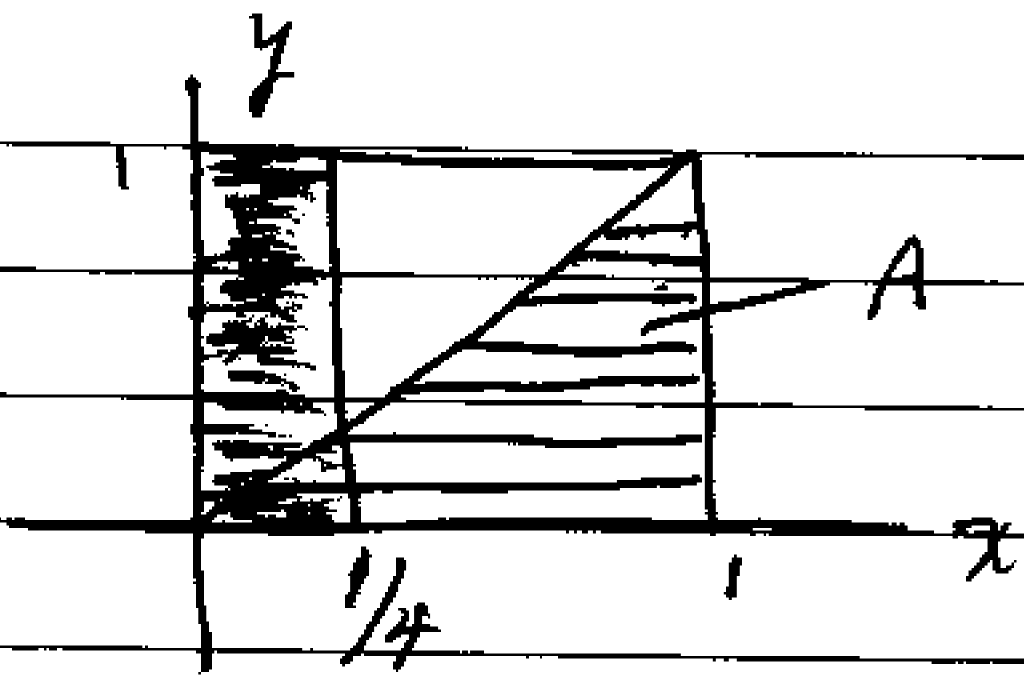
yes

15)



$$P[A/B) = \frac{P(AB)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

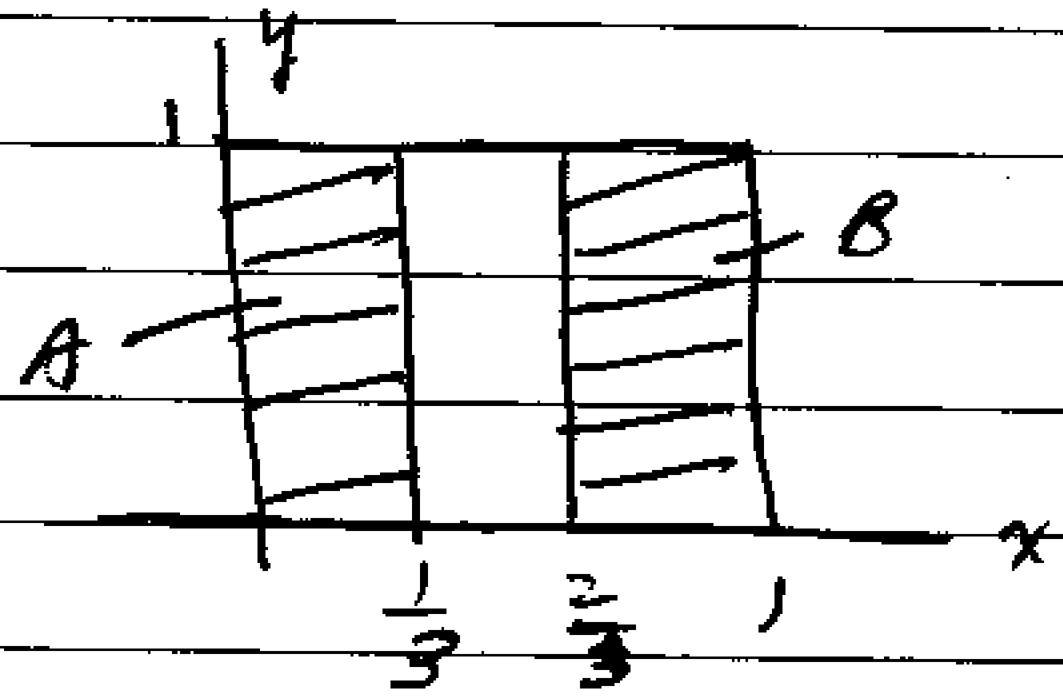
yes



$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{\frac{1}{2}(\frac{1}{4})(\frac{1}{4})}{1/4} = \frac{1}{8} \neq P(A) = \frac{1}{2}$$

No

16)



$P(AB) = 0$

$P(A)P(B) = \frac{1}{9}$

17) $A = \{(x,y,z) : 0 \le x \le \frac{1}{2}\}$

$B = \{(x,y,z) : 0 \le y \le \frac{1}{2}\}$

$C = \{(x,y,z) : 0 \le z \le \frac{1}{2}\}$

$P(AB) = P(0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2}) = 1/4$

$= P(A)P(B)$ etc.

$P(ABC) = P(0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2}, 0 \le z \le \frac{1}{2})$

$= 1/8 = P(A)P(B)P(C)$

18) $P(ABC) = P(A)P(B)P(C)$ all events

Let $C = S \Rightarrow ABS = AB$

$$P(\underbrace{ABC}_{AB}) = P(A)P(B)\underbrace{P(C)}_{P(S)=1}$$

$\Rightarrow P(AB) = P(A)P(B)$ etc.

19) $A_3 = \{\text{road floods}\}$
$A_2 = \{\text{sewer overflows}\}$
$A_1 = \{\text{rain}\}$

$$P(A_3) = P(A_3 \mid A_1 A_2)\, P(A_2 \mid A_1)\, P(A_1)$$
$$= P(A_3 \mid A_2)\, P(A_2 \mid A_1)\, P(A_1)$$

↑ since flooding is only dependent on sewer overflowing

$$= (0.3)(0.5)(0.2) = 0.03$$

20) $P[AB] = P(1) = 1/4 = P(A)P(B)$
$P[AC] = P(1) = 1/4 = P(A)P(C)$
$P(BC) = P(1) = 1/4 = P(B)P(C)$
Yes
$P(ABC) = P(1) = \frac{1}{4} \neq P(A)P(B)P(C) = \frac{1}{8}$
No

21) $P(BC) = P(3,4) = 1/3 \neq P(B)P(C) = \left(\frac{2}{3}\right)^2 = 4/9$
Not pairwise ind. $\Rightarrow$ not ind.

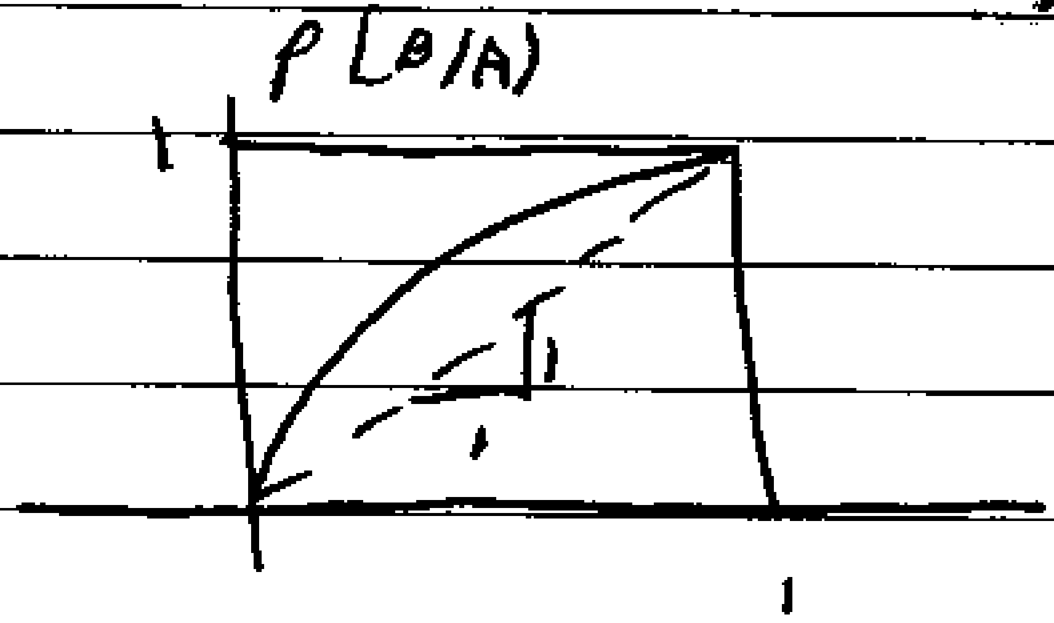
22) $$\text{odds} = \frac{P[R_2 \mid B_1]}{P(R_2^c \mid B_1)} = \frac{P(R_2 \mid B_1)}{1 - P(R_2 \mid B_1)}$$

$$P(R_2 \mid B_1) = \frac{P[(B_1, R_2)]}{P(B_1)} = \frac{\frac{2 \cdot 4}{6 \cdot 5}}{2/6} = 4/5$$

$$\text{odd} = \frac{4/5}{1 - 4/5} = 4$$ (after 1st draw have 4R, 1B)

23) $P(B|A) = \dfrac{0.99p}{0.99p + 0.2(1-p)} = \dfrac{99p}{99p + 20(1-p)}$

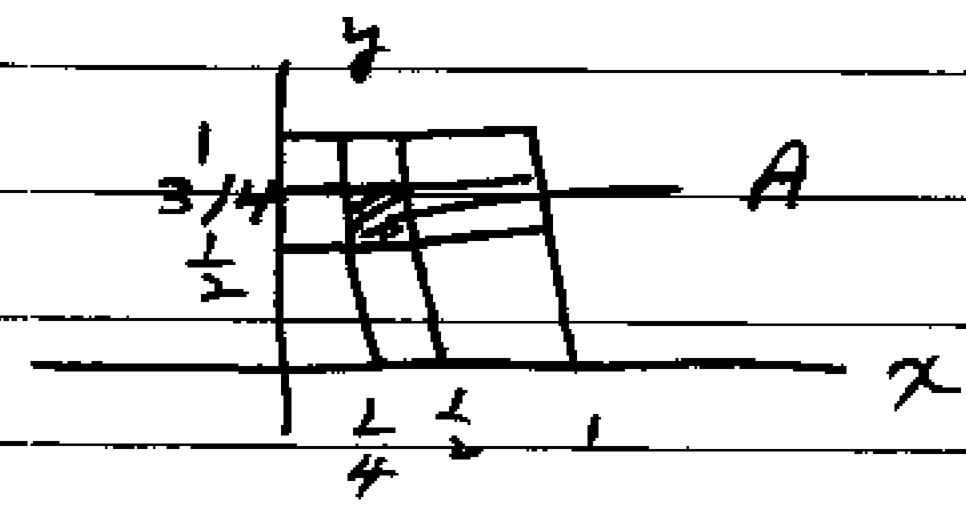
$= \dfrac{99p}{79p + 20}$ $\quad p = P[B]$



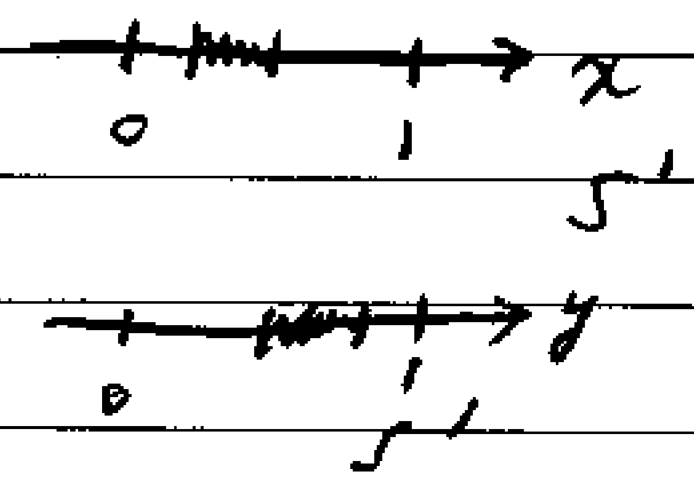
$P[B/A] > P(B)$
Otherwise test is useless.

24) $S^2 = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$

$P(A) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$



$P(A) = P(\frac{1}{4} \le x \le \frac{1}{2})$
$\cdot P(\frac{1}{2} \le y \le 3/4)$



25) Possible runs are

1) H H H H H T - - - -

2) T H H H H H T - - -

3) - T H H H H H T - -

4) - - T H H H H H T -

5) - - - T H H H H H T

6) - - - - T H H H H H

"-" indicates H or T

Probabilities are for each sequence

$$P_1 = 2^4/2^{10}$$
$$P_2 = 2^3/2^{10}$$
$$P_3 = P_4 = P_5 = P_2$$
$$P_6 = P_1$$

$$p = \sum_{i=1}^{6} P_i = \frac{2 \cdot 2^4 + 4 \cdot 2^3}{2^{10}} = \frac{1}{16} = 0.0625$$

$$\hat{p} = 0.0660$$

```
% probprob4_25.m
%
rand('state',0)
nreal=1000;
hit=0;
for k=1:nreal
   x=floor(rand(10,1)+0.5);
   for i=1:6
      if i==1&prod(x(1:5))==1&x(6)==0
         hit=hit+1;
      elseif i==6&prod(x(6:10))==1&x(5)==0
         hit=hit+1;
      elseif i>1&i<6&prod(x(i:i+4))==1&x(i-1)==0&x(i+5)==0
         hit=hit+1;
      end
   end
   end
   prob=hit/nreal
```

26) Assuming she is guessing $p = \frac{1}{2}$.

$$P(k=8) = \binom{10}{8}\left(\frac{1}{2}\right)^{10} = \frac{\frac{10!}{2!\,8!}}{1024}$$

$$= \frac{45}{1024} = 0.0439$$

Unlikely she is guessing

27) $P[k=3] = \binom{10}{3} p^3 (1-p)^7$

Must maximize $p^3(1-p)^7$ over $0 \le p \le 1$.

$g(p) = p^3(1-p)^7$
$g'(p) = p^3(1-p)^6(-7) + 3p^2(1-p)^7 = 0$

$\Rightarrow -7p + 3(1-p) = 0$
$-7p + 3 - 3p = 0 \Rightarrow p = 3/10$

If $p = 3/10$, then for an ∞ sequence of Bernoulli trials we will obtain 3 successes for every 10 trials. Hence, for 10 trials it is most probable to have 3 successes.

28) We can "lump together" a failure and don't know so that we have a Bernoulli sequence with $p = 1/2$.

$P[k=3] = \binom{5}{3}\left(\frac{1}{2}\right)^5 = \frac{5 \cdot 4}{2} \frac{1}{32} = 5/16$

29) $\hat{P}[k=3] = 0.3085$

```
% probprob4_29.m
%
clear all
rand('state',0)
hit=0;nreal=10000
for i=1:nreal
   for j=1:5
   u=rand(1,1);
   if u<=0.5
      x(j,1)=1;
   elseif u>0.5&u<=0.75
      x(j,1)=0;
   elseif u>0.75
      x(j,1)=0;
   end
   end
   if sum(x)==3
      hit=hit+1;
   end
   end
   prob=hit/nreal
```

30) Call a success one step forward. To move 2 steps forward in 10 steps we must have

$$\underset{\text{steps forward}}{a} + b = 10$$
$$a - \underset{\text{steps backward}}{b} = 2 \Rightarrow a = 6, \quad b = 4$$

$$P[k=6] = \binom{10}{6}\left(\frac{1}{2}\right)^{10} = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}\left(\frac{1}{2}\right)^{10}$$

$$= 105/512$$

31) $$\sum_{k=1}^{\infty} P[k] = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \frac{1}{1-(1-p)} = 1$$

32) Need $(k-1)$ successes followed by failure

$$\Rightarrow P[k] = p^{k-1}(1-p) \qquad k = 1, 2, \ldots$$

33) In k trials we must have the outcome $(f, f, \ldots, f, s, f, \ldots, f, s)$ with $k-2$ failures

This outcome has prob $(1-p)^{k-2}p^2$ but there are $(k-1)$ outcomes like this - consider where 1^{st} success occurs.

$$P[k] = (k-1)(1-p)^{k-2}p^2 \qquad k = 2, 3, \ldots$$

34) P(first success at $k = m+l$ | first m failures)

$$= \frac{P[(\overbrace{f, f, \ldots f}^{m+l-1}, s)]}{P[(\underbrace{f, f, \ldots, f}_{m})]} = \frac{(1-p)^{m+l-1}p}{(1-p)^m}$$

$$= (1-p)^{l-1}p \qquad l = 1, 2, \ldots$$

35) $$P(2R, 2B, 1W) = \binom{5}{2, 2, 1}(0.4)^2(0.4)^2(0.2)^1$$

$$= \frac{5!}{2!\,2!\,1!}\left(\frac{2}{5}\right)^4\frac{1}{5} = 30 \cdot \frac{16}{5^5}$$

$$= 96/625 = 0.1536$$

36) $$\binom{M}{k_1}\binom{M-k_1}{k_2} = \frac{M!}{(M-k_1)!\,k_1!}\;\frac{(M-k_1)!}{(M-k_1-k_2)!\,k_2!}$$

$$= \frac{M!}{k_1!\,k_2!\,(\underbrace{M-k_1-k_2}_{k_3})!}$$

37) From (4.19) for $N = 2$

$$p(k_1, k_2) = \binom{M}{k_1, k_2} p_1^{k_1} p_2^{k_2} \qquad k_1 + k_2 =$$

$$= \frac{M!}{k_1!\, k_2!} p_1^{k_1} p_2^{k_2}$$

$$= \frac{M!}{k_1!\,(M-k_1)!} p_1^{k_1} p_2^{M-k}$$

$$= \binom{M}{k_1} p_1^{k_1} p_2^{M-k}$$

But letting $k_1 = k$ and $p_1 = p$ we have

$$p(k, M-k) = \binom{M}{k} p^k (1-p)^{M-k}$$

since $p_1 + p_2 = 1$

38) $$p(k_1, k_2, k_3) = \binom{6}{k_1, k_2, k_3} \left(\tfrac{1}{3}\right)^{k_1} \left(\tfrac{1}{3}\right)^{k_2} \left(\tfrac{1}{3}\right)^{k_3}$$

$$= \underbrace{\binom{6}{k_1, k_2, k_3}} \left(\tfrac{1}{3}\right)^{k_1+k_2+k_3=6}$$

maximize this over

$k_1 + k_2 + k_3 = 6$

$$\binom{6}{k_1, k_2, k_3} = \frac{6!}{k_1!\, k_2!\, (6-k_1-k_2)!}$$

where $k_1 = 0, 1, \ldots, 6$

$k_2 = 0, 1, \ldots 6$

and $k_1 + k_2 \leq 6$

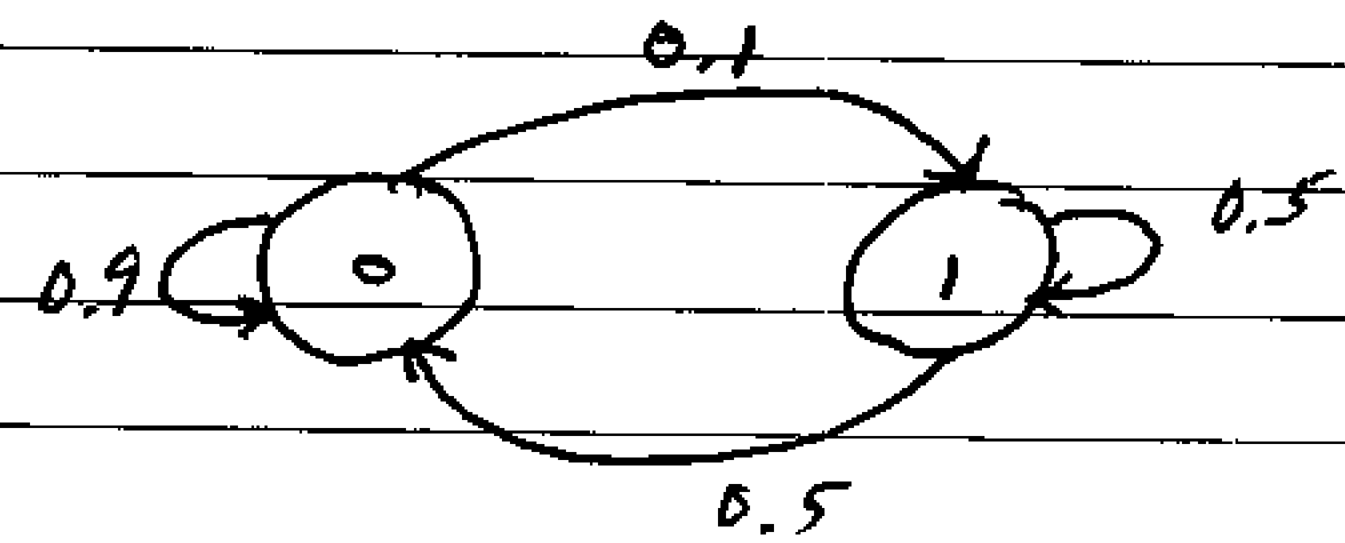
X =

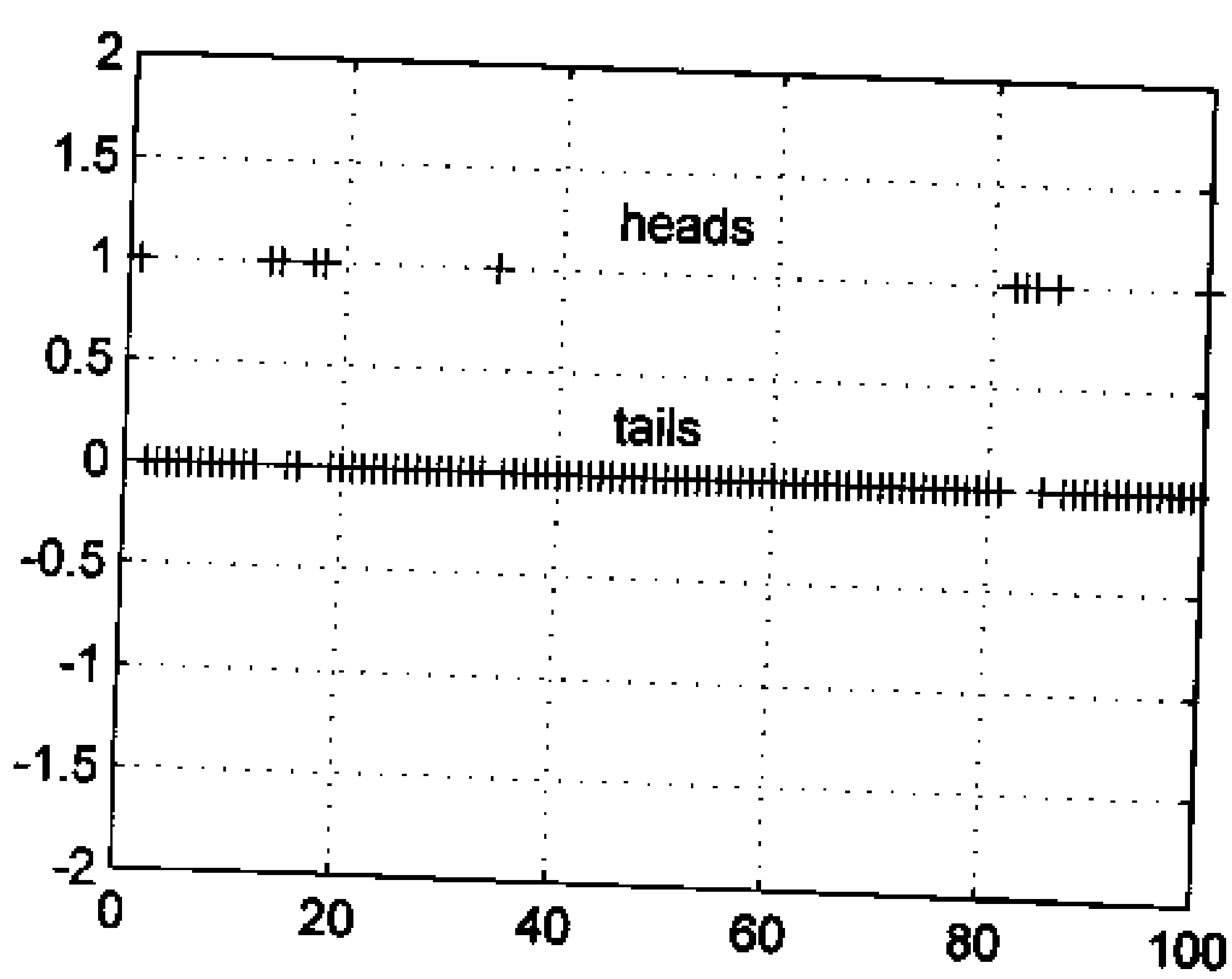
0	0	1
0	1	6
0	2	15
0	3	20
0	4	15
0	5	6
0	6	1
1	0	6
1	1	30
1	2	60
1	3	60
1	4	30
1	5	6
2	0	15
2	1	60
2	2	90
2	3	60
2	4	15
3	0	20
3	1	60
3	2	60
3	3	20
4	0	15
4	1	30
4	2	15
5	0	6
5	1	6
6	0	1

```
% probprob4_38.m
%
clear all
X=[];
for k1=0:6
   for k2=0:6
      if k1+k2<=6
         den(k1+1,k2+1)=prod(1:k1)*prod(1:k2)*prod(1:6-k1-k2);
         X=[X;k1 k2 720./den(k1+1,k2+1)];
      end
   end
end
X
```

Maximum for $k_1 = k_2 = k_3 = 2$ as expected

39)





Once the process is in state 0 it is highly likely to stay in that state - hence the runs of 0's

```
% probfig4_39.m
%
clear all
rand('state',0)
p=0.1;
x=zeros(100,1);
x(1)=floor(rand(1,1)+0.5);
for i=2:100
   if x(i-1)==0
      if rand(1,1)>1-p
         x(i)=1;
      end
   else
      x(i)=floor(rand(1,1)+0.5);
   end
end
plot([1:100]',x,'+')
grid
axis([0 100 -2 2])
text(45,1.25,'heads')
text(45,0.25,'tails')
```

40) $P[0,1,1,0] = P[0]P[1|0]P[1|1]P[0|1]$

$P[0] = 3/4$

$P[1|0] = 1/4$

$P(1|1) = \frac{1}{2}$

$P[0|1] = \frac{1}{2}$

$P(0,1,1,0) = 3/4(1/4)(\frac{1}{2})(\frac{1}{2}) = 3/64$

41) $P[0,1,0,1,0] = P[0]P(1|0)P[0|1]P(1|0)P[0|1]$

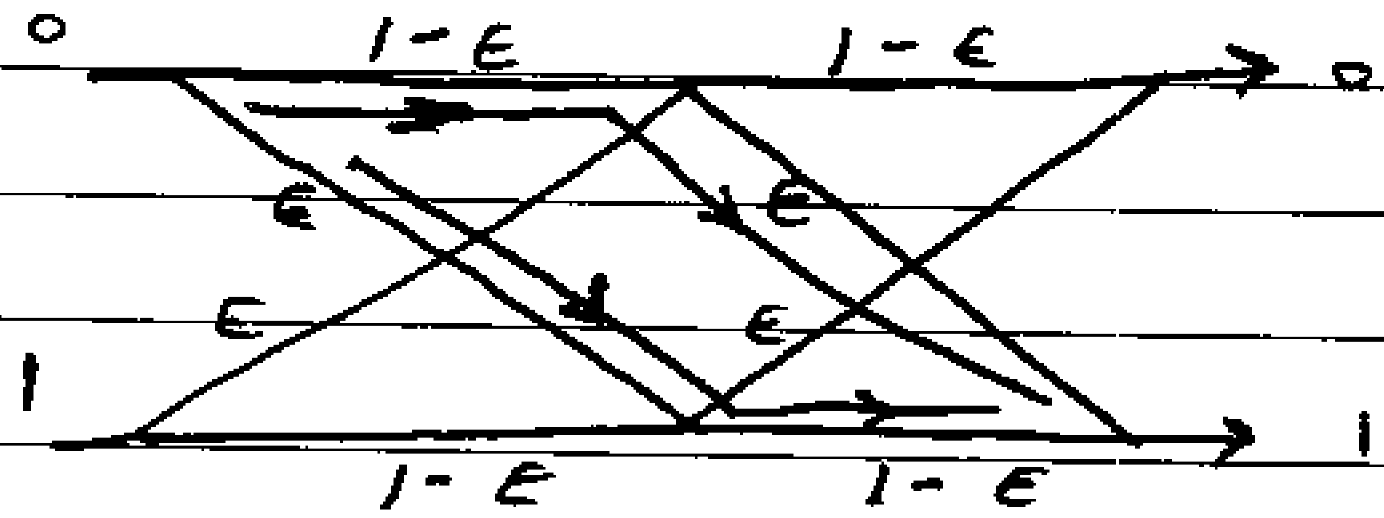
$P[0] = 1/2$

$P[1|0] = 1-P[0|0] = 3/4$

$P[0|1] = 3/4$

$P[0,1,0,1,0] = \frac{1}{2}(3/4)(3/4)(3/4)(3/4)$

$= 81/512$

42) 

Two paths are shown leading to an error when a 0 is transmitted

$$P_{e_2} = P(\text{error}\,|\,0)\,P(0) + P(\text{error}\,|\,1)\,P(1)$$
$$= P(\text{error}\,|\,0)\,\tfrac{1}{2} + P(\text{error}\,|\,1)$$

By symmetry $P(\text{error}\,|\,0) = P(\text{error}\,|\,1)$

$$\Rightarrow P_{e_2} = P(\text{error}\,|\,0)$$
$$= (1-\epsilon)\epsilon + \epsilon(1-\epsilon) = 2\epsilon - \epsilon^2$$
$$= 2\epsilon(1-\epsilon)$$

For a single section $P_{e_1} = \epsilon$. For $\epsilon < \frac{1}{2}$
$$P_{e_2} = \epsilon\, 2(1-\epsilon) > \epsilon = P_{e_1}$$

43) Each outcome is a simple event $\Rightarrow$ add probs.

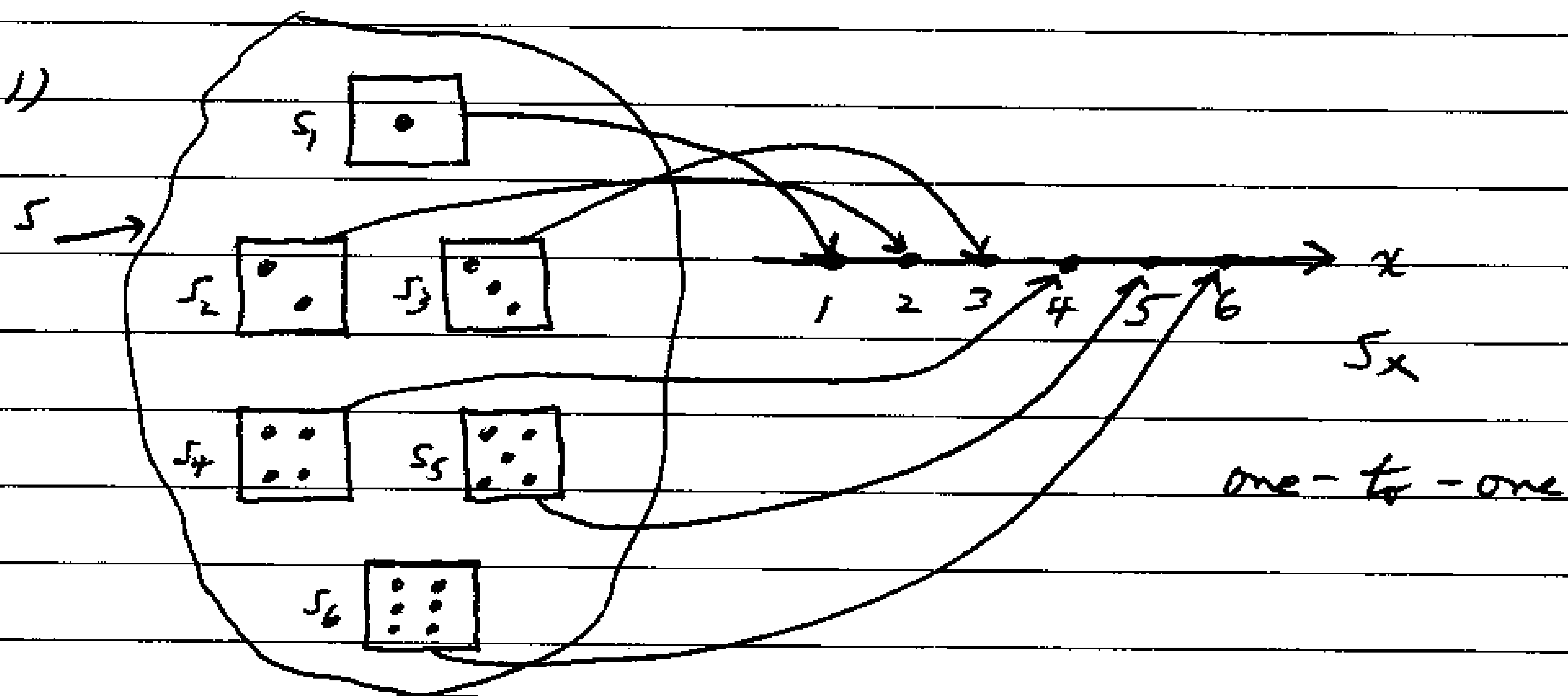
$$P[0,1,0,0) + P(0\,0\,0\,0) =$$
$$P(0)\,P(1|0)\,P(0|1)\,P(0|0) + P(0)\,P(0|0)\,P(0|0)\,P(0|0)$$

$$= 5/8\,(1/4)(\tfrac{1}{2})(3/4) + 5/8\,(3/4)^3$$

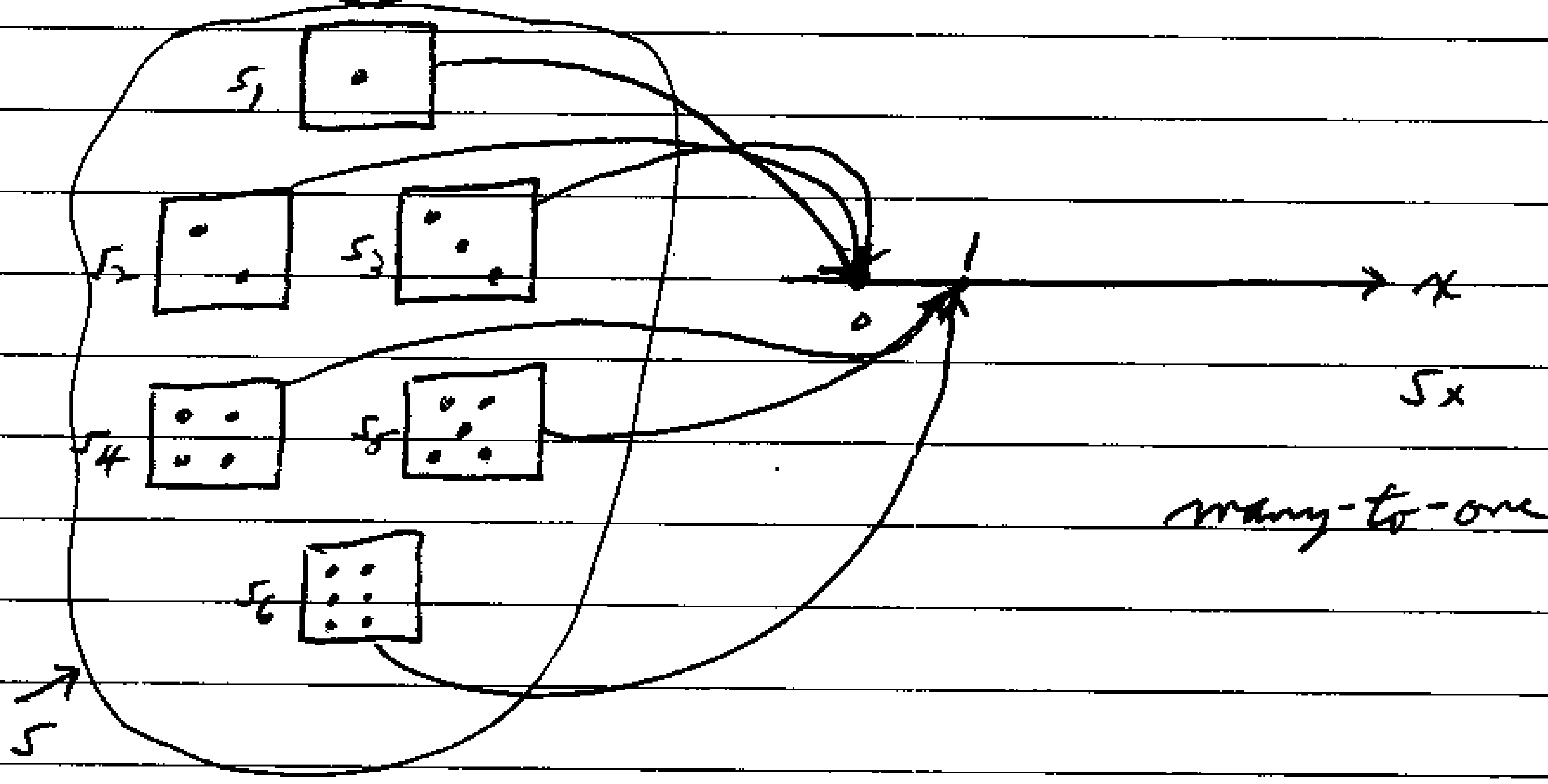
$$= \frac{15}{256} + \frac{135}{512} = \frac{165}{512}$$

Chapter 5

1)



2)



3) $S_X = \{1, 4, 9, \ldots, 100\}$

$$p_X(x_i) = \begin{cases} \frac{1}{10} & x_i = 1 \\ \frac{1}{10} & x_i = 4 \\ \vdots & \\ \frac{1}{10} & x_i = 100 \end{cases}$$

4) $S_X = \{0, 1, 4, 9\}$

$$p_X[x_i] = \begin{cases} 1/7 & x_i = 0 \\ 2/7 & x_i = 1 \\ 2/7 & x_i = 4 \\ 2/7 & x_i = 9 \end{cases}$$

using (5.2).

5) X = fine $X(s_i) = 50i$ where $s_i = i$

$$p_X[x_i] = P[X(s_i) = x_i]$$
$$= P[s_i = x_i/50]$$
$$= \left(\tfrac{1}{2}\right)^{x_i/50} \qquad x_i = 50, 100, \ldots$$

$$P[X > 1000] = \sum_{\{x_i : x_i = 1050, 1100, \ldots\}} \left(\tfrac{1}{2}\right)^{x_i/50}$$

$$= \sum_{i=21}^{\infty} \left(\tfrac{1}{2}\right)^{i} = \frac{\left(\tfrac{1}{2}\right)^{21}}{1 - \tfrac{1}{2}} = \left(\tfrac{1}{2}\right)^{20}$$

6) $\displaystyle\sum_{k=2}^{\infty} p_X[k] = 1 \Rightarrow \sum_{k=2}^{\infty} \alpha p^k = 1$

For $p_X[k] \geq 0 \Rightarrow \overset{k=2}{\alpha p^2 \geq 0} \Rightarrow \alpha \geq 0$

$\qquad\qquad\qquad \overset{k=3}{\alpha p^3 \geq 0} \Rightarrow p \geq 0$

For sum to be finite $p < 1$.

$$\sum_{k=2}^{\infty} \alpha p^k = \alpha \frac{p^2}{1-p} = 1 \Rightarrow \alpha = \frac{1-p}{p^2}$$

Thus, $0 \leq p < 1$ and $\alpha = \dfrac{1-p}{p^2}$

7) Let $Q[k] = p[k]/p[k-1]$

$$Q[k] = \frac{\binom{M}{k}}{\binom{M}{k-1}} \quad \frac{p^k}{p^{k-1}} \quad \frac{(1-p)^{M-k}}{(1-p)^{M}}$$

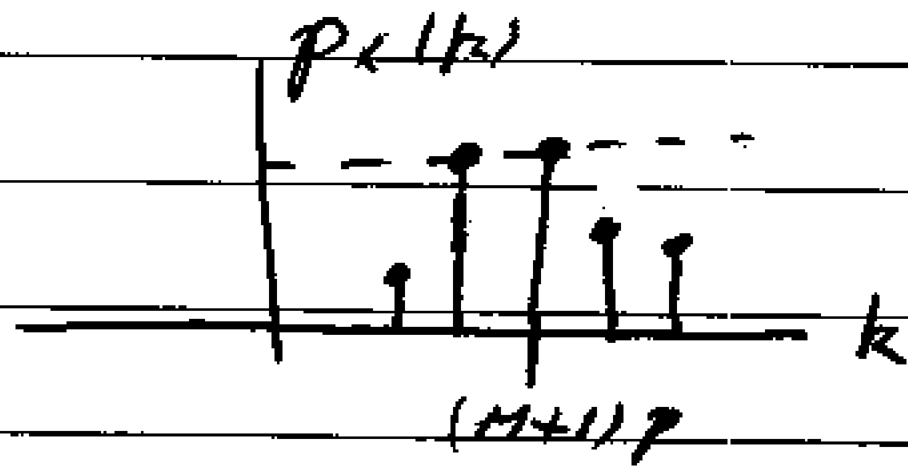
$$= \frac{\dfrac{M!}{(M-k)!\,k!}}{\dfrac{M!}{(M-k+1)!\,(k-1)!}} \, \frac{p}{1-p} = \frac{(M-k+1)}{k} \, \frac{p}{1-p}$$

$$= 1 + \frac{(M-k+1)p - k(1-p)}{k(1-p)} = 1 + \frac{(M+1)p - k}{k(1-p)}$$

If $(M+1)p \geq k \Rightarrow Q(k) \geq 1$

$(M+1)p < k \Rightarrow Q(k) < 1$

Thus,



$$\frac{p_x((M+1)p)}{p_x((M+1)p-1)} = 1$$

Maximum at $k = (M+1)p$ and $(M+1)p - 1$

For $(M+1)p$ not an integer the location of the maximum will be unique or only 1 maximum.

8) p = prob of success = $1/100$

P[success at k^{th} trial] $= (1-p)^{k-1} p$ geometric law

$$p_x(k) = (0.99)^{k-1}(0.01) \qquad k = 1, 2, \ldots$$

P(at least 19 failures) $= P[k \geq 20]$

$$= \sum_{k=20}^{\infty} p_x(k) = \sum_{k=20}^{\infty} (1-p)^{k-1} p$$

$$= \frac{p}{1-p}\,\frac{(1-p)^{20}}{1-(1-p)} = (1-p)^{19} = (0.99)^{19}$$

$$= 0.8262$$

9) $$P[X \geq 4] = \sum_{k=4}^{\infty} (1-p)^{k-1} p$$

$$= \frac{p}{1-p} \sum_{k=4}^{\infty} (1-p)^k = \frac{p}{1-p}\,\frac{(1-p)^4}{1-(1-p)}$$

$$= (1-p)^3 = 0.75^3 = 0.4219$$

From simulation $\hat{P}[X \geq 4] = 0.4280$

```
% probprob5_9.m
%
clear all
rand('state',0)
ntrials=1000;
x=zeros(ntrials,1);
for i=1:ntrials
   k=0;
   while x(i)==0
      k=k+1;
      if rand(1,1)<0.25
         x(i,1)=k;
      end
   end
end
number=length(find(x>3));
probest=number/ntrials
meanest=mean(x)
```

10) Average value $= 4.0490$ (see code for Problem 5.9)

This average value $= 1/p$

If there is a success with a prob of 0.25, then on the average it takes 4 tries for a success.

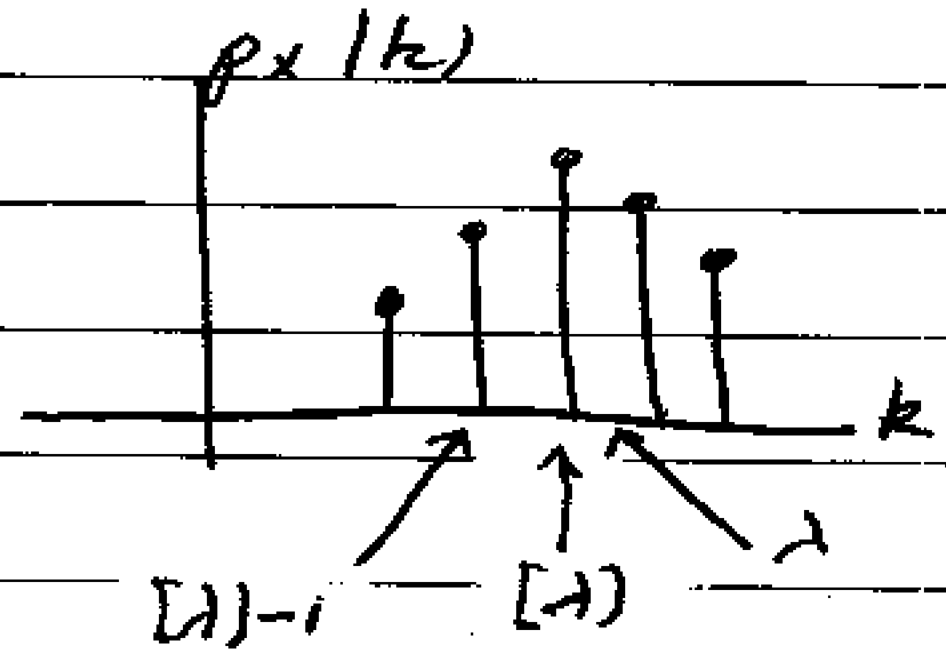
11) $$\Phi[k] = \frac{p_X[k]}{p_X[k-1]} = \frac{e^{-\lambda}\lambda^k/k!}{e^{-\lambda}\lambda^{k-1}/(k-1)!}$$

$$\begin{aligned} &= \lambda/k > 1 \quad \text{for } \lambda > k \text{ or } k < \lambda \\ & < 1 \quad \text{for } \lambda < k \text{ or } k > \lambda \\ & = 1 \quad \text{for } \lambda = k \end{aligned}$$

Maxima at $k = \lambda - 1$ and λ if λ an integer
Maximum at $k = [\lambda]$ if λ not an integer
since

$$\frac{p_X[k]}{p_X[k-1]} \begin{cases} > 1 & k < \lambda \text{ or } k \le [\lambda] \\ < 1 & k > \lambda \text{ or } k \ge [\lambda]+1 \end{cases}$$

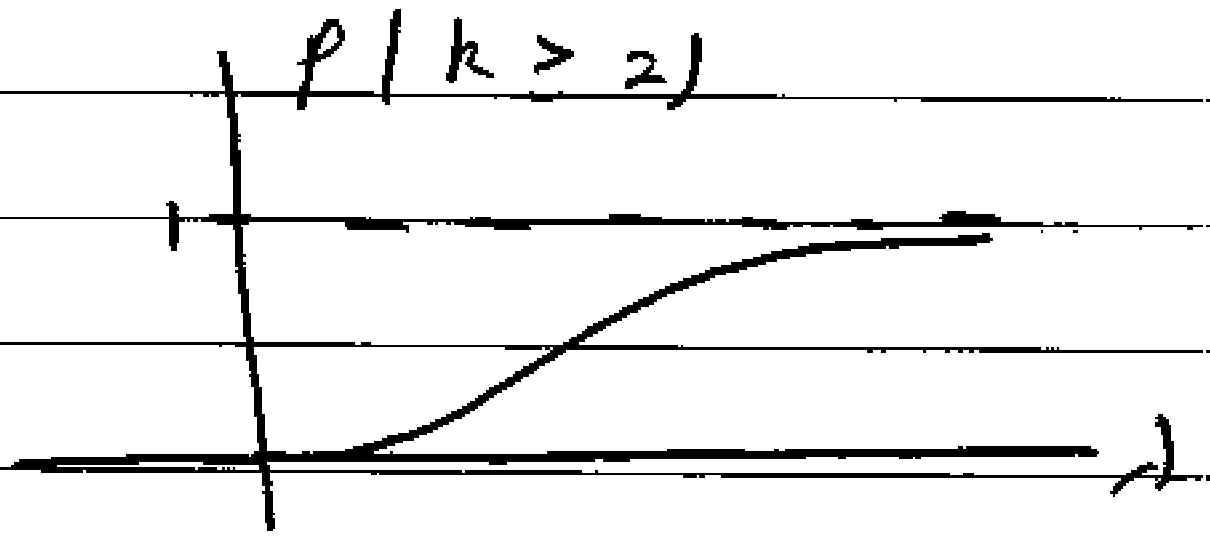
$$p_X[k] \begin{cases} > p_X[k-1] & k \le [\lambda] \\ < p_X[k-1] & k \ge [\lambda]+1 \end{cases}$$



12) $$P[X \ge 2] = \sum_{k=2}^{\infty} e^{-\lambda}\lambda^k/k!$$

$$1 = e^{-\lambda}\sum_{k=0}^{1} \lambda^k/k!$$

$$= 1 - e^{-\lambda}(1+\lambda) = 1 - e^{-\lambda} - \lambda e^{-\lambda}$$



Considering the Poisson as approx. to the binomial, as λ increases $\Rightarrow p$ increases $\Rightarrow$ greater chance of more successes

13) Average value from simulation = 5.0310
From Chapter 6 true average value = $\lambda = 5$

```
% probprob5_13.m
%
clear all
rand('state',0)
M=100;p=0.05;
for i=1:1000
   x=floor(rand(M,1)+0.05);
   b(i,1)=sum(x);
end
mean(b)
```

14) $p_X[5] = \binom{100}{5} 0.01^5 \, 0.99^{95} = 0.0029$

$$\lambda = Mp = 100(0.01) = 1$$

$$\hat{p}_X[5] = e^{-\lambda} \lambda^5/5! = e^{-1} \frac{1^5}{5!} = 0.0031$$

15) $f(u) = (1 + x/u)^u$

$$\ln f(u) = u \ln(1 + x/u)$$

$$\lim_{u \to \infty} \ln f(u) = \lim_{u \to \infty} \frac{\ln(1 + x/u)}{1/u}$$

$$\lim_{u\to\infty} \frac{\frac{1}{1+x/u}(-xu^{-2})}{u^{-2}} = \lim_{u\to\infty} \frac{x}{1+x/u} = x$$

$\Rightarrow$ $f(u) \to e^x$ as $u \to \infty$

16) Poisson: $p_X(k) = e^{-1} 1^k / k!$ $k = 0, 1, \ldots$

binomial: $p_X(k) = \binom{100}{k}(0.01)^k (0.99)^{100-k}$

$k = 0, 1, \ldots, 100$

```
» probprob5_16

ans =
```

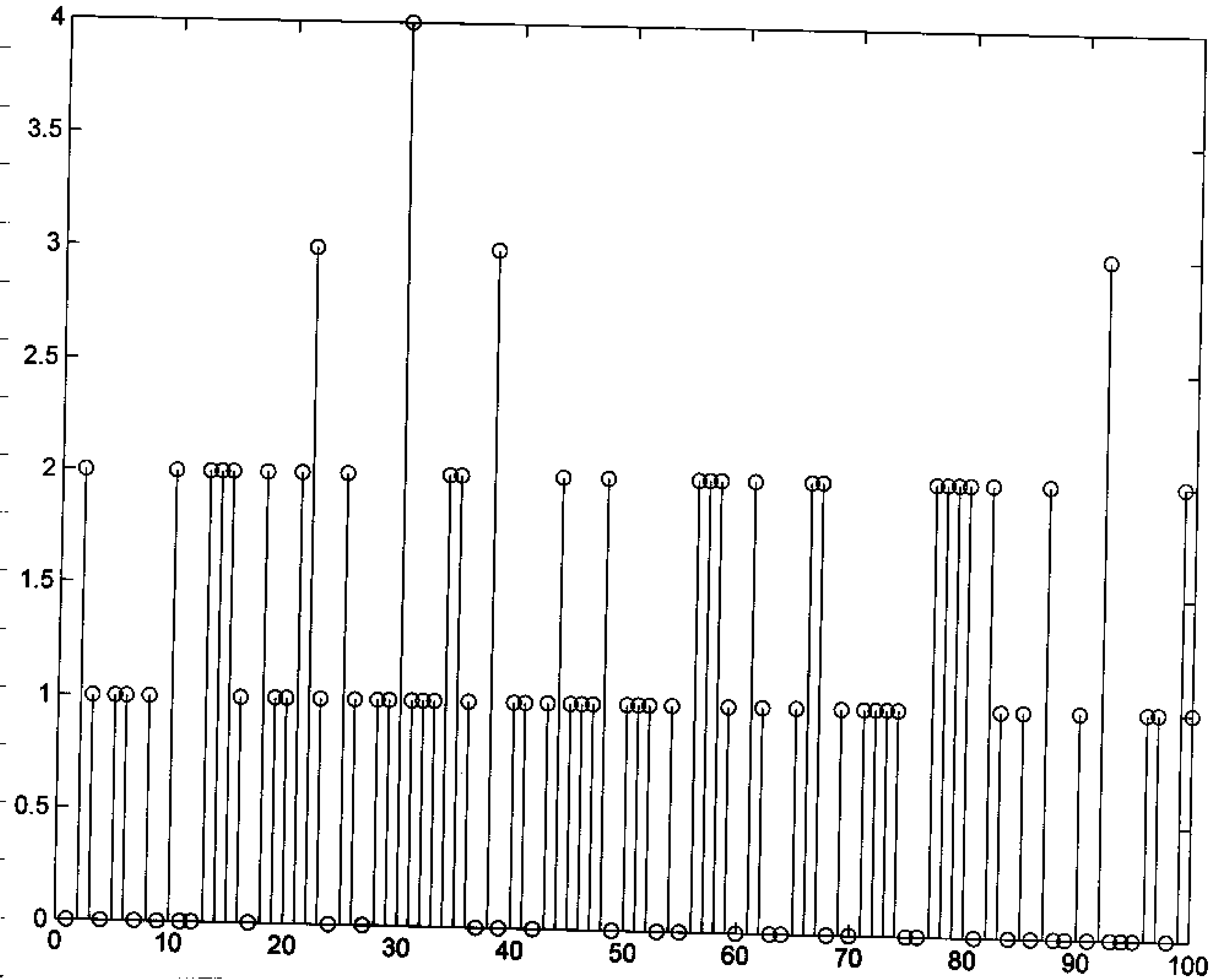
k	bin.	Poisson
0	0.3660	0.3679
1.0000	0.3697	0.3679
2.0000	0.1849	0.1839
3.0000	0.0610	0.0613
4.0000	0.0149	0.0153
5.0000	0.0029	0.0031
6.0000	0.0005	0.0005
7.0000	0.0001	0.0001
8.0000	0.0000	0.0000
9.0000	0.0000	0.0000
10.0000	0.0000	0.0000

```
%probprob5_16.m
%
clear all
p=0.01;M=100;
lambda=M*p;
for k=0:M
   Pbin(k+1,1)=(prod(1:M)/(prod(1:M-k)*prod(1:k)))*p^k*(1-p)^(M-k);
   Ppois(k+1,1)=exp(-lambda)/prod(1:k);
end
kk=[0:10]';
[kk';Pbin(1:11)';Ppois(1:11)']'
```

17) To check results, for a Poisson PMF

$$p_X(3) = e^{-1}\frac{1^3}{3!} = 0.0613$$

From simulation $\hat{p}_X(3) = 0.0607$




```
%probprob5_17.m
%
clear all
rand('state',0)
p=0.01;M=100;
ntrials=10000;
for i=1:ntrials
   x=floor(rand(M,1)+p);
   y(i,1)=sum(x);
end
stem([1:100]',y(1:100))
count=0;
for i=1:ntrials
   if y(i)==3
      count=count+1;
   end
end
Probest=count/ntrials
```

18) See Problem 5.17 solution

19) $X \sim \text{Ber}(p)$

$$Y = -X = \begin{cases} 0 & X = 0 \\ -1 & X = 1 \end{cases}$$

$$\Rightarrow p_Y(k) = \begin{cases} p & k = -1 \\ 1-p & k = 0 \end{cases}$$

20) $X \sim \text{Pois}(\lambda)$

$$Y = 2X = 0, 2, 4, \ldots$$

$$p_Y(k) = e^{-\lambda} \frac{\lambda^{k/2}}{(k/2)!} \qquad k = 0, 2, 4, \ldots$$

21) $Y = \sin \pi X$

x	y
-1	0
$-\frac{1}{2}$	-1
0	0
$\frac{1}{2}$	1
1	0

$$p_Y(0) = p_X(-1) + p_X(0) + p_X(1) = \frac{1}{2} + \frac{1}{8} + \frac{1}{16} = {}^{11}/_{16}$$

$$p_Y(k) = \begin{cases} {}^{1}/_{4} & k = -1 \\ {}^{11}/_{16} & k = 0 \\ {}^{1}/_{16} & k = 1 \end{cases}$$

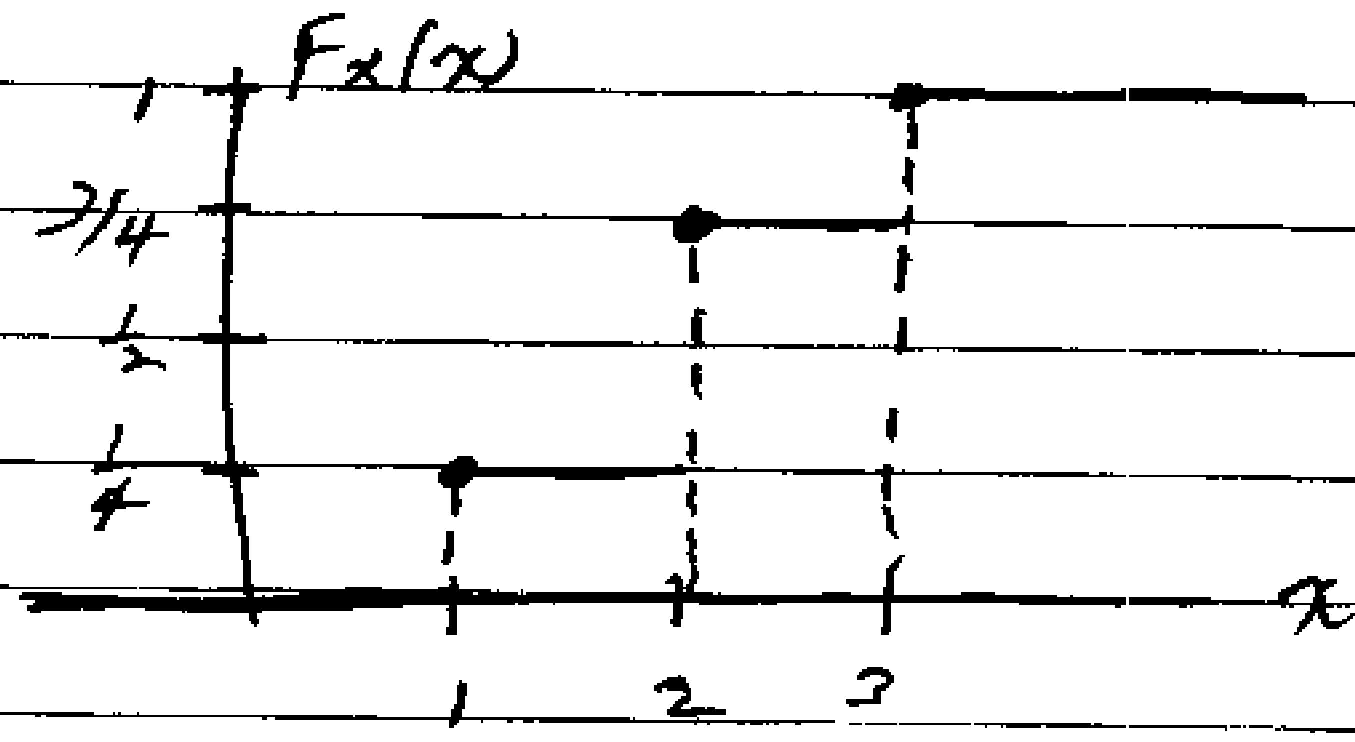
22) $g(0) = 1$

$g'(x) = e^{x} \qquad g''(x) = e^{x}$ etc

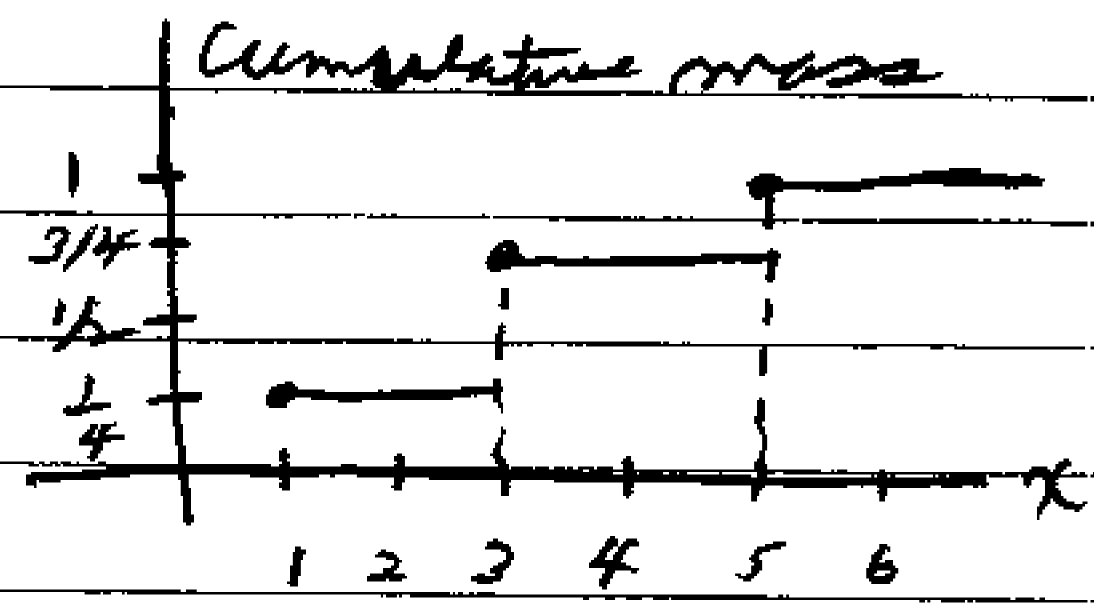
$$g^{(n)}(x) = e^{x} \Rightarrow g^{(n)}(0) = 1$$

$$e^{x} = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^{n} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

2.3)

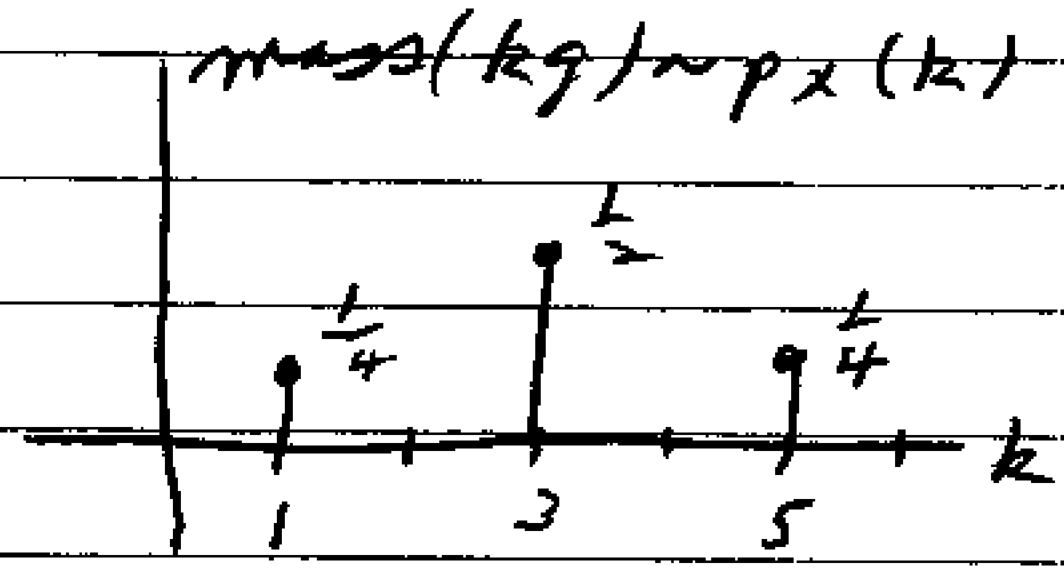


2.4)



This is a CDF where masses are analogous to prob. masses.

The analogous PMF is



2.5) $Y = -X$

$= 0 \quad X = 0$

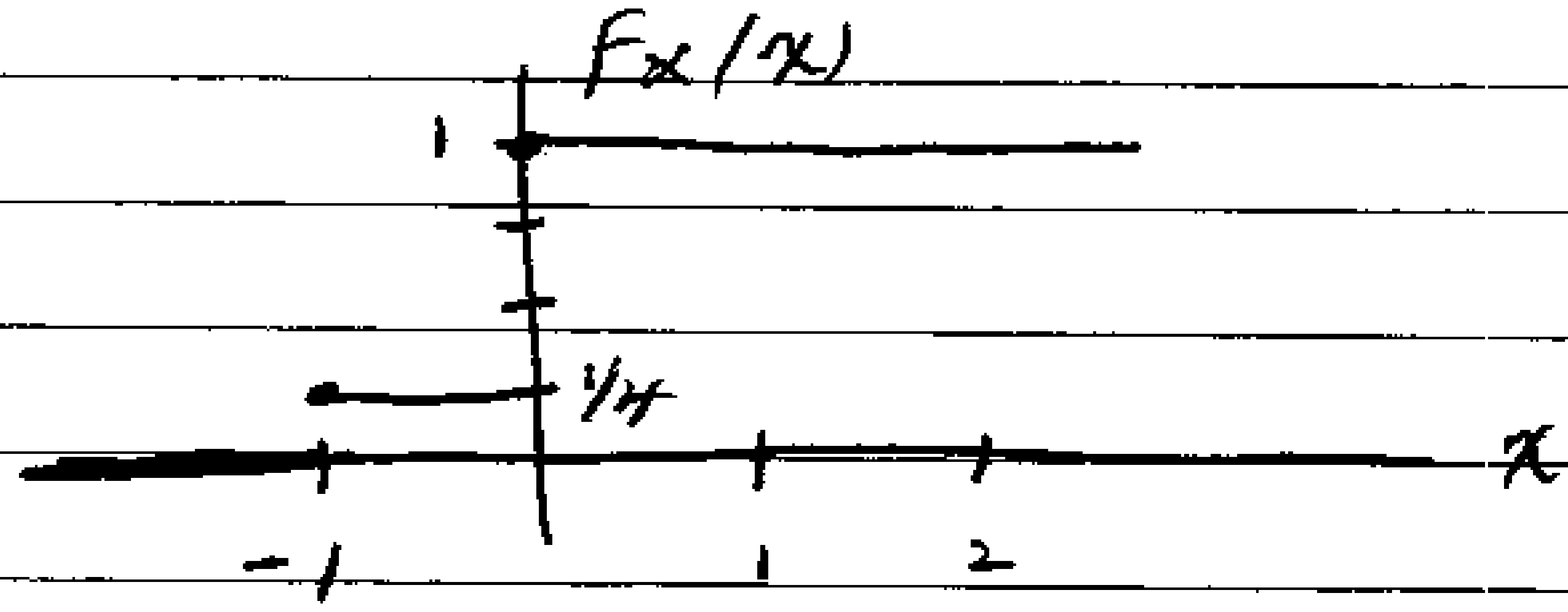
$\;\; -1 \quad X = 1$

$p_Y(k) = 1-p \quad k = 0$

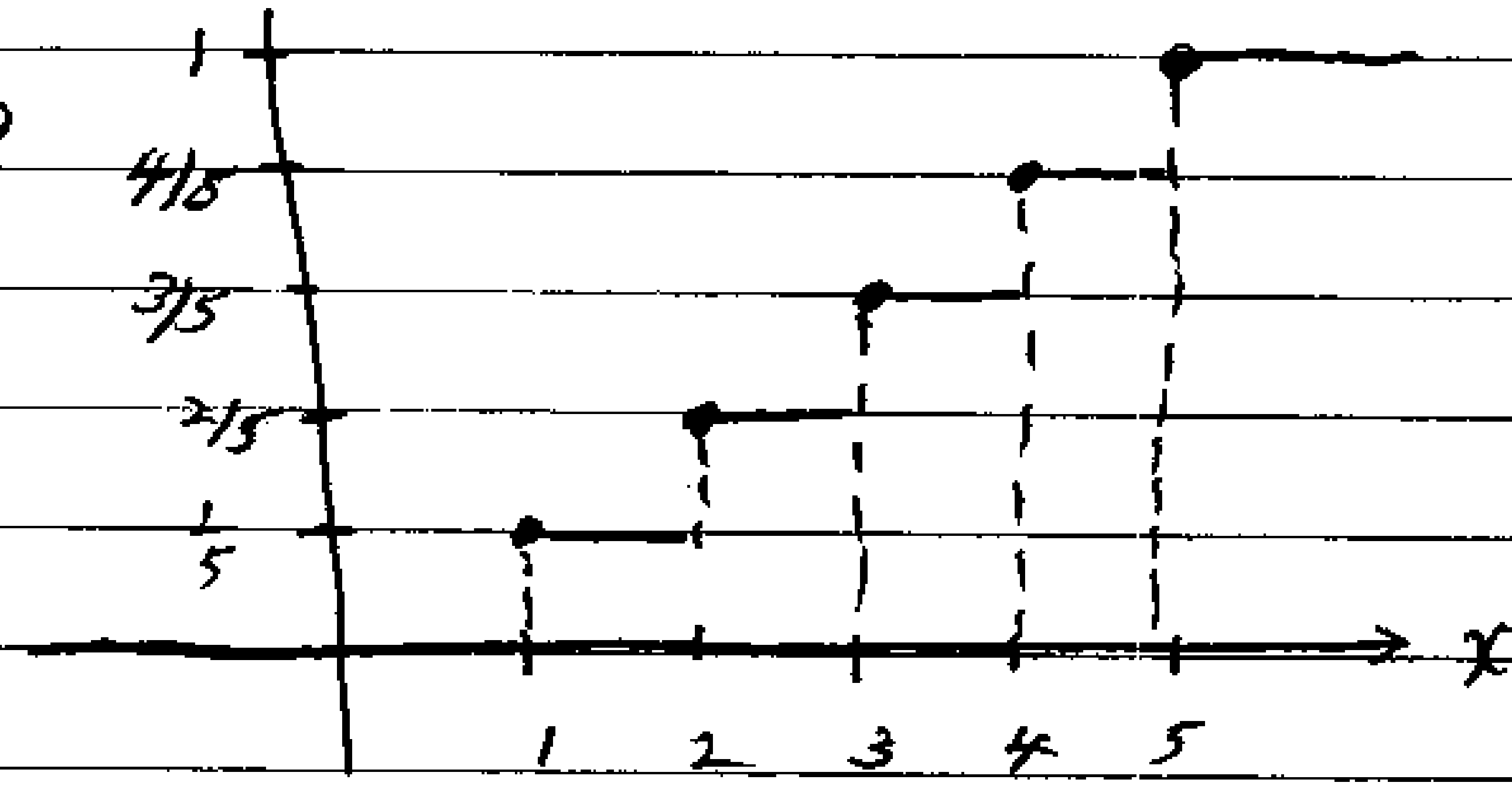
$\qquad p \quad k = -1$

$= 3/4 \quad k = 0$

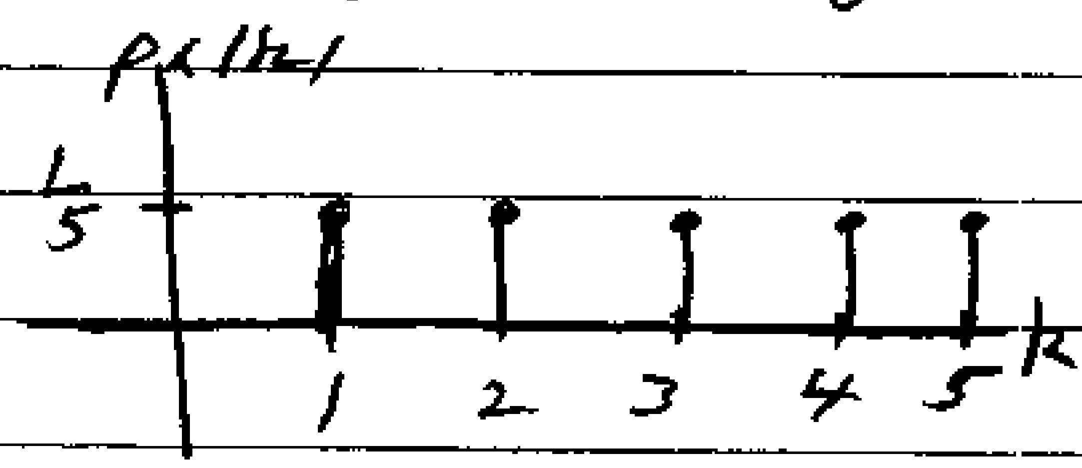
$\;\; 1/4 \quad k = -1$



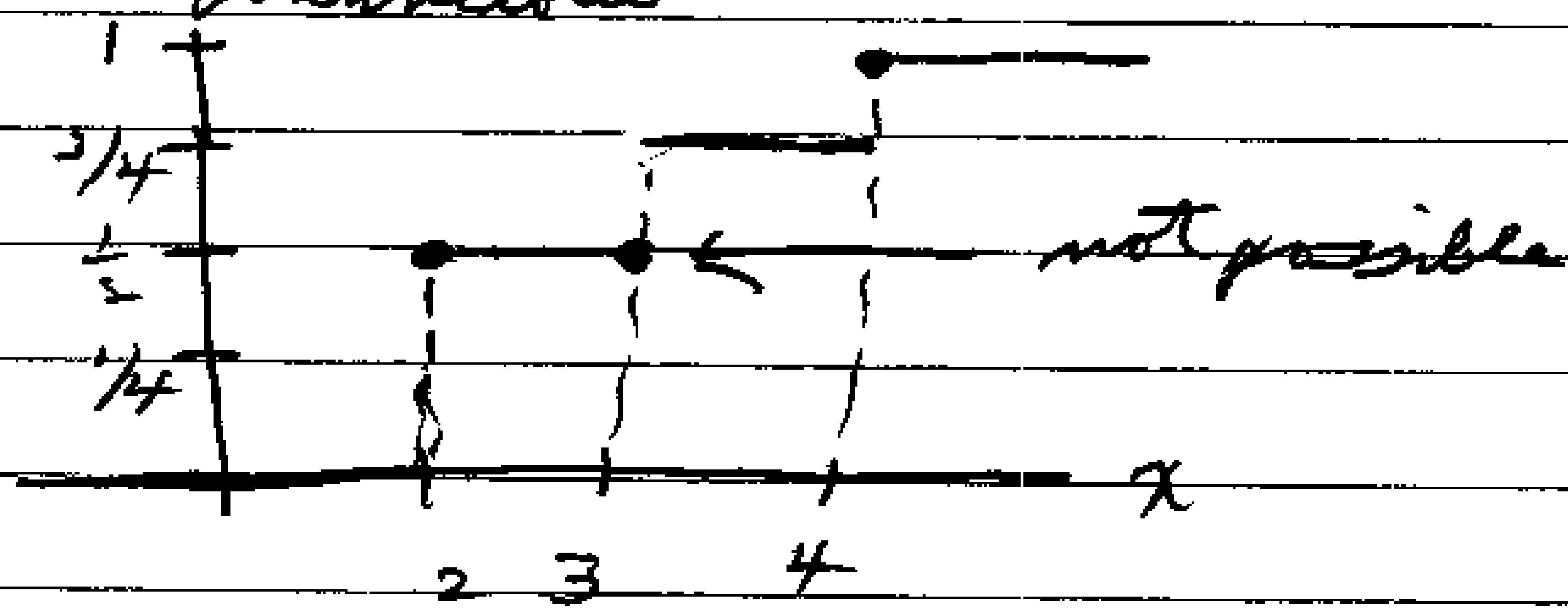
26)



$p_X(k)$ given by jumps or $p_X(x) = F_X(x^+) - F_X(x^-)$



27) No since $F_X(3) = \frac{1}{2}$ which is left-continuous



Just set $F_X(3) = 3/4$.

28) $P[2 < X \leq 4] = F_X(4^+) - F_X(2^-)$

$= 0.9375 - 0.5 = 0.4375$

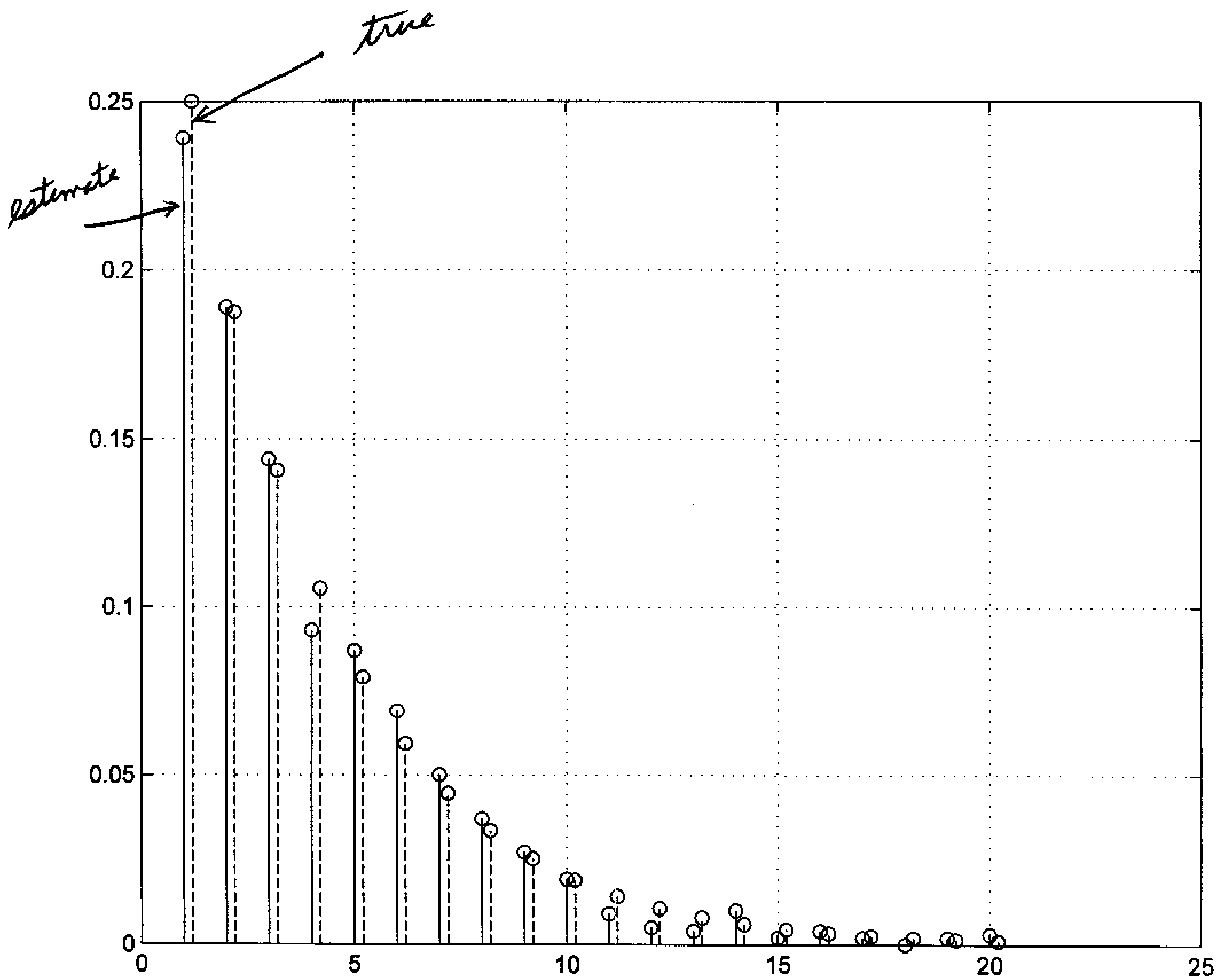
29) Need to show that if $x_2 \geq x_1$, then $e^{x_2} \geq e^{x_1}$. Assume $x_2 \geq x_1$, then

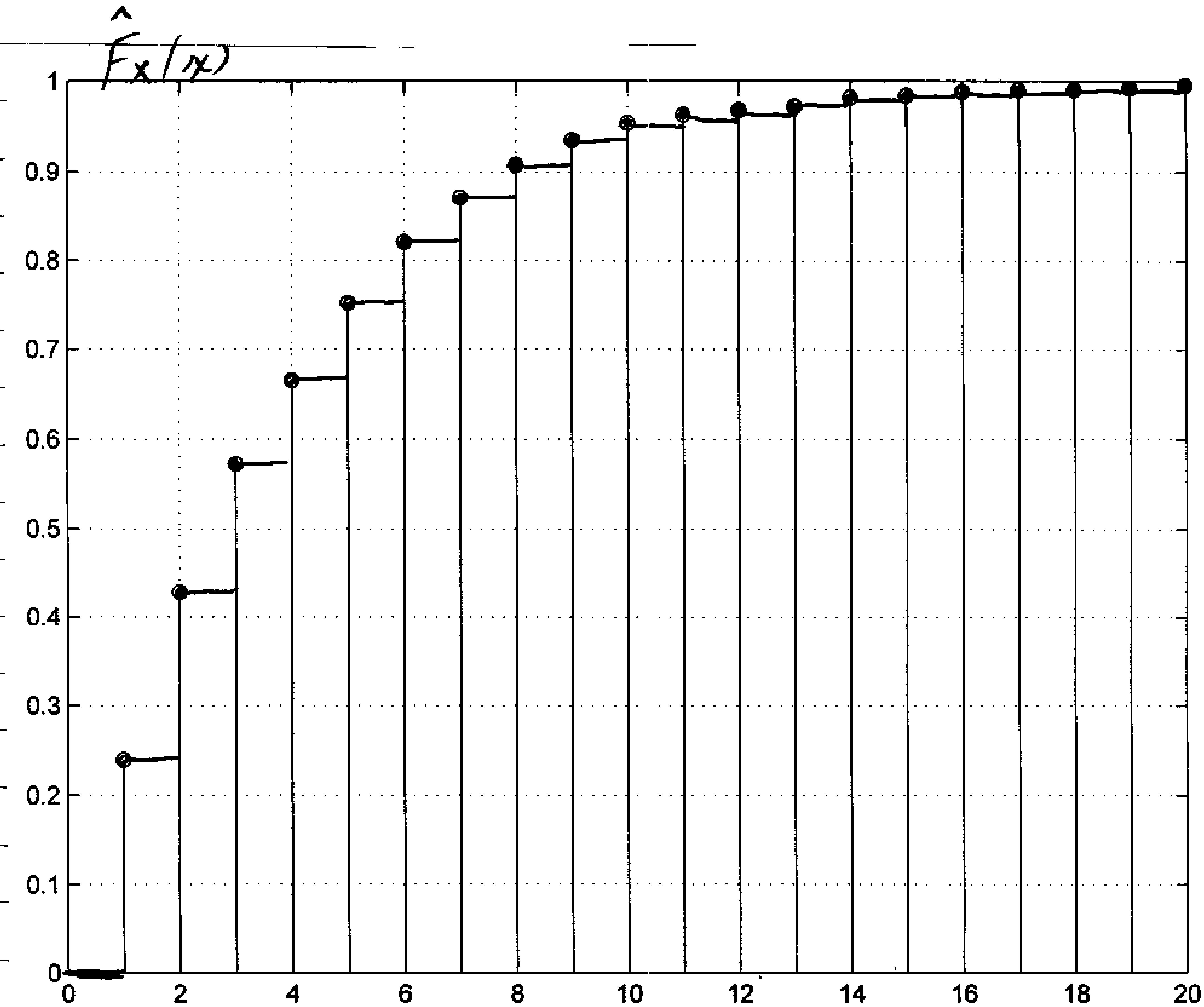
$$\frac{e^{x_2}}{e^{x_1}} = e^{x_2 - x_1} \geq 1 \quad \text{since } x_2 - x_1 \geq 0$$

$\Rightarrow \quad e^{x_2} \geq e^{x_1}$

30)

```
%probprob5_30.m
%
clear all
rand('state',0)
p=0.25;
ntrials=1000;x=zeros(ntrials,1);
for i=1:ntrials
   success=0;k=0;
while success==0
   if rand(1,1)<p
      success=1;
      k=k+1;
   else
      k=k+1;
   end
end
x(i,1)=k;
end
for i=1:20
   pX(i,1)=length(find(x-i==0))/ntrials;
end
figure
stem([1:20]',pX,'-')
hold on
stem([1:20]'+0.2,p*(1-p).^([1:20]'-1),'--')
grid
c(1,1)=pX(1);
for i=2:20
   c(i,1)=c(i-1,1)+pX(i);
end
hold off
figure
stem([1:20]',c)
grid
```


true
estimate
0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25



31) $\lambda = 1(T) = T$ T in sec

$$P[X > 100] = \sum_{k=101}^{\infty} p_X(k)$$

$$= \sum_{k=101}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= \sum_{k=101}^{\infty} e^{-60} \frac{60^k}{k!}$$

$$= 1 - \sum_{k=0}^{100} e^{-60} \frac{60^k}{k!}$$

$$= 8.68 \times 10^{-7}$$ (need Computer To evaluate)

Chapter 6

1) $$CM = \frac{\sum_{i=1}^{4} m_i x_i}{\sum_{i=1}^{4} m_i} = \frac{\sum_{i=1}^{4} 10 x_i}{\sum_{i=1}^{4} 10}$$

$$= \sum_{i=1}^{4} x_i \frac{1}{4} = \frac{1}{4}(36) = 9\,m$$

Analogous to $E[X]$ Let $p_X[i] = \dfrac{m_i}{\sum_{i=1}^{4} m_i}$

$$= \frac{10}{40} = \frac{1}{4}$$

PMF analogous to fraction of mass.
This is identical to example in Sect 6.3.

2) $$E[X] = \sum_{k=0}^{9} k\, p_X[k] = \frac{1}{10} \sum_{k=0}^{9} k = \frac{1}{10}\left[10/2(9+0)\right]$$

$$= 4\tfrac{1}{2}$$

3) $$\sum_{k=1}^{6} p_X[k] = 1 \qquad \text{Let } p = P[X=1]$$

$$3p + 6p = 1 \Rightarrow p = 1/9$$

$$E[X] = \sum_{k=1}^{6} k\, p_X[k] = \sum_{k=1}^{3} k p + \sum_{k=4}^{6} k\, 2p$$

$$= \frac{1}{9}(6) + 2/9\,(15) = 4$$

4) $$E[X] = 1 \cdot 2/3 + 0 \cdot 1/3 = 2/3$$

5) $$E[X] = \sum_{k=0}^{\infty} k\, p_X[k] = \sum_{k=0}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\left[\frac{d}{d\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right]\lambda$$

$$= e^{-\lambda}\left(\frac{d}{d\lambda} e^{\lambda}\right)\lambda = e^{-\lambda} e^{\lambda}\lambda = \lambda$$

6) $$\lim_{N_U \to \infty} \sum_{k=1}^{\infty} k\, p_X[k] = \frac{2}{\pi} \lim_{N_U \to \infty} \sum_{k=1}^{\infty} \underbrace{k/k^2}_{1/k} = \infty$$

limit not finite $\Rightarrow$
$E[X]$ does not exist since $E[X] = \infty - \infty = ?$

$$E[X] = \lim_{N \to \infty} \sum_{k=-N}^{N} k\, p_X[k] = \frac{2}{\pi} \lim_{N \to \infty} \sum_{k=-N}^{N} k/k^2$$

$$= \frac{2}{\pi} \lim_{N \to \infty} \underbrace{\sum_{k=-N}^{N} 1/k}_{\substack{=0 \\ \text{for all } N}} = \frac{2}{\pi} \lim_{N \to \infty} 0 = 0$$

7) $$E[X] = \sum_{k=-\infty}^{\infty} k\, p_X[k]$$

$$E[X] - m = \sum_{k=-\infty}^{\infty} (k-m)\, p_X[k]$$

Let $n = k - m$

$$E[X] - m = \sum_{n=-\infty}^{\infty} n\, p_X[m+n]$$

$$= \underbrace{\sum_{n=-\infty}^{-1} n\, p_X[m+n]}_{\sum_{n=1}^{\infty} (-n) \underbrace{p_X[m-n]}_{p_X[m+n]}} + \sum_{n=1}^{\infty} n\, p_X[m+n]$$

$$= 0$$

8) geometric PMF   9) $p_X(k) = \begin{cases} -1 & k=-M \\ 1 & k=M \end{cases}$

$E(X) = 0$ all M

10) $p_X(k) = \frac{1}{5} \quad k = 0, 1, 2, 3, 4$

$$Y = g(X) = \sin \frac{\pi}{2} X$$

$$= \begin{cases} 0 & X = 0 \\ 1 & 1 \\ 0 & 2 \\ -1 & 3 \\ 0 & 4 \end{cases}$$

$$\Rightarrow p_Y(k) = \begin{cases} \frac{1}{5} & k=-1 \\ 3/5 & k=0 \\ 1/5 & k=1 \end{cases}$$

$$E(Y) = \sum_{k=-1}^{1} k\, p_Y(k) = (-1)\frac{1}{5} + 0\left(3/5\right) + 1\left(\frac{1}{5}\right)$$

$$= 0$$

$$E(g(X)) = \sum_i g(x_i)\, p_X(x_i)$$

$$= \sum_{k=0}^{4} \sin \frac{\pi}{2} k \; \frac{1}{5} = \frac{1}{5}(0+1+0+(-1)+0)$$

$$= 0$$

11) $E[a_1 g_1(X) + a_2 g_2(X)] = \sum_i [a_1 g_1(x_i) + a_2 g_2(x_i)] p_X(x_i)$

$$= a_1 \sum_i g_1(x_i) p_X(x_i) + a_2 \sum_i g_2(x_i) p_X(x_i)$$

$$= a_1 E[g_1(X)] + a_2 E[g_2(X)]$$

12) $$E(X^2) = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p$$

$$\frac{d^2}{d(1-p)^2} \sum_{k=1}^{\infty} (1-p)^k = \sum_{k=1}^{\infty} k(k-1)(1-p)^{k-2}$$

$$= \sum_{k=1}^{\infty} k^2 (1-p)^{k-2} - \sum_{k=1}^{\infty} k(1-p)^{k-2}$$

$$= \frac{1}{1-p} \sum_{k=1}^{\infty} k^2 (1-p)^{k-1}$$

$$- \frac{1}{1-p} \underbrace{\sum_{k=1}^{\infty} k(1-p)^{k-1}}_{\frac{E(X)}{p}}$$

$$\Rightarrow \frac{d^2}{d(1-p)^2} \sum_{k=1}^{\infty} (1-p)^k = \frac{1}{1-p} \frac{E(X^2)}{p} - \frac{1}{1-p} \frac{E(X)}{p}$$

But $\sum_{k=1}^{\infty} (1-p)^k = \frac{1-p}{p}$

$E(X) = \frac{1}{p}$ from Sect 6.4.3

$$\Rightarrow \frac{d^2}{d(1-p)^2} \sum_{k=1}^{\infty} (1-p)^k = \frac{d^2}{dq^2} \sum_{k=1}^{\infty} q^k$$

$$= \frac{d^2}{dq^2} \frac{q}{1-q} = \frac{d}{dq} \frac{(1-q) - q(-1)}{(1-q)^2}$$

$$= \frac{d}{dq} (1-q)^{-2} = \frac{-2(-1)}{(1-q)^3}$$

$$= 2/p^3$$

Finally, $\frac{2}{p^3} = \frac{1}{(1-p)p} E(X^2) - \frac{1}{(1-p)p^2}$

$$E(X^2) = \left(\frac{2}{p^3} + \frac{1}{p^2(1-p)}\right) p(1-p)$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p}$$

$$= \frac{2(1-p)+p}{p^2} = \frac{2-p}{p^2}$$

$$= \frac{2}{p^2} - \frac{1}{p}$$

13) $E(X^2) = \sum_i x_i^2 p_X(x_i)$

$$E^2(X) = \left(\sum_i x_i p_X(x_i)\right)^2$$

Not the same. Only true if

$\mathrm{var}(X) = E[X^2] - E^2(X) = 0$

True for $p_X(k) = \begin{cases} 1 & k=0 \\ 0 & \text{otherwise} \end{cases}$

as an example or if X = constant

14) Best predictor $= E(X)$

$$= 1\cdot\tfrac{1}{8} + 2\cdot\tfrac{2}{8} + 3\cdot\tfrac{4}{8} + 4\cdot\tfrac{1}{8}$$

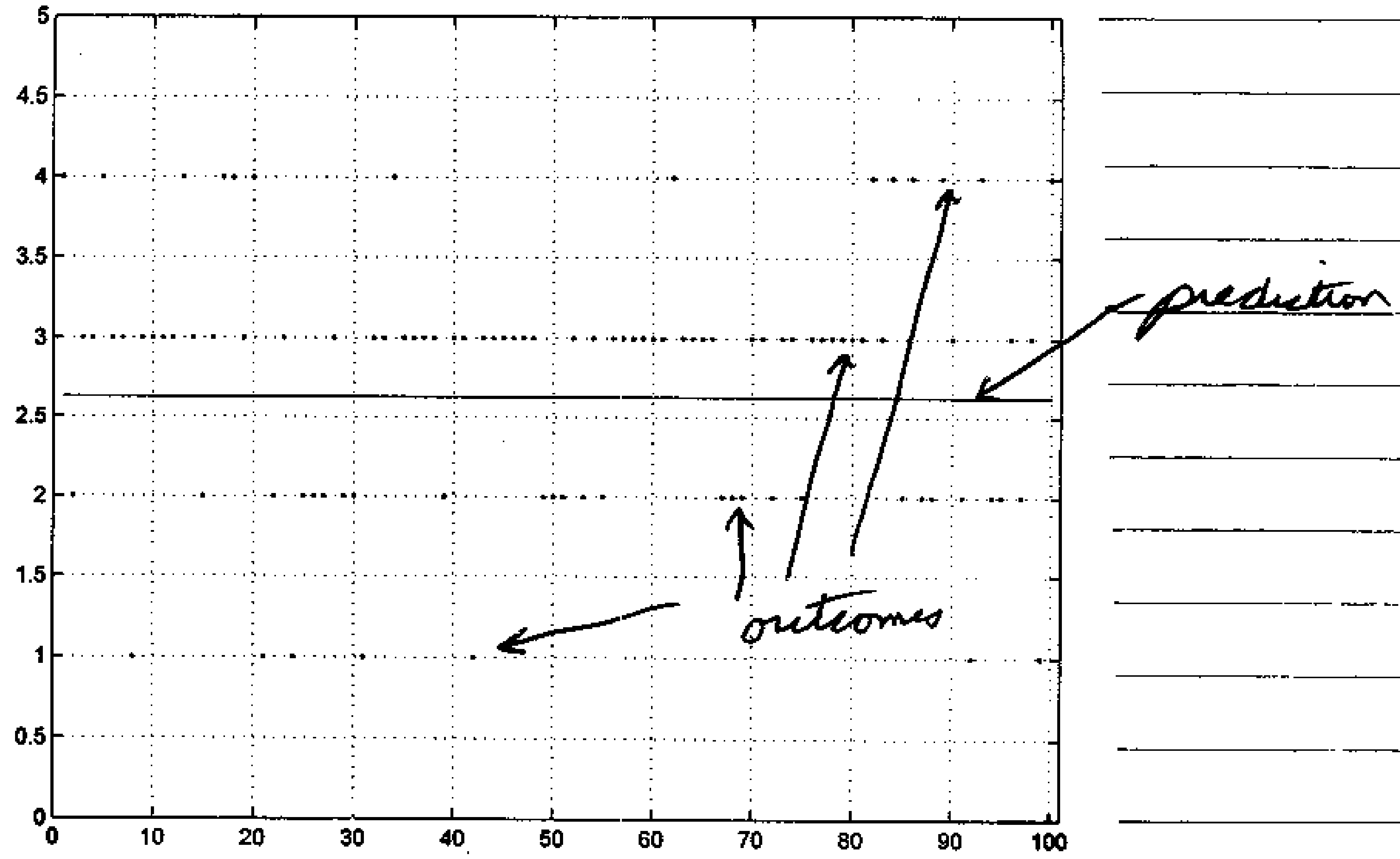
$$= 21/8$$

$$mse_{MIN} = \sum_{k=1}^{4} (k - E(X))^2 p_X(k)$$

$$= (-13/8)^2 \tfrac{1}{8} + (-5/8)^2\, 2/8 + (3/8)^2 \tfrac{4}{8} + (11/8)^2 \tfrac{1}{8}$$

$$= \frac{169+50+36+121}{512} = \frac{376}{512} = 0.7343$$

15) $\widehat{mse}_{min} = 0.7371$




```
% probprob6_15.m
%
rand('state',0)
x=zeros(10000,1);
xi=[1 2 3 4]';pX=[1/8 2/8 4/8 1/8]';
x=PMFdata(10000,xi,pX);
truemean=21/8;
mse=mean((x-truemean).^2)
figure
plot([1:100]',x(1:100),'.')
grid
axis([0 101 0 5])
hold on
plot([1:100]',truemean*ones(100,1))
hold off
```

16) a) and c) have same variance since PMFs are just shifted. b) is largest since the values at $k = 6\ 7$ have been shifted from $k = 1\ 2$ of PMF at a). This places more probability at ends of PMF and hence more

probability of being far away from the mean.

17) $$\left.\begin{aligned} \mathrm{var}(Y) &= \mathrm{var}(aX+b) \\ &= \mathrm{var}(aX) \\ &= a^2\,\mathrm{var}(X) \end{aligned}\right\}\ \text{see Prop. 6.2}$$

18) $$\mathrm{var}(X) = \underbrace{\sum_{k=0}^{\infty} k^2 p_X(k)}_{E(X^2)} - \underbrace{E^2(X)}_{\lambda^2}$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!}$$

$$\frac{d}{d\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} k \frac{\lambda^{k-1}}{k!}$$

$$\frac{d^2}{d\lambda^2} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^{k-2}}{k!}$$

$$= \sum_{k=0}^{\infty} k^2 \frac{\lambda^{k-2}}{k!} - \sum_{k=0}^{\infty} k \frac{\lambda^{k-2}}{k!}$$

$$= \frac{1}{\lambda^2} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} - \frac{1}{\lambda^2} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!}$$

$$= \frac{1}{\lambda^2} e^{\lambda} E(X^2) - \frac{1}{\lambda^2} \underbrace{E(X)}_{\lambda} e^{\lambda}$$

$$= \frac{e^{\lambda}}{\lambda^2} E(X^2) - \frac{1}{\lambda} e^{\lambda}$$

But $\displaystyle\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \Rightarrow \frac{d^2}{d\lambda^2} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$

$$e^{\lambda} = \frac{e^{\lambda}}{\lambda^2} E[X^2] - \frac{1}{\lambda} e^{\lambda}$$

$$E[X^2] = (1 + 1/\lambda)\lambda^2 = \lambda^2 + \lambda$$

$$\mathrm{var}(X) = E[X^2] - \lambda^2 = \lambda$$

19) $E[X] = 9/2$ see Prob. 6.2 solution

$$E[X^2] = \sum_{k=0}^{9} k^2 \frac{1}{10}$$

$$= \frac{1}{10} \frac{9(10)(19)}{6} \quad \text{from Section 6.6}$$

$$= 57/2$$

$$\mathrm{var}(X) = 57/2 - (9/2)^2 = \frac{114 - 81}{4} = 33/4$$

20) $\phi_X(\omega) = e^{\lambda(e^{j\omega} - 1)}$

$$E[X^2] = \frac{1}{j^2} \frac{d^2}{d\omega^2} e^{\lambda(e^{j\omega}-1)} \Big|_{\omega=0}$$

$$= -\frac{d}{d\omega} e^{\lambda(e^{j\omega}-1)} \lambda e^{j\omega}(j) \Big|_{\omega=0}$$

$$= -\lambda j \frac{d}{d\omega} e^{\lambda(e^{j\omega}-1)+j\omega} \Big|_{\omega=0}$$

$$= -\lambda j\, e^{\lambda(e^{j\omega}-1)+j\omega} [\lambda e^{j\omega}(j) + j] \Big|_{\omega=0}$$

$$= -\lambda j [\lambda j + j] = \lambda(\lambda+1) = \lambda^2 + \lambda$$

21) $\mathrm{var}(C) = E[(C - E[C])^2] = E[(c-c)^2]$

$$= E[0^2] = 0$$

$$\mathrm{var}(X+c) = E[(X+c-E[X+c])^2]$$

$$= E[(X+c-E[X]-c)^2] = E[(X-E[X])^2]$$
$$= \mathrm{var}(X)$$

$$\mathrm{var}(cX) = E[(cX-E[cX])^2]$$
$$= E[(cX-cE[X])^2]$$
$$= E[c^2(X-E[X])^2]$$
$$= c^2\,\mathrm{var}(X)$$

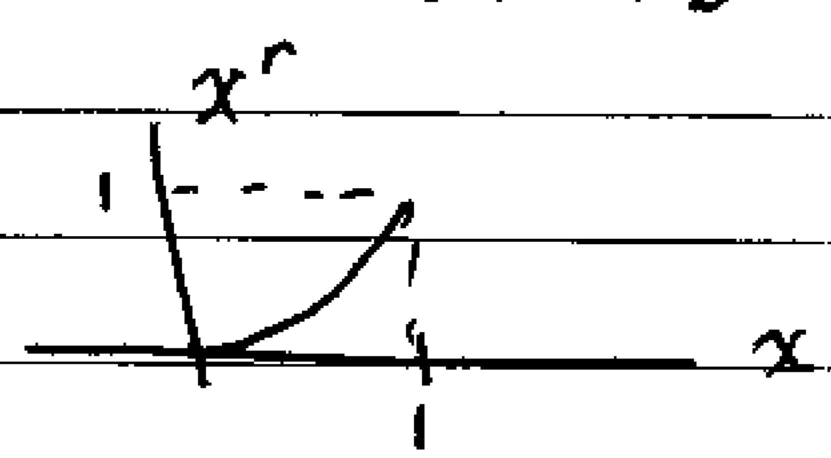
22) $\mathrm{var}(X) = 0 \Rightarrow E[(X-E(X))^2] = 0$

$$\sum_i (x_i - E[X])^2 \underbrace{p_X[x_i]}_{>0} = 0$$

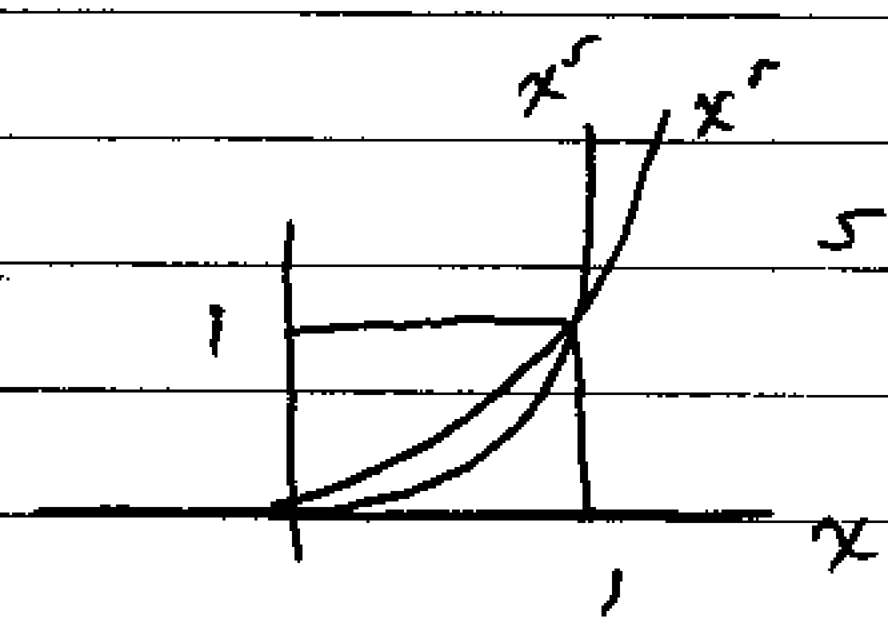
$\Rightarrow$ $x_i - E[X] = 0$ for all i

If there is more than one x_i, this cannot be true. Hence there is only one x_i and $x_i = E[X]$ or $X = E[X] =$ constant

23) a)



b) $|x|^r = \underbrace{|x|^{r-s}}_{\leq 1 \text{ for } r<s} |x|^s$



$s > r$

c) For $|x| \leq 1$

$$\underbrace{|x|^r \leq 1}_{\text{from a}} \leq |x|^s + 1$$

For $|x| > 1$ $\underbrace{|x^r| \leq |x|^s}_{\text{from b}} \leq |x|^s + 1$

d) $E[|X|^r] = \sum_i |x_i|^r p_X[x_i]$

$\leq \sum_i (|x_i|^s + 1) p_X[x_i]$ from (c)

$= \underbrace{\sum_i |x_i|^s p_X[x_i]}_{\text{assumed } < \infty} + \underbrace{\sum_i p_X[x_i]}_{=1}$

$< \infty$

24) $E[X] = -1(\frac{1}{4}) + 1(3/4) = \frac{1}{2}$

$\text{var}(X) = E[X^2] - E^2[X]$

$= (-1)^2 \frac{1}{4} + (1)^2 \, 3/4 - (\frac{1}{2})^2$

$= \frac{1}{4} + 3/4 - \frac{1}{4} = 3/4$

25) $E[X^n] = \sum_{k=-\infty}^{\infty} k^n p_X[k]$

$= \sum_{k=-\infty}^{-1} k^n p_X[k] + \sum_{k=1}^{\infty} k^n p_X[k]$

$= \sum_{k=1}^{\infty} (-k)^n p_X(-k) + \sum_{k=1}^{\infty} k^n p_X(k)$

$= \sum_{k=1}^{\infty} ((-1)^n k^n \underbrace{p_X[-k]}_{p_X[k]} + k^n p_X[k])$

$= \sum_{k=1}^{\infty} \underbrace{[(-1)^n + 1]}_{=0 \text{ for } n \text{ odd}} k^n p_X[k] = 0$

26) $E[(X - E[X])^n] = E[\sum_{k=0}^{n} \binom{n}{k} X^k (-E[X])^{n-k}]$

$= \sum_{k=0}^{n} \binom{n}{k} E[X^k] (-1)^{n-k} E^{n-k}[X]$

$$= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} E^{n-k}[X]\, E[X^k]$$

27) $$\phi_Y(\omega) = E[e^{j\omega Y}] = E[e^{j\omega(aX+b)}]$$

$$= E[e^{j\omega b} e^{j\omega a X}] = E[(\cos\omega b + j\sin\omega b)\, e^{ja\omega X}]$$

$$= \cos\omega b\, E[e^{ja\omega X}] + j\sin\omega b\, E[e^{j\omega a X}]$$

$$= (\cos\omega b + j\sin\omega b)\, E[e^{ja\omega X}]$$

$$= e^{j\omega b}\, \phi_X(a\omega)$$

$$E[Y] = \frac{1}{j} \frac{d\phi_Y(\omega)}{d\omega}\Big|_{\omega=0}$$

$$= \frac{1}{j} \frac{d}{d\omega} e^{j\omega b} \phi_X(a\omega)\Big|_{\omega=0}$$

$$= \frac{1}{j}\left[e^{j\omega b} \phi_X'(a\omega)(a) + jb\, e^{j\omega b} \phi_X(a\omega)\right]\Big|_{\omega=0}$$

$$= \frac{1}{j}\Big[1 \cdot \underbrace{\phi_X'(0)}_{jE[X]}\, a + jb\, \underbrace{\phi_X(0)}_{=1}\Big]$$

$$= aE[X] + b$$

28) $$\phi_X(\omega) = E[e^{j\omega X}]$$

$$= \sum_{k=-2}^{2} e^{j\omega k} \frac{1}{5}$$

$$= \frac{1}{5}\left(e^{-j2\omega} + e^{-j\omega} + 1 + e^{j\omega} + e^{j2\omega}\right)$$

$$= \frac{1}{5}(1 + 2\cos\omega + 2\cos 2\omega)$$

29) $$\frac{d\phi_x(\omega)}{d\omega} = \frac{d}{d\omega}\left[(pe^{j\omega}+q)^M\right] \qquad q=1-p$$

$$= M(pe^{j\omega}+q)^{M-1}pe^{j\omega}j$$

$$\frac{d^2\phi_x(\omega)}{d\omega^2} = M\{(M-1)(pe^{j\omega}+q)^{M-2}(pe^{j\omega}j)^2 + (pe^{j\omega}+q)^{M-1}pe^{j\omega}j^2\}$$

$$\left.\frac{d\phi_x(\omega)}{d\omega}\right|_{\omega=0} = Mpj$$

$$\left.\frac{d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0} = M[(M-1)(pj)^2 + pj^2]$$

$$E[x] = \frac{1}{j}\left.\frac{d\phi_x(\omega)}{d\omega}\right|_{\omega=0} = Mp$$

$$E[x^2] = \frac{1}{j^2}\left.\frac{d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0}$$

$$= M[(M-1)p^2+p]$$

$$\text{var}(x) = E[x^2] - E^2[x] =$$
$$= M^2p^2 - Mp^2 + Mp - (Mp)^2$$
$$= Mp - Mp^2$$
$$= Mp(1-p)$$

30) $$\frac{d}{d\omega}e^{\lambda(e^{j\omega}-1)} = e^{\lambda(e^{j\omega}-1)}\lambda e^{j\omega}j$$

$$\frac{d^2 e^{\lambda(e^{j\omega}-1)}}{d\omega^2} = e^{\lambda(e^{j\omega}-1)}(\lambda e^{j\omega}j)^2 + e^{\lambda(e^{j\omega}-1)}\lambda e^{j\omega}(j)^2$$

$$\left.\frac{d\,\phi_x(w)}{dw}\right|_{w=0} = \lambda j$$

$$\Rightarrow E(x) = \frac{1}{j}(\lambda j) = \lambda$$

$$\left.\frac{d^2\,\phi_x(w)}{dw^2}\right|_{w=0} = (\lambda j)^2 + \lambda j^2$$

$$E(x^2) = \frac{1}{j^2}(\lambda^2 j^2 + \lambda j^2) = \lambda^2 + \lambda$$

$$\text{var}(x) = E(x^2) - E^2(x) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

31) $$p_x(k) = \int_{-\pi}^{\pi} \underbrace{\phi_x(\omega)}_{\cos\omega} e^{-j\omega k}\,\frac{d\omega}{2\pi}$$

Can do integration to find $p_x(k)$.
Easier way is to recognize

$$\phi_x(\omega) = \sum_{k=-\infty}^{\infty} p_x(k)\, e^{j\omega k}$$

$$= \underbrace{p_x(-1)}_{\frac{1}{2}} e^{-j\omega} + \underbrace{p_x(1)}_{\frac{1}{2}} e^{j\omega}$$

$$= \cos\omega$$

32) True mean $= \frac{1}{2}$
True var $= 3/4$

$\hat{E}[X] = 0.5000$

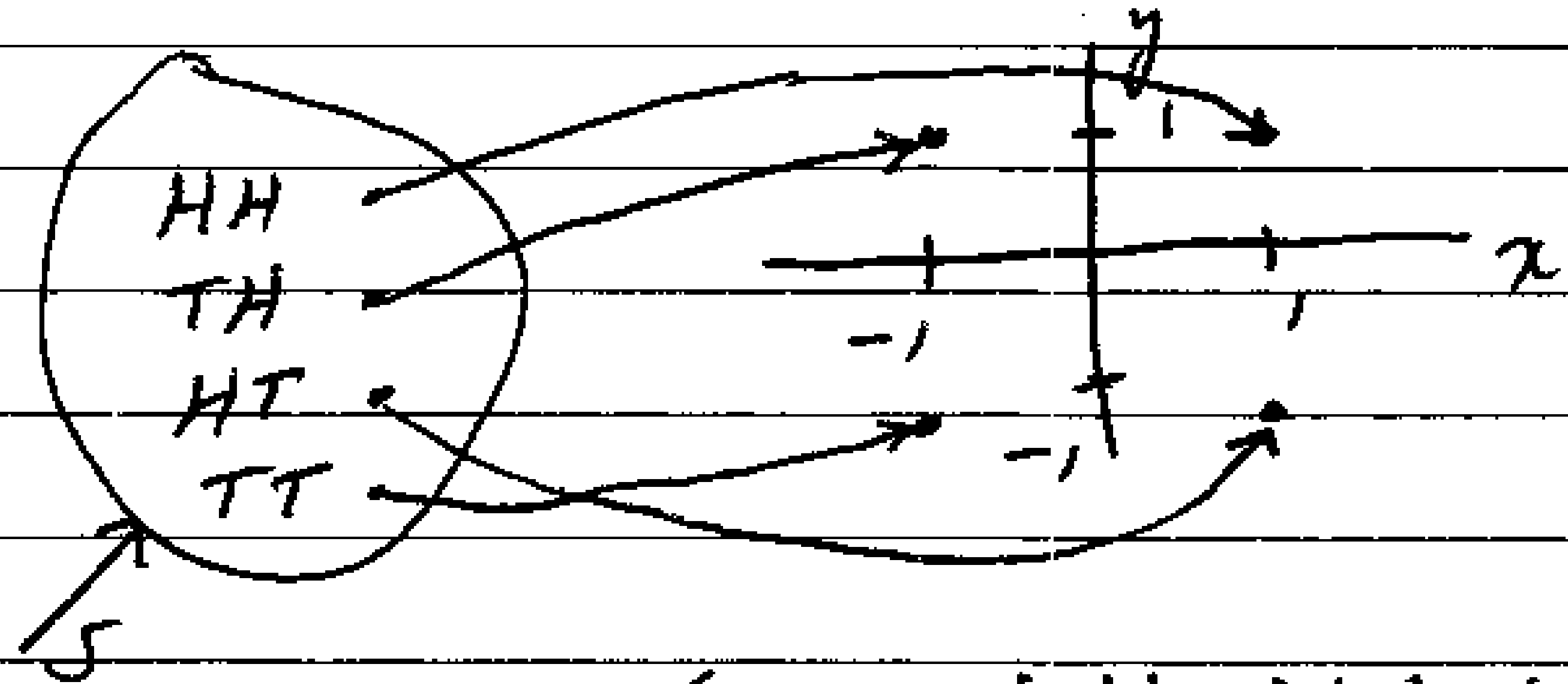
$\hat{\text{var}}(X) = 0.7500$

```
% probprob6_32.m
%
clear all
rand('state',0)
xi=[-1 1]';pX=[1/4 3/4]';
x=PMFdata(1000,xi,pX);
meanest=mean(x)
varest=mean(x.^2)-meanest^2
```

Chapter 7

1) $S_{X,Y} = \{(i,j):\ i = 1, 2, \ldots, 8;\ j = 1, 2, \ldots, 8\}$

2)

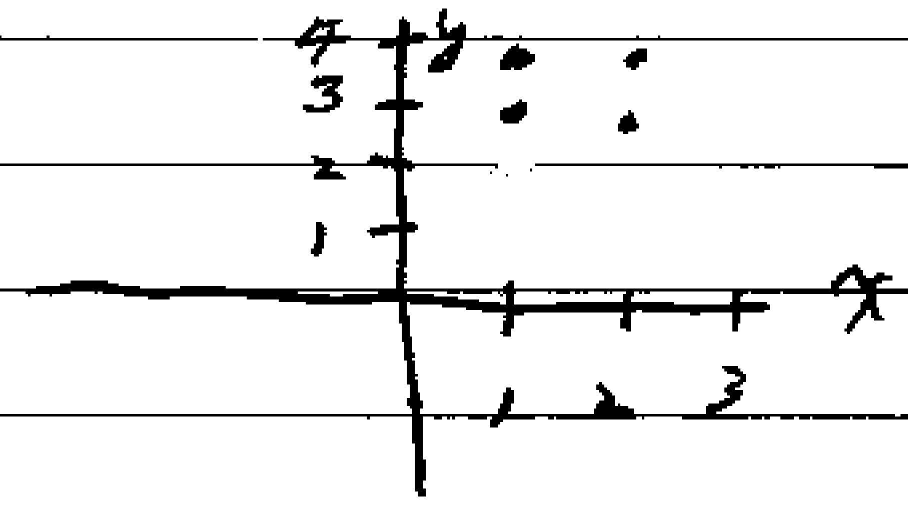


$$S_{X,Y} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix} \right\}$$

3) $S = \{(p,n), (p,d), (n,p), (n,d), (d,p), (d,n)\}$

$S_{X,Y} = \{(1,5), (1,10), (5,1), (5,10), (10,1), (10,5)\}$

4)

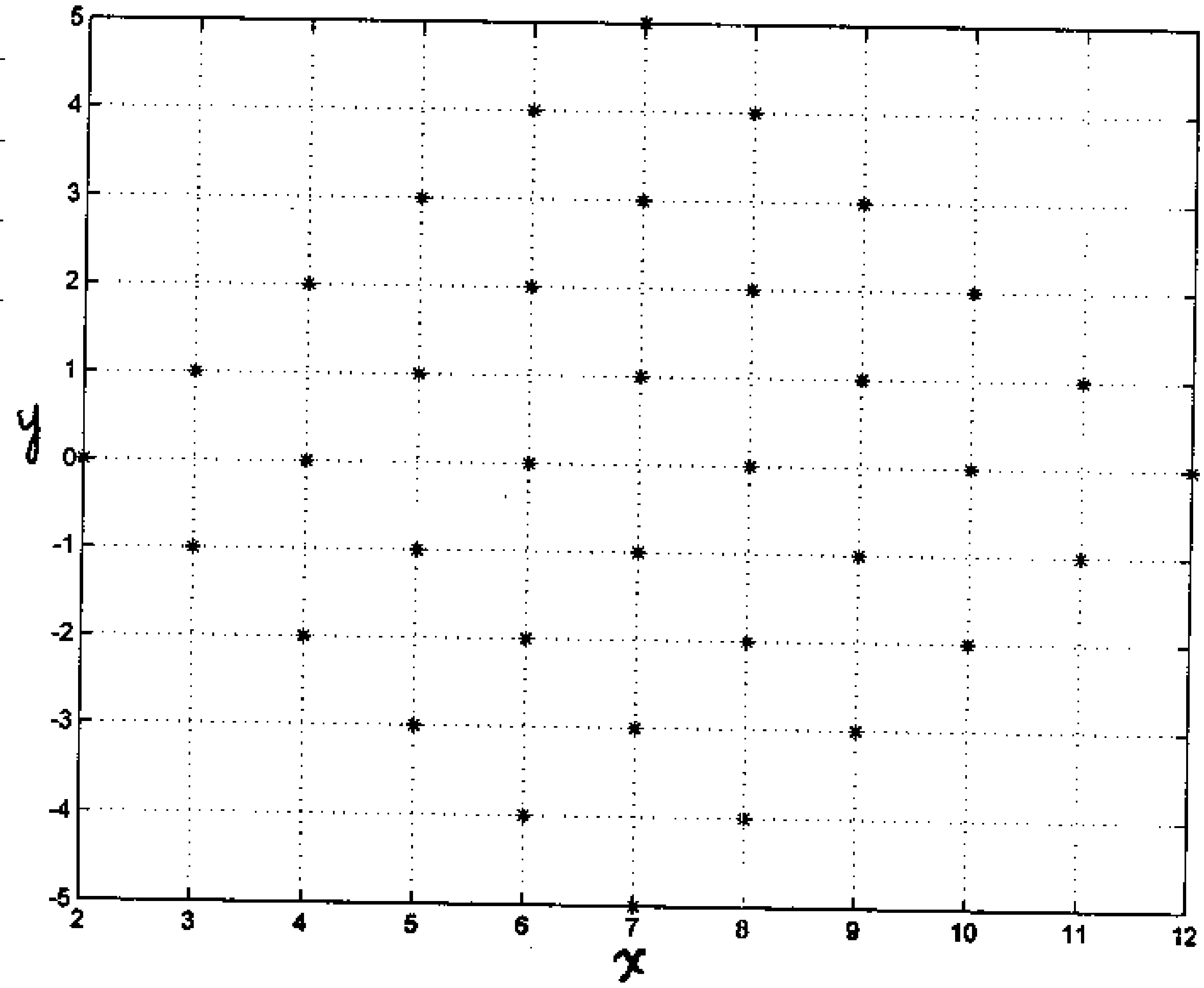


$N_{XY} = 4 = N_X N_Y$

5) X = sum, Y = difference

die 1 \ die 2	1	2	3	4	5	6
1	(2,0)	(3,-1)	(4,-2)	(5,-3)	(6,-4)	(7,-5)
2	(3,1)	(4,0)	(5,-1)	(6,-2)	(7,-3)	(8,-4)
3	(4,2)	(5,1)	(6,0)	(7,-1)	(8,-2)	(9,-3)
4	(5,3)	(6,2)	(7,1)	(8,0)	(9,-1)	(10,-2)
5	(6,4)	(7,3)	(8,2)	(9,1)	(10,0)	(11,-1)
6	(7,5)	(8,4)	(9,3)	(10,2)	(11,1)	(12,0)

$N_{XY} = 36$



6) $$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c(1-p_1)^i(1-p_2)^j = c\sum_{i=1}^{\infty}(1-p_1)^i\sum_{j=1}^{\infty}(1-p_2)^j$$

$$= c\,\frac{1-p_1}{1-(1-p_1)}\,\frac{1-p_2}{1-(1-p_2)} = c\,\frac{(1-p_1)(1-p_2)}{p_1 p_2} = 1$$

$$\Rightarrow \quad c = \frac{p_1 p_2}{(1-p_1)(1-p_2)}$$

$$p_{X,Y}[i,j] = p_1(1-p_1)^{i-1}\,p_2(1-p_2)^{j-1}$$

7) $$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\tfrac{1}{2}\right)^{i+j} = \sum_{i=0}^{\infty}\left(\tfrac{1}{2}\right)^i\sum_{j=0}^{\infty}\left(\tfrac{1}{2}\right)^j$$

$$= \frac{1}{1-\frac{1}{2}}\cdot\frac{1}{1-\frac{1}{2}} = 4 \quad \text{No}$$

8) $S = \{(H,H),\ (H,T),\ (T,H),\ (T,T)\}$

$S_{X,Y} = \left\{ \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix} \right\}$

$p_{X,Y}[i,j] = \frac{1}{4}$

9) Using (7.2) with $P[\{s_k\}] = 1/36$

$$p_{X,Y}[i,j] = \begin{cases} 1/36 & i=1, j=1 \\ 2/36 & i=1, j=2 \\ 3/36 & i=1, j=3 \\ 4/36 & i=1, j=4 \\ 5/36 & i=1, j=5 \\ 6/36 & i=1, j=6 \\ 1/36 & i=0, j=2 \\ 2/36 & i=0, j=3 \\ 3/36 & i=0, j=4 \\ 4/36 & i=0, j=5 \\ 5/36 & i=0, j=6 \end{cases}$$

Could perform a computer simulation

10)
$$P[\{s_k\}] = \frac{1}{N_X N_Y} = \frac{1}{20}$$

$$P[(X,Y) \in A] = \sum_{\{(i,j):(x_i,y_j)\in A\}} \underbrace{p_{X,Y}[x_i,y_j]}_{= P[\{s_k\}]}$$

$$= \sum_{\{(i,j):(i,j)\in A\}} 1/20$$

$$= 4/20 = 1/5$$

11) $$P[A] = \sum_{i=1}^{3} \sum_{j=2}^{\infty} \left(\tfrac{1}{2}\right)^{i+j} = \sum_{i=1}^{3} \left(\tfrac{1}{2}\right)^{i} \sum_{j=2}^{\infty} \left(\tfrac{1}{2}\right)^{j}$$

$$= \left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right) \frac{\frac{1}{4}}{1-\frac{1}{2}} = \tfrac{7}{8}\left(\tfrac{1}{2}\right) = 7/16$$

12) $p_X(i) = \frac{1}{8} + \frac{1}{4} = 3/8 \qquad i=0$

$\frac{1}{8} + \frac{1}{4} = 3/8 \qquad i=1$

$\frac{1}{8} + \frac{1}{8} = 2/8 \qquad i=2$

$p_Y(j) = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} \qquad j=0$

$\frac{1}{8} \qquad j=1$

$\frac{1}{4} + \frac{1}{4} + \frac{1}{8} = \frac{5}{8} \qquad j=2$

13) $p_X(i) = \sum_{j=1}^{\infty} p^2(1-p)^{i+j-2} \qquad i=1,2,\ldots$

$$= p^2(1-p)^{i-2} \underbrace{\sum_{j=1}^{\infty} (1-p)^j}_{\frac{1-p}{p}}$$

$$= p(1-p)^{i-1}$$

and $p_Y(j) = p(1-p)^{j-1}$ (interchange i and j)

14) $p_X(i) = \sum_{j=1}^{6} p_{X,Y}(i,j) = 6\left(\frac{1}{36}\right) \qquad i=1,2,\ldots,6$

$= 1/6$

same for $p_Y(j)$

15) $\sum_{i=0}^{1}\sum_{j=0}^{10} c\binom{10}{j}\left(\tfrac{1}{2}\right)^{10} = 2c \underbrace{\sum_{j=0}^{10}\binom{10}{j}\left(\tfrac{1}{2}\right)^{10}}_{\text{bin}(10,\frac{1}{2})} = 2c = 1$

$\Rightarrow \quad c = \frac{1}{2}$

$$p_X(i) = \sum_{j=0}^{10} \tfrac{1}{2}\binom{10}{j}\left(\tfrac{1}{2}\right)^{10} = \tfrac{1}{2}\underbrace{\sum_{j=0}^{10}\binom{10}{j}\left(\tfrac{1}{2}\right)^{10}}_{=1} = \tfrac{1}{2}$$

$$p_Y(j) = \sum_{i=0}^{1} \frac{1}{2}\binom{10}{j}\left(\frac{1}{2}\right)^{10} \qquad j = 0, 1, \ldots, 10$$

$$= \binom{10}{j}\left(\frac{1}{2}\right)^{10}$$

16)

	$j=0$	$j=1$	$p_X(i)$
$i=0$	$\frac{1}{4}$	0	$\frac{1}{4}$
$i=1$	$\frac{1}{8}$	$\frac{5}{8}$	$3/4$
$p_Y(j)$	$\frac{3}{8}$	$\frac{5}{8}$	

17) a) $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$= P(A) \qquad A = \{(x',y') : x' \le x, y' \le y\}$

$\Rightarrow \quad 0 \le F_{X,Y}(x,y) \le 1$

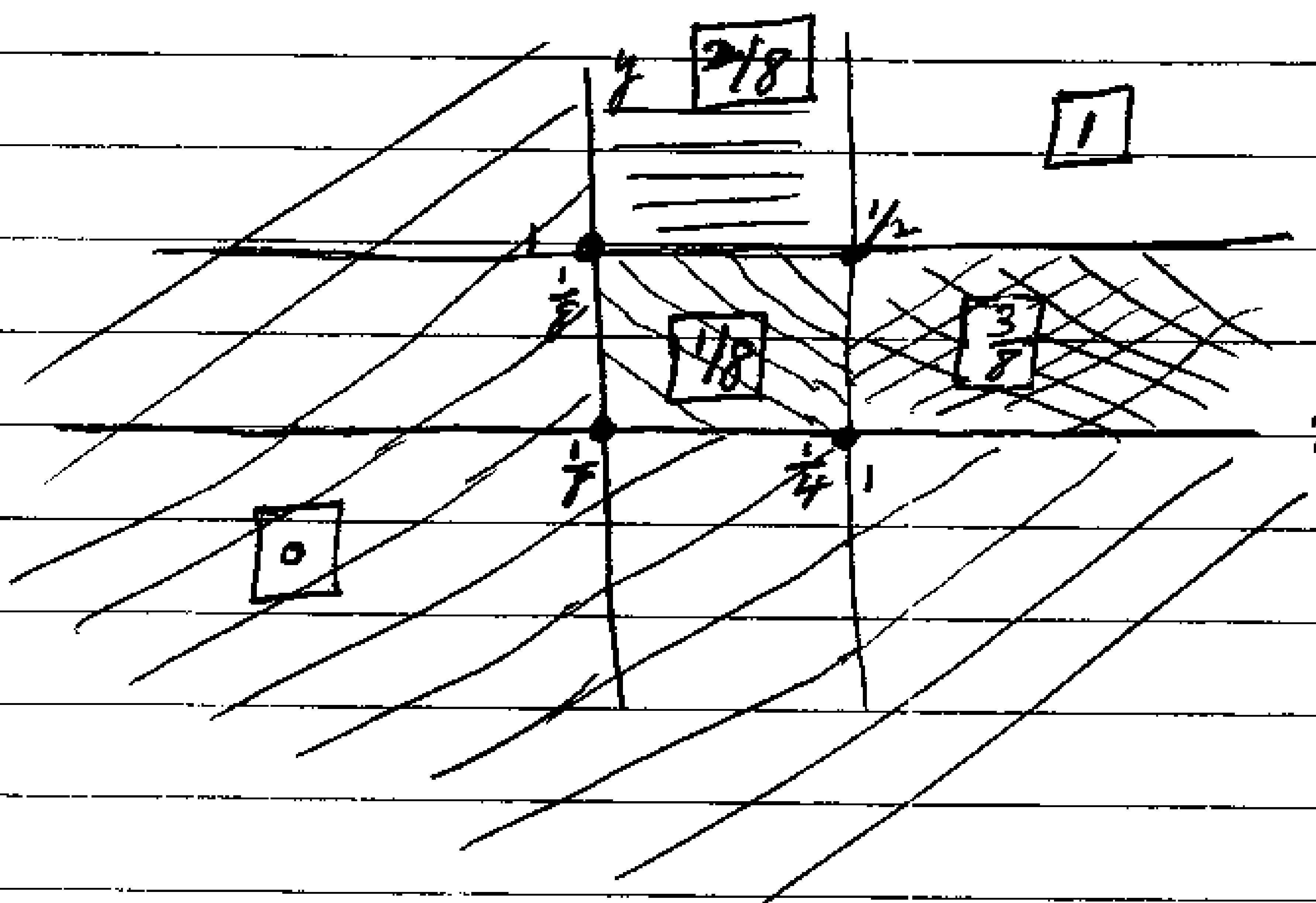
b) $F_{X,Y}(-\infty, -\infty) = P(X \le -\infty, Y \le -\infty)$

$= P(\phi) = 0$

$F_{X,Y}(+\infty, +\infty) = P(X \le \infty, Y \le \infty)$

$= P(S) = 1$

18)



$\square$ indicates CDF value

19) $p_{X,Y}(0,0) = \frac{1}{4} \quad p_X(0) = \frac{1}{4} \quad p_Y(0) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$p_{X,Y}(0,0) \ne p_X(0)\,p_Y(0)$ No

20) $$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \sum_{\{(x_i,y_j):\, x_i \le x,\, y_j \le y\}} \sum p_{X,Y}(x_i, y_j)$$

$$= \sum_{\{x_i:\, x_i \le x\}} \sum_{\{y_j:\, y_j \le y\}} \underbrace{p_{X,Y}(x_i, y_j)}_{p_X(x_i)\, p_Y(y_j)}$$

$$= \sum_{\{x_i:\, x_i \le x\}} p_X(x_i) \sum_{\{y_j:\, y_j \le y\}} p_Y(y_j)$$

$$= F_X(x)\, F_Y(y)$$

21) $p_X(0) = a + b \qquad p_X(1) = 1 - (a+b)$

$p_Y(0) = a + c \qquad p_Y(1) = 1 - (a+c)$

For independence

$a = (a+b)(a+c) \qquad (0,0)$

$b = (a+b)(1-(a+c)) \qquad (0,1)$

$c = (1-(a+b))(a+c) \qquad (1,0)$

$d = (1-(a+b))(1-(a+c)) \qquad (1,1)$

$$\Rightarrow \quad \frac{a}{b} = \frac{c}{d} \quad \text{or} \quad ad = bc$$

22) $$p_{X,Y}(i,j) = p_X(i)\, p_Y(j)$$

$$= p_X^{\,i}(1-p_X)^{1-i}\, p_Y^{\,j}(1-p_Y)^{1-j}$$

$$= p_X^{\,i}\, p_Y^{\,j}(1-p_X)^{1-i}(1-p_Y)^{1-j}$$

$i = 0, 1; \; j = 0, 1$

23) $p_{X}(i,j) = \underbrace{\binom{10}{i}\left(\frac{1}{2}\right)^{10}}_{bin(10,\frac{1}{2})} \underbrace{\binom{11}{j}\left(\frac{1}{2}\right)^{11}}_{bin(11,\frac{1}{2})}$

Yes

24) $\left.\begin{array}{l} W = X \\ Z = X - Y \end{array}\right\} \Rightarrow \begin{array}{lll} k = i & \text{or} & i = k \\ l = i - j & \text{or} & j = k - l \end{array}$

But i and j take on all integer values
$\Rightarrow$ k and l " " " " "

$p_{W,Z}[k,l] = p_{X,Y}[k, k-l]$ mapping is one-to-one

$p_Z[l] = \sum_{k=-\infty}^{\infty} p_{X,Y}[k, k-l] = \sum_{k=-\infty}^{\infty} p_X[k]\, p_Y[k-l]$

Now if X, Y take on values $0, 1, \ldots$ then
$i = 0, 1, \ldots$; $j = 0, 1, \ldots \Rightarrow k = i = 0, 1, \ldots$

$\Rightarrow p_Z[l] = \sum_{k=0}^{\infty} p_{X,Y}[k, k-l] = \sum$

and $j = k - l = 0, 1, \ldots \Rightarrow k \geq l$

$\Rightarrow p_Z[l] = \begin{array}{ll} \sum_{k=l}^{\infty} p_{X,Y}[k, k-l] & \text{if } l \geq 0 \\ \sum_{k=0}^{\infty} p_{X,Y}[k, k-l] & l < 0 \end{array}$

and $l = i - j = \ldots, -1, 0, 1, \ldots$

$p_Z[l] = \sum_{k=l}^{\infty} \underbrace{p_{X,Y}[k, k-l]}_{p_X[k]\, p_Y[k-l]} \qquad l = 0, 1, \ldots$

$= \sum_{k=0}^{\infty} \underbrace{p_{X,Y}[k, k-l]}_{p_X[k]\, p_Y[k-l]} \qquad l = \ldots, -2, -1$

25) $$p_Z(l) = \sum_{k=\max(0,l)}^{\infty} p_X(k)\, p_Y(k-l)$$

$$= \sum_{k=\max(0,l)}^{\infty} e^{-\lambda_X} \frac{\lambda_X^{k}}{k!}\, e^{-\lambda_Y} \frac{\lambda_Y^{k-l}}{(k-l)!}$$

For $l < 0$

$$= \sum_{k=0}^{\infty} e^{-(\lambda_X+\lambda_Y)} \frac{(\lambda_X\lambda_Y)^k}{k!\,(k-l)!} \lambda_Y^{-l}$$

$$= \frac{e^{-(\lambda_X+\lambda_Y)}}{\lambda_Y^{\,l}} \sum_{k=0}^{\infty} \frac{(\lambda_X\lambda_Y)^k}{k!\,(k-l)!}$$

For $l \geq 0$

$$p_Z(l) = \frac{e^{-(\lambda_X+\lambda_Y)}}{\lambda_Y^{\,l}} \sum_{k=l}^{\infty} \frac{(\lambda_X\lambda_Y)^k}{k!\,(k-l)!}$$

26)

	$j=0$	$j=1$
$i=0$	0	1
$i=1$	1	1

entries are Z

$$p_Z(k) = \begin{cases} 3/8 & k=0 \\ 5/8 & k=1 \end{cases}$$

27) $$p_Z(j) = \sum_{i=-\infty}^{\infty} p_X(i)\, p_Y(j-i)$$

$$p_X[i] = \left(\tfrac{1}{2}\right)^i \left(1-\tfrac{1}{2}\right)^{1-i} = \tfrac{1}{2} \qquad i = 0, 1$$

$$p_Y[i] = \tfrac{1}{2} \qquad i = 0, 1$$

$$\Rightarrow \quad Z = 0, 1, 2$$

$$p_Z[j] = \sum_{i=0}^{j} p_X[i]\, p_Y[j-i] \quad \text{since } i \ge 0,\ j-i \ge 0$$

$$p_Z[0] = \sum_{i=0}^{0} p_X[i]\, p_Y[0-i] = p_X[0]\, p_Y[0] = \tfrac{1}{4}$$

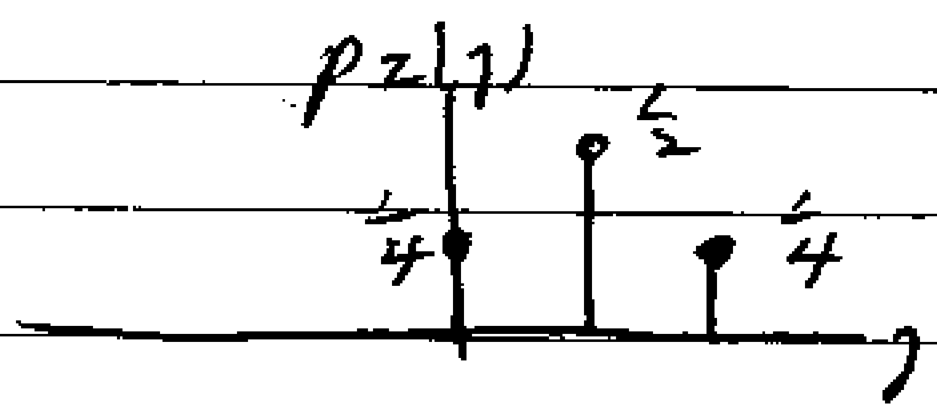
$$p_Z[1] = \sum_{i=0}^{1} p_X[i]\, p_Y[1-i]$$

$$= p_X[0]\, p_Y[1] + p_X[1]\, p_Y[0] = \tfrac{1}{2}$$

$$p_Z[2] = \sum_{i=0}^{2} p_X[i]\, p_Y[2-i]$$

$$= p_X[0]\, \overset{0}{\cancel{p_Y[2]}} + p_X[1]\, p_Y[1] + \overset{0}{\cancel{p_X[2]}}\, p_Y[0]$$

$$= 1/4$$



var(X + Y) = var(X) + var(Y)
due to independence
⇒ width increases

28) $E_{X,Y}[g(X)] = \sum_i \sum_j g(x_i)\, p_{X,Y}[x_i, y_j]$

$$= \sum_i g(x_i) \underbrace{\sum_j p_{X,Y}[x_i, y_j]}_{p_X[x_i]} = E_X[g(X)]$$

No

29) $E_{X,Y}[a\,g(X) + b\,h(Y)] =$

$$\sum_i \sum_j [a\,g(x_i) + b\,h(y_j)]\, p_{X,Y}[x_i, y_j]$$

$$= \sum_i \sum_j a\,g(x_i)\, p_{X,Y}[x_i, y_j] + \sum_i \sum_j b\,h(y_j)\, p_{X,Y}[x_i, y_j]$$

$$= a \sum_i g(x_i) \underbrace{\sum_j p_{X,Y}[x_i, y_j]}_{p_X[x_i]} + b \sum_j h(y_j) \underbrace{\sum_i p_{X,Y}[x_i, y_j]}_{p_Y(y_j)}$$

$$= a\,E_X[g(X)] + b\,E_Y[h(Y)]$$

30) $E_{X,Y}[g(X)\,h(Y)] = \sum_i \sum_j g(x_i)\, h(y_j)\, \underbrace{p_{X,Y}[x_i, y_j]}_{= p_X p_Y}$

$$= \sum_i g(x_i)\, p_X[x_i] \sum_j h(y_j)\, p_Y[y_j]$$

$$= E_X[g(X)]\, E_Y[h(Y)]$$

31) $\mathrm{var}(X-Y) = E_{X,Y}[((X-Y) - E_{X,Y}[X-Y])^2]$

$$= E_{X,Y}[((X - E_X[X]) - (Y - E_Y[Y]))^2]$$

$$= E_{X,Y}[(X - E_X[X])^2 - 2(X - E_X[X])(Y - E_Y[Y]) + (Y - E_Y[Y])^2]$$

$$= \mathrm{var}(X) + \mathrm{var}(Y) - 2\,\mathrm{cov}(X,Y)$$

They are equal if X, Y are uncorrelated or $\mathrm{cov}(X,Y) = 0$.

32) $Cov(x,y) = E_{x,y}[xy] - E_x[x]E_y[y]$

$E_x[x] = \frac{1}{2}$, $E_y[y] = \frac{1}{2}$

$E_{x,y}[xy] = 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} = \frac{1}{4}$

$Cov(x,y) = \frac{1}{4} - (\frac{1}{2})^2 = 0$

Must be zero since x,y are independent

33) $E_x[x] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E_y[y] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

$E_{x,y}[xy] = 0 \cdot 3/8 + 0 \cdot \frac{1}{8} + 0 \cdot \frac{1}{8} + 1 \cdot 3/8 = 3/8$

$Cov(x,y) = 3/8 - (\frac{1}{2})^2 = \frac{1}{8}$

34) a) $Cov(x,y) = E_{x,y}[(x - E_x[x])(y - E_y[y])]$

$= E_{x,y}[xy - xE_y[y] - E_x[x]y + E_x[x]E_y[y]]$

$= E_{x,y}[xy] - \underbrace{E_{x,y}[x]}_{E_x[x]}E_y[y] - E_x[x]\underbrace{E_{x,y}[y]}_{E_y[y]} + E_x[x]E_y[y]$

$= E_{x,y}[xy] - E_x[x]E_y[y]$

b) $Cov(x,x) = E_{x,x}[(x - E_x[x])(x - E_x[x])]$

$= E_x[(x - E_x[x])^2] = var(x)$

c) $Cov(y,x) = E_{x,y}[(y - E_y[y])(x - E_x[x])]$

$= E_{x,y}[(x - E_x[x])(y - E_y[y])]$

$= Cov(x,y)$

d) $Cov(cx,y) = E_{x,y}[(cx - E_x[cx])(y - E_y[y])]$

$= E_{X,Y}[c(X - E_X[X])(Y - E_Y[Y])]$

$= c E_{X,Y}[(X - E_X[X])(Y - E_Y[Y])] = c\, \mathrm{cov}(X, Y)$

e) Same proof

f) $\mathrm{cov}(X, X+Y) = E_{X,Y}[(X - E_X[X])(X + Y - E_{X,Y}[X+Y])]$

$= E_{X,Y}[(X - E_X[X])(X - \underbrace{E_{X,Y}[X]}_{E_X[X]} + Y - \underbrace{E_{X,Y}[Y]}_{E_Y[Y]})]$

$= E_{X,Y}[(X - E_X[X])(X - E_X[X])]$
$+ E_{X,Y}[(X - E_X[X])(Y - E_Y[Y])]$

$= \mathrm{cov}(X, X) + \mathrm{cov}(X, Y)$

g) Same proof

35) $\mathrm{cov}(W, Z) = \mathrm{cov}(X, aX + Y)$
$= \mathrm{cov}(X, aX) + \mathrm{cov}(X, Y)$
$= a\, \mathrm{cov}(X, X) + \mathrm{cov}(X, Y)$ Problem 7.34
$= \dfrac{-\mathrm{cov}(X, Y)}{\mathrm{var}(X)} \mathrm{var}(X) + \mathrm{cov}(X, Y) = 0$

36) $a = -\mathrm{cov}(X, Y)/\mathrm{var}(X)$

From Problem 7.33 $\mathrm{cov}(X, Y) = 1/8$
$\mathrm{var}(X) = E_X[X^2] - E_X^2[X]$
$= (1 \cdot \frac{1}{2}) - (\frac{1}{2})^2 = 1/4$

$a = -1/8 / 1/4 = -\frac{1}{2} \Rightarrow Z = -\frac{1}{2} X + Y$

$$\begin{aligned} \operatorname{cov}(W, Z) &= \operatorname{cov}(X, -\tfrac{1}{2}X + Y) \\ &= -\tfrac{1}{2}\operatorname{var}(X) + \operatorname{cov}(X, Y) \\ &= -\tfrac{1}{2}(\tfrac{1}{4}) + \tfrac{1}{8} = 0 \end{aligned}$$

37) $p_{X,Y} = p_X p_Y \Rightarrow$ independent

$\Rightarrow \operatorname{cov}(X, Y) = 0$

38) $E_{X,Y}\left[\left(Y - \left(\tfrac{14}{11}X - \tfrac{1}{11}\right)\right)^2\right]$

$= E_{X,Y}\left[\left(Y - \tfrac{14}{11}X + \tfrac{1}{11}\right)\left(Y - \tfrac{14}{11}X + \tfrac{1}{11}\right)\right]$

$= E_{X,Y}\Big(Y^2 - \tfrac{14}{11}XY + \tfrac{1}{11}Y - \tfrac{14}{11}XY + \left(\tfrac{14}{11}\right)^2 X^2 - \tfrac{14}{121}X$
$\quad + \tfrac{1}{11}Y - \tfrac{14}{121}X + \tfrac{1}{121}\Big)$

$= E_Y(Y^2) - \tfrac{28}{11}E_{X,Y}(XY) + \left(\tfrac{14}{11}\right)^2 E_X(X^2)$
$\quad - \tfrac{28}{121}E_X(X) + \tfrac{2}{11}E_Y(Y) + \tfrac{1}{121}$

$= \tfrac{7}{2} - \tfrac{28}{11}\cdot\tfrac{11}{4} + \left(\tfrac{14}{11}\right)^2 \tfrac{9}{4} - \tfrac{28}{121}\cdot\tfrac{5}{4} + \tfrac{2}{11}\cdot\tfrac{3}{2} + \tfrac{1}{121}$

$= 3/22$

39) $mse_{min} = \operatorname{var}(Y)(1 - \rho_{X,Y}^2)$

$= 5/4\left(1 - \dfrac{(7/8)^2}{\tfrac{11}{16}\cdot\tfrac{5}{4}}\right)$

$= 5/4 - \dfrac{98}{88} = 3/22$

40)

	$j=0$	$j=1$	$p_X(i)$
$i=0$	$\frac{1}{8}$	$\frac{1}{4}$	$3/8$
$i=1$	$\frac{1}{4}$	$\frac{3}{8}$	$5/8$
$p_Y(j)$	$3/8$	$5/8$	

$E_Y(Y) = 5/8$

$\operatorname{var}(Y) = E_Y(Y^2) - E_Y^2(Y)$

$= 5/8 - 5/8^2 = 15/64$

Optimal prediction $= E_Y(Y) = 5/8$

$mse_{MIN} = var(Y) = 15/64$

$E_X(X) = 5/8 \qquad var(X) = var(Y) = 15/64$

$E_{X,Y}(XY) = 3/8$

$cov(X,Y) = 3/8 - (5/8)^2 = -1/64$

$$\hat{Y} = E_Y(Y) + \frac{cov(X,Y)}{var(X)}(x - E_X(X))$$

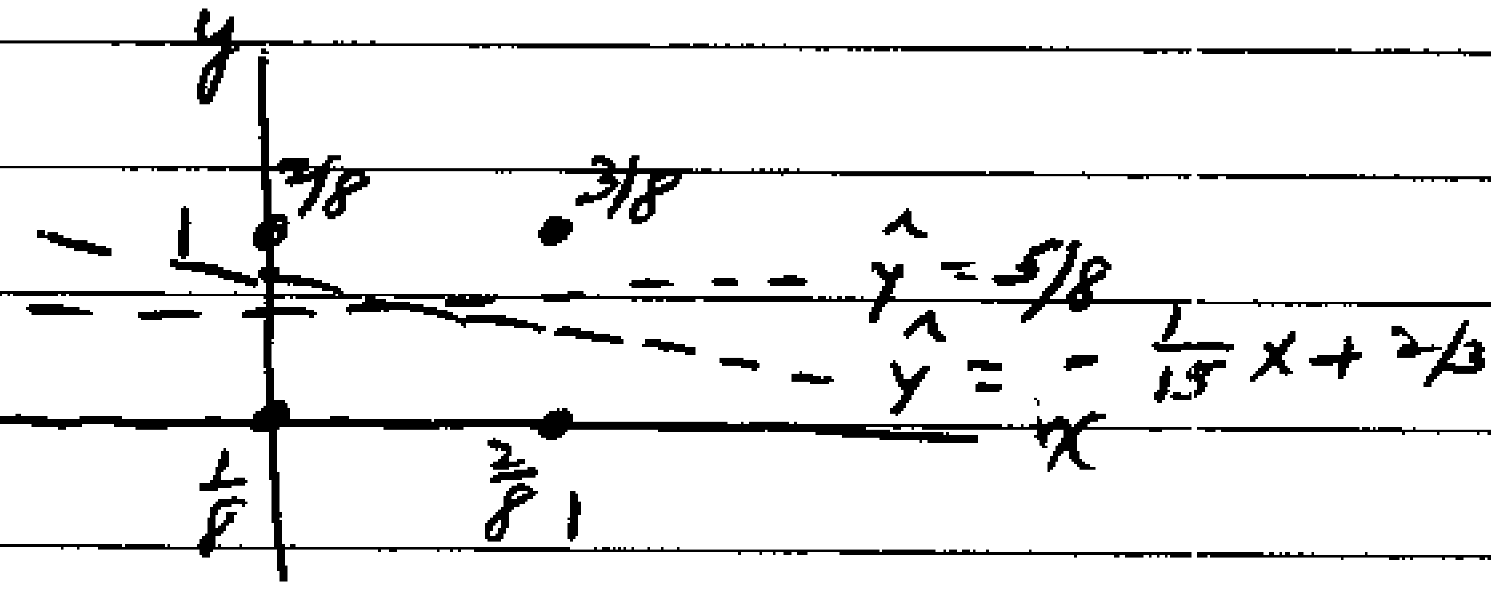
$$= 5/8 - \frac{1/64}{15/64}(x - 5/8)$$

$$= -\frac{1}{15}x + 5/8 + \frac{1}{24} = -\frac{1}{15}x + 2/3$$

$$\rho_{X,Y} = \frac{-1/64}{\sqrt{15/64 \cdot 15/64}} = -\frac{1}{15}$$

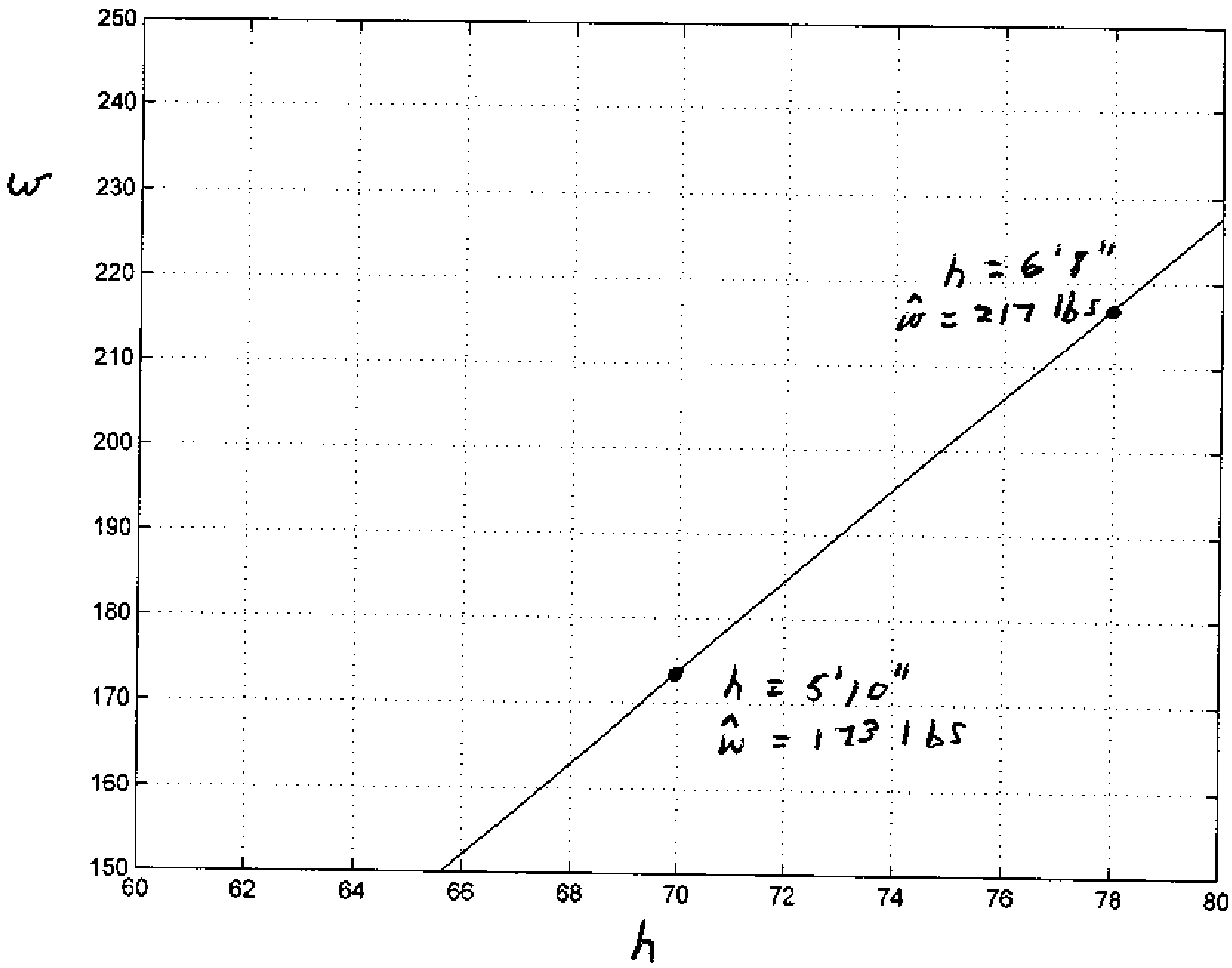
$$mse_{MIN} = var(Y)(1 - \rho_{X,Y}^2) = 15/64(1 - (\tfrac{1}{15})^2) < \frac{15}{64}$$

$$= 7/30$$



41) $\hat{\rho}_{HW} = 0.7583$

Yes, they are positively correlated.




```
% probprob7_41.m
%
clear all
P=[0.08 0.04 0.02 0 0;0.06 0.12 0.06 0.02 0;0 0.06 0.14 0.06 0;0 0.02 0.06 0.10 0.04;0 0 0
0.08 0.04];
for i=1:5
   pH(i,1)=sum(P(i,:));
end
for j=1:5
   pW(j,1)=sum(P(:,j));
end
W=[115 145 175 205 235]';
H=[62 66 70 74 78]';
EW=W'*pW;EH=H'*pH;
EHW=0;
for i=1:5
   for j=1:5
      EHW=EHW+H(i)*W(j)*P(i,j);
   end
end
cov=EHW-EH*EW;
EW2=(W.^2)'*pW;
varW=EW2-EW^2;
EH2=(H.^2)'*pH;
varH=EH2-EH^2;
a=cov/varH
b=EW-a*EH
HH=[60:80]';
Wpred=a*HH+b;
figure
plot(HH,Wpred)
rho=cov/sqrt(varW*varH)
grid
axis([60 80 150 250])
```

Prediction at $h = 80$ is poor. Due to fact that on the average most of the population has heights near 70" (prob = 0.26) while for heights near 80", prob = 0.12 (see Table 4.1). Hence, MSE attempts to minimize errors for higher prob. events, which occur for heights near 70".

4.2) $E_X\left[\frac{X - E_X(X)}{\sqrt{var(X)}}\right] = \frac{1}{\sqrt{var(X)}} E_X(X - E_X(X))$

$= \frac{1}{\sqrt{var(X)}}(E_X(X) - E_X(X)) = 0$

$var\left(\frac{X - E_X(X)}{\sqrt{var(X)}}\right) = var\left(\frac{X}{\sqrt{var(X)}} + \underset{\uparrow\text{ constant}}{c}\right)$

$= var\left(\frac{X}{\sqrt{var(X)}}\right) = \left(\frac{1}{\sqrt{var(X)}}\right)^2 var(X) = 1$

4.3) $\rho_{W,Z} = \frac{cov(W,Z)}{\sqrt{var(W)var(Z)}} = \frac{cov(X, X+N)}{\sqrt{var(X)var(X+N)}}$

$= \frac{var(X) + cov(X,N)}{\sqrt{var(X)[var(X) + var(N)]}}$ see Prob 7.43 also X, N are uncorrelated

$= \frac{var(X)}{\sqrt{var(X)(var(X) + var(N))}}$

$= \sqrt{\frac{var(X)}{var(X) + var(N)}} = \sqrt{\frac{\eta}{\eta + 1}}$

$\eta = E_X(X^2)/E_N(N^2)$ = "signal-to-noise ratio" (SNR)

As SNR increases, $\eta \to \infty \Rightarrow \rho_{W,Z} \to 1$
$\Rightarrow$ perfect prediction.

44) $var(X+Y) = var(X) + var(Y) + 2\,cov(X,Y)$

$= var(X) + var(Y) + 2\rho_{X,Y}\sqrt{var(X)\,var(Y)}$

$< var(X) + var(Y)$ for $\rho_{X,Y} < 0$
$> var(X) + var(Y)$ for $\rho_{X,Y} > 0$

45) $\rho_{X,Y} = \dfrac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}} = \dfrac{cov(X,aX+b)}{\sqrt{var(X)\,var(aX+b)}}$

$= \dfrac{a\,cov(X,X) + cov(X,b)}{\sqrt{var(X)\,a^2\,var(X)}}$ see Prob 7.34

But $cov(X,b) = E_X[(X - E_X[X])(b - E[b])] = 0$

$\rho_{X,Y} = \dfrac{a\,var(X)}{|a|\,var(X)} = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases}$

4.6) From Table 6.1

$\phi_X(\omega) = p e^{j\omega} + (1-p) = \frac{1}{2}e^{j\omega} + \frac{1}{2}$

$\phi_Z(\omega) = \phi_X(\omega)\,\phi_Y(\omega)$

$= \left(\frac{1}{2}e^{j\omega} + \frac{1}{2}\right)\left(\frac{1}{2}e^{j\omega} + \frac{1}{2}\right)$

$= \frac{1}{4}\left(e^{j\omega} + 1\right)^2$

$= \frac{1}{4}\left(e^{j2\omega} + 2e^{j\omega} + 1\right)$

But $\mathcal{F}^{-1}\{e^{jn\omega}\} = 1$
$\qquad 0 \quad$ otherwise

$$p_Z(k) = \mathcal{F}^{-1}\left\{\tfrac{1}{4} + \tfrac{1}{2}e^{j\omega} + \tfrac{1}{4}e^{j2\omega}\right\}$$

$$= \begin{cases} \tfrac{1}{4} & k=0 \\ \tfrac{1}{2} & k=1 \\ \tfrac{1}{4} & k=2 \end{cases}$$

Same as before using convolution.

47) From Table 6.1

$$\phi_X(\omega) = [p e^{j\omega} + (1-p)]^M$$

$$\phi_Z(\omega) = \phi_X(\omega)\,\phi_Y(\omega)$$
$$= (p e^{j\omega} + (1-p))^{M_X}\,[p e^{j\omega} + (1-p)]^{M_Y}$$

$$= [p e^{j\omega} + (1-p)]^{M_X + M_Y}$$

$\Rightarrow X + Y \sim bin(M_X + M_Y, p)$

X is sum of outcomes of M_X ind. Bernoulli trials
Y " " " " " M_Y ind " "

$X + Y$ = sum of $M_X + M_Y$ ind Bernoulli trials
$\sim bin(M_X + M_Y, p)$

48)

true	$j=0$	$j=1$
$i=0$	$\frac{1}{8}$	$\frac{1}{8}$
$i=1$	$\frac{1}{4}$	$\frac{1}{2}$

est.	$j=0$	$j=1$
$i=0$	0.1190	0.1310
$i=1$	0.2410	0.5090

```
% probprob7_48.m
%
clear all
rand('state',0)
M=1000;
 for m=1:M
 u=rand(1,1);
 if u<=1/8
 x(m,1)=0;y(m)=0;
 elseif u>1/8&u<=1/4
 x(m,1)=0;y(m,1)=1;
 elseif u>1/4&u<1/2
 x(m,1)=1;y(m,1)=0;
 else
 x(m,1)=1;y(m,1)=1;
 end
end
p00=0;p01=0;p10=0;p11=0;
for m=1:M
   if x(m)==0&y(m)==0
      p00=p00+1/M;
   elseif x(m)==0&y(m)==1
      p01=p01+1/M;
   elseif x(m)==1&y(m)==0
      p10=p10+1/M;
   elseif x(m)==1&y(m)==1
      p11=p11+1/M;
   end
end
p00
p01
p10
p11
pX0=p00+p01
pX1=p10+p11
pY0=p00+p10
pY1=p01+p11
```

49)

	$j=0$	$j=1$	$p_X(i)$
$i=0$	$1/8$	$1/8$	$1/4$
$i=1$	$1/4$	$1/2$	$3/4$
$p_Y(j)$	$3/8$	$5/8$	

$E_X[X] = 3/4 \quad E_Y[Y] = 5/8$

$E_{XY}[XY] = \frac{1}{2}$

$\mathrm{cov}(X,Y) = \frac{1}{2} - 3/4(5/8)$

$= 1/32$

$\mathrm{var}(X) = 3/4 - (3/4)^2 = 3/16$

$\mathrm{var}(Y) = 5/8 - (5/8)^2 = \frac{15}{64}$

$$\rho_{X,Y} = \frac{1/32}{\sqrt{3/16 \cdot 15/64}} = \frac{\frac{1}{32}(32)}{\sqrt{45}}$$

$$= \frac{1}{3\sqrt{5}} = \frac{\sqrt{5}}{15} = 0.1490$$

$$\hat{\rho}_{X,Y} = 0.1497$$

```
% probprob7_49.m
%
clear all
rand('state',0)
M=100000;
 for m=1:M
 u=rand(1,1);
 if u<=1/8
 x(m,1)=0;y(m)=0;
 elseif u>1/8&u<=1/4
 x(m,1)=0;y(m,1)=1;
 elseif u>1/4&u<1/2
 x(m,1)=1;y(m,1)=0;
 else
 x(m,1)=1;y(m,1)=1;
 end
end
EX=mean(x);EY=mean(y);
cov=mean(x.*y)-EX*EY;
EX2=mean(x.^2);EY2=mean(y.^2);
varX=EX2-EX^2;varY=EY2-EY^2;
rho=cov/sqrt(varX*varY)
```

50) $p_Z[k] = \sum_{i=-\infty}^{\infty} p_X[k-i]\,p_Y[i] \qquad k=2,3,\ldots$

$$= \sum_{i=-\infty}^{\infty} (1-p)^{k-i-1} p\, u[k-i-1](1-p)^{i-1} p\, u[i-1]$$

need $k-i-1 \geq 0 \Rightarrow k \geq i+1$ or $i \leq k-1$

$\underline{\text{and}}$ $i-1 \geq 0 \Rightarrow i \geq 1$

$\Rightarrow 1 \leq i \leq k-1 \qquad k=2,3,\ldots$

$$p_Z[k] = \sum_{i=1}^{k-1} p^2(1-p)^{k-2} = p^2(k-1)(1-p)^{k-2}$$

$k=2,3,\ldots$

$$p_Z[k] = \tfrac{1}{4}(k-1)\left(\tfrac{1}{2}\right)^{k-2}$$

$$= (k-1)\left(\tfrac{1}{2}\right)^{k} \qquad k=2,3,\ldots$$

```
» probprob7_50

ans = True     est.

         0         0
    0.2500    0.2570
    0.2500    0.2600
    0.1875    0.1830
    0.1250    0.1070
    0.0781    0.0750
    0.0469    0.0420
    0.0273    0.0340
    0.0156    0.0140
    0.0088    0.0150
    0.0049    0.0060
    0.0027    0.0020
    0.0015    0.0030
    0.0008    0.0010
    0.0004    0.0010
    0.0002         0
    0.0001         0
    0.0001         0
    0.0000         0
    0.0000         0
```

```
% probprob7_50.m
%
clear all
rand('state',0)
M=1000;
p=1/2;
for i=1:M
x(i,1)=0;success=0;
while success ==0
   if rand(1,1)<p
      success=1;
      x(i)=x(i)+1;

   else
      x(i)=x(i)+1;
   end
end
end
for i=1:M
y(i,1)=0;success=0;
while success ==0
   if rand(1,1)<p
      success=1;
      x(i)=x(i)+1;

   else
    y(i)=y(i)+1;
   end
end
end
z=x+y;
for i=1:20
   pz(i,1)=p^2*(1-p)^(i-2)*(i-1);
end
for i=1:20
   pzhat(i,1)=length(find(z-i==0))/M;
end
[pz pzhat]
```

51) $\text{cov}(X,Y) = E_{XY}[XY] - E_X[X]E_Y[Y]$

$= 1 \cdot 1 \cdot 3/8 - (\frac{1}{2})^2 = 1/8$

$\text{var}(X) = E_X[X^2] - E^2[X] = 1^2 \cdot \frac{1}{2} - (\frac{1}{2})^2 = \frac{1}{4}$

$a = -\text{cov}(X,Y)/\text{var}(X) = -\frac{1/8}{1/4} = -\frac{1}{2}$

$W = X$

$Z = -\frac{1}{2}X + Y$

$\widehat{\text{cov}}(X,Y) = 0.1279 \qquad \widehat{\text{cov}}(W,Z) = 0.0030$

```
% probprob7_51.m
%
clear all
rand('state',0)
M=1000;
 for m=1:M
 u=rand(1,1);
 if u<=3/8
 x(m,1)=0;y(m,1)=0;
 elseif u>3/8&u<=1/2
 x(m,1)=0;y(m,1)=1;
 elseif u>1/2&u<=5/8
 x(m,1)=1;y(m,1)=0;
 else
 x(m,1)=1;y(m,1)=1;
 end
end
EX=mean(x);EY=mean(y);
covXY=mean(x.*y)-EX*EY
varX=mean(x.^2)-EX^2;
w=x;z=-x/2+y;
covWZ=mean(w.*z)-mean(w)*mean(z)
```

Chapter 8

1) $p_{Y|X}[j|i] = \ ?$ $\quad i = 0, 1 ; \ j = 0, 1$

$$p_X[i] = \begin{cases} \frac{1}{2} & i = 0 \\ \frac{1}{2} & i = 1 \end{cases}$$

Pick from B $(X = 0)$

$$p_{Y|X}[j|0] = \begin{cases} \frac{1}{3/2} = 2/3 & j = 0 \quad \text{lower quad.} \\ \frac{1/2}{3/2} = 1/3 & j = 1 \quad \text{upper quad.} \end{cases}$$

Pick from A $(X = 1)$

$$p_{Y|X}[j|1] = \begin{cases} \frac{1/2}{3/2} = \frac{1}{3} & j = 0 \quad \text{lower quad.} \\ \frac{1}{3/2} = \frac{2}{3} & j = 1 \quad \text{upper quad.} \end{cases}$$

$$P[Y = 0] = p_{Y|X}[0|0]p_X[0] + p_{Y|X}[0|1]p_X[1]$$
$$= 2/3(1/2) + \frac{1}{3}(\frac{1}{2}) = \frac{1}{2}$$

2) $p_{Y|X}[j|0] = 1$ $\quad j = 0 ; \ i = 0$

$p_{Y|X}[j|1] = 1/6$ $\quad j = 1, 2, 3, 4, 5, 6 ; \ i = 1$

$$P[Y = 1] = p_{Y|X}[1|0]p_X[0] + p_{Y|X}[1|1]p_X[1]$$

$$= 0 \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$$

3) $p_{Y|X}[j|i] = \dfrac{p_{X,Y}[i,j]}{p_X[i]}$ $\quad j = 0, 1, 2, 3 \quad i = 0, 1$

All outcomes are

0, 0, 0	1, 0, 0
0, 0, 1	1, 0, 1
0, 1, 0	1, 1, 0
0, 1, 1	1, 1, 1
↑ toss 1	↑ toss 1

First toss = tail

$$p_{Y|X}[j|0] = \begin{cases} \frac{1}{8}/\frac{1}{2} = \frac{1}{4} & j=0 \\ 2/8/\frac{1}{2} = \frac{1}{2} & j=1 \\ 1/8/\frac{1}{2} = \frac{1}{4} & j=2 \end{cases}$$

First toss = head

$$p_{Y|X}(j|1) = \begin{cases} \frac{1}{8}/\frac{1}{2} = \frac{1}{4} & j=1 \\ \frac{2}{8}/\frac{1}{2} = \frac{1}{2} & j=2 \\ \frac{1}{8}/\frac{1}{2} = \frac{1}{4} & j=3 \end{cases}$$

4) $$\sum_{j=-\infty}^{\infty} p_{Y|X}(y_j|x_i) = \sum_{j=-\infty}^{\infty} \frac{p_{X,Y}(x_i,y_j)}{p_X(x_i)}$$

$$= \frac{\sum_{j=-\infty}^{\infty} p_{X,Y}(x_i,y_j)}{p_X(x_i)} = \frac{\sum_{j=-\infty}^{\infty} p_{X,Y}(x_i,y_j)}{\sum_{j=-\infty}^{\infty} p_{X,Y}(x_i,y_j)} = 1$$

5) a) No, doesn't sum to 1 (not geometric PMF)

b) No, $x_i = -\frac{1}{2}$ yields

$$p_{Y|X}[0|-\tfrac{1}{2}] = \binom{N}{0} 1 \cdot (1-(-\tfrac{1}{2}))^N$$

$$= (3/2)^N > 1 \quad \text{for } N=1$$

$$p_{Y|X}[N|-\tfrac{1}{2}] = \binom{N}{N}(-\tfrac{1}{2})^j < 0 \quad \text{for odd}$$

c) No, $\sum_{j=2}^{\infty} p_{Y|X}[j|x_i] \to \infty$

6) $p_X[i] = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \quad i=0$

$\frac{1}{3} + \frac{1}{6} = \frac{1}{2} \quad i=1$

$p_Y(j) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \quad j=0$

$\frac{1}{3} + \frac{1}{6} = \frac{1}{2} \quad j=1$

$$p_{Y|X}(j|1) = \frac{p_{X,Y}(1,j)}{p_X(1)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \quad j=0$$

$\frac{1}{3} \quad j=1$

$$p_{Y|X}(j|0) = \frac{p_{X,Y}(0,j)}{p_X(0)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \quad j=0$$

$2/3 \quad j=1$

$$p_{X|Y}[i|0] = \frac{p_{X,Y}(i,0)}{p_Y[0]} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} \quad i=0$$

$\frac{2}{3} \quad i=1$

$$p_{X|Y}(i|1) = \frac{p_{X,Y}(i,1)}{p_Y(1)} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \quad i=0$$

$\frac{1}{3} \quad i=1$

7) $p_{Y|X}(j|0) = \frac{1}{18} N_j$

From Table 8.1 - nonbolded entries

$$p_{X|Y}(j|0) = \begin{cases} \frac{1}{18}\cdot 2 & j=3 \\ \frac{1}{18}\cdot 4 & 5 \\ \frac{1}{18}\cdot 6 & 7 \\ \frac{1}{18}\cdot 4 & 9 \\ \frac{1}{18}\cdot 2 & 11 \end{cases}$$

8) $$p_{Y|X}[j|i] = \frac{p_{X,Y}[i,j]}{p_X[i]}$$

$$= \frac{\frac{1}{10}}{p_X[i]} \quad \text{for all } i, j$$

$$= \frac{\frac{1}{10}}{\frac{1}{2}} = \frac{1}{5} \qquad \begin{matrix} i = 1, 2 \\ j = 0, 1, 2, 3, 4 \end{matrix}$$

$$p_{X|Y}[i|j] = \frac{p_{X,Y}[i,j]}{p_Y[j]}$$

$$= \frac{1/10}{2/10} = \frac{1}{2} \qquad \begin{matrix} i = 1, 2 \\ j = 0, 1, 2, 3, 4 \end{matrix}$$

Given $X = 1$ the reduced sample space is the first column of dots and since they are all equally likely, each dot has prob $= 1/5$. Similarly, for $X = 2$ and for given Y.

9) Needed for cond. PMF to sum to one

10) No, $p_{X,Y} = p_{Y|X}\, p_X = p_{X|Y}\, p_Y$

need this or

11) $P[\text{failure at use } 11, 12, \ldots] = P[Y > 10]$

$$= \sum_{j=11}^{\infty} p_Y[j]$$

$$p_Y(j) = \sum_{i=1}^{3} p_{Y|X}(j|i)\underbrace{p_X(i)}_{1/3} \qquad j = 1, 2, \ldots$$

$$= \frac{1}{3}\left[(0.99)^{j-1}0.01 + (0.9)^{j-1}0.1 + (0.8)^{j-1}0.2\right]$$

$$P(Y > 10) = \sum_{j=11}^{\infty} \frac{1}{3}\left[(0.99)^{j-1}0.01 + (0.9)^{j-1}0.1 + (0.8)^{j-1}0.2\right]$$

But $\displaystyle\sum_{j=11}^{\infty} q^{j-1}(1-q) = (1-q)\frac{q^{10}}{1-q} = q^{10}$

$$P(Y>10) = \frac{1}{3}\left[0.99^{10} + 0.9^{10} + 0.8^{10}\right] = 0.4535$$

12) $\displaystyle p_{Y|X}(j|i) = \frac{p_{XY}(i,j)}{p_X(i)}$

$$p_X(i) = 4/10,\ \frac{2}{10},\ \frac{4}{10} \qquad i = 1, 2, 3$$

$$p_{Y|X}(j|1) = \frac{\frac{1}{10}}{\frac{4}{10}},\ \frac{\frac{1}{10}}{\frac{4}{10}},\ \frac{\frac{2}{10}}{\frac{4}{10}} = \frac{1}{4}, \frac{1}{4}, \frac{1}{2} \qquad j = 1, 2, 3$$

$$p_{Y|X}(j|2) = \frac{\frac{1}{20}}{\frac{2}{10}},\ \frac{\frac{1}{20}}{\frac{2}{10}},\ \frac{\frac{1}{10}}{\frac{2}{10}} = \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$$

$$p_{Y|X}(j|3) = \frac{\frac{3}{10}}{\frac{4}{10}},\ \frac{\frac{1}{20}}{\frac{4}{10}},\ \frac{\frac{1}{20}}{\frac{4}{10}} = \frac{3}{4}, \frac{1}{8}, \frac{1}{8}$$

$$p_Y[j] = \frac{1}{10} + \frac{1}{20} + \frac{3}{10} = 9/20 \qquad j = 1$$

$$\frac{1}{10} + \frac{1}{20} + \frac{1}{20} = 4/20 \qquad j = 2$$

$$\frac{2}{10} + \frac{1}{10} + \frac{1}{20} = 7/20 \qquad j = 3$$

No $p_{Y|X} \neq p_Y$. Also obvious since

$p_{Y|X}(j|i)$ is different for various values of i.

13)

	$y_1 = -\frac{1}{\sqrt{2}}$	$y_2 = 0$	$y_3 = 1/\sqrt{2}$	$p_X(x_i)$
$x_1 = 0$	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$x_2 = 1/\sqrt{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
$x_3 = \sqrt{2}$	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$p_Y(y_j)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	

$$p_{Y|X}(y_j|x_i) = \frac{p_{X,Y}(x_i, y_j)}{p_X(x_i)}$$

$$= 0, 1, 0 \qquad x_i = 0$$
$$\frac{1}{2}, 0, \frac{1}{2} \qquad x_i = 1/\sqrt{2}$$
$$0, 1, 0 \qquad x_i = \sqrt{2}$$

No, $p_{Y|X} \neq p_Y$

	$j = 0$	$j = 1$	$p_X(x_i)$
$i = 0$	$1/4$	$1/4$	$\frac{1}{2}$
$i = 1$	$1/4$	$1/4$	$\frac{1}{2}$
$p_Y(y_j)$	$\frac{1}{2}$	$\frac{1}{2}$	

$$p_{Y|X}(y_j|x_i) = \frac{1}{2}, \frac{1}{2} \qquad x_i = 0$$
$$\frac{1}{2}, \frac{1}{2} \qquad x_i = 1$$

Yes, $p_{Y|X} = p_Y$

14) Let $B = \{s : X(s) = x_i\}$

$$P(A|B) = P(Y \in A \mid X = x_i)$$

But $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{P(\{s : s \in A\} \cap \{s : X(s) = x_i\})}{P(\{s : X(s) = x_i\})}$

by definition $P(\{s : X(s) = x_i\}) = p_X(x_i)$

$$\{s : s \in A\} = \bigcup_j \{s : Y(s) = y_j\} = \bigcup_j A_j$$

$P(\bigcup_j A_j \cap B) = P(\bigcup_j A_j B)$ distributive

But A_j's are disjoint $\Rightarrow$ $A_j B$'s are disjoint

$$P(\bigcup_j A_j B) = \sum_j P(A_j B)$$

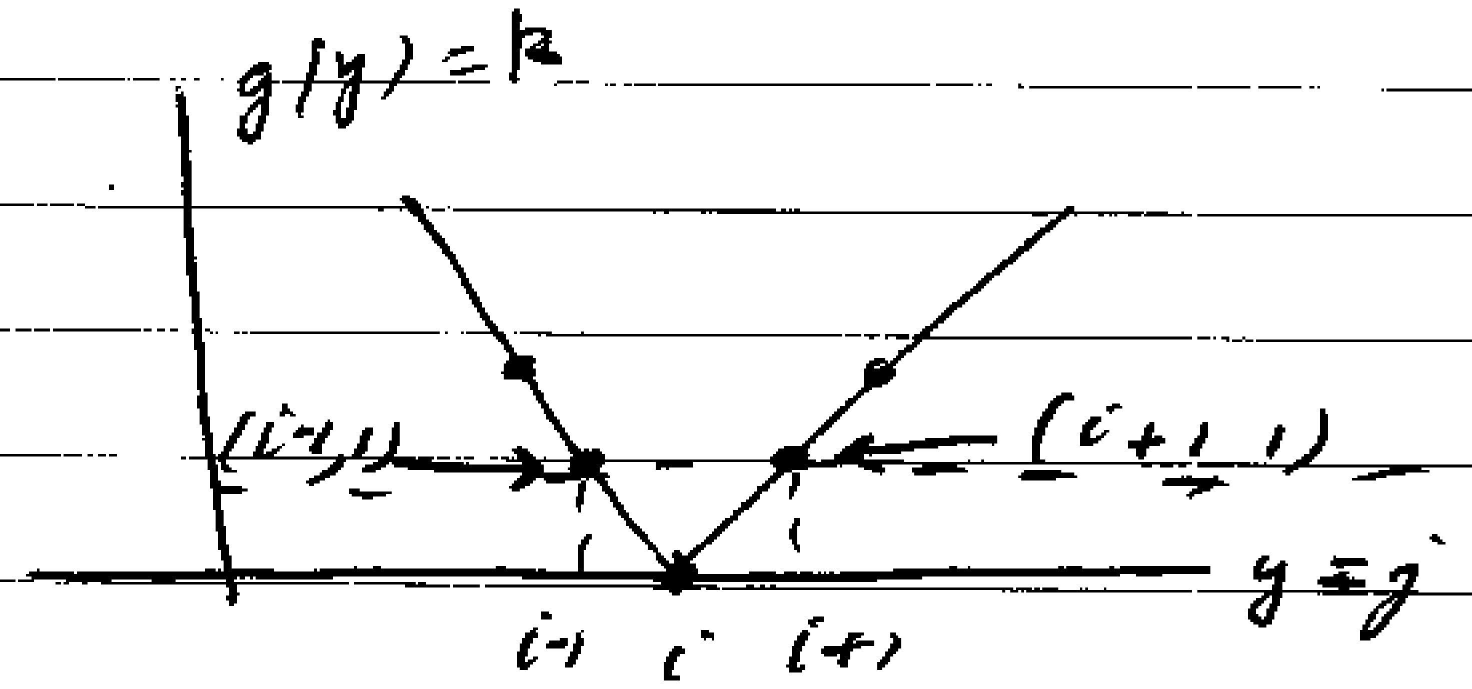
and $P(A_j B) = P(\{s : s \in A_j, s \in B\})$

$= p_{X,Y}(x_i, y_j)$ definition

$P(A|B) = \dfrac{\sum_j p_{X,Y}(x_i, y_j)}{p_X(x_i)}$ $\Rightarrow$

$$P(A \mid X = x_i) = \sum_{\{j : y_j \in A\}} p_{Y|X}(y_j \mid x_i)$$

15) $Z|(X = i) = |Y - i|$



Conditionally on $X = i$ the transformed

$$p_{Z|X}[k|i] = p_{Y|X}[i|i] \quad \text{for } k = 0$$

$$p_{Y|X}[i-k|i] + p_{Y|X}[i+k|i] \quad \text{for } k \neq 0$$

Inverse transformation is from k to j or $j = i-k$ and $j = i+k$ for $k \neq 0$ and $j = i$ for $k = 0$

$$p_{Z|X}(k|i) = \left. \begin{array}{ll} p_Y[i] & k=0 \\ p_Y(i-k) + p_Y(i+k) & k \neq 0 \end{array} \right\} \text{independence}$$

$$p_Z(k) = \sum_{i=0}^{\infty} p_{Z|X}(k|i)\, p_X(i)$$

$$= \sum_{i=0}^{\infty} p_Y(i)\, p_X(i) \qquad k = 0$$

$$= \sum_{i=0}^{\infty} (p_Y(i-k) + p_Y(i+k))\, p_X(i)$$

$$i' = i-k \qquad = \sum_{i'=-k}^{\infty} \underbrace{p_Y(i')}_{=0 \text{ for } i' < 0} p_X(i'+k) + \sum_{i=0}^{\infty} p_X(i) p_Y(i+k)$$

$$= \sum_{i=0}^{\infty} [p_Y(i)\, p_X(i+k) + p_X(i) p_Y(i+k)] \quad k = 1, 2, \ldots$$

16) Need $P[\,|X-Y| \le 2]$

$$p_Z(k) = \sum_{i=0}^{\infty} \left(\tfrac{1}{2}\right)^{2i+2} \qquad k=0$$

$$= \tfrac{1}{4} \sum_{i=0}^{\infty} \left(\tfrac{1}{4}\right)^{i} = \tfrac{1}{4}\,\frac{1}{1-\frac{1}{4}} = \tfrac{1}{3}$$

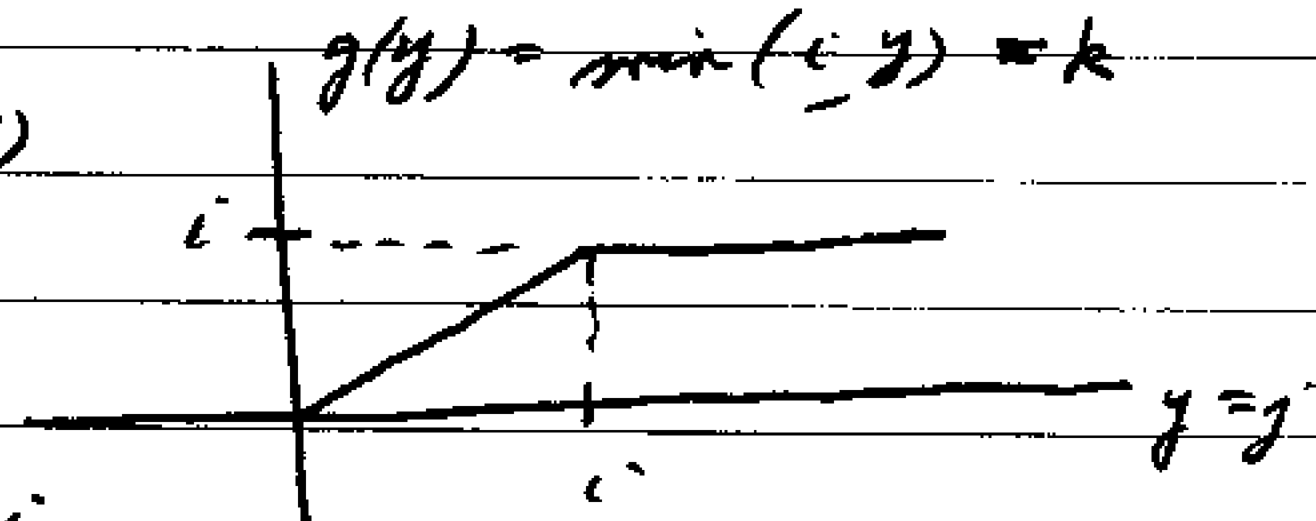
$$p_Z(k) = \sum_{i=0}^{\infty} \left[\left(\tfrac{1}{2}\right)^{2i+2+k} + \left(\tfrac{1}{2}\right)^{2i+2+k}\right] \qquad k=1,2,\ldots$$

$$= 2\left(\tfrac{1}{2}\right)^{k+2} \sum_{i=0}^{\infty} \left(\tfrac{1}{4}\right)^{i} = \left(\tfrac{1}{2}\right)^{k+1} 4/3$$

$$= 2/3 \left(\tfrac{1}{2}\right)^{k} \qquad k=1,2,\ldots$$

$$P[\,|X-Y| \le 2] = p_Z(0) + p_Z(1) + p_Z(2)$$
$$= 1/3 + \tfrac{1}{3} + \tfrac{1}{6} = 5/6$$

17) $Z|(X=i) = \min(i, Y)$



$$p_{Z|X}[k|i] = p_{Y|X}[k|i] \quad \text{for } k<i$$

$$= \sum_{j=i}^{\infty} p_{Y|X}[j|i] \qquad k=i$$

$$0 \qquad k>i$$

or
$$p_{Z|X}[k|i] = \begin{cases} p_Y(k) & k<i \\ \sum_{j=i}^{\infty} p_Y[j] & k=i \\ 0 & k>i \end{cases}$$

$$p_Z(k) = \sum_{i=0}^{\infty} p_{Z|X}(k|i)\, p_X(i)$$

$$= \sum_{i=k}^{\infty} p_{Z|X}(k|i)\, p_X(i)$$

$$= \underbrace{p_{Z|X}(k|k)}_{p_X(k)} + \sum_{i=k+1}^{\infty} p_{Z|X}(k|i)\, p_X(i)$$

$$= \sum_{j=k}^{\infty} p_Y(j)\, p_X(k) + \sum_{i=k+1}^{\infty} p_Y(k)\, p_X(i)$$

$$= p_X(k) \sum_{j=k}^{\infty} p_Y(j) + p_Y(k) \sum_{j=k+1}^{\infty} p_X(j)$$

(8) $$P(\underbrace{Y > X}_{A}) = \sum_i P(A \mid X = i)\, p_X(i)$$

$$= \sum_i P(Y > X \mid X = i)\, p_X(i)$$

$$= \sum_i P(Y > i \mid X = i)\, p_X(i)$$

$$= \sum_i \left[\sum_{j=i+1}^{\infty} p_{Y|X}(j|i) \right] p_X(i)$$

$$= \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} p_{X,Y}(i,j)$$

$$= \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} p_1 p_2 (1-p_1)^i (1-p_2)^j$$

$$= \sum_{i=0}^{\infty} p_1 (1-p_1)^i p_2 \underbrace{\sum_{j=i+1}^{\infty} (1-p_2)^j}_{\frac{(1-p_2)^{i+1}}{p_2}}$$

$$= \sum_{i=0}^{\infty} p_1 (1-p_1)^i (1-p_2)^{i+1}$$

$$= p_1(1-p_2)\sum_{i=0}^{\infty}[(1-p_1)(1-p_2)]^i$$

$$= \frac{p_1(1-p_2)}{1-(1-p_1)(1-p_2)}$$

19) From Prob 8.6

$$p_{Y|X}[j|1] = \begin{cases} 2/3 & j=0 \\ 1/3 & j=1 \end{cases}$$

$$p_{X|Y}[j|0] = \begin{cases} 1/3 & j=0 \\ 2/3 & j=1 \end{cases}$$

$$E_{Y|X}[Y|1] = 1\cdot\tfrac{1}{3} = \tfrac{1}{3}$$
$$E_{Y|X}[Y|0] = 1\cdot 2/3 = 2/3$$

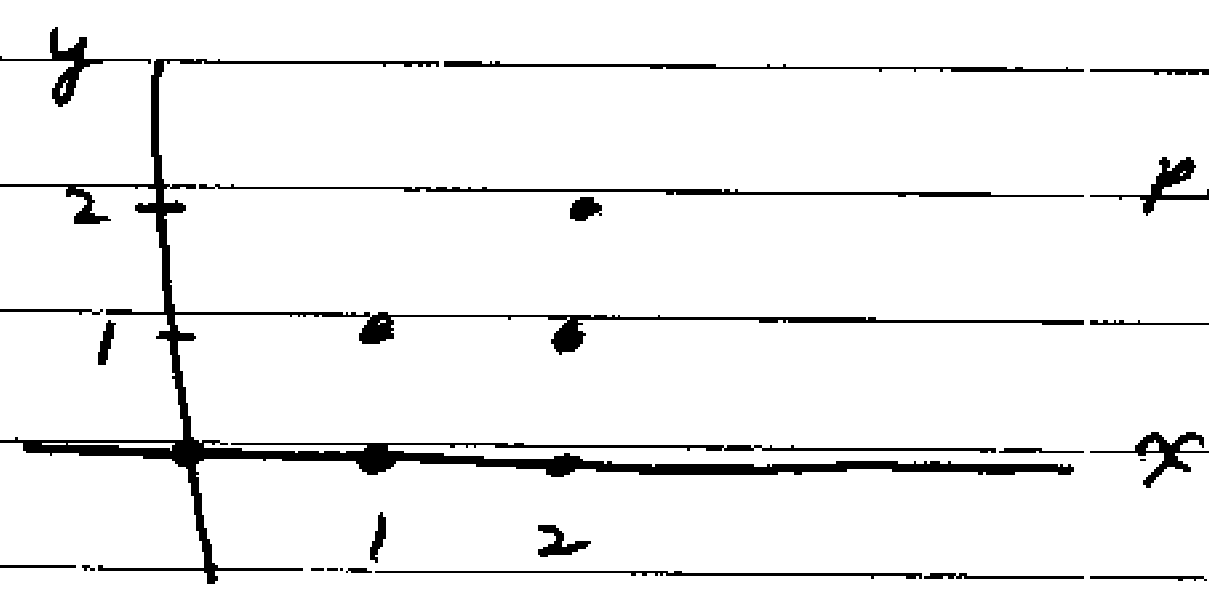
20) $$p_X[i] = \sum_{j=0}^{\infty} p_{X,Y}[i,j] = \sum_{j=0}^{\infty}\left(\tfrac{1}{2}\right)^{i+1} e^{-\lambda}\frac{\lambda^j}{j!}$$

$$= \left(\tfrac{1}{2}\right)^{i+1} \quad \text{since } \sum_{j=0}^{\infty}\lambda^j/j! = e^{\lambda}$$

$$p_{Y|X}[j|i] = p_{X,Y}[i,j]/p_X[i]$$

$$= e^{-\lambda}\frac{\lambda^j}{j!} \quad \sim \text{Poisson}$$

$E_{Y|X}[Y|i] = \lambda$ for all i since X and Y are independent

21) 

$p_{X,Y}[i,j] = 1/6$

$$p_X[i] = \begin{cases} 1/6 & i=0 \\ 2/6 & i=1 \\ 3/6 & i=2 \end{cases}$$

$$p_{Y|X}[j|0] = \begin{cases} 1 & j=0 \\ 0 & j=1 \\ 0 & j=2 \end{cases}$$

$$p_{Y|X}[j|1] = \begin{cases} \frac{1}{2} & j=0 \\ \frac{1}{2} & j=1 \\ 0 & j=2 \end{cases}$$

$$p_{Y|X}[j|2] = \begin{cases} \frac{1}{3} & j=0 \\ \frac{1}{3} & j=1 \\ \frac{1}{3} & j=2 \end{cases}$$

$E_{Y|X}[Y|0] = 0$
$E_{Y|X}[Y|1] = \frac{1}{2}$
$E_{Y|X}[Y|2] = 1$

22) $\mathrm{var}(Y|x_i) = E_{Y|X}[Y^2|x_i] - E^2_{Y|X}[Y|x_i]$

$\mathrm{var}(Y|0) = 0^2 \cdot 1 - 0^2 = 0$
$\mathrm{var}(Y|1) = (0^2 \cdot \frac{1}{2} + 1^2 \cdot \frac{1}{2}) - (\frac{1}{2})^2 = \frac{1}{4}$
$\mathrm{var}(Y|2) = 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{3} - 1^2 = 2/3$

variance increases with i since cond. PMF is uniform over $i+1$ points

23) $\mathrm{var}(Y|x_i) = \sum_j (y_j - E_{Y|X}[Y|x_i])^2 p_{Y|X}[y_j|x_i]$

$= \sum_j y_j^2 p_{Y|X}[y_j|x_i] - 2\sum_j y_j E_{Y|X}[Y|x_i] p_{Y|X}[y_j|x_i]$

$+ \sum_j E^2_{Y|X}[Y|x_i] p_{Y|X}[y_j|x_i]$

$= E_{Y|X}[Y^2|x_i] - 2E_{Y|X}[Y|x_i] \underbrace{\sum_j y_j p_{Y|X}[y_j|x_i]}_{E_{Y|X}[Y|x_i]}$

$+ E^2_{Y|X}[Y|x_i] \underbrace{\sum_j p_{Y|X}[y_j|x_i]}_{=1}$

$= E_{Y|X}[Y^2|x_i] - E^2_{Y|X}[Y|x_i]$

24) From Prob 8.21

$p_Y[j] = \begin{cases} 1/2 & j=0 \\ 1/3 & j=1 \\ 1/6 & j=2 \end{cases}$

$\Rightarrow \quad E_Y[Y] = 0\cdot\frac{1}{2} + 1\cdot\frac{1}{3} + 2\cdot\frac{1}{6} = 2/3$

From Prob. 8.21

$p_X[i] = \begin{cases} 1/6 & i=0 \\ 1/3 & i=1 \\ 1/2 & i=2 \end{cases}$

and $E_{Y|X}[Y|0] = 0$

$E_{Y|X}[Y|1] = \frac{1}{2}$

$E_{Y|X}[Y|2] = 1$

$\Rightarrow \quad E_Y[Y] = 0\left(\frac{1}{6}\right) + \frac{1}{2}\left(\frac{1}{3}\right) + 1\left(\frac{1}{2}\right) = 2/3$

25) $$E_Y[g(Y)] = \sum_j g(y_j)\, p_Y(y_j)$$

$$= \sum_j g(y_j) \sum_i p_{Y|X}[y_j | x_i]\, p_X(x_i)$$

$$= \sum_i \underbrace{\left[\sum_j g(y_j)\, p_{Y|X}(y_j | x_i)\right]}_{E_{Y|X}[g(Y) | x_i]} p_X(x_i)$$

$$= E_X\{E_{Y|X}[g(Y) | X]\}$$

26) $$\phi_Y(\omega) = E_Y[e^{j\omega Y}]$$

$$= E_M\{E_{Y|M}[e^{j\omega Y} | M]\}$$

$$= E_M\{\underbrace{[pe^{j\omega} + (1-p)]}_{a}{}^M\}$$

$$= \sum_{m=0}^{\infty} a^m \frac{e^{-\lambda}\lambda^m}{m!} = e^{-\lambda} \sum_{m=0}^{\infty} \frac{(a\lambda)^m}{m!}$$

$$= e^{-\lambda} e^{a\lambda} = e^{\lambda(a-1)}$$

$$= e^{\lambda[pe^{j\omega} + (1-p) - 1]}$$

$$= e^{\lambda p(e^{j\omega} - 1)} \quad \sim \text{Pois}(\lambda p)$$

27) $$h(c) = \sum_j (y_j - c)^2 p_{Y|X}(y_j | x_i)$$

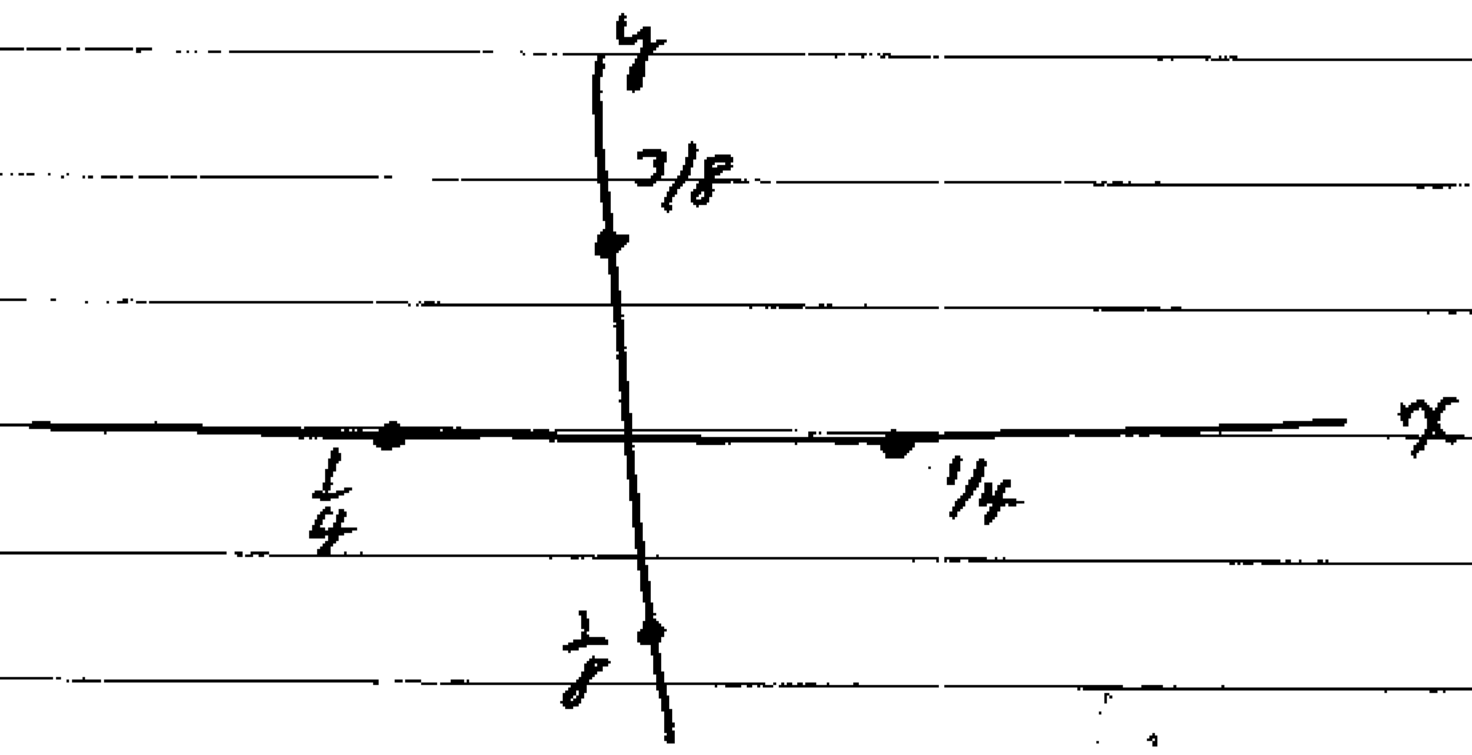
$$\frac{dh}{dc} = -2 \sum_j (y_j - c)\, p_{Y|X}(y_j | x_i) = 0$$

$$\sum_j y_j\, p_{Y|X}(y_j | x_i) = c \underbrace{\sum_j p_{Y|X}(y_j | x_i)}_{=1}$$

$$c = \sum_j y_j \, p_{Y|X}(y_j \mid x_i)$$

$$= E_{Y|X}(Y \mid x_i)$$

28)



$$p_X[i] = \begin{cases} \frac{1}{4} & i=-1 \\ \frac{1}{2} & i=0 \\ \frac{1}{4} & i=1 \end{cases}$$

$$p_{Y|X}[j \mid -1] = \begin{cases} 0 & j=-1 \\ 1 & j=0 \\ 0 & j=1 \end{cases}$$

$$p_{Y|X}(j \mid 0) = \begin{cases} \frac{1}{4} & j=-1 \\ 0 & j=0 \\ 3/4 & j=1 \end{cases}$$

$$p_{Y|X}(j \mid 1) = \begin{cases} 0 & j=-1 \\ 1 & j=0 \\ 0 & j=1 \end{cases}$$

$$E_{Y|X}[Y \mid -1] = 0$$
$$E_{Y|X}(Y \mid 0) = 1/2$$
$$E_{Y|X}(Y \mid 1) = 0$$

Also, $$\hat{Y} = E_Y[Y] + \frac{\mathrm{cov}(X,Y)}{\mathrm{var}(X)}(X - \underbrace{E_X(X)}_{=0})$$

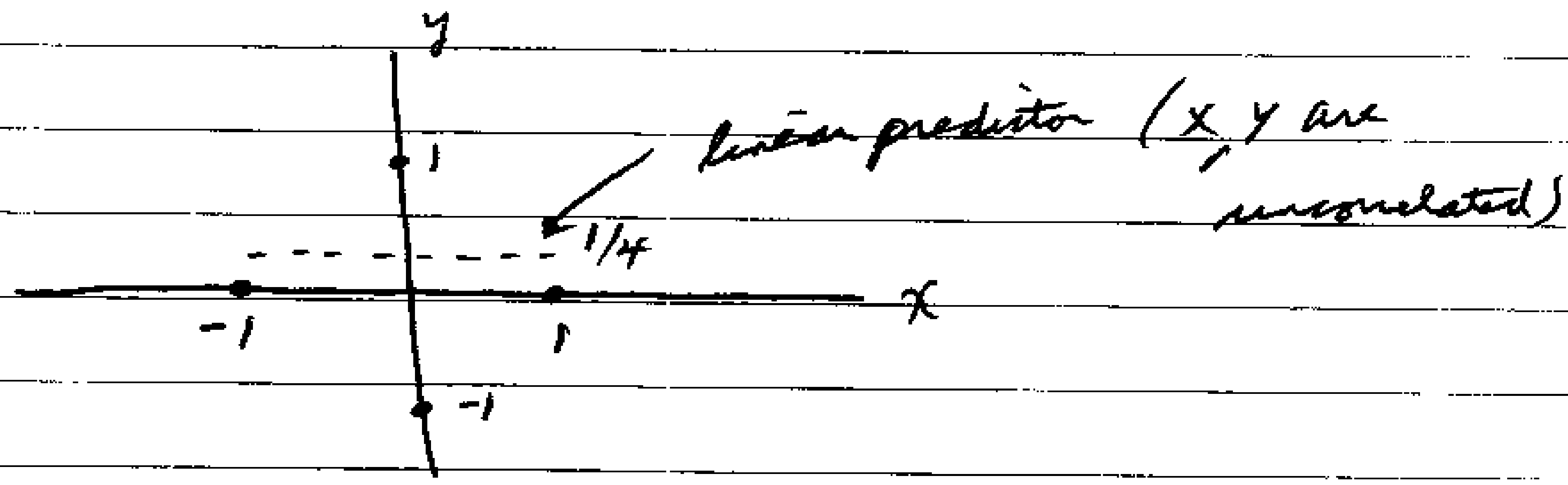
$$p_Y[j] = \begin{cases} \frac{1}{8} & j = -1 \\ \frac{1}{2} & j = 0 \\ 3/8 & j = 1 \end{cases}$$

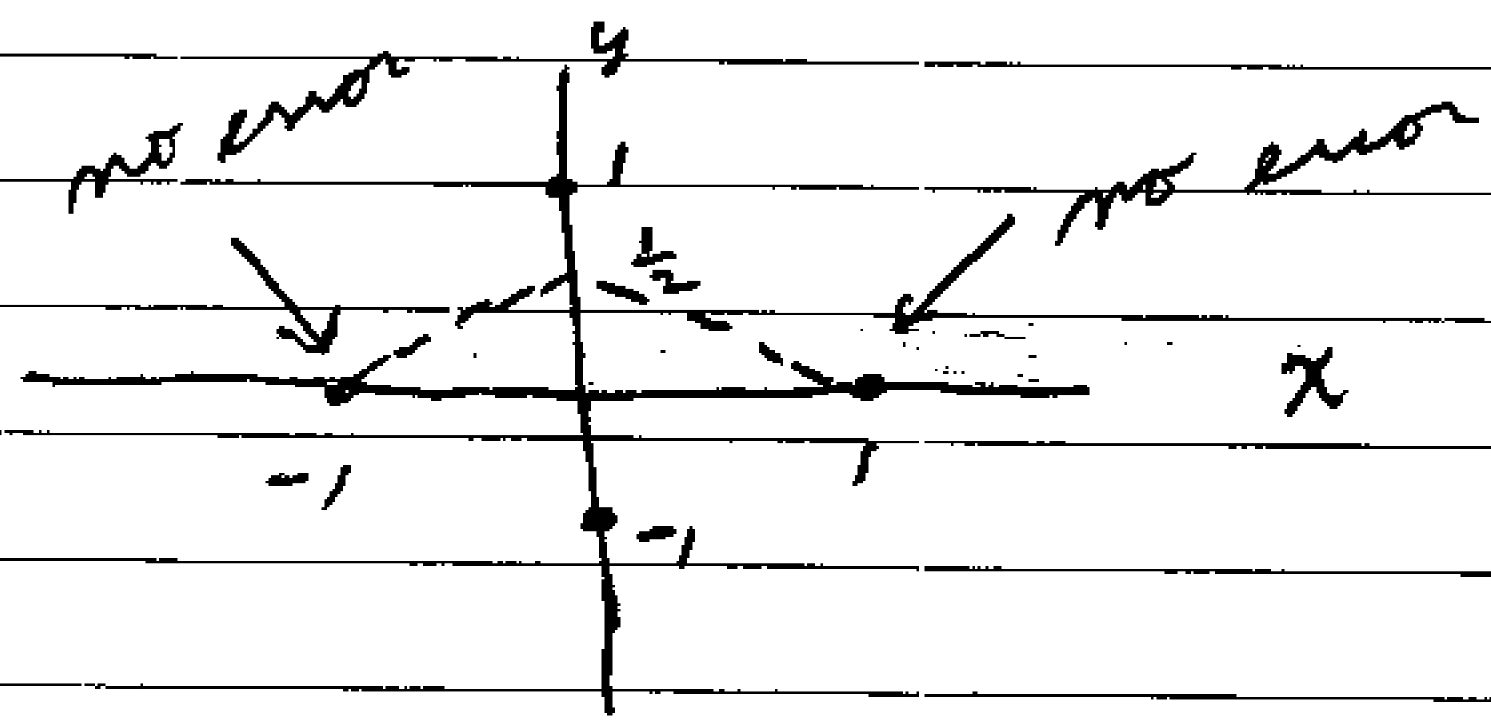
$\Rightarrow E_Y[Y] = 1/4$

$\mathrm{var}(X) = E_X[X^2] = (-1)^2 \frac{1}{4} + 1^2 \cdot \frac{1}{4} = \frac{1}{2}$

$\mathrm{cov}(X,Y) = E_{X,Y}[XY] = 0$ since X or $Y = 0$

$\Rightarrow \hat{Y} = E_Y[Y] = 1/4$





		True PMF
29)	$\hat{p}_{X,Y}[0,0] = 0.1176$	0.1250
	$\hat{p}_{X,Y}[0,1] = 0.1276$	0.1250
	$\hat{p}_{X,Y}[1,0] = 0.2508$	0.2500
	$\hat{p}_{X,Y}[1,1] = 0.5040$	0.5000

30) From Example 8.6

$$p_{Y|X}(j|0) = \begin{cases} \frac{1}{2} & j=0 \\ \frac{1}{2} & j=1 \end{cases}$$

$$p_{Y|X}(j|1) = \begin{cases} \frac{1}{3} & j=0 \\ \frac{2}{3} & j=1 \end{cases}$$

$$\Rightarrow E_{Y|X}[Y|0] = \frac{1}{2}$$

$$E_{Y|X}[Y|1] = 2/3$$

$$\hat{E}_{Y|X}[Y|0] = 0.5204$$

$$\hat{E}_{Y|X}[Y|1] = 0.6677$$

```
% probprob8_30.m
%
clear all
rand('state',0)
M=10000;

          for m=1:M
            ux=rand(1,1);
            uy=rand(1,1);
            if ux<=1/4; % Refer to px[i]
              x(m,1)=0;
              if uy<=1/2 % Refer to py|x[j|0]
                y(m,1)=0;
              else
                y(m,1)=1;
              end
            else
              x(m,1)=1;  % Refer to px[i]
              if uy<=1/3 % Refer to py|x[j|1]
                y(m,1)=0;
              else
                y(m,1)=1;
              end
            end
          end
          EYx0=0;EYx1=0;
          countx0=0;countx1=0;
          for m=1:M
            if x(m)==0
               EYx0=EYx0+y(m);
               countx0=countx0+1;
            else
               EYx1=EYx1+y(m);
               countx1=countx1+1;
            end
          end
          EYx0=EYx0/countx0
          EYx1=EYx1/countx1
```

31) From Problem 8.30

$$E_{Y|X}[Y|0] = \frac{1}{2}$$
$$E_{Y|X}[Y|1] = 2/3$$

$$\Rightarrow E_Y[Y] = E_{Y|X}[Y|0]\,p_X[0] + E_{Y|X}[Y|1]\,p_X[1]$$

$$p_X[0] = \frac{1}{4} \quad p_X[1] = 3/4$$

$$E_Y[Y] = \frac{1}{2}\left(\frac{1}{4}\right) + \frac{2}{3}\left(\frac{3}{4}\right) = 5/8$$

$$\Rightarrow \hat{E_Y}[Y] = 0.6316$$

```
% probprob8_31.m
%
clear all
rand('state',0)
M=10000;

          for m=1:M
            ux=rand(1,1);
            uy=rand(1,1);
            if ux<=1/4; % Refer to px[i]
              x(m,1)=0;
              if uy<=1/2 % Refer to py|x[j|0]
                y(m,1)=0;
              else
                y(m,1)=1;
              end
            else
              x(m,1)=1;  % Refer to px[i]
              if uy<=1/3 % Refer to py|x[j|1]
                y(m,1)=0;
              else
                y(m,1)=1;
              end
            end
          end
          EYx0=0;EYx1=0;px0=0;px1=0;
          countx0=0;countx1=0;
          for m=1:M
             if x(m)==0
                px0=px0+1/M;
                EYx0=EYx0+y(m);
                countx0=countx0+1;
             else
                EYx1=EYx1+y(m);
                countx1=countx1+1;
                px1=px1+1/M;
             end
          end
          EYx0=EYx0/countx0;
          EYx1=EYx1/countx1;
          EY=EYx0*px0+EYx1*px1
```

32) $i = N/2$

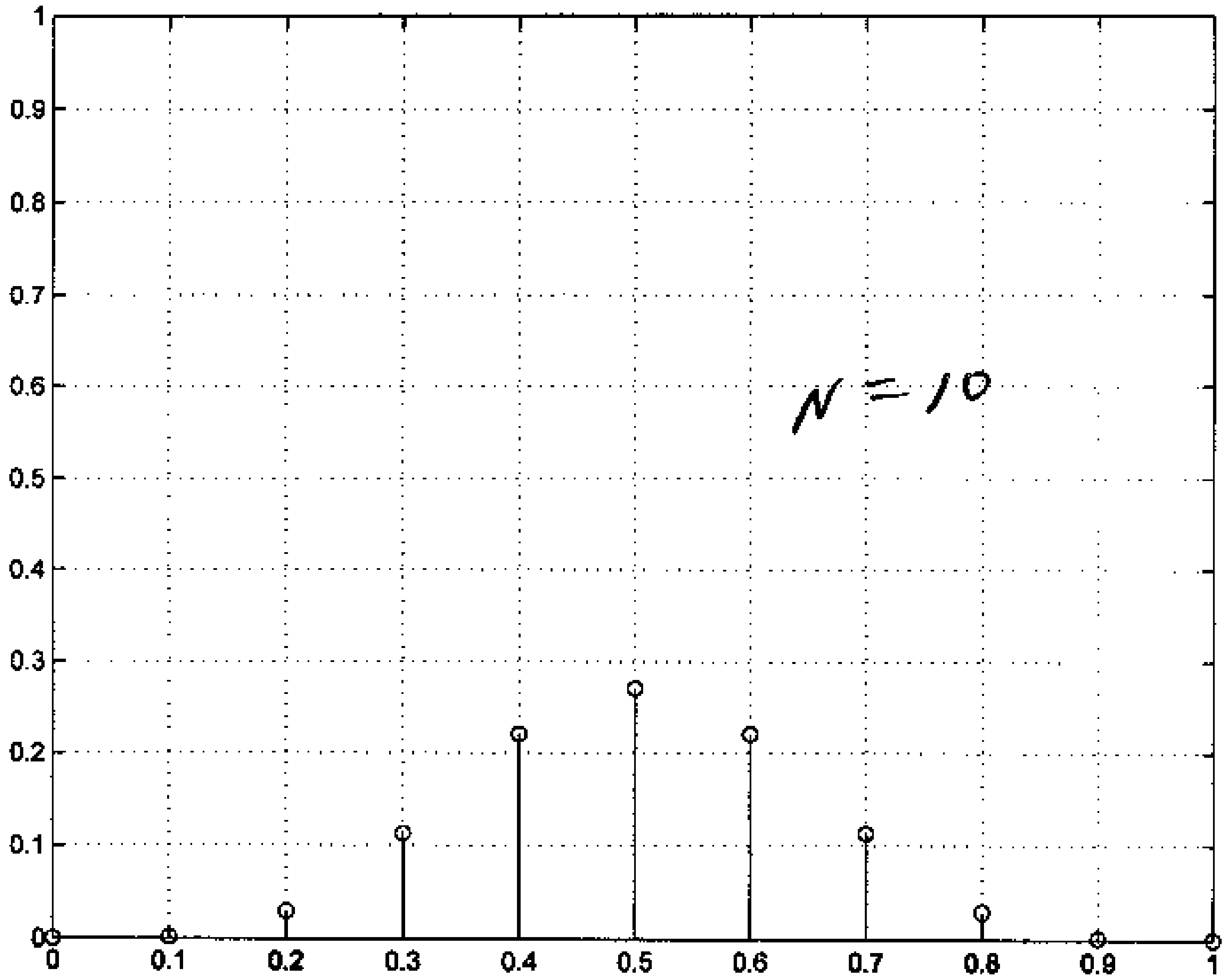
$$p_{Y|X}[y_j|i] = \frac{y_j^i(1-y_j)^{N-i}}{\sum_{j=0}^{M} y_j^i (1-y_j)^{N-i}} \qquad y_j = 0, \tfrac{1}{M}, \ldots, M \quad i = 0, 1, \ldots, N$$

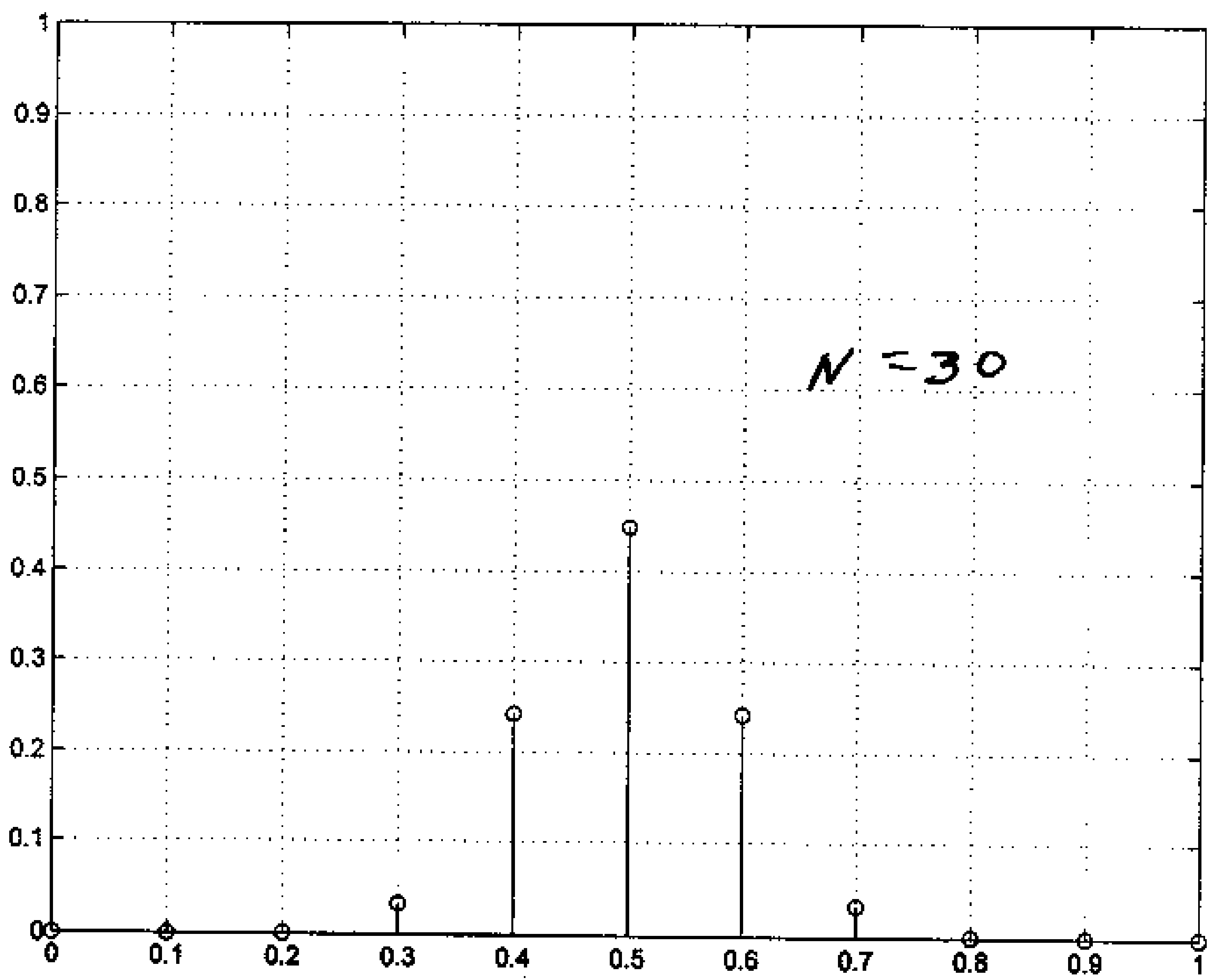
$$= \frac{y_j^{N/2}(1-y_j)^{N/2}}{\sum_{j=0}^{11} y_j^{N/2}(1-y_j)^{N/2}}$$

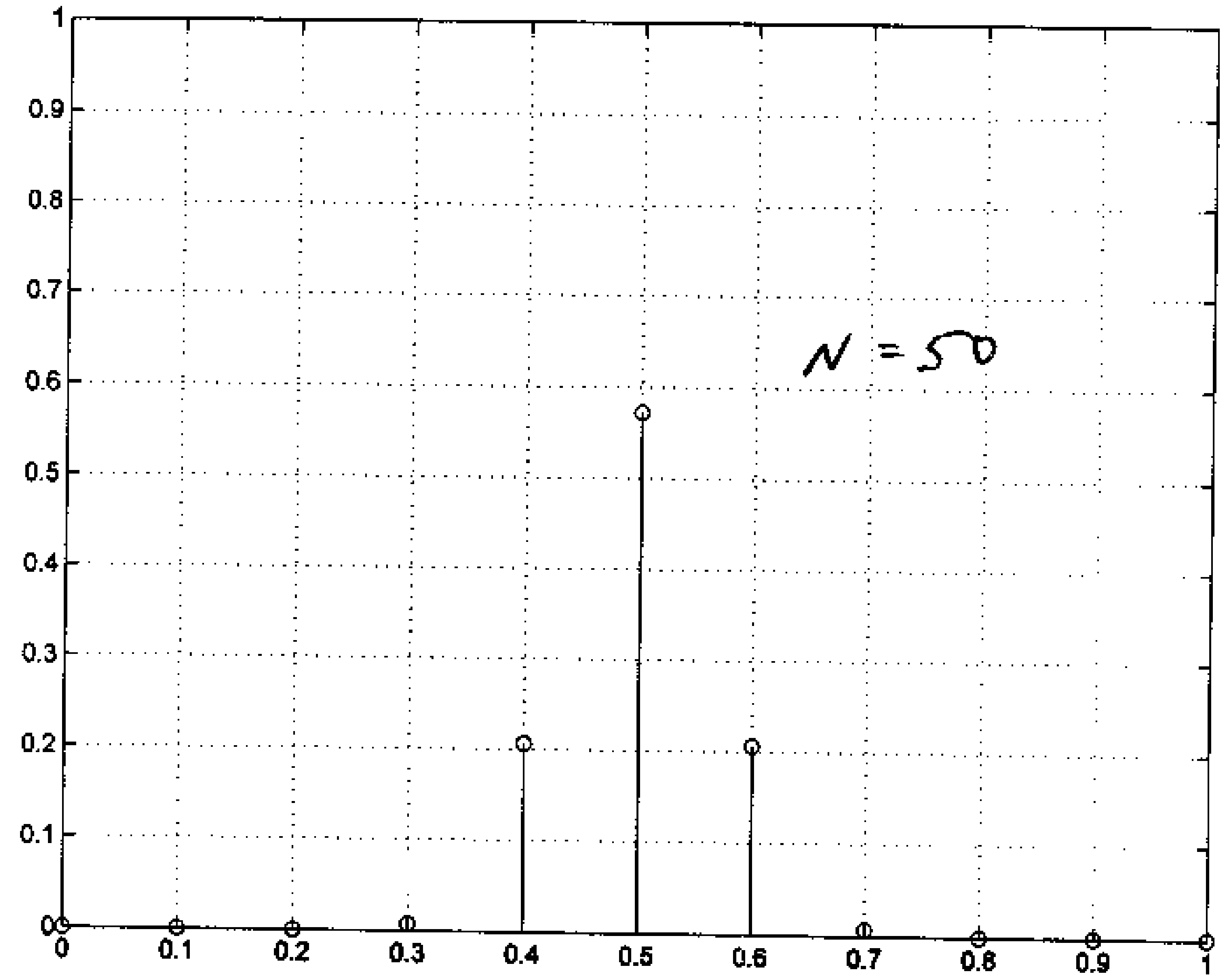
$$= \frac{[y_j(1-y_j)]^{N/2}}{\sum_{j=0}^{11} [y_j(1-y_j)]^{N/2}} \qquad y_j = 0, \tfrac{1}{11}, \ldots, 1$$

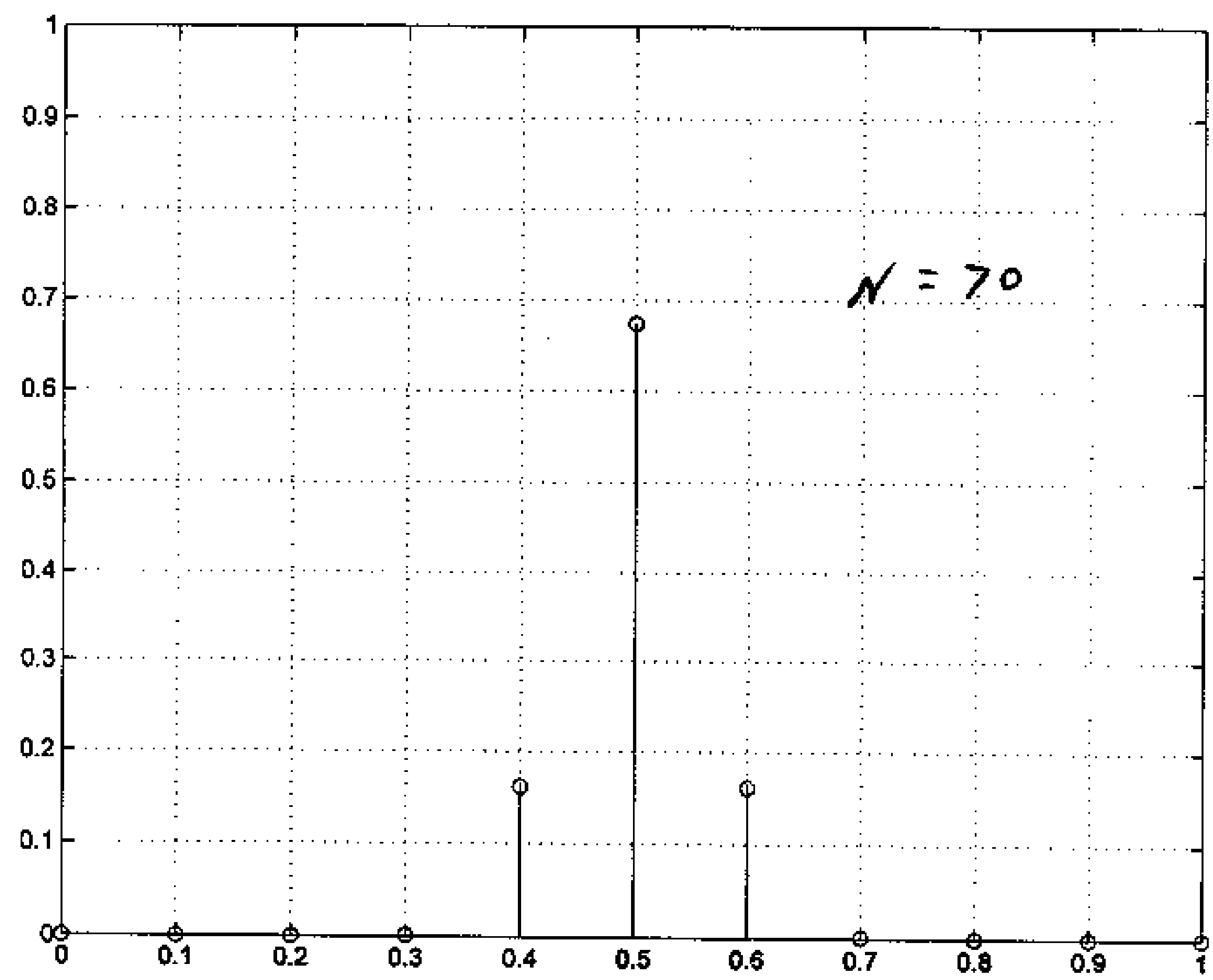
See next pages. Since $i = N/2$ or we observe $N/2$ heads in N tosses, we expect to learn that the coin is fair. Thus the posterior PMF is concentrated about $1/2$.

```
% probprob8_32.m
%
clear all
M=11;
y=[0:0.1:1]';
for N=10:20:70
   pYX=(y.*(1-y)).^(N/2);
   pYX=pYX/sum(pYX);
   figure
   stem(y,pYX)
   axis([0 1 0 1])
   grid
end
```


N = 10
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N = 30
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N = 50


N = 70

Chapter 9

1) $N = 3 \quad p_1 = 0.3, \; p_2 = 0.2, \; p_3 = 0.5$
$M = 7$

where

X_1 = number of days fishing = 3
X_2 = " " " visiting = 2
X_3 = " " " gardening = 2

$$p_{X_1, X_2, X_3}[k_1, k_2, k_3] = \binom{M}{k_1, k_2, k_3} p_1^{k_1} p_2^{k_2} p_3^{k_3}$$

$$p_{X_1, X_2, X_3}[3, 2, 2] = \binom{7}{3, 2, 2} (0.3)^3 (0.2)^2 (0.5)^2$$

$$= \frac{7!}{3!\,2!\,2!} (0.3)^3 (0.2)^2 (0.5)^2$$

$$= 0.0567$$

2) $N = 3, \; M = 4, \; p_1 = 0.2, \; p_2 = 0.4, \; p_3 = 0.4$

$$p_{X_1, X_2, X_3}[k_1, k_2, k_3] = \binom{M}{k_1, k_2, k_3} p_1^{k_1} p_2^{k_2} p_3^{k_3}$$

$$= \binom{4}{k_1, k_2, k_3} 0.2^{k_1} \, 0.4^{k_2} \, 0.4^{k_3}$$

where $k_1 + k_2 + k_3 = 4$

$$\Rightarrow p_{X_1, X_2, X_3}[k_1, k_2, \underbrace{k_3}_{4-k_1-k_2}] = \binom{4}{k_1, k_2, 4-k_1-k_2} \cdot 0.2^{k_1} \, 0.4^{k_2} \, 0.4^{4-k_1-k_2}$$

Compute for all k_1, k_2 where $0 \leq k_1 + k_2 \leq 4$

```
» probprob9_2

X =
```

k_1	k_2	p_{X_1,X_2,X_3}
0	0	0.0256
0	1.0000	0.1024
0	2.0000	0.1536
0	3.0000	0.1024
0	4.0000	0.0256
1.0000	0	0.0512
1.0000	1.0000	0.1536
1.0000	2.0000	0.1536
1.0000	3.0000	0.0512
2.0000	0	0.0384
2.0000	1.0000	0.0768
2.0000	2.0000	0.0384
3.0000	0	0.0128
3.0000	1.0000	0.0128
4.0000	0	0.0016

```
sum =

    1.0000
```

```
%probprob9_2.m
%
sum=0;
X=[];
for k1=0:4
   for k2=0:4
      if k1+k2<=4
         p=24/(prod(1:k1)*prod(1:k2)*prod(1:4-k1-k2));
         p=p*0.2^k1*0.4^k2*0.4^(4-k1-k2);
         X=[X;k1 k2 p];
         sum=sum+p;
      end
   end
end
X
sum
```

3) $$\underbrace{(a_1+a_2+a_3)}_{b}{}^M = \sum_{k_1=0}^{M} \frac{M!}{k_1!(M-k_1)!} a_1^{k_1} \cdot \sum_{k_2=0}^{M-k_1} \frac{(M-k_1)!}{k_2!(M-k_1-k_2)!} a_2^{k_2} a_3^{M-k_1-k_2}$$

$$= \sum_{k_1=0}^{M} \sum_{k_2=0}^{M-k_1} \frac{M!}{k_1!\,k_2!\,\underbrace{(M-k_1-k_2)}_{k_3}!} a_1^{k_1} a_2^{k_2} a_3^{\underbrace{M-k_1-k_2}_{k_3}}$$

and note that

$$\sum_{k_1=0}^{M}\sum_{k_2=0}^{M-k_1} = \sum_{\substack{k_1=0 \\ k_1+k_2+k_3=M}}^{M}\sum_{k_2=0}^{M}\sum_{k_3=0}^{M}$$

4) $$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=-1}^{1} \tfrac{1}{8}\left(\tfrac{1}{2}\right)^{k_1}\left(\tfrac{1}{4}\right)^{k_2}$$

$$= 3\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \tfrac{1}{8}\left(\tfrac{1}{2}\right)^{k_1}\left(\tfrac{1}{4}\right)^{k_2}$$

$$= 3/8 \underbrace{\sum_{k_1=0}^{\infty}\left(\tfrac{1}{2}\right)^{k_1}}_{\frac{1}{1-\frac{1}{2}}} \underbrace{\sum_{k_2=0}^{\infty}\left(\tfrac{1}{4}\right)^{k_2}}_{\frac{1}{1-\frac{1}{4}}}$$

$$= \tfrac{3}{8}(2)(4/3) = 1 \qquad \text{yes}$$

5) $$p_{X_1}[k_1] = \sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}(1-a)(1-b)(1-c)\,a^{k_1}b^{k_2}c^{k_3}$$

$$= (1-a)(1-b)(1-c)\,a^{k_1} \underbrace{\sum_{k_2=0}^{\infty} b^{k_2}}_{\frac{1}{1-b}} \underbrace{\sum_{k_3=0}^{\infty} c^{k_3}}_{\frac{1}{1-c}}$$

$$= (1-a)\,a^{k_1} \qquad k_1 = 0, 1, \ldots$$

Similarly, $p_{X_2}[k_2] = (1-b)\,b^{k_2} \qquad k_2 = 0, 1, \ldots$

$p_{X_3}[k_3] = (1-c)\,c^{k_3} \qquad k_3 = 0, 1, \ldots$

6) $$p_{X_1,X_2,X_3}[k_1,k_2,k_3] = \underbrace{\binom{M}{k_1, M-k_1} p_1^{k_1} p_2^{M-k_1}}_{\text{bin}(M,p_1)}$$

$$\cdot \underbrace{(1-p_3)\,p_3^{k_3}}_{\text{sums to } 1 \;\Rightarrow\; \text{PMF}}$$

Yes (x_1, x_2) is independent of x_3

7) $p_{\underline{y}}(\underline{y}) = p_{\underline{x}}(\underline{A}^{-1}\underline{y})$

Replace $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ by $\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}$ which are integers

$$\underline{A}^{-1}\underline{l} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \underline{l} = \begin{bmatrix} l_1 \\ l_2 - l_1 \\ l_3 - l_2 \end{bmatrix}$$

Since $k_1 = 0, 1, \ldots$
$k_2 = 0, 1, \ldots$
$k_3 = 0, 1, \ldots$

for $p_{\underline{x}}(\underline{A}^{-1}\underline{l})$ to be nonzero, we must have
$l_1 \geq 0$, $l_2 - l_1 \geq 0$, $l_3 - l_2 \geq 0$ or
$l_1 = 0, 1, \ldots$; $l_2 = l_1, l_1 + 1, \ldots$;
$l_3 = l_2, l_2 + 1, \ldots$

$$p_{\underline{y}}(\underline{l}) = e^{-(\lambda_1 + \lambda_2 + \lambda_3)} \frac{\lambda_1^{l_1} \lambda_2^{l_2 - l_1} \lambda_3^{l_3 - l_2}}{l_1!\,(l_2 - l_1)!\,(l_3 - l_2)!}$$

for $l_1 = 0, 1, \ldots$
$l_2 = l_1, l_1 + 1, \ldots$
$l_3 = l_2, l_2 + 1, \ldots$

8)
$$\int_{-\pi}^{\pi} e^{j\omega k} \frac{d\omega}{2\pi} = \int_{-\pi}^{\pi} \cos \omega k \frac{d\omega}{2\pi} + j \int_{-\pi}^{\pi} \sin \omega k \frac{d\omega}{2\pi}$$

$$= \frac{\sin \omega k}{2\pi k} \Big|_{-\pi}^{\pi} + j \frac{-\cos \omega k}{2\pi k} \Big|_{-\pi}^{\pi}$$

$$= \frac{\sin \pi k}{2\pi k} - \frac{\sin(-\pi k)}{2\pi k} - j\left(\frac{\cos \pi k}{2\pi k} - \frac{\cos(-\pi k)}{2\pi k}\right)$$

$$= 0 + 0 - j\left[\frac{\cos \pi k}{2\pi k} - \frac{\cos \pi k}{2\pi k}\right) \qquad k \neq 0$$

$$= 0 \qquad k \neq 0$$

For $k = 0$ $\displaystyle\int_{-\pi}^{\pi} e^{j\omega(0)} \frac{d\omega}{2\pi} = \int_{-\pi}^{\pi} 1 \frac{d\omega}{2\pi} = 1$

9) $\phi_{x_i}(\omega) = e^{\lambda_i(e^{j\omega}-1)}$

$$\phi_Y(\omega) = \prod_{i=1}^{N} \phi_{x_i}(\omega) = \prod_{i=1}^{N} e^{\lambda_i(e^{j\omega}-1)}$$

$$= e^{\sum_1^N \lambda_i(e^{j\omega}-1)}$$

$\Rightarrow Y \sim \text{Pois}\left(\sum_{i=1}^{N} \lambda_i\right)$

10) $E_{\underline{x}}[\bar{x}] = \frac{1}{N} \sum_{i=1}^{N} E_{\underline{x}}[x_i]$

$$= \frac{1}{N} \sum_{i=1}^{N} E_{x_i}[x_i] = \frac{1}{N} \sum_{i=1}^{N} E_x[x]$$

$$= E_x[x]$$

$$\text{var}(\bar{x}) = \text{var}\left(\frac{1}{N} \sum_{i=1}^{N} x_i\right)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \text{var}(x_i) = \frac{1}{N^2} \sum_{i=1}^{N} \text{var}(x)$$

$$= \text{var}(x)/N$$

As $N \to \infty$ $\text{var}(\bar{x}) \to 0 \Rightarrow$ PMF becomes concentrated about mean, $E_x[x]$.

11) For $x_i \sim \text{Ber}(p)$

$$\bar{x} \to E_x(x) = p$$

Thus, average number of successes in a long run of independent Bernoulli Trials is p. Or probability of success on a single trial is equal to relative frequency of successes.

12) $\rho_{x_1,x_2} = 0 \qquad \rho_{x_1,x_3} = \frac{1}{\sqrt{1 \cdot 4}} = \frac{1}{2}$

$$\rho_{x_2,x_3} = \frac{2}{\sqrt{2 \cdot 4}} = 1/\sqrt{2}$$

13) $\text{cov}(x_1, x_2) = \text{cov}(U, 2U) = 2\,\text{var}(U) = 2$
$\text{var}(x_2) = \text{var}(2U) = 4\,\text{var}(U) = 4$

$$\underline{C}_x = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\rho_{x_1,x_2} = \frac{2}{\sqrt{1 \cdot 4}} = 1$$

$$\det(\underline{C}_x) = 1 \cdot 4 - 2^2 = 0$$

No, only positive semidefinite

14) $\text{var}\left(\sum_{i=1}^{N} a_i x_i\right) = E_{\underline{x}}\left[\left(\sum a_i x_i - E_{\underline{x}}\left(\sum a_i x_i\right)\right)^2\right]$

$$= E_{\underline{x}}\left[\left(\sum a_i x_i - \sum a_i E_{\underline{x}}[x_i]\right)^2\right]$$

$$= E_{\underline{x}}\left[\left(\sum a_i (x_i - E_{x_i}[x_i])\right)^2\right]$$

$$= E_{\underline{x}}\left[\sum_i a_i (x_i - E_{x_i}[x_i]) \sum_j a_j (x_j - E_{x_j}[x_j])\right]$$

$$= E_{\underline{x}}\left[\sum_i \sum_j a_i a_j (x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right]$$

$$= \sum_i \sum_j a_i a_j E_{x_i x_j}\left[(x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right]$$

$$= \sum_i \sum_j a_i a_j \operatorname{cov}(x_i, x_j) = \underline{a}^T \underline{C}_x \underline{a}$$

15) $\operatorname{var}(X_1 + X_2 + X_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} \operatorname{cov}(x_i, x_j)$

$$= 13$$

16) Yes, $\underline{C}_x = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ Valid $\underline{C}_x$ since $(\underline{C}_x)_{11} > 0$ $(\underline{C}_x)_{22} > 0$ $\det(\underline{C}_x) > 0$

$\operatorname{var}(X_1 + X_2) = 1 + (-1) + (-1) + 2 = 1 = \operatorname{var}(X_1)$

17) a) No, $\det < 0$ b) No, $\operatorname{var}(X_1) < 0$
c) yes d) No, not symmetric

18) $\det(\underline{C}_x) = \det\left(\begin{bmatrix} \operatorname{var}(X_1) & \operatorname{cov}(X_1, X_2) \\ \operatorname{cov}(X_1, X_2) & \operatorname{var}(X_2) \end{bmatrix}\right)$

$$= \operatorname{var}(X_1)\operatorname{var}(X_2) - \operatorname{cov}^2(X_1, X_2) \geq 0$$

$$1 - \underbrace{\frac{\operatorname{cov}^2(X_1, X_2)}{\operatorname{var}(X_1)\operatorname{var}(X_2)}}_{\rho^2_{X_1, X_2}} \geq 0 \Rightarrow \rho^2_{X_1, X_2} \leq 1$$

$$|\rho_{X_1, X_2}| \leq 1$$

19) $$\underline{y} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{A} \underline{x}$$

$$E_{\underline{y}}[\underline{y}] = \underline{A}\, E_{\underline{x}}[\underline{x}] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

20) $$\underline{C}_y = \underline{A}\, \underline{C}_x\, \underline{A}^T$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$$

21) $$E_{\underline{x}}\left[\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} E_{x_1}[x_1] \\ E_{x_2}[x_2] \end{bmatrix}\right)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{pmatrix} E_{x_1}[x_1] \\ E_{x_2}[x_2] \end{pmatrix}\right)^T\right]$$

$$= E_{\underline{x}}\left[\begin{bmatrix} x_1 - E_{x_1}[x_1] \\ x_2 - E_{x_2}[x_2] \end{bmatrix} \begin{bmatrix} x_1 - E_{x_1}[x_1] & x_2 - E_{x_2}[x_2] \end{bmatrix}\right]$$

$$= E_{\underline{x}}\begin{bmatrix} (x_1 - E_{x_1}[x_1])^2 & (x_1 - E_{x_1}[x_1])(x_2 - E_{x_2}[x_2]) \\ (x_2 - E_{x_2}[x_2])(x_1 - E_{x_1}[x_1]) & (x_2 - E_{x_2}[x_2])^2 \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{var}(x_1) & \mathrm{cov}(x_1, x_2) \\ \mathrm{cov}(x_2, x_1) & \mathrm{var}(x_2) \end{bmatrix} = \underline{C}_x$$

22) $\underline{y} = \underline{A}\,\underline{x} \Rightarrow y_i = \sum_{j=1}^{N} [\underline{A}]_{ij} \cdot x_j$

$$E_{y_i}[y_i] = E_{\underline{x}}\left(\sum_{j=1}^{N} (\underline{A})_{ij} \cdot x_j \right)$$

$$= \sum_{j=1}^{N} [\underline{A}]_{ij} \cdot \underbrace{E_{\underline{x}}[x_j]}_{E_{x_j}[x_j]}$$

$$[E_{\underline{y}}[\underline{y}]]_i = \sum_{j=1}^{N} [\underline{A}]_{ij} [E_{\underline{x}}[\underline{x}]]_j$$

23) $[\underline{A}\,\underline{G}(\underline{x})\underline{A}^T]_{il} = \sum_k \sum_j [\underline{A}]_{ij} [\underline{G}(\underline{x})]_{jk} [\underline{A}^T]_{kl}$

$$E_{\underline{x}}[[\underline{A}\,G(\underline{x})\underline{A}^T]_{il}] = \sum_k \sum_j [\underline{A}]_{ij} \cdot E_{\underline{x}}([G(\underline{x})]_{jk})[\underline{A}^T]_{kl}$$

$$= \underline{A}\, E_{\underline{x}}[\underline{G}(\underline{x})]\underline{A}^T$$ from $[\underline{D}]_{il}$ expression

24) $[\underline{A}\,\underline{b}_1 \quad \underline{A}\,\underline{b}_2] = \left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_1^{(1)} \\ b_2^{(1)} \end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_1^{(2)} \\ b_2^{(2)} \end{pmatrix} \right]$

$$= \begin{bmatrix} a_{11} b_1^{(1)} + a_{12} b_2^{(1)} & a_{11} b_1^{(2)} + a_{12} b_2^{(2)} \\ a_{21} b_1^{(1)} + a_{22} b_2^{(1)} & a_{21} b_1^{(2)} + a_{22} b_2^{(2)} \end{bmatrix}$$

$$\underline{A}\,(\underline{b}_1 \quad \underline{b}_2) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_1^{(1)} & b_1^{(2)} \\ b_2^{(1)} & b_2^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} b_1^{(1)} + a_{12} b_2^{(1)} & a_{11} b_1^{(2)} + a_{12} b_2^{(2)} \\ a_{21} b_1^{(1)} + a_{22} b_2^{(1)} & a_{21} b_1^{(2)} + a_{22} b_2^{(2)} \end{bmatrix}$$

$$[c_1 \underline{d}_1 \quad c_2 \underline{d}_2] = \left[c_1 \begin{bmatrix} d_1^{(1)} \\ d_2^{(1)} \end{bmatrix} \quad c_2 \begin{bmatrix} d_1^{(2)} \\ d_2^{(2)} \end{bmatrix} \right]$$

$$= \begin{bmatrix} c_1 d_1^{(1)} & c_2 d_1^{(2)} \\ c_1 d_2^{(1)} & c_2 d_2^{(2)} \end{bmatrix}$$

$$(\underline{d}_1 \ \underline{d}_2) \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} d_1^{(1)} & d_1^{(2)} \\ d_2^{(1)} & d_2^{(2)} \end{bmatrix} \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 d_1^{(1)} & c_2 d_1^{(2)} \\ c_1 d_2^{(1)} & c_2 d_2^{(2)} \end{bmatrix}$$

25)

```
» CX=[26 6;6 26];
» [V Lambda]=eig(CX)

V =   [ v1        v2 ]

    0.7071    0.7071
   -0.7071    0.7071


Lambda =  Λ

    20     0
     0    32
```

Note: MATLAB 6.5, R13 will give you

$$\underline{V} = \begin{bmatrix} -0.7071 & 0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$$

Either one is correct.

26)

```
» CX=[10 6;6 20];
» [V Lambda]=eig(CX)

V =

    0.9056    0.4242
   -0.4242    0.9056


Lambda =

    7.1898         0
         0   22.8102
```

See previous note.

$$\underline{A} = \underline{V}^T$$

$$= \begin{bmatrix} 0.9056 & -0.4242 \\ 0.4242 & 0.9056 \end{bmatrix}$$

$var(Y_1) = 7.1898$

$var(Y_2) = 22.8102$

27) $\|\underline{y}\| = \|\underline{U}\,\underline{x}\|^2 = \sqrt{(\underline{U}\,\underline{x})^T(\underline{U}\,\underline{x})}$

$$= \sqrt{\underline{x}^T \underbrace{\underline{U}^T\underline{U}}_{=\underline{I}} \underline{x}} = \sqrt{\underline{x}^T\underline{x}} = \|\underline{x}\|$$

28) $\phi_{X_1,X_2,\ldots,X_N}(\omega_1,\omega_2,\ldots,\omega_N) = E_{X_1,\ldots,X_N}\left[e^{j(\omega_1 X_1+\ldots+\omega_N X_N)}\right]$

$= E_{X_1,\ldots,X_N}\left[e^{j\omega_1 X_1}\cdots e^{j\omega_N X_N}\right]$

$= E_{X_1,\ldots,X_N}\left[e^{j\omega_1 X_1}\right]\cdots E_{X_1,\ldots,X_N}\left[e^{j\omega_N X_N}\right]$ independence

$= E_{X_1}\left[e^{j\omega_1 X_1}\right]\cdots E_{X_N}\left[e^{j\omega_N X_N}\right]$

$= \phi_{X_1}(\omega_1)\cdots\phi_{X_N}(\omega_N)$

If $\phi_{\underline{X}}(\underline{\omega})$ factors, then

$p_{X_1,\ldots,X_N}(k_1,\ldots,k_N) = \int\cdots\int \phi_{X_1}(\omega_1)\cdots\phi_{X_N}(\omega_N)$

$\cdot\, e^{-j\omega_1 k_1}\cdots e^{-j\omega_N k_N}\,\frac{d\omega_1}{2\pi}\cdots\frac{d\omega_N}{2\pi}$

$= \int \phi_{X_1}(\omega_1)\, e^{-j\omega_1 k_1}\frac{d\omega_1}{2\pi}\cdots\int\phi_{X_N}(\omega_N)\, e^{-j\omega_N k_N}\frac{d\omega_N}{2\pi}$

$= p_{X_1}[k_1]\cdots p_{X_N}[k_N] \Rightarrow X_i$'s are independent

29) $E_{\underline{U}}[X_n] = \sum_{i=1}^{n} E_{U_i}[U_i] = \sum_{i=1}^{n} -(1-p)+p$

$= n(2p-1) = \frac{1}{2}n$

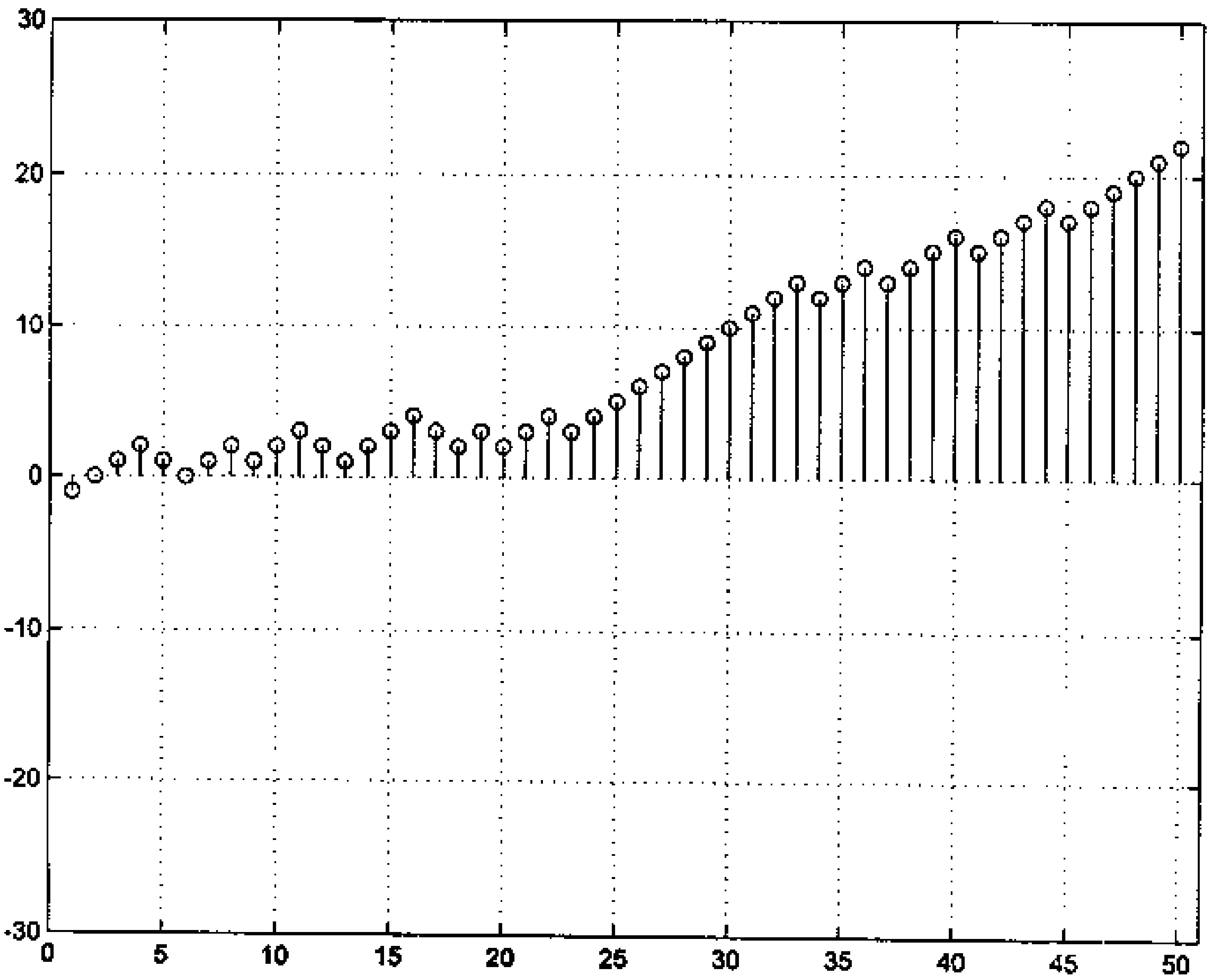
$\mathrm{var}(X_n) = \sum_{i=1}^{n}\mathrm{var}(U_i) = \sum_{i=1}^{n}\left(E_{U_i}[U_i^2] - E_{U_i}^2[U_i]\right)$

$$= n\left[1-(2p-1)^2\right] = n\left[1-\left(\tfrac{1}{2}\right)^2\right]$$

$$= \tfrac{3}{4}\, n$$

Outcomes should go to ∞.

30)



```
% probprob9_30.m
%
clear all
rand('state',0)
N=50;
for n=1:N
   if n==1
     if rand(1,1)<=3/4
        x(1)=1;
     else
        x(1)=-1;
        end
   else
      if rand(1,1)<=3/4
         p=1;
      else
         p=-1;
         end
      x(n)=x(n-1)+p;
   end
end
figure
stem([1:N]',x)
grid
axis([ 0 N+1 -30 30])
```

In Figure 9.3b we had $p = \frac{1}{2}$ so that $E[X_n] = 0$ but $\text{var}(X_n) = n$ so that even there the outcomes will become large with n.

31) $$(\underline{C}_X)_{ij} = \left[E_{\underline{X}}[\underline{X}\underline{X}^T] - E_{\underline{X}}[\underline{X}]E_{\underline{X}}[\underline{X}]^T\right]_{ij}$$

$$= \left[E_{\underline{X}}[\underline{X}\underline{X}^T]\right]_{ij} - \left[E_{\underline{X}}[\underline{X}]E_{\underline{X}}[\underline{X}]^T\right]_{ij}$$

But $[E_{\underline{x}}(\underline{x}\underline{x}^T)]_{ij} = E_{\underline{x}}(x_i x_j)$

$= E_{x_i x_j}(x_i x_j)$

and $(E_x(\underline{x}) E_x(\underline{x})^T)_{ij} = (E_{\underline{x}}(\underline{x}))_i (E_x(x))_j$

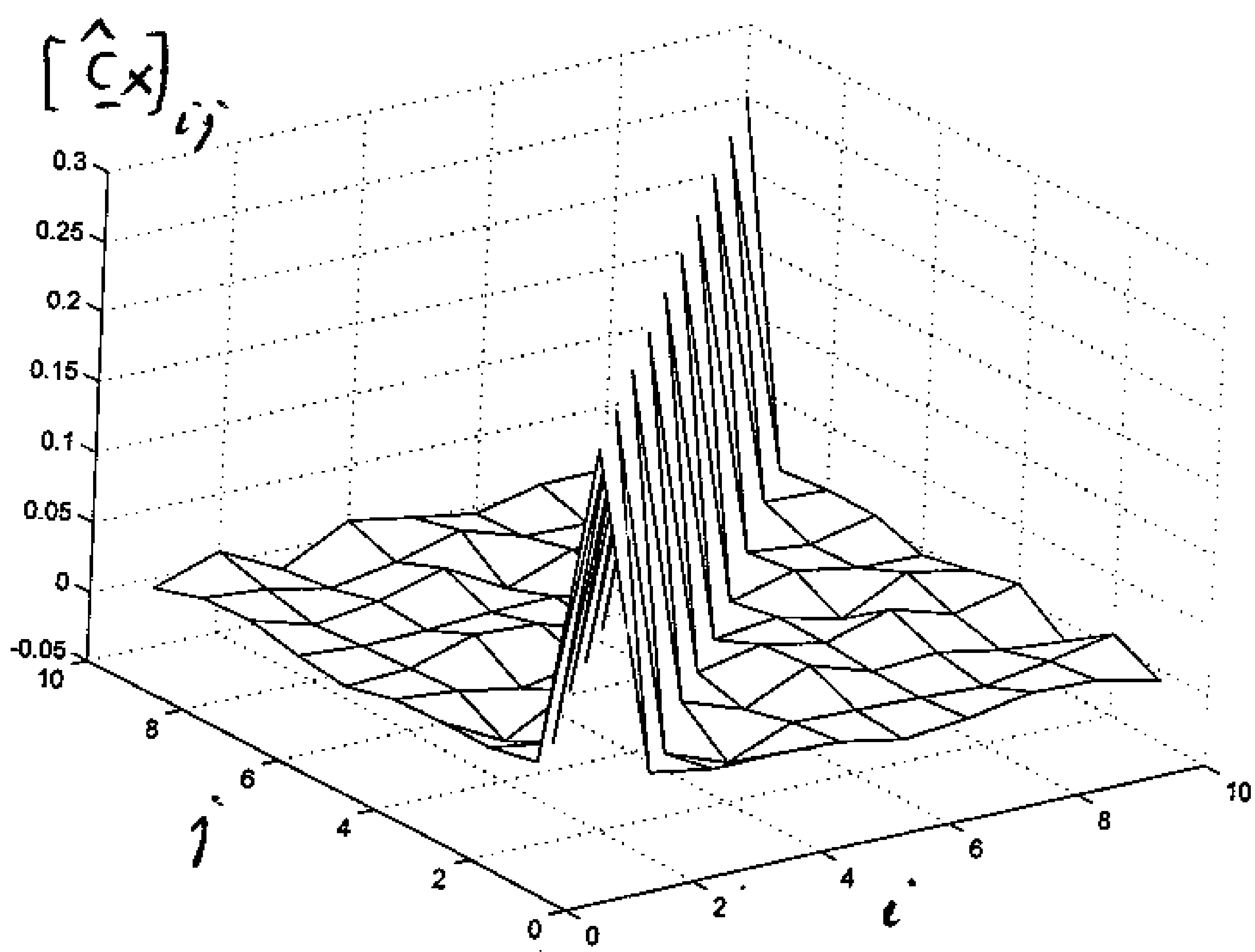
$= E_{x_i}(x_i) E_{x_j}(x_j)$

since $[\underline{a}\,\underline{b}^T]_{ij} = a_i b_j$

$\Rightarrow [\underline{C}_x]_{ij} = E_{x_i x_j}(x_i x_j) - E_{x_i}(x_i) E_{x_j}(x_j)$

$= \mathrm{cov}(x_i, x_j)$

32)



Appears to be diagonal.

```
% probprob9_32.m
%
clear all
rand('state',0)
N=10;M=1000;
CX=zeros(N,N);meanx=zeros(N,1);
for m=1:M
   x=floor(rand(N,1)+0.5);
   meanx=meanx+x/M;
   CX=CX+x*x'/M;
end
CX=CX-meanx*meanx'
mesh(CX)
```

33) $E[(X_3-(X_1+X_2))^2] = \mathrm{var}(X_3-X_1-X_2)$

$= \mathrm{var}(\underline{a}^T\underline{X}) \qquad \underline{a} = [-1\ -1\ \ 1]^T$

$= \underline{a}^T \underline{C}_X \underline{a}$

$$= [-1\ -1\ \ 1]\underbrace{\begin{bmatrix} 4 & 1 & 5 \\ 1 & 4 & 5 \\ 5 & 5 & 10 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} = 0$$

34) $\underline{C} = E_{\underline{X}}[\underline{X}\underline{X}^T]$ zero mean

$= E_{\underline{U}}[\underline{B}\underline{U}(\underline{B}\underline{U})^T] = E_{\underline{U}}[\underline{B}\underline{U}\underline{U}^T\underline{B}^T]$

$= \underline{B}\underbrace{E_{\underline{U}}[\underline{U}\underline{U}^T]}_{\underline{I}}\underline{B}^T = \underline{B}\underline{B}^T = \underline{V}\sqrt{\underline{\Lambda}}(\underline{V}\sqrt{\underline{\Lambda}})^T$

$= \underline{V}\sqrt{\underline{\Lambda}}\sqrt{\underline{\Lambda}}^T\underline{V}^T$

$$= \underline{V}\sqrt{\underline{\Lambda}}\sqrt{\underline{\Lambda}}\,\underline{V}^T \qquad (\underline{\Lambda} \text{ is diagonal})$$

$$= \underline{V}\,\underline{\Lambda}\,\underline{V}^T = \underline{C}_x$$

35) $\det\left(\begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}\right) = 0$

$$(4-\lambda)^2 - 1 = 0 \Rightarrow \lambda^2 - 8\lambda + \quad = 0$$

$$(\lambda-3)(\lambda-5) = 0$$

$$\lambda_1 = 3,\ \lambda_2 = 5 \Rightarrow \underline{\Lambda} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

$$(\underline{C}_x - \lambda\,\underline{I})\,\underline{v} = \underline{0}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underline{0} \Rightarrow v_1 = -v_2$$

$$\underline{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underline{0} \Rightarrow v_1 = v_2$$

$$\underline{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow \underline{V} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\underline{B} = \underline{V}\sqrt{\underline{\Lambda}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix} = \begin{bmatrix} \sqrt{3/2} & \sqrt{5/2} \\ -\sqrt{3/2} & \sqrt{5/2} \end{bmatrix}$$

36)

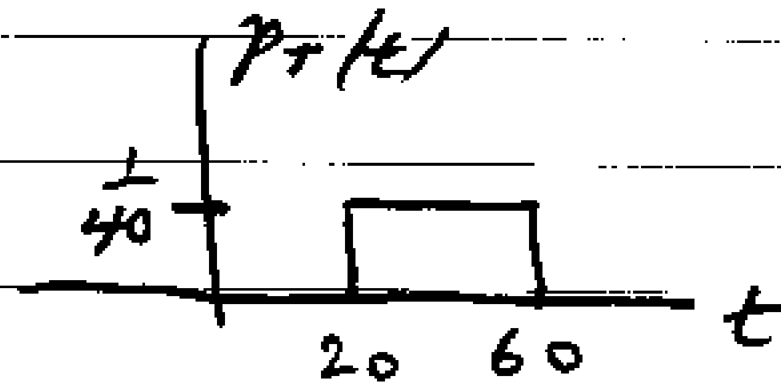
```
» probprob9_36

CX =

    4.0693    0.9996
    0.9996    3.9300
```

```
%  probprob9_36.m
%
clear all
rand('state',0)
B=[sqrt(3/2) sqrt(5/2);-sqrt(3/2) sqrt(5/2)];
M=1000;
for m=1:M
   r1=rand(1,1);
   if r1<0.5
      u(1,m)=-1;
   else
      u(1,m)=1;
   end
   r2=rand(1,1);
   if r2<0.5
      u(2,m)=-1;
   else
      u(2,m)=1;
   end
   x(:,m)=B*u(:,m);
end
CX=zeros(2,2);meanx=zeros(2,1);
for m=1:M
   meanx=meanx+x(:,m)/M;
   CX=CX+x(:,m)*x(:,m)'/M;
end
CX=CX-meanx*meanx'
```

Chapter 10

1) a) c b) d c) d d) d

2) $P[\hat{T} = 30] = \int_{29.75}^{30.25^-} p_T(t)\,dt$



$= \frac{1}{40}(0.5) = \frac{1}{80}$

PDF is more general for any physical measurement - no information is lost due to roundoff errors

3) $M([0, 0.2]) = \int_0^{0.2} \frac{1}{2} x\,dx = \frac{1}{4} x^2 \Big|_0^{0.2}$

$= \frac{1}{4}(0.2)^2 = 0.01$

$M([1.8, 2]) = \frac{1}{4} x^2 \Big|_{1.8}^{2} = \frac{1}{4}(4 - 3.24)$
$= 0.19$

4) a) No, $\int_{-\infty}^{\infty} p_X(x)\,dx \neq 1$

b) Yes c) No, $p_X(x) < 0$

5) $\int_{-5}^{5} c(1 - |x|/5)\,dx = 2c \int_0^5 (1 - \frac{1}{5}x)\,dx$

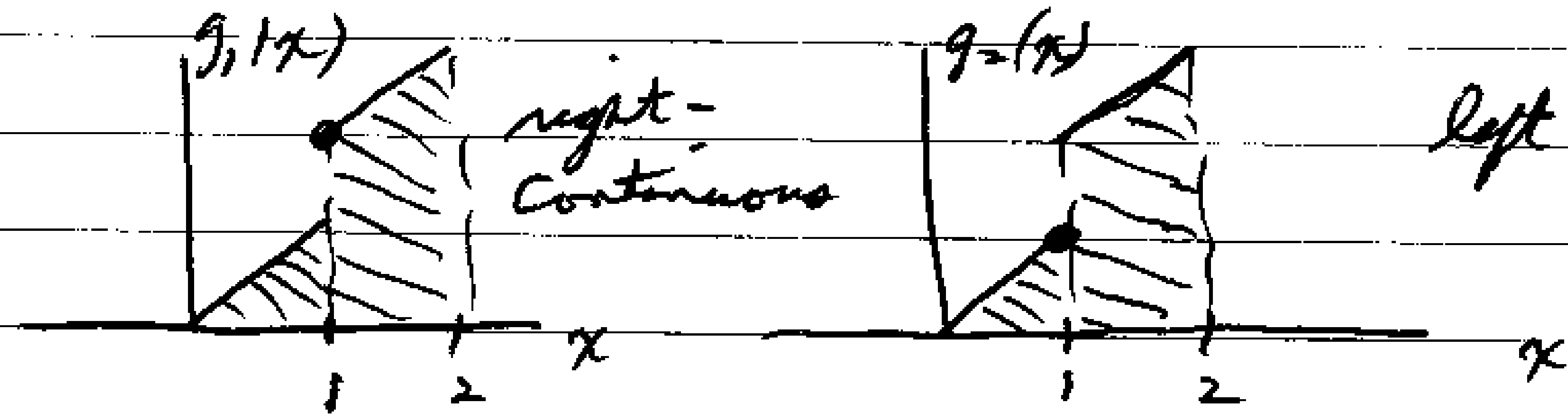
$= 2c\left[x - \frac{1}{10}x^2 \Big|_0^5\right) = 2c(5 - 2.5) = 5c = 1$

$\Rightarrow c = 1/5$

6) $\alpha_1 \geq 0,\ \alpha_2 \geq 0$ so $p_X(x) \geq 0$

$\alpha_1 + \alpha_2 = 1$ so $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$

7)



Same area - can change value of any point without changing area

8) $$P[X \leq 365) = \int_0^{365} 0.001\, e^{-0.001x}\,dx$$

$$= -e^{-0.001x}\Big|_0^{365} = 1 - e^{-0.365}$$

$$= 0.305$$

9) $x = r\cos\theta$

$y = r\sin\theta$

$$\left|\frac{\partial(x\, y)}{\partial(r\, \theta)}\right| = r$$

$$I^2 = \int_0^{\infty}\int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{1}{2}r^2}\, r\, d\theta\, dr$$

$$= \int_0^{\infty} r\, e^{-\frac{1}{2}r^2} \underbrace{\int_0^{2\pi} \frac{1}{2\pi}\, d\theta}_{=1}\, dr$$

$$= -e^{-\frac{1}{2}r^2}\Big|_0^{\infty} = 0 - (-1) = 1$$

$\Rightarrow I = 1$ since $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \geq 0$

10) $$P(x > \mu + a\sigma) = \int_{\mu+a\sigma}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Let $u = \frac{x-\mu}{\sigma}$ $\quad du = \frac{1}{\sigma} dx$

$$= \int_a^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}u^2} \sigma\, du$$

$$= \int_a^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = Q(a)$$

Using Q.m we have

$P(x > \mu + \sigma) = 0.1587$
$P(x > \mu + 2\sigma) = 0.0228$
$P(x > \mu + 3\sigma) = 0.0013$

11) $P(X \le med) = P(x \le \mu)$

$$= \int_{-\infty}^{\mu} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Let $u = \frac{x-\mu}{\sigma}$ $\quad du = \frac{1}{\sigma} dx$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}u^2} \sigma\, du$$

$$= \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$= \frac{1}{2}$ due to symmetry

12) $V = IR = R \sim N(1, 0.1)$

$\Rightarrow \quad V \sim N(1, 0.1)$

$$P(0.99 \le V \le 1.01) = \int_{0.99}^{1.01} \frac{1}{\sqrt{2\pi(0.1)}} e^{-\frac{1}{2(0.1)}(v-1)^2} dv$$

Let $u = (v-1)/\sqrt{0.1}$

$$= \int_{-\frac{0.01}{\sqrt{0.1}}}^{\frac{0.01}{\sqrt{0.1}}} \frac{1}{\sqrt{2\pi(0.1)}} e^{-\frac{1}{2}u^2} \sqrt{0.1}\, du$$

$$= \int_{-0.01/\sqrt{0.1}}^{0.01/\sqrt{0.1}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= \Phi\left(\frac{0.01}{\sqrt{0.1}}\right) - \Phi\left(\frac{-0.01}{\sqrt{0.1}}\right)$$

$$= 1 - Q\left(\frac{0.01}{\sqrt{0.1}}\right) - \left(1 - Q\left(-\frac{0.01}{\sqrt{0.1}}\right)\right)$$

$$= 1 - 2Q\left(\frac{0.01}{\sqrt{0.1}}\right) = 0.0252 \quad \text{from Q.m}$$

13) $P_{clip} = P(X > 3 \text{ or } X < -3)$

$$= \int_{-\infty}^{-3} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} dx + \int_{3}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2} dx$$

Let $u = x - \mu$

$$P_{clip} = \int_{-\infty}^{-3-\mu} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du + \int_{3-\mu}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= \Phi(-3-\mu) + 1 - \Phi(3-\mu)$$

$$\frac{dPdp}{d\mu} = -p_X(-3-\mu) + p_X(3-\mu) = 0$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-3-\mu)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(3-\mu)^2}$$

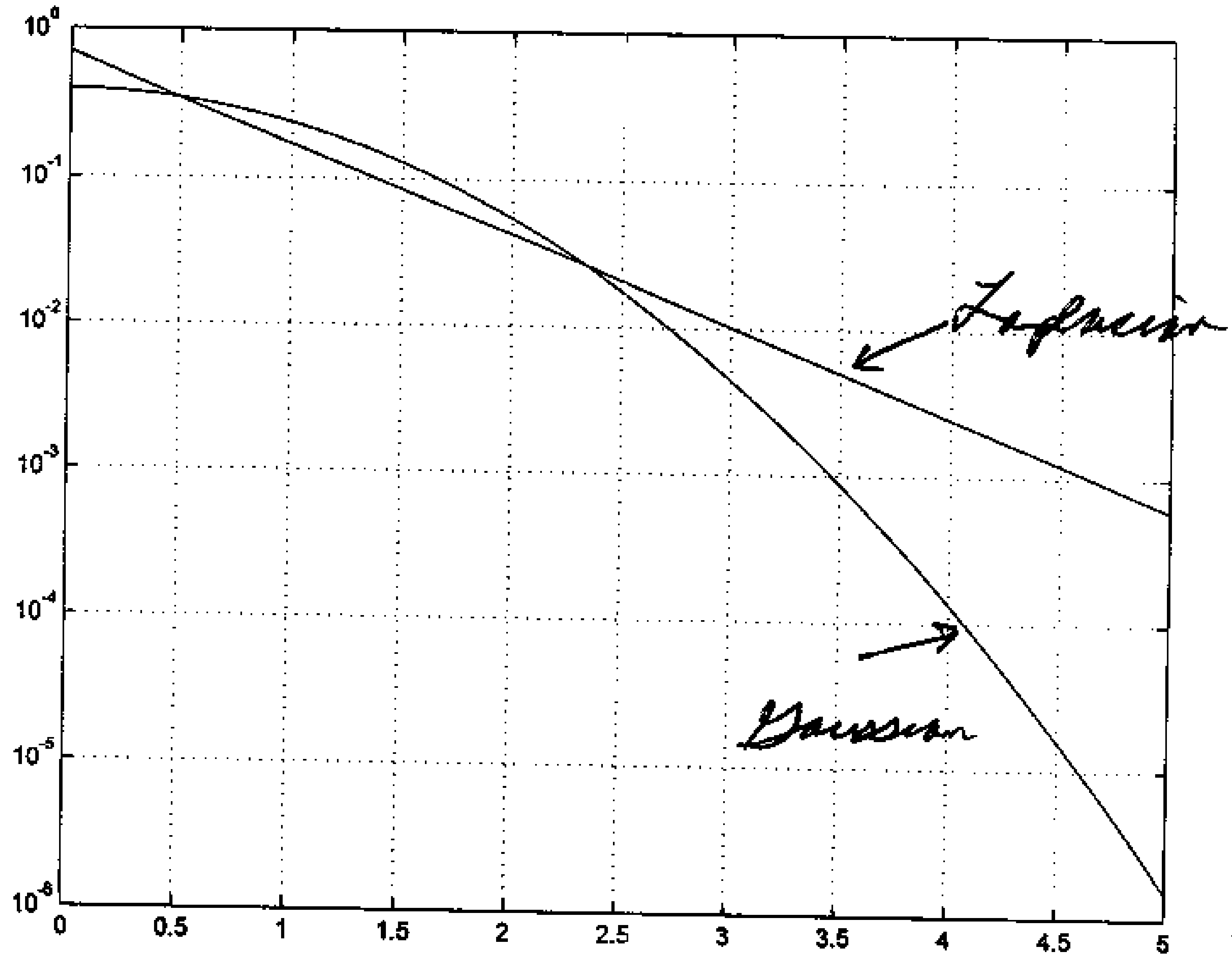
$$(3+\mu)^2 = (3-\mu)^2 \quad \Rightarrow \quad \mu = 0$$

(4) $P[X>3] = \int_3^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = Q(3) = 0.0013$

$$P[X>3] = \int_3^\infty \frac{1}{\sqrt{2}} e^{-\sqrt{2}x} dx$$

$$= -\frac{1}{2} e^{-\sqrt{2}x}\Big|_3^\infty = \frac{1}{2} e^{-3\sqrt{2}} = 0.0072$$

Laplacian is larger due to heavier "tails"



15) $\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} dx = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+x^2} dx$

$= \frac{2}{\pi} \arctan x \Big|_0^{\infty} = \frac{2}{\pi} \arctan \infty - \frac{2}{\pi} \arctan 0$

$= \frac{2}{\pi}(\pi/2) - 0 = 1$

16) $\Gamma(z+1) = \int_0^{\infty} \underbrace{t^z}_{u} \underbrace{e^{-t} dt}_{dv}$ $\qquad v = -e^{-t}$

$du = z t^{z-1} dt$

$= -t^z e^{-t} \Big|_0^{\infty} - \int_0^{\infty} -e^{-t} z t^{z-1} dt$

$= 0 + z \int_0^{\infty} t^{z-1} e^{-t} dt = z\Gamma(z)$

17) For $N=1$ $\quad p_X(x) = \lambda e^{-\lambda x}$

$P[X \le 5] = \int_0^5 \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^5$

$= 1 - e^{-5\lambda} = 1 - e^{-5(0.1)} = 0.393$

For $N=2$ $\quad p_X(x) = \frac{\lambda^2}{\Gamma(2)} x e^{-\lambda x}$

$= \lambda^2 x e^{-\lambda x}$

$P[X \le 5] = \int_0^5 \lambda^2 x e^{-\lambda x} dx$

$= \lambda^2 \left(\frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \Big|_0^5 \right)$

$= \lambda^2 \left(\frac{5 e^{-5\lambda}}{-\lambda} - \frac{e^{-5\lambda}}{\lambda^2} - 0 + 1/\lambda^2 \right)$

$= 0.1^2 \left(\frac{5 e^{-0.5}}{-0.1} - \frac{e^{-0.5}}{0.01} + \frac{1}{0.01} \right) = 0.090$

18) $M([0,x_0]) = \int_0^{x_0} \frac{1}{2} x \, dx = \frac{1}{4} x^2 \Big|_0^{x_0} = \frac{1}{4} x_0^2$

This is the cumulative mass from 0 to x_0.
At $x_0 = 2$ we have $M([0,2]) = 1$ kg.
In effect it is the function analogous to CDF.

19) $F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+t^2)} \, dt$

$$= \frac{1}{\pi} \arctan t \Big|_{-\infty}^{x} = \frac{1}{\pi} \arctan x - \frac{1}{\pi} \underbrace{\arctan(-\infty)}_{-\pi/2}$$

$$= \frac{1}{2} + \frac{1}{\pi} \arctan x$$

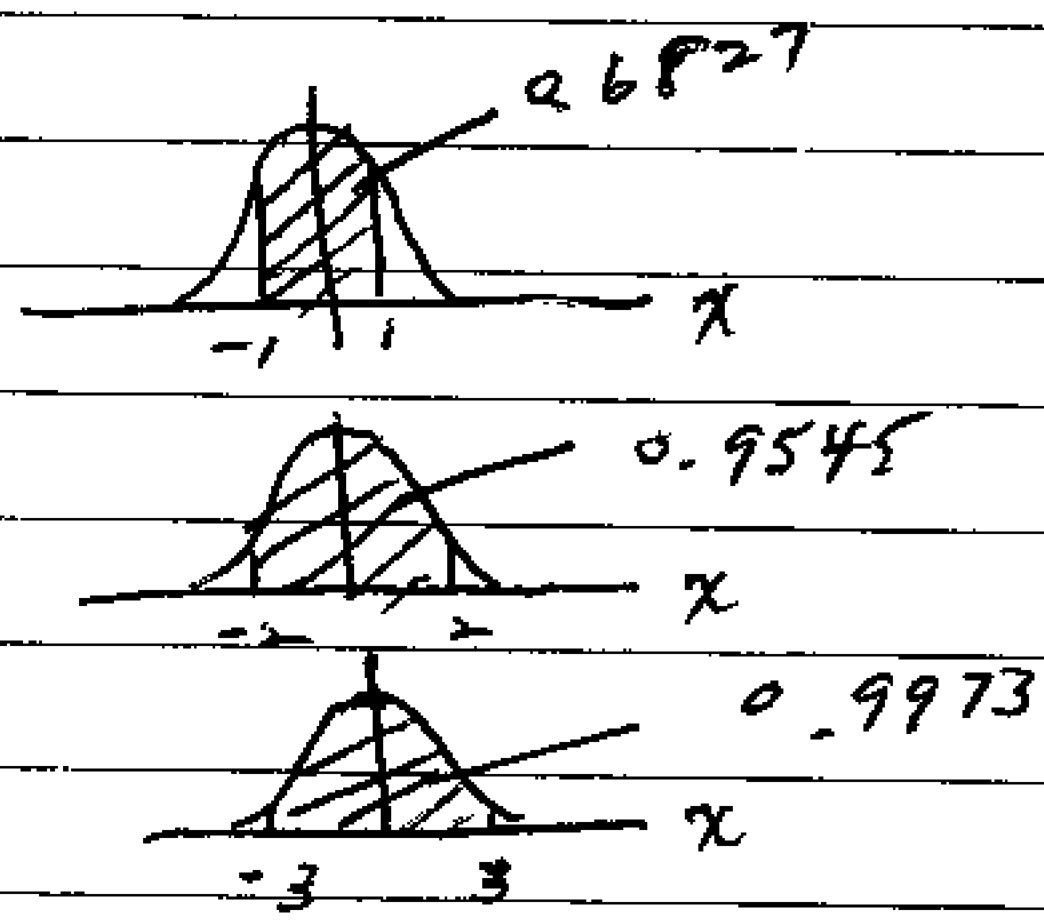
20) $P(|X| \le a) = \int_{-a}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx$

$$= \Phi(a) - \Phi(-a)$$

$$= [1 - Q(a)) - [1 - Q(-a))$$

$$= 1 - Q(a) - \cancel{1} + (1 - Q(a))$$

$$= 1 - 2Q(a)$$



$= 0.6827 \quad a = 1$

$0.9545 \quad a = 2$

$0.9973 \quad a = 3$

21) $P[1 \le X \le 2] = Q(1) - Q(2) = 0.1359$

For $M = 1000$ expect 136 outcomes in $[1, 2)$

From simulation obtain 116.

```
% probprob10_21.m
%
clear all
randn('state',0)
x=randn(1000,1);
count=0;
for i=1:1000
   if x(i)>=1&x(i)<=2
      count=count+1;
   end
end
count
```

22) $F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(t-\mu)^2} dt$

Let $u = \frac{t-\mu}{\sigma}$ $\quad du = \frac{1}{\sigma} dt$

$$F_X(x) = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}u^2} \sigma du$$

$$= \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

23) $F_X(x) = \int_{-\infty}^{x} p_X(t) dt$

$$F_X(-x) = \int_{-\infty}^{-x} p_X(t) dt$$

Let $t' = -t$

$$F_X(-x) = \int_{\infty}^{x} p_X(-t')(-dt')$$

$$= \int_{x}^{\infty} p_X(-t') dt'$$

$$= \int_{x}^{\infty} p_X(t') dt' \quad \text{Symmetric PDF}$$

$= 1 - F_x(x)$

Yes

24) $P(X > a) = \int_a^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$

Let $u = \frac{x-\mu}{\sigma}$ $du = \frac{1}{\sigma} dx$

$$= \int_{\frac{a-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}u^2} \sigma \, du$$

$$= \int_{\frac{a-\mu}{\sigma}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du = Q\left(\frac{a-\mu}{\sigma}\right)$$

25) $Q(-\infty) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$

$Q(\infty) = \int_\infty^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 0$

$Q(0) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$

$$= \frac{1}{2} \left(\int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \right.$$ symmetry

$$\left. + \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx \right)$$

$$= \frac{1}{2} [1] = \frac{1}{2}$$

$Q(-x) = \int_{-x}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$

Let $u = -t$

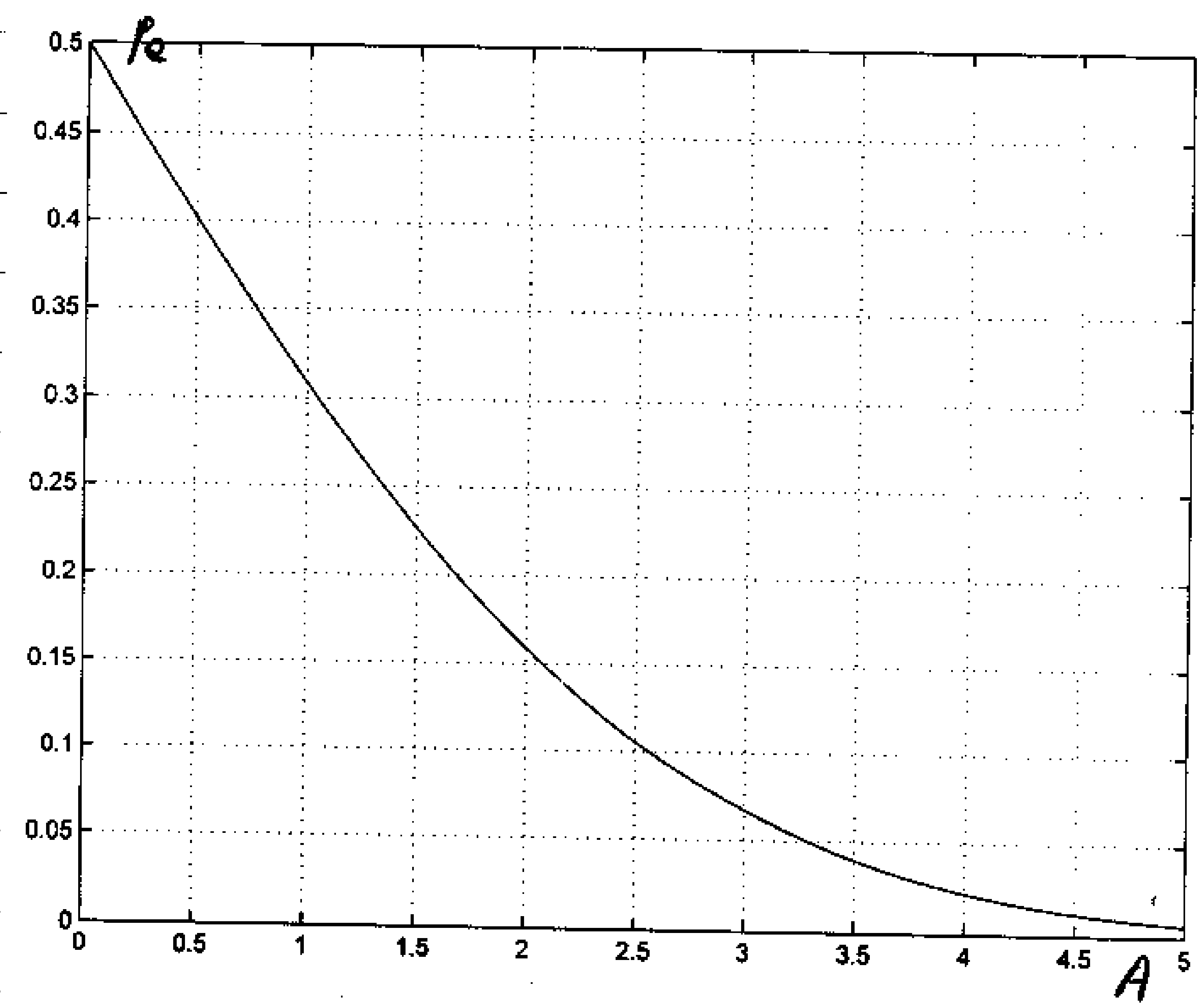
$$= \int_x^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} (-du)$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du$$

$$= 1 - Q(x)$$

26)




```
% probprob10_26.m
%
clear all
A=[0:0.01:5]';
Pe=Q(A/2);
plot(A,Pe)
grid
```

27) $P[X > 4] = Q(4) = 3.1671 \times 10^{-5}$

$\hat{P}[X > 4] = 2.3 \times 10^{-5}$ based on 10^6 realizations

Need at least $100/P[X > x]$ realizations
For $P[X > 7] = 1.2 \times 10^{-12}$ need about
10^{14} realizations!

```
% probprob10_27.m
%
clear all
randn('state',0)
M=10^6;
x=randn(M,1);
count=0;
for i=1:M
   if x(i)>4
      count=count+1;
   end
end
Pest=count/M
```

28) $P[X > 70000] = Q\left(\frac{70000-50000}{10^4}\right)$

$= Q(2) = 0.0228$

or about 2.3%

29) $P[T > 8] = Q\left(\frac{8-7}{1}\right) = Q(1) = 0.1587$

30) $P_E = Q\left(\frac{700-500}{70}\right) = Q\left(\frac{200}{70}\right) = Q(2.857)$

$= 0.0021$

$P_W = Q\left(\frac{700-525}{60}\right) = Q\left(\frac{175}{60}\right) = Q(2.916)$

$= 0.0018$

Eastern U.S. has higher probability due
to wider spread of scores.

31) $P[5 \le T \le 6] = Q(5-7) - Q(6-7) = Q(-2) - Q(-1)$
$\uparrow N(7,1)$ $= 0.1359$ using Q.m

32) $P[x > 2] = Q(2) = 0.0228$ using Q.m

33) $F_X(-\infty) = \frac{1}{1+e^{\infty}} = 0$

$F_X(\infty) = \frac{1}{1+e^{-\infty}} = 1$

$$\frac{dF_X(x)}{dx} = -(1+e^{-x})^{-2}\, e^{-x}(-1)$$

$$= \frac{e^{-x}}{(1+e^{-x})^2} \geq 0 \quad \text{all } x$$

$\Rightarrow$ monotonically increasing

34) $P[0 \leq X \leq 1] = F_X(1) = \frac{2}{\pi}\arctan 1$

$$= \frac{2}{\pi}(\pi/4) = \frac{1}{2}$$

35) $P[a \leq X \leq b] = \int_a^b p_X(x)\,dx$

$$= \int_{-\infty}^{b} p_X(x)\,dx - \int_{-\infty}^{a} p_X(x)\,dx$$

$$= F_X(b) - F_X(a)$$

36) $X \sim N(65, 38)$

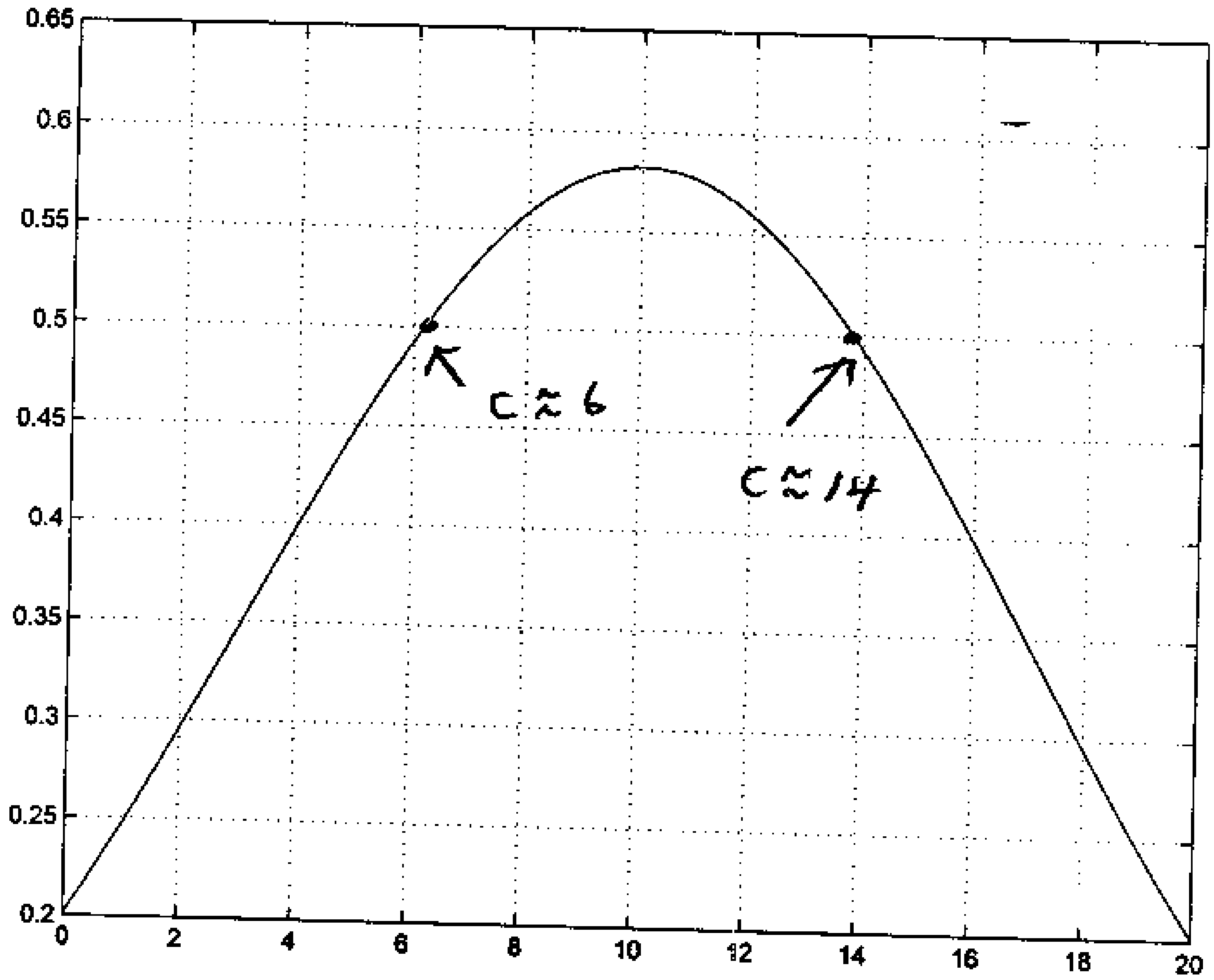
$Y = X + c \sim N(65+c, 38)$ from Ex 10.5

$P[70 \leq Y \leq 80] = 0.5$

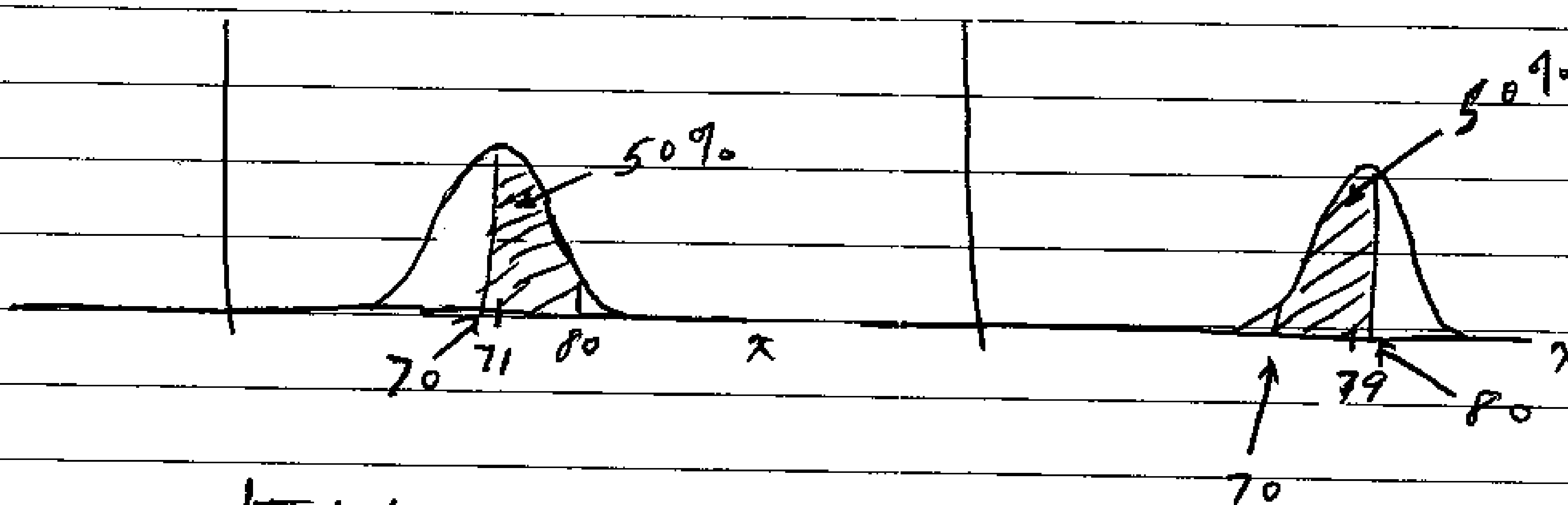
$$Q\left(\frac{70-(65+c)}{\sqrt{38}}\right) - Q\left(\frac{80-(65+c)}{\sqrt{38}}\right) = 0.5$$

$$Q\left(\frac{5-c}{\sqrt{38}}\right) - Q\left(\frac{15-c}{\sqrt{38}}\right) = 0.5$$

Plot left-hand-side vs c.



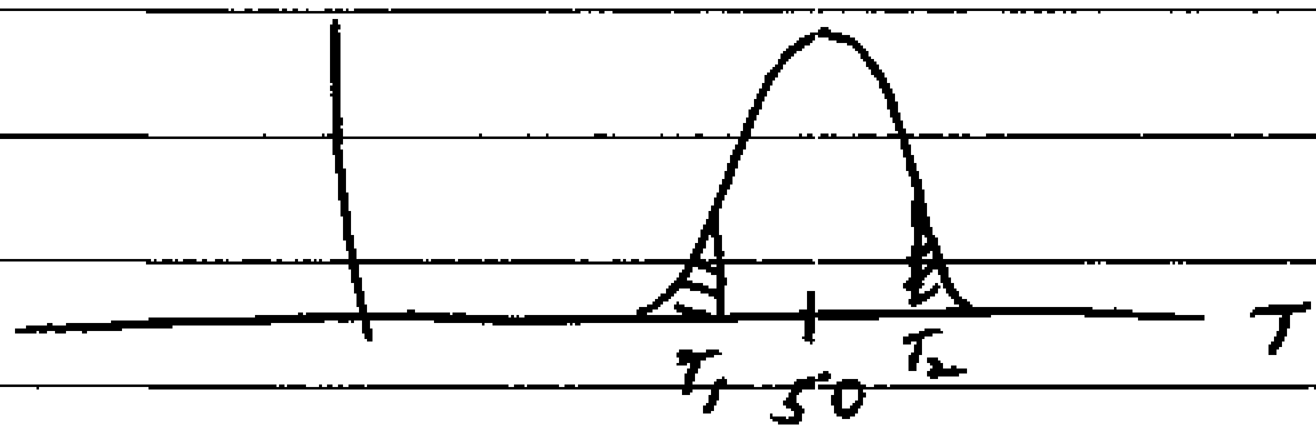

```
%probprob10_36.m
clear all
c=[0:0.1:20]';
P=Q((5-c)/sqrt(38))-Q((15-c)/sqrt(38));
plot(c,P)
grid
```



Students would prefer c = 14.

37) $P[T_1 \le T \le T_2] = 0.95$

$$Q\left(\frac{T_1-50}{\sqrt{10}}\right) - Q\left(\frac{T_2-50}{\sqrt{10}}\right) = 0.95$$



Choose $50 - T_1 = T_2 - 50 \Rightarrow T_2 = 100 - T_1$

$$\underbrace{Q\left(\frac{T_1-50}{\sqrt{10}}\right)}_{x} - Q\left(\frac{50-T_1}{\sqrt{10}}\right) = 0.95$$

$Q(x) - Q(-x) = 0.95$

$1 - 2Q(x) = 0.95 \Rightarrow Q(x) = 0.025$

$\Rightarrow x = 1.96$ use Qinv.m

$T_1 = \sqrt{10}\ 1.96 + 50 = 56.19$

$T_2 = 43.8$

or $T_2 \approx 44°$

$T_1 \approx 56°$

38) $\frac{d F_X(x)}{dx} = 0 \qquad x < 1$

$= \frac{d}{dx}(x-1) = 1 \qquad 1 < x < 2$

$= 0 \qquad x > 2$

at $x = 1, 2$ no derivative exists due To slope change

39) $y = e^x$ $\quad y = g(x) = e^x \Rightarrow x = \ln y = g^{-1}(y)$

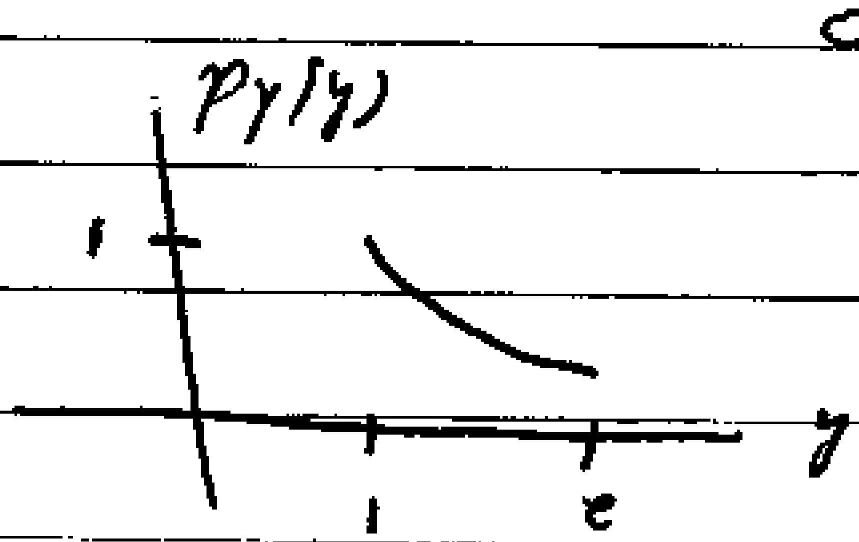
$$p_Y(y) = p_X(g^{-1}(y)) \left| \frac{d g^{-1}(y)}{dy} \right|$$

$$= \begin{cases} p_X(\ln y)\, \frac{1}{y} & y > 0 \\ 0 & y \le 0 \end{cases}$$

For $X \sim U(0,1)$ $\quad e^0 < y < e^1$

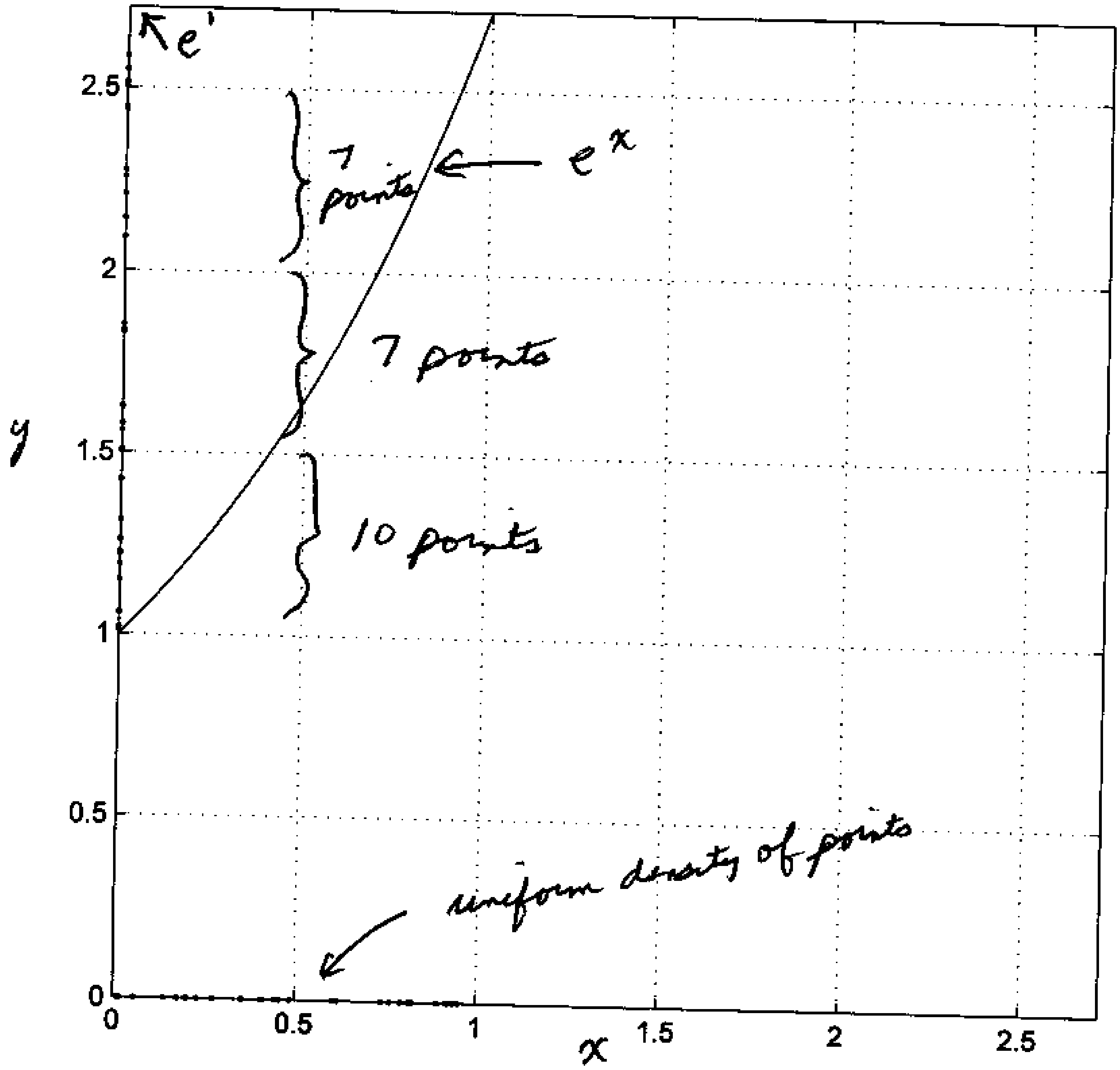
$$\Rightarrow p_Y(y) = 1 \cdot \frac{1}{y} \quad 1 < y < e$$

$$= \begin{cases} \frac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$



density decreases as y increases

```
% probprob10_39.m
%
clear all
rand('state',0)
M=30;
x=rand(M,1);
y=exp(x);
xx=[zeros(M,1);x];
yy=[y;zeros(M,1)];
plot(xx,yy,'.')
xxx=[0:0.01:8]';yyy=exp(xxx);
line(xxx,yyy)
axis([0 exp(1) 0 exp(1)])
axis('square')
grid
```



40) $y = x^4 + 1 \qquad x \geq 0 \Rightarrow y \geq 1$

$x = (y-1)^{1/4} = g^{-1}(y)$

$p_Y(y) = p_X(g^{-1}(y)) \left| \dfrac{d g^{-1}(y)}{dy} \right|$

$= \lambda e^{-\lambda (y-1)^{1/4}} \left| \frac{1}{4} (y-1)^{-3/4} \right|$

$= \dfrac{\lambda}{4 (y-1)^{3/4}} e^{-\lambda (y-1)^{1/4}} \qquad y \geq 1$

$0 \qquad y < 1$

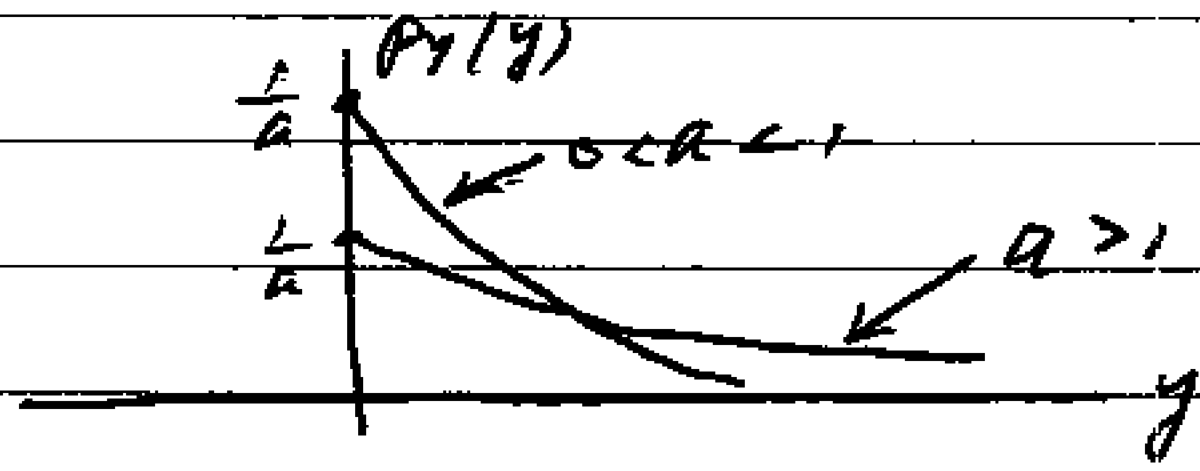
41) $p_Y(y) = p_X\left(\frac{y-b}{a}\right)\left|\frac{1}{a}\right| \qquad b < y < a+b$

$$= \frac{1}{a} \qquad \underset{\overset{\uparrow}{2}}{b} < y < \underbrace{a+b}_{6}$$

$\Rightarrow \quad a = 4 \quad b = 2$

42) $p_Y(y) = p_X(y/a)\left|\frac{1}{a}\right|$

$X \sim \exp(1) \Rightarrow p_Y(y) = e^{-y/a}\,\frac{1}{a} \qquad a > 0$ and $y \geq 0$



$a>1$ expands PDF

$a<1$ compresses PDF

If a random variable is scaled by $a > 1$, the values are magnified and thus more spread out. Hence probability of Y large increases. Vice-versa for $0 < a < 1$.

43) $y = |x| \qquad g_1^{-1}(y) = y \qquad g_2^{-1}(y) = -y$

$$p_Y(y) = p_X(g_1^{-1}(y))\left|\frac{dg_1^{-1}(y)}{dy}\right|$$

$$+ p_X(g_2^{-1}(y))\left|\frac{dg_2^{-1}(y)}{dy}\right|$$

$$= p_X(y)|1| + p_X(-y)|-1|$$

$$= p_X(y) + p_X(-y)$$

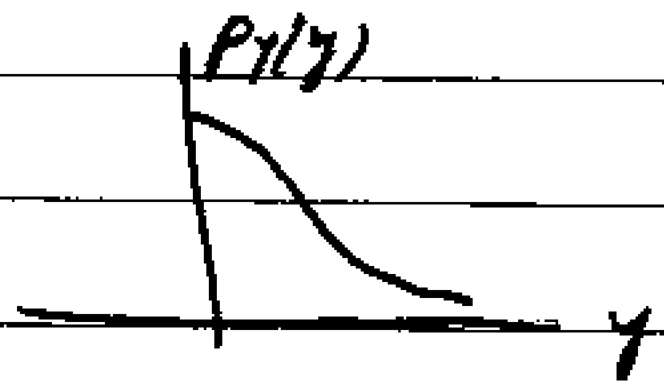
For $X \sim N(0,1)$

$$p_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-y)^2}$$

$$= \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

$$= \sqrt{2/\pi}\, e^{-\frac{1}{2}y^2} \qquad y \geq 0$$

$$0 \qquad y < 0$$



44) $F_Y(y) = P(Y \leq y) = P(e^X \leq y)$

$$= P(X \leq \ln y) \qquad y > 0$$

$$= \int_{-\infty}^{\ln y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

$$\frac{d F_Y(y)}{dy} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\ln y)^2} \frac{1}{y}$$

$$p_Y(y) = \frac{1}{\sqrt{2\pi}\, y} e^{-\frac{1}{2}(\ln y)^2} \qquad y > 0$$

$$0 \qquad y \leq 0$$

45) $P(Y > 2) = P(|X| > 2) = 2P(X > 2)$

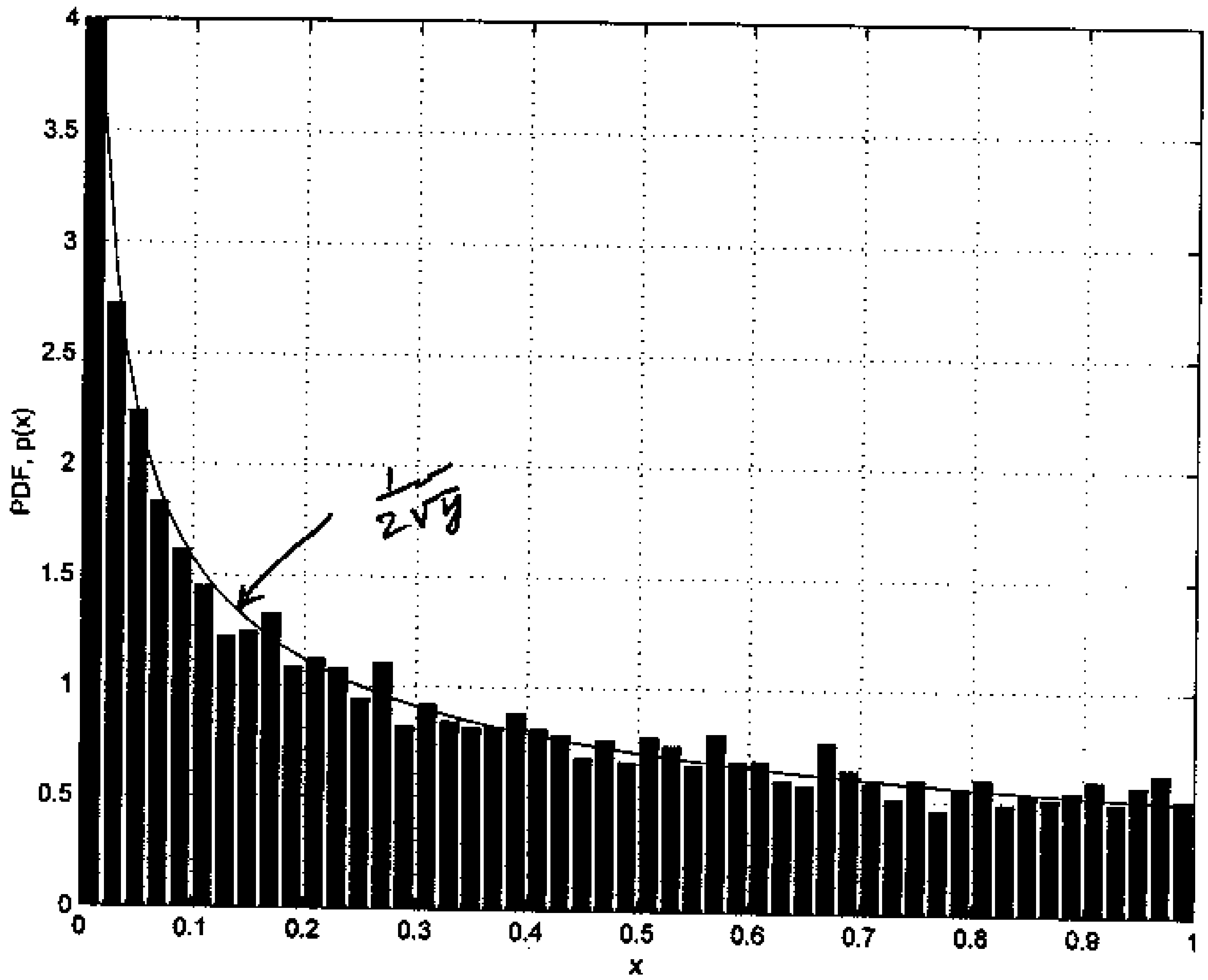
$$= 2Q(2) = 0.0455$$

46) $y = x^2 \Rightarrow$
$x = \sqrt{y}$

$$p_Y(y) = p_X(\sqrt{y}) \frac{1}{2\sqrt{y}} \qquad 0 < y < 1$$

$$= 1 \cdot \frac{1}{2\sqrt{y}}$$

$$p_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$




```
% probprob10_46.m
%
clear all
rand('state',0)
x=rand(10000,1);
y=x.^2;
pdf(y,10000,50,0,1,4)
hold on
xx=[0.01:0.01:1]';
plot(xx,1./(2*sqrt(xx)))
hold off
```

47) $$p_X[k] = \begin{cases} 1-p & k=0 \\ p & k=1 \end{cases}$$

$$p_X(x) = (1-p)\,\delta(x) + p\,\delta(x-1)$$

$$F_X(x) = \int_{-\infty}^{x^+} p_X(t)\,dt$$

$$= \int_{-\infty}^{x^+} \left[(1-p)\,\delta(t) + p\,\delta(t-1)\right] dt$$

$$= (1-p)\,u(x) + p\,u(x-1)$$

4f) $$I = \int_{-\infty}^{\infty} \sum_{i=1}^{N} a_i\,\delta(x-i)\,dx$$

$$= \sum_{i=1}^{N} a_i \underbrace{\int_{-\infty}^{\infty} \delta(x-i)\,dx}_{=1 \text{ for all } i}$$

$$= \sum_{i=1}^{N} a_i$$

Can use impulses to represent a point for which the probability is nonzero. Hence the usual value of an integral $\int_{-\epsilon}^{\epsilon} g(x)\,dx$ which is zero as $\epsilon \to 0$ can be converted to a nonzero value by using impulses. Hence, a PDF, which is <u>integrated</u> to find a probability, can be used for continuous <u>and</u> discrete random variables.

49) $\delta(x-2)$ not included due to $\bar{\sum}$

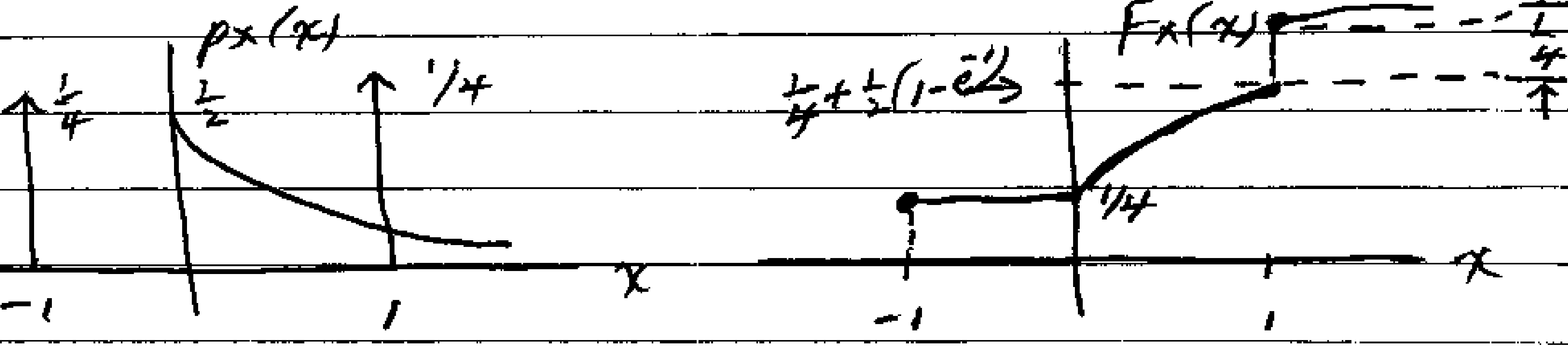
$\delta(x-4)$ " "

$\delta(x-3/2)$ is included

Integral $= 1/8$

Integrand integrates to 1 over $-\infty < x < \infty$ and strengths of impulses > 0. Could be a PDF. Hence, integral is $P[1 \le X < 2]$.

50)



$$F_X(x) = \begin{cases} 0 & x < -1 \\ 1/4 & -1 \le x < 0 \\ \frac{1}{4} + \frac{1}{2}(1 - e^{-x}) & 0 \le x < 1 \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-x}) = 1 - \frac{1}{2} e^{-x} & x \ge 1 \end{cases}$$

or $F_X(x) = \frac{1}{4} u(x+1) + \frac{1}{2}(1 - e^{-x}) u(x) + \frac{1}{4} u(x-1)$

51)

$$P(-2 \le X < 2) = \int_{2^-}^{2^+} p_X(x)\, dx$$

$$= \frac{1}{2} + \int_0^2 \frac{1}{2} e^{-x} dx = \frac{1}{2} - \frac{1}{2} e^{-x} \Big]_0^2$$

$$= \frac{1}{2} - \frac{1}{2}(e^{-2} - 1) = 1 - \frac{1}{2} e^{-2}$$

$$P[-1 \le X \le 1] = \int_{-1^-}^{1^+} p_X(x)\, dx$$

$$= \frac{1}{2} + \int_0^1 \frac{1}{2} e^{-x} dx$$

$$= \frac{1}{2} - \frac{1}{2} e^{-x} \Big|_0^1 = \frac{1}{2} - \frac{1}{2}(e^{-1} - 1)$$

$$= 1 - \frac{1}{2} e^{-1}$$

$$P[-1 < X \le 1] = \int_{-1^+}^{1^+} p_X(x)\,dx$$

$$= \int_{-1^-}^{1^+} p_X(x)\,dx - 1/4$$

$$= 3/4 - \tfrac{1}{2} e^{-1}$$

$$P[-1 < X < 1] = \int_{-1^+}^{1^-} p_X(x)\,dx$$

$$= \int_0^1 \tfrac{1}{2} e^{-x}\,dx$$

$$= -\tfrac{1}{2} e^{-x}\Big]_0^1 = -\tfrac{1}{2}(e^{-1}-1)$$

$$= \tfrac{1}{2}(1-e^{-1})$$

$$P[-1 \le X < 1] = \int_{-1^-}^{1^-} p_X(x)\,dx = \int_{-1^-}^{1^+} p_X(x)\,dx - \tfrac{1}{4}$$

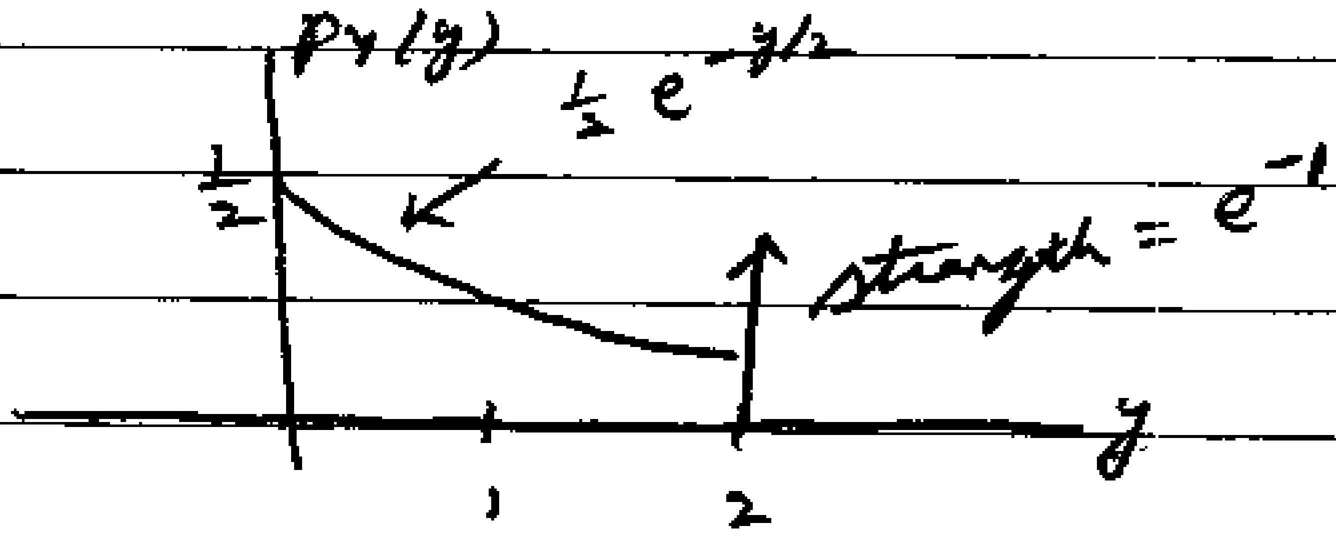
$$= 3/4 - \tfrac{1}{2} e^{-1}$$

52) $$p_Y(y) = p_X(g^{-1}(y)) \left|\frac{d g^{-1}(y)}{dy}\right| + P[X \ge 1]\,\delta(y-2)$$

$$= \underbrace{p_X(y/2)\,\tfrac{1}{2}}_{\text{for } 0 \le y < 2} + P[X \ge 1]\,\delta(y-2)$$

$$P[X \ge 1] = \int_1^\infty e^{-x}\,dx = -e^{-x}\Big]_1^\infty = e^{-1}$$

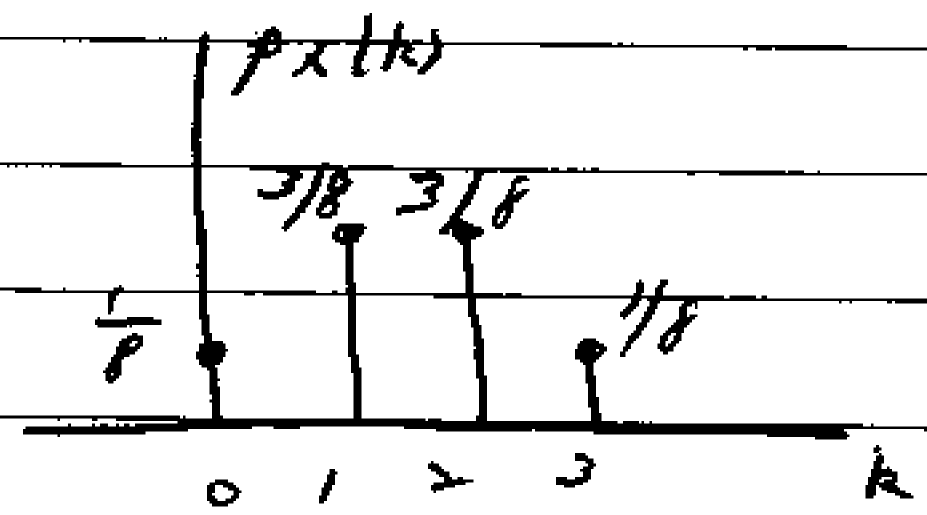
$$p_Y(y) = \begin{cases} e^{-y/2}\,\tfrac{1}{2} & 0 \le y < 2 \\ e^{-1}\,\delta(y-2) & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

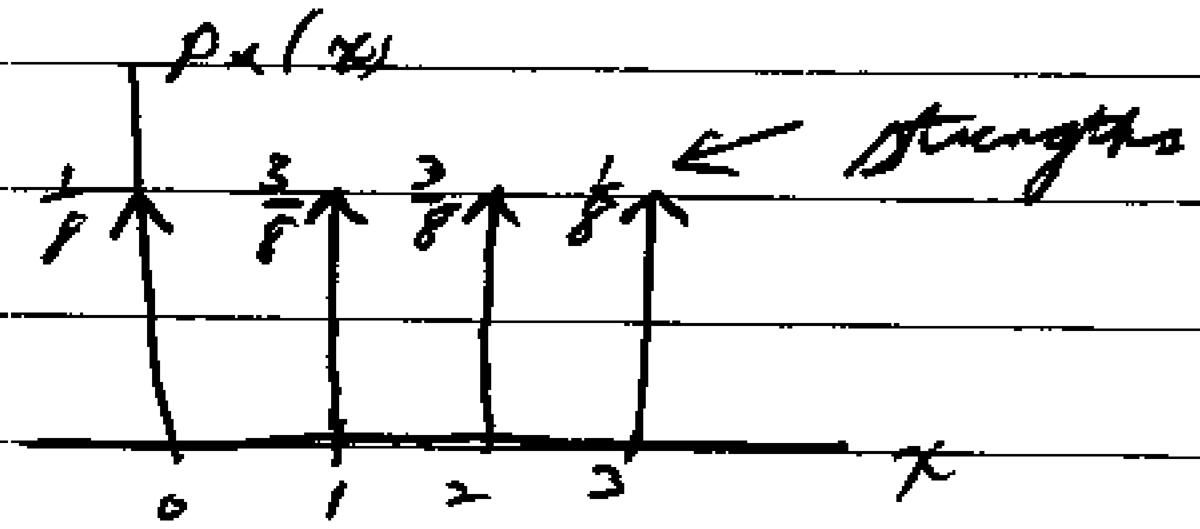


53) $P(X=k) = \binom{3}{k}\left(\frac{1}{2}\right)^3$

$$p_X(k) = \begin{cases} \frac{1}{8} & k=0 \\ \frac{3}{8} & k=1 \\ \frac{3}{8} & k=2 \\ \frac{1}{8} & k=3 \end{cases}$$

$$p_X(x) = \frac{1}{8}\delta(x) + \frac{3}{8}\delta(x-1) + \frac{3}{8}\delta(x-2) + \frac{1}{8}\delta(x-3)$$





54) $F_X(x) = \int_0^x t e^{-\frac{1}{2}t^2} dt \qquad x \geq 0$

$$= -e^{-\frac{1}{2}t^2}\Big|_0^x = 1 - e^{-\frac{1}{2}x^2}$$

$$y = 1 - e^{-\frac{1}{2}x^2} \Rightarrow 1-y = e^{-\frac{1}{2}x^2}$$

$$x^2 = -2\ln(1-y) \Rightarrow x = \sqrt{2\ln\frac{1}{1-y}}$$

$$g(v) = \sqrt{2\ln\frac{1}{1-v}}$$

55) $F_X(x) = \int_0^x 2(1-t)\,dt \qquad 0 \leq x \leq 1$

$$= 2t - t^2 \Big|_0^x = 2x - x^2$$

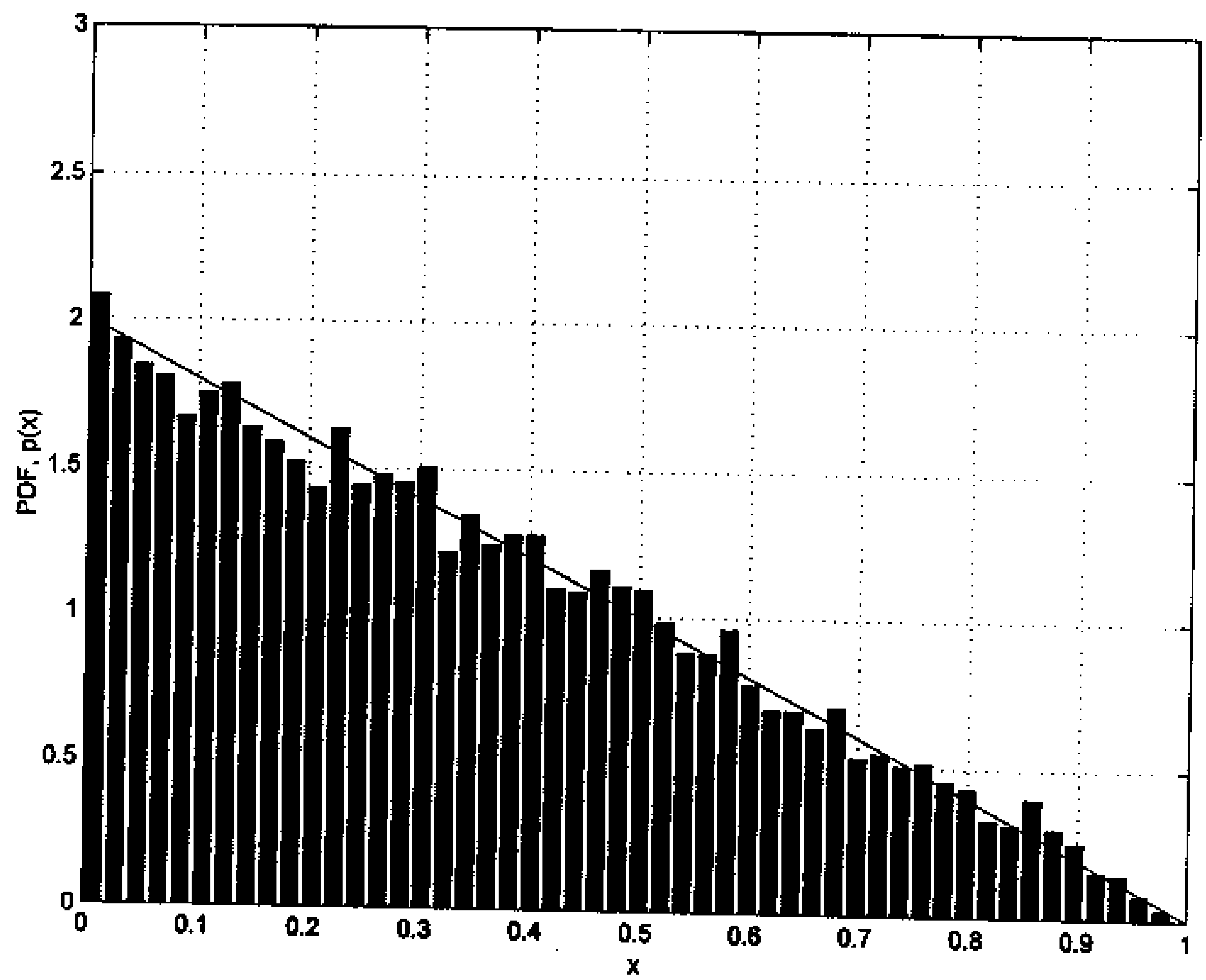
$$y = 2x - x^2 \Rightarrow x^2 - 2x + y = 0$$

$$x = \frac{2 \pm \sqrt{4-4y}}{2} = 1 \pm \sqrt{1-y}$$

Since $0 \le x \le 1$ must have $x = 1 - \sqrt{1-y}$

$$g(u) = 1 - \sqrt{1-u}$$

56)




```
% probprob10_56.m
%
clear all
rand('state',0)
u=rand(10000,1);
x=1-sqrt(1-u);
pdf(x,10000,50,0,1,3)
hold on
xx=[0.01:0.01:1]';
plot(xx,2-2*xx)
hold off
```

57) $\frac{d e^x}{dx} = e^x \geq 0$ increasing

$\frac{d \ln x}{dx} = \frac{1}{x} \geq 0$ for $x > 0$ increasing

$\frac{d\, 1/x}{dx} = -\frac{1}{x^2} \leq 0$ for $x > 0$ decreasing

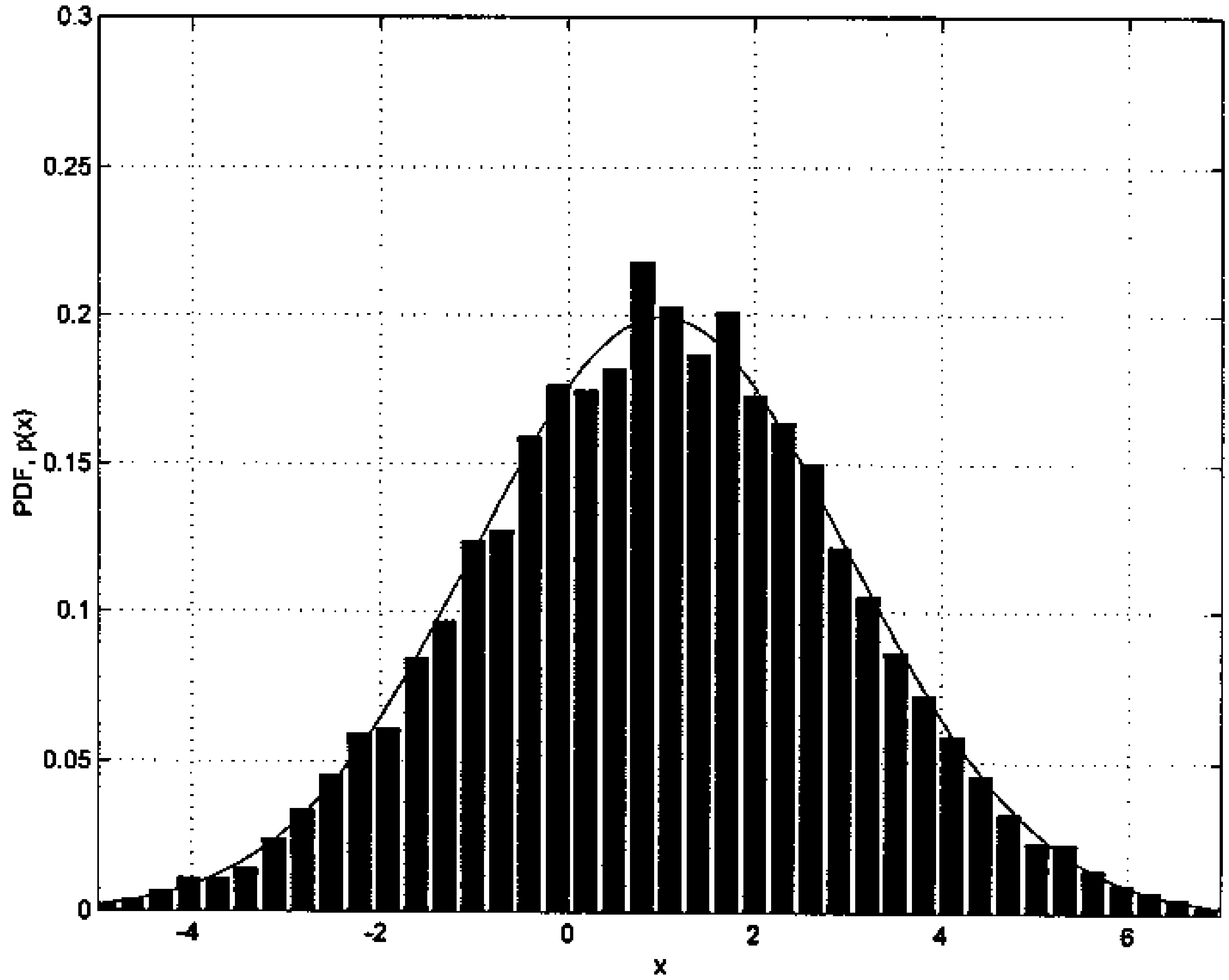
58) $\ln x$ is monotonically increasing (see Prob 5.57). Thus if $x \geq x_0$, then $\ln x \geq \ln x_0$ by definition.

No $1/x$ is monotonically decreasing, if $x \geq x_0$, $1/x \leq 1/x_0$.

59) Almost the same except at $y = 0$. True PDF has a discontinuity at $y = 0$. In Figure 2.10 we used a bin from -0.25 to 0.25. To see the discontinuity we would need a much narrower bin, with many more realizations.

60)

```
% probprob10_60.m
%
clear all
randn('state',0)
x=2*randn(10000,1)+1;
pdf(x,10000,50,-5,7,0.3)
hold on
xx=[-5:0.01:7]';
ptrue=(1/sqrt(2*pi*4))*exp(-0.5*((xx-1)/2).^2);
plot(xx,ptrue)
hold off
```



61) $y = x^3 \Rightarrow x = y^{1/3}$

$g^{-1}(y) = y^{1/3} \qquad 0 < y < 1$

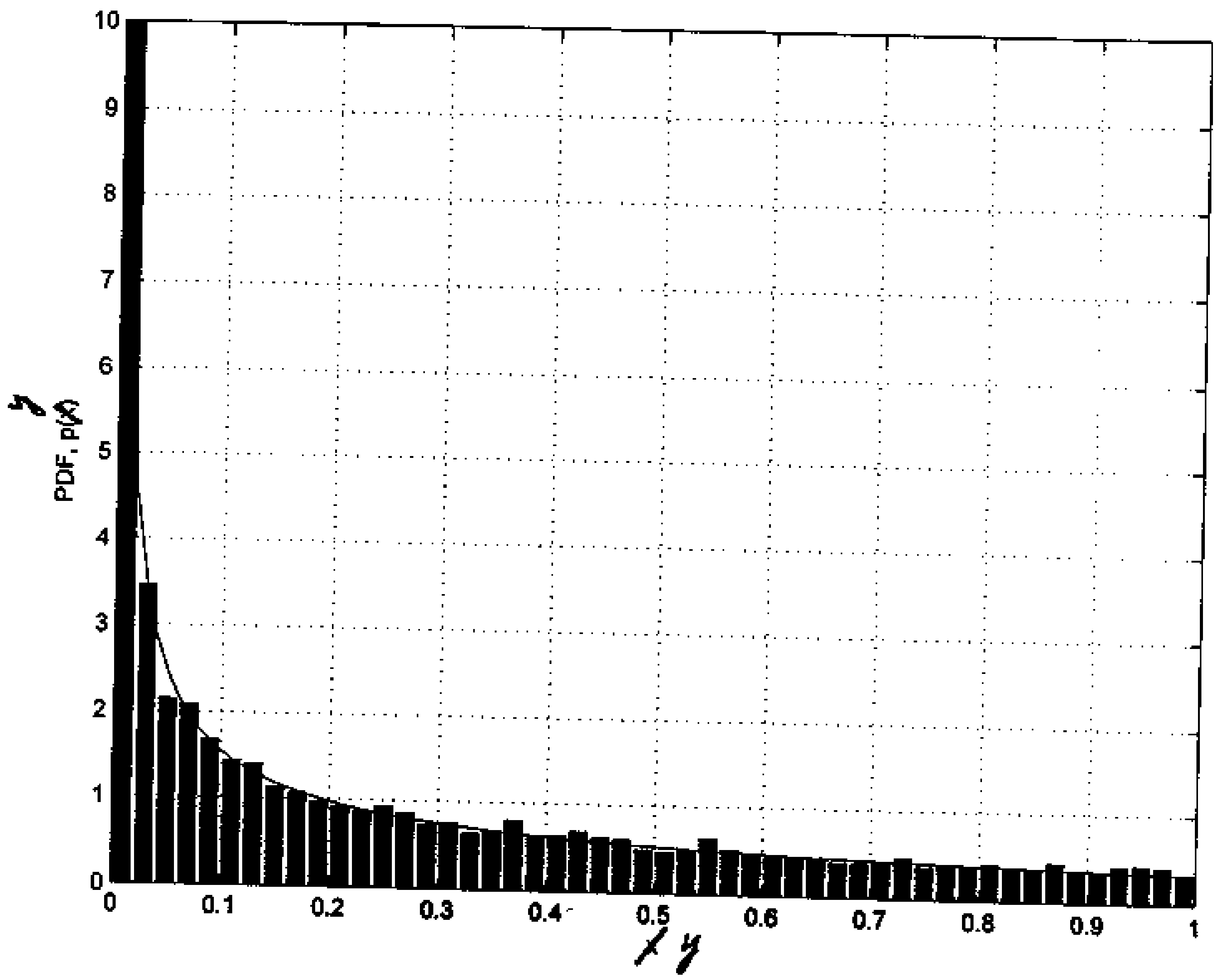
$$p_Y(y) = p_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

$$= 1 \cdot \frac{1}{3} y^{-2/3}$$

$$= \frac{1}{3y^{2/3}} \qquad 0 < y < 1$$

$$0 \qquad \text{otherwise}$$

```
% probprob10_61.m
%
clear all
rand('state',0)
x=rand(10000,1);
y=x.^3;
pdf(y,10000,50,0,1,10)
hold on
xx=[0:0.01:1]';
ptrue=(1/3)*xx.^(-2/3);
plot(xx,ptrue)
hold off
```



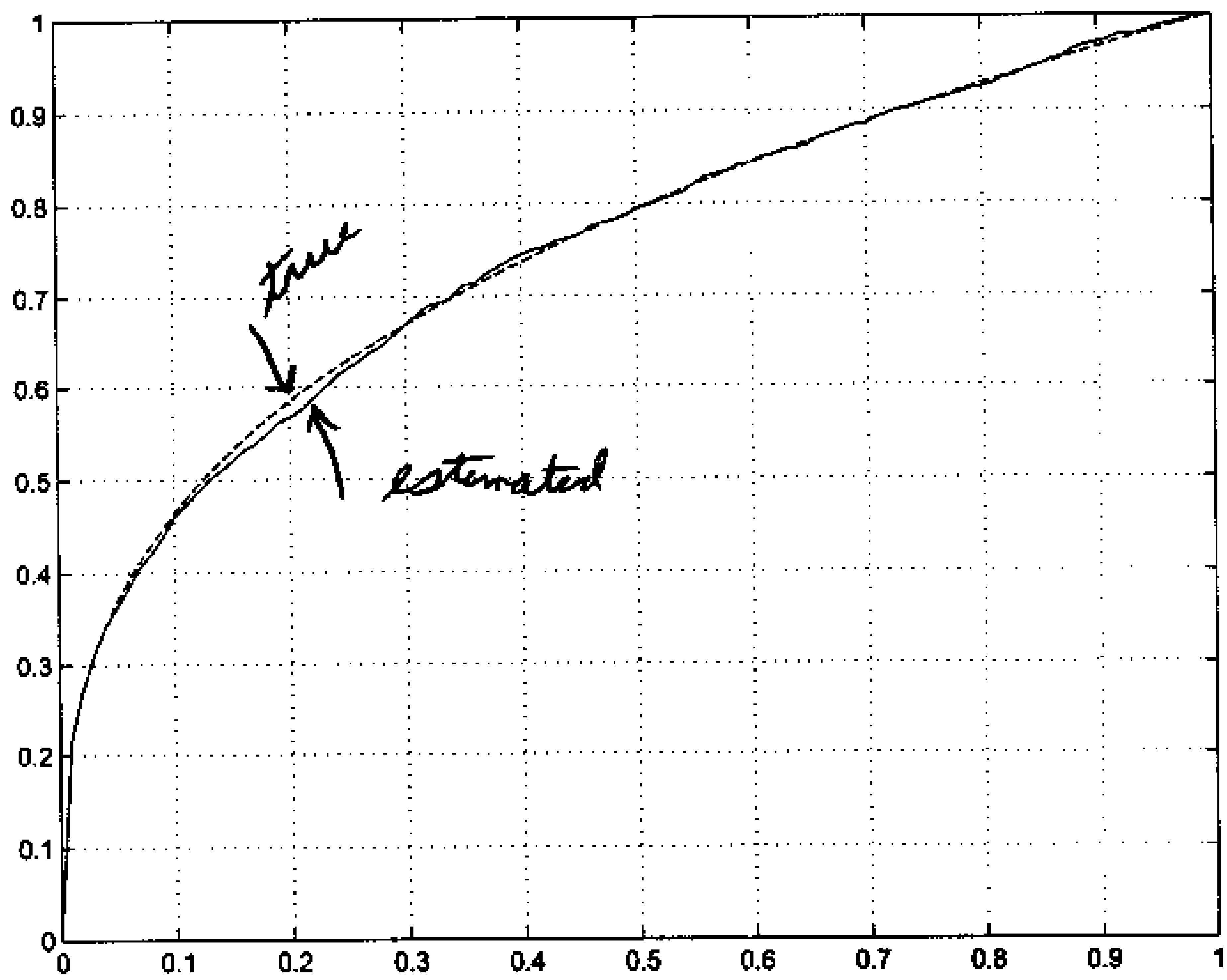
62) $F_Y(y) = \int_0^y \frac{1}{3t^{2/3}}\,dt$

$= t^{1/3}\Big|_0^y = y^{1/3} \qquad 0 \le y \le 1$

$0 \qquad 0 < y$

$1 \qquad y > 1$

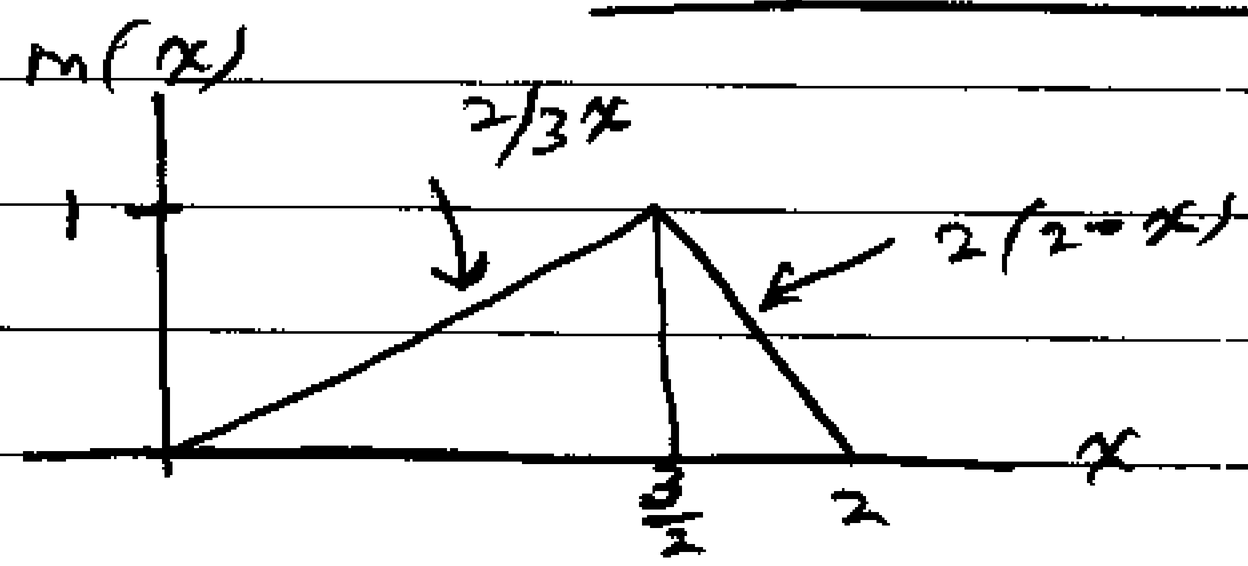
```
% probprob10_62.m
%
clear all
rand('state',0)
M=1000;
x=rand(M,1);
y=x.^3;
b=[0:0.01:1]';
CDF=zeros(length(b),1);
for i=1:length(b)
   for k=1:M
      if y(k)<b(i)
         CDF(i)=CDF(i)+1;
      end
   end
   end
   CDF=CDF/M;
   CDFtrue=b.^(1/3);
 plot(b,CDF,'-',b,CDFtrue,'--')
grid
```



Converges very quickly to true CDF
due to integration or summing of PDF values.

Chapter 11

1)



$$\text{Total mass} = \int_0^2 m(x)\,dx$$
$$= \tfrac{1}{2}(3/2)(1) + \tfrac{1}{2}(\tfrac{1}{2})(1) = 1$$

$$CM = \int_0^2 x\,m(x)\,dx = \int_0^{3/2} \tfrac{2}{3} x^2\,dx$$
$$+ \int_{3/2}^2 2(2-x)\,x\,dx$$

$$= \tfrac{2}{9} x^3 \Big|_0^{3/2} + \left[2x^2 - \tfrac{2}{3} x^3 \Big|_{3/2}^{2} \right)$$

$$= 2/9(27/8) + \left[8 - \tfrac{2}{3}(8) - 2(9/4) + \tfrac{2}{3}\left(\tfrac{27}{8}\right) \right)$$

$$= 3/4 + \left[8/3 - 9/2 + 9/4 \right] = 7/6$$

2) $$\int_{-\infty}^{\infty} x\,p_x(x)\,dx = \int_{-\infty}^{a} x\,p_x(x)\,dx + \int_a^{\infty} x\,p_x(x)\,dx$$

$u = x - a \Rightarrow$ $$= \int_{-\infty}^{0} (u+a)\,p_x(a+u)\,du$$

$u = a - x \Rightarrow$ $$-\int_0^{-\infty} (a-u)\,\underbrace{p_x(a-u)}_{p_x(a+u)}\,du$$

$$= \int_{-\infty}^{0} (u+a)\,p_x(a+u)\,du + \int_{-\infty}^{0} (a-u)\,p_x(a+u)\,du$$

$$= \int_{-\infty}^{0} a\,p_x(a+u)\,du + \underbrace{\int_{-\infty}^{0} a\,p_x(a+u)\,du}_{\text{Let } v = -u}$$

$$= \int_{-\infty}^{0} a\,p_x(a+u)\,du$$
$$+ \int_0^{\infty} a\,p_x(a-v)\,dv$$

$$= \int_{-\infty}^{0} a\, p_X(a-u)\,du + \int_{0}^{\infty} a\, p_X(a-u)\,du$$

$$= a \int_{-\infty}^{\infty} p_X(a-u)\,du = a$$

3) $$p_X(x) = p_U(g^{-1}(x)) \left| \frac{d g^{-1}(x)}{dx} \right|$$

$$x = \tan(\pi(u - \tfrac{1}{2}))$$

$$\Rightarrow \quad u - \tfrac{1}{2} = \frac{1}{\pi} \arctan x$$

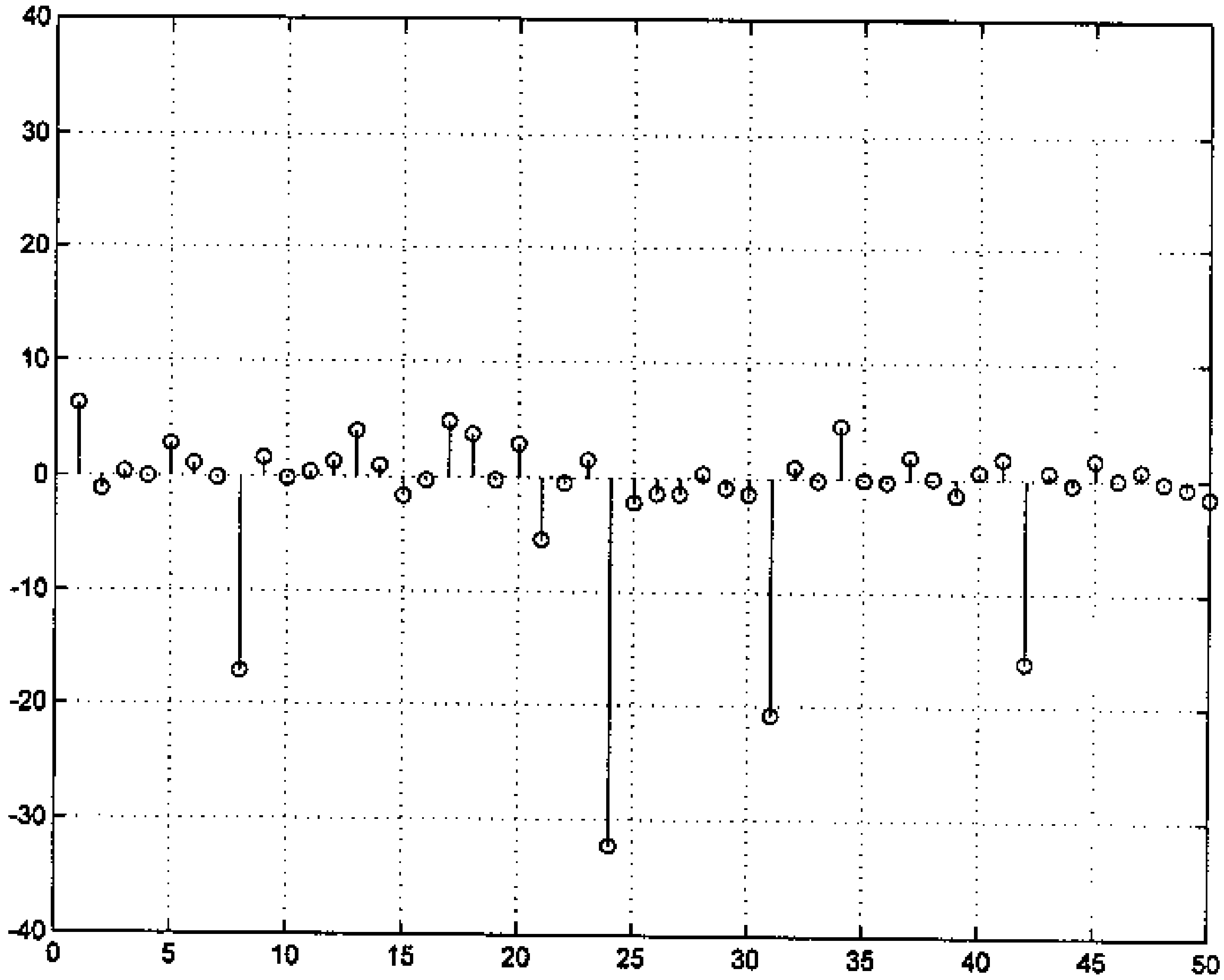
$$u = \frac{1}{2} + \frac{1}{\pi} \arctan x$$

$$g^{-1}(x) = \frac{1}{2} + \frac{1}{\pi} \arctan x$$

$$\frac{d g^{-1}(x)}{dx} = \frac{1}{\pi} \frac{1}{1+x^2}$$

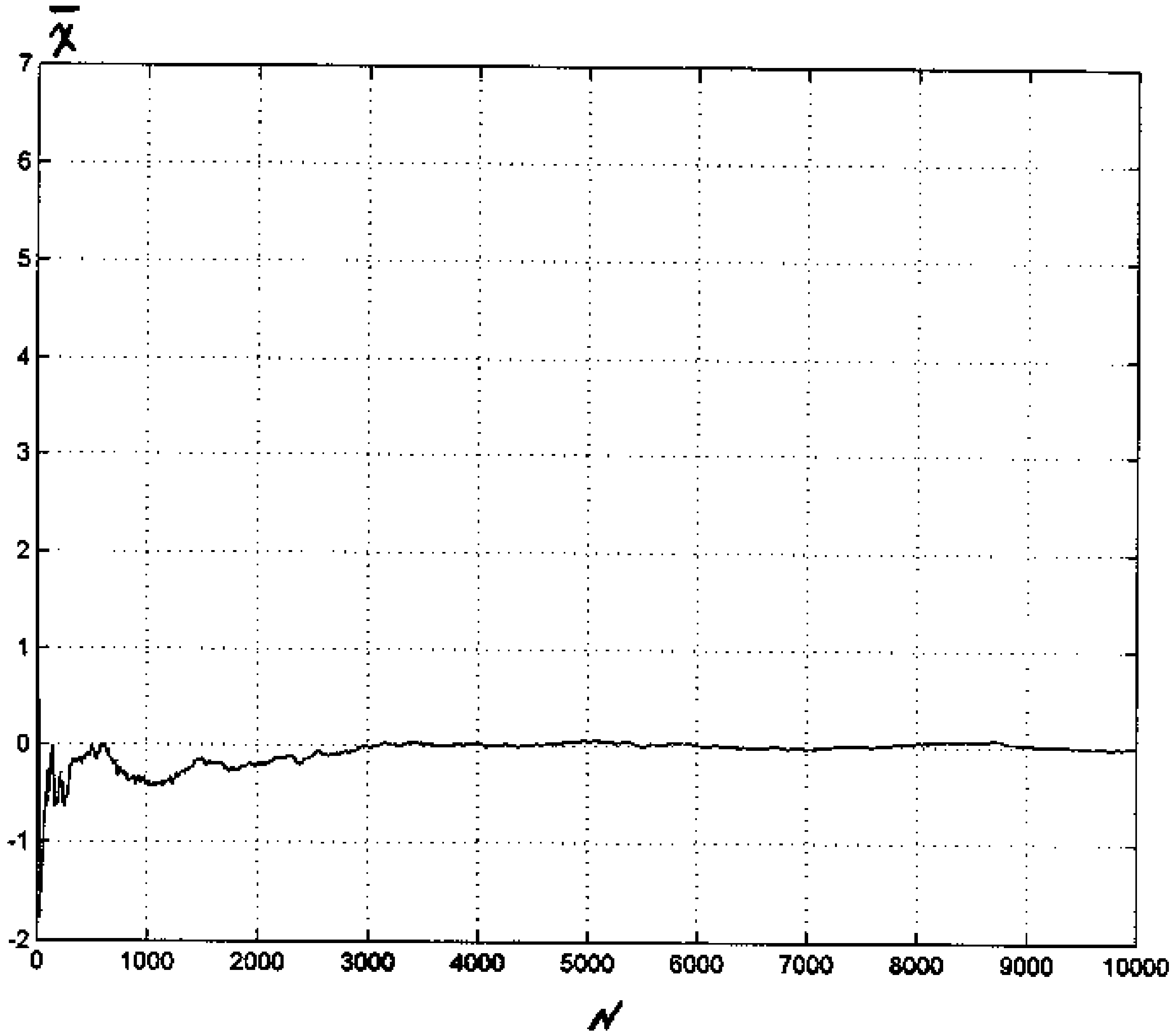
$$p_X(x) = 1 \cdot \frac{1}{\pi} \frac{1}{1+x^2} \qquad -\infty < x < \infty$$

```
% probprob11_3.m
%
clear all
rand('state',0)
N=50;
n=[1:N]';
x=tan(pi*(rand(N,1)-0.5));
figure
stem(n,x)
grid
axis([0 50 -40 40])
```



4)

```
% probprob11_4.m
%
clear all
rand('state',0)
N=10000;
n=[1:N]';
x=tan(pi*(rand(N,1)-0.5));

xmax=50;
for i=1:N
   if x(i)>xmax
      x(i)=xmax;
   elseif x(i)<-xmax
      x(i)=-xmax;
   end
end
for k=1:N
   meanest(k,1)=mean(x(1:n(k)));
end
plot(n,meanest)
grid
```



5) $mse(\cdot) = E[(X-b)^2]$

$$= E[X^2 - 2bX + b^2]$$

$$= E[X^2] - 2bE[X] + b^2$$

$$\frac{d\,mse(b)}{db} = 0 - 2E[X] + 2b = 0$$

$$\Rightarrow b_{opt} = E[X] \Rightarrow \hat{X} = E[X]$$

6) $$\int_{-\infty}^{\infty} g(x)dx = \underbrace{\int_{-\infty}^{0} g(x)dx}_{\text{Let } u=-x} + \int_{0}^{\infty} g(x)dx$$

$$= \int_{\infty}^{0} g(-u)(-du) + \int_{0}^{\infty} g(x)dx$$

$$= \int_{0}^{\infty} g(-x)dx + \int_{0}^{\infty} g(x)dx$$

$$= \int_0^\infty \underbrace{(g(x)+g(-x))}\,dx$$

$$= 2g(x) \quad \text{if } g(x) \text{ is even}$$
$$= 0 \quad \text{if } g(x) \text{ is odd}$$

$$E[X] = \int_{-\infty}^{\infty} \underbrace{x\,p(x)}_{g(x)}\,dx$$

$$g(-x) = -x\,p(-x) = -x\,p(x) = -g(x)$$
$\Rightarrow$ integrand is odd $\Rightarrow E[X] = 0$

$$1 = \int_{-\infty}^{\infty} p(x)\,dx = 2\int_0^\infty p(x)\,dx$$
$$\Rightarrow \int_0^\infty p(x)\,dx = 1/2$$

7) $$\int x e^{ax}\,dx = \underbrace{x}_{u}\,\underbrace{\frac{1}{a}e^{ax}}_{v} - \int \underbrace{\frac{1}{a}e^{ax}}_{v}\,\underbrace{dx}_{du}$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax}$$

8) $$E[X] = \int_a^b x\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\Big|_a^b$$

$$= \frac{1}{b-a}\left(\frac{1}{2}(b^2-a^2)\right) = \frac{1}{2}\,\frac{(b+a)(b-a)}{b-a}$$

$$= \frac{1}{2}(a+b)$$

By Problem 11.2 PDF is symmetric about midpoint of (a,b) interval $\Rightarrow E[X] = \frac{1}{2}(a+b)$

9) $$\int_{x_{MIN}}^{x_{MAX}} x_{MIN}\,p_X(x)\,dx \le \int_{x_{MIN}}^{x_{MAX}} x\,p_X(x)\,dx \le \int_{x_{MIN}}^{x_{MAX}} x_{MAX}\,p_X(x)\,dx$$

$$\underbrace{x_{MIN}\int_{x_{MIN}}^{x_{MAX}} p_X(x)\,dx}_{=1} \le E[X] \le \underbrace{x_{MAX}\int_{x_{MIN}}^{x_{MAX}} p_X(x)\,dx}_{=1}$$

$$x_{MIN} \le E[X] \le x_{MAX}$$

10) $$SNR = \frac{E^2[X]}{var(X)} = \frac{E^2[A+U]}{var(A+U)}$$

$$= \frac{E^2[A]}{var(U)} = \frac{A^2}{1/12} = 12A^2$$

For $SNR = 1000$ $\quad 12A^2 = 1000 \Rightarrow A = \pm 9.12$

11) $E[X] = 1/\lambda = 1000 \Rightarrow \lambda = 0.001$

$$P[X > 2000] = \int_{2000}^{\infty} \lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_{2000}^{\infty}$$

$$= e^{-2000\lambda} = e^{-2} = 0.1353$$

12) $E[X] = 10 = 1/\lambda \Rightarrow \lambda = 0.1$

$$P[X < 1] = \int_0^1 \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^1$$

$$= 1 - e^{-\lambda} = 1 - e^{-0.1} = 0.0951$$

13) 7 minutes

14) $$E[x] = \int_0^\infty x \frac{1}{2^{N/2}\Gamma(N/2)} x^{\frac{N}{2}-1} e^{-\frac{1}{2}x} dx$$

$$= \frac{1}{2^{N/2}\Gamma(N/2)} \int_0^\infty \frac{1}{2^{\frac{N+2}{2}}\Gamma(\frac{N+2}{2})} x^{\frac{N+2}{2}-1} e^{-\frac{1}{2}x} dx \cdot 2^{\frac{N+2}{2}} \Gamma\left(\frac{N+2}{2}\right)$$

$$= \frac{2\,\Gamma(\frac{N+2}{2})}{\Gamma(N/2)} \int_0^\infty \underbrace{\frac{1}{2^{M/2}\Gamma(M/2)} x^{M/2-1} e^{-\frac{1}{2}x} dx}_{\chi^2_M \text{ PDF}}$$

$$= 2\,\frac{\Gamma(N/2+1)}{\Gamma(N/2)} \cdot 1 = 2\,\frac{(N/2)\,\Gamma(N/2)}{\Gamma(N/2)}$$

$$= N$$

15) $$E[x] = \int_0^\infty x \frac{\lambda^N}{(N-1)!} x^{N-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^N}{(N-1)!} \int_0^\infty x^{(N+1)-1} e^{-\lambda x} dx$$

$$= \frac{\lambda^N}{(N-1)!} \int_0^\infty \frac{\lambda^{N+1}}{N!} x^{(N+1)-1} e^{-\lambda x} dx \cdot \frac{N!}{\lambda^{N+1}}$$

$$= \frac{N}{\lambda} \int_0^\infty \underbrace{\frac{\lambda^M}{(M-1)!} x^{M+1} e^{-\lambda x} dx}_{\text{Erlang PDF with parameter } M}$$

$$= \frac{N}{\lambda} \cdot 1 = N/\lambda$$

16) $$E(X) = \int_0^\infty \underbrace{x}_{u} \underbrace{\frac{x}{\sigma^2} e^{-\frac{1}{2}x^2/\sigma^2} dx}_{dv}$$

$$du = dx \qquad v = -e^{-\frac{1}{2}x^2/\sigma^2}$$

$$E(X) = -x e^{-\frac{1}{2}x^2/\sigma^2}\Big|_0^\infty - \int_0^\infty - e^{-\frac{1}{2}x^2/\sigma^2} dx$$

$$= 0 + \int_0^\infty e^{-\frac{1}{2}x^2/\sigma^2} dx$$

$$= \frac{1}{2} \int_{-\infty}^\infty e^{-\frac{1}{2}x^2/\sigma^2} dx$$

$$= \frac{1}{2}\sqrt{2\pi\sigma^2} = \sqrt{\pi\sigma^2/2}$$

17) For Gaussian mode $= \mu$ since

$$e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \le 1$$

and $= 1$ if $x = \mu$ $\Rightarrow$ mode = mean

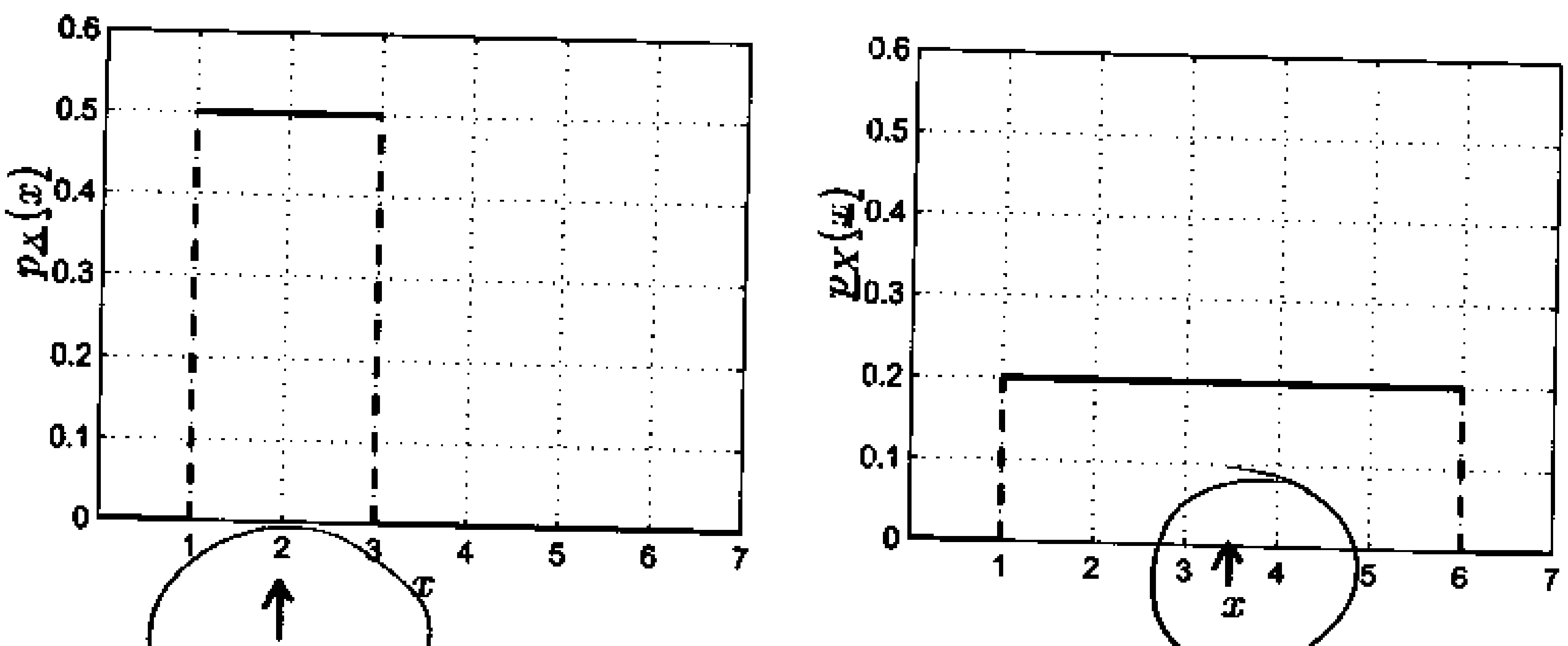
For Rayleigh

$$\frac{d}{dx} x/\sigma^2 e^{-\frac{1}{2}x^2/\sigma^2} = (x/\sigma^2)(-x/\sigma^2) e^{-\frac{1}{2}x^2/\sigma^2} + \frac{1}{\sigma^2} e^{-\frac{1}{2}x^2/\sigma^2} = 0$$

$$\Rightarrow \frac{x^2}{\sigma^4} = \frac{1}{\sigma^2} \Rightarrow x^2 = \sigma^2 \text{ or } x = \sigma$$

mode $\neq$ mean

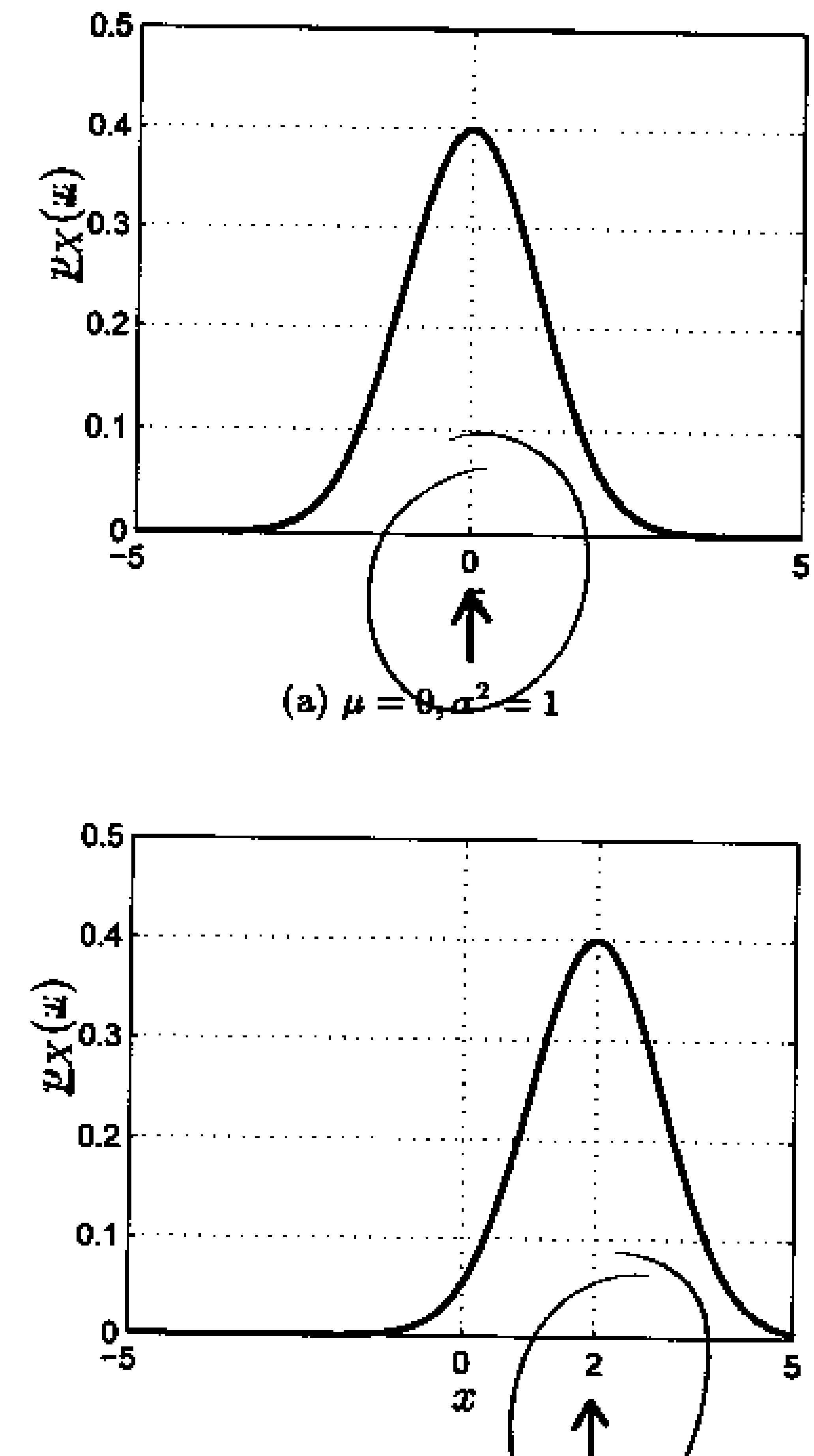
18) See next pages



(a) $a = 1, b = 3$

(b) $a = 1, b = 6$

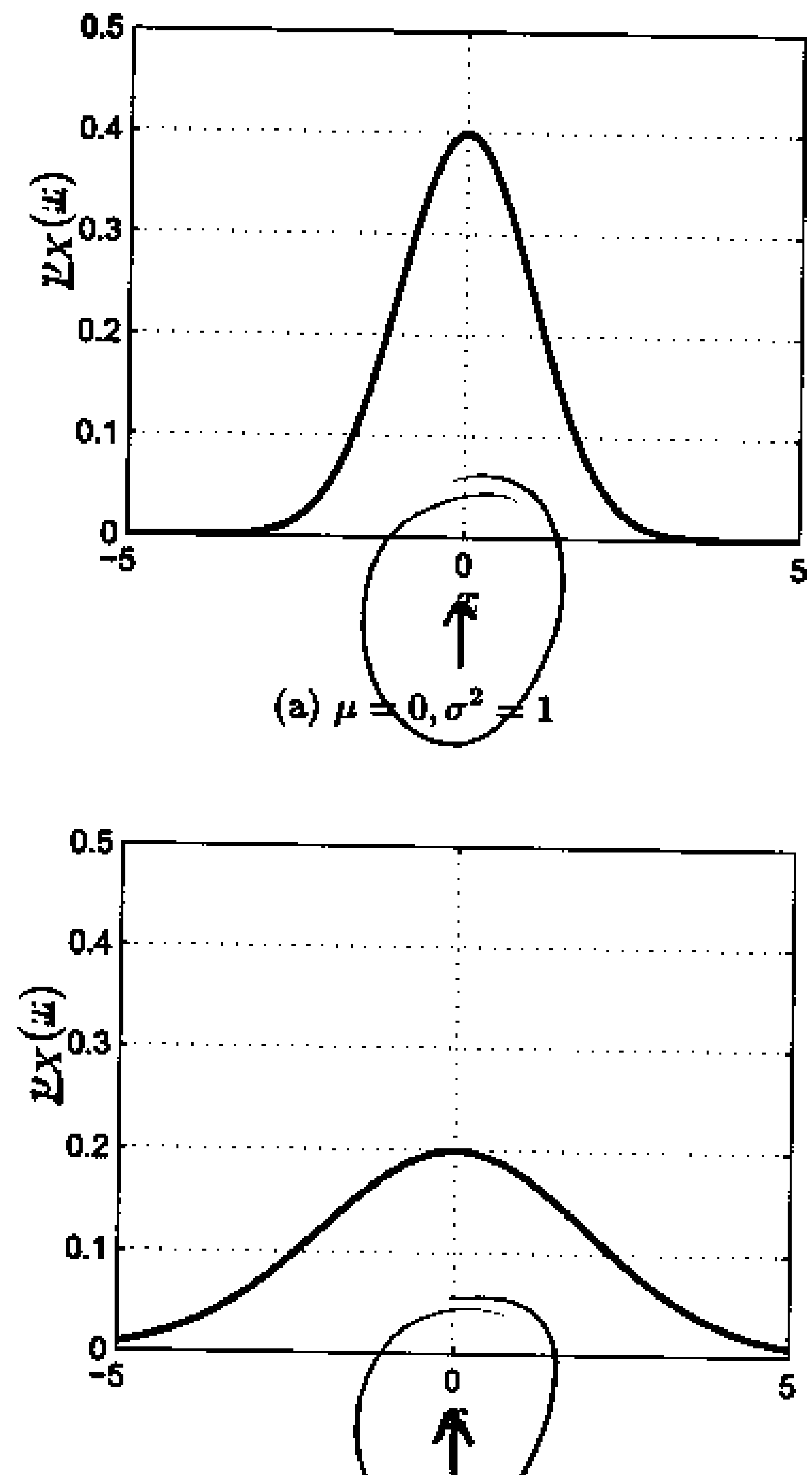
Figure 10.7: Examples of Uniform PDF.



(a) $\mu = 0, \sigma^2 = 1$

(c) $\mu = 2, \sigma^2 = 1$

Figure 10.8: Examples of Gaus



(a) $\mu = 0, \sigma^2 = 1$

(c) $\mu = 0, \sigma^2 = 2$

Figure 10.9: Examples of Gau

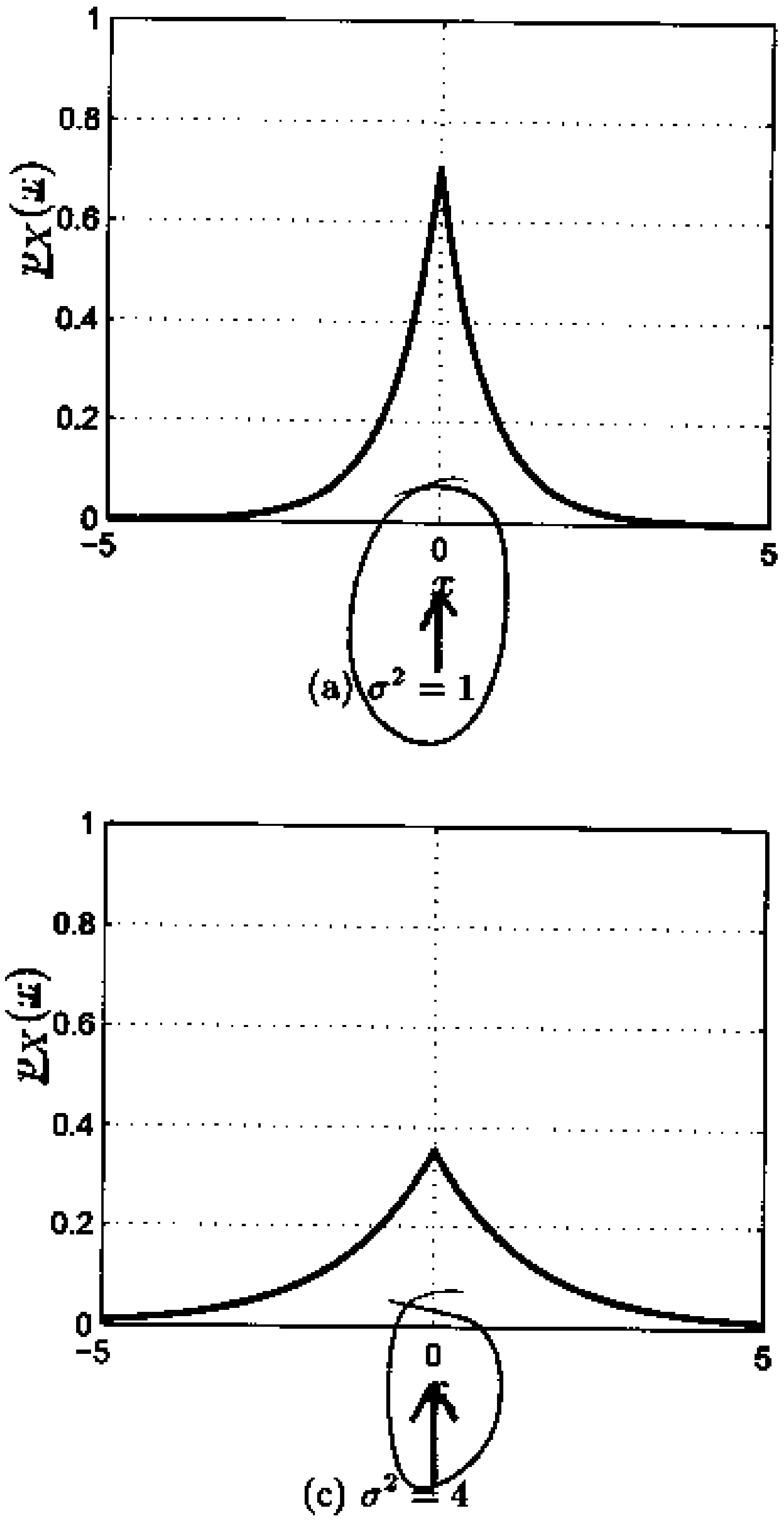


Figure 10.10: Examples of Lapl

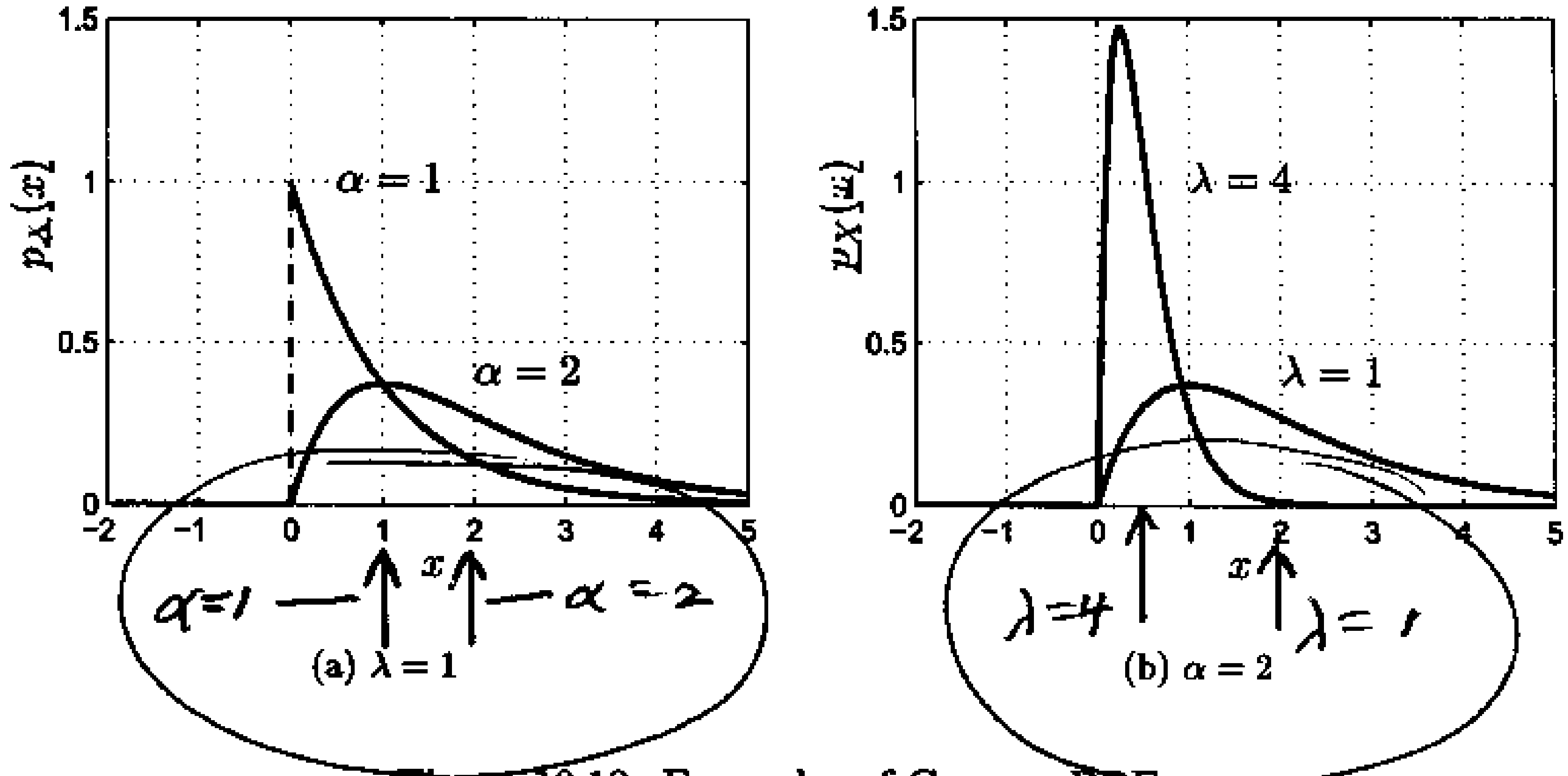


Figure 10.12: Examples of Gamma PDF.

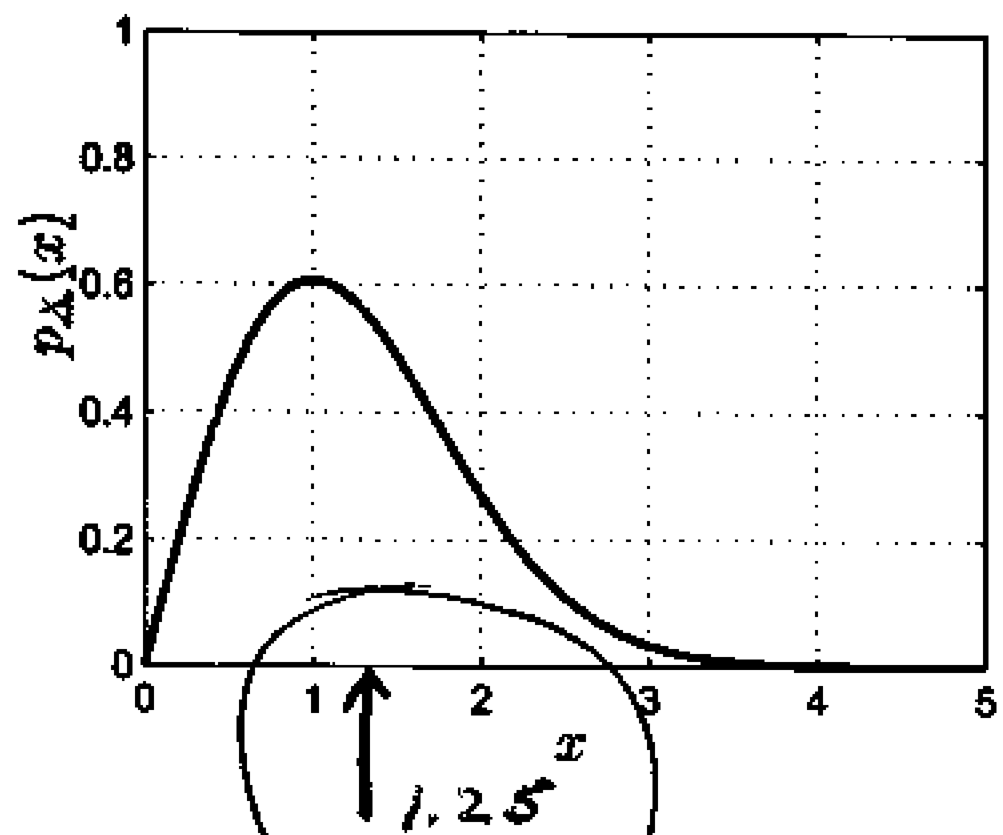


Figure 10.13: Rayleigh PDF with $\sigma^2 = 1$.

19) $E[X] = \sqrt{\pi\sigma^2/2} = 10 \Rightarrow \sigma^2 = \frac{2}{\pi}(100)$

$$P[X \le 1] = \int_0^1 x/\sigma^2 \, e^{-\frac{1}{2}x^2/\sigma^2} dx$$

$$= -e^{-\frac{1}{2}x^2/\sigma^2}\Big|_0^1 = 1 - e^{-\frac{1}{2\sigma^2}}$$

$$= 1 - e^{-\frac{1}{2}(\pi/200)} = 0.0078$$

20) 2.8) 1/2

2.9) 1

2.10) 1

2.11) 1

21) $\sqrt{\text{mean of } U} = \sqrt{1/2}$ for $U \sim U(0,1)$

$E[\sqrt{U}] = \int_0^1 \sqrt{u} \cdot 1 \, du = 2/3 \, u^{3/2} \Big|_0^1 = 2/3$

No

22) $E[s(t_0)] = E[\cos(2\pi F_0 t + \phi)]$

$= \int_0^{2\pi} \cos(2\pi F_0 t + \phi) \frac{1}{2\pi} d\phi$

$= \frac{\sin(2\pi F_0 t + \phi)}{2\pi} \Big|_0^{2\pi}$

$= \frac{\sin(2\pi F_0 t + 2\pi) - \sin(2\pi F_0 t)}{2\pi} = 0$

$E[s^2(t_0)] = \int_0^{2\pi} \cos^2(2\pi F_0 t + \phi) \frac{1}{2\pi} d\phi$

$= \int_0^{2\pi} \left[\frac{1}{2} + \frac{1}{2}\cos(4\pi F_0 t + 2\phi)\right] \frac{1}{2\pi} d\phi$

$= \frac{1}{2} + \frac{1}{4\pi} \frac{\sin(4\pi F_0 t + 2\phi)}{2} \Big|_0^{2\pi}$

$= \frac{1}{2} + \frac{1}{8\pi}\left[\sin(4\pi F_0 t + 4\pi) - \sin(4\pi F_0 t)\right]$

$= \frac{1}{2}$

Yes, coincides with notion of root-mean-square (RMS) voltage or power since RMS voltage $= \frac{1}{\sqrt{2}}$

23) $E[X^2] = \text{var}(X) + E^2(X) = \sigma^2 + \mu^2$

24) $E[(2X+1)^2] = 4E[X^2) + 4E(X) + 1$
$= 4(\sigma^2+\mu^2) + 4\mu + 1$
$= 4\sigma^2 + 4\mu^2 + 4\mu + 1$

25) $E[I_A(X)] = \int_A 1 \cdot p_X(x)\,dx = P(A)$

$E[I_A^2(X)] = \int_A 1^2 \cdot p_X(x)\,dx = P(A)$

$\text{var}(I_A(X)) = E[I_A^2(X)] - E^2[I_A(X)]$
$= P(A) - P^2(A) = P(A)(1-P(A))$

This is actually a Bernoulli random variable.

26) $Y = 0 \quad X < 0$
$\quad\;\; X \quad X \geq 0$

$E[Y^2] = \int_0^\infty y^2 p_Y(y)\,dy$

$= \int_0^\infty x^2 p_X(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 p_X(x)\,dx$

$= \frac{1}{2}E[X^2) = \frac{1}{2}\sigma^2 \quad$ since $E[X) = 0$

X just as likely to be negative as positive. Thus power is due to sum of positive and negative voltage contributions. Hence, rectifier cuts off half the power.

27) $E[X^2) = 0^2 \cdot \frac{1}{2} + \int_0^\infty x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}\,dx$

$= \sigma^2/2$

PDF for output of half-wave rectifier.

28) $$E(x) = \int_0^\infty \left(1-(1-e^{-\lambda x})\right) dx$$

$$= \int_0^\infty e^{-\lambda x}\, dx = \left. \frac{-e^{-\lambda x}}{\lambda} \right|_0^\infty$$

$$= 0 - (-1/\lambda) = 1/\lambda$$

$$E(x) = \int_0^\infty \underbrace{\int_x^\infty px(t)\,dt}_{v}\ \underbrace{dx}_{dV}$$

$V = x$

$\Rightarrow\ dv = -px(x)\,dx$ since

$$\frac{d}{dx}\int_x^\infty px(t)\,dt = \frac{d}{dx}\left[1-\int_0^x px(t)\,dt\right]$$

$$= -px(x) \quad \text{(Fundamental theorem of Calculus)}$$

$$E(x) = \left. \int_x^\infty px(t)\,dt\ x \right|_0^\infty - \int_0^\infty x(-px(x)\,dx)$$

$$\lim_{x\to\infty} x\int_x^\infty px(t)\,dt - \underbrace{\int_0^\infty px(t)\,dt}_{1}\cdot 0 + \int_0^\infty x\,px(x)\,dx$$

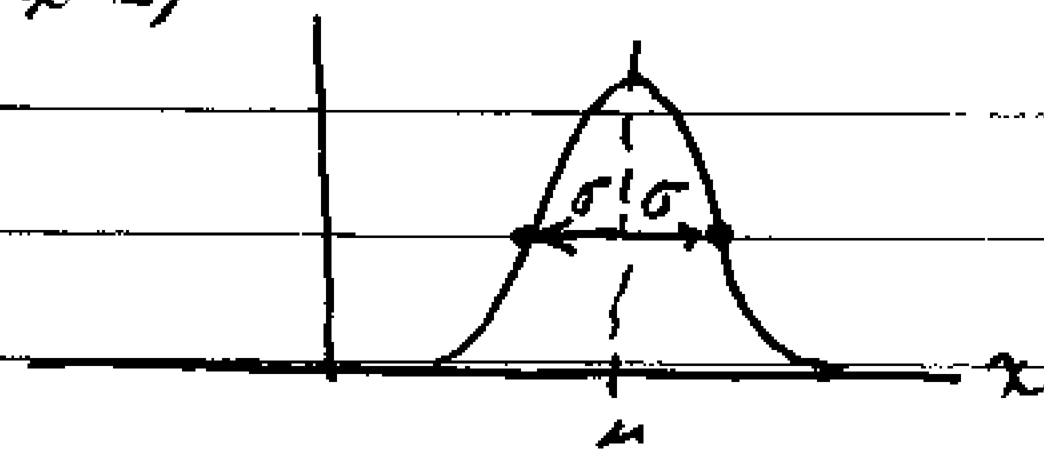
$$= \int_0^\infty x\,px(x)\,dx$$

Note: $$\lim_{x\to\infty} \frac{\int_x^\infty px(t)\,dt}{1/x} \overset{\text{L'Hospital}}{=} \lim_{x\to\infty} \frac{-px(x)}{-1/x^2}$$

$$= \lim_{x\to\infty} x^2 px(x) = 0 \quad \text{or else}$$

$E(x)$ will not exist, ie, Cauchy for example.

29) $p_X(x) = \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}}}_{c} \, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$



$$\frac{d p_X(x)}{dx} = c e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \cdot \left[-\frac{1}{\sigma^2}(x-\mu)\right]$$

$$\frac{d^2 p_X(x)}{dx^2} = c e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \left[-\frac{1}{\sigma^2}(x-\mu)\right]^2 - c e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \frac{1}{\sigma^2}$$

$$= c e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \left(\left(\frac{1}{\sigma^4}(x-\mu)^2 \right) - \frac{1}{\sigma^2} \right)$$

at $x = \mu \pm \sigma$

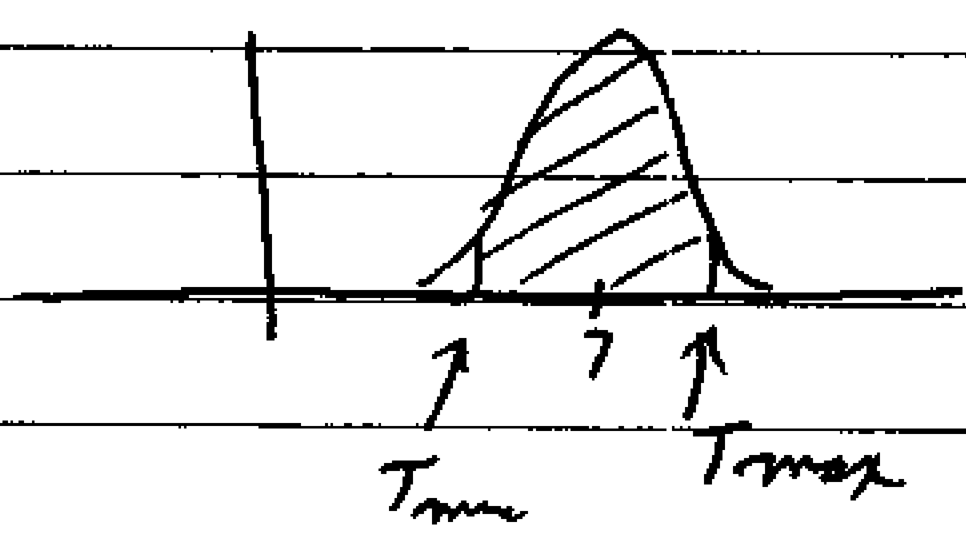
$$\frac{d^2 p_X(x)}{dx^2} = c e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \left(\frac{\sigma^2}{\sigma^4} - \frac{1}{\sigma^2} \right) = 0$$

30) $T \sim N(7, 1)$ See (1.2)

$$P(T_{min} \le T \le T_{max}) = 0.95$$

$$\Phi(T_{min} - 7) - \Phi(T_{max} - 7) = 0.95$$

Let $T_{max} - 7 = x$, $T_{min} - 7 = -x$

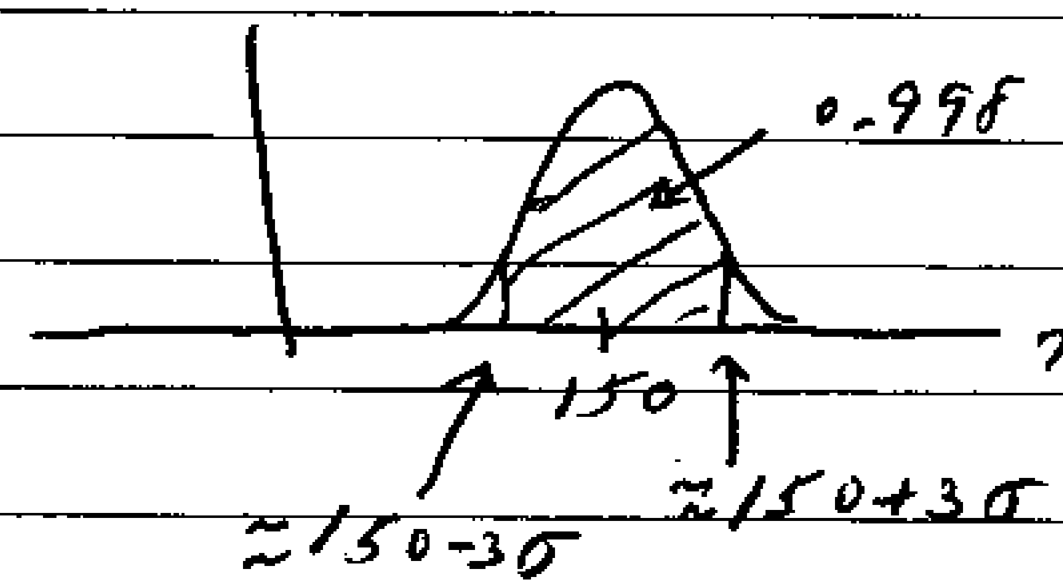


$$\Phi(-x) - \Phi(x) = 0.95$$

$$1 - Q(x) - Q(x) = 0.95$$
$$2Q(x) = 0.05$$
$$Q(x) = 0.025$$
$$x = Q^{-1}(0.025) = 1.96 \quad \text{from Qinv.m}$$

$$T_{min} = 5.04 \quad T_{max} = 8.96 \quad \text{or} \approx \mu \pm 2\sigma$$

31) $W \sim N(150, 30^2)$
$\sigma = 30$
$150 \pm 3\sigma =$
$[60, 240]$



32) $g_1(X) = X$
$g_2(X) = X$
$$\text{var}(g_1(X) + g_2(X)) = \text{var}(2X) = 4\,\text{var}(X)$$
$$= 2\,[\text{var}(g_1(X)) + \text{var}(g_2(X))]$$
$$\neq \text{var}(g_1(X)) + \text{var}(g_2(X))$$

33) $$\text{SNR} = \frac{E^2[X]}{\text{var}(X)} = \frac{(1/\lambda)^2}{1/\lambda^2} = 1$$

As mean increases, the PDF spreads out so that the variance increases. Not so for $N(\mu, \sigma^2)$ since for an increase in μ, PDF shifts but does not spread out.

34) PDF for symmetric about $x = 0 \Rightarrow E(X) = 0$ by Prob 11.2

$$\text{var}(X) = E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\sigma^2}} e^{-\sqrt{\frac{2}{\sigma^2}}|x|}\,dx$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\sigma^2}}\, x^2 e^{-\sqrt{2/\sigma^2}\,x}\, dx$$

$$= \frac{2}{\sqrt{2\sigma^2}} \int_0^\infty \underbrace{x^2}_{U} \cdot \underbrace{e^{-\sqrt{2/\sigma^2}\,x}\, dx}_{dV}$$

$$du = 2x\,dx \quad V = \frac{e^{-\sqrt{2/\sigma^2}\,x}}{-\sqrt{2/\sigma^2}}$$

$$= \frac{2}{\sqrt{2\sigma^2}} \Big[\underbrace{x^2 \frac{e^{-\sqrt{2/\sigma^2}\,x}}{-\sqrt{2/\sigma^2}} \Big|_0^\infty}_{=0} - \int_0^\infty \frac{e^{-\sqrt{2/\sigma^2}\,x}}{-\sqrt{2/\sigma^2}}\, 2x\, dx \Big]$$

$$= \frac{2}{\sqrt{2\sigma^2}} \frac{2}{\sqrt{2/\sigma^2}} \int_0^\infty x\, e^{-\sqrt{2/\sigma^2}\,x}\, dx$$

$$= 2\int_0^\infty \underbrace{x}_{U} \underbrace{e^{-\sqrt{2/\sigma^2}\,x}\, dx}_{dV}$$

$$= 2\Big[\underbrace{x \frac{e^{-\sqrt{2/\sigma^2}\,x}}{-\sqrt{2/\sigma^2}} \Big|_0^\infty}_{=0} - \int_0^\infty \frac{e^{-\sqrt{2/\sigma^2}\,x}}{-\sqrt{2/\sigma^2}}\, dx \Big]$$

$$= 2\int_0^\infty \sqrt{\sigma^2/2}\; e^{-\sqrt{2/\sigma^2}\,x}\, dx = 2\sqrt{\sigma^2/2}\; \frac{e^{-\sqrt{2/\sigma^2}}}{-\sqrt{2/\sigma^2}} \Big|_0^\infty$$

$$= 2\sqrt{\sigma^2/2}\; \frac{1}{\sqrt{2/\sigma^2}} = \frac{\sqrt{2\sigma^2}}{\sqrt{2/\sigma^2}} = \sigma^2$$

35) $$E(x^3) = \int_{-\infty}^\infty x^3 \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

Let $u = x - \mu$

$$= \int_{-\infty}^\infty (u+\mu)^3 \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}u^2}\, du$$

$$= \int_{-\infty}^{\infty} (u^3 + 3u^2\mu + 3u\mu^2 + \mu^3)\, N(0, \sigma^2)\, du$$

$$= 0 + 3\mu\sigma^2 + 0 + \mu^3$$

$$= 3\mu\sigma^2 + \mu^3$$

since for $N(0, \sigma^2) \Rightarrow \int_{-\infty}^{\infty} \underbrace{x^n}_{\text{odd}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}x^2}}_{\text{even}} dx = 0$

for n odd

Third central moment is

$$E((x-\mu)^3) = \int_{-\infty}^{\infty} (x-\mu)^3 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Let $u = x - \mu$

$$= \int_{-\infty}^{\infty} \underbrace{u^3}_{\text{odd}} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}u^2}}_{\text{even}} du = 0$$

(See Problem 11.6)

36) $E(X) = \frac{1}{2}\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} dx$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+1)^2} dx$$

$$= \frac{1}{2} \underbrace{E(X_1)}_{N(1,1)} + \frac{1}{2} \underbrace{E(X_2)}_{N(-1,1)} = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) = 0$$

$\mathrm{var}(X) = E(X^2)$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-1)^2} dx$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x+1)^2} dx$$

$$= \frac{1}{2} E(X_1^2) + \frac{1}{2} E(X_2^2)$$
$$= \frac{1}{2}(var(X_1) + E^2(X_1)) + \frac{1}{2}(var(X_2) + E^2(X_2))$$
$$= \frac{1}{2}(1 + 1^2) + \frac{1}{2}(1 + (-1)^2) = 2$$

37) $E(x^n) = \int_{-\infty}^{\infty} x^n p_X(x)\,dx$

$$= \int_{-\infty}^{0} x^n p_X(x)\,dx + \int_0^{\infty} x^n p_X(x)\,dx$$

Let $u = -x$

$$= \int_{\infty}^{0} (-u)^n p_X(-u)(-du) + \int_0^{\infty} x^n p_X(x)\,dx$$

$$= \int_0^{\infty} (-x)^n \underbrace{p_X(-x)}_{p_X(x)}\,dx + \int_0^{\infty} x^n p_X(x)\,dx$$

$$= \int_0^{\infty} [(-x)^n + (x)^n]\, p_X(x)\,dx = 0 \quad n \text{ odd}$$

38) $E(x^n) = \frac{1}{j^n} \left.\frac{d^n \phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$

From Table 11.1 $\phi_X(\omega) = \frac{1}{\omega^2+1}$ for $\sigma^2 = 2$

$$\phi_X(\omega) = (\omega^2+1)^{-1}$$
$$\phi_X'(\omega) = \frac{-2\omega}{(\omega^2+1)^2}$$

Continuing with be difficult as the derivatives get more complicated. Here's a Trick!

$$\frac{1}{\omega^2+1} = \frac{\left[\frac{1}{\omega-j} - \frac{1}{\omega+j}\right]}{2j}$$

Now $\phi_X'(\omega) = -\frac{1}{2j}(\omega-j)^{-2} + \frac{1}{2j}\cdot(\omega+j)^{-2}$

$$\phi_x''(\omega) = \frac{2}{2j}(\omega - j)^{-3} - \frac{2}{2j}(\omega + j)^{-3}$$

$$\phi_x'''(\omega) = -\frac{2\cdot 3}{2j}(\omega - j)^{-4} + \frac{2\cdot 3}{2j}(\omega + j)^{-4}$$

$$\Rightarrow \phi_x^{(n)}(\omega) = \frac{(-1)^n n!}{2j}(\omega - j)^{-n-1} - \frac{(-1)^n n!}{2j}(\omega + j)^{-n-1}$$

$$\phi_x^{(n)}(0) = \frac{(-1)^n n!}{2j}(-j)^{-n-1} - \frac{(-1)^n n!}{2j} j^{-n-1}$$

$$= \frac{(-1)^n n!}{2j}(-1)(-1)^{-n} j^{-n-1} - \frac{(-1)^n n!}{2j} j^{-n-1}$$

$$\frac{1}{j^n}\phi_x^{(n)}(0) = \frac{1}{j^n}\frac{(-1)^n n!}{2j} j^{-n-1} \underbrace{\left((-1)^n(-1) - 1\right)}$$

$= 0 \quad n$ odd
$= -2 \quad n$ even

For n even

$$= \frac{-2(-1)^n n!}{2 j^{2n+2}} = \frac{-2(j^{2n})\, n!}{2 j^{2n+2}}$$

$$= \frac{-n!}{j^2} = n! \Rightarrow E[X^n] = 0 \quad n \text{ odd}$$
$$\qquad\qquad\qquad\qquad\qquad\quad n! \quad n \text{ even}$$

39) $\phi_X(\omega) = e^{-\frac{1}{2}\sigma^2\omega^2}$

$$E[X] = \frac{1}{j^2}\frac{d^2\phi_X(\omega)}{d\omega^2}$$

$$\frac{d\phi_X(\omega)}{d\omega} = -\sigma^2\omega e^{-\frac{1}{2}\sigma^2\omega^2}$$

$$\frac{d^2\phi_X(\omega)}{d\omega^2} = (-\sigma^2\omega)^2 e^{-\frac{1}{2}\sigma^2\omega^2} - \sigma^2 e^{-\frac{1}{2}\sigma^2\omega^2}$$

$$\left.\frac{d^2\phi_x(\omega)}{d\omega^2}\right|_{\omega=0} = -\sigma^2$$

$$E(x^2) = \frac{1}{j^2}(-\sigma^2) = \sigma^2$$

40) $$E(e^{j\omega x}) = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} e^{j\omega x}\, dx$$

Let $x \to -u$

$$= \int_{\infty}^{-\infty} \frac{1}{\pi(1+u^2)} e^{-j\omega u} (-du)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi(1+u^2)} e^{-j\omega u}\, du$$

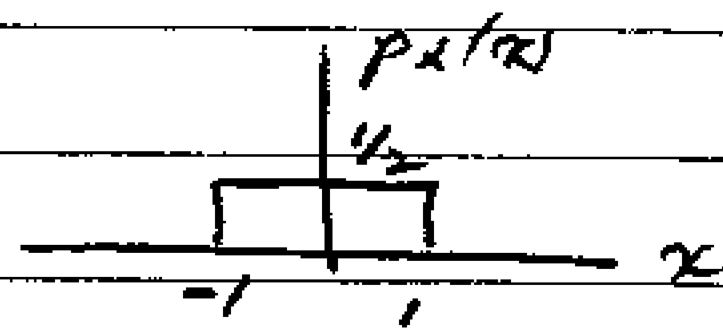
$$= 2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{1+\underset{\uparrow \omega}{u^2}} e^{-j\overset{x}{\omega}\overset{\omega}{u}} \frac{d\overset{\omega}{u}}{2\pi}}_{\frac{1}{2} e^{-|\omega|}}$$

$$= e^{-|\omega|}$$

41) If $a = -1$, $b = 1$

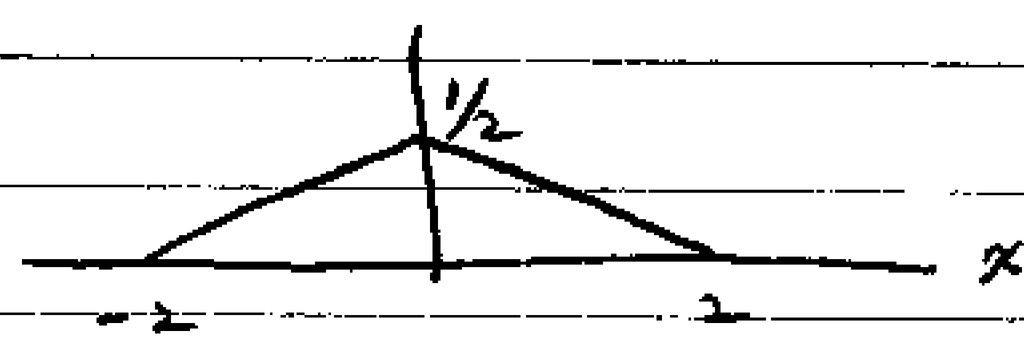
$$\phi_x(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{j\omega(1-(-1))} = \frac{2j \sin\omega}{2j\omega}$$

$$= \frac{\sin\omega}{\omega}$$



$$\Rightarrow \mathcal{F}^{-1}\{\phi_x^2(\omega)\} = p_x(x) \star p_x(x)$$

=



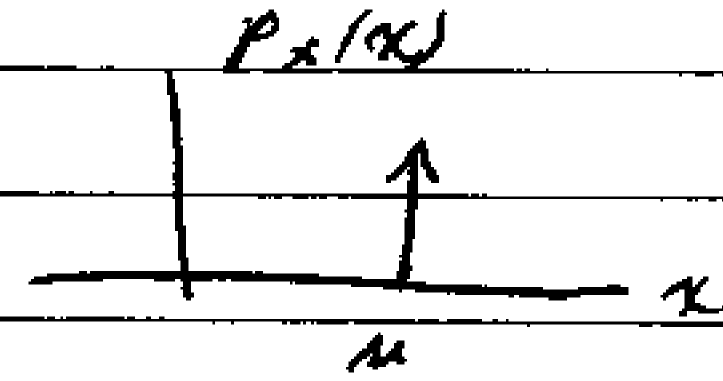
42) $\phi_X^{(n)}(\omega) = e^{j\omega\mu - 1/n\,\omega^2/2}$

As $n \to \infty$, $\phi_X^{(n)}(\omega) \to e^{j\omega\mu}$

and

$$\mathcal{F}^{-1}\{e^{j\omega\mu}\} = \delta(x-\mu)$$

$$\Rightarrow p_X^{(n)}(x) \to \delta(x-\mu)$$



43) For $X \sim \Gamma(\alpha, \lambda)$

$$\phi_X(\omega) = \frac{1}{(1-j\omega/\lambda)^\alpha} \quad \text{from Table 10.1}$$

$\alpha = N/2$, $\lambda = 1/2$ for χ^2_N

$$\Rightarrow \phi_{\chi^2_N}(\omega) = \frac{1}{(1-j2\omega)^{N/2}}$$

$$\frac{d\phi_{\chi^2_N}(\omega)}{d\omega} = -\frac{N}{2}(-2j)\frac{1}{(1-j2\omega)^{N/2+1}}$$

$$\frac{d^2\phi_{\chi^2_N}(\omega)}{d\omega^2} = -\frac{N}{2}(-2j)\left(-\frac{N}{2}-1\right)(-2j)\frac{1}{(1-j2\omega)^{N/2+2}}$$

$$\left.\frac{d\phi_{\chi^2_N}(\omega)}{d\omega}\right|_{\omega=0} = \frac{N}{2}(2)j = Nj$$

$$\left.\frac{d^2\phi_{\chi^2_N}(\omega)}{d\omega^2}\right|_{\omega=0} = \frac{N}{2}\left(\frac{N}{2}+1\right)(-4) = -N(N+2)$$

$\Rightarrow E(X) = N \qquad E(X^2) = N(N+2)$

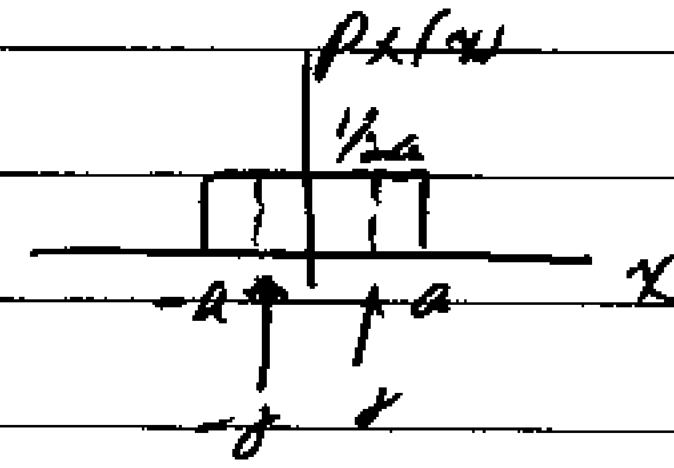
$\mathrm{var}(X) = N(N+2) - N^2 = 2N$

44) $P(|X - E[X]| > \gamma) \le \frac{var(X)}{\gamma^2} = \frac{1}{2}$

$$\frac{var(X)}{\gamma^2} = \frac{1}{2} \Rightarrow \gamma^2 = 2\, var(X)$$

$$\text{or } \gamma = \sqrt{2\, var(X)}$$

45) $P(|X| > \gamma) = 0 \quad \text{if } \gamma > a$



$$= 2(a-\gamma)\frac{1}{2a}$$

$$= 1 - \gamma/a \qquad 0 \le \gamma \le a$$

For $a = 2$

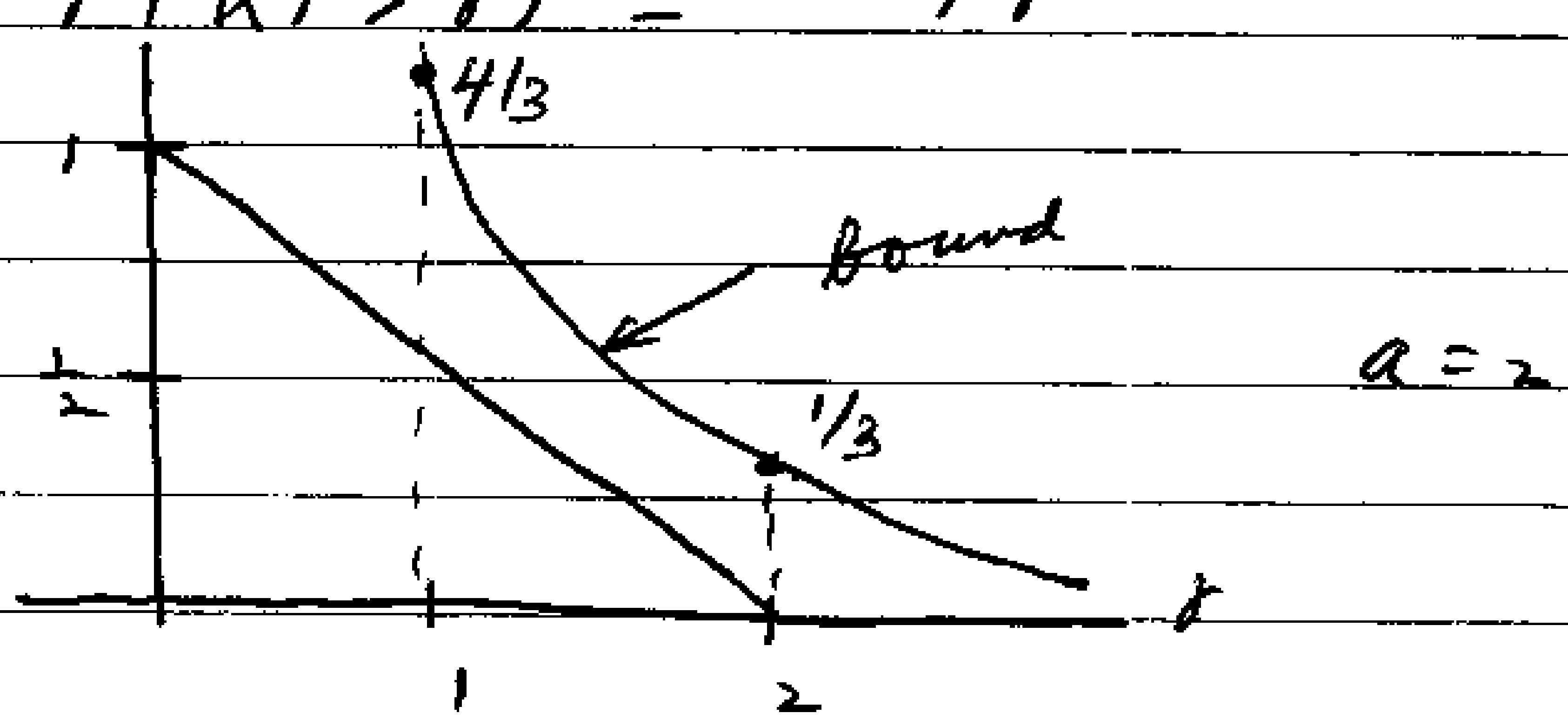
$$P(|X| > \gamma) = 0 \qquad \gamma > 2$$
$$1 - \gamma/2 \qquad 0 \le \gamma \le 2$$

Bound is $P(|X - \underbrace{E[X]}_{=0}| > \gamma) \le \frac{var(X)}{\gamma^2}$

$$P(|X| > \gamma) \le var(X)/\gamma^2$$

$$var(X) = \frac{(2a)^2}{12} = \frac{4^2}{12} = 4/3$$

$$P(|X| > \gamma) \le 4/3/\gamma^2$$



46) Using inverse probability integral transformation method

$$p_X(x) = x e^{-\frac{1}{2}x^2} \quad x \geq 0$$
$$0 \quad x < 0$$

$$F_X(x) = \int_0^x t e^{-\frac{1}{2}t^2} dt = -e^{-\frac{1}{2}t^2}\Big|_0^x$$
$$y = 1 - e^{-\frac{1}{2}x^2}$$

$$-\frac{1}{2}x^2 = \ln(1-y) \Rightarrow x = \sqrt{2\ln\frac{1}{1-y}}$$

or $x = \sqrt{2\ln\frac{1}{1-u}}$ for $U \sim U(0,1)$

$E(X) = \sqrt{\pi/2} = 1.2533$ from
$\mathrm{var}(X) = 2 - \pi/2 = 0.4292$ Table 11.1

$\hat{E}(X) = 1.2538$
$\widehat{\mathrm{var}}(X) = 0.4269$

```
% probprob11_46.m
%
clear all
rand('state',0)
u=rand(1000,1);
x=sqrt(-2.*log(1-u));
meanest=mean(x)
EX2=mean(x.^2);
varest=EX2-meanest^2
meantrue=sqrt(pi/2)
vartrue=2-pi/2
```

47) $Q(7) = 1.2 \times 10^{-12}$
Need about 10^{14} realizations

$\hat{Q}(7) = 1.0947 \times 10^{-12}$

```
% probprob11_47.m
%
clear all

rand('state',0) % sets random number generator to
                % initial value
M=100000;gamma=7;% change M for different estimates
u=rand(M,1);    % generates M U(0,1) realizations
x=-log(1-u);    % generates M exp(1) realizations
k=0;
for i=1:M       % computes estimate of P[X>gamma]
   if x(i)>gamma
      k=k+1;
      y(k,1)=(1/sqrt(2*pi))*exp(-0.5*x(i)^2+x(i)); % computes weights
                                                   % for estimate
   end
end
Qest=sum(y)/M  % final estimate of P[X>gamma]

Qtrue=Q(gamma)
```

Chapter 12

1) $P[\text{outermost ring}] = \dfrac{\text{area of ring}}{\text{Total area}}$

$$= \frac{\pi(1)^2 - \pi(\ \ )^2}{\pi(1)^2} = 1-(3/4)^2 = 7/16$$

$$P[\text{outermost ring}] = \iint_A \frac{1}{\pi}\,dx\,dy$$

$$= \int_0^{2\pi}\int_{3/4}^{1} \frac{1}{\pi}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{\pi}\, r^2/2 \Big|_{3/4}^{1} d\theta = \frac{1}{2\pi}\int_0^{2\pi}(1-9/16)\,d\theta$$

$$= 7/16$$

2)

```
%  probprob12_2.m
%
clear all
rand('state',0)
randn('state',0)
m=0;
while m<100
   u1=2*(rand(1,1)-0.5);u2=2*(rand(1,1)-0.5);
   if u1^2+u2^2<=1
      m=m+1;
      x(m,1)=u1;y(m,1)=u2;
   end
end
plot(x,y,'*')
grid
axis([-1.2 1.2 -1.2 1.2])
axis('square')
hold on
xx=[-1:0.01:1]';yy1=sqrt(1-xx.^2);yy2=-yy1;
line(xx,yy1,'linewidth',1.5)
line(xx,yy2,'linewidth',1.5)
hold off
```

All retained points are equally likely to be anywhere in the unit circle since points within the square are equally likely to be anywhere in the square, including the unit circle.

3) $P[0 \le R \le 0.5] = \dfrac{\pi(0.5)^2}{\pi(1)^2} = 1/4$

No since R is not a uniform random variable but is $R = \sqrt{X^2+Y^2}$ for $X \sim U(-1,1)$, $Y \sim U(-1,1)$

4) $$V = \iint_A g(x,y)\,dx\,dy$$

$$= \iint_A h\,dx\,dy = \int_0^{2\pi}\int_0^r h\,r'\,dr'\,d\theta$$

$$= \int_0^r r' \int_0^{2\pi} h\,d\theta \quad dr' = \int_0^r 2\pi h\,r'\,dr'$$

$$= 2\pi h\,\tfrac{1}{2}r'^2\Big|_0^r = 2\pi h\left(\tfrac{1}{2}r^2\right) = \pi r^2 h$$

5) $$P[X^2+Y^2 \le 1] = \iint_{\{(x,y):\,x^2+y^2\le 1\}} p_{X,Y}(x,y)\,dx\,dy$$

$$= \iint_{\{(x,y):\,x^2+y^2\le 1\}} \underbrace{\tfrac{1}{2}}_{p_X}\cdot\underbrace{\tfrac{1}{2}}_{p_Y}\,dx\,dy$$

$$= \tfrac{1}{4}\pi(1)^2 = \pi/4$$

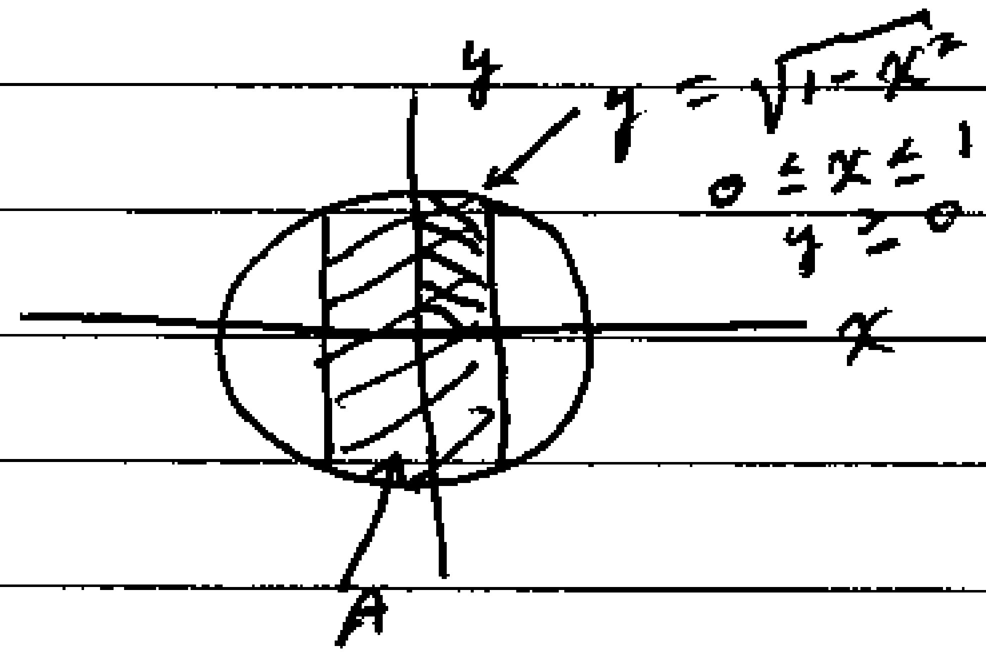
(or just use geometrical arguments as in Section 12.3)

$$\Rightarrow \pi = 4P[X^2+Y^2 \le 1]$$

$\hat{\pi} = 3.1140$

```
%probnprob12_5.m
%
rand('state',0)
ntrials=10000;
x1=2*(rand(ntrials,1)-0.5);
x2=2*(rand(ntrials,1)-0.5);
count=0;
for i=1:ntrials
   if x1(i)^2+x2(i)^2<1
      count=count+1;
   end
end
piest=4*(count/ntrials)
```

6) $P\{|X| \le \frac{1}{2}\} = \iint_A p_{X,Y}(x,y)\,dy\,dx$



By symmetry $P\{|X| \le \frac{1}{2}\} = 4 \int_0^{\frac{1}{2}} \int_0^{\sqrt{1-x^2}} \frac{1}{\pi}\,dy\,dx$

$= \frac{4}{\pi} \int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$

$= \frac{4}{\pi} \left\{ \frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \arcsin x \Big|_0^{\frac{1}{2}} \right)$

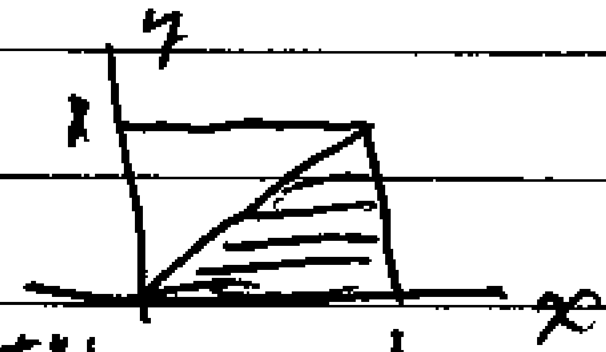
$= \frac{4}{\pi} \left\{ \frac{1}{4}\sqrt{3/4} + \frac{1}{2} \underbrace{\arcsin \frac{1}{2}}_{\pi/6} - 0 - 0 \right)$

$= \frac{\sqrt{3}}{2\pi} + \frac{1}{3}$

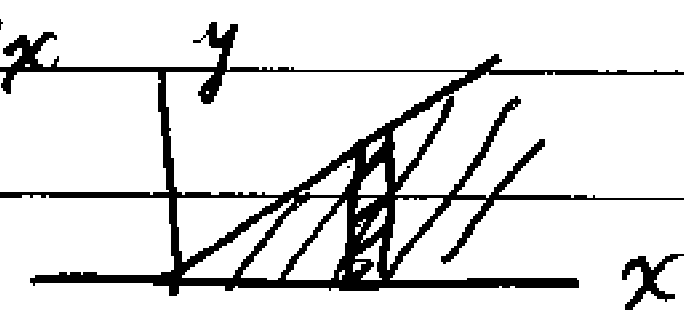
7) $$\int_0^1\int_0^1 \frac{c}{\sqrt{xy}}\,dx\,dy = c\int_0^1 x^{-\frac{1}{2}}dx\int_0^1 y^{-\frac{1}{2}}dy$$

$$= c\, 2\sqrt{x}\Big|_0^1\, 2\sqrt{y}\Big|_0^1 = 4c \Rightarrow c = 1/4$$

8) $$P[Y \le X] = \frac{1}{2}$$



9) $$P[Y \le X] = \int_0^\infty\int_0^x e^{-(x+y)}\,dy\,dx$$

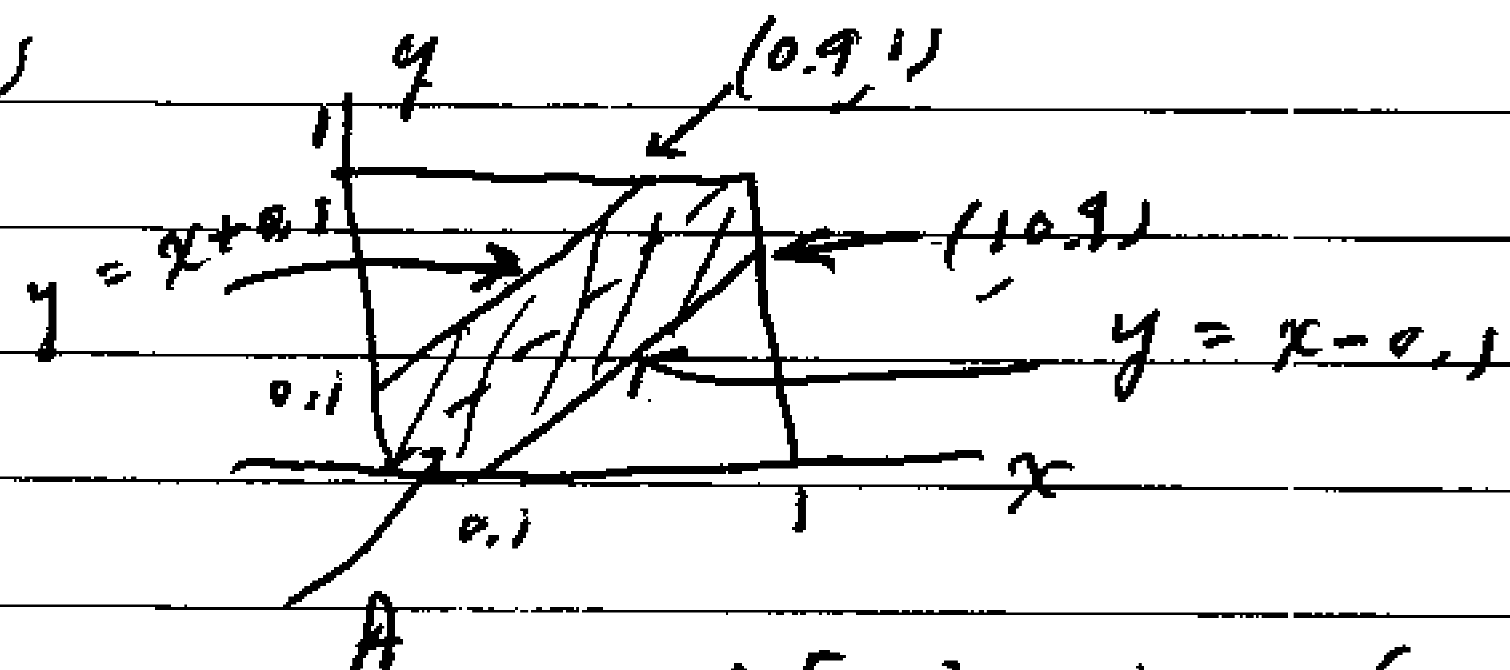


$$= \int_0^\infty e^{-x}\; -e^{-y}\Big|_0^x\, dx$$

$$= \int_0^\infty e^{-x}(1-e^{-x})\,dx$$

$$= -e^{-x} + \frac{1}{2}e^{-2x}\Big|_0^\infty = 1 - \frac{1}{2} = \frac{1}{2}$$

10)



$$A = \{(x,y): |y-x| \le 0.1\}$$

$$P[A] = 1 - \frac{2}{2}(0.9)^2 = 0.19$$

$$\hat{P}(A) = 0.1872$$

```
%probprob12_10.m
%
rand('state',0)
ntrials=10000;
x1=rand(ntrials,1);
x2=rand(ntrials,1);
count=0;
for i=1:ntrials
   if abs(x1(i)-x2(i))<=0.1
      count=count+1;
   end
end
prob=count/ntrials
```

11) 0

12) $y^2 - 2\rho x y + (x^2 - 1) = 0$

$$y = \frac{2\rho x \pm \sqrt{4\rho^2 x^2 - 4(x^2-1)}}{2}$$

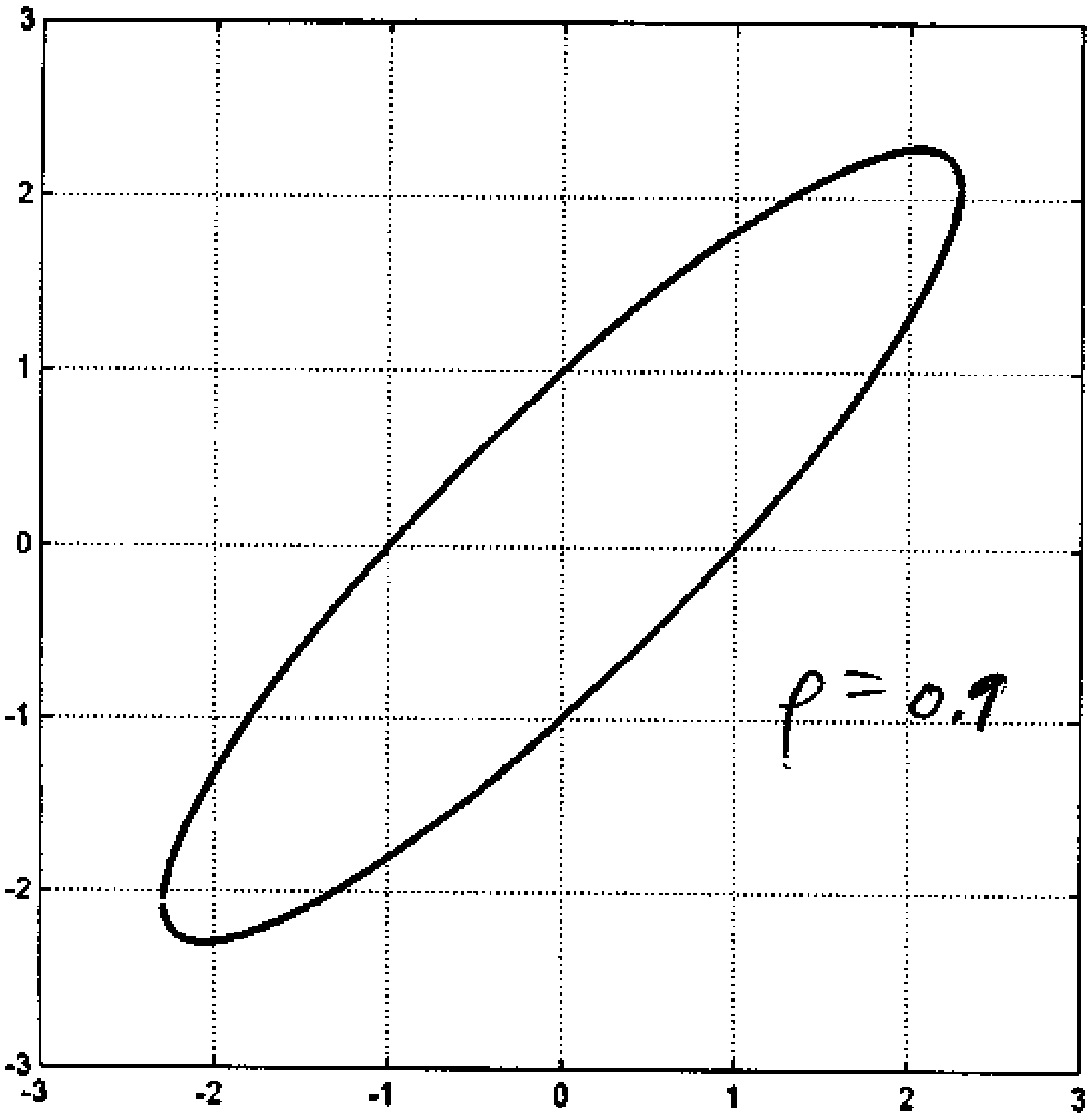
$$= \rho x \pm \sqrt{\rho^2 x^2 - x^2 + 1}$$

$$= \rho x \pm \sqrt{\underbrace{1 + x^2(\rho^2 - 1)}_{\geq 0}}$$

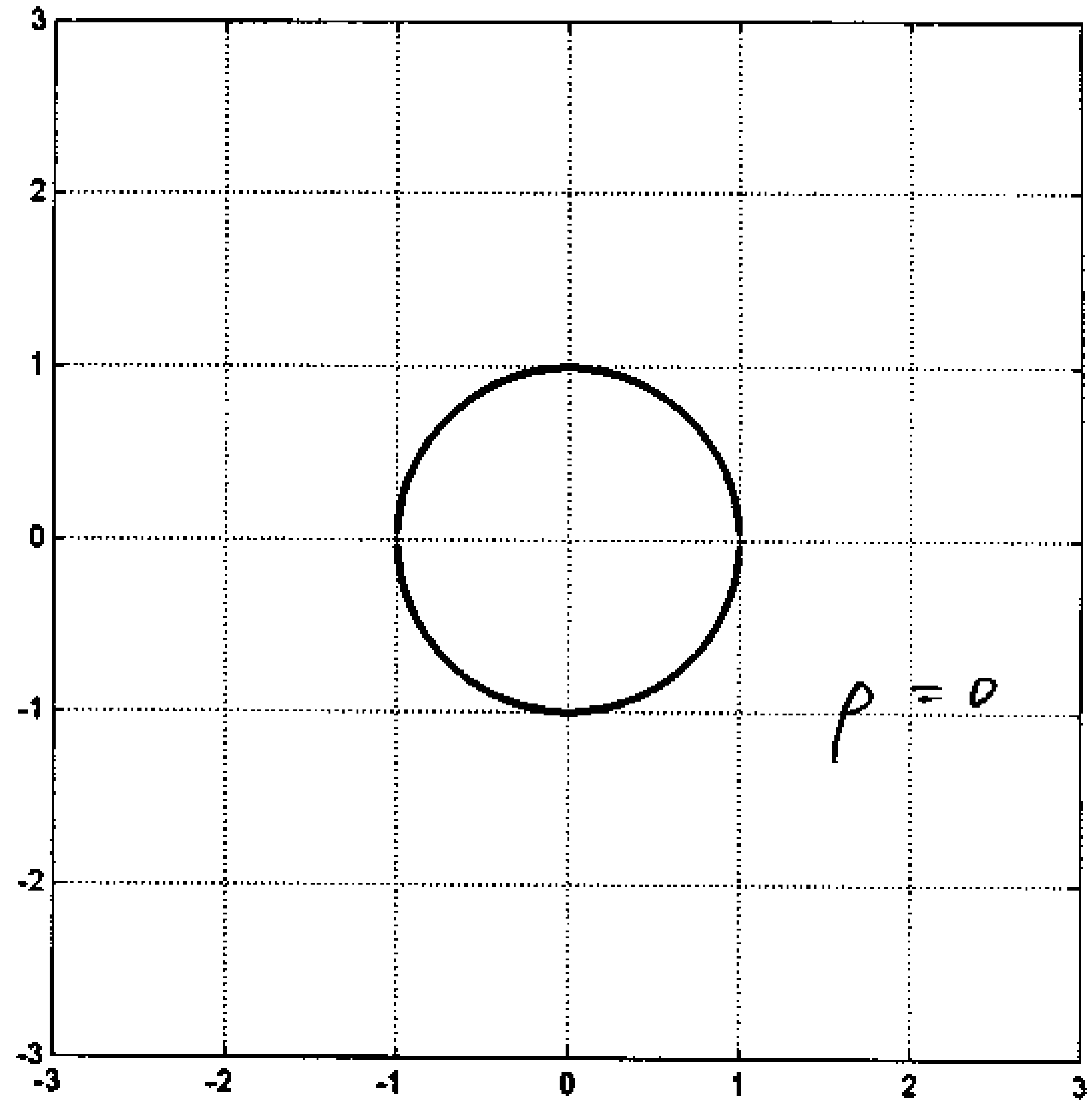
for real roots

$$\Rightarrow 1 + x^2 \underbrace{(\rho^2 - 1)}_{\leq 0} \geq 0 \Rightarrow x^2 \leq \frac{-1}{\rho^2 - 1} = \frac{1}{1-\rho^2}$$

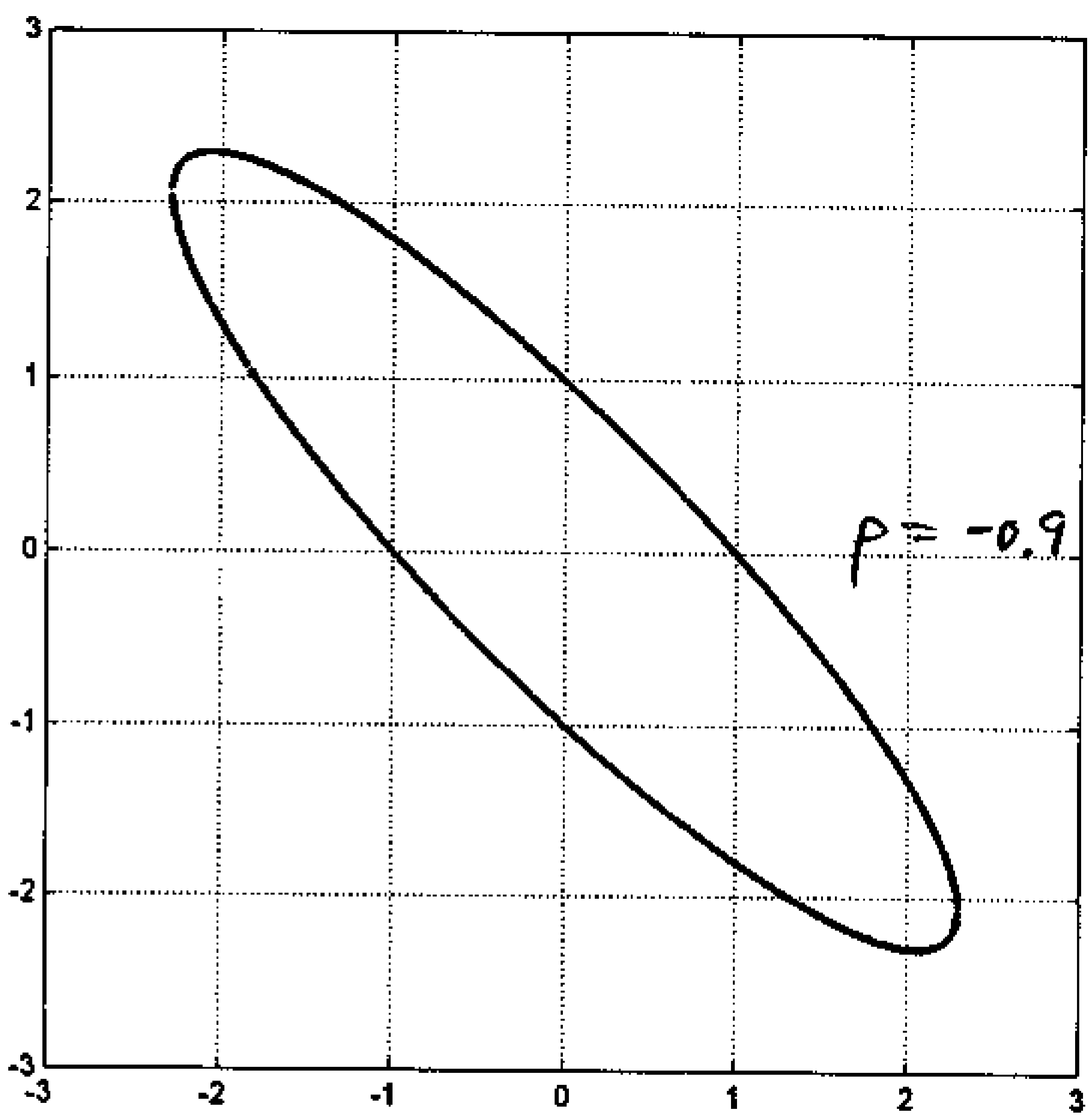
```
%probprob12_12.m
%
clear all
rho=-0.9;
xmax=1/(1-rho^2);
x=[-xmax:0.001:xmax]';
k=0;
for i=1:length(x)
  if (1-rho^2)*x(i)^2<=1
      k=k+1;
      z(1,k)=x(i);z(2,k)=rho*x(i)+sqrt(1-(1-rho^2)*x(i)^2);
       k=k+1;
       z(1,k)=x(i);z(2,k)=rho*x(i)-sqrt(1-(1-rho^2)*x(i)^2);
       end
  end
figure
 plot(z(1,:),z(2,:),'.')
 axis([-3 3 -3 3])
  axis('square')
  grid
```



3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3
$\rho = 0.9$

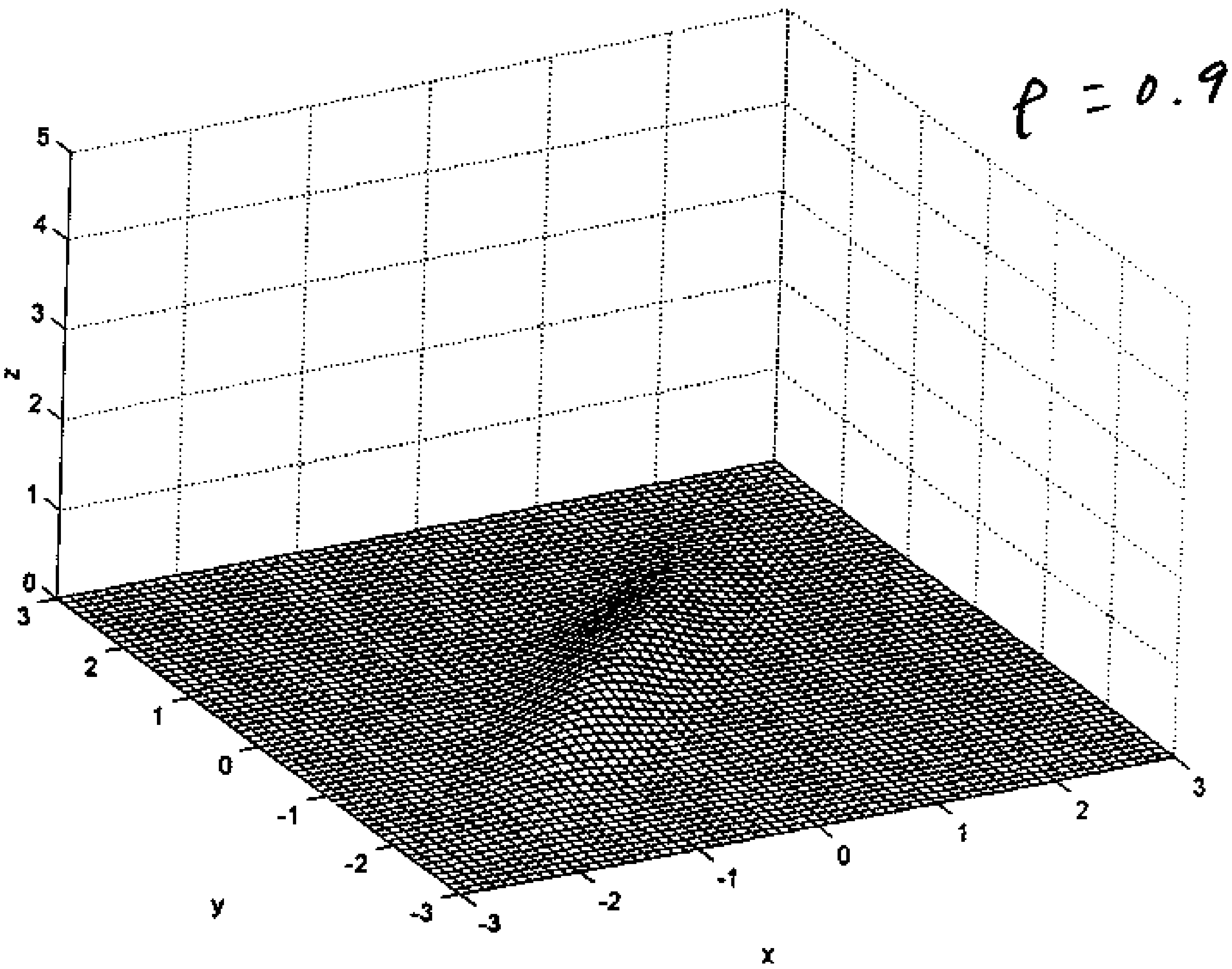


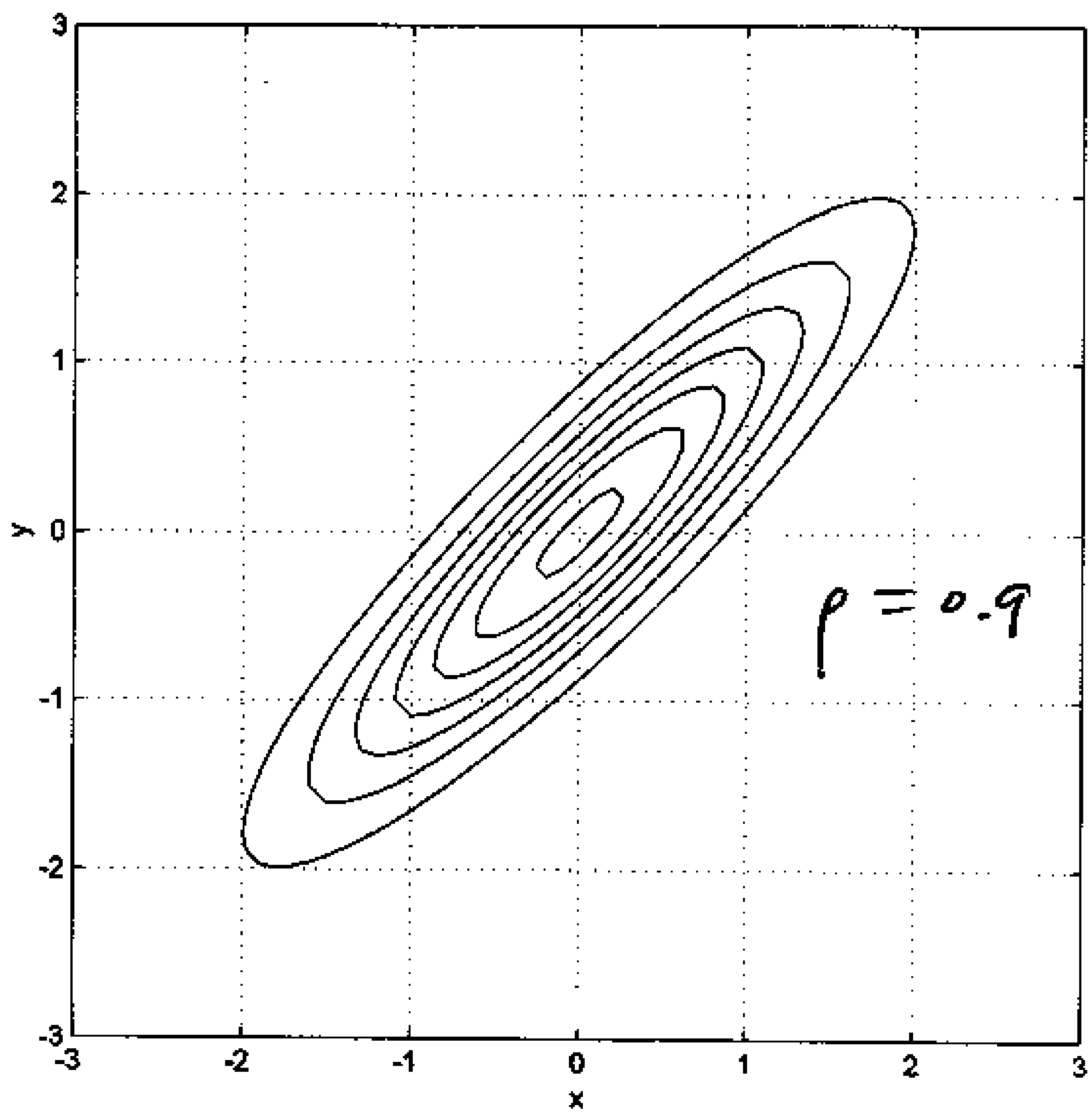
3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3
$\rho = 0$

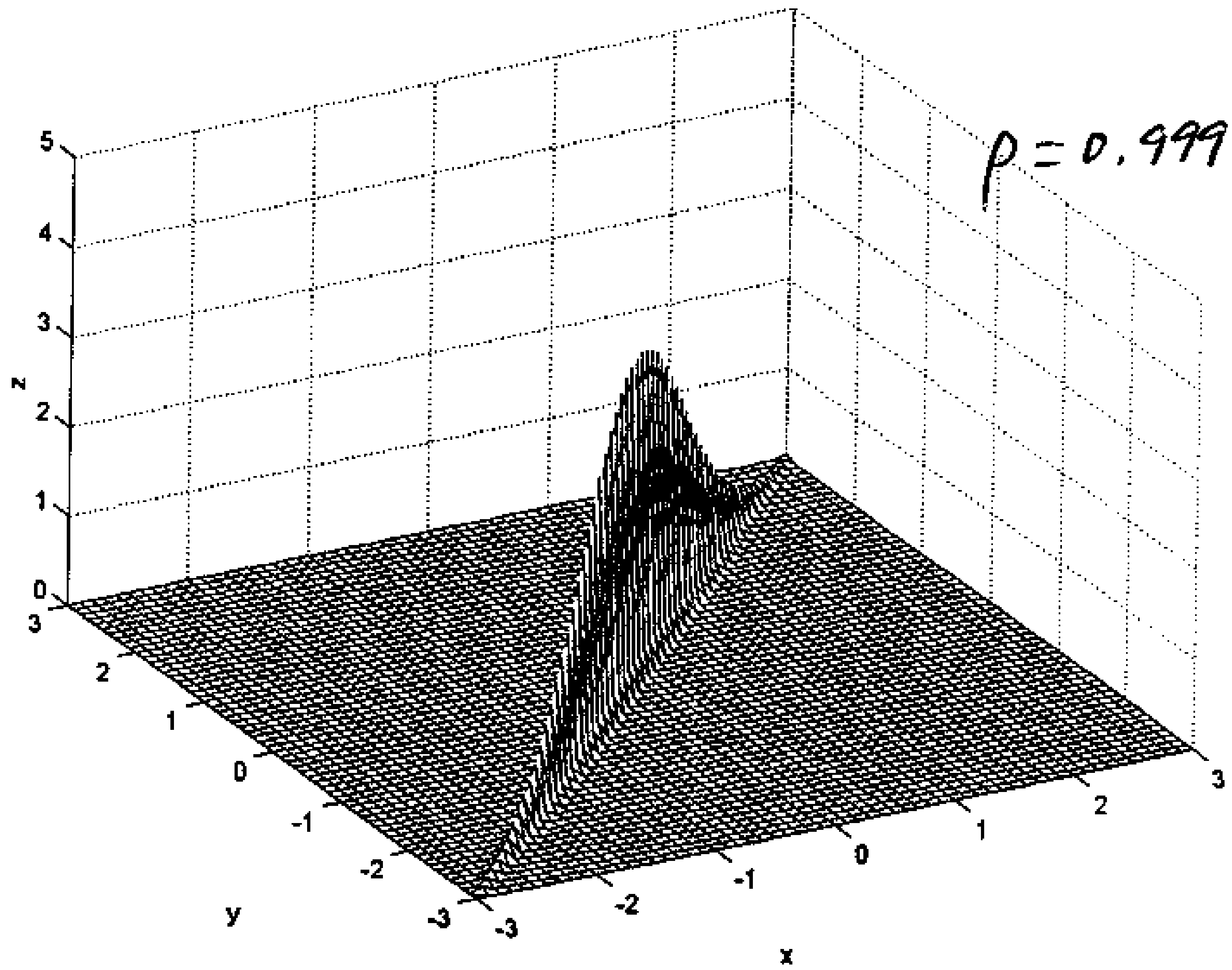


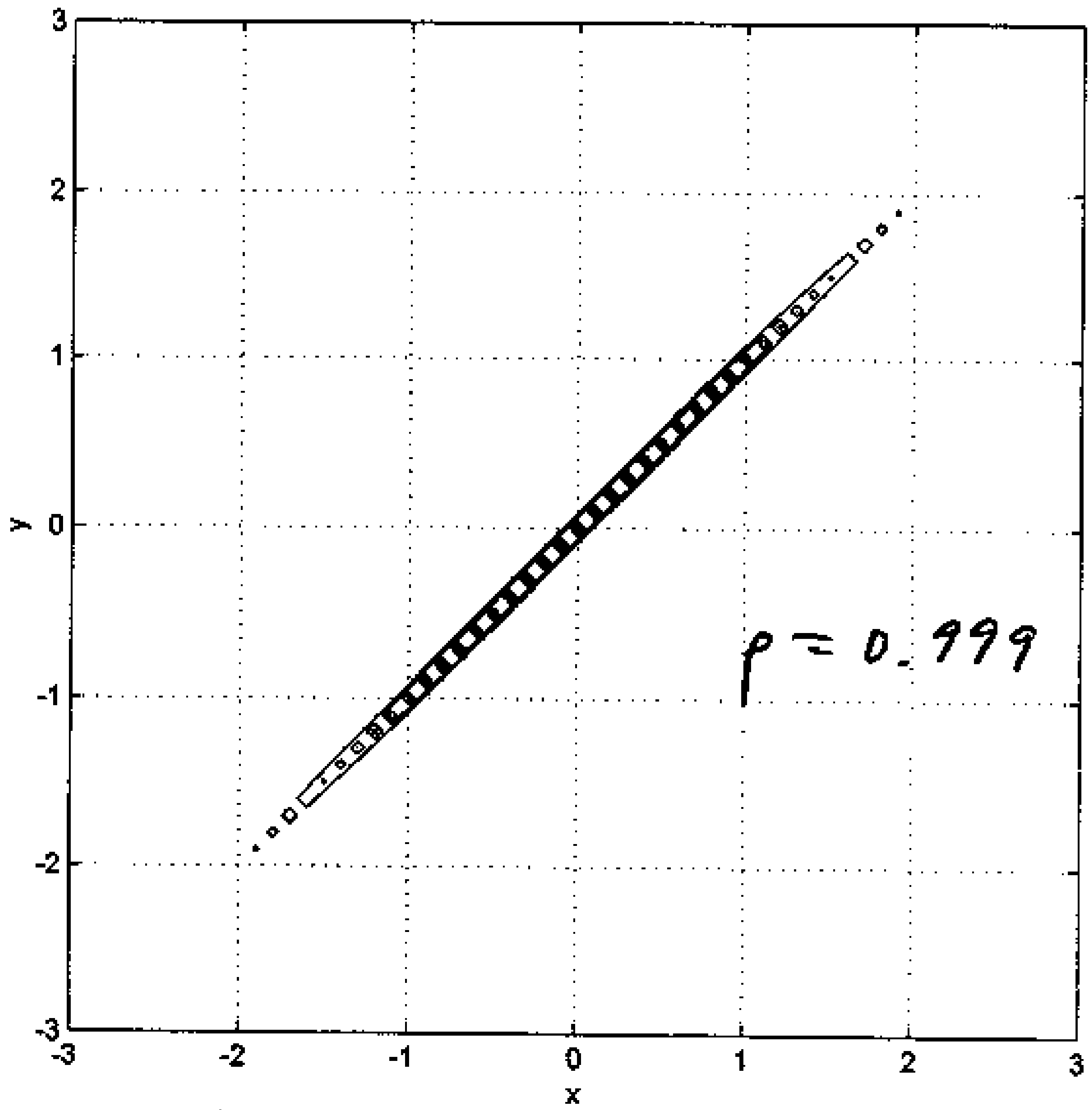
3
2
1
0
-1
-2
-3
-3 -2 -1 0 1 2 3
$\rho = -0.9$

13) PDF becomes concentrated at $y = x$ as $\rho \to 1$.
We have perfect prediction as $\rho \to 1$ using
$\hat{y} = \rho x = \rho$






ρ = 0.999
z
y
x


ρ = 0.999
y
x

```
% probprob12_13
%
clear all
x=[-3:0.1:3]';y=x;
rho=0.999;
for i=1:length(x)
   for j=1:length(y)
      Q=x(i)^2-2*rho*x(i)*y(j)+y(j)^2;
      pxy(i,j)=(1/(2*pi*sqrt(1-rho^2)))*exp(-(1/(2*(1-rho^2)))*Q);
   end
   end
figure
colormap(gray(1))

mesh(x,y,pxy')
axis([-3 3 -3 3 0 5])
view(-30,36)
xlabel('x')
ylabel('y')
zlabel('z')

figure
colormap(gray(1))

[C,H]=contour(x,y,pxy);

axis('square')
xlabel('x')
ylabel('y')
grid
```

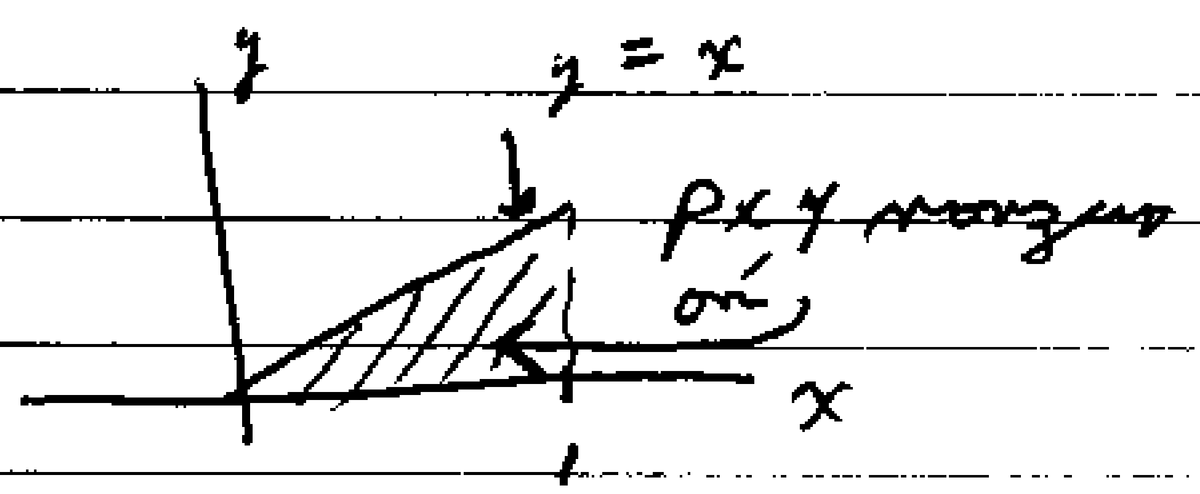
14) $p_x(x) = \int_{-\infty}^{\infty} p_{x,y}(x,y)\,dy$

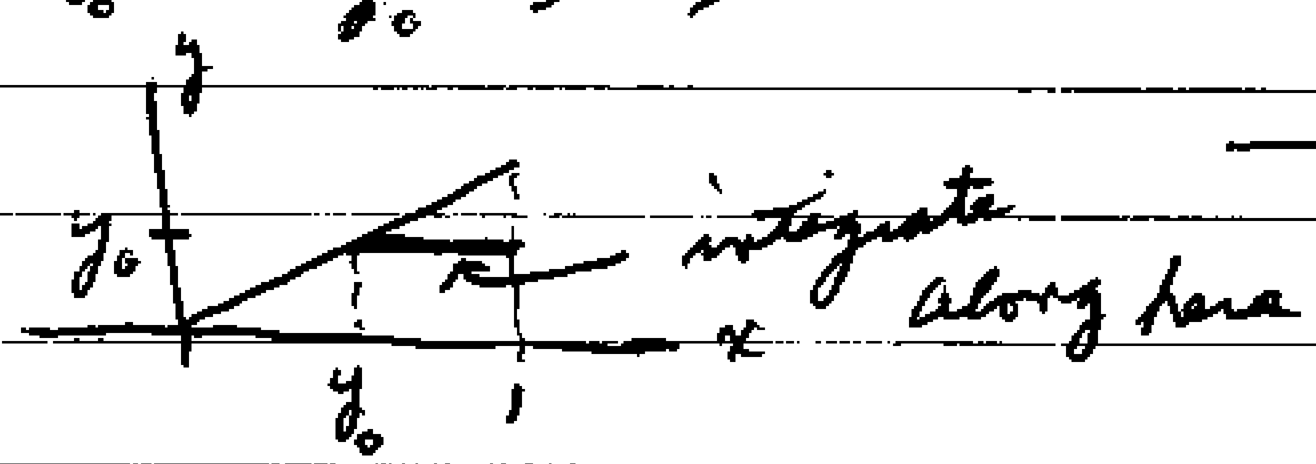
$$= \int_0^{\infty} e^{-(x+y)}\,dy \qquad \text{for } x \geq 0$$

$$= e^{-x}\left(-e^{-y}\right)\Big|_0^{\infty} = e^{-x}$$

$$p_x(x) = e^{-x}\,u(x)$$
$$p_y(y) = e^{-y}\,u(y)$$

15) $p_y(y_0) = \int_{y_0}^{1} p_{x,y}(x,y)\,dx$





$$p_Y(y_0) = \int_{y_0}^{1} 2\,dx = 2(1-y_0)$$

$$p_Y(y) = \begin{cases} 2(1-y) & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(x) = \int_0^x p_{X,Y}(x,y)\,dy \quad \text{by similar argument}$$

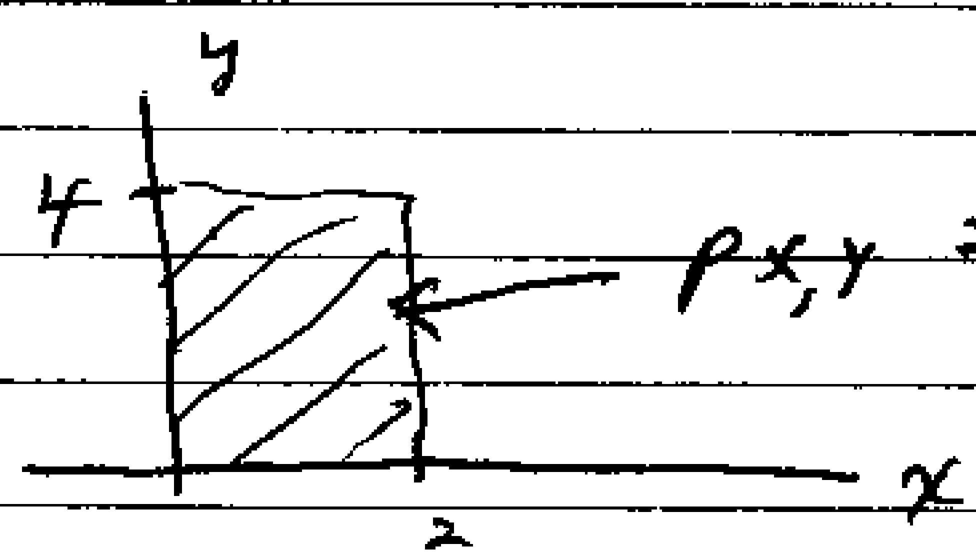
$$= \int_0^x 2\,dy = \begin{cases} 2x & 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

16) $$Q = x^2 - 2\rho xy + y^2$$
$$= \underbrace{(y-\rho x)^2}_{\geq 0} + \underbrace{(1-\rho^2)x^2}_{>0} > 0$$

17) Can't unless x, y are independent

18) $$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$$

$$= \begin{cases} \tfrac{1}{8} & 0<x<2,\ 0<y<4 \\ 0 & \text{otherwise} \end{cases}$$

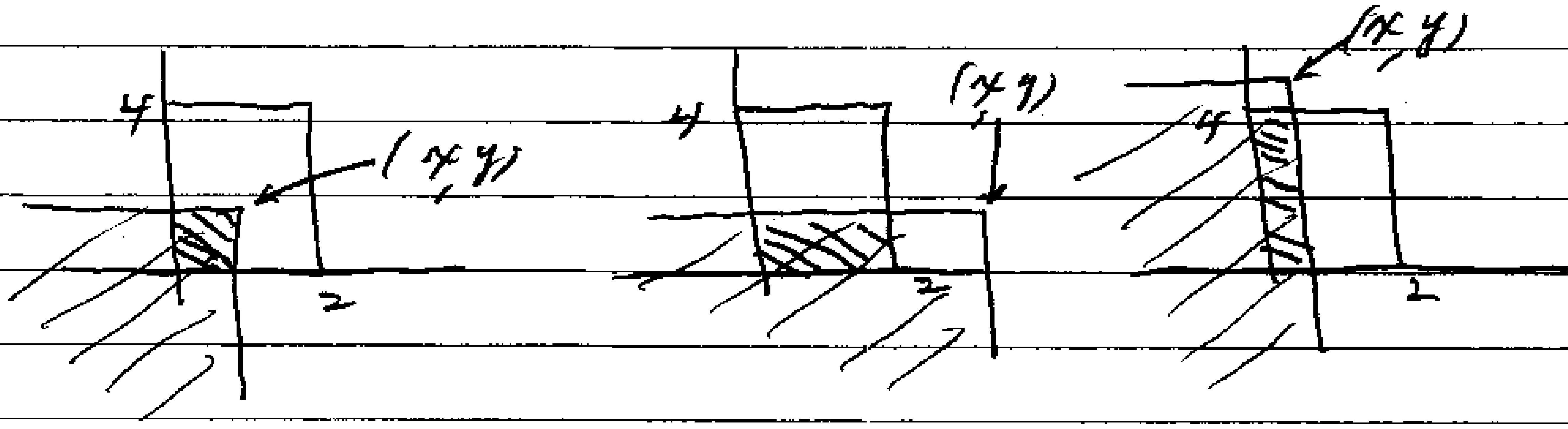


$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p_{X,Y}(t,u)\,dt\,du$$

For $x<0$ or $y<0$ $\quad F_{X,Y}(x,y) = 0$

For $x \geq 2,\ y \geq 4$ $\quad F_{X,Y}(x,y) = 1$

0 otherwise



$0 \le x < 2$ $\quad$ $x \ge 2$ $\quad$ $0 \le x < 2$

$0 \le y < 4$ $\quad$ $0 \le y < 4$ $\quad$ $y \ge 4$

$F_{X,Y}$ is double-cross hatched area $\times 1/8$

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ \frac{1}{8}xy & 0 \le x < 2,\ 0 \le y < 4 \\ \frac{1}{4}y & x \ge 2,\ 0 \le y < 4 \\ \frac{1}{2}x & 0 \le x < 2,\ y \ge 4 \\ 1 & x \ge 2,\ y \ge 4 \end{cases}$$

19) $F_{X,Y}(x,y) = \int_0^y\int_0^x t u e^{-\frac{1}{2}(t^2+u^2)}\,dt\,du$

$$= \int_0^y u e^{-\frac{1}{2}u^2}\,du \int_0^x t e^{-\frac{1}{2}t^2}\,dt$$

$$= -e^{-\frac{1}{2}u^2}\Big|_0^y \cdot -e^{-\frac{1}{2}t^2}\Big|_0^x$$

$$= (1-e^{-\frac{1}{2}y^2})(1-e^{-\frac{1}{2}x^2}) \qquad x \ge 0,\ y \ge 0$$

$$0 \qquad \text{otherwise}$$

$F_{X,Y}(-\infty,-\infty) = 0$

$F_{X,Y}(+\infty,+\infty) = 1 \cdot 1 = 1$

$F_{X,Y}(x,\infty) = 1 - e^{-\frac{1}{2}x^2} \qquad x \ge 0$

$= 0$ for $x < 0$

$= F_X(x)$ since X, Y are independent and each one is Rayleigh

Similarly, $F_{X,Y}(\infty, y) = F_Y(y)$

As x and y increase, $e^{-\frac{1}{2}x^2}, e^{-\frac{1}{2}y^2}$ decrease
$\Rightarrow F_{X,Y}$ increases

Continuous since exponential functions are continuous. Also, $F_{X,Y}(0,0) = 0$ so $F_{X,Y}$ is also continuous at $x = y = 0$.

20) a) $A = \{x : a \le x \le b\}$, $B = \{y : c \le y \le d\}$

$$P(X \in A, Y \in B) = \int_c^d \int_a^b p_{X,Y}(x,y)\, dx\, dy$$

$$= \int_c^d \int_a^b \underbrace{p_X(x) p_Y(y)}_{\text{by assumption}}\, dx\, dy$$

$$= \underbrace{\int_a^b p_X(x)\, dx}_{P(X \in A)} \underbrace{\int_c^d p_Y(y)\, dy}_{P(X \in B)}$$

b) $P[x_0 - \frac{\Delta x}{2} \le X \le x_0 + \Delta x/2,\ y_0 - \frac{\Delta y}{2} \le Y \le y_0 + \Delta y/2]$

$$= \int_{y_0 - \frac{\Delta y}{2}}^{y_0 + \Delta y/2} \int_{x_0 - \Delta x/2}^{x_0 + \Delta x/2} p_{X,Y}(x,y)\, dx\, dy \qquad (1)$$

$= P(x_0 - \frac{\Delta x}{2} \le X \le x_0 + \Delta x/2)\, P(y_0 - \Delta y/2 \le Y \le y_0 + \Delta y/2)$
by assumption

$$= \int_{x_0-\Delta x/2}^{x_0+\Delta x/2} p_X(x)\,dx \int_{y_0-\Delta y/2}^{y_0+\Delta y/2} p_Y(y)\,dy$$

$$= \int_{y_0-\frac{\Delta y}{2}}^{y_0+\Delta y/2}\int_{x_0-\Delta x/2}^{x_0+\Delta x/2} p_X(x)\,p_Y(y)\,dx\,dy \qquad (2)$$

From (1) and (2) as $\Delta x \to 0$, $\Delta y \to 0$

$$p_{X,Y}(x_0, y_0) = p_X(x_0)\,p_Y(y_0)$$

and since x_0, y_0 were arbitrary $\Rightarrow$

$$p_{X,Y}(x, y) = p_X(x)\,p_Y(y)$$

2.11 a) If $p_{X,Y} = p_X\,p_Y$

$$F_{X,Y}(x, y) = \int_{-\infty}^{y}\int_{-\infty}^{x} p_{X,Y}(t, u)\,dt\,du$$

$$= \int_{-\infty}^{y}\int_{-\infty}^{x} p_X(t)\,p_Y(u)\,dt\,du$$

$$= \int_{-\infty}^{x} p_X(t)\,dt \int_{-\infty}^{y} p_Y(u)\,du$$

$$= F_X(x)\,F_Y(y)$$

b) If $F_{X,Y} = F_X\,F_Y$

$$p_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x\,\partial y}$$

$$= \frac{\partial^2 F_X(x)\,F_Y(y)}{\partial x\,\partial y}$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial F_X(x)\,F_Y(y)}{\partial y}\right]$$

$$= \underbrace{\frac{\partial}{\partial x} F_X(x)}_{p_X(x)} \underbrace{\frac{\partial F_Y(y)}{\partial y}}_{p_Y(y)}$$

22) $Z = X + Y$

where $E[X] = E[Y] = 0$ for a Laplacian PDF

$$E[Z^2] = E_{X,Y}[(X+Y)^2] = E_X[X^2] + E_{X,Y}[2XY] + E_Y[Y^2]$$

$$= E_X[X^2] + 2\underbrace{E_X[X]E_Y[Y]}_{\text{due to ind.}} + E_Y[Y^2]$$

$$= E_X[X^2] + 0 + E_Y[Y^2]$$

$= var(X) + var(Y)$ since $E_X[X] = E_Y[Y] = 0$

$$= 2\sigma^2$$

23) $\lambda = 0.001$

$P[X \le 2000, Y \le 2000] = P[X \le 2000] \cdot P[Y \le 2000]$ one bulb does not affect other

$$= \left[\int_0^{2000} 0.001\, e^{-0.001x}\, dx\right]^2$$

$$= \left[-e^{-0.001x}\Big|_0^{2000}\right]^2$$

$$= (1 - e^{-2})^2 \approx 0.747$$

24) Yes

$$p_{X,Y}(x,y) = [2e^{-2x}u(x)][3e^{-3y}u(y)]$$

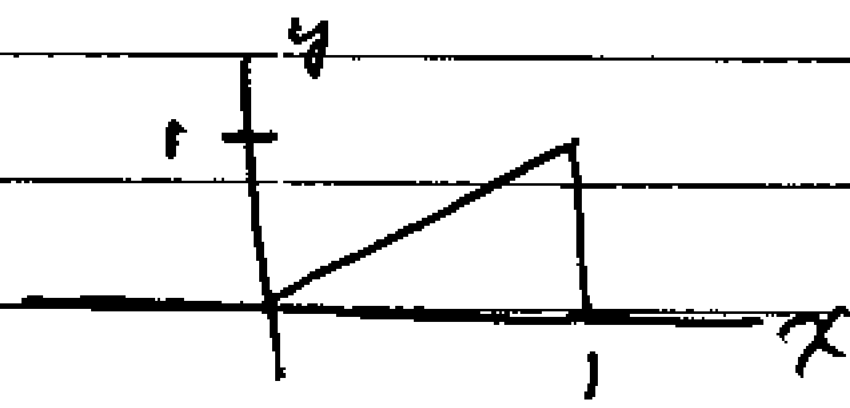
$$P(0<X<2,\ 0<Y<1) = P(0<X<2)\,P(0<Y<1)$$

$$= \int_0^2 2e^{-2x}\,dx \int_0^1 3e^{-3y}\,dy$$

$$= -e^{-2x}\Big|_0^2 \cdot -e^{-3y}\Big|_0^1$$

$$= (1-e^{-4})(1-e^{-3})$$

25) No, can't factor since



$$p_{X,Y}(x,y) = \begin{cases} 2 & 0 \le x < 1,\ 0 \le y < x \\ 0 & \text{otherwise} \end{cases}$$

can't factor

$$= 2\,\underbrace{[u(x)-u(x-1)]}_{\substack{0\le x<1 \text{ to} \\ \text{be nonzero}}}\,\underbrace{[u(y)-u(y-x)]}_{\substack{0\le y<x \text{ to} \\ \text{be nonzero}}}$$

26) $P(X-Y>0) = ?$

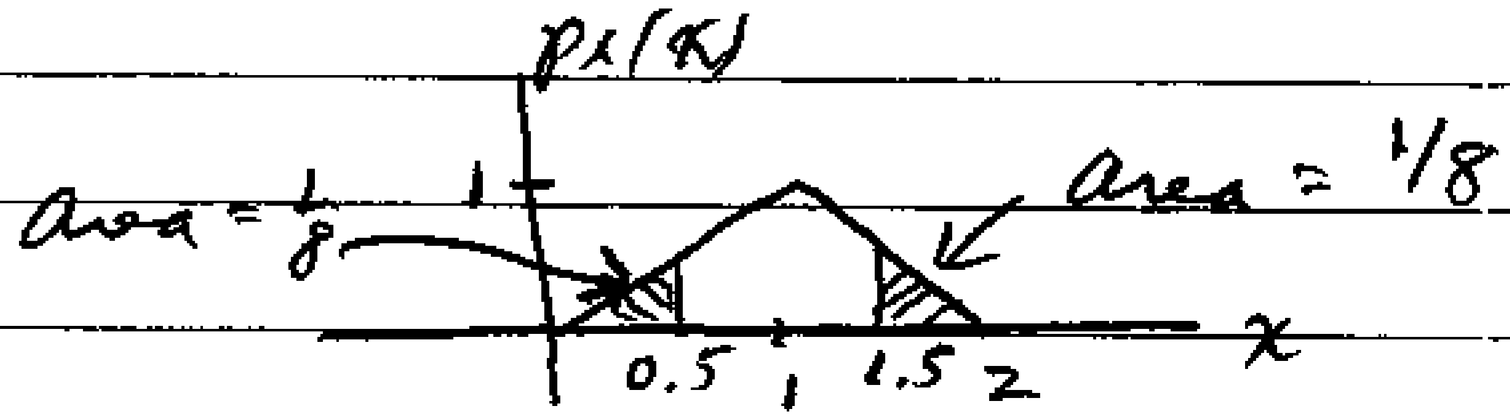
$$X-Y \sim N(\mu_X-\mu_Y,\ \sigma_X^2+\sigma_Y^2)$$
$$= N(20-100,\ 1500+100)$$
$$= N(-80,\ 1600)$$

$$P(X-Y>0) = Q\left(\frac{0-(-80)}{\sqrt{1600}}\right) = Q\left(\frac{80}{40}\right)$$

$$= Q(2) = 0.0228$$

27) $X = U_1 + U_2$ $\quad \left.\begin{array}{l} U_1 \sim U(0,1) \\ U_2 \sim U(0,1) \end{array}\right\}$ independent

From Fig 12.15



The probability of a single outcome in $[0, 0.5)$ is $1/8$ and similarly for $[1.5, 2)$ it is $1/8$. Hence, the probability of an outcome in $[0, 0.5)$ is $1/8$, in $[0.5, 1.5)$ is $6/8$, and in $[1.5, 2)$ is $1/8$. Since there are 3 possible outcomes we use the trinomial PMF with

X_1 = number of outcomes in $[0, 0.5)$
X_2 = " " " " $[0.5, 1.5)$
X_3 = " " " " $[1.5, 2)$

$k_1 = 500,\ k_2 = 0,\ k_3 = 500,\ M = 1000$

$$\text{Prob} = \binom{1000}{500, 0, 500} \left(\tfrac{1}{8}\right)^{500} (6/8)^0 (1/8)^{500}$$

$$= \frac{1000!}{500!\,500!\,0!} \left(\tfrac{1}{8}\right)^{1000} = \binom{1000}{500} \left(\tfrac{1}{8}\right)^{1000}$$

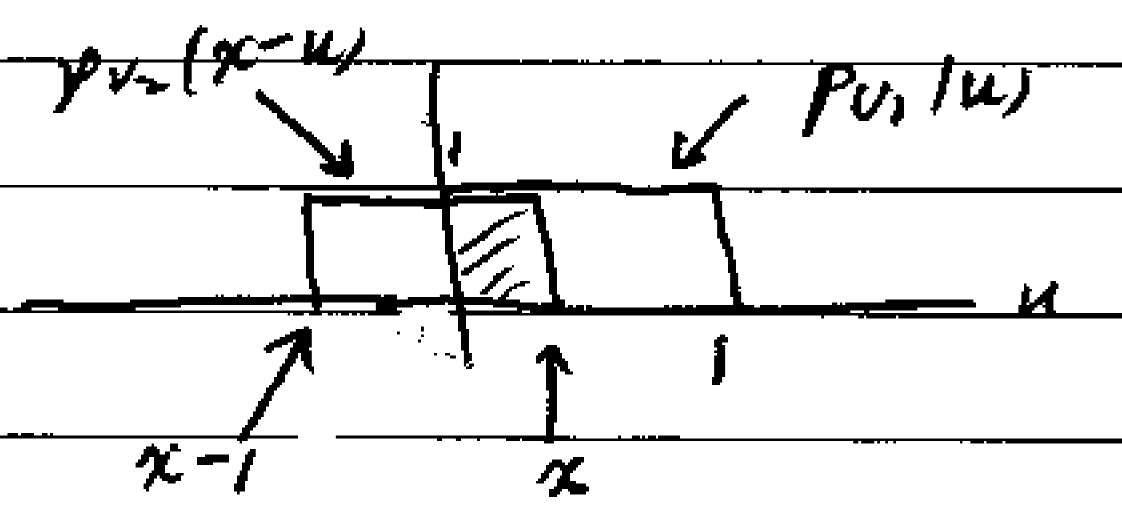
To evaluate this

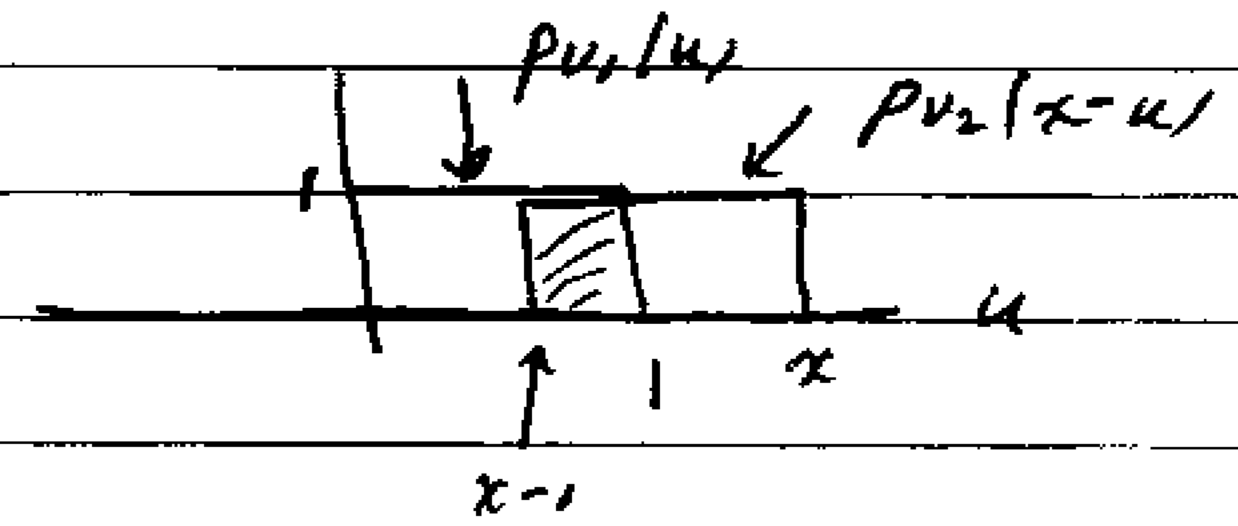
$$\log P = \log 1000! - 2 \log 500! + 1000 \log 1/8$$
$$= \sum_{i=1}^{1000} \log i - 2 \sum_{i=1}^{500} \log i - 1000 \log 8$$

$$= -603.6582 \Rightarrow P = 10^{-604} \cdot 10^{0.3418}$$

$$\approx 2.2 \times 10^{-604}$$

28) $p_X(x) = \int_{-\infty}^{\infty} p_{V_1}(u)\, p_{V_2}(x-u)\, du$





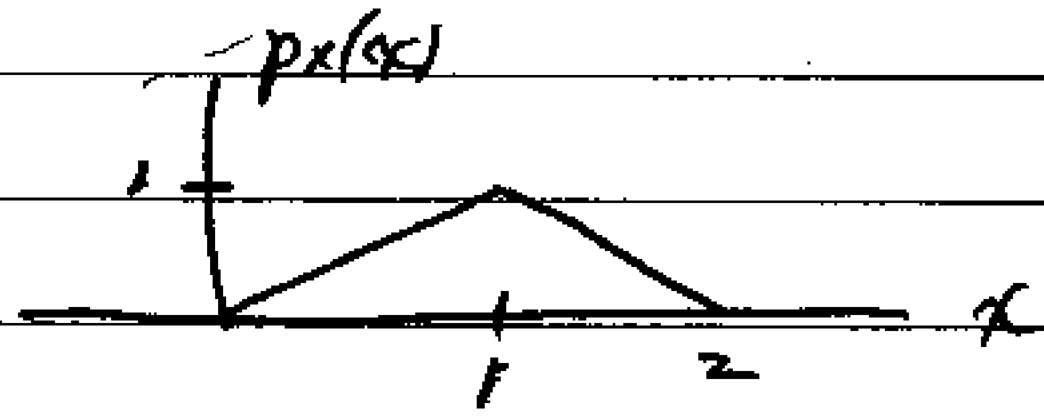
For $0 \leq x < 1$

$$p_X(x) = \int_0^x 1 \cdot 1\, du = x$$

For $1 \leq x \leq 2$

$$p_X(x) = \int_{x-1}^{1} 1 \cdot 1\, dx = 2 - x$$

$p_X(x) = 0$ otherwise



29) Ratio of areas $= \dfrac{BH}{1 \cdot 1} = BH$

$$\sin\theta = \frac{H}{d - \left(\frac{c}{a}\right) b}$$

$$\cos(\pi/2 - \theta) = \frac{a}{B} \Rightarrow \sin\theta = a/B$$

$$\frac{H}{d - \left(\frac{c}{a}\right) b} = \frac{a}{B} \Rightarrow BH = ad - bc$$

30) $P[\text{bullseye}] = P[\sqrt{X^2+Y^2} \leq \frac{1}{4}]$

But $\sqrt{X^2+Y^2} \sim \text{Rayleigh}(1/64)$

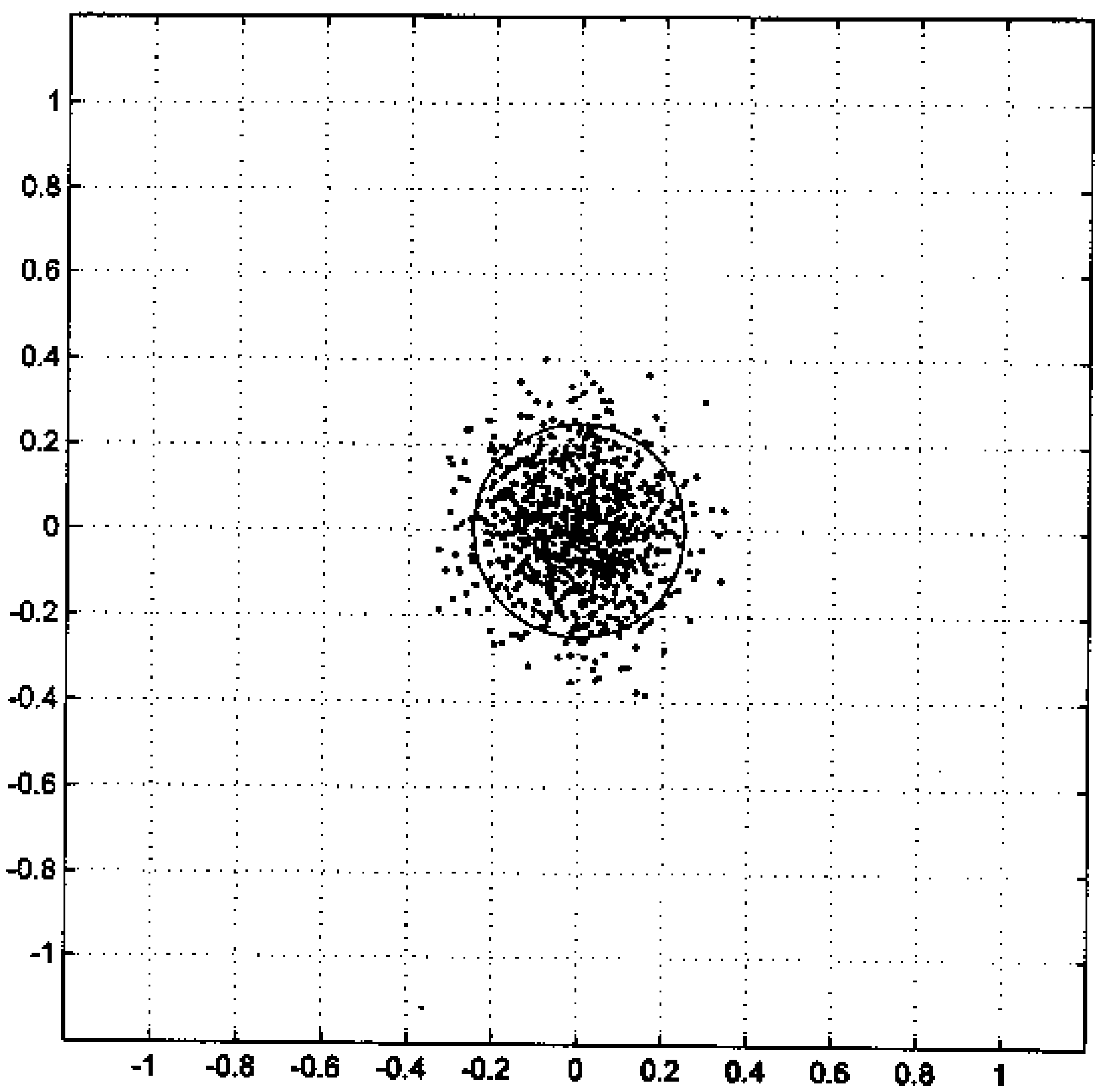
$$P[\text{bullseye}] = \int_0^{1/4} \frac{r}{\sigma^2} e^{-\frac{1}{2} r^2/\sigma^2} dr$$

$$= -e^{-\frac{1}{2}\frac{r^2}{\sigma^2}}\Big|_0^{1/4} = 1 - e^{-\frac{1}{2}\frac{(\frac{1}{4})^2}{1/64}}$$

$$= 1 - e^{-2}$$

$$= 0.8646$$

$$\hat{P}[\text{bullseye}] = 0.8730$$




```
%  probprob12_30.m
%
clear all
randn('state',0)
M=1000;
x=(1/8)*randn(M,1);y=(1/8)*randn(M,1);
count=0;
for i=1:M
   if x(i)^2+y(i)^2<=0.25^2
      count=count+1;
   end
end
probest=count/M
plot(x,y,'.')
axis([-1.2 1.2 -1.2 1.2])
axis('square')
grid
x=[-0.25:0.01:0.25]';y1=sqrt(0.25^2-x.^2);
y2=-y1;
hold on
plot(x,y1)
plot(x,y2)
hold off
```

31) $(1-\rho^2)\sigma_W^2\sigma_Z^2 \Rightarrow \det\begin{bmatrix}\sigma_W^2 & \rho\sigma_W\sigma_Z \\ \rho\sigma_Z\sigma_W & \sigma_Z^2\end{bmatrix}$

$$= \sigma_W^2\sigma_Z^2 - \rho^2\sigma_W^2\sigma_Z^2$$

$$= \sigma_W^2\sigma_Z^2(1-\rho^2)$$

$$\frac{\dfrac{w^2}{\sigma_w^2} - \dfrac{2\rho w z}{\sigma_w \sigma_z} + \dfrac{z^2}{\sigma_z^2}}{1-\rho^2} =$$

$$\frac{[w \;\; z] \begin{bmatrix} 1/\sigma_w^2 & -\rho/\sigma_w\sigma_z \\ -\dfrac{\rho}{\sigma_w\sigma_z} & 1/\sigma_z^2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}}{1-\rho^2}$$

transpose of cofactor matrix

$$= \frac{\begin{bmatrix} w \\ z \end{bmatrix}^T \begin{bmatrix} \sigma_z^2 & -\rho\sigma_w\sigma_z \\ -\rho\sigma_z\sigma_w & \sigma_w^2 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}}{\underbrace{(1-\rho^2)\sigma_w^2\sigma_z^2}_{\det(\underline{C})}}$$

$$= \begin{bmatrix} w \\ z \end{bmatrix}^T \underline{C}^{-1} \begin{bmatrix} w \\ z \end{bmatrix}$$

32) $z = h(x, y)$

$$\approx h(x_0, y_0) + \left.\frac{\partial h}{\partial x}\right|_{\substack{x = x_0 \\ y = y_0}} (x - x_0)$$

$$+ \left.\frac{\partial h}{\partial y}\right|_{\substack{x = x_0 \\ y = y_0}} (y - y_0)$$

$$\Rightarrow \begin{bmatrix} w \\ z \end{bmatrix} \approx \begin{bmatrix} g(x_0, y_0) \\ h(x_0, y_0) \end{bmatrix} + \left.\begin{bmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} \end{bmatrix}\right|_{\substack{x = x_0 \\ y = y_0}} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Since $w = g(x,y)$, $z = h(x,y)$, the matrix is

$$\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = \text{Jacobian matrix.}$$

33) $\underline{G} = \begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix}$

$$\underline{G}^{-1} = \frac{\begin{bmatrix} 1 & -2 \\ -2 & 2 \end{bmatrix}}{2-4} = \begin{bmatrix} -\frac{1}{2} & 1 \\ 1 & -1 \end{bmatrix}$$

$$\det(\underline{G}^{-1}) = \tfrac{1}{2} - 1 = -\tfrac{1}{2}$$

$$p_{W,Z}(w,z) = p_{X,Y}\left(\underline{G}^{-1}\begin{bmatrix} w \\ z \end{bmatrix}\right) |\det(\underline{G}^{-1})|$$

$$= p_{X,Y}\left(-\tfrac{1}{2}w + z,\ w - z\right) \left|-\tfrac{1}{2}\right|$$

$$= \left(\tfrac{1}{4}\right)^2 e^{-\frac{1}{2}\left(\left|-\frac{1}{2}w + z\right| + |w - z|\right)} \tfrac{1}{2}$$

$$= \frac{1}{32} e^{-\frac{1}{2}\left(\left|z - \frac{1}{2}w\right| + |w - z|\right)}$$

$$-\infty < w < \infty$$
$$-\infty < z < \infty$$

34) $w = xy$

$z = y/x \Rightarrow x^2 = w/z \Rightarrow x = \sqrt{w/z}$

$y^2 = wz \Rightarrow y = \sqrt{wz}$

Since $x \geq 0$, $y \geq 0 \Rightarrow w \geq 0$, $z \geq 0$ and these are only solutions.

From (12.22)

$$\frac{\partial(x,y)}{\partial(w,z)} = \begin{bmatrix} 1/\sqrt{z}\ \frac{1}{2\sqrt{w}} & \sqrt{w}\left(-\frac{1}{2} z^{-3/2}\right) \\ \sqrt{z}\ \frac{1}{2\sqrt{w}} & \sqrt{w}\ \frac{1}{2\sqrt{z}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2\sqrt{wz}} & -\frac{1}{2}\sqrt{w}/z^{3/2} \\ \frac{1}{2}\sqrt{z/w} & \frac{1}{2}\sqrt{w/z} \end{bmatrix}$$

$$\det\left(\frac{\partial(x,y)}{\partial(w,z)}\right) = \frac{1}{4z} + \frac{1}{4}\frac{1}{z}$$

$$= \frac{1}{2z}$$

$$p_{w,z}(w,z) = e^{-(\sqrt{w/z}+\sqrt{wz})}\frac{1}{2z} \qquad w > 0,\ z \geq 0$$

$$0 \qquad \text{otherwise}$$

35) Area ratio $= \left|\det\left(\frac{\partial(w,z)}{\partial(x,y)}\right)\right|$

$$= \left|\det\begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}\right| = \left|\det\begin{bmatrix} 2x & 10y \\ -10x & 2y \end{bmatrix}\right|$$

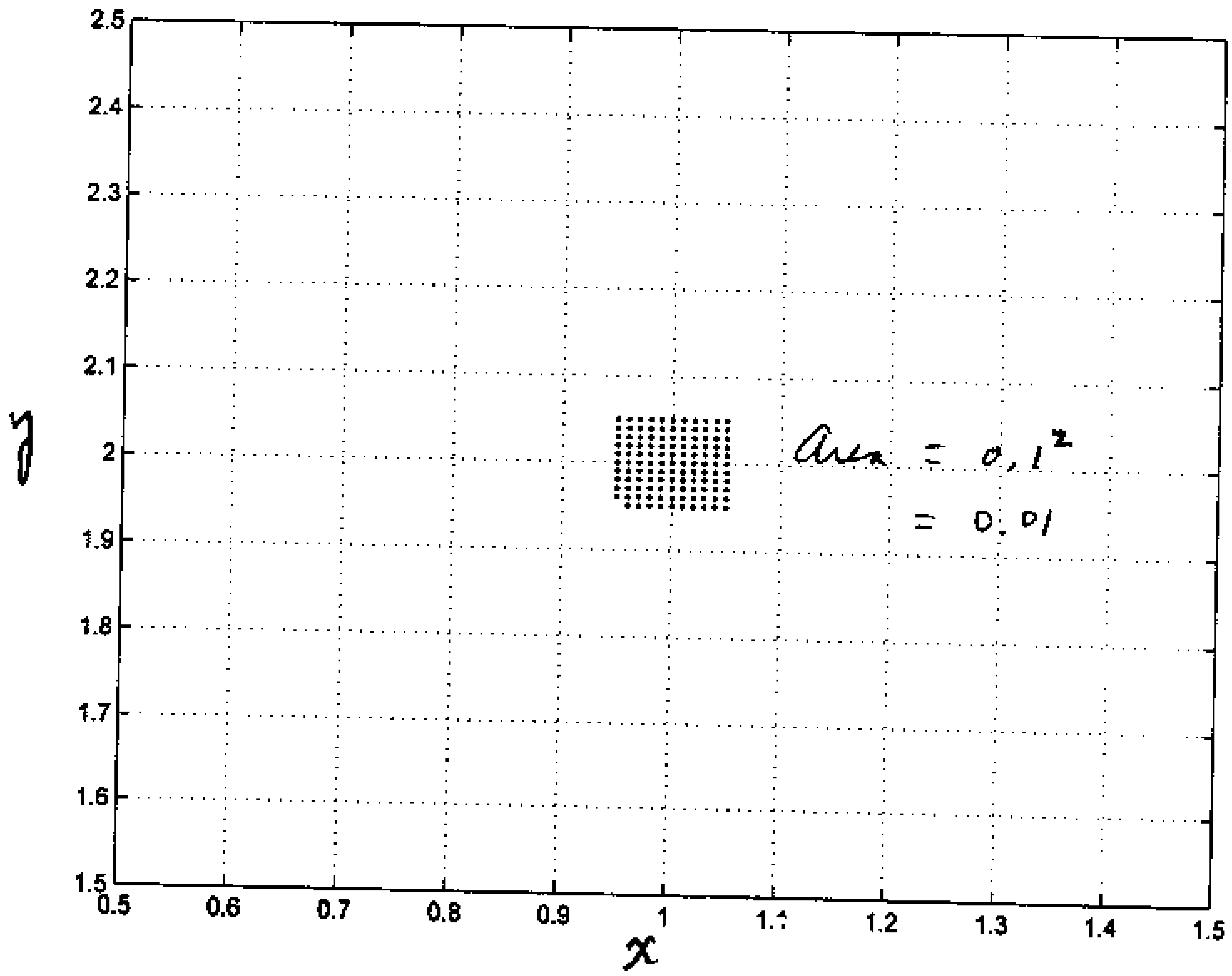
$$= |4xy + 100xy| = |104xy|$$

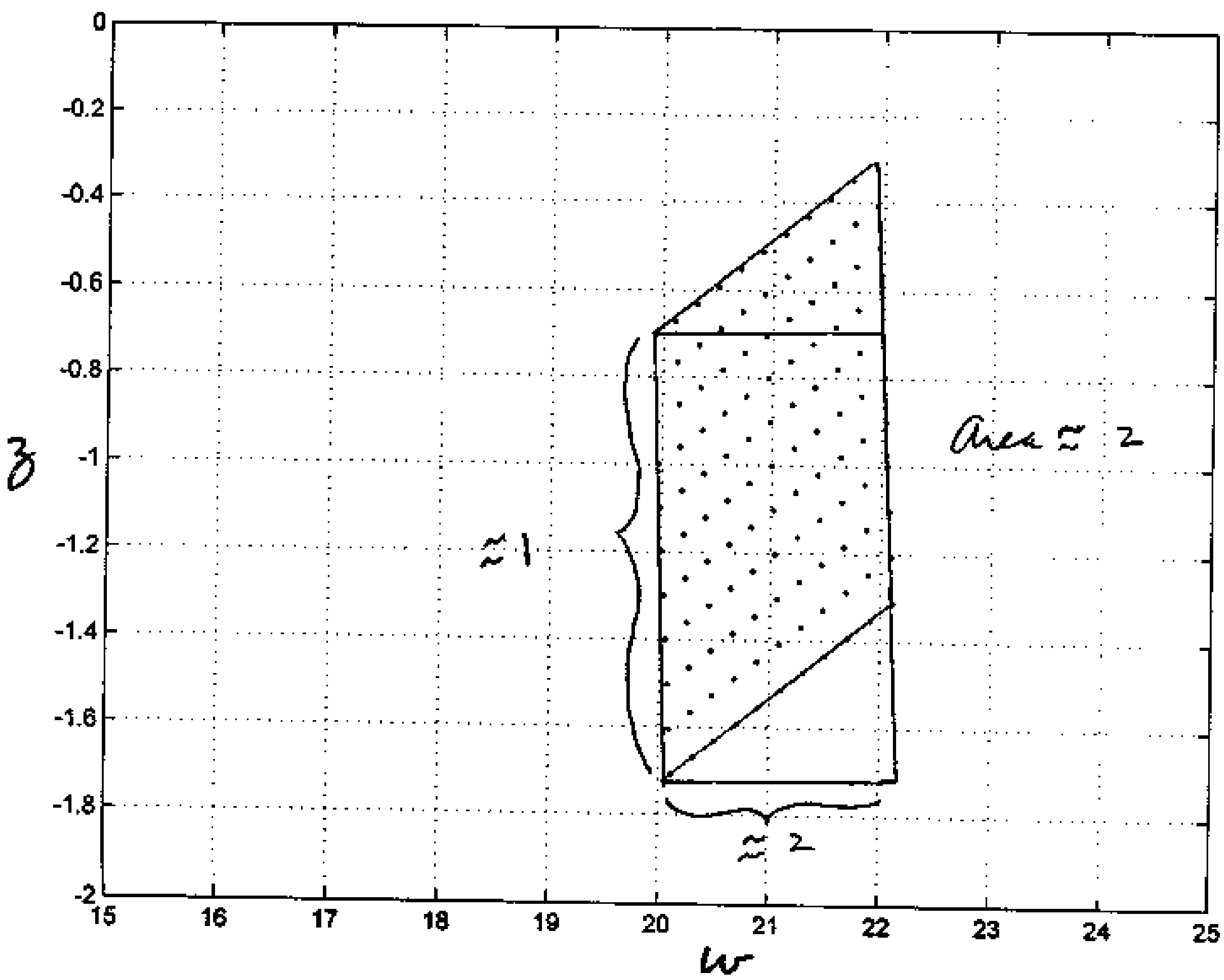
at $x = 1,\ y = 2$

Area ratio $= |104(2)| = 208$

Area of square = 0.01 (see Figure)
Area of parallelogram = 2 (see Figure)

Area ratio = $2/0.01$ = 200





```
%  probprob12_35.m
%
clear all
u1=[0.95:0.01:1.05]';
u2=[1.95:0.01:2.05]';
k=0;
for i=1:length(u1)
   for j=1:length(u2)
      k=k+1;
     x(:,k)=[u1(i) u2(j)]';
      w(:,k)=[u1(i)^2+5*u2(j)^2 -5*u1(i)^2+u2(j)^2]';
   end
end
figure
plot(x(1,i),x(2,i),'.')
hold on
for i=2:k
   plot(x(1,i),x(2,i),'.')
end
axis([0.5 1.5 1.5 2.5])
grid
hold off
figure
plot(w(1,i),w(2,i),'.')
hold on
for i=2:k
   plot(w(1,i),w(2,i),'.')
end
grid
axis([15 25 -2 0])
hold off
```

36) $p_w(w) = \int_{-\infty}^{\infty} \frac{1}{2\pi \det^{1/2}(\underline{\underline{C}})}$

$$\cdot\, e^{-\frac{1}{2}\underbrace{\begin{Bmatrix} w-\mu_w \\ z-\mu_z \end{Bmatrix}^T \underline{\underline{C}}^{-1} \begin{Bmatrix} w-\mu_w \\ z-\mu_z \end{Bmatrix}}_{q}}\, dz$$

Let $\bar{w} = w - \mu_w$

$\bar{z} = z - \mu_z$

$$q = [\bar{w} \;\; \bar{z}] \begin{bmatrix} \sigma_w^2 & \rho\sigma_w\sigma_z \\ \rho\sigma_w\sigma_z & \sigma_z^2 \end{bmatrix}^{-1} \begin{Bmatrix} \bar{w} \\ \bar{z} \end{Bmatrix}$$

$$= [\bar{w} \;\; \bar{z}] \frac{\begin{bmatrix} \sigma_z^2 & -\rho\sigma_w\sigma_z \\ -\rho\sigma_w\sigma_z & \sigma_w^2 \end{bmatrix}}{\sigma_w^2\sigma_z^2(1-\rho^2)} \begin{Bmatrix} \bar{w} \\ \bar{z} \end{Bmatrix}$$

$$= \frac{1}{\sigma_W^2 \sigma_Z^2 (1-\rho^2)} \left(\sigma_Z^2 \bar{w}^2 - 2\rho \sigma_W \sigma_Z \bar{w} \bar{z} + \sigma_W^2 \bar{z}^2 \right)$$

$$= \frac{1}{\sigma_Z^2 (1-\rho^2)} \left[\bar{z}^2 - 2\rho \frac{\sigma_Z}{\sigma_W} \bar{w} \bar{z} + \frac{\sigma_Z^2}{\sigma_W^2} \bar{w}^2 \right]$$

$$= \frac{1}{\sigma_Z^2 (1-\rho^2)} \left[\left(\bar{z} - \rho \frac{\sigma_Z}{\sigma_W} \bar{w} \right)^2 - \rho^2 \frac{\sigma_Z^2}{\sigma_W^2} \bar{w}^2 + \frac{\sigma_Z^2}{\sigma_W^2} \bar{w}^2 \right)$$

$$= \frac{1}{\sigma_Z^2 (1-\rho^2)} \left(\bar{z} - \rho \frac{\sigma_Z}{\sigma_W} \bar{w} \right)^2 + \bar{w}^2 / \sigma_W^2$$

$$p_W(w) = \frac{1}{2\pi \det^{1/2}(\underline{C})} e^{-\frac{1}{2} \bar{w}^2 / \sigma_W^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma_Z^2 (1-\rho^2)} \left(\bar{z} - \rho \frac{\sigma_Z}{\sigma_W} \bar{w} \right)^2} d\bar{z}}_{= \sqrt{2\pi \sigma_Z^2 (1-\rho^2)}}$$

$$= \frac{\sqrt{\sigma_Z^2 (1-\rho^2)}}{\sqrt{2\pi} \sqrt{\sigma_W^2 \sigma_Z^2 (1-\rho^2)}} e^{-\frac{1}{2} \bar{w}^2 / \sigma_W^2}$$

$$= \frac{1}{\sqrt{2\pi \sigma_W^2}} e^{-\frac{1}{2\sigma_W^2} (w - \mu_W)^2}$$

$\Rightarrow \quad W \sim N(\mu_W, \sigma_W^2)$

and similarly $\quad Z \sim N(\mu_Z, \sigma_Z^2)$

37) $\quad X \sim N(1, 3) \quad Y \sim N(2, 2)$

38) $\quad \begin{bmatrix} W \\ Z \end{bmatrix} \sim N(\underline{G}\underline{\mu}, \underline{G}\underline{C}_{X,Y}\underline{G}^T)$

$$\underline{G}\,\underline{\mu} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\underline{G}\,\underline{C}_{x,y}\,\underline{G}^T = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 \\ 5 & 14 \end{bmatrix}$$

39) $z = y/x$

$w = x$

$\Rightarrow \quad x = w$

$y = xz = wz$

$$\frac{\partial(x,y)}{\partial(w,z)} = \begin{bmatrix} 1 & 0 \\ z & w \end{bmatrix}$$

$$\left| \det \frac{\partial(x,y)}{\partial(w,z)} \right| = |w|$$

$$p_{w,z}(w,z) = p_{x,y}(w, wz)\,|w|$$
$$= p_x(w)\,p_y(wz)\,|w|$$

$$p_z(z) = \int_{-\infty}^{\infty} p_x(w)\,p_y(wz)\,|w|\,dw$$

40) $z = xy$

$w = x$

$\Rightarrow \quad x = w$

$y = z/x = z/w$

$$\frac{\partial(x,y)}{\partial(w,z)} = \begin{bmatrix}1 & 0\\ -z/w^2 & 1/w\end{bmatrix}$$

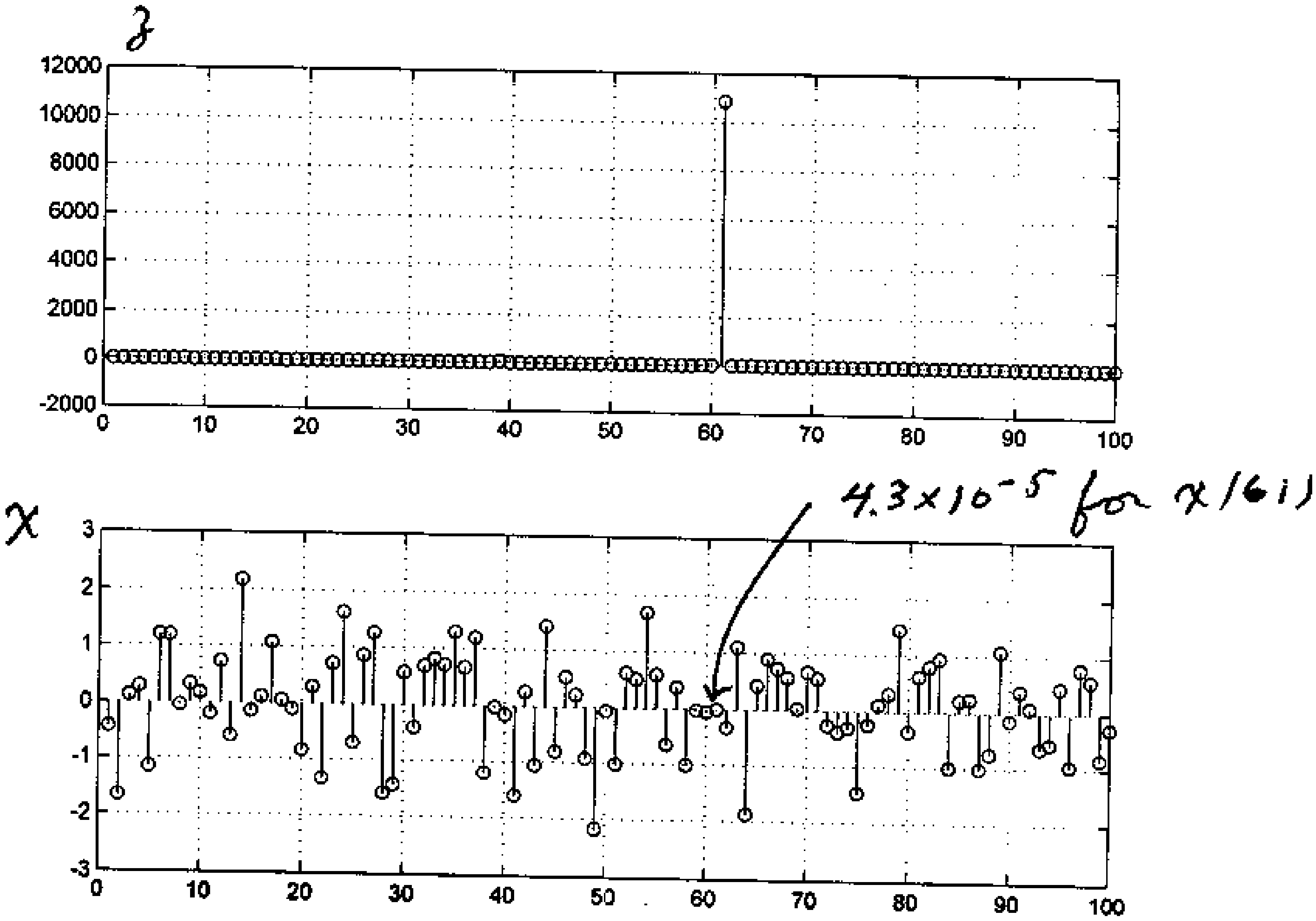
$$\left|\det \frac{\partial(x,y)}{\partial(w,z)}\right| = |1/w|$$

$$p_{W,Z}(w,z) = p_{X,Y}(w, z/w)\,|1/w|$$

$$p_Z(z) = \int_{-\infty}^{\infty} p_{X,Y}(w, z/w)\left|\frac{1}{w}\right| dw$$

$$= \int_{-\infty}^{\infty} p_X(w)\,p_Y(z/w)\,\frac{1}{|w|}\, dw$$

41)



Large but infrequent values of a Cauchy random variable occur when $X \approx 0$.

```
%  probprob12_41.m
%
clear all
randn('state',0)
M=100;
x=randn(M,1);y=randn(M,1);
z=y./x;
subplot(2,1,1)
stem(z)
grid
subplot(2,1,2)
stem(x)
grid
[maxz,I]=max(z)
x(I)
```

42) Let $A = r\cos\theta$

$B = r\sin\theta$

$$s(t) = r\cos\theta\cos 2\pi F_0 t + r\sin\theta\sin 2\pi F_0 t$$
$$= r\cos(2\pi F_0 t - \theta)$$
$$= \sqrt{A^2+B^2}\cos(2\pi F_0 t - \arctan B/A)$$

43) $D = \sqrt{(V_x t)^2 + (V_y t)^2}$

$= \sqrt{V_x^2 + V_y^2}$ in 1 sec

But $\left.\begin{array}{l} V_x \sim N(0,10) \\ V_y \sim N(0,10) \end{array}\right\}$ independent

$\Rightarrow \sqrt{V_x^2 + V_y^2} \sim$ Rayleigh (10)

$$E[D] = \sigma\sqrt{\pi/2} = \sqrt{10}\sqrt{\pi/2} = \sqrt{5\pi} \approx 3.96\ m$$

44) $F_Z(z) = P[Z \le z] = P[X^2+Y^2 \le z]$

$$= \iint_{\{(x,y):\, x^2+y^2 \le z\}} p_{x,y}(x,y)\, dx\, dy$$

$$= \iint_{\{(x,y):\, x^2+y^2 \le z\}} \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}\, dx\, dy$$

$$= \int_0^{\sqrt{z}} \int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{1}{2}r^2}\, r\, d\theta\, dr$$

$$= \int_0^{\sqrt{z}} r e^{-\frac{1}{2}r^2}\, dr = -e^{-\frac{1}{2}r^2}\Big|_0^{\sqrt{z}}$$

$$= 1 - e^{-\frac{1}{2}z} \qquad z \ge 0$$

$$p_Z(z) = \frac{d F_Z(z)}{dz} = \begin{cases} \frac{1}{2} e^{-\frac{1}{2}z} & z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow Z \sim \chi_2^2$

45) $\operatorname{cov}(W, Z) = \operatorname{cov}(X+Y, X-Y)$

$= E_{X,Y}[(X+Y)(X-Y)]$ zero means

$= E_X[X^2] - E_Y[Y^2] = 1 - 1 = 0$

$\Rightarrow$ uncorrelated but not necessarily independent

46) $E_{X,Y}[X+Y] = E_X[X] + E_Y[Y] = 1 + 1 = 2$

$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X,Y)$

$= 2 + 2 + 2(1) = 6$

47) $\underline{C} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Need to find modal matrix $\underline{V}$

Then $\underline{G} = \underline{V}^T$

$$\det(\underline{C} - \lambda \underline{I}) = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 3$$

$$(\underline{C} - \lambda_1 \underline{I})\underline{v}_1 = \underline{0} \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} [\underline{v}_1]_1 \\ [\underline{v}_1]_2 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \underline{v}_1 = \frac{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{\sqrt{2}}$$

$$(\underline{C} - \lambda_2 \underline{I})\underline{v}_2 = \underline{0} \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} [\underline{v}_2]_1 \\ [\underline{v}_2]_2 \end{pmatrix} = \underline{0}$$

$$\Rightarrow \underline{v}_2 = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\sqrt{2}}$$

$$\underline{V} = [\underline{v}_1 \ \underline{v}_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

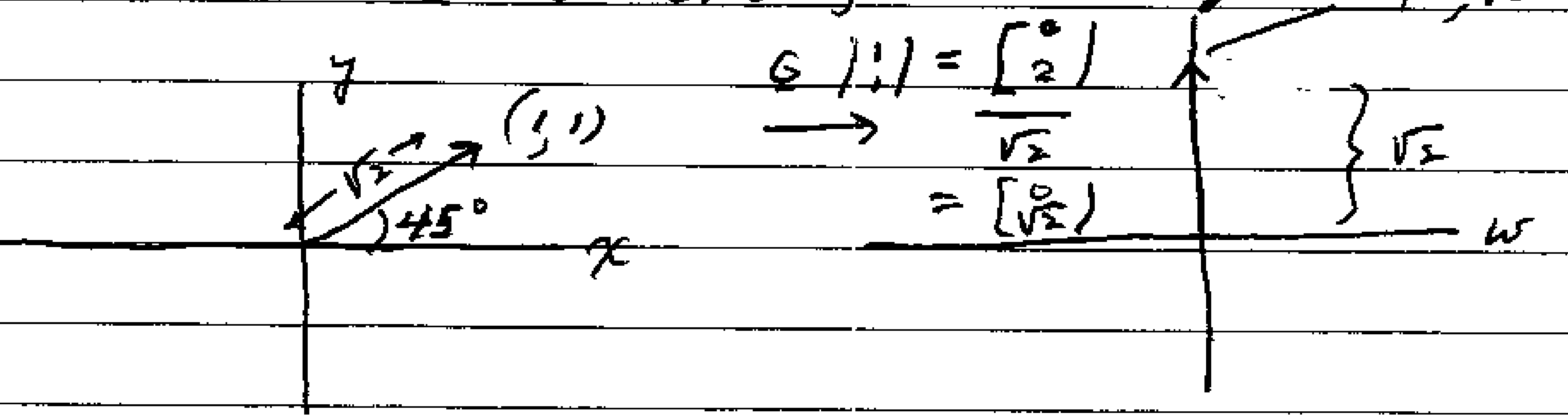
$$\underline{G} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

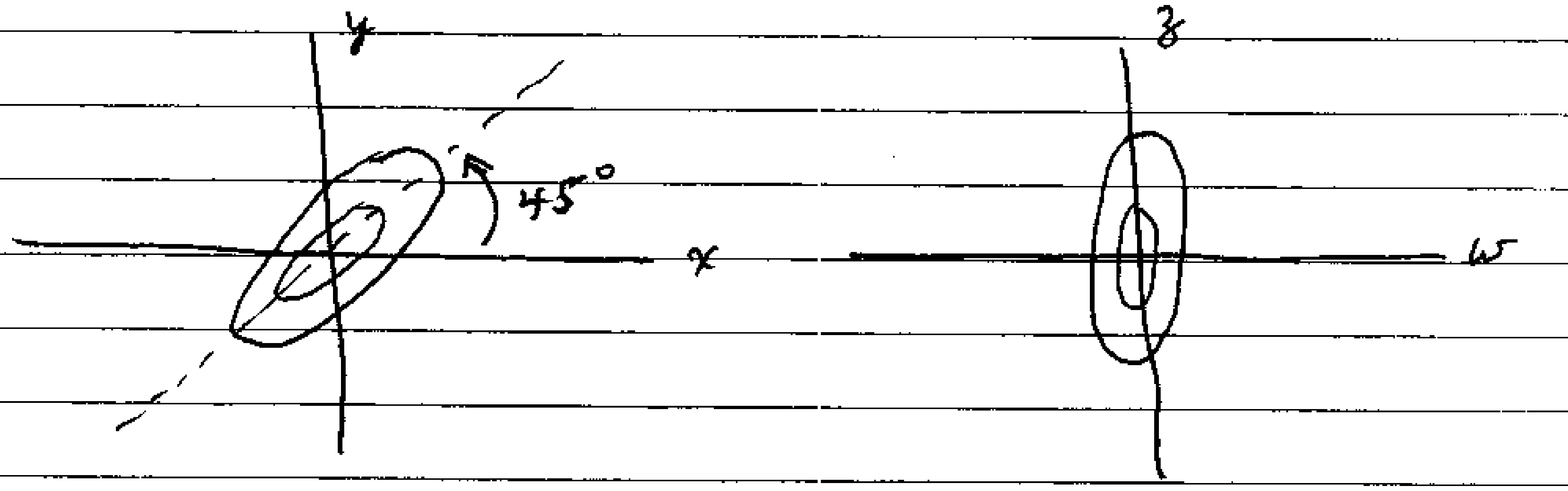
48) $$\underline{G}\,\underline{C}\,\underline{G}^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

$$= \begin{bmatrix} 1/\sqrt{2}\,(a-b) & \frac{1}{\sqrt{2}}(a+b) \\ 1/\sqrt{2}\,(b-a) & \frac{1}{\sqrt{2}}(a+b) \end{bmatrix}$$

$$= \begin{bmatrix} a-b & 0 \\ 0 & a+b \end{bmatrix}$$

$$\underline{G} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{where } \theta = 45^\circ$$





X, Y are correlated

W, Z are uncorrelated (also independent)

49)

$\rho = 0.9$ from $-2\rho xy$ term

$\hat{Y} = \rho x = 0.9x$ $\quad (E_Y[Y] = 0,\ E_X[X] = 0,$ $\ \mathrm{var}(X) = 1,\ \mathrm{cov}(X, Y) = \rho)$

$Y - \hat{Y} = Y - 0.9X$

But X, Y are jointly Gaussian and thus $Y - 0.9X = \begin{bmatrix} -0.9 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = G \begin{bmatrix} X \\ Y \end{bmatrix}$ is a Gaussian random variable using Theorem 12.7.1

But $\underline{G}\begin{bmatrix}X\\Y\end{bmatrix} \sim N(\underline{G}\underline{\mu}, \underline{G}\underline{C}\underline{G}^T)$

$\underline{\mu} = \begin{bmatrix}0\\0\end{bmatrix} \quad \underline{C} = \begin{bmatrix}1 & 0.9\\0.9 & 1\end{bmatrix}$

$\Rightarrow \underline{G}\begin{bmatrix}X\\Y\end{bmatrix} \sim N\left(0, [-0.9\ \ 1]\underbrace{\begin{bmatrix}1 & 0.9\\0.9 & 1\end{bmatrix}\begin{bmatrix}-0.9\\1\end{bmatrix}}_{\begin{bmatrix}0\\0.19\end{bmatrix}}\right)$

$= N(0, 0.19)$

Note that $mse_{min} = var(Y)(1-\rho^2)$
$= 1\cdot(1-0.9^2) = 0.19$

50) $X = V + N \sim N(1, 3)$ since V, N are independent

Let $Y = V$ so that
$X = Y + N \quad Y \sim N(1,1), \ N \sim N(0,2)$

$\hat{Y} = E_Y(Y) + \frac{cov(X,Y)}{var(X)}(x - E_X(X))$

$= 1 + \frac{cov(X,Y)}{3}(x-1)$

But $Cov(X,Y) = E_{XY}(XY) - E_X(X)E_Y(Y)$
$= E[(Y+N)Y] - E[X]E[Y]$
and Y ($=V$) and N are independent

$= E[Y^2] + E[Y]E[N] - E[X]E[Y]$
$= E[V^2] + E[V]E[N] - E[X]E[Y]$
$= (1+1) + 1\cdot 0 - 1\cdot 1 = 1$

$$\hat{y} = 1 + \tfrac{1}{3}(x-1) = \tfrac{1}{3}x + 2/3$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} V+N \\ V \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\underline{G}} \underbrace{\begin{bmatrix} V \\ N \end{bmatrix}}_{\sim N\left(\underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\underline{\mu}}, \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\underline{C}}\right)}$$

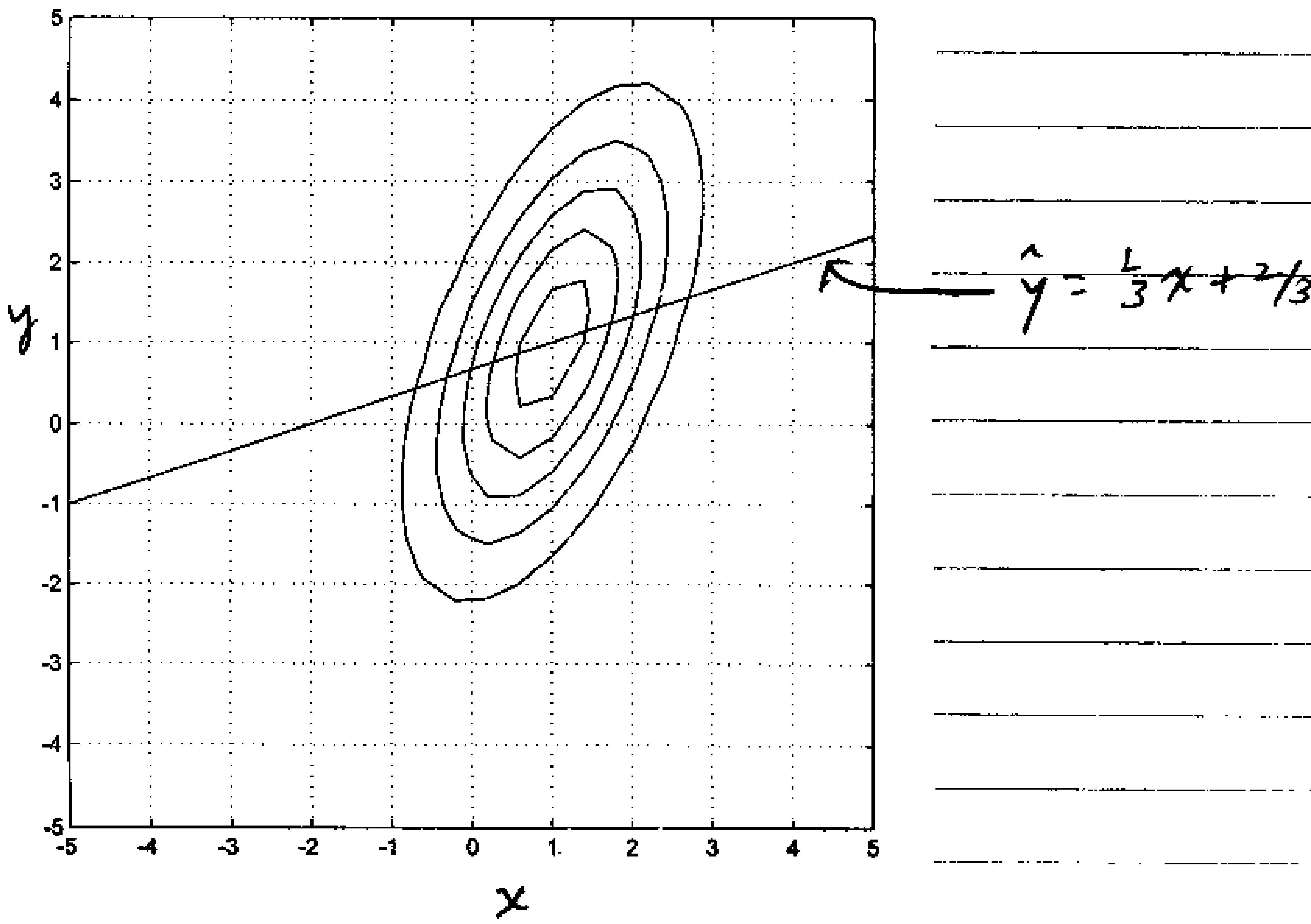
$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} \sim N(\underline{G}\,\underline{\mu}, \underline{G}\,\underline{C}\,\underline{G}^T)$$

$$\underline{G}\,\underline{\mu} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\underline{G}\,\underline{C}\,\underline{G}^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}}$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\right)$$



```
% probprob12_50
%
clear all
C=[3 1;1 1];
xmax=5;
x=[-xmax:0.4:xmax]';y=x;
for i=1:length(x)
   for j=1:length(y)
      xx=[x(i)-1 y(j)-1]';
            Q=xx'*inv(C)*xx;
      pxy(i,j)=(1/(2*pi*sqrt(det(C))))*exp(-0.5*Q);
   end
   end
figure
colormap(gray(1))
[C,H]=contour(x,y,pxy);
axis('square')
grid
hold on
plot(x,x/3+2/3)
```

51) $Z = X + Y$

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(x)\, p_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x)^2} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\underbrace{(x^2+(z-x)^2)}_{\varphi}} dx$$

$$\varphi = x^2 + z^2 - 2xz + x^2$$

$$= 2(x^2 - xz + \tfrac{1}{4}z^2) - \tfrac{1}{2}z^2 + z^2$$

$$= 2(x - \tfrac{1}{2}z)^2 + \tfrac{1}{2}z^2$$

$$p_Z(z) = \frac{1}{2\pi} e^{-1/4 z^2} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2\cdot 1/2}(x-\frac{1}{2}z)^2} dx}_{\sqrt{2\pi(\frac{1}{2})}}$$

$$= \frac{1}{\sqrt{2\pi\cdot 2}} e^{-\frac{1}{2\cdot 2}z^2}$$

$$\sim N(0, 2)$$

52) $Z = X + Y \sim N(0, 4)$ $\quad$ X, Y are independent (from Example 12.15)

$$P(Z > 2) = Q\left(\frac{2-0}{2}\right) = Q(1) = 0.1587$$

53) $Z = X + Y$

$$\phi_Z(\omega) = \phi_X(\omega)\,\phi_Y(\omega)$$

$$= \frac{1}{(1 - j\omega/\lambda)^{\alpha_X}} \; \frac{1}{(1 - j\omega/\lambda)^{\alpha_Y}}$$

$$= \frac{1}{(1 - j\omega/\lambda)^{\alpha_X + \alpha_Y}}$$

$$\Rightarrow \quad Z \sim \Gamma(\alpha_X + \alpha_Y, \lambda)$$

54) $W = \sqrt{-2 \ln X} \cos 2\pi Y$

$Z = \sqrt{-2 \ln X} \sin 2\pi Y$

$$w^2 + z^2 = -2\ln x \Rightarrow x = e^{-\frac{1}{2}(w^2+z^2)}$$

$$\frac{z}{w} = \tan 2\pi y \Rightarrow y = \frac{1}{2\pi} \arctan z/w$$

$$\frac{\partial(x, y)}{\partial(w, z)} = \begin{bmatrix} -w\,e^{-\frac{1}{2}(w^2+z^2)} & -z\,e^{-\frac{1}{2}(w^2+z^2)} \\ \frac{1}{2\pi}\,\frac{-z/w^2}{1+(z/w)^2} & \frac{1}{2\pi}\,\frac{1/w}{1+(z/w)^2} \end{bmatrix}$$

$$\det \frac{\partial(x, y)}{\partial(w, z)} = -\frac{1}{2\pi}\,\frac{e^{-\frac{1}{2}(w^2+z^2)}}{1+(z/w)^2} - \frac{1}{2\pi}\,\frac{z^2/w^2}{1+(z/w)^2} \cdot e^{-\frac{1}{2}(w^2+z^2)}$$

$$= -\frac{1}{2\pi}\,e^{-\frac{1}{2}(w^2+z^2)}\Bigg[\underbrace{\frac{1}{1+(z/w)^2} + \frac{(z/w)^2}{1+(z/w)^2}}_{=1}\Bigg]$$

$$\left| \det \frac{\partial(x,y)}{\partial(w,z)} \right| = \frac{1}{2\pi} e^{-\frac{1}{2}(w^2+z^2)}$$

$$p_{W,Z}(w,z) = p_{X,Y}(g^{-1}(w,z), h^{-1}(w,z)) \left| \det \frac{\partial(x,y)}{\partial(w,z)} \right|$$

But $X \sim U(0,1)$, $Y \sim U(0,1)$ and independent

$\Rightarrow \quad p_{X,Y} = 1$

$$\Rightarrow \quad p_{W,Z}(w,z) = \frac{1}{2\pi} e^{-\frac{1}{2}(w^2+z^2)}$$

or $W \sim N(0,1)$, $Z \sim N(0,1)$ and independent

55) $\underline{V}^T \underline{C}\, \underline{V} = \underline{\Lambda}$ where $\underline{V}^T = \underline{V}^{-1}$ since $\underline{V}$ is orthogonal matrix

$$C = (\underline{V}^T)^{-1} \underline{\Lambda}\, V^{-1}$$
$$= \underline{V}\, \underline{\Lambda}\, \underline{V}^T$$

Let $\underline{\Lambda} = \sqrt{\underline{\Lambda}}\sqrt{\underline{\Lambda}}$ $\quad \underline{\Lambda}$ is diagonal

$$\Rightarrow \quad \underline{C} = \underline{V}\sqrt{\underline{\Lambda}}\sqrt{\underline{\Lambda}}\, V^T = \underbrace{V\sqrt{\underline{\Lambda}}}_{\underline{G}}\,(\underline{V}\sqrt{\underline{\Lambda}})^T$$

From Example 9.4

$$\underline{V} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\underline{\Lambda} = \begin{bmatrix} 20 & 0 \\ 0 & 32 \end{bmatrix}$$

$$\sqrt{\underline{\Lambda}} = \begin{bmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{32} \end{bmatrix}$$

$$\underline{G} = \underline{V}\sqrt{\underline{\Lambda}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{pmatrix} \sqrt{20} & 0 \\ 0 & \sqrt{32} \end{pmatrix}$$

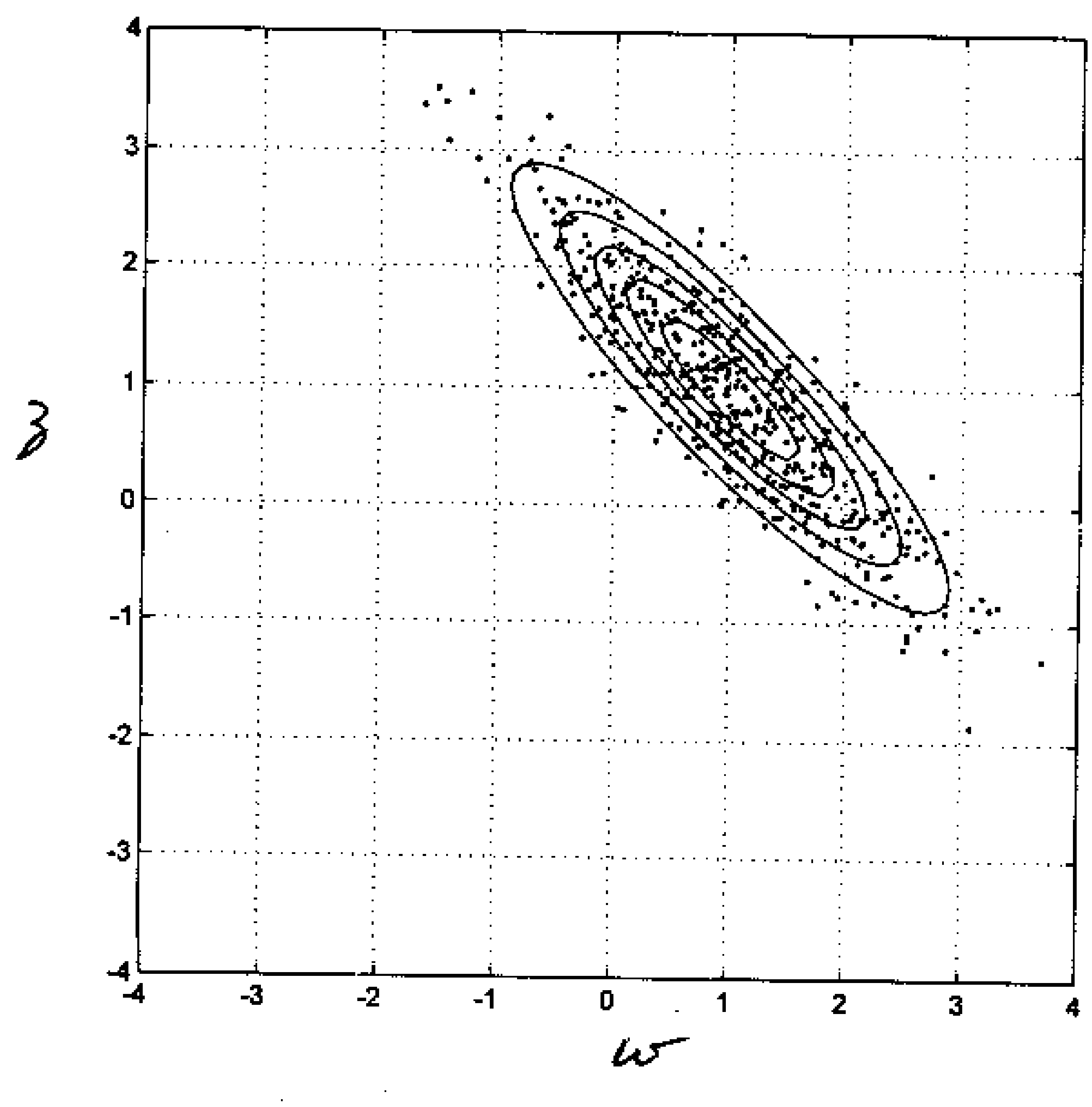
$$= \boxed{\begin{matrix} \sqrt{10} & 4 \\ -\sqrt{10} & 4 \end{matrix}}$$

$$\underline{G}\,\underline{G}^T = \begin{bmatrix} \sqrt{10} & 4 \\ -\sqrt{10} & 4 \end{bmatrix} \begin{bmatrix} \sqrt{10} & -\sqrt{10} \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 26 & 6 \\ 6 & 26 \end{bmatrix} = \underline{C}$$

56) Using (12.53) with $\mu_W = \mu_Z = 1$, $\rho = -0.9$, $\sigma_W^2 = \sigma_Z^2 = 1$

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.9 & \sqrt{0.19} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $\left.\begin{matrix} X \sim N(0,1) \\ Y \sim N(0,1) \end{matrix}\right\}$ independent



```
% probprob12_56.m
%
clear all
figure
xx=[-4:0.1:4]';yy=xx;
C=[1 -0.9;-0.9 1];
for i=1:length(xx)
   for j=1:length(yy)
      xv=[xx(i) yy(j)]';
            Q=(xv-1)'*inv(C)*(xv-1);
      pxy(i,j)=(1/(2*pi*sqrt(det(C))))*exp(-0.5*Q);
   end
   end
colormap(gray(1))
contour(xx,yy,pxy,5);
axis('square')
axis([-4 4 -4 4])
hold on
randn('state',0)
G=[1 0;-0.9 sqrt(1-0.9^2)];
M=500;
for m=1:M
   x=randn(1,1);y=randn(1,1);
   wz=G*[x y]'+[1 1]';
   WZ(:,m)=wz;
   plot(wz(1),wz(2),'.')
end
grid
hold off
```

Chapter 13

1) Theoretical prob $\simeq 0.25$ This is only approximate since the area in Figure 13.16 is not rectangular. Simulation results are $\hat{p} = 0.2238$

```
% probprob13_1.m
%
clear all
rand('state',0)
M=10000;
for m=1:M
   r=2;
   while r>1
      xx=2*(rand(1,1)-0.5);yy=2*(rand(1,1)-0.5);
      r=sqrt(xx^2+yy^2);
   end
   x(m,1)=xx;y(m,1)=yy;
end
Deltax2=0.2;
xx=[];yy=[];
   for i=1:length(x)
      if abs(x(i))<Deltax2
         xx=[xx;x(i)];
         yy=[yy;y(i)];

                end
   end
count=0;
 for i=1:length(xx)
   if sqrt(xx(i)^2+yy(i)^2)<=0.25
      count=count+1;
          end
       end
       prob=count/length(xx)
```

For $\Delta x/2 = 0.01$ and $M = 100000$ we get $\hat{p} = 0.2671$, which is closer to 0.25.

2) $$\int_0^\infty c e^{-y/x} dy = \left. \frac{c e^{-y/x}}{-1/x} \right|_0^\infty = cx = 1$$

$$\Rightarrow \quad c = 1/x$$

$$p_{Y|X}(y|x) = \begin{cases} \frac{1}{x} e^{-y/x} & y \ge 0,\ x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Valid conditional PDF

3) Yes, choose $X = x$ and then generate a $N(\mu, 1)$ random variable with $\mu = x$

4) $p_X(x) = \int_0^x p_{X,Y}(x,y)\,dy \qquad x \ge 0$

$$= \int_0^x e^{-(x+y)}\,dy$$

$$= e^{-x}\left[-e^{-y}\Big|_0^x\right] = e^{-x}(1-e^{-x})$$

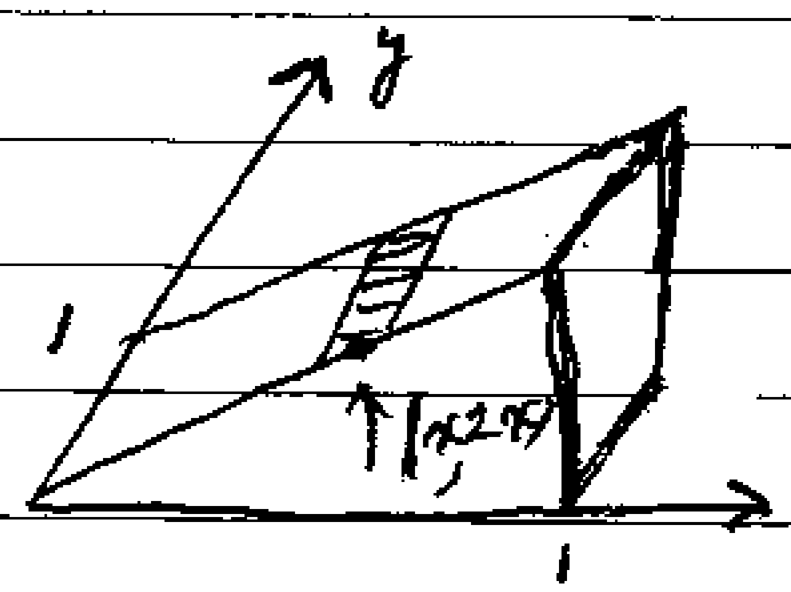
$$= e^{-x} - e^{-2x}$$

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

$$= \frac{e^{-(x+y)}}{e^{-x} - e^{-2x}}$$

$$= \frac{e^{-y}}{1-e^{-x}} \qquad 0 \le y \le x, \qquad x \ge 0$$

5)



$$p_{Y|X}(y|x) = \frac{2x}{2x} = 1$$

for $0 < y < 1$

and $0 < x < 1$

Since $p_{X,Y}$ is constant for a fixed x, $p_{Y|X}$ should be a uniform PDF over its range, which is $0<y<1$. Hence,

$$p_{Y|X}(y|x) = 1 \qquad 0<y<1 \quad , \quad 0<x<1$$

6) $$P[a \le Y \le b \mid x-\Delta x/2 \le X \le x-\Delta x/2] =$$

$$\frac{P[x-\frac{\Delta x}{2} \le X \le x+\Delta x/2 \, , \, a \le Y \le b]}{P[x-\frac{\Delta x}{2} \le X \le x+\Delta x/2]} =$$

$$= \frac{\int_a^b \int_{x-\Delta x/2}^{x+\Delta x/2} p_{X,Y}(x,y)\,dx\,dy}{\int_{x-\Delta x/2}^{x+\Delta x/2} p_X(x)\,dx}$$

$$= \int_a^b \frac{\int_{x-\Delta x/2}^{x+\Delta x/2} p_{X,Y}(x,y)\,dx}{\int_{x-\Delta x/2}^{x+\Delta x/2} p_X(x)\,dx}\,dy$$

As $\Delta x \to 0$

$$= \int_a^b \frac{p_{X,Y}(x,y)\,\Delta x}{p_X(x)\,\Delta x}\,dy$$

$$= \int_a^b \frac{p_{X,Y}(x,y)}{p_X(x)}\,dy = \int_a^b p_{Y|X}(y|x)\,dy$$

7) $$P[Y > \tfrac{1}{2} \mid X=0] = \int_{\frac{1}{2}}^{\infty} p_{Y|X}(y|0)\,dy$$

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$$

$$= \int_0^1 2x\,dy = 2x$$

$$\Rightarrow p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)} = \frac{2x}{2x} = 1$$

$$0 < y < 1, \quad 0 < x < 1$$

$$P(Y > \tfrac{1}{2} \mid X = 0) = \int_{1/2}^{1} 1 \cdot dy = \tfrac{1}{2}$$

8) $p_{X,Y}(x,y) = p_{Y|X}(y|x)\,p_X(x)$

$$= \left(\frac{1}{x}\right) \cdot 1 = 1/x \qquad \begin{matrix} 0 < y < x \\ 0 < x < 1 \end{matrix}$$

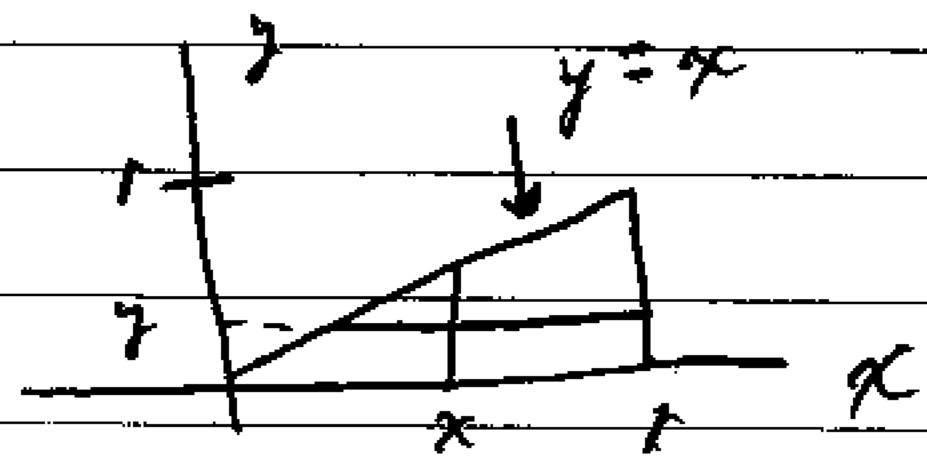
$$p_Y(y) = \int_y^1 1/x\,dx = \ln x \Big|_y^1 =$$

$$= 0 - \ln y = \ln 1/y \qquad 0 < y < 1$$

9) From (13.5) $Y|(X=x) \sim N(\rho x, 1-\rho^2)$
Also by interchanging X and Y
$X|(Y=y) \sim N(\rho y, 1-\rho^2)$

Have same functional form but will not be true in general since
$p_{X|Y} = p_{Y|X}\, p_X / p_Y$. p_X and p_Y will not have same functional form in general

10)



$$p_{X,Y}(x,y) = 2 \qquad \begin{matrix} 0 < x < 1 \\ 0 < y < x \end{matrix}$$

$$p_X(x) = \int p_{X,Y}(x,y)\,dy$$

$$= \int_0^x 2\,dy = 2y\Big|_0^x = 2x \qquad 0<x<1$$

$$p_Y(y) = \int p_{X,Y}(x,y)\,dx$$

$$= \int_y^1 2\,dx = 2(1-y) \qquad 0<y<1$$

$$p_{Y|X}(y|x) = \frac{2}{2x} = \frac{1}{x} \qquad 0<y<x,\ 0<x<1$$

$$p_{X|Y}(x|y) = \frac{2}{2(1-y)} = \frac{1}{1-y} \qquad y<x<1,\ 0<y<1$$

11) 1) $p_{Y|X} = \dfrac{p_{X,Y}}{p_X} = \dfrac{p_{X,Y}}{\int p_{X,Y}\,dy}$

Same for $p_{X|Y}$

2) $p_{X|Y} = \dfrac{p_{X,Y}}{p_Y} = \dfrac{p_{Y|X}\,p_X}{p_Y}$

3) $p_{Y|X} = \dfrac{p_{X,Y}}{p_X} = \dfrac{p_{X|Y}\,p_Y}{p_X}$

$$= \frac{p_{X|Y}\,p_Y}{\int p_{X,Y}\,dy} = \frac{p_{X|Y}\,p_Y}{\int p_{X|Y}\,p_Y\,dy}$$

4) from definitions

5) $p_Y = \int p_{X,Y}\,dx = \int p_{Y|X}\,p_X\,dx$

12) $Z \mid (X = x) = \frac{Y}{X} \Big| (X = x) = \frac{Y}{x} \Big| (X = x)$

But $Y/x \sim N(0, 1/x^2)$ since $Y \sim N(0,1)$

$$p_Z(z) = \int_{-\infty}^{\infty} p_{Z|X}(z|x)\, p_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi/x^2}}\, e^{-\frac{1}{2 \cdot 1/x^2} z^2} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |x|\, e^{-\frac{1}{2}x^2(z^2+1)}\, dx$$

$$= \frac{1}{2\pi}\, 2 \underbrace{\int_0^{\infty} x\, e^{-\frac{1}{2}x^2(z^2+1)}\, dx}_{\left.\frac{e^{-\frac{1}{2}x^2(z^2+1)}}{-(z^2+1)}\right|_0^{\infty}}$$

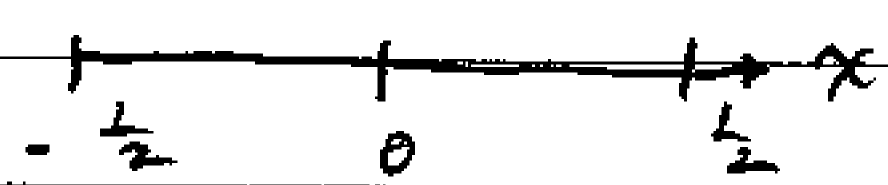
$$= \frac{1}{\pi}\, \frac{1}{z^2+1} \qquad -\infty < z < \infty$$

13) $Z = X + Y \Rightarrow Z \mid (X = x) = x + Y$

$$p_{Z|X}(z|x) = p_{Y|X}(z-x) = p_Y(z-x)$$

$$p_Z(z) = \int_{-\infty}^{\infty} p_{Z|X}(z|x)\, p_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} p_Y(z-x)\, p_X(x)\, dx$$

14) 

$P[\text{player 2 wins}] = P[\,|X_2| < |X_1|\,]$

$= P[\,|X_2| - |X_1| < 0\,]$

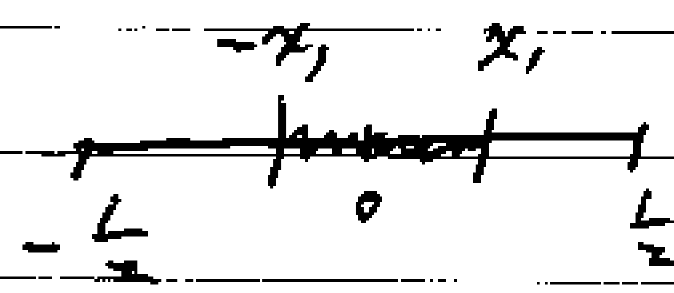
$$= \int P[\,|X_2| - |X_1| < 0 \mid X_1 = x_1)\, p_{X_1}(x_1)\, dx_1$$

$$= \int P(|X_2| < |x_1| \mid X_1 = x_1)\, p_{X_1}(x_1)\, dx_1$$

$$= \int P[\,|X_2| < |x_1|)\, \underbrace{p_{X_1}(x_1)}_{=1 \text{ for } |x_1| < \frac{L}{2}}\, dx_1$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} P(|X_2| < |x_1|)\, dx_1$$

$$= \int_{-\frac{L}{2}}^{\frac{L}{2}} 2|x_1|\, dx_1$$



$$= 2\int_0^{\frac{L}{2}} 2x_1\, dx_1 = 2\, x_1^2 \Big|_0^{\frac{L}{2}}$$

$$= \frac{L}{2}$$

15) From (13.4)

$$P[Y \in A \mid X = x) = \int_A p_{Y|X}(y|x)\, dy$$

$$\int_{-\infty}^{\infty} P[Y \in A \mid X = x)\, p_X(x)\, dx =$$

$$\int_{-\infty}^{\infty} \int_A p_{Y|X}(y|x)\, dy\, p_X(x)\, dx$$

$$= \int_A \int_{-\infty}^{\infty} p_{Y|X}(y|x)\, p_X(x)\, dx\, dy$$

$$= \int_A \underbrace{\int_{-\infty}^{\infty} p_{X,Y}(x,y)\, dx}_{p_Y(y)}\, dy$$

$$= P[Y \in A]$$

16) $P[V > 10 \mid R = 11) = \int_{10}^{\infty} p_{V|R}(v|r)\,dv$

But $V = iR + E = R + E$

$V|(R=r) = R+E)|(R=r) = E + r \,|(R=r)$

$= E + 11 \,|\,(R=11) = E + 11$

$\sim N(11, 1)$ $\uparrow$ <u>if</u> R and E are independent

$$P[V > 10 \mid R = 11) = \int_{10}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(v-11)^2}\,dv$$

$$= Q(10-11) = Q(-1)$$

17) $$\frac{dJ(\hat{y}(x))}{d\hat{y}} = \int_{-\infty}^{\infty} 2(y - \hat{y}(x))\,p_{Y|X}(y|x)\,dy = 0$$

$$\int_{-\infty}^{\infty} y\,p_{Y|X}(y|x)\,dy = \int_{-\infty}^{\infty} \hat{y}(x)\,p_{Y|X}(y|x)\,dy$$

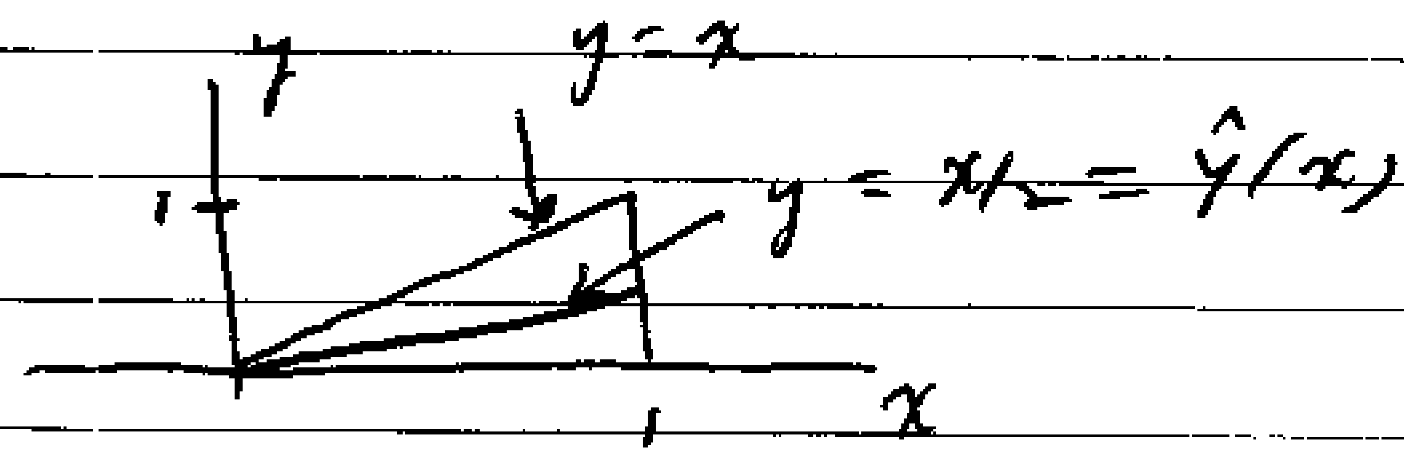
$$= \hat{y}(x) \underbrace{\int_{-\infty}^{\infty} p_{Y|X}(y|x)\,dy}_{=1}$$

$$\hat{y}(x) = E_{Y|X}[Y|x]$$

18) $p_{Y|X}(y|x) = 1/x \qquad 0<y<x,\ 0<x<1$

$$E_{Y|X}[Y|x] = \int_0^x y\,\frac{1}{x}\,dy = \frac{1}{x}\,y^2/2\Big|_0^x$$

$$= x/2$$



19) $E_x[E_{y|x}[y|x]] =$

$$\int E_{y|x}[y|x]\, p_x(x)\, dx$$

$$= \iint y\, p_{y|x}(y|x)\, dy\, p_x(x)\, dx$$

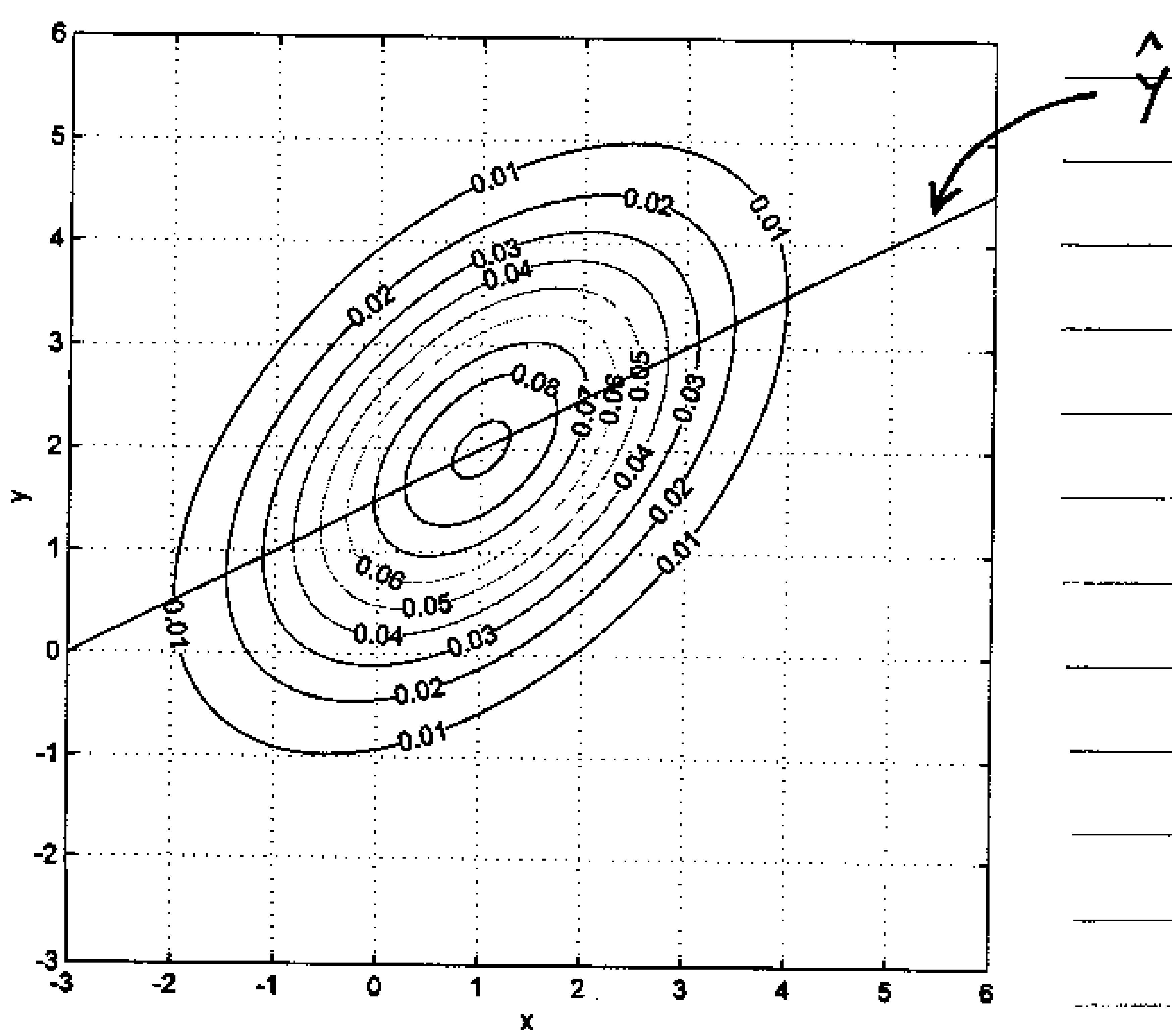
$$= \iint y\, p_{x,y}(x,y)\, dx\, dy$$

$$= \int y \underbrace{\int p_{x,y}(x,y)\, dx}_{p_y(y)}\, dy$$

$$= E_y[y]$$

20) $$\hat{y} = E_y[y] + \frac{cov(x,y)}{var(x)}(x - E_x[x])$$

$$= 2 + \frac{1}{2}(x-1) = \frac{1}{2}x + 3/2$$



For this joint PDF $\hat{Y}(x)$ is the most probable value of y when x is observed or the maximum over y of $p_{Y|X}(y|x)$. (not true in general).

```
% probprob13_20
%
clear all
C=[2 1;1 2];xmax=5;
x=[-3:0.1:6]';y=x;
yhat=2+(1/2)*(x-1);
for i=1:length(x)
   for j=1:length(y)
      xx=[x(i)-1 y(j)-2]';
         Q=xx'*inv(C)*xx;
      pxy(i,j)=(1/(2*pi*sqrt(det(C))))*exp(-0.5*Q);
   end
   end
figure
[C,H]=contour(x,y,pxy');
axis('square')
xlabel('x')
ylabel('y')
grid
clabel(C,H)
hold on
plot(x,yhat)
hold off
```

21) $E_X[X] = E_{\sigma^2}[E_{X|\sigma^2}[X^2|\sigma^2]]$ since $E_X[X]=0$

$= E_{\sigma^2}[\mathrm{var}(X)] = E_{\sigma^2}[\sigma^2]$

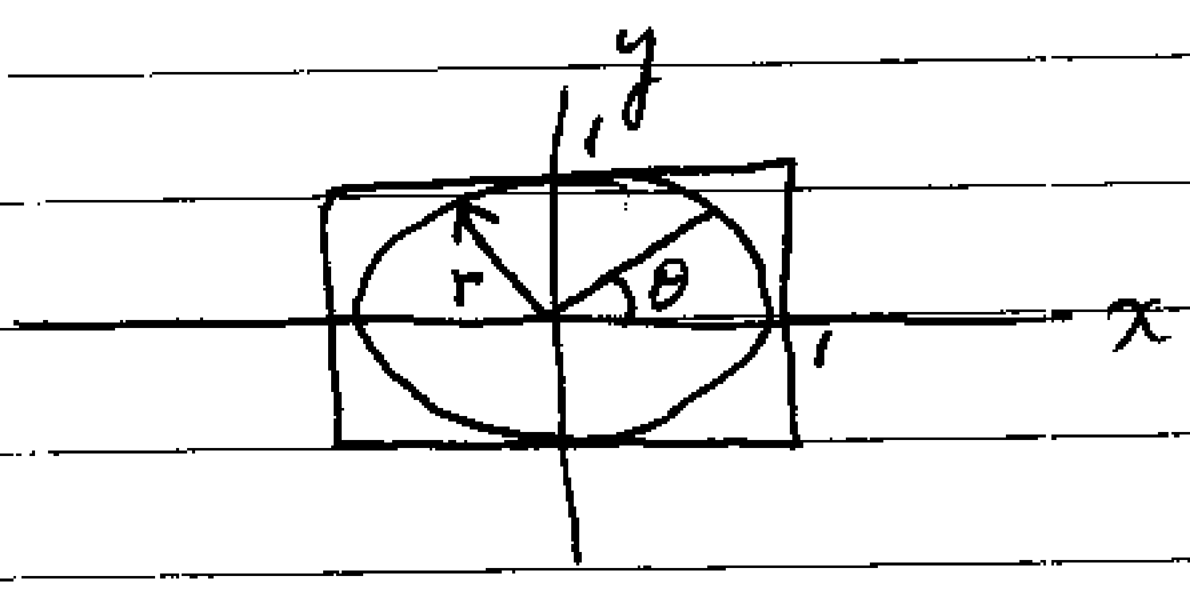
$= \int_0^1 \sigma^2 \, d\sigma^2 = \frac{1}{2}(\sigma^2)^2 \Big|_0^1 = \frac{1}{2}$

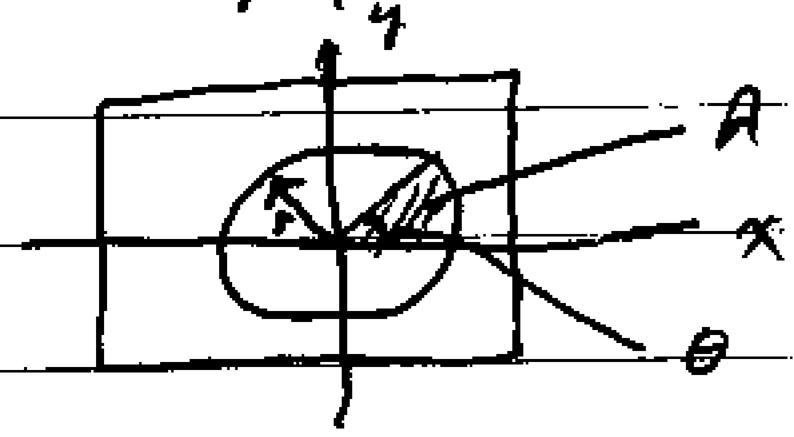
22) $E_{Y|X}[Y|x_0] = \rho x_0 = 0.9(1) = 0.9$

$\hat{E}_{Y|X}[Y|x_0] = 0.9165$

```
% probprob13_22.m
%
clear all
randn('state',0)
rho=0.9;
M=10000;
for i=1:M
   u=randn(1,1);v=randn(1,1);
   xx(i,1)=u;
   yy(i,1)=rho*u+sqrt(1-rho^2)*v;
end
deltax2=0.1;
x=1;
m=0;
for i=1:M
   if abs(xx(i)-x)<=deltax2
      m=m+1;
      xkeep(m,1)=xx(i);ykeep(m,1)=yy(i);
   end
end
mean(ykeep)
```

23.1 $\quad P[R \le r, \Theta \le \theta \mid R \le 1] = \dfrac{P[R \le r, \Theta \le \theta]}{P[R \le 1]}$





$P[R \le 1] = P[(x,y) \text{ in circle of radius } 1]$

$= \dfrac{\pi r^2}{4} = \pi/4$

$P[R \le r, \Theta \le \theta] = P[A] = \dfrac{(\theta/2\pi)(\pi r^2)}{4}$

$= \dfrac{\theta r^2}{8} \qquad r \le 1$

$P[R \le r, \Theta \le \theta \mid R \le 1] = \dfrac{\theta r^2/8}{\pi/4} = \dfrac{\theta r^2}{2\pi}$

$0 \le r \le 1$

$0 \le \theta < 2\pi$

$$p_{R,\Theta}(r,\theta) = \frac{\partial^2 P(R \le r, \Theta \le \theta \mid R \le 1)}{\partial r \partial \theta}$$

$$= \frac{\partial}{\partial r}\left[\frac{\partial}{\partial \theta}\frac{\theta r^2}{2\pi}\right) = \frac{\partial}{\partial r}\frac{r^2}{2\pi}$$

$$= r/\pi$$

Let $x = r\cos\theta$ over the unit circle $\}$ since $0 \le r \le 1$,
$y = r\sin\theta$ unit circle $\}$ $0 \le \theta < 2\pi$

$$p_{X,Y}(x,y) = p_{R,\Theta}(r,\theta)\left|\det\frac{\partial(r,\theta)}{\partial(x,y)}\right|$$

$$= \frac{p_{R,\Theta}(g^{-1}(x,y), h^{-1}(x,y))}{\left|\det\frac{\partial(x,y)}{\partial(r,\theta)}\right|}$$

$$= \frac{r/\pi}{r} = \frac{1}{\pi} \quad \text{over the unit circle}$$

$$0 \quad \text{otherwise}$$

24) $$p_{Y|X}(y|x) = \frac{\partial F_{Y|X}(y|x)}{\partial y}$$

$$= \frac{\partial}{\partial y}\left[1 - e^{e^{\frac{x-m}{\ell}}(1-e^{y/\ell})}\right) \quad y \ge 0$$

$$= -e^{e^{\frac{x-m}{\ell}}(1-e^{y/\ell})} e^{\frac{x-m}{\ell}}(-e^{y/\ell})\frac{1}{\ell}$$

$$= \frac{1}{\ell} e^{e^{\frac{x-m}{\ell}}(1-e^{y/\ell}) + \frac{x+y-m}{\ell}}$$

Let $x = 50$ and $\pi = 75$

$E_{Z|X}[Z \mid X = 50] = 77.45$

$E_{Z|X}[Z \mid X = 75] = 84.57$

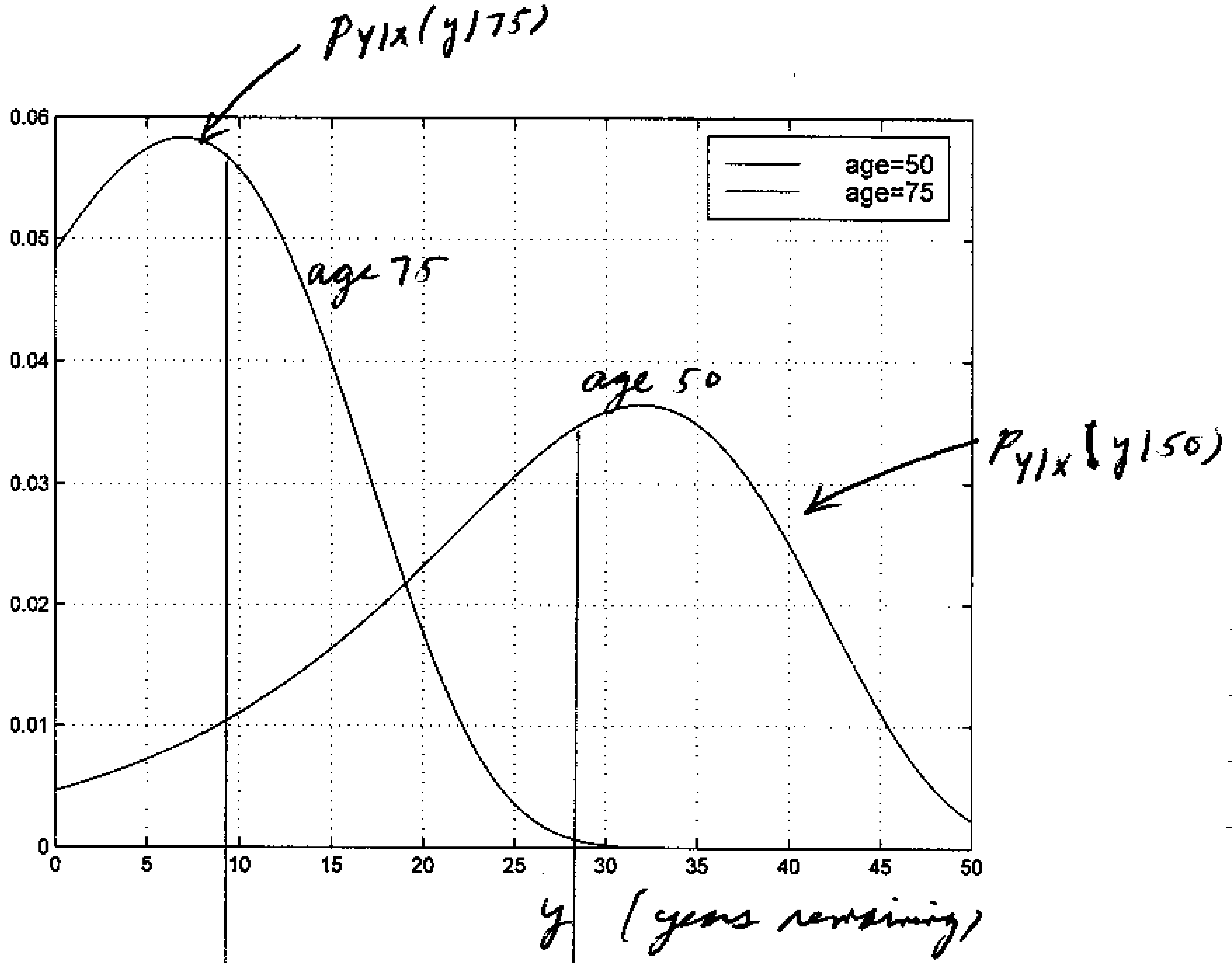
where $Z = X + Y$ and

$$E_{Z|X}[Z \mid x] = E_{X,Y|X}[X + Y \mid x]$$

$$= E_{Y|X}[x + Y \mid x]$$

$$= \underbrace{E_{Y|X}[Y \mid x]}_{\text{see plot}} + x$$

```
% probprob13_24.m
%
clear all
randn('state',0)
x=50; m=81.95;l=10.6;
y=[0:0.1:50]';
e1=exp((x-m)/l);e2=(x+y-m)/l;
p1=(1/l)*exp(  e1*(1-exp(y/l))+e2  );
x=75;
e1=exp((x-m)/l);e2=(x+y-m)/l;
p2=(1/l)*exp(  e1*(1-exp(y/l))+e2  );
plot(y,p1,y,p2)
grid
legend('age=50','age=75')
E1=sum(y.*p1)*0.1+50
E2=sum(y.*p2)*0.1+75
```



$E_{Y|X}[y|75] = 9.57$ $E_{Y|X}[y|50] = 27.45$

25) $F_{Y|X}(y|x) = 1 - e^{e^{\frac{x-m}{2}}(1-e^{y/\alpha})}$ $y \geq 0$, $x \geq 0$

$F_{Y|X}(0|x) = 1 - e^{e^{\frac{x-m}{2}}(1-1)}$

$= 1 - e^{0} = 0$

$F_{Y|X}(\infty|x) = 1 - e^{\underbrace{e^{\frac{x-m}{2}}}_{>0}\underbrace{(1-e^{\infty})}_{-\infty}}$

$= 1 - e^{-\infty} = 1$

$F_{Y|X}(y|x)$ is monotonically increasing with $0 \le y < \infty$ since as y increases

$e^{y/\lambda}$ increases $\Rightarrow$ $1-e^{y/\lambda}$ is negative and becomes more negative.

$e^{\frac{x-m}{\lambda}} > 0 \Rightarrow e^{\frac{x-m}{\lambda}}(1-e^{y/\lambda})$ becomes more negative $\Rightarrow$

$F_{Y|X}(y|x)$ increases

Chapter 14

1) $$E_Y[Y] = E_{\underline{x}}[x_1 + x_2 + x_3)$$
$$= E_{x_1}[x_1) + E_{x_2}[x_2) + E_{x_3}[x_3)$$
$$= 1 + 2 + 3 = 6$$

$$\mathrm{var}(Y) = \sum_{i=1}^{3}\sum_{j=1}^{3} \mathrm{cov}(x_i, x_j) = \text{sum of all}$$

$$\text{elements in } \underline{C} = 11/2$$

2) $$P[x_1^2 + x_2^2 > R^2) = \iint_{\{(x_1,x_2):\, x_1^2+x_2^2>R^2\}} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x_1^2+x_2^2)} dx_1\, dx_2$$

$$= \int_R^{\infty}\int_0^{2\pi} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}r^2} r\, d\theta\, dr$$

$$= \int_R^{\infty} \frac{1}{\sigma^2} r e^{-\frac{1}{2\sigma^2}r^2} dr = -e^{-\frac{1}{2\sigma^2}r^2}\Big|_R^{\infty}$$

$$= e^{-\frac{1}{2}R^2/\sigma^2}$$

For $\sigma^2 = 1,\ R = 1 \Rightarrow P = e^{-\frac{1}{2}} = 0.6065$

From simulation $- \hat{P} = 0.6080$

```
% probprob14_2.m
%
clear all
randn('state',0)
x1=randn(1000,1);
x2=randn(1000,1);
y=x1.^2+x2.^2;
count=0;
for i=1:1000
   if y(i)>1
      count=count+1;
   end
end
Probest=count/1000
Probtrue=exp(-0.5)
```

3) $P[X_1^2 + X_2^2 + X_3^2 \leq R^2] =$

$$1 - 2Q(R) - \sqrt{2/\pi}\, R\, e^{-\frac{1}{2}R^2}$$

Let $R^2 = y \Rightarrow$

$$P[Y \leq y] = 1 - 2Q(\sqrt{y}) - \sqrt{2/\pi}\,\sqrt{y}\, e^{-\frac{1}{2}y}$$

$$p_Y(y) = -2\frac{d}{dy}Q(\sqrt{y}) - \sqrt{2/\pi}\,\frac{d}{dy}\left(\sqrt{y}\,e^{-\frac{1}{2}y}\right)$$

$$= -2\frac{d}{dy}\left(1 - \Phi(\sqrt{y})\right) - \sqrt{2/\pi}\,\frac{d}{dy}\left(\sqrt{y}\,e^{-\frac{1}{2}y}\right)$$

$$= +2\,\frac{1}{\sqrt{2\pi}}\,e^{-\frac{1}{2}y}\,\frac{1}{2\sqrt{y}} - \sqrt{2/\pi}\left[-\frac{1}{2}\sqrt{y}\,e^{-\frac{1}{2}y} + \frac{1}{2\sqrt{y}}\,e^{-\frac{1}{2}y}\right]$$

$$= \sqrt{2/\pi}\,\frac{1}{2}\sqrt{y}\,e^{-\frac{1}{2}y} = \underbrace{\frac{1}{2^{3/2}\Gamma(3/2)}\,y^{3/2-1}e^{-\frac{1}{2}y}}_{\chi^2_3 \text{ PDF}}$$

4) $P[X > \frac{1}{2}] = 0.05$

$$\int_{\frac{1}{2}}^{\infty} \lambda e^{-\lambda x}\,dx = 0.05$$

$$-e^{-\lambda x}\Big|_{\frac{1}{2}}^{\infty} = 0.05$$

$$e^{-\lambda/2} = 0.05$$

$$\Rightarrow \lambda = 2\ln 20 \approx 6$$

But $Y = X_1 + X_2 + X_3 \sim$ Erlang (see (10.13))
(assumes X_i's are independent)

$$p_Y(y) = \begin{cases} \frac{\lambda^3}{2} y^2 e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

$$P(Y > 3/2) = \int_{3/2}^{\infty} \frac{\lambda^3}{2} y^2 e^{-\lambda y}\, dy$$

But $\int y^2 e^{-\lambda y}\, dy = -e^{-\lambda y}\left[\frac{y^2}{\lambda} + \frac{2y}{\lambda^2} + \frac{2}{\lambda^3} \right)$

$$P(Y > 3/2) = \frac{\lambda^3}{2} e^{-3/2\lambda}\left[\frac{(3/2)^2}{\lambda} + \frac{2(3/2)}{\lambda^2} + \frac{2}{\lambda^3} \right)$$

$$= 108\, e^{-9}\left[\frac{9/4}{6} + \frac{3}{36} + \frac{2}{108} \right) = 0.0062$$

5)
$$x_1 = r\cos\theta\sin\phi$$
$$x_2 = r\sin\theta\sin\phi$$
$$x_3 = r\cos\phi$$

$$\det\left(\frac{\partial(x_1 x_2 x_3)}{\partial(r,\theta,\phi)}\right) = \det\begin{bmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{bmatrix}$$

$$= \cos\phi \begin{vmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} - r\sin\phi \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi \end{vmatrix}$$

$$= \cos\phi\,(-r^2\sin^2\theta\sin\phi\cos\phi - r^2\cos^2\theta\sin\phi\cos\phi)$$
$$- r\sin\phi\,(r\cos^2\theta\sin^2\phi \quad + r\sin^2\theta\sin^2\phi)$$

$$= -r^2\cos\phi\sin\phi\cos\phi - r\sin\phi\, r\sin^2\phi$$

$$= -r^2\sin\phi \Rightarrow \left|\det\left(\frac{\partial(x_1 x_2 x_3)}{\partial(r,\theta,\phi)}\right)\right| = r^2\sin\phi$$

6) $P\{X_1 > 150, X_2 > 150, X_3 > 150, X_4 > 150\}$

$= P\{X > 150\}^4$ (assume independence)

$= Q\left(\frac{150-150}{\sqrt{30}}\right)^4 = Q(0)^4 = \frac{1}{16}$

7) Let $\bar{p}(x,y;\rho)$ = standard bivariate Gaussian PDF

$$p_{X,Y}(x,y) = \tfrac{1}{2}\bar{p}(x,y;\rho) + \tfrac{1}{2}\bar{p}(x,y;-\rho)$$

$$p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} \tfrac{1}{2}\bar{p}(x,y;\rho) + \tfrac{1}{2}\bar{p}(x,y;-\rho)\,dy$$

$$= \tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} + \tfrac{1}{2}\tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Same for $p_Y(y)$.

$\mathrm{cov}(X,Y) = E_{X,Y}[XY]$ since $E_X[X] = E_Y[Y] = 0$
and $X \sim N(0,1)$, $Y \sim N(0,1)$

$$\mathrm{cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\left[\tfrac{1}{2}\bar{p}(x,y;\rho) + \tfrac{1}{2}\bar{p}(x,y;-\rho)\right] dx\,dy$$

$$= \tfrac{1}{2}\iint xy\,\bar{p}(x,y;\rho)\,dx\,dy$$

$$+ \tfrac{1}{2}\iint xy\,\bar{p}(x,y;-\rho)\,dx\,dy$$

$$= \tfrac{1}{2}\rho + \tfrac{1}{2}(-\rho) = 0$$

8) $\phi_{\underline{y}}(\underline{\omega}) = E_{\underline{x}}[e^{j\underline{\omega}^T \underline{G}\,\underline{x}}]$

$$= E_{\underline{x}}[e^{j(\underbrace{\underline{G}^T\underline{\omega}}_{\underline{\omega}'})^T\underline{x}}]$$

$$= e^{-\frac{1}{2}\underline{\omega}'^T\underline{C}\,\underline{\omega}'} = e^{-\frac{1}{2}\underline{\omega}^T\underbrace{\underline{G}\,\underline{C}\,\underline{G}^T}_{\underline{C}_y}\underline{\omega}}$$

$\Rightarrow \underline{y} \sim N(\underline{0}, \underline{G}\,\underline{C}\,\underline{G}^T)$

9) From Problem 14.8

$\underline{y} \sim N(\underline{0}, \underline{G}\,\underline{C}\,\underline{G}^T)$

Here $\underline{G} = [1\ 1\ 1] \Rightarrow \underline{y}$ is a scalar

$$\Rightarrow \underline{G}\,\underline{C}\,\underline{G}^T = [1\ 1\ 1]\begin{bmatrix}\sigma_1^2 & 0 & 0\\ 0 & \sigma_2^2 & 0\\ 0 & 0 & \sigma_3^2\end{bmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \sigma_1^2 + \sigma_2^2 + \sigma_3^2$$

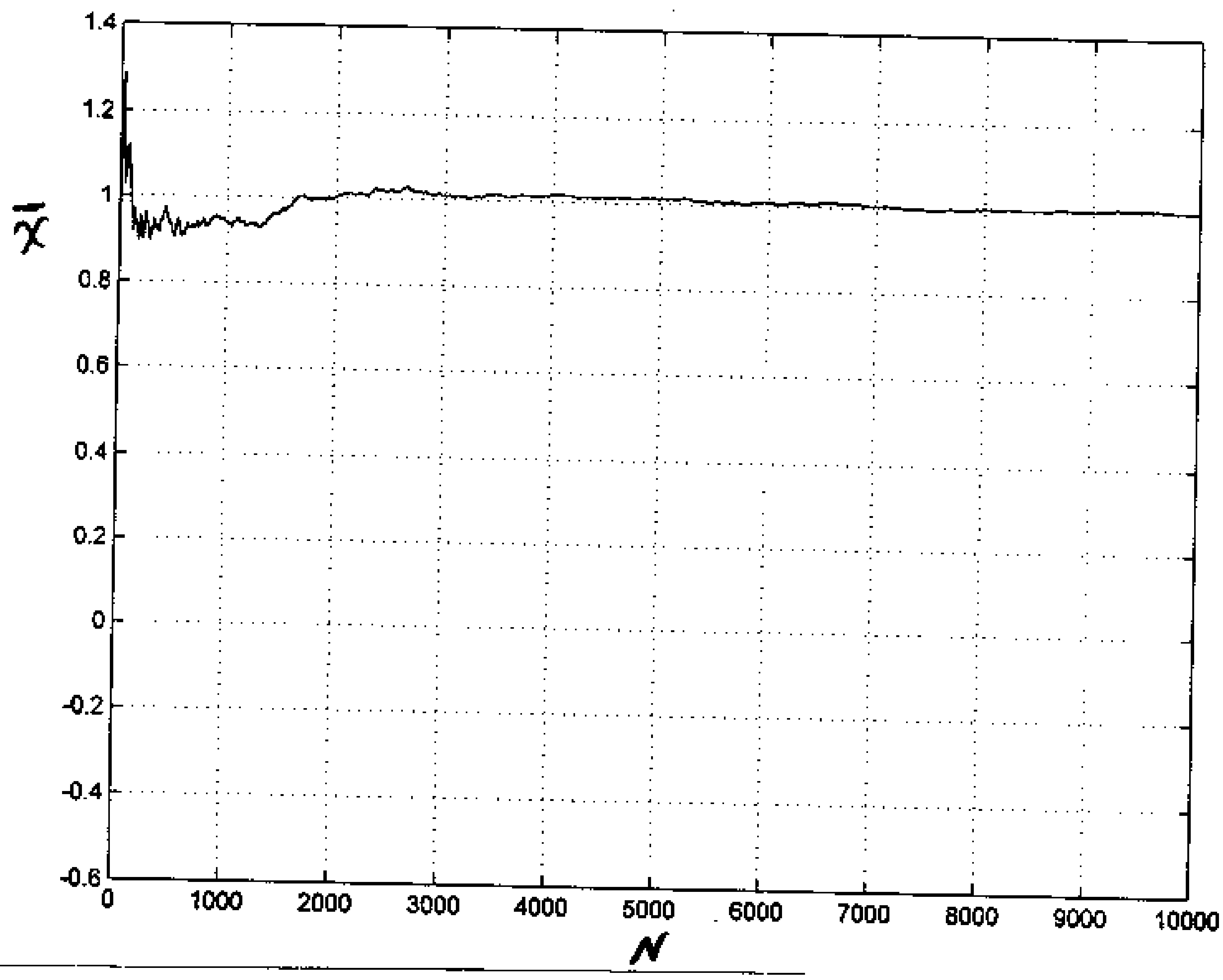
$y \sim N(0, \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

10) $\underline{a}^T\underline{C}_x\underline{a} = \sum_{i=1}^{N}\sum_{j=1}^{N} a_i \underbrace{[\underline{C}_x]_{ij}}_{=0 \text{ for } i \neq j} a_j$

$$= \sum_{i=1}^{N} a_i [\underline{C}_x]_{ii}\, a_i$$

$$= \sum_{i=1}^{N} a_i^2 \operatorname{var}(x_i)$$

11) As $N \to \infty$, $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \to 1 = E_x[x]$




```
% probprob14_11.m
%
clear all
randn('state',0)
x=1+sqrt(2)*randn(10000,1);
for N=1:10000
   xmean(N,1)=mean(x(1:N));
end
plot([1:10000]',xmean)
grid
```

(2) $E_{\underline{x}}[\bar{x}] = E_{\underline{x}}\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E_{\underline{x}}[x_i]$

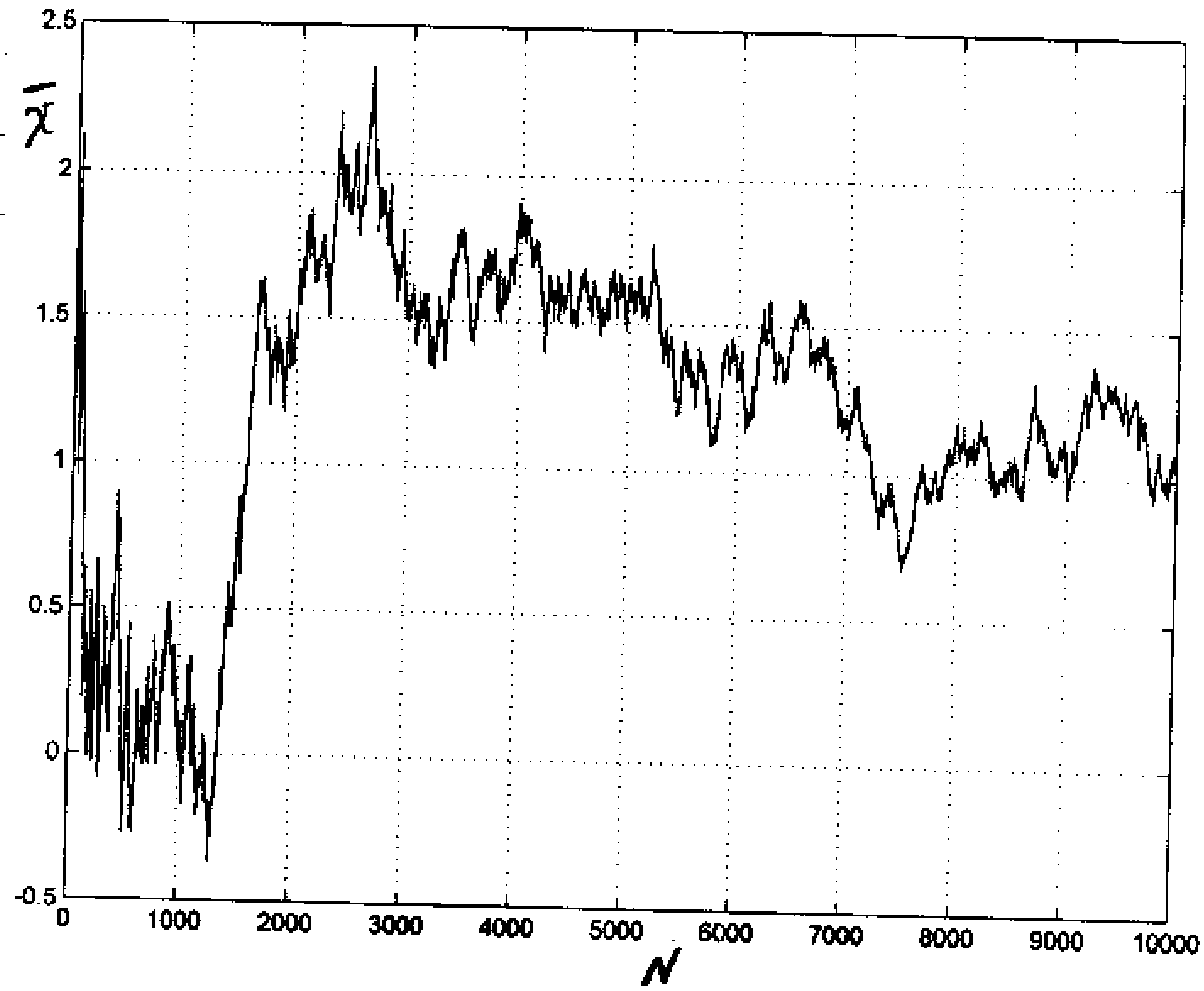
$$= \frac{1}{N}\sum_{i=1}^{N} E_{x_0}[x_i] = \mu$$

$$var(\bar{x}) = \frac{1}{N^2}\sum_{i=1}^{N} var(x_i) = \frac{1}{N^2}\sum_{i=1}^{N} i\sigma^2$$

$$= \frac{\sigma^2}{N^2}\frac{N}{2}(N+1) = \frac{\sigma^2}{2}\frac{N+1}{N} \to \sigma^2/2$$

as $N \to \infty$. $\bar{X}$ does not converge to μ

For $\mu = 1$, $\sigma^2 = 1$




```
% probprob14_12.m
%
clear all
randn('state',0)
for i=1:10000
   x(i,1)=1+sqrt(i)*randn(1,1);
end
for N=1:10000
   xmean(N,1)=mean(x(1:N));
end
plot([1:10000]',xmean)
grid
```

13) $\phi_{X_1,\ldots,X_N}(\omega_1,\ldots,\omega_N) = E_{X_1,\ldots,X_N}\left[e^{j(\omega_1 x_1+\ldots+\omega_N x_N)}\right]$

$$= \int \cdots \int e^{j(\omega_1 x_1 + \cdots + \omega_N x_N)} p_{x_1, \ldots, x_N}(x_1, \ldots, x_N) \, dx_1 \ldots dx_N$$

$$= \int \cdots \int e^{j\omega_1 x_1} \cdots e^{j\omega_N x_N} p_{x_1}(x_1) \cdots p_{x_N}(x_N) \, dx_1 \ldots dx_N$$

$$= \int e^{j\omega_1 x_1} p_{x_1}(x_1) \, dx_1 \cdots \int e^{j\omega_N x_N} p_{x_N}(x_N) \, dx_N$$

$$= \phi_{x_1}(\omega_1) \cdots \phi_{x_N}(\omega_N)$$

14) $\phi_{x_1, \ldots, x_N}(\omega_1, \ldots, \omega_N) = E_{x_1, \ldots, x_N}\left[e^{j(\omega_1 x_1 + \cdots + \omega_N x_N)}\right]$

$$\phi_{x_1, \ldots, x_N}(\omega_1, \omega_2, 0, \ldots, 0) = E_{x_1, \ldots, x_N}\left[e^{j(\omega_1 x_1 + \omega_2 x_2)}\right]$$

$$= E_{x_1, x_2}\left[e^{j(\omega_1 x_1 + \omega_2 x_2)}\right]$$

$$= \phi_{x_1, x_2}(\omega_1, \omega_2)$$

15) Let $\underline{x} = \underline{z} + \underline{\mu} \Rightarrow \underline{z} = \underline{x} - \underline{\mu} \sim N(\underline{0}, \underline{C})$

$$\phi_{\underline{x}}(\underline{\omega}) = E_{\underline{x}}\left[e^{j\underline{\omega}^T \underline{x}}\right] = E_{\underline{z}}\left[e^{j\underline{\omega}^T(\underline{z} + \underline{\mu})}\right]$$

$$= e^{j\underline{\omega}^T \underline{\mu}} \phi_{\underline{z}}(\underline{\omega}) = e^{j\underline{\omega}^T \underline{\mu}} e^{-\frac{1}{2}\underline{\omega}^T \underline{C} \underline{\omega}}$$

$$= e^{j\underline{\omega}^T \underline{\mu} - \frac{1}{2}\underline{\omega}^T \underline{C} \underline{\omega}}$$

$$\phi_{\underline{y}}(\underline{\omega}) = E_{\underline{y}}\left[e^{j\underline{\omega}^T \underline{y}}\right]$$

$$= E_{\underline{x}}\left[e^{j\underline{\omega}^T \underline{G}\underline{x}}\right]$$

$$= E_{\underline{x}}\left[e^{j(\underline{G}^T\underline{\omega})^T\underline{x}}\right]$$

$$= \phi_{\underline{x}}(\underline{G}^T\underline{\omega}) = e^{j(\underline{G}^T\underline{\omega})^T\underline{\mu} - \frac{1}{2}(\underline{G}^T\underline{\omega})^T\underline{C}(\underline{G}^T\underline{\omega})}$$

$$= e^{j\underline{\omega}^T\underbrace{\underline{G}\underline{\mu}}_{\underline{\mu}'} - \frac{1}{2}\underline{\omega}^T\underbrace{\underline{G}\underline{C}\underline{G}^T}_{\underline{C}'}\underline{\omega}}$$

$$\Rightarrow \underline{y} \sim N(\underline{\mu}', \underline{C}') = N(\underline{G}\underline{\mu}, \underline{G}\underline{C}\underline{G}^T)$$

16) From Problem 14.15 with $\underline{G} = \underline{e}_i^T$

$$y \sim N(\underline{e}_i^T\underline{\mu}, \underline{e}_i^T\underline{C}\underline{e}_i) = N(\mu_i, \sigma_i^2)$$

But $y = \underline{e}_i^T\underline{x} = x_i \Rightarrow x_i \sim N(\mu_i, \sigma_i^2)$

17) $\phi_{x_i^2}(\omega) = E_{x_i}\left[e^{j\omega x_i^2}\right]$

$$= \int_{-\infty}^{\infty} e^{j\omega x^2}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\,dx$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2)\underbrace{(1-2j\omega)}_{1/c}}\,dx$$

$$= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi c}}e^{-\frac{1}{2}x^2/c}\,dx\;\sqrt{c}$$

$$= \sqrt{c} = \frac{1}{\sqrt{1-2j\omega}}$$

$$\phi_y(\omega) = \left(\frac{1}{\sqrt{1-2j\omega}}\right)^N \quad \text{for } y = \sum_{i=1}^{N} x_i$$

$$= \frac{1}{(1-2j\omega)^{N/2}}$$

From Table 11.1

$$Y \sim \Gamma(\underbrace{N/2}_{\alpha}, \underbrace{\tfrac{1}{2}}_{\lambda}) = \chi^2_N \quad \text{(see Section 10.5.6)}$$

18) $\phi_{X_i}(\omega) = \dfrac{\lambda}{\lambda - j\omega}$ from Table 11.1

$$\phi_Y(\omega) = \left(\frac{\lambda}{\lambda - j\omega}\right)^N \qquad Y = \sum_{i=1}^{N} X_i$$

$$= \frac{\lambda^N}{(\lambda - j\omega)^N} = \frac{1}{(1 - j\omega/\lambda)^N}$$

$$\Rightarrow Y \sim \Gamma(\underbrace{N}_{\alpha}, \lambda) \qquad \text{from Table 11.1}$$

$\Rightarrow$ Erlang (see Section 10.5.6)

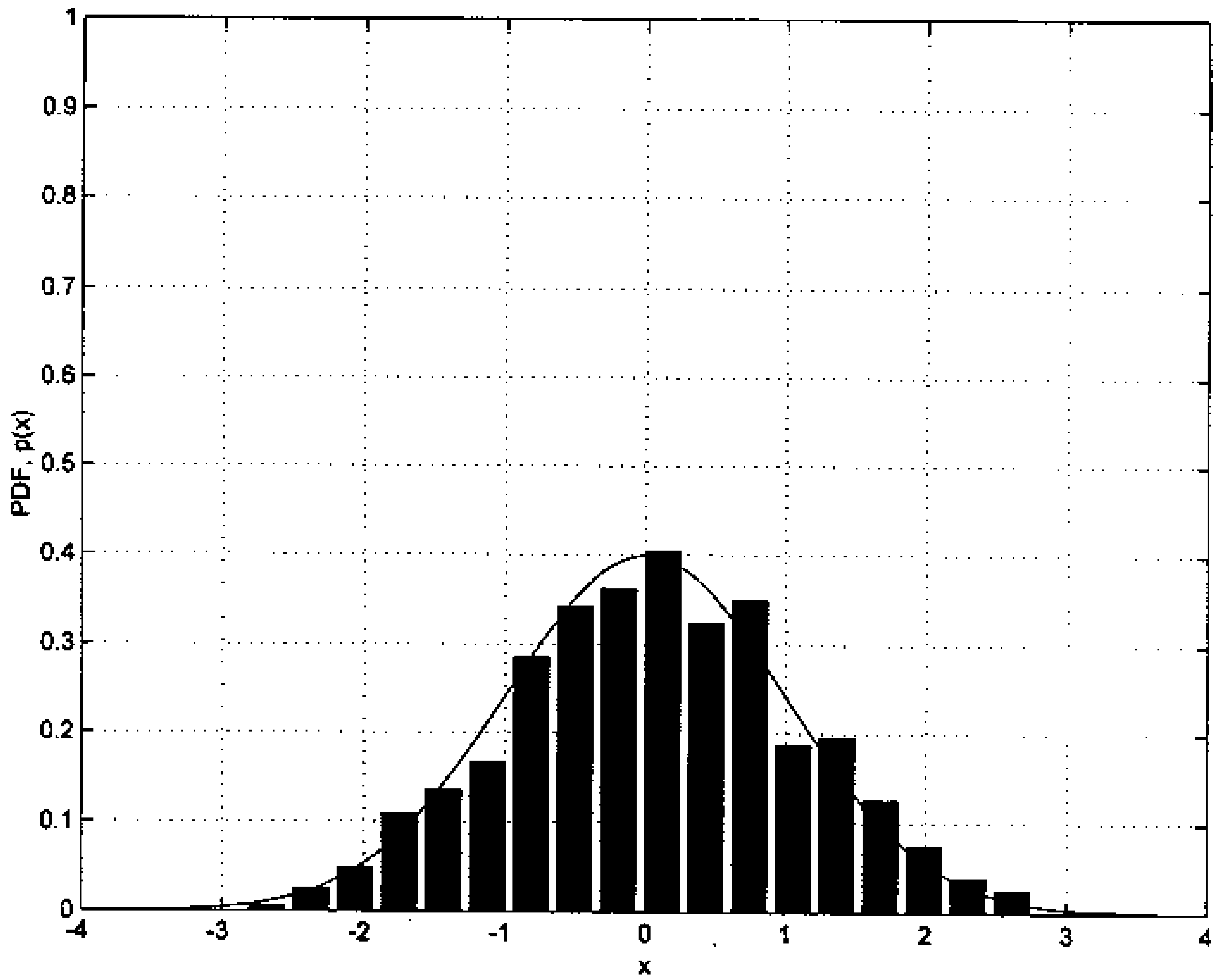
19) $$E[Y] = E\left[\sum_{i=1}^{12}\left(U_i - \tfrac{1}{2}\right)\right]$$

$$= \sum_{i=1}^{12}\underbrace{\left(E[U_i] - \tfrac{1}{2}\right)}_{=0} = 0$$

$$\operatorname{var}(Y) = \operatorname{var}\left(\sum_{i=1}^{12}\left(U_i - \tfrac{1}{2}\right)\right)$$

$$= \sum_{i=1}^{12} \operatorname{var}\left(U_i - \tfrac{1}{2}\right) = \sum_{i=1}^{12} \operatorname{var}(U_i)$$

$$= \sum_{i=1}^{12} \frac{1}{12} = 1$$




```
% probprobl4_19.m
%
clear all
rand('state',0)
for i=1:1000
   x(i,1)=sum(rand(12,1)-0.5);
end
pdf(x,1000,20,-4,4,1)
hold on
xx=[-4:0.01:4]';
pgauss=(1/sqrt(2*pi))*exp(-0.5*xx.^2);
plot(xx,pgauss)
```

20) If we average, $\bar{V} = \frac{1}{3}(V_1 + V_2 + V_3)$

$$E[\bar{V}] = \frac{1}{3}(100+100+100) = 100$$

$$\mathrm{var}(\bar{V}) = \frac{1}{9}(\mathrm{var}(V_1) + \mathrm{var}(V_2) + \mathrm{var}(V_3))$$

$$= \frac{1}{9}(1+10+5) = 16/9 > 1$$

Better to use V_1.

21) $p = 2 \Rightarrow$ use (14.27)

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}}_{\underline{C}} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\underline{C} \Rightarrow \underline{C}^{-1} = \frac{\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}}{5}$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/5 \\ 4/5 \end{bmatrix}$$

$$\hat{x}_3 = a_1 x_1 + a_2 x_2 = -\frac{1}{5}(1) + \frac{4}{5}(2)$$

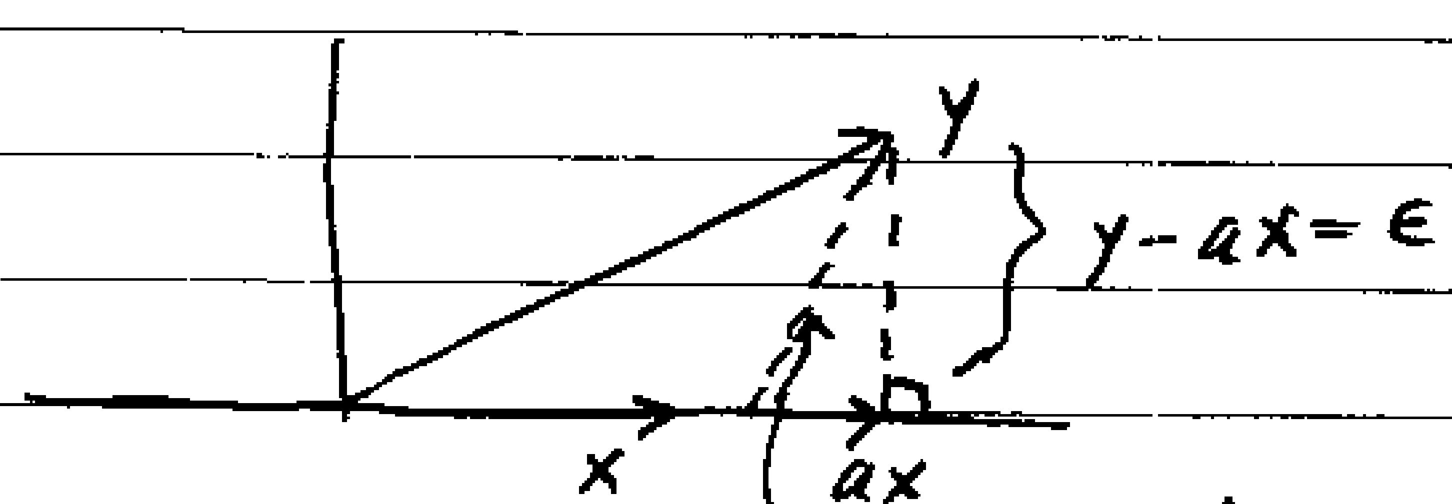
$$= 7/5$$

22) $E_{X,Y}[\epsilon X] = E_{X,Y}[(Y - aX)X] = 0$

$$E_{X,Y}[XY] = a E_{X,Y}[X^2]$$

$$a_{OPT} = \frac{E_{X,Y}[XY]}{E_X[X^2]} = \frac{\text{cov}(X,Y)}{\text{var}(X)}$$

since X, Y are zero mean



for other value of $a \neq a_{OPT}$

23) $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \rho^2 \\ \rho \end{bmatrix}$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \frac{\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}}{1-\rho^2} \begin{bmatrix} \rho^2 \\ \rho \end{bmatrix} = \frac{\begin{bmatrix} 0 \\ -\rho^3+\rho \end{bmatrix}}{1-\rho^2}$$

$$= \begin{bmatrix} 0 \\ \rho \end{bmatrix}$$

$$\hat{X}_3 = \rho X_2$$

$$E(\epsilon X_1) = E((X_3 - \rho X_2) X_1) = E(X_1 X_3) - \rho E(X_1 X_2)$$

$$= \rho^2 - \rho \cdot \rho = 0$$

Once X_2 is used the error becomes uncorrelated with X_1. Thus the final prediction cannot depend on X_1

24) $E_{X_1, X_2, X_3}[(X_3 - a_{1opt} X_1 - a_{2opt} X_2)^2] =$

$$E_{X_1, X_2, X_3}[(X_3 - a_{1opt} X_1 - a_{2opt} X_2) X_3]$$

$$- \underbrace{E_{X_1, X_2, X_3}[(X_3 - a_{1opt} X_1 - a_{2opt} X_2)(a_{1opt} X_1 + a_{2opt} X_2)]}$$

$$= E_{X_1, X_2, X_3}[\epsilon(a_{1opt} X_1) + \epsilon(a_{2opt} X_2)]$$

$$= a_{1opt} \underbrace{E_{X_1, X_2, X_3}(\epsilon X_1)}_{=0} + a_{2opt} \underbrace{E_{X_1, X_2, X_3}(\epsilon X_2)}_{=0}$$

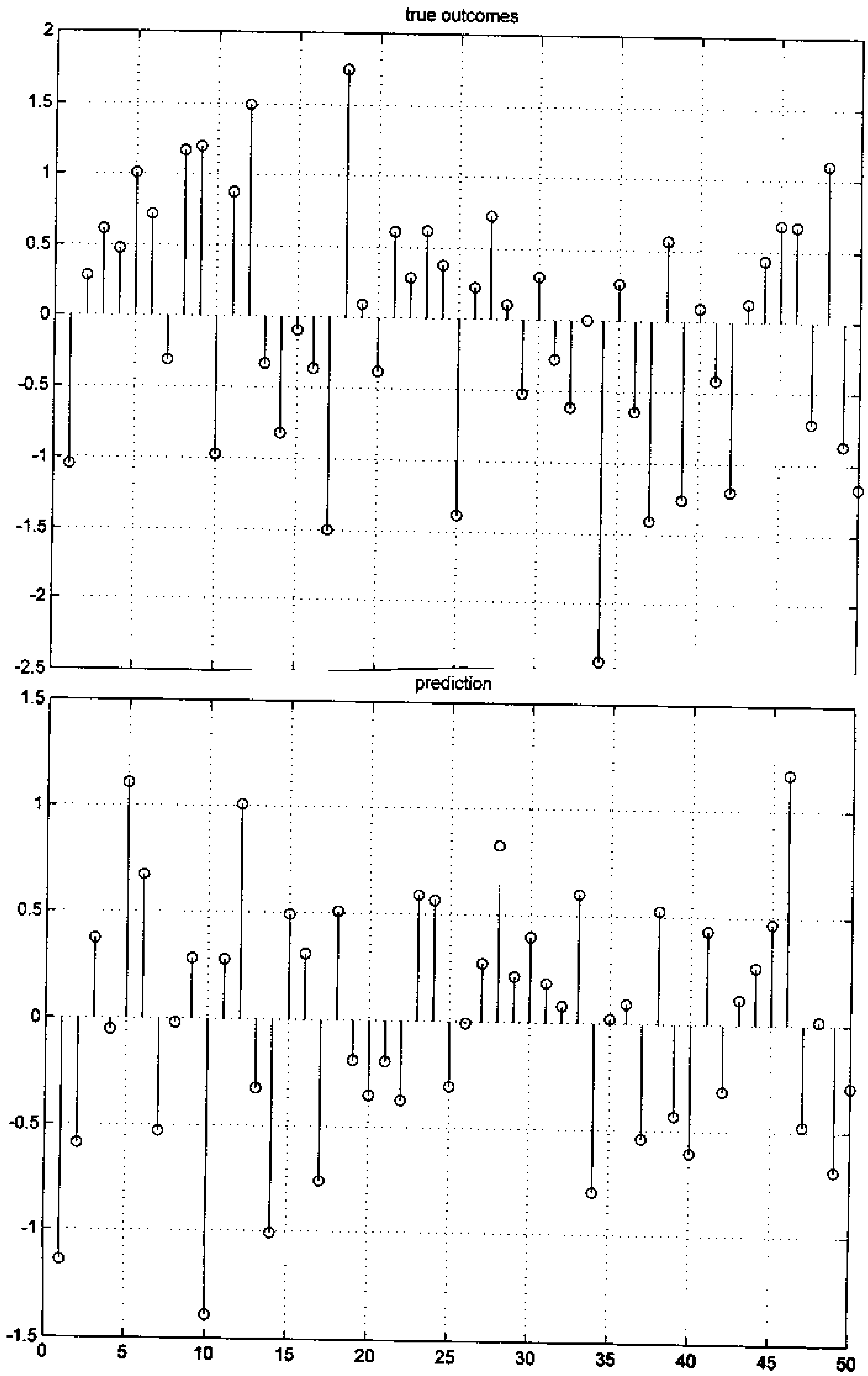
by orthogonality principle

$$\begin{pmatrix} a_{1\,opt} \\ a_{2\,opt} \end{pmatrix} = \begin{pmatrix} -1/5 \\ 4/5 \end{pmatrix}$$

$$mse_{min} = 1 - (-1/5)(\tfrac{1}{3}) - \tfrac{4}{5}(\tfrac{2}{3})$$

$$= 1 + \tfrac{1}{15} - \tfrac{8}{15} = 8/15 = 0.5333$$

25)



$mse_{MIN} = 0.5333$

$\hat{mse}_{MIN} = 0.5407$

```
% probprob14_25
%
clear all
randn('state',0)
C=[1 2/3 1/3;2/3 1 2/3;1/3 2/3 1];
G=chol(C)'; % perform Cholesky decomposition
            % MATLAB uses C=A'*A so G=A'
M=5000;
for m=1:M % generate realizations of x
   u=[randn(1,1) randn(1,1) randn(1,1)]';
   x=G*u;
   x3(m,1)=x(3);
   x3est(m,1)=-0.2*x(1)+0.8*x(2);
end
figure
stem([1:50]',x3est(1:50))
title('prediction')
grid
figure
stem([1:50]',x3(1:50))
title('true outcomes')
grid
mse=mean((x3-x3est).^2)
```

26) We say a signal is present if $\bar{x} > A/2$

$$P[\bar{x} > A/2 ; \underset{\uparrow \text{ true } \mathcal{H}}{\mathcal{H}_{S+N}}) = \int_{A/2}^{\infty} p_{\bar{x}}(\bar{x})\, d\bar{x}$$

But $\bar{x}$ is Gaussian since it is a weighted sum of independent Gaussian random variables. From Example 14.4

$E_{\underline{x}}[\bar{x}] = A \qquad \text{var}(\bar{x}) = \sigma^2/N$

$\Rightarrow \quad \bar{x} \sim N(A, \sigma^2/N)$

$$P(\bar{x} > A/2;\, \mathcal{H}_{S+W}) = \Phi\left(\frac{A/2 - A}{\sqrt{\sigma^2/N}}\right)$$

$$= \Phi\left(-\frac{A/2}{\sqrt{\sigma^2/N}}\right) = \Phi\left(-\sqrt{N}\,\frac{A/2}{\sqrt{\sigma^2}}\right)$$

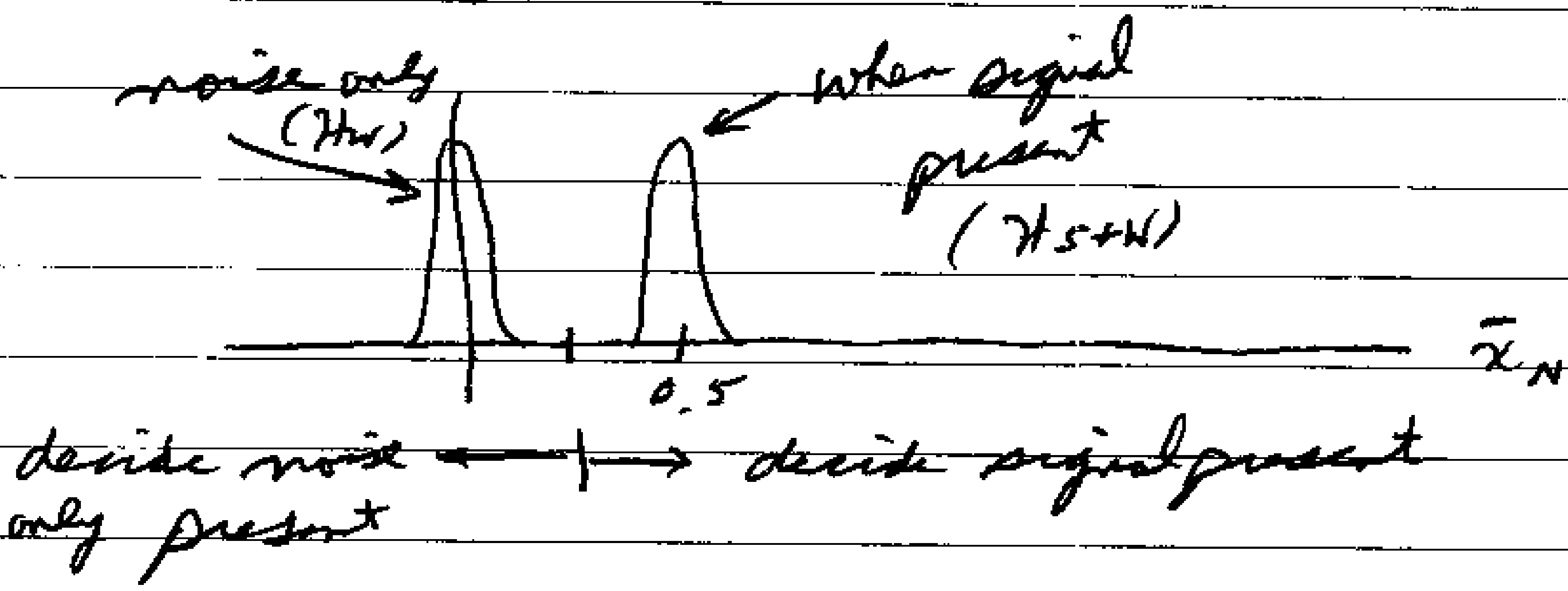
$$\to 1 \quad \text{since} \quad -\sqrt{N}\,\frac{A/2}{\sqrt{\sigma^2}} \to -\infty \text{ as } N \to \infty$$

27) For all N $\quad E[\bar{x}_N] = 0$ if $\mathcal{H}_W$ true

$= 0.5$ if $\mathcal{H}_{S+W}$ true

also, $\text{var}(\bar{x}_N) = 1/N \to 0$ as $N \to \infty$

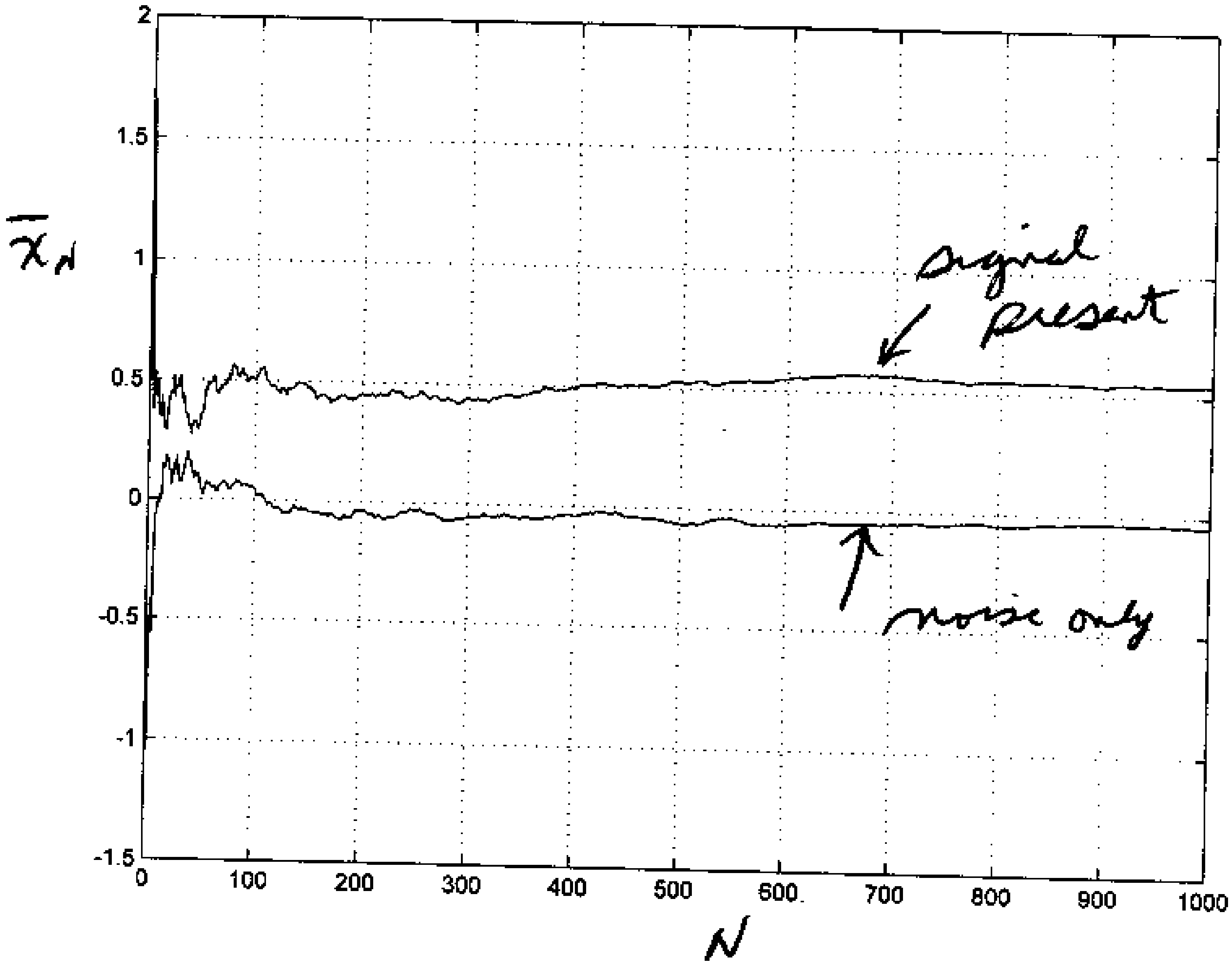
as $N \to \infty$



We never make a mistake as $N \to \infty$
because PDFs do not overlap!

Also, note that $\bar{x}_N \to E[\bar{x}_N] = \begin{cases} 0 & \text{under } \mathcal{H}_W \\ 0.5 & \text{under } \mathcal{H}_{S+W} \end{cases}$

See next page

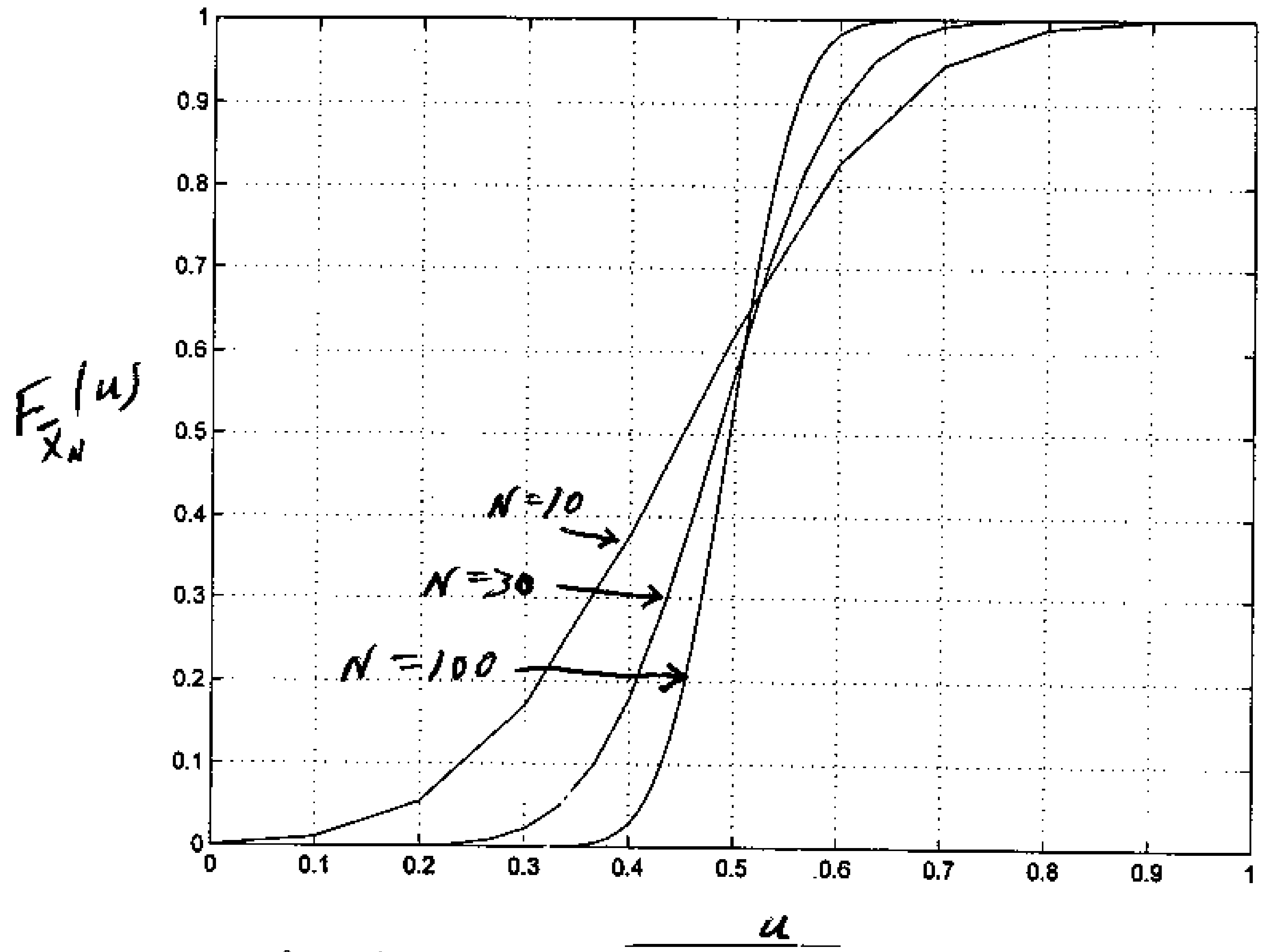



```
% probprob14_27
%
clear all
randn('state',0)
N=1000;A=0.5;
n=[0:N-1]';
x0=randn(N,1);
x1=A+randn(N,1);
for N1=1:N
   x0mean(N1,1)=mean(x0(1:N1));
   x1mean(N1,1)=mean(x1(1:N1));
end
plot([1:1000]',x0mean,[1:1000]',x1mean)
grid
```

Chapter 15

1) As $N \to \infty$ $F_{\bar{X}_N}(u) \to 1$ for $u > 0.5$
$\qquad\qquad\qquad\qquad\quad 0$ for $u < 0.5$

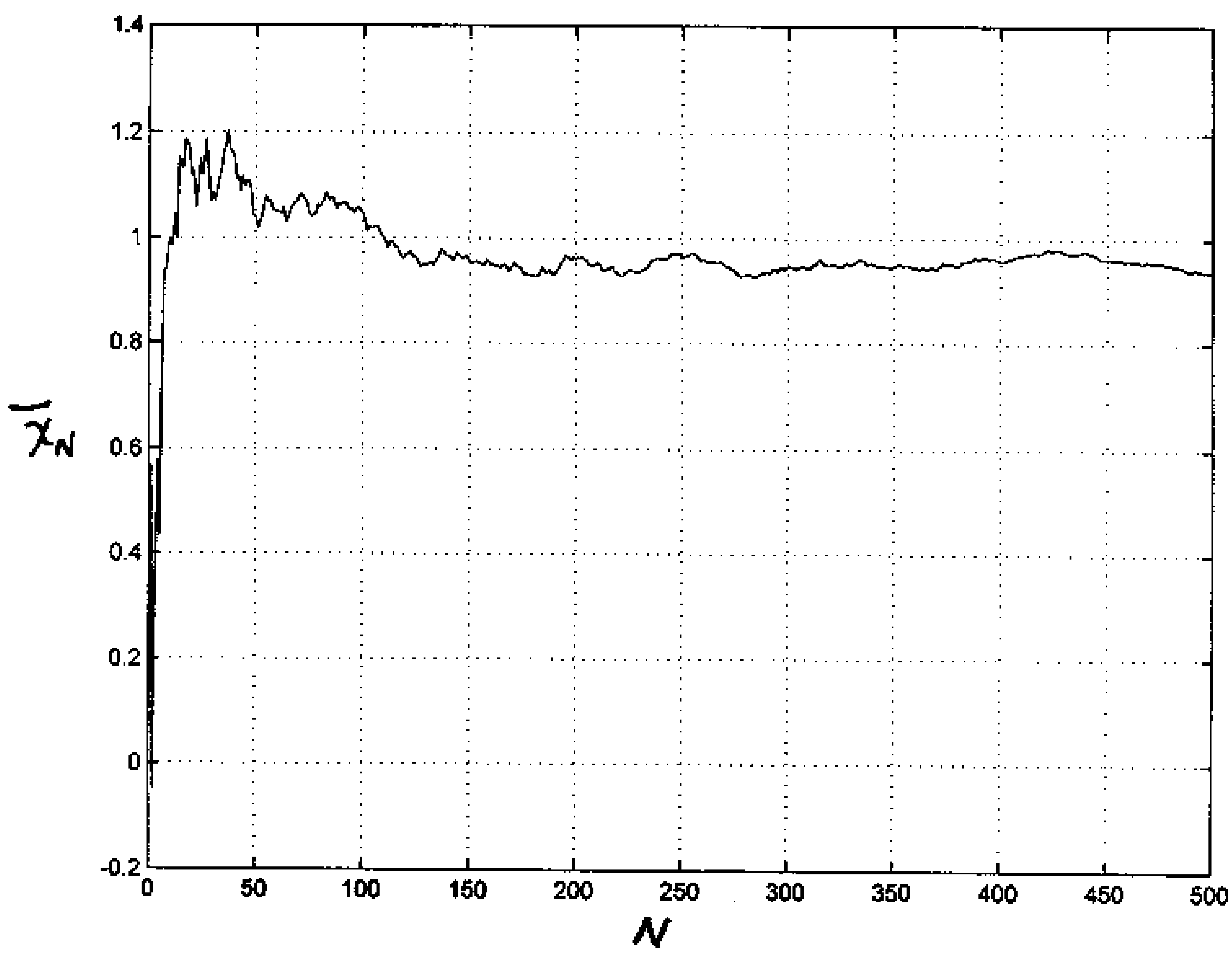
or a unit step beginning at $u = 0.5$




```
% probprob15_1.m
%
clear all
N=10;
u=[0:N]'/N;
for k=0:N
    p(k+1,1)=comb(N,round(N*u(k+1)))*0.5^(N);
  if k==0
    F(k+1,1)=p(k+1);
  else
    F(k+1,1)=F(k,1)+p(k+1);
  end
end
figure
plot(u,F)
grid
axis([0 1 0 1])
hold on
N=30;
u=[0:N]'/N;
for k=0:N
    p(k+1,1)=comb(N,round(N*u(k+1)))*0.5^(N);
  if k==0
    F(k+1,1)=p(k+1);
  else
    F(k+1,1)=F(k,1)+p(k+1);
  end
end
```

```
plot(u,F)
grid
axis([0 1 0 1])
N=100;
u=[0:N]'/N;
for k=0:N
      p(k+1,1)=comb(N,round(N*u(k+1)))*0.5^(N);
  if k==0
      F(k+1,1)=p(k+1);
   else
      F(k+1,1)=F(k,1)+p(k+1);
   end
end

plot(u,F)
grid
axis([0 1 0 1])
hold off
```

2) Should converge (with high probability) to $E_{x_i}[x_i] = 1$, by law of large numbers.



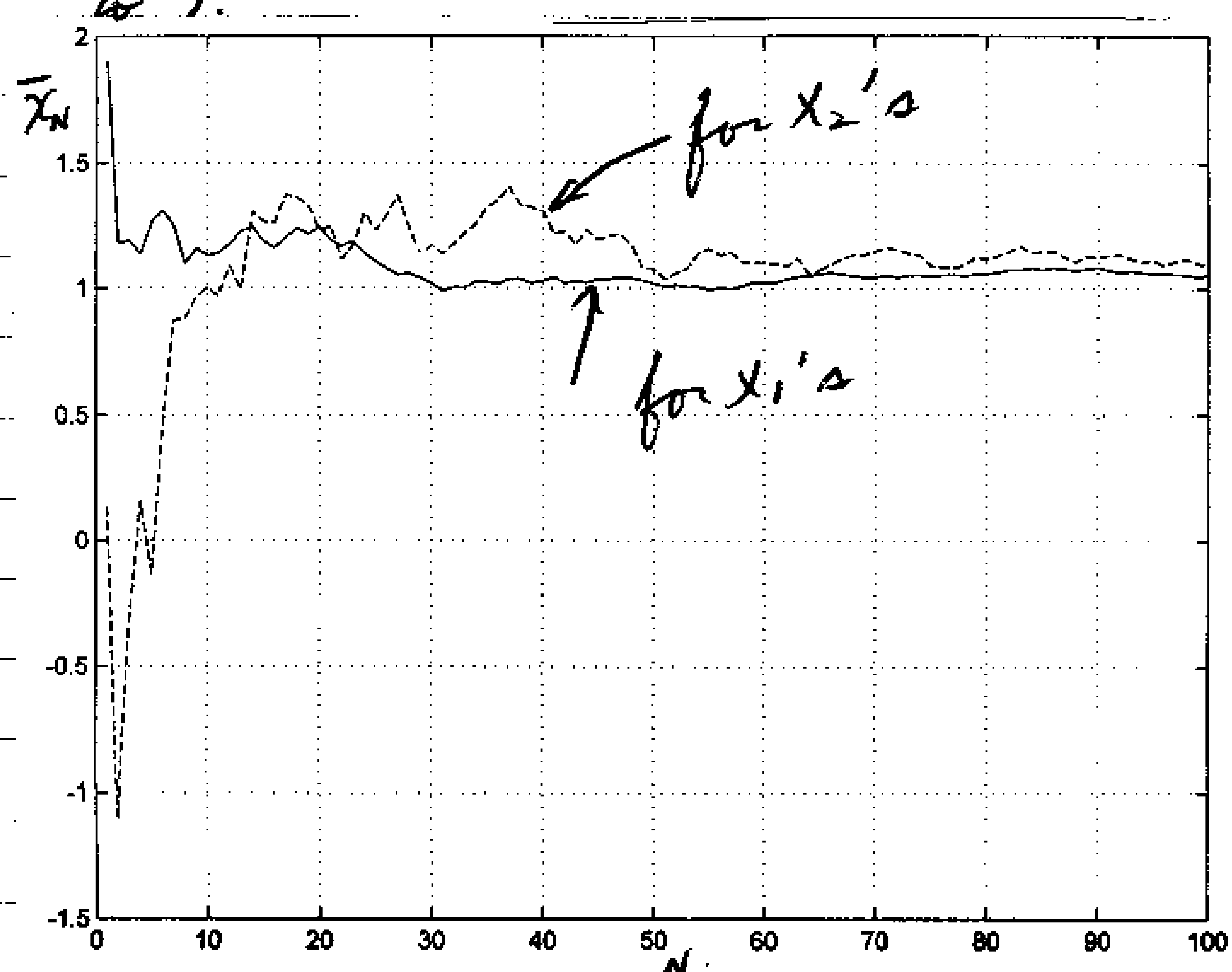
```
% probprob15_2.m
%
clear all
randn('state',0)
M=500;
x=randn(M,1)+1;
for N=1:M
   xmean(N,1)=mean(x(1:N));
end
plot([1:500]',xmean)
grid
```

3) $\operatorname{var}(\bar{X}_N) = \operatorname{var}(X_i)/N$

$$\operatorname{var}(\bar{X}_N) = \frac{\frac{1}{12}2^2}{N} = \frac{1}{3N} \quad \text{if } X_i \sim \mathcal{U}(0,2)$$

$$= \frac{4}{N} \quad \text{if } X_i \sim N(1,4)$$

$\Rightarrow$ uniform random variables have $\frac{1}{12}$ of the variance of Gaussian and so should converge faster. Note both have $E(X) = 1$ so they both should converge to 1.



```
% probprob15_3.m
%
clear all
randn('state',0)
rand('state',0)
M=100;
x1=2*rand(M,1);
x2=2*randn(M,1)+1;
for N=1:M
   x1mean(N,1)=mean(x1(1:N));
   x2mean(N,1)=mean(x2(1:N));
end
plot([1:M]',x1mean,'-',[1:M]',x2mean,'--')
grid
```

4) $E(Y_N) = \sum_{i=1}^{N} \alpha_i E_{X_i}(X_i) = 0$

$$\mathrm{var}(Y_N) = \sum_{i=1}^{N} \mathrm{var}(\alpha_i X_i) = \sum_{i=1}^{N} \alpha_i^2 \mathrm{var}(X_i)$$

$$= \sum_{i=1}^{N} \alpha_i^2$$

Must have $\lim_{N\to\infty} \sum_{i=1}^{N} \alpha_i^2 = 0$

Could have $\alpha_i = 1/N^{3/4}$

5) $X_N = X_{N-1} + U_N$

$X_1 = U_1$

$X_2 = X_1 + U_2 = U_1 + U_2$

$\vdots$

$X_N = \sum_{i=1}^{N} U_i$

$$E(X_N) = N E_{U_i}[U_i] = N\left(-1\cdot\tfrac{1}{2} + 1\cdot\tfrac{1}{2}\right) = 0$$

$$\mathrm{var}(X_N) = \sum_{i=1}^{N} \mathrm{var}(U_i) = N\,\mathrm{var}(U_i) \to \infty$$

as $N\to\infty$ $\Rightarrow$ X_N does not converge

6) $\underset{\underline{X}}{E}\left[\frac{1}{N}\sum_{i=1}^{N} x_i^2\right] = \frac{1}{N}\sum_{i=1}^{N} \underset{x_i}{E}[x_i^2]$

assume x_i's are identically distributed

$\Rightarrow \quad = \frac{1}{N}\sum_{i=1}^{N} E_x[x^2] = E_x[x^2]$

on average get second moment

$\mathrm{var}\left(\frac{1}{N}\sum_{i=1}^{N} x_i^2\right) = \frac{1}{N^2}\mathrm{var}\left(\sum_{i}^{N} x_i^2\right)$

assume independence of x_i's $\Rightarrow$
x_i^2 are independent $\Rightarrow$ x_i^2 are also uncorrelated

$\Rightarrow \quad = \frac{1}{N^2}\sum_{1}^{N}\mathrm{var}(x_i^2) = \frac{1}{N^2}\sum_{1}^{N}\mathrm{var}(x^2)$

$= \mathrm{var}(x^2)/N \to 0$ if $\mathrm{var}(x^2) < \infty$

or since $E_x[x^2]$ is assumed to exist need
$\mathrm{var}(x^2) = E_x[x^4] - E_x^2[x^2] < \infty$ or $E_x[x^4] < \infty$

7) $E_{\underline{x}}\left[1/\sqrt{N}\sum_{1}^{N} x_i\right] = \frac{1}{\sqrt{N}} N E_x[x] = \sqrt{N} E_x[x] \to \infty$
as $N \to \infty$ No. Even if $E_x[x] = 0$
~~the~~ so that $E_{\underline{x}}\left[\frac{1}{\sqrt{N}}\sum_{1}^{N} x_i\right] = 0$ we have

$\mathrm{var}\left(\frac{1}{\sqrt{N}}\sum_{1}^{N} x_i\right) = \frac{1}{N} N\,\mathrm{var}(x_i) = \mathrm{var}(x) \not\to 0$
as $N \to \infty$

8) $P[|Y_N| > \epsilon] = P[|Y_N| > \epsilon \,/\, \frac{x_N}{\sqrt{N}} > 0.1]\,P(\frac{x_N}{\sqrt{N}} > 0.1)$
$+ P[|Y_N| > \epsilon \,/\, \frac{x_N}{\sqrt{N}} \le 0.1]\,P[\frac{x_N}{\sqrt{N}} \le 0.1]$

$$= P\left[\left|\frac{x_N}{\sqrt{N}}+1\right| > \epsilon\right] P\left[\frac{x_N}{\sqrt{N}} > 0.1\right]$$

$$+ P\left[\left|\frac{x_N}{\sqrt{N}}\right| > \epsilon\right] P\left[\frac{x_N}{\sqrt{N}} \le 0.1\right]$$

But $y = \frac{x_N}{\sqrt{N}} \sim N(0, 1/N)$

$$z = \frac{x_N}{\sqrt{N}} + 1 \sim N(1, 1/N)$$

$P[|Y_N| > \epsilon] =$

$$[1 - P[-\epsilon \le Z \le \epsilon]]\, P[Y > 0.1] + [1 - P[-\epsilon \le Y \le \epsilon]]\, P[Y \le 0.1]$$

$$= \left[1 - \left(Q\left(\frac{-\epsilon-1}{\sqrt{1/N}}\right) - Q\left(\frac{\epsilon-1}{\sqrt{1/N}}\right)\right)\right] Q\left(0.1/\sqrt{1/N}\right)$$

$$+ \left[1 - \left(Q\left(\frac{-\epsilon}{\sqrt{1/N}}\right) - Q\left(\frac{\epsilon}{\sqrt{1/N}}\right)\right)\right]\left[1 - Q\left(0.1/\sqrt{1/N}\right)\right]$$

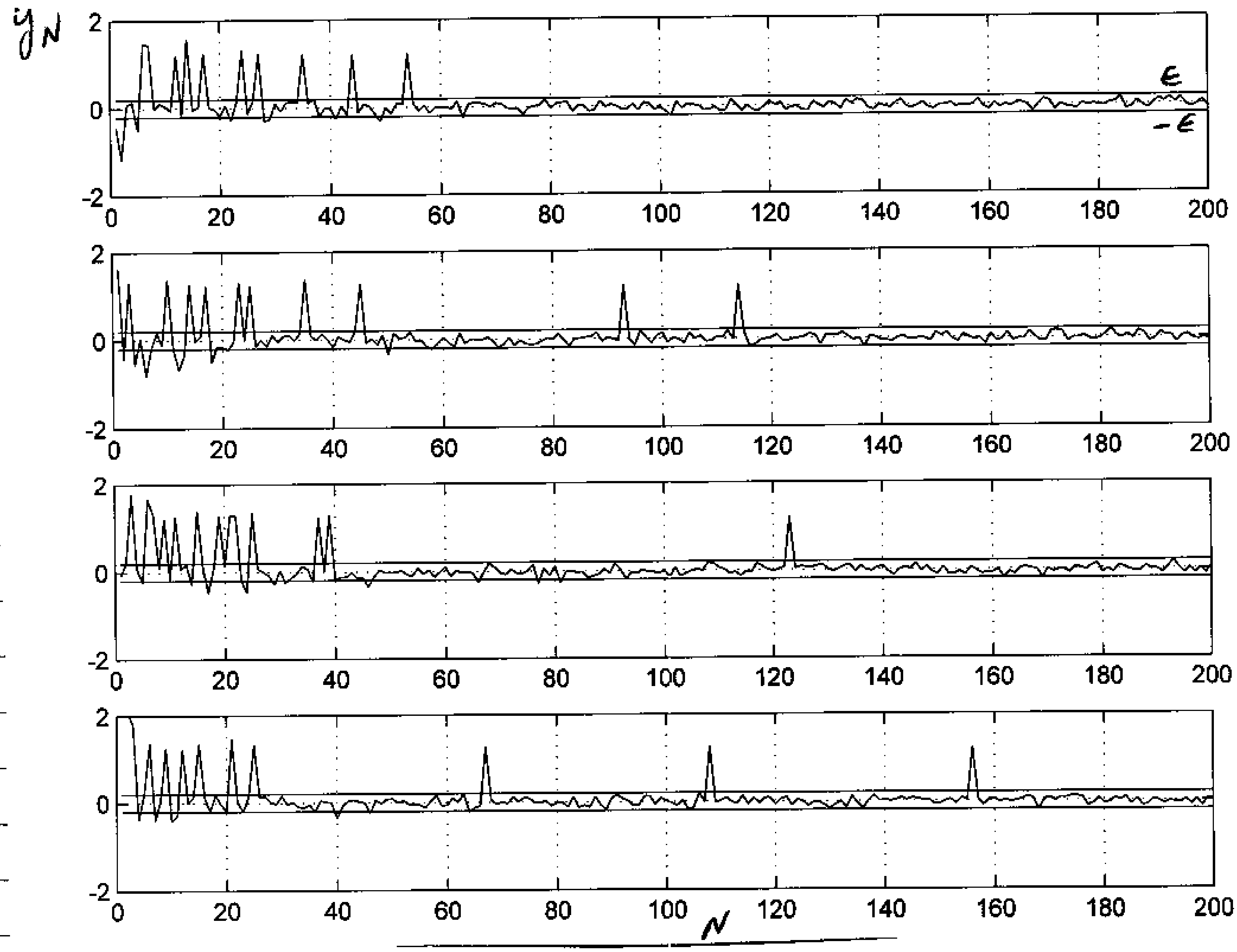
$$= \left[Q(\sqrt{N}(1+\epsilon)) - Q(\sqrt{N}(\epsilon-1))\right] Q(0.1\sqrt{N})$$
$$+ \left[Q(\sqrt{N}\,\epsilon) + Q(\sqrt{N}\,\epsilon)\right]\left[1 - Q(0.1\sqrt{N})\right]$$

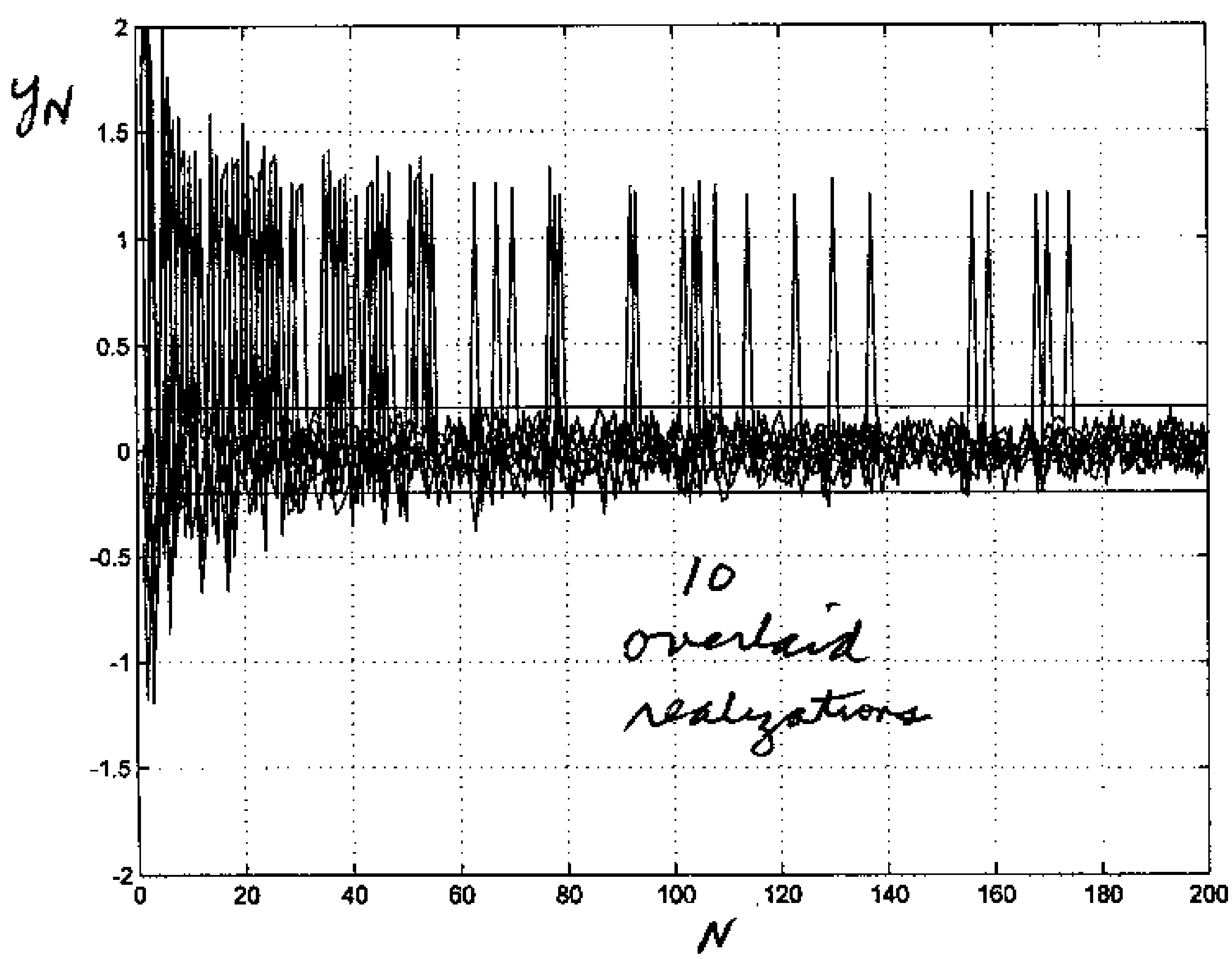
as $N \to \infty$, $Q(0.1\sqrt{N}) \to 0$ and
$Q(\sqrt{N}\,\epsilon) \to 0 \Rightarrow$

$P[|Y_N| > \epsilon] \to 0$

See next page. Cannot say that a given sequence converges - only that the probability of a value outside a band $[-\epsilon, \epsilon]$ is small as $N \to \infty$.

4 realizations





```
% probprob15_8.m
%
randn('state',0)
clear all
N=200;
for m=1:10
   for n=1:N
   y=randn(1,1)/sqrt(n);
   if y>0.2
      x(n,m)=y+1;
   else
      x(n,m)=y;
        end

end
end
figure
for i=1:4
subplot(4,1,i)
   plot([1:N]',x(:,i))
   line([1:N]',0.2*ones(N))
   line([1:N]',-0.2*ones(N))
   axis([0 N -2 2])
   grid
end
figure
plot([1:N]',x(:,1))
line([1:N]',0.2*ones(N))
   line([1:N]',-0.2*ones(N))
   axis([0 N -2 2])
   grid
   for i=2:10
      line([1:N]',x(:,i));
   end
```

9) $P\left(\sum_{1}^{100} R_i > 1030\right)$

$\sum_{1}^{100} R_i \sim N(1000, 200)$ by central limit theorem

$$P\left(\sum_{1}^{100} R_i > 1030\right) \simeq Q\left(\frac{1030-1000}{\sqrt{200}}\right)$$

$$= Q\left(30/\sqrt{200}\right) = Q\left(30/10\sqrt{2}\right)$$

$$= Q\left(\frac{3}{\sqrt{2}}\right)$$

$$= 0.0169$$

10) $$\bar{X}_{2N} = \frac{1}{2N}\left[\sum_1^N x_i + \sum_1^N x_i\right]$$

$$= \frac{1}{N}\sum_1^N x_i$$

as $N \to \infty$, $E[\bar{X}_{2N}] = E[X]$

$var(\bar{X}_{2N}) = var(X_i)/N \to 0$

but will converge slower since $var(\bar{X}_{2N})$ does not decrease as $1/2N$ but only as $1/N$.

Law of large numbers holds

By similar argument the central limit theorem also holds.

11) Yes since Erlang is equivalent to sum of IID exponential random variables

12) $$E[Y] = 100\,E[X^2] = 100\,(var(X) + E^2(X))$$

$$= 100\,(8 + 16) = 2400$$

$$var(Y) = 100\,var(X^2)$$

$$= 100\,[E[X^4] - E^2[X^2]]$$

But $E[X^2] = 24$

$E[X^4] = E[(V-4)^4]$ where $V \sim N(0, 8)$

$$= \sum_{k=0}^{4} \binom{4}{k} E[V^k]\,(-4)^{4-k}$$

$$E(v^k) = 0 \qquad k \text{ odd}$$

$$E(x^4) = (-4)^4 + \binom{4}{2} E(v^2)(-4)^2 + E(v^4)$$

$$= (-4)^4 + 6\, var(v)(16) + 3\, var(v)^2$$

$$= 256 + 96(8) + 3(8)^2 = 1216$$

$$var(Y) = 100(1216 - 24^2) = 640$$

$$Y \sim N(2400,\ 64000)$$

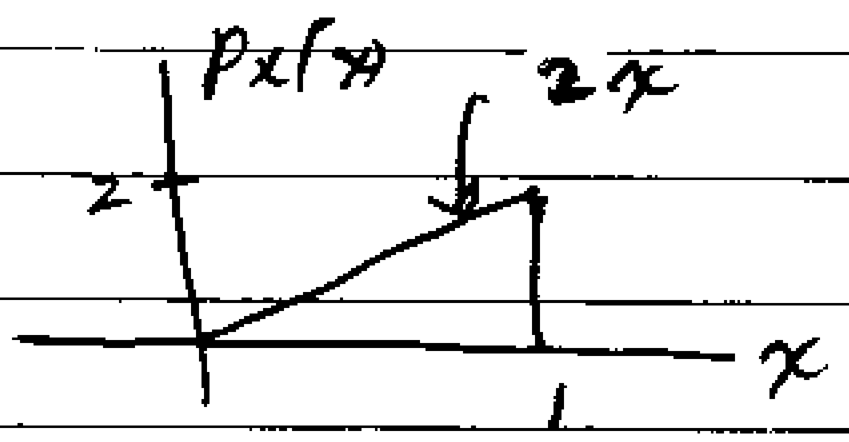
13) $E(X) = 2$

$var(X) = 2^2/12 = 1/3$

$E(Y) = 2000 \qquad var(Y) = 1000/3$

$\Rightarrow \; Y \sim N(2000,\ 1000/3)$

14)



$$E(X) = \int_0^1 x\, 2x\, dx = \int_0^1 2x^2\, dx = \tfrac{2}{3} x^3 \Big|_0^1 = 2/3$$

$$E(X^2) = \int_0^1 x^2\, 2x\, dx = \int_0^1 2x^3\, dx = \tfrac{1}{2} x^4 \Big|_0^1 = \tfrac{1}{2}$$

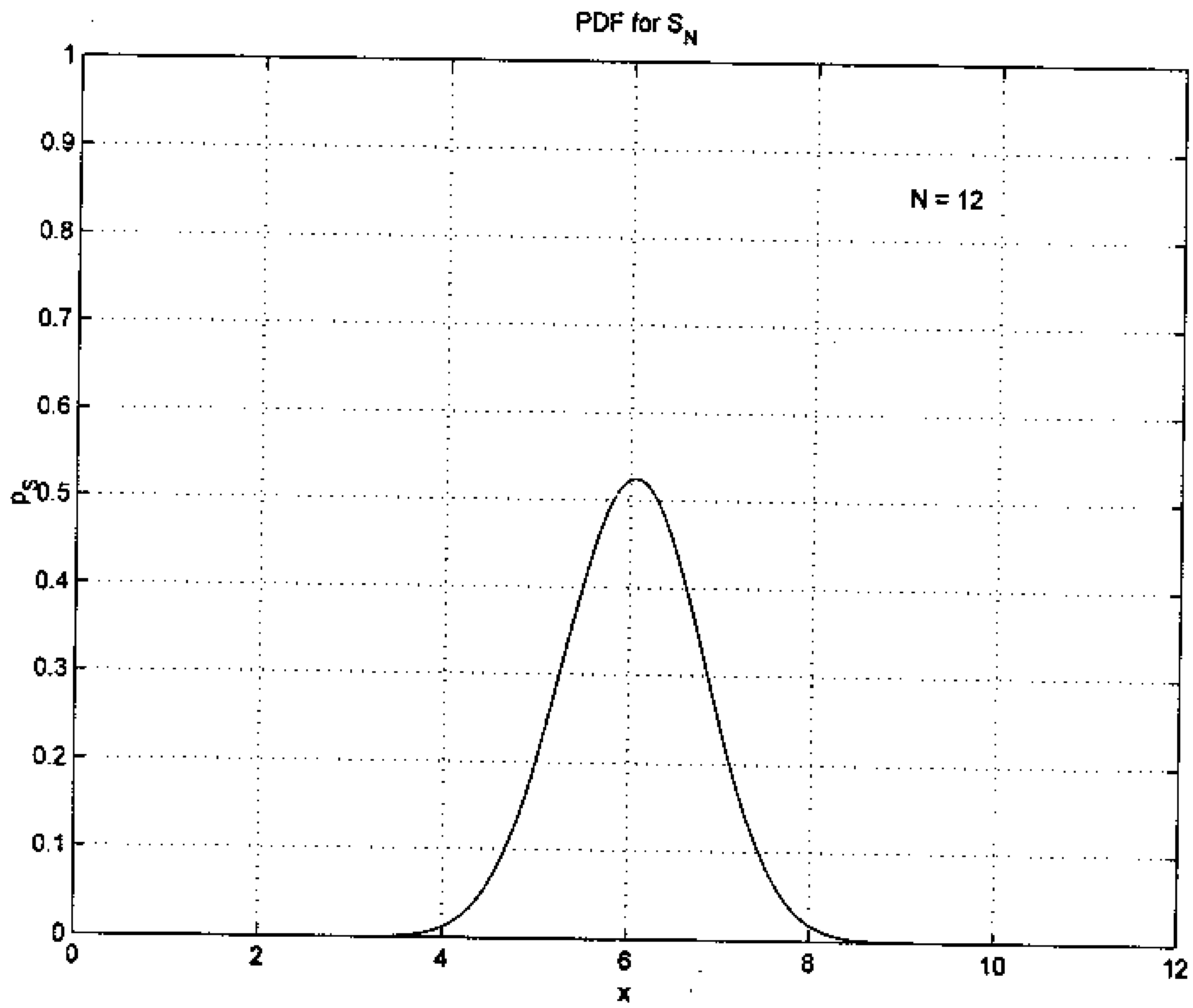
$$var(X) = \tfrac{1}{2} - (2/3)^2 = 9/18 - 8/18 = 1/18$$

$$S_{10} \sim N\left(\frac{20}{3}, 5/9\right)$$

$$P(S_{10} > 7) = Q\left(\frac{7 - 20/3}{\sqrt{5/9}}\right) = Q\left(\frac{1/3}{\frac{1}{3}\sqrt{5}}\right)$$

$$= Q(1/\sqrt{5}) = 0.3274$$

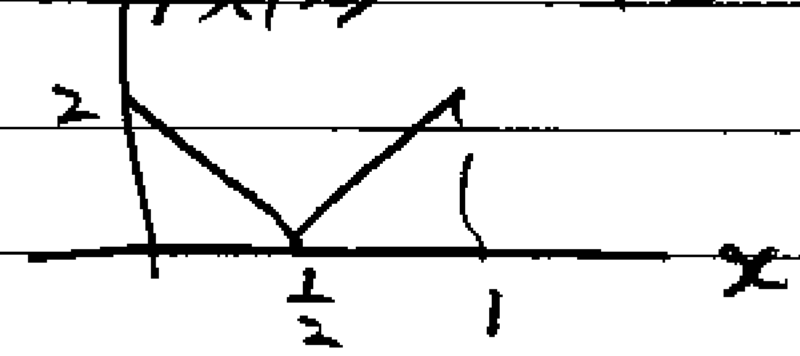
15)



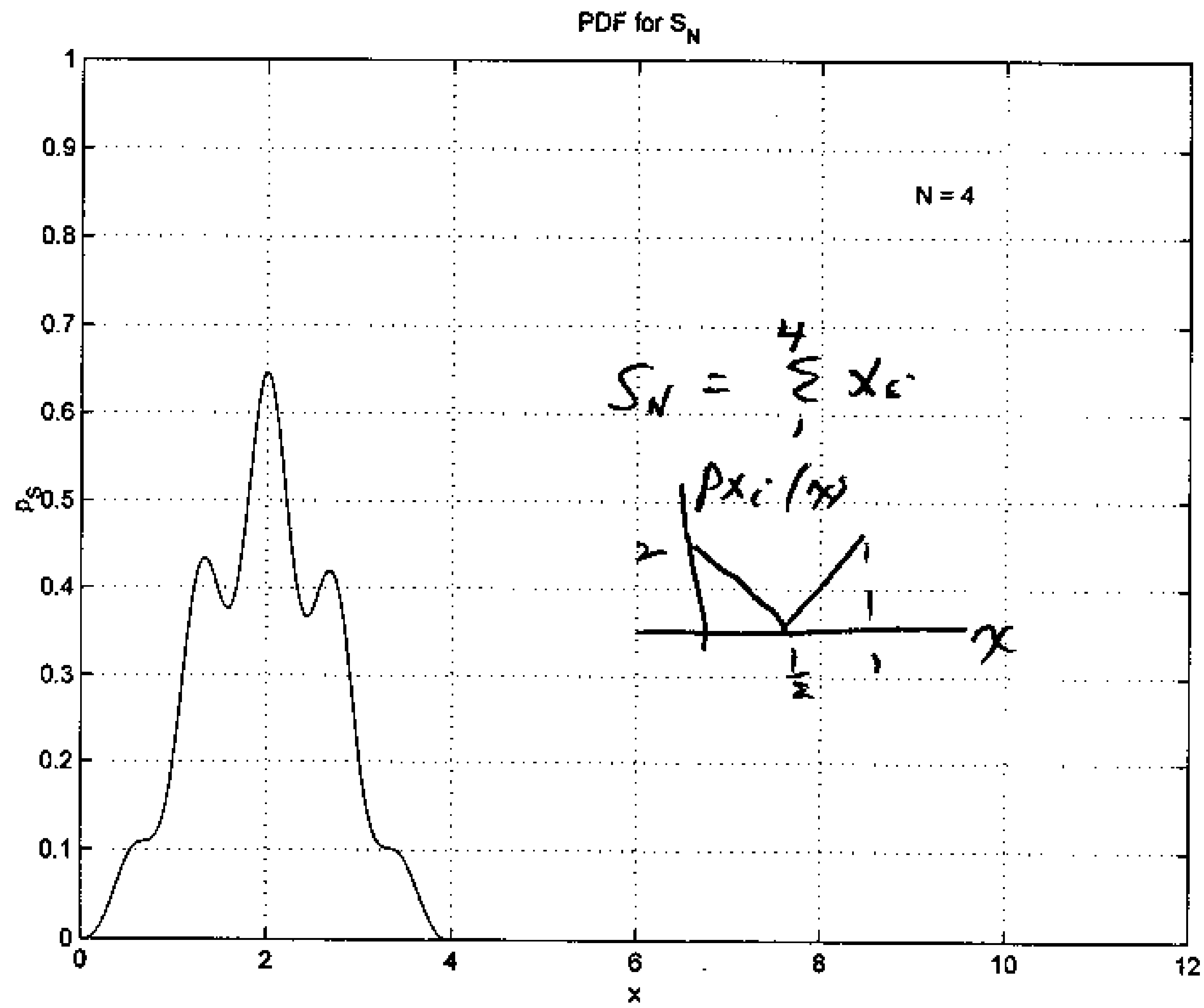
```
% probprob15_15.m
%
%  This program demonstrates the central limit theorem.  It determines the
%  PDF for the sum S_N of N IID random variables. Each marginal PDF is assumed
%  to be nonzero over the interval (0,1). The repeated convolution integral is
%  implemented using a discrete convolution.  The plots of the PDF of S_N as
%  N increases are shown successively (press carriage return for next plot).
%
% clt_demo.m
clear all
delu=0.005;
u=[0:delu:1-delu]'; % p_X defined on interval [0,1)
p_X=(pi/2)*sin(pi*u);
x=[u;u+1]; % increase abcissa values since repeated convolution increases
           % nonzero width of output
p_S=zeros(length(x),1);
N=12; % number of random variables summed
for j=1:length(x) % start discrete convolution
   for i=1:length(u)
      if j-i>0&j-i<=length(p_X)
         p_S(j)=p_S(j)+p_X(i)*p_X(j-i)*delu;
      end

   end
end
plot(x,p_S) % plot results for N=2
grid
axis([0 N 0 1])
xlabel('x')
ylabel('p_S')
title('PDF for S_N')
text(0.75*N,0.85,'N = 2')
for n=3:N
   pause
      x=[x;u+n-1]; % increase abcissa values since repeated convolution increases
                   % nonzero width of output
      p_S=[p_S;zeros(length(u),1)];
      g=zeros(length(p_S),1);
   for j=1:length(x) %start discrete convolution
   for i=1:length(u)
      if j-i>0
         g(j,1)=g(j,1)+p_X(i)*p_S(j-i)*delu;
      end

   end
end
p_S=g; % plot results for N=3,4,...,12
plot(x,p_S)
grid
axis([0 N 0 1])
xlabel('x')
ylabel('p_S')
title('PDF for S_N')
text(0.75*N,0.85,['N = 'num2str(n)])
end
```

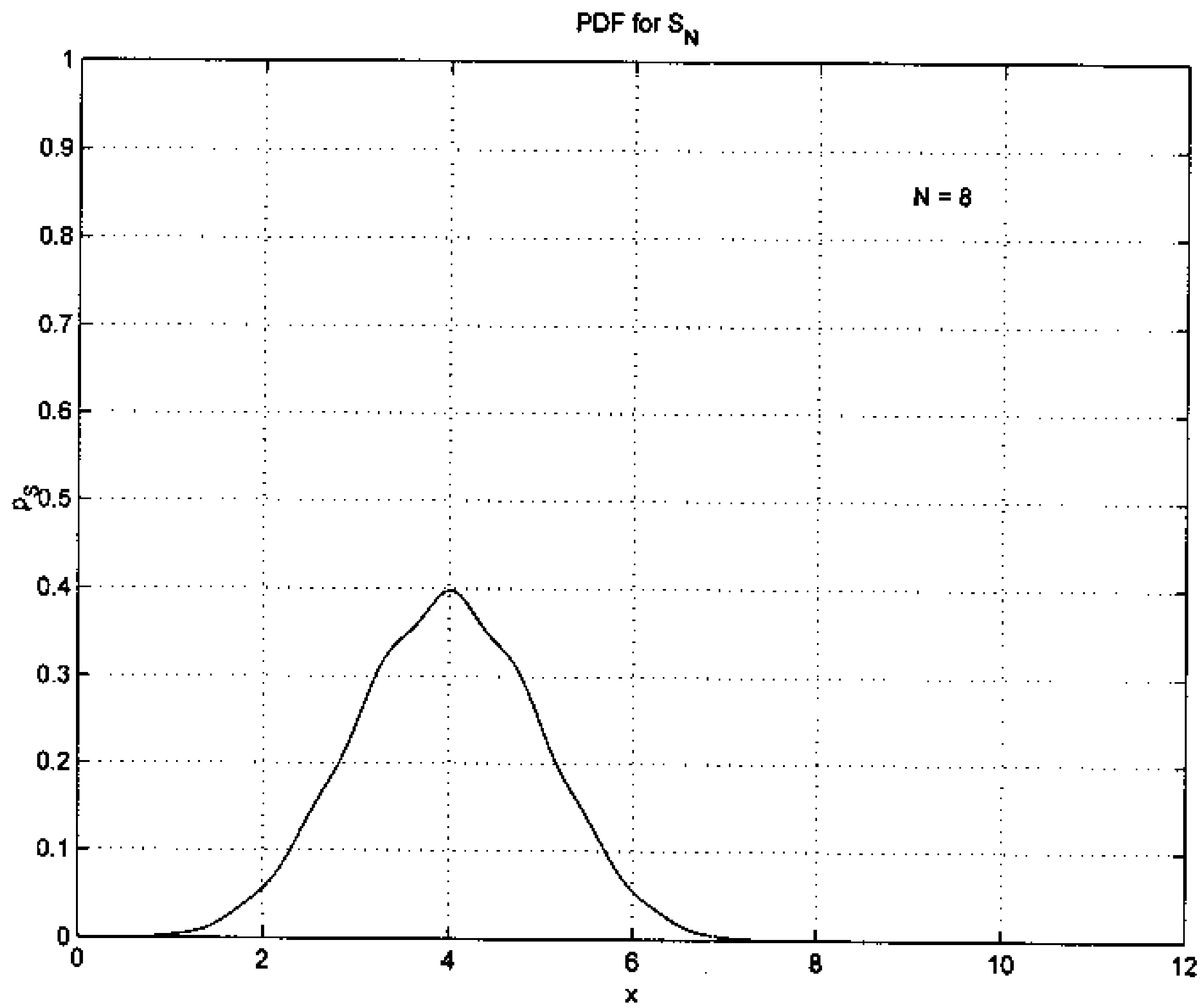
16) The uniform PDF converges faster than for

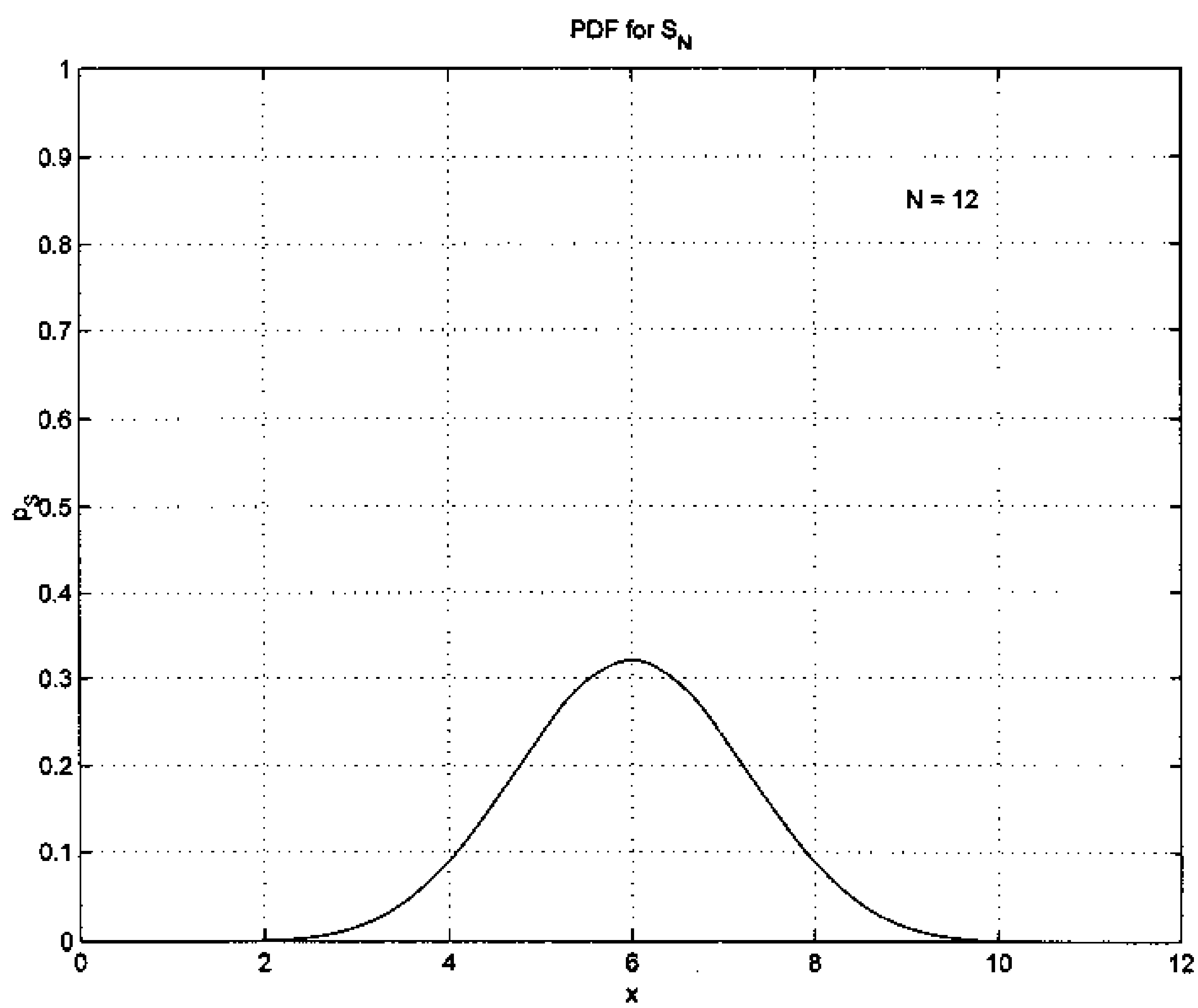
$p_X(x) = |2 - 4x|$



since the latter is very non-Gaussian.
Compare below figures to Figure 15.7




PDF for S_N
N = 8
p_S
x


PDF for S_N
N = 12
p_S
x

```
% probprob15_16.m
%
%  This program demonstrates the central limit theorem.  It determines the
%  PDF for the sum S_N of N IID random variables. Each marginal PDF is assumed
%  to be nonzero over the interval (0,1). The repeated convolution integral is
%  implemented using a discrete convolution.  The plots of the PDF of S_N as
%  N increases are shown successively (press carriage return for next plot).
%
% clt_demo.m
clear all
delu=0.005;
u=[0:delu:1-delu]'; % p_X defined on interval [0,1)
%p_X=ones(length(u),1);
p_X=abs(2-4*u);
x=[u;u+1]; % increase abcissa values since repeated convolution increases
           % nonzero width of output
p_S=zeros(length(x),1);
N=12; % number of random variables summed
for j=1:length(x) % start discrete convolution
   for i=1:length(u)
      if j-i>0&j-i<=length(p_X)
         p_S(j)=p_S(j)+p_X(i)*p_X(j-i)*delu;
      end

   end
end
plot(x,p_S) % plot results for N=2
grid
axis([0 N 0 1])
xlabel('x')
ylabel('p_S')
title('PDF for S_N')
text(0.75*N,0.85,'N = 2')
for n=3:N
   pause
      x=[x;u+n-1]; % increase abcissa values since repeated convolution increases
                   % nonzero width of output
      p_S=[p_S;zeros(length(u),1)];
      g=zeros(length(p_S),1);
   for j=1:length(x) %start discrete convolution
   for i=1:length(u)
      if j-i>0
         g(j,1)=g(j,1)+p_X(i)*p_S(j-i)*delu;
      end

   end
end
p_S=g; % plot results for N=3,4,...,12
plot(x,p_S)
grid
axis([0 N 0 1])
xlabel('x')
ylabel('p_S')
title('PDF for S_N')
text(0.75*N,0.85,['N = 'num2str(n)])
end
```

17) From Table 11.1 for $\Gamma(\alpha, \lambda)$

$$\phi_X(\omega) = \frac{1}{(1 - j\omega/\lambda)^{\alpha}}$$

Let $\alpha = N/2$, $\lambda = \frac{1}{2} \Rightarrow \chi^2_N$

$$\phi_{Y_N}(\omega) = \frac{1}{(1 - 2j\omega)^{N/2}}$$

$$Z_N = \frac{Y_N - E[Y_N]}{\sqrt{\mathrm{var}(Y_N)}} = \frac{Y_N - N}{\sqrt{2N}}$$

$$\phi_{Z_N}(\omega) = E[e^{j\omega Z_N}]$$

$$= E_{Y_N}\left[e^{j\omega\left(\frac{Y_N - N}{\sqrt{2N}}\right)}\right]$$

$$= e^{-j\omega N/\sqrt{2N}} \underbrace{E_{Y_N}\left[e^{j\omega/\sqrt{2N}\, Y_N}\right]}_{\phi_{Y_N}(\omega/\sqrt{2N})}$$

$$= e^{-j\omega\sqrt{N/2}} \cdot \frac{1}{(1 - 2j\omega/\sqrt{2N})^{N/2}}$$

$$= \frac{e^{-j\omega\sqrt{N/2}}}{(1 - j\omega\sqrt{2/N})^{N/2}}$$

$$\ln \phi_{Z_N}(\omega) = -j\omega\sqrt{N/2} - N/2 \ln\left(1 - j\omega\sqrt{2/N}\right)$$

as $N \to \infty$, $-j\omega\sqrt{2/N} \to 0 \Rightarrow$

$$\ln \phi_{Z_N}(\omega) \to -j\omega\sqrt{N/2} - N/2\left[-j\omega\sqrt{2/N} - \frac{1}{2}\left(j\omega\sqrt{2/N}\right)^2\right]$$

$$= -j\omega\sqrt{N/2} + j\omega\sqrt{N/2} + N/2\left(\tfrac{1}{2}\right)\left(-\omega^2\, 2/N\right)$$

$$= -\tfrac{1}{2}\omega^2$$

$$\Rightarrow \quad \phi_{Z_N}(\omega) \to e^{-\frac{1}{2}\omega^2} \quad \text{or}$$

$$Z_N \to N(0,1)$$

18) $$S_N = \sum_{i=1}^{100} V_i \sim N(0, 100 \cdot 0.1)$$

$$= N(0, 10)$$

This is exact since sum of IID Gaussian random variables is Gaussian.

$$P[S_N > 5] = Q(5/\sqrt{10})$$

$$= 0.0569$$

19) $$P[\,|\bar{X}_N - \tfrac{1}{2}| \le 0.01] = 0.99$$

$$\bar{X}_N \sim N\left(\tfrac{1}{2}, \tfrac{1}{12N}\right)$$ by central limit theorem

$$Y_N = \bar{X}_N - \tfrac{1}{2} \sim N\left(0, \tfrac{1}{12N}\right)$$

$$P[\,|\bar{X}_N - \tfrac{1}{2}| \le 0.01] = P[\,|Y_N| \le 0.01]$$

$$= 1 - P[\,|Y_N| > 0.01] = 1 - 2P[Y_N > 0.01]$$

$$= 1 - 2Q\left(\frac{0.01}{\sqrt{1/12N}}\right) = 0.99$$

$$Q\left(\frac{0.01}{\sqrt{1/12N}}\right) = 0.005$$

From Qinv.m

$$0.01\sqrt{12N} = 2.5758$$

$$\Rightarrow \quad N = 5529$$

20) $W_i \sim U(3, 7)$ in ozs

$$P\left[\sum_1^{15000} W_i \le 4000(16)\right] = ?$$

$$E[W_i] = 5 \qquad var(W_i) = 4^2/_{12} = 4/3$$

$\Rightarrow$ by central limit theorem

$$Y = \sum_1^{15000} W_i \sim N(75000, 20000)$$

$$P[Y \le 64000] = 1 - Q\left(\frac{64000 - 75000}{\sqrt{20000}}\right)$$

$$= 1 - Q(-77.78) \approx 0$$

21) Let $X_i = \begin{matrix} 1 & \text{if pill effective for } i^{th} \text{ patient} \\ 0 & \text{if not} \end{matrix}$

$S_N = \sum_1^{160} X_i$ = number of patients for which pill is effective

$$P[S_N \ge 125] = ?$$

But $S_N \sim N(160p, 160p(1-p))$ by central limit theorem

$$= N(120, 30)$$

$$P[S_N \ge 125] = Q\left(\frac{125-120}{\sqrt{30}}\right) = Q(5/\sqrt{30})$$

$$= 0.1807$$

22) a Gaussian, since $S_N = \sum_{i=1}^{N} x_i$ will be exactly Gaussian for any $N \geq 1$

23) $\hat{p} \pm 1.96\sqrt{\frac{\hat{p}(1-\hat{p})}{N}} =$

$0.75 \pm 1.96\sqrt{\frac{0.75(0.25)}{1000}} = 0.75 \pm 0.027$

$= [0.723, 0.777]$

No

24)

```
% probprob15_24.m
%
clear all
sum=0;N=1000;
for k=490:510
   x=0;
   for i=N:-1:(N-k+1)
      x=x+log(i);
   end
   p=0;
   for i=1:k
      p=p+log(i);
   end
   y=x-p-N*log(2);
   sum=sum+exp(y);
end
sum
```

25) $\frac{1}{\sqrt{N}} = 0.01 \Rightarrow N = 10{,}000$ for conservative estimate

26) For drug

$\hat{p}_D = 4/5$

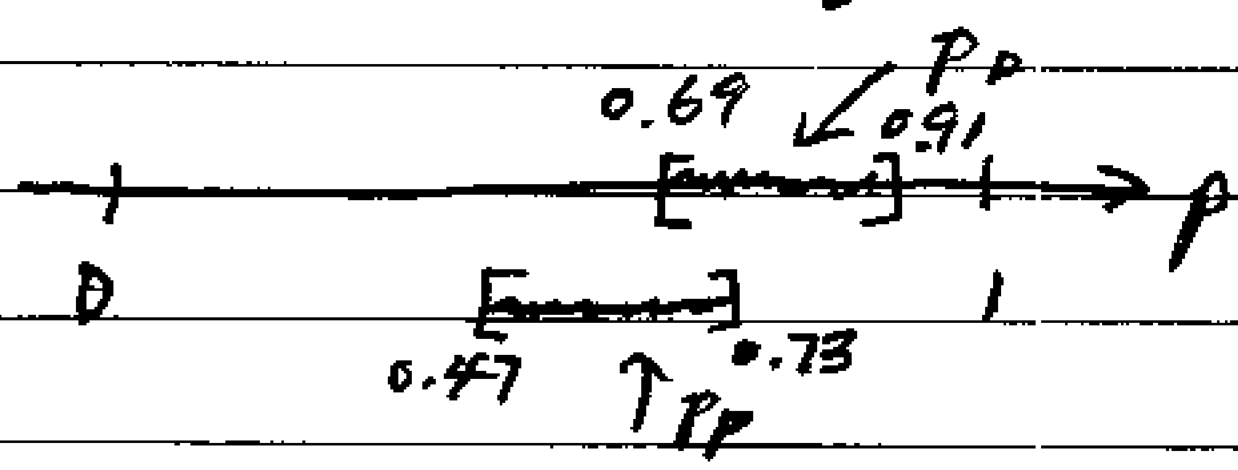
For placebo $\hat{p}_P = 3/5$

$$\hat{p}_D \pm 1.96\sqrt{\hat{p}_D(1-\hat{p}_D)/50} = 0.8 \pm 0.11$$

$$\hat{p}_P \pm 1.96\sqrt{\hat{p}_P(1-\hat{p}_P)/50} = 0.6 \pm 0.13$$

For drug we have [0.69, 0.91] for confidence interval and for placebo we have [0.47, 0.73]

Can't conclude drug helps since



Could have true value of $p = 0.7$ for both groups.

Chapter 16

1) a) temperature at noon in °F
b) expense in dollars and cents
c) time in hours and minutes leaving house

We do not know beforehand the outcomes and thus can consider the sequence of numbers observed as random.

2) Let the sides be labeled a, b, c, d, e, f denoting the 1, 2, 3, 4, 5, 6 dot pattern

$$S = \{(a, c, d, e, \ldots), (f, a, b, c, \ldots), \ldots\}$$
$$S_X = \{(1, 3, 4, 5, \ldots), (6, 1, 2, 3, \ldots), \ldots\}$$

Yes, assuming the tosses are independent we have

$$P_{X[n_1], \ldots, X[n_N]}(x_1, \ldots, x_N) = \prod_{i=1}^{N} \left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)^N$$

3) $0.875 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.111000\ldots$
$0.625 = \frac{1}{2} + \frac{1}{8} = 0.101000\ldots$

Since x can be any number in $[0, 1]$ and each number is distinct, its binary representation is also distinct. Thus the number of sequences $(b_1, b_2, \ldots)$ must be uncountable.

4) $P\{(1,0,1,0,\ldots,1,0)\} = \prod_{i=1}^{50} p \prod_{i=1}^{50} (1-p)$

$$= p^{50}(1-p)^{50}$$

$$P\{(1,0,1,0,1,0,\ldots)\} = \lim_{M\to\infty} p^{M/2}(1-p)^{M/2} = 0$$

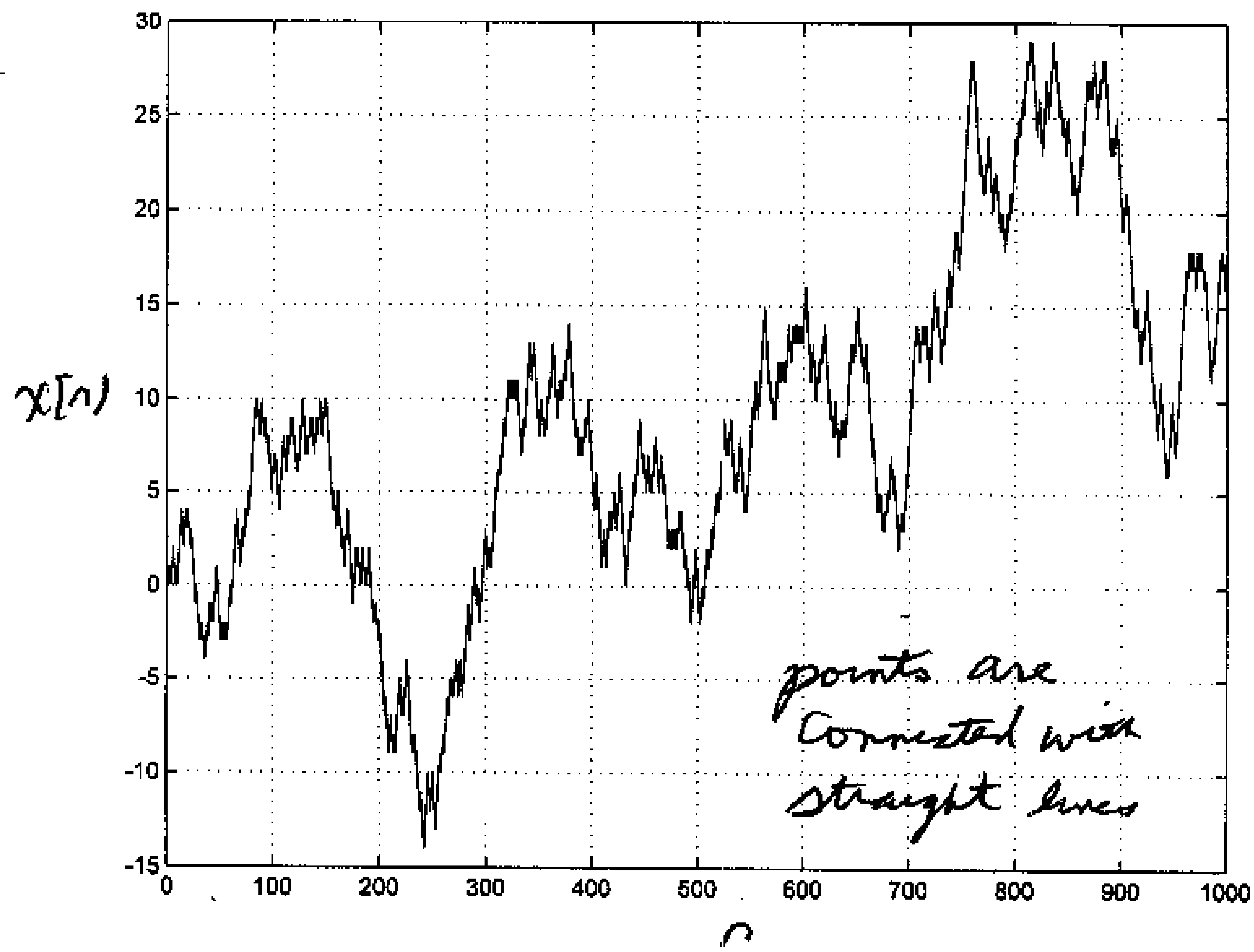
5) a) CTCV

b) DTDV

c) DTCV

d) CTDV

6)



As n becomes large $x[n]$ becomes more random (due to more IID random variables being summed).

```
%probprob16_6.m
%
clear all
rand('state',0)
N=1000;
for n=1:N
    if n==1
        u(1)= 2*floor(rand(1,1)+0.5)-1;
        x(1)=u(1);
    else
        u(n)=2*floor(rand(1,1)+0.5)-1;
        x(n)=x(n-1)+u(n);
    end
end
plot([1:N]',x)
grid
```

7) $E[x[n]] = \sum_{i=0}^{n} E[u[i]] = \sum_{i=0}^{n} (-1)\frac{1}{4} + 1(3/4)$

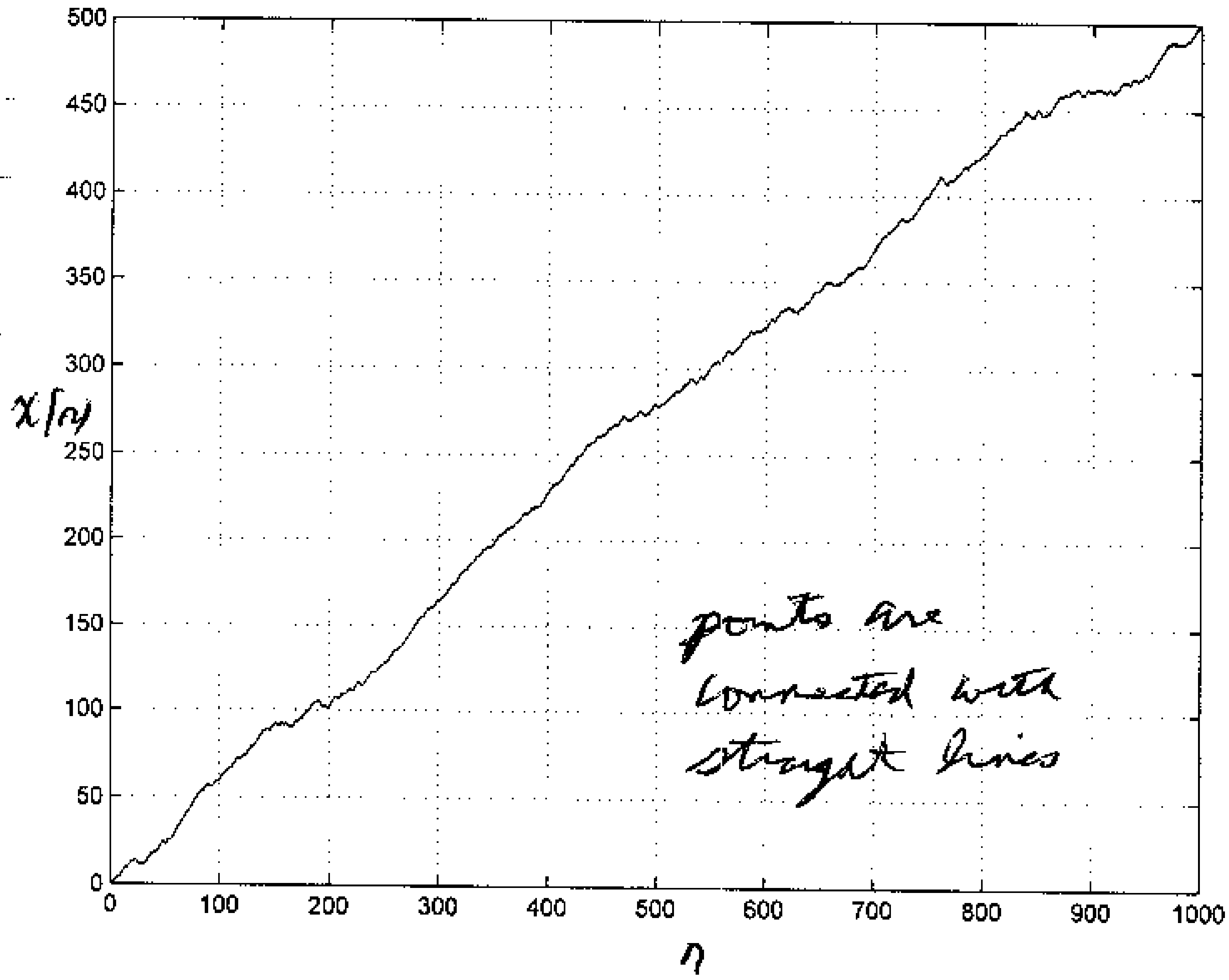
$= \frac{n+1}{2}$

$\text{var}[x[n]] = \sum_{i=0}^{n} \text{var}(u[i])$

$= \sum_{i=0}^{n} \left[E[u^2(i)] - E^2[u[i]]\right]$

$= \sum_{i=0}^{n} \left[1 - \left(\frac{1}{2}\right)^2\right] = 3/4\,(n+1)$

As n becomes large, $x[n] \to \infty$ due to mean of $\frac{n+1}{2}$.




```
%probprob16_7.m
%
clear all
rand('state',0)
N=1000;
for n=1:N
   if n==1
      u(1)= 2*floor(rand(1,1)+0.75)-1;
      x(1)=u(1);
   else
      u(n)=2*floor(rand(1,1)+0.75)-1;
      x(n)=x(n-1)+u(n);
   end
end
plot([1:N]',x)
grid
```

8) Marginal PDF/PMF does not depend on n.

$$E[x[100]] = E[x[10]] = 10$$

$$\text{var}(x[100]) = \text{var}(x[10]) = 1$$

9) $P[x[0] > 1, x[1] > 1, x[2] > 1] =$

$$\prod_{n=0}^{2} P[x[n] > 1] = P[x[0] > 1]^3$$

$$= \left(\int_1^\infty e^{-x}\,dx\right)^3 = \left[-e^{-x}\Big|_1^\infty\right]^3$$

$$= e^{-3}$$

10) Yes, since functions of independent random variables are independent or if X_1, X_2 are independent, then so are $g(X_1), g(X_2)$. Also if X_1, X_2 have the same marginal PDF/PMF, then so do $g(X_1), g(X_2)$.

11) $y[n] = (-1)^n x[n]$ $\qquad x[n] \sim \text{Ber}(\frac{1}{2})$

$\uparrow$ IID

$Y[n]$'s are still independent $\Rightarrow$ joint PMF = product of marginals

But $y[n_0] = (-1)^{n_0} x[n_0]$

$y[0] = x[0]$

$y[1] = -x[1]$

$$p_{Y[0]}[y_1] = \begin{cases} 1/2 & y_1 = 0 \\ 1/2 & y_1 = 1 \end{cases}$$

$$p_{Y[1]}[y_1] = \begin{cases} 1/2 & y_1 = 0 \\ 1/2 & y_1 = -1 \end{cases}$$

$\Rightarrow \rho_Y[0] \neq \rho_Y[1]$

not stationary

12) $\mu_X[n] = E[X[n]] = a^{|n|}E[U[n]] = 0$

$C_X[n_1, n_2] = E[X[n_1]X[n_2]]$

$= a^{|n_1|}a^{|n_2|}E[U[n_1]U[n_2]]$

$= a^{|n_1|+|n_2|}\sigma^2\delta[n_1 - n_2]$

not stationary $C_X[n_1+n_0, n_2+n_0] \neq C_X[n_1, n_2]$

Let $Y[n] = a^{-|n|}X[n]$

$\Rightarrow Y[n] = U[n]$ where $U[n]$ is WGN and is IID $\Rightarrow Y[n]$ is stationary

13) An increment is of the form

$$\sum_{i=n_1}^{n_2} U[n]$$

For nonoverlapping increments, they are independent but not stationary since for example

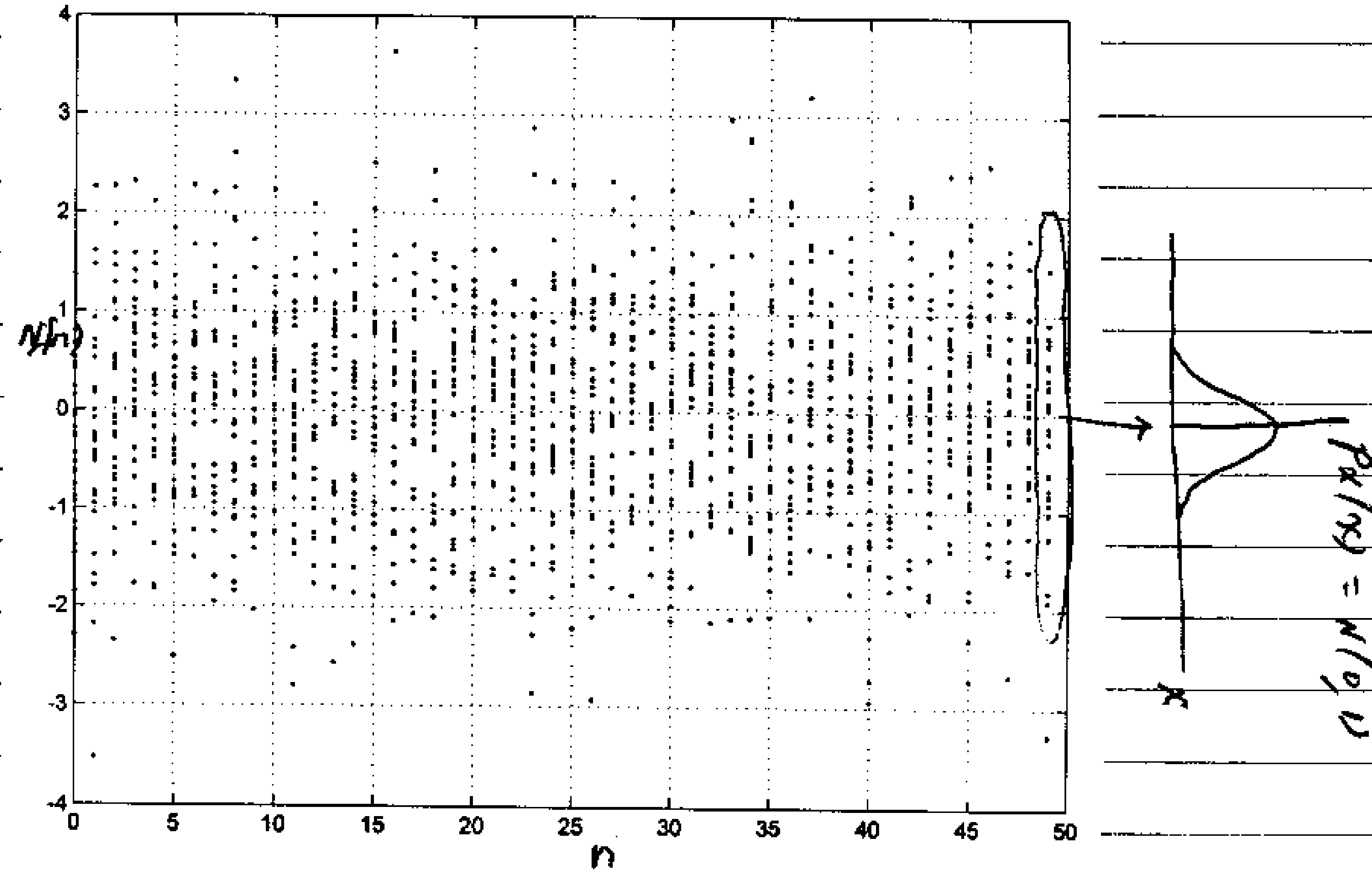
$\text{var}(U[0]+U[1])$ $\text{var}(U[2]+U[3])$

$= 1 + \frac{1}{2}$ $= \frac{1}{4} + \frac{1}{8}$

$= 3/2$ $= 3/8$

14)

```
%probprob16_14.m
%
clear all
rand('state',0)
N=50;
figure
plot([0:N-1]',randn(N,1),'.')
grid
hold on
for i=1:30
  plot([0:N-1]',randn(N,1),'.')
end
```



Turning the plot counter clockwise 90° produces the realization of a $N(0,1)$ random variable (see PDF plot above).

15) $\underline{C} = \sigma^2 \underline{I} = \begin{bmatrix} \sigma^2 & & \underline{0} \\ & \sigma^2 & \\ \underline{0} & & \ddots \ \sigma^2 \end{bmatrix}$ $N \times N$

$\det(\underline{C}) = \sigma^{2N}$

$\underline{C}^{-1} = \frac{1}{\sigma^2} \underline{I} = \begin{bmatrix} 1/\sigma^2 & & \underline{0} \\ & 1/\sigma^2 & \\ \underline{0} & & \ddots \ 1/\sigma^2 \end{bmatrix}$

$\underline{x}^T \underline{C}^{-1} \underline{x} = [x_1\ x_2 \dots x_N] \begin{bmatrix} 1/\sigma^2 & & 0 \\ & 1/\sigma^2 & \\ 0 & & \ddots \ 1/\sigma^2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$

$= \sum_{i=1}^{N} x_i^2 / \sigma^2$

$$p_{\underline{x}}(\underline{x}) = \frac{1}{(2\pi)^{N/2} \det^{1/2}(\underline{C})} e^{-\frac{1}{2}\underline{x}^T \underline{C}^{-1} \underline{x}}$$

$$= \frac{1}{(2\pi)^{N/2} (\sigma^{2N})^{\frac{1}{2}}} e^{-\frac{1}{2} \sum_{i=1}^{N} x_i^2/\sigma^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{N} x_i^2}$$

16) $\mu_x[n] = E[x[n]] = 0$ since $a = -\sqrt{3}$

$b = \sqrt{3}$

$E[x[n]] = \frac{1}{2}(a+b)$

$c_x[n_1, n_2] = 0 \quad n_1 \neq n_2$

due to independence

For $n_1 = n_2 = n \Rightarrow c_x[n_1, n_2] = \text{var}(x[n])$

$$= \frac{1}{12}\left(\sqrt{3} - (-\sqrt{3})\right)^2 = 1$$

$c_x[n_1, n_2] = \delta[n_2 - n_1]$

Same mean and covariance sequences as for WGN with variance $\sigma^2 = 1$.

Similar to property that random variables with different PDFs can have same means.

17) $x[n] = \frac{1}{N} \sum_{i=0}^{N-1} u[n-i]$.

Since $E[x[n]] = 0$ for all n

$$\text{cov}(x[0], x[1]) = E[x[0]\,x[1]]$$

$$= E\left[\frac{1}{N^2}\sum_i\sum_j u[0-i]\,u[1-j]\right]$$

$$= \frac{1}{N^2}\sum_i\sum_j \underbrace{E[u[-i]\,u[1-j]]}_{\sigma_u^2\,\delta[i-j+1]}$$

$$= \frac{\sigma_u^2}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\underbrace{\delta[i-j+1]}_{\text{must have } j = i+1 \text{ to contribute to sum}}$$

$$= \frac{\sigma_u^2}{N^2}(N-1)$$

$$\begin{array}{cc} & j \rightarrow \\ i \downarrow & \begin{bmatrix} (0,0) & \boxed{(0,1)} & (0,2) \\ (1,0) & (1,1) & \boxed{(1,2)} \\ (2,0) & (2,1) & (2,2) \end{bmatrix} \end{array}$$

for $N=3$

$\text{var}(x[n]) =$

$$\frac{1}{N^2}E\left[\sum_i u[n-i]\sum_j u[n-j]\right] \Rightarrow N-1 \text{ entries}$$

$$= \frac{1}{N^2}\sum_i\sum_j \underbrace{E[u[n-i]\,u[n-j]]}_{\sigma_u^2\,\delta[i-j]}$$

$$= \frac{1}{N^2}N\sigma_u^2 = \sigma_u^2/N \quad \text{for } n=0 \text{ and } n=1$$

$$\rho_{x[0],x[1]} = \frac{\sigma_u^2/N^2\,(N-1)}{\sqrt{\sigma_u^2/N\;\sigma_u^2/N}} = \frac{N-1}{N} = 1-\frac{1}{N}$$

As N increases the samples become more heavily correlated or smoothed.

18) $P[U[n] > 3] = Q(3) = 0.0013$

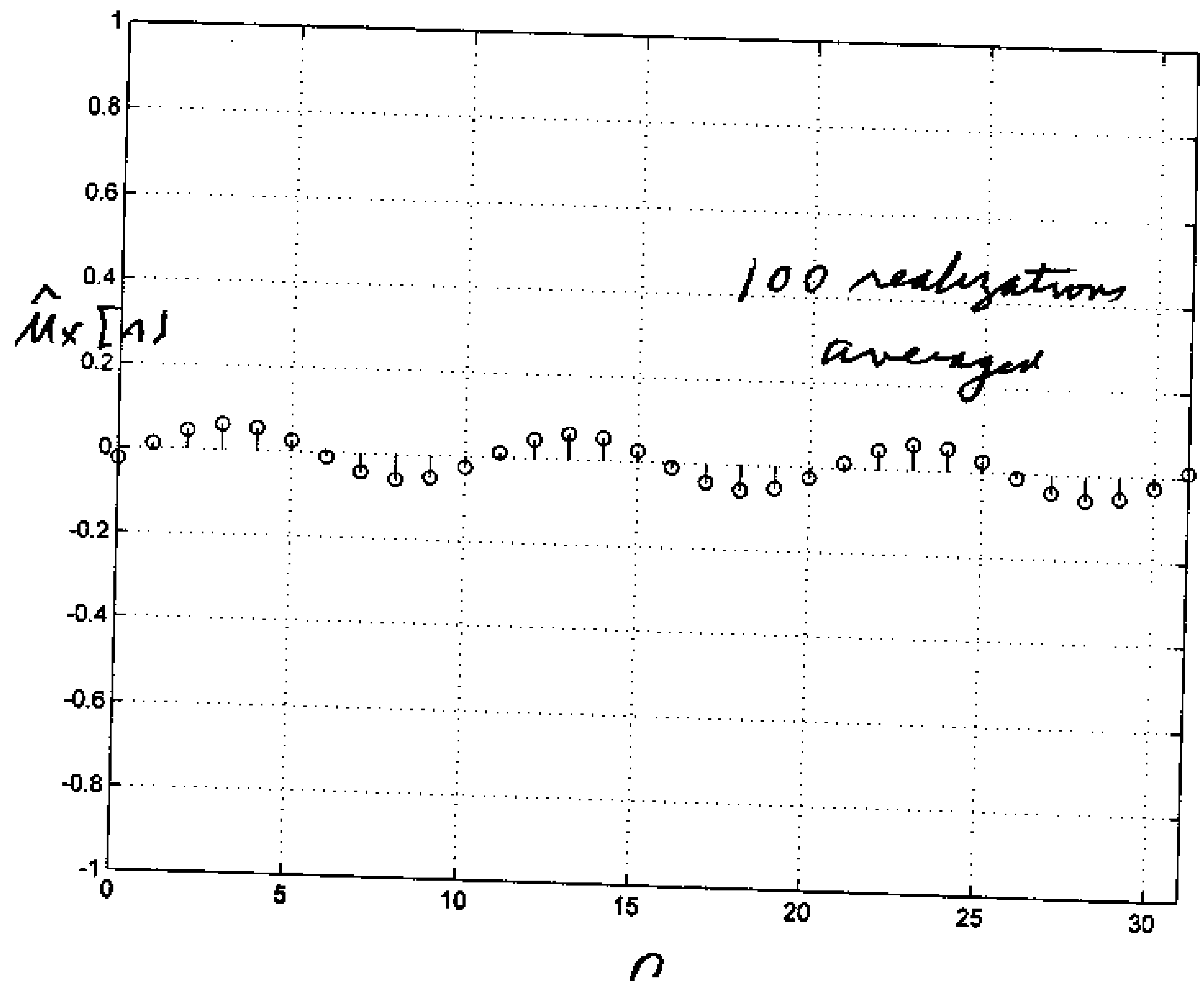
$$X[n] = \tfrac{1}{2}\big(\underbrace{U[n]}_{N(0,1)} + \underbrace{U[n-1]}_{N(0,1)}\big) \sim N(0, \tfrac{1}{2})$$

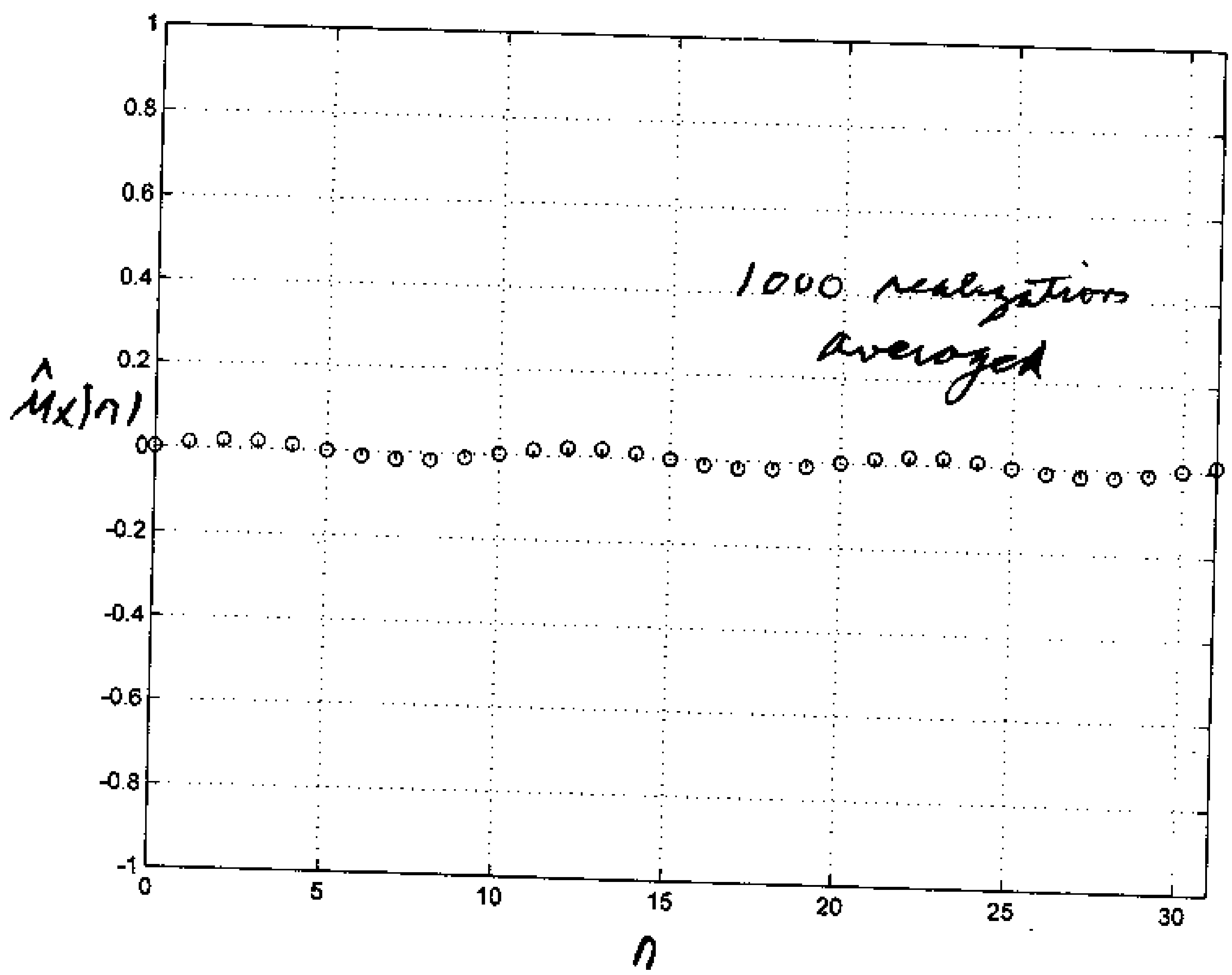
$$P[X[n] > 3] = Q\left(3/\sqrt{1/2}\right) = 0.0000011$$

Much less probable that MA random process will exhibit large values ⇒ will appear smoother when $x[n]$ plotted than $u[n]$.

19)

```
% probprob16_19.m
%
clear all
rand('state',0)
n=[0:31]';N=length(n);
xmean=zeros(N,1);
nreal=1000;
for i=1:nreal
s=cos(2*pi*0.1*n+2*pi*rand(1,1));
xmean=xmean+s/nreal;
end
figure
stem(n,xmean)
grid
axis([0 31 -1 1])
```


100 realizations
averaged
$\hat{\mu}_x[n]$
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30


1000 realizations
averaged
$\hat{\mu}_x[n]$
n
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30

20) $2\cos(2\pi(0.1))\,x[n-1]-x[n-2]$

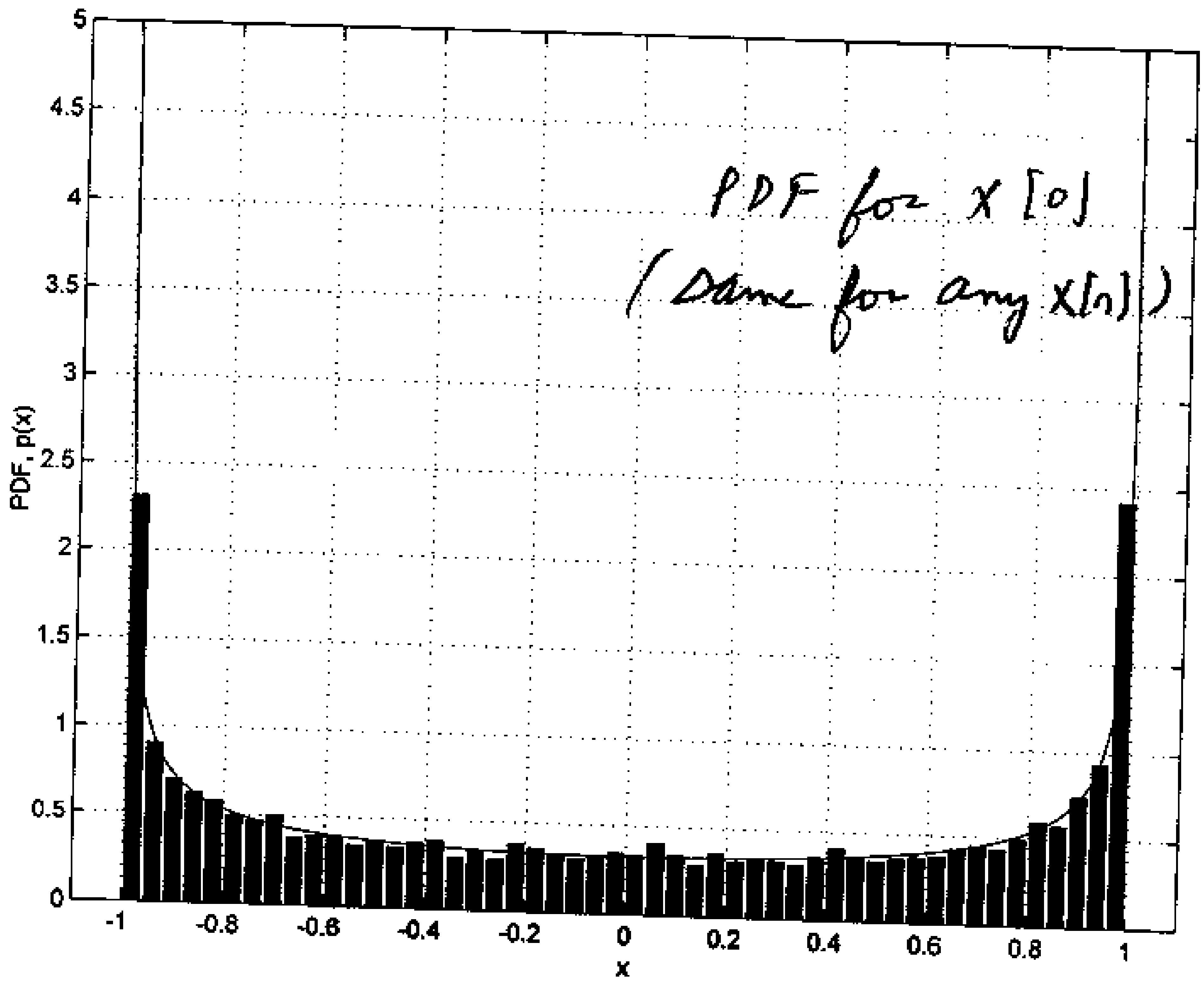
$= 2\cos(2\pi(0.1))\cos(2\pi(0.1)(n-1)) - \cos[2\pi(0.1)(n-2)]$

$= \cos[2\pi(0.1)n] + \cos[2\pi(0.1)(n-2)] - \cos[2\pi(0.1)(n-2)]$

$= \cos[2\pi(0.1)n] = x[n]$

Also, $x[n-2] = 2\cos(2\pi(0.1))x[n-1] - x[n]$ $\quad n \le 1$

21)




```
% probprob16_21.m
%
clear all
rand('state',0)
x(:,1)=cos(2*pi*rand(10000,1));
pdf(x,10000,50,-1.1,1.1,5)
hold on
xx=[-0.999:0.001:0.999]';
pdftrue=(1/pi)*(1./sqrt(1-xx.^2));
plot(xx,pdftrue)
hold off
```

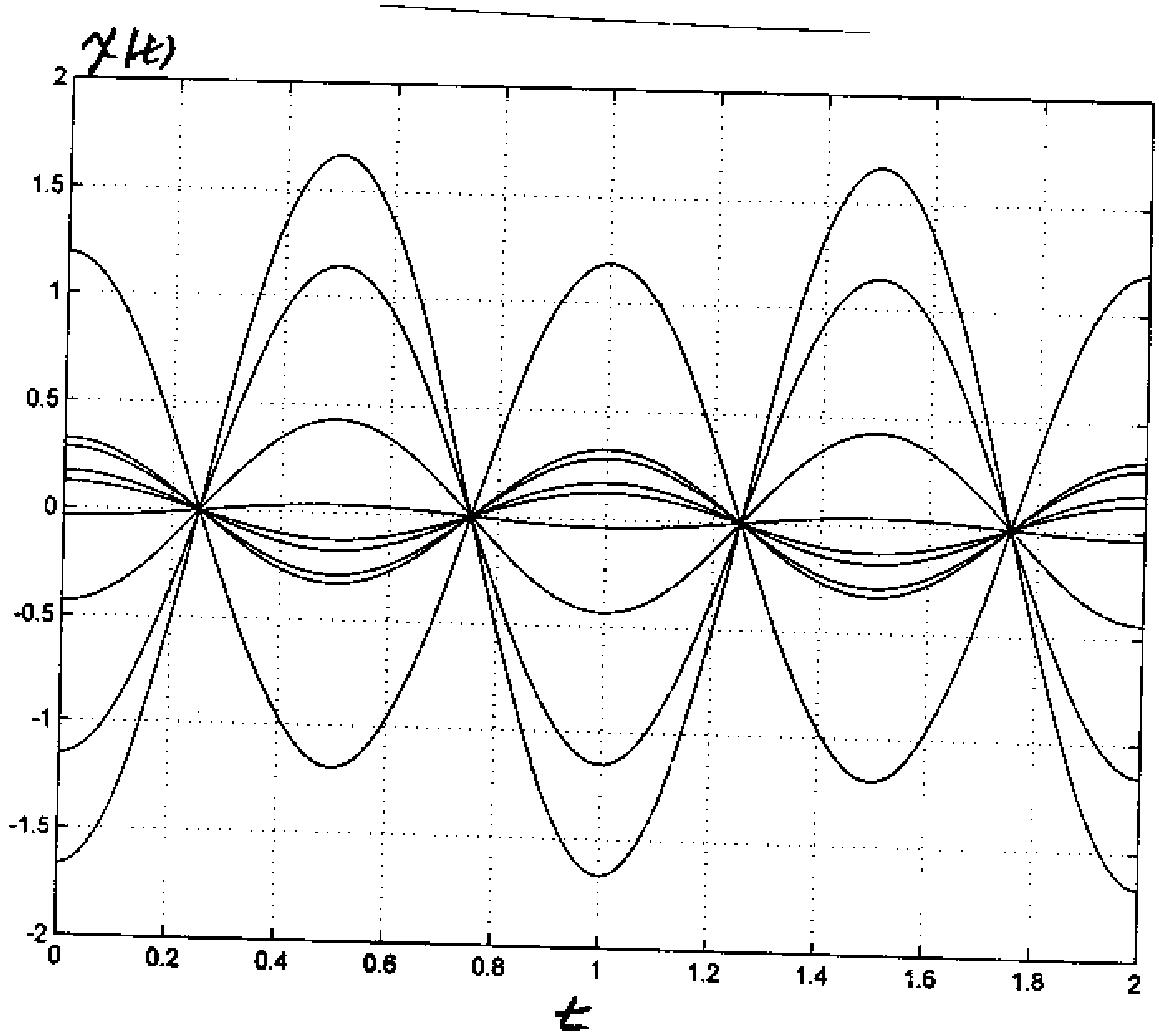
22) $\mu_X(t) = E[X(t)] = E[A]\cos 2\pi t = 0$

$$C_X(t_1,t_2) = E[X(t_1)X(t_2)] \qquad E[X(t)] = 0$$

$$= E[A^2 \cos 2\pi t_1 \cos 2\pi t_2]$$

$$= E[A^2]\cos 2\pi t_1 \cos 2\pi t_2 = \cos 2\pi t_1 \cos 2\pi t_2$$

Not stationary




```
% probprob16_22.m
%
clear all
randn('state',0)
t=[0:0.001:2]';
plot(t,randn(1,1)*cos(2*pi*t));
grid
hold on
for i=1:9
   plot(t,randn(1,1)*cos(2*pi*t));
end
hold off
```

2.3) $\mu_X[n] = E[v[n]] = E[v[n]] + \sin 2\pi f_0 n$
$= 0 + \sin 2\pi f_0 n = \sin 2\pi f_0 n$

$c_X[n_1, n_2] = E[(x[n_1] - \mu_X[n_1])(x[n_2] - \mu_X[n_2])]$

$= E[v[n_1]v[n_2]] = \sigma_v^2 \delta[n_2 - n_1]$

since $v[n]$ is WGN.

2.4) $E[y[n]] = E[x[n]] - E[x[n-1]] = \mu - \mu = 0$

However $y[n]$ now has correlated samples. For example,

$E[y[0]y[1]] = E[(x[0] - x[-1])(x[1] - x[0])]$
$= E[x[0]x[1]] - E[x^2[0]]$
$- E[x[-1]x[1]] + E[x[-1]x[0]]$

$= [E[x[0]x[1]] - \mu^2] - [E[x^2[0]] - \mu^2]$
$- [E[x[-1]x[1]] - \mu^2] + E[x[-1]x[0]] - \mu^2]$

$= \operatorname{cov}(x[0], x[1]) - \operatorname{var}(x[0]) - \operatorname{cov}(x[-1], x[1])$
$+ \operatorname{cov}(x[-1], x[0])$

$= 0 - 1 - 0 + 0 = -1 \Rightarrow$ not IID

2.5) $c_X[n_1, n_2] = E[x[n_1]x[n_2]]$ $\quad E[x[n]] = 0$

$= E[(h[0]v[n_1] + h[1]v[n_1 - 1])$
$(h[0]v[n_2] + h[1]v[n_2 - 1])]$

$$= h^2[0]E\{v[n_1]v[n_2]\} + h[0]h[1]E\{v[n_1]v[n_2-1]\}$$
$$+ h[1]h[0]E\{v[n_1-1]v[n_2]\}$$
$$+ h^2[1]E\{v[n_1-1]v[n_2-1]\}$$

$$= h^2[0]\sigma_v^2\delta[n_2-n_1] + h[0]h[1]\sigma_v^2\delta[n_2-1-n_1]$$
$$+ h[1]h[0]\sigma_v^2\delta[n_2-n_1+1] + h^2[1]\sigma_v^2\delta[n_2-n_1]$$

$$= \sigma_v^2(h^2[0]+h^2[1])\delta[n_2-n_1]$$
$$+ h[0]h[1]\sigma_v^2\delta[n_2-n_1-1]$$
$$+ h[1]h[0]\sigma_v^2\delta[n_2-n_1+1]$$

$$c_x[\underset{n_1}{0},\underset{n_2}{1}] = h[0]h[1]\sigma_v^2$$

this term only is nonzero

$$c_x[\underset{n_1}{9},\underset{n_2}{10}] = h[0]h[1]\sigma_v^2$$

26) $c_x[n_1,n_2] = E\{x[n_1]x[n_2]\}$ $\qquad E\{x[n]\}=0$

$$= E\left\{\sum_{i=0}^{n_1} v[i]\sum_{j=0}^{n_2} v[j]\right\}$$

$$= \sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \underbrace{E\{v[i]v[j]\}}_{\sigma_v^2\delta[j-i]}$$

For $n_1 \geq n_2$ for each value of j there will be a value for i which is identical. Hence, or as an example for $n_1 = 3$, $n_2 = 2$

$$\begin{array}{cc} & j \rightarrow \\ i \downarrow & \begin{bmatrix} \boxed{(0,0)} & (0,1) \\ (1,0) & \boxed{(1,1)} \\ (2,0) & (2,1) \end{bmatrix} \end{array}$$

terms for which $i = j$

$\Rightarrow$ number of terms $= \min(n_1, n_2)$

$$c_x(n_1, n_2) = \sigma_u^2 \min(n_1, n_2)$$

27) Should obtain

$$c_x(n_1, n_2) = \begin{cases} 1/2 & n_1 = n_2 \\ 1/4 & |n_2 - n_1| = 1 \\ 0 & |n_2 - n_1| > 1 \end{cases}$$

$$\Rightarrow \quad c_x(1,1) = 1/2$$
$$c_x(1,2) = 1/4$$
$$c_x(1,3) = 0$$

```
» eval('probprob16_27')

c_x_11 =

    0.5057


c_x_12 =

    0.2595


c_x_13 =

   -0.0016
```

```
% probprob16_27.m
%
clear all
randn('state',0)
nreal=10000;
for i=1:nreal
   for n=1:11
        u(n,1)=randn(1,1);
      if n==1
         u0=randn(1,1);
         x(n,i)=0.5*(u(n,1)+u0);
      else
         x(n,i)=0.5*(u(n,1)+u(n-1,1));
      end
   end
   end
   c_x_11=0;c_x_12=0;c_x_13=0;
   for i=1:nreal
      c_x_11=c_x_11+x(2,i)*x(2,i)/nreal;
      c_x_12=c_x_12+x(2,i)*x(3,i)/nreal;
      c_x_13=c_x_13+x(2,i)*x(4,i)/nreal;
   end
   c_x_11
   c_x_12
   c_x_13
```

28) $C_x[n_1, n_2] = \frac{1}{2}\cos(2\pi(0.1)(n_2 - n_1))$

$$\Rightarrow \operatorname{cov}(x[0], x[10]) = \frac{1}{2}\cos(2\pi(0.1)(10)) = \frac{1}{2}$$

$$\Rightarrow \hat{X}[10] = \frac{\operatorname{cov}(x[0], x[10])}{\operatorname{var}(x[0])} x[0] \quad \text{since } E[x[0]] = 0$$

$$= \frac{\operatorname{cov}(x[0], x[10])}{\operatorname{cov}(x[0], x[0])} x[0]$$

$$= \frac{1/2}{1/2} x[0]$$

$$\hat{X}[10] = x[0] \qquad (\text{Note: } \rho_{x[0], x[10]} = 1)$$

There is no error since $x[10] = x[0]$

See Figure 16.10 for some examples.

29) $$\hat{y} = E[Y] + \frac{\operatorname{cov}(X, Y)}{\operatorname{var}(X)}(X - E[X])$$

$X \to x[n_0]$

$Y \to X[n_0 + k]$

$$\Rightarrow \hat{X}[n_0 + k] = E[x[n_0 + k]] + \frac{\operatorname{cov}(x[n_0], x[n_0 + k])}{\operatorname{var}(x[n_0])} \cdot (x[n_0] - E(x[n_0]))$$

$$\hat{x}[n_0+k] = \mu_x[n_0+k] + \frac{c_x[n_0, n_0+k]}{c_x[n_0, n_0]}$$

$$\cdot (x[n_0] - \mu_x[n_0])$$

$$\hat{x}[n_0+k] = \cos 2\pi f_0(n_0+k) + \frac{0.9^k}{1} (x[n_0] - \cos 2\pi f_0 n_0)$$

as $k \to \infty$

$$\hat{x}[n_0+k] = \cos 2\pi f_0(n+k) = \mu_x[n_0+k]$$

since there is no correlation between $x[n_0]$ and $x[n_0+k]$.

30) $E[x[n]] = E[A]\, s[n] = 0$

$\Rightarrow \mu_x[n] = 0$ all n

$$c_x[n_1, n_2] = E[x[n_1]x[n_2]]$$
$$= E[A^2]\, s[n_1]\, s[n_2]$$
$$= s[n_1]\, s[n_2]$$

31) $\mu_x[n] = E[x[n]] = E[A\, v[n]]$

$= E[A]\, E[v[n]] = 0$

$$c_x[n_1, n_2] = E[x[n_1]x[n_2]]$$
$$= E[A v[n_1] A v[n_2]]$$
$$= E[A^2]\, E[v[n_1] v[n_2]]$$
$$= \sigma_A^2\, \sigma_v^2\, \delta[n_2 - n_1] \Rightarrow$$ white noise random process

32) $$\frac{\partial J(b)}{\partial b} = \frac{\partial}{\partial b} \sum_{n=0}^{N-1} (x[n] - b)^2$$

$$= -2\sum_{n=0}^{N-1}(x[n]-b) = 0$$

$$\Rightarrow \sum_{n=0}^{N-1} x[n] = \sum_{n=0}^{N-1} b = Nb$$

$$\hat{b} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$$

33) Let $\underline{H} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix}$ $N\times 2$ $\quad \underline{x} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$

$$\underline{H}^T\underline{H}\begin{bmatrix} b \\ a \end{bmatrix} = \underline{H}^T\underline{x}$$

$$\underline{H}^T\underline{H} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & N-1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix} = \begin{bmatrix} N & \Sigma n \\ \Sigma n & \Sigma n^2 \end{bmatrix}$$

$$\underline{H}^T\underline{x} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & N-1 \end{bmatrix}\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} = \begin{bmatrix} \Sigma x[n] \\ \Sigma n\, x[n] \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} = \underbrace{(\underline{H}^T\underline{H})^{-1}\underline{H}^T}_{\underline{G}}\underline{x}$$

From Theorem 12.7.1

$$\begin{bmatrix} \hat{b} \\ \hat{a} \end{bmatrix} \sim N(\underline{G}\underline{\mu}, \underline{G}\underline{C}\underline{G}^T)$$

where $\underline{x} \sim N(\underline{\mu}, \underline{C})$

But $\mu_x[n] = b$ for all n and $\underline{C} = \sigma^2\underline{I}$ (for WGN)

$$\Rightarrow \underline{\mu} = \underline{H}\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix} \qquad \underline{G}\underline{\mu} = (\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{H}\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$\underset{\sim}{G}\,\underset{\sim}{C}\,\underset{\sim}{G}^T = (\underline{H}^T\underline{H})^{-1}\underline{H}^T \sigma^2 \underline{I}\left[(\underline{H}^T\underline{H})^{-1}\underline{H}^T\right]^T$$

$$= \sigma^2(\underline{H}^T\underline{H})^{-1}\underline{H}^T\underline{H}\,\underbrace{(\underline{H}^T\underline{H})^{-1^T}}_{\substack{\left[(\underline{H}^T\underline{H})^T\right]^{-1} \\ \left[\underline{H}^T\underline{H}\right]^{-1}}}$$

$$= \sigma^2(\underline{H}^T\underline{H})^{-1}$$

Thus $\begin{bmatrix}\hat{b}\\ \hat{a}\end{bmatrix} \sim N\left(\begin{bmatrix} b \\ 0\end{bmatrix}, \sigma^2(\underline{H}^T\underline{H})^{-1}\right)$

The PDF of $\hat{a}$ is Gaussian with mean zero and variance the $(2,2)$ element of $\sigma^2(\underline{H}^T\underline{H})^{-1}$

$$\text{var}(\hat{a}) = \sigma^2\left[\begin{bmatrix} N & \Sigma n \\ \Sigma n & \Sigma n^2\end{bmatrix}^{-1}\right]_{22}$$

$$= \frac{\sigma^2 N}{N\Sigma n^2 - (\Sigma n)^2}$$

$$= \frac{\sigma^2}{\sum_{n=0}^{N-1} n^2 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)^2}$$

For $\sigma^2 = 10.05$ and $N = 108$ this is

$\text{var}(\hat{a}) = 9.57 \times 10^{-5}$ so that

$$\hat{a} \sim N(0, 9.57\times 10^{-5})$$

$$P[\hat{a} > 0.0173] = Q\left(\frac{0.0173}{\sqrt{9.57\times10^{-5}}}\right) = 0.0385$$

34) $$P\{\hat{a} > 0.0173\} = Q\left(\frac{0.0173}{\sqrt{\mathrm{var}(\hat{a})}}\right) \leq 10^{-6}$$

$$\Rightarrow \quad \frac{0.0173}{\sqrt{\mathrm{var}(\hat{a})}} = 4.7534$$

$$\Rightarrow \quad \mathrm{var}(\hat{a}) = 1.3246 \times 10^{-5}$$

$$= \frac{10.05}{\sum_{n=0}^{N-1} n^2 - \frac{1}{N}\left(\sum_{n=0}^{N-1} n\right)^2}$$

$N = 209$

Chapter 17

1) Since it is IID $\Rightarrow$ stationary $\Rightarrow$ WSS

$$\mu_X[n] = E[X[n]] = -1\cdot(1-p) + 1(p) = 2p-1 = \mu$$

$$E[X[n]X[n+k]] = \begin{cases} E[X[n]]E[X[n+k]] & k \neq 0 \\ E[X^2[n]] & k = 0 \end{cases}$$

$$= \mu^2 \quad k \neq 0$$
$$= (-1)^2(1-p) + 1^2\cdot p = 1 \quad k = 0$$

$$\Rightarrow \quad r_X[k] = \delta[k] + \mu^2(1-\delta[k])$$

2) $E[X[n]] = a_0 E[U[n]] + a_1 E[U[n-1]] = 0$

$$E[X[n]X[n+k]] = E[(a_0 U[n] + a_1 U[n-1])(a_0 U[n+k] + a_1 U[n+k-1])]$$
$$= a_0^2\,\delta[k] + a_0 a_1\,\delta[k-1] + a_1 a_0\,\delta[k+1] + a_1^2\,\delta[k]$$

Since $E[U[n_1]U[n_2]] = \begin{cases} 0 & n_1 \neq n_2 \\ \sigma_U^2 = 1 & n_1 = n_2 \end{cases}$

$$r_X[k] = (a_0^2 + a_1^2)\,\delta[k] + a_0 a_1\,\delta[k-1] + a_1 a_0\,\delta[k+1]$$

$X[n]$ is WSS

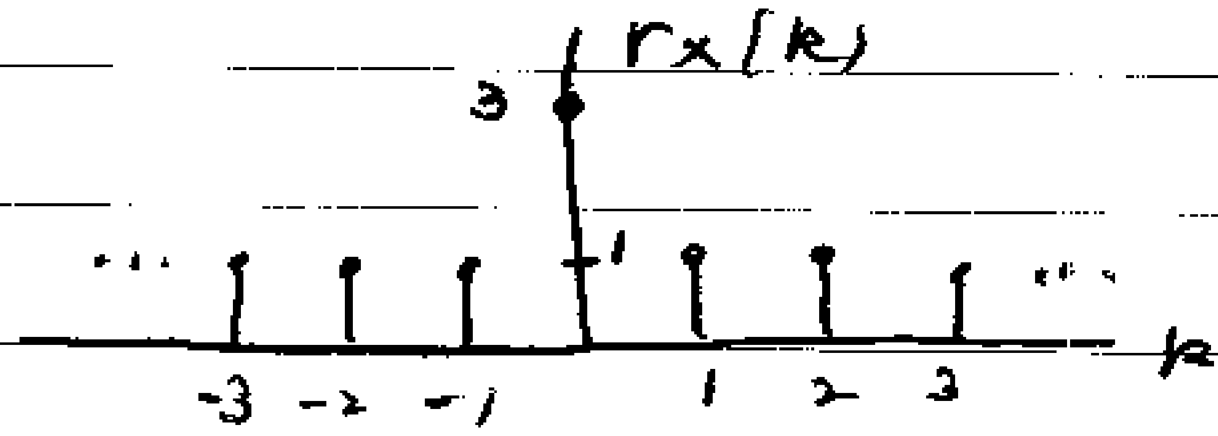
3) $E[X[n]] = E[A]\cos 2\pi f_0 n = 0$

$$E[X[n]X[n+k]] = E[A^2 \cos 2\pi f_0 n \cos 2\pi f_0(n+k)]$$
$$= E[A^2]\left(\tfrac{1}{2}\cos 2\pi f_0 k + \tfrac{1}{2}\cos(4\pi f_0 n + 2\pi f_0 k)\right)$$

is a function of n $\Rightarrow$ not WSS

4) $r_X[k] = c_X[n, n+k] + \mu^2$

$$= 2\,\delta[k] + 1^2 = 2\,\delta[k] + 1$$



5) $E[x[n]] = 0$ all n

$$r_x[k] = E[x[n]\,x[n+k]]$$
$$= E[x[n]]\,E[x[n+k]] \quad k \neq 0 \text{ (independence)}$$
$$= 0 \quad k \neq 0$$
$$= E[x^2[n]] \quad k = 0$$
$$= \mathrm{var}(x[n]) = 1 \quad \text{all } n$$

$\Rightarrow \mu_x[n] = 0$

$r_x[k] = \delta[k]$

$\Rightarrow$ WSS but not stationary since $p_{x[0]} \neq p_{x[1]}$ for example

6) $E[z[n]] = E[x[n]] + E[y[n]]$

$= \mu_x + \mu_y$ due to WSS of $x[n]$, $y[n]$

$$E[z[n]\,z[n+k]] = E[(x[n]+y[n])(x[n+k]+y[n+k])]$$
$$= \underbrace{E[x[n]\,x[n+k]]}_{r_x[k]} + \underbrace{E[x[n]\,y[n+k]]}_{= \mu_x \mu_y}$$
$$+ \underbrace{E[y[n]\,x[n+k]]}_{= \mu_y \mu_x} + \underbrace{E[y[n]\,y[n+k]]}_{r_y[k]}$$

$$r_z[k] = r_x[k] + r_y[k] + 2\mu_x \mu_y$$

$\mu_Z = \mu_X + \mu_Y$

$r_Z[k] = r_X[k] + r_Y[k] + 2\mu_X\mu_Y$

$\Rightarrow$ WSS

7) $E[Z[n]] = E[X[n]Y[n]] = E[X[n]]E[Y[n]]$ (ind)

$= \mu_X\mu_Y$

$E[Z[n]Z[n+k]] = E[X[n]Y[n]X[n+k]Y[n+k]]$

$= E[X[n]X[n+k]]\,E[Y[n]Y[n+k]]$

by independence

$= r_X[k]\,r_Y[k]$ ($X[n], Y[n]$ are WSS)

$\Rightarrow$ WSS

8) $r_X[k] = \left(\frac{1}{2}\right)^{|k|}$ all k

$r_X[0] = 1 > 0$ Prop 17.1

$r_X[-k] = \left(\frac{1}{2}\right)^{|-k|} = \left(\frac{1}{2}\right)^{|k|} = r_X[k]$ Prop 17.2

$|r_X[k]|_{MAX} = 1 = r_X[0]$

$\Rightarrow |r_X[k]| = \left(\frac{1}{2}\right)^{|k|} \le 1 = r_X[0]$

Prop 17.3

9) $r_X[0] > 0 \Rightarrow a > 0$

$|r_X[k]|_{MAX} = r_X[0] = a \Rightarrow |b| \le 1$

10) $\rho_{X[n],X[n+P]} = \dfrac{r_X[P]}{r_X[0]} = 1$ since

$r_X[k+P] = r_X[k]$ for all k, let $k=0$

$\Rightarrow$ $X[n+P]$ is perfectly predictable from $\hat{X}[n] = X[n]$

For randomly phased sinusoid

$x[n+P] = x[n+10] = \cos[2\pi(0.1)(n+10)+\Theta]$

$= \cos[2\pi(0.1)n+\Theta] = x[n]$

Yes

11) $$\rho_{x[n],x[n+k]} = \frac{\operatorname{cov}(x[n],x[n+k])}{\sqrt{\operatorname{var}(x[n])\operatorname{var}(x[n+k])}}$$

$$= \frac{E[x[n]x[n+k]] - E[x[n]]E[x[n+k]]}{\sqrt{\operatorname{var}(x[n])\operatorname{var}(x[n+k])}}$$

$$= \frac{r_x[k]-\mu^2}{\sqrt{(r_x[0]-\mu^2)(r_x[0]-\mu^2)}} = \frac{r_x[k]-\mu^2}{r_x[0]-\mu^2}$$

12) a) O.K.

b) No, $r_x[0] < 0$

c) OK

d) No, $r_x[0]$ not maximum

e) No, $r_x[1] \neq r_x[-1]$

f) OK

13) From (17.3) with $\mu_x[n] = 0$

$$\hat{x}[n_2] = \frac{c_x[n_1,n_2]}{c_x[n_1,n_1]}x[n_1]$$

Since $x[n]$ is WSS $\Rightarrow$ $c_x[n_1,n_2] = r_x[n_2-n_1]$

$$\Rightarrow \hat{x}[n_2] = \frac{r_x[n_2-n_1]}{r_x[0]}x[n_1]$$

But $r_x[k] = \frac{1}{2}\cos[2\pi(0.1)k]$

$$\hat{x}[n_2] = \cos[2\pi(0.1)(n_2-n_1)]\, x[n_1]$$

$$\hat{x}(1) = \cos[2\pi(0.1)]\, x[0]$$
$$\hat{x}[10] = \cos[2\pi(0.1)(10)]\, x[0] = x[0]$$

Since $\rho_{x[0],x[1]} = \frac{r_x[1]}{r_x[0]} = \cos[2\pi(0.1)] \neq \pm 1$

$x[1]$ cannot be perfectly predicted but since

$$\rho_{x[0],x[10]} = \frac{r_x[10]}{r_x[0]} = 1$$

$\Rightarrow$ $x[10]$ can be perfectly predicted. In fact,

$$x[10] = \cos[2\pi(0.1)(10) + \Theta]$$
$$= \cos(\Theta) = x[0] \quad (x[n] \text{ is periodic with } P = 10)$$

14)

$$\hat{x}[n_0+k_0] = \rho_{x[n_0],x[n_0+k_0]}\, x[n_0]$$

$$= \frac{r_x[k_0]}{r_x[0]}\, x[n_0]$$

$$= a^{|k_0|}\, x[n_0]$$

$$mse_{min} = E[(x[n_0+k_0] - \hat{x}[n_0+k_0])^2]$$
$$= E[(x[n_0+k_0] - a^{|k_0|} x[n_0])^2]$$
$$= r_x[0] - 2a^{|k_0|} r_x[k_0] + a^{2|k_0|} r_x[0]$$
$$= r_x[0]\left(1 + a^{2|k_0|} - 2a^{|k_0|} \frac{r_x[k_0]}{r_x[0]}\right)$$

$$= r_x[0]\left(1 + a^{2|k_0|} - 2a^{|k_0|}a^{|k_0|}\right)$$

$$= r_x[0]\left(1 - a^{2|k_0|}\right)$$

$$= \frac{\sigma_v^2}{1-a^2}\left(1 - a^{2|k_0|}\right)$$

As k_0 increases, mse_{min} also increases since there is less correlation between samples.

15) $r_x[0] = E\left(x^2[n]\right)$

$$= E\left(\sum_k a^k v[n-k] \sum_\ell a^\ell v[n-\ell]\right)$$

$$= \sum_k \sum_\ell a^k a^\ell \underbrace{E\left(v[n-k]\, v[n-\ell]\right)}_{r_v[k-\ell] = \sigma_v^2 \delta[k-\ell]}$$

$$= \sigma_v^2 \sum_{k=0}^{\infty} a^{2k} = \sigma_v^2 \frac{1}{1-a^2}$$

16) $E\left(x[n]\,x[n+k]\right) = E\left[\sum_{i=0}^{\infty} a^i v[n-i] \sum_{j=0}^{\infty} a^j v[n+k-j]\right]$

$$= \sum_i \sum_j a^{i+j} \underbrace{E\left(v[n-i]\, v[n+k-j]\right)}_{r_v[k+i-j] = \sigma_v^2 \delta[k+i-j]}$$

$$= \sum_{i=0}^{\infty} a^{i+k+i} \sigma_v^2 \qquad \text{(since } j = k+i$$

and $i \geq 0$, $j \geq 0$, $j = k+i$ only valid for $k \geq 0$)

$$= \sigma_v^2 a^k \sum_{i=0}^{\infty} a^{2i} = \frac{\sigma_v^2}{1-a^2} a^k \quad k \geq 0$$

17) $x[0] = v[0]$

$x[1] = a\,x[0] + v[1] = a\,v[0] + v[1]$

$x[2] = a\,x[1] + v[2] = a^2 v[0] + a\,v[1] + v[2]$

etc

$$\Rightarrow \quad x[n] = \sum_{k=0}^{n} a^k v[n-k]$$

$$E[x[n]] = \sum_{k=0}^{n} a^k \underbrace{E[v[n-k]]}_{=0} = 0$$

$$\text{var}(x[n]) = E[x^2[n]]$$

$$= E\left[\sum_{k=0}^{n}\sum_{l=0}^{n} a^k a^l v[n-k]\,v[n-l]\right]$$

$$= \sum_{k=0}^{n}\sum_{l=0}^{n} a^k a^l \underbrace{E[v[n-k]\,v[n-l]]}_{r_v[l-k] = \sigma_v^2 \delta[l-k]}$$

$$= \sigma_v^2 \sum_{k=0}^{n} a^{2k} = \sigma_v^2 \frac{1-a^{2(n+1)}}{1-a^2}$$

depends on n $\Rightarrow$ not WSS

As $n \to \infty$, $\text{var}(x[n]) \to \dfrac{\sigma_v^2}{1-a^2} = r_x[0]$

Only WSS as $n \to \infty$.

18) Let $x[-1] = I$ where $E[I] = 0$

$$\text{var}(I) = \frac{\sigma_v^2}{1-a^2}$$

$x[0] = a\,x[-1] + v[0]$

$$E[x[0]] = a\underbrace{E[I]}_{=0} + E[v[0]] = 0$$

$$\text{var}(x[0]) = E[x^2[0]]$$

$$= E[(aI + v[0])^2]$$

$$= a^2 E[I^2] + 2a\,\underbrace{E[I v[0]]}_{=0} + E[v^2[0]]$$

$$= a^2 \frac{\sigma_v^2}{1-a^2} + \sigma_v^2$$

$$= \frac{a^2\sigma_v^2 + (1-a^2)\sigma_v^2}{1-a^2}$$

$$= \frac{\sigma_v^2}{1-a^2}$$

$\Rightarrow$ $x[0]$ has same variance as if we had

$$x[n]\Big|_{n=0} = \sum_{k=0}^{\infty} a^k v[n-k]\Big|_{n=0}$$

$$= \Bigg(\underbrace{\sum_{k=0}^{n} a^k v[n-k]}_{\substack{\text{if } x[-1]=0, \text{ get} \\ \text{only this term}}} + \underbrace{\sum_{k=n+1}^{\infty} a^k v[n-k]}_{\substack{I \text{ compensates} \\ \text{for this} \\ \text{missing term}}} \Bigg)\Bigg|_{n=0}$$

Letting $x[-1] = I$ yields same mean and variance as if AR process started at $n = -\infty$.

19) $$\underline{R}_x = \begin{bmatrix} 1 & -7/8 & 0 \\ -7/8 & 1 & -7/8 \\ 0 & -7/8 & 1 \end{bmatrix}$$

$\underline{R}_x^{(1)}$, $\underline{R}_x^{(2)}$, $\underline{R}_x^{(3)}$

$$\det(\underline{R}_x^{(1)}) = 1 > 0$$

$$\det(\underline{R}_x^{(2)}) = 1\cdot 1 - (-7/8)(-7/8) = 15/64 > 0$$

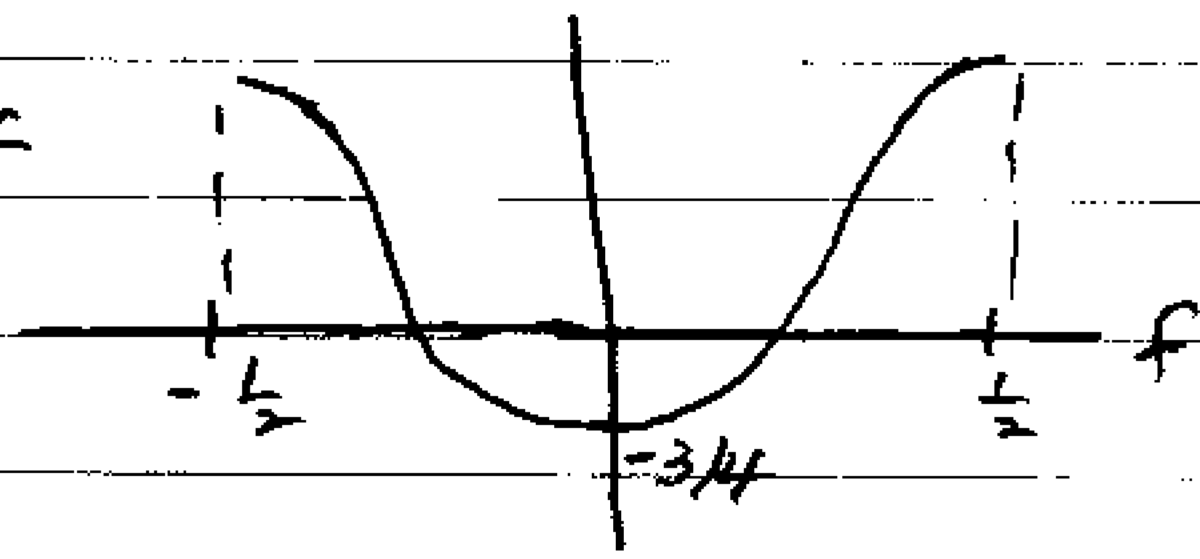
$$\det(\underline{R}_x^{(3)}) = 1\cdot\det\begin{bmatrix}1 & -7/8\\ -7/8 & 1\end{bmatrix} + 7/8\det\begin{bmatrix}-7/8 & 0\\ -7/8 & 1\end{bmatrix}$$

$$= 1 - 49/64 + 7/8(-7/8) = -34/64 < 0$$

$\Rightarrow$ $\underline{R}_x$ is not positive semidefinite

$$\sum_{k=-1}^{1} r[k]e^{-j2\pi fk} = -7/8\, e^{j2\pi f} + 1 - 7/8\, e^{-j2\pi f}$$

$$= 1 - 7/4\cos 2\pi f$$



Not ≥ 0 $\Rightarrow$ cannot be a PSD

For Fourier transform to be nonnegative $r_x[k]$ must be positive semidefinite.

20) $E[x[n]] = E[\frac{1}{2}(v[n]+v[n-1])]$

$$= \frac{1}{2}\mu + \frac{1}{2}\mu = \mu$$

$$r_x[k] = E[x[n]x[n+k]]$$

$$= E[\tfrac{1}{2}(v[n]+v[n-1])\,\tfrac{1}{2}(v[n+k]+v[n+k-1])]$$

$$= \tfrac{1}{4}(r_v[k]+r_v[k-1]+r_v[k+1]+r_v[k])$$

$\Rightarrow$ WSS

21) From Problem 17.20

$$r_x[k] = \tfrac{1}{2}r_v[k] + \tfrac{1}{4}r_v[k-1] + \tfrac{1}{4}r_v[k+1]$$

But $r_v[k] = E[v[n]v[n+k]]$

$$= \sigma_v^2 + \mu^2 \qquad k=0$$

$$= \mu^2 \qquad k \neq 0$$

$$\Rightarrow \quad r_U[k] = \mu^2 + \sigma_U^2 \delta[k]$$

$$r_x[k] = \mu^2 + \frac{\sigma_U^2}{2}\delta[k] + \frac{\sigma_U^2}{4}\delta[k-1] + \frac{\sigma_U^2}{4}\delta[k+1]$$

$$\mathrm{var}(\hat{\mu}_N) = \frac{1}{N}\sum_{k=-(N-1)}^{N-1}\left(1-\frac{|k|}{N}\right)\left(\frac{\sigma_U^2}{2}\delta[k] + \frac{\sigma_U^2}{4}\delta[k-1] + \frac{\sigma_U^2}{4}\delta[k+1]\right)$$

$$= \frac{1}{N}\left[\frac{\sigma_U^2}{2} + (1-1/N)\frac{\sigma_U^2}{4} + (1-1/N)\frac{\sigma_U^2}{4}\right]$$

$$= \frac{\sigma_U^2}{2N} + \frac{N-1}{N^2}\frac{\sigma_U^2}{4} + \frac{N-1}{N^2}\frac{\sigma_U^2}{4} \to 0 \text{ as } N \to \infty$$

2.2) $$\mathrm{var}(\hat{\mu}_N) = \frac{1}{N}\sum_{k=-k_0}^{k_0}\left(1-\frac{|k|}{N}\right)(r_x[k]-\mu^2)$$

$$\leq \frac{1}{N}\sum_{k=-k_0}^{k_0}(1)(\underbrace{|r_x[k]|}_{\leq r_x[0]} + \mu^2)$$

$\Rightarrow$ sum is finite $\Rightarrow \mathrm{var}(\hat{\mu}_N) \leq C/N \to 0$ as $N \to \infty$

2.3) $$\mathrm{var}(\hat{\mu}_N) = \mathrm{var}\left(\frac{1}{N}\sum_{n=0}^{N-1} x[n]\right)$$

$$= \mathrm{var}\left(\sum_{n=0}^{N-1} x[n]\right)/N^2$$

$$= \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} \mathrm{cov}(x[i], x[j])$$

$$= \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} [\underline{C}_x]_{ij}$$

$$= \frac{1}{N^2}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} (r_x[i-j]-\mu^2)$$

But $\sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g[i-j] = \sum_{k=-(N-1)}^{N-1} (N-|k|)\, g[k]$

$$\mathrm{var}(\hat{\mu}_N) = \frac{1}{N^2} \sum_{k=-(N-1)}^{N-1} (N-|k|)\,(r_x[k]-\mu^2)$$

$$= \frac{1}{N} \sum_{k=-(N-1)}^{N-1} \left(1-\frac{|k|}{N}\right)(r_x[k]-\mu^2)$$

24) $$\mathrm{var}(\hat{\mu}_N) = \frac{1}{N} \sum_{k=-(N-1)}^{N-1} \left(1-\frac{|k|}{N}\right)(1-0)$$

$$= \frac{1}{N^2} \sum_{k=-(N-1)}^{N-1} (N-|k|)$$

$$= \frac{1}{N^2}\left[2\sum_{k=1}^{N-1} (N-k) + N\right]$$

$$= \frac{1}{N^2}\left[2N(N-1) - 2\sum_{k=1}^{N-1} k + N\right]$$

$$= \frac{1}{N^2}\left[2N^2 - N - 2\left(\frac{N-1}{2}\right)(N)\right]$$

$$= \frac{1}{N^2}\left[2N^2 - N - N(N-1)\right] = 1$$

25) $$\hat{\mu}_N = \frac{1}{N}\sum_{n=0}^{N-1} \cos(2\pi(0.1)n+\theta)$$ for some realization $\Theta = \theta$

$\rightarrow 0$ due to hint

$$\hat{r}_x[k] = \frac{1}{N-k}\sum_{n=0}^{N-1-k} \cos(2\pi(0.1)n+\theta) \cdot \cos[2\pi(0.1)(n+k)+\theta]$$

$$= \frac{1}{N-k}\sum_{n=0}^{N-1} \frac{1}{2}\cos[4\pi(0.1)n + \underbrace{2\pi(0.1)k+2\theta}_{\phi}] + \frac{1}{2}\cos[2\pi(0.1)k]$$

$\rightarrow \frac{1}{2}\cos[2\pi(0.1)k]$ since limit of first term is zero by hint

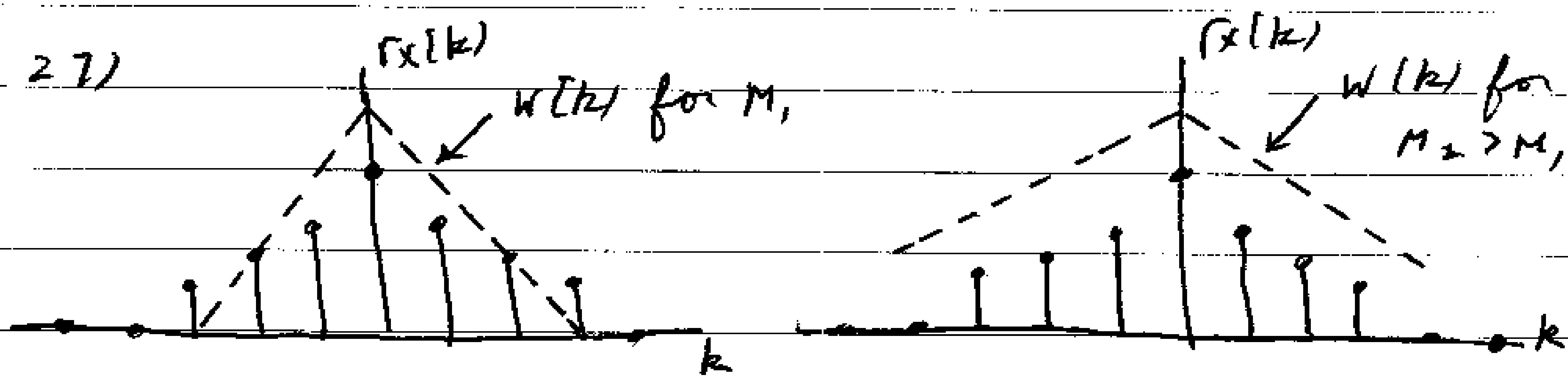
26) $$\sum_{m=-1}^{1}\sum_{n=-1}^{1} g[m-n] = \sum_{m=-1}^{1} \left(g[m+1] + g[m] + g[m-1] \right)$$

$$= g[0] + g[-1] + g[-2] + g[1] + g[0] + g[-1] + g[2] + g[1] + g[0] =$$

sum of elements of $\begin{bmatrix} g[0] & g[-1] & g[-2] \\ g[1] & g[0] & g[-1] \\ g[2] & g[1] & g[0] \end{bmatrix}$

$$= \sum_{k=-2}^{2} (3-|k|)\, g[k]$$

27)

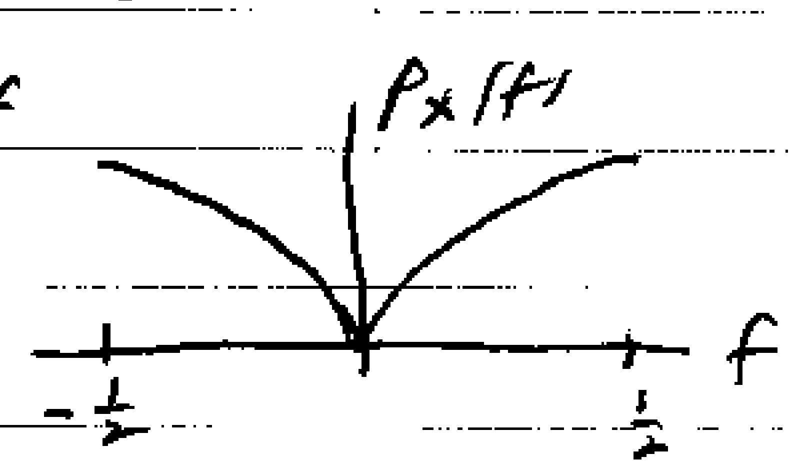


As $M \to \infty$ values of $r_x[k]$ that are significant (not near zero) get multiplied by $w[k]$ where $w[k] \to 1$.

28) $r_x[k] = 2\sigma_v^2 \delta[k] - \sigma_v^2 \delta[k-1] - \sigma_v^2 \delta[k+1]$

from (17.10)

$$\Rightarrow \quad P_x(f) = 2\sigma_v^2 - \sigma_v^2 e^{-j2\pi f} - \sigma_v^2 e^{j2\pi f}$$
$$= 2\sigma_v^2 - 2\sigma_v^2 \cos 2\pi f$$
$$= 2\sigma_v^2 (1 - \cos 2\pi f)$$

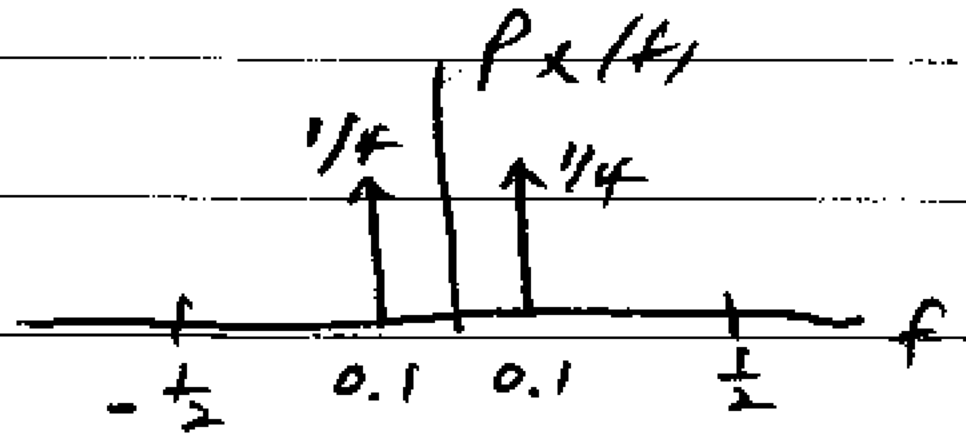


$P_x(0) = 0 \Rightarrow$ little power at lower frequencies

29) $r_X[k] = \frac{1}{2}\cos(2\pi(0.1)k)$

$$= \frac{1}{4} e^{j2\pi(0.1)k} + \frac{1}{4} e^{-j2\pi(0.1)k}$$

$$P_X(f) = \frac{1}{4}\delta(f-0.1) + \frac{1}{4}\delta(f+0.1) \qquad |f| \le \frac{1}{2}$$



all power at $f = 0.1$

30) $r_X[k] = E[AU[n]AU[n+k]]$

$$= E[A^2]E[U[n]U[n+k]]$$

since A and $U[n]$ are independent

$$= \sigma_A^2 \sigma_U^2 \delta[k]$$

$P_X(f) = \sigma_A^2\sigma_U^2 \Rightarrow X[n]$ is white noise

31) Not WSS $\Rightarrow$ PSD can't be defined

32) From Problem 17.2

$$r_X[k] = (a_0^2 + a_1^2)\delta[k] + a_0 a_1 \delta[k-1] + a_1 a_0 \delta[k+1]$$

$$P_X(f) = a_0^2 + a_1^2 + a_0 a_1 e^{-j2\pi f} + a_1 a_0 e^{j2\pi f}$$

$$= a_0^2 + a_1^2 + 2a_0 a_1 \cos 2\pi f$$

33) Since $X[n]$ is IID, the samples are uncorrelated. Also, the mean is zero.

Thus $r_X[k] = 0 \qquad k \ne 0$

$$= E[X^2[n]] \qquad k = 0$$

or $r_x[k] = \delta[k]$ since $E[x^2[n]] = 1$

$P_x(f) = 1$ $\quad x[n]$ is white noise

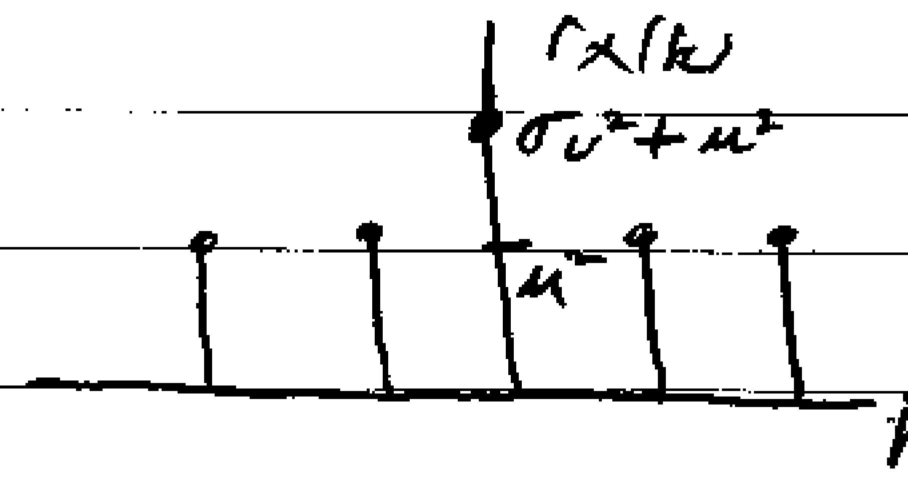
34) $r_x[k] = E[(v[n]+\mu)(v[n+k]+\mu)]$

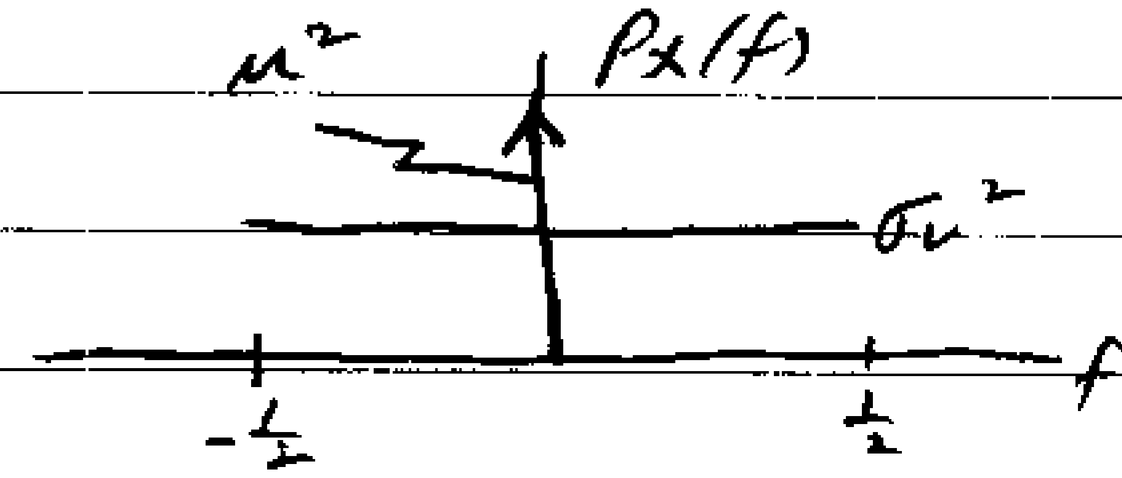
$= E[v[n]v[n+k]] + \mu^2 \quad (E[v[n]] = 0)$

$= r_v[k] + \mu^2$

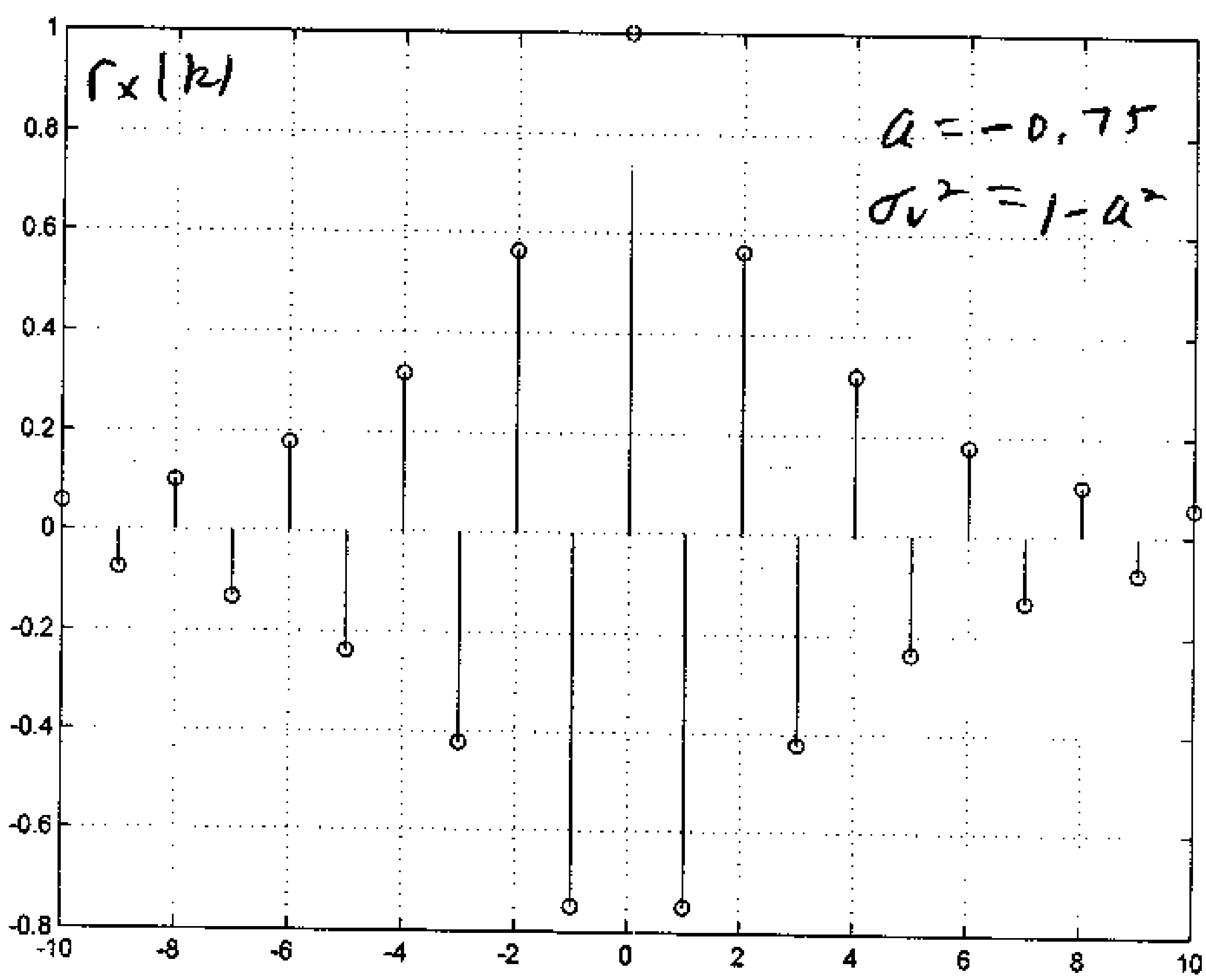
$= \sigma_v^2\delta[k] + \mu^2$ def. of white noise

$P_x(f) = \sigma_v^2 + \mu^2\delta(f)$

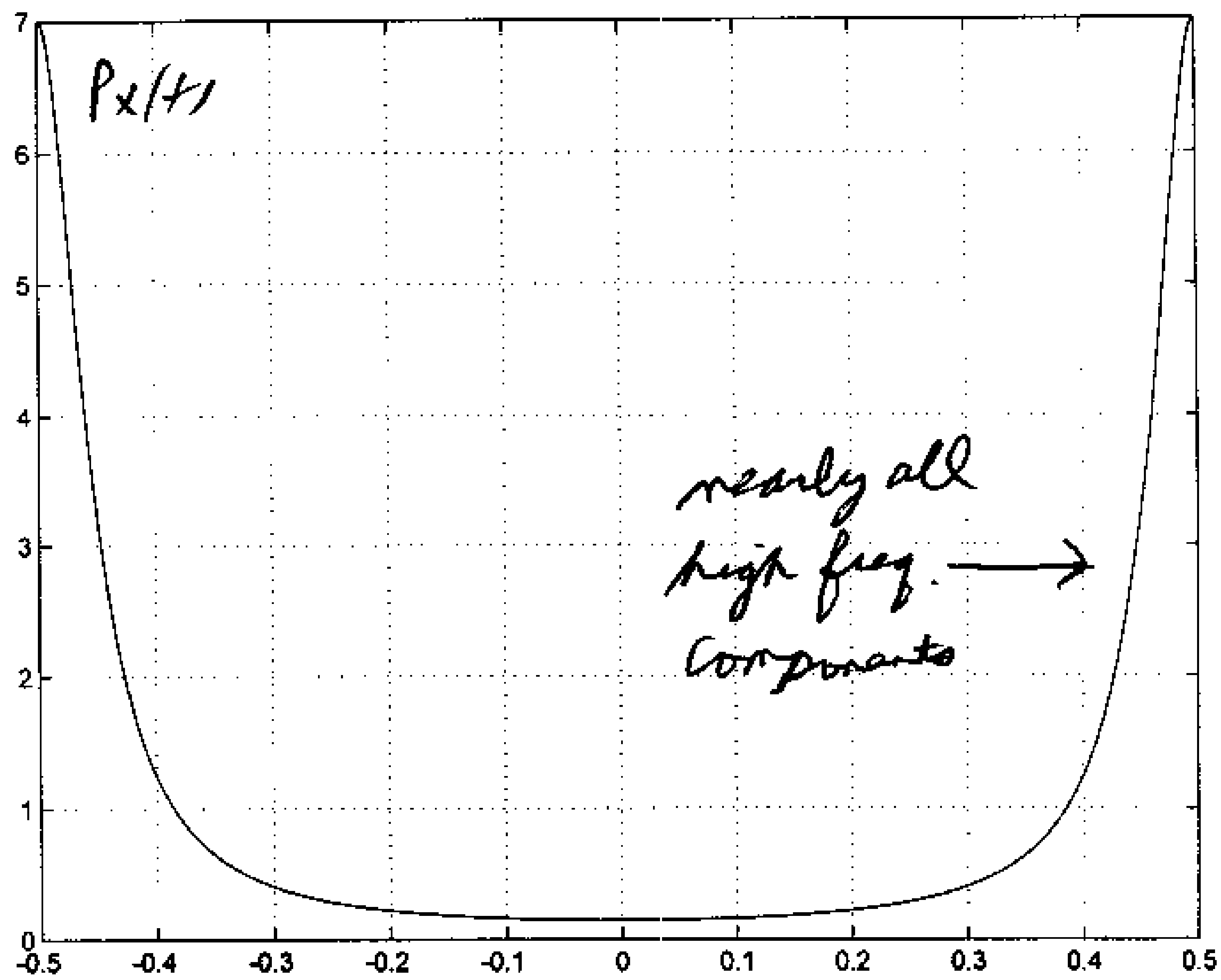




35)



ACS varies rapidly, a realization of $x[n]$ would also exhibit a rapid set of transitions from positive to negative and vice-versa.

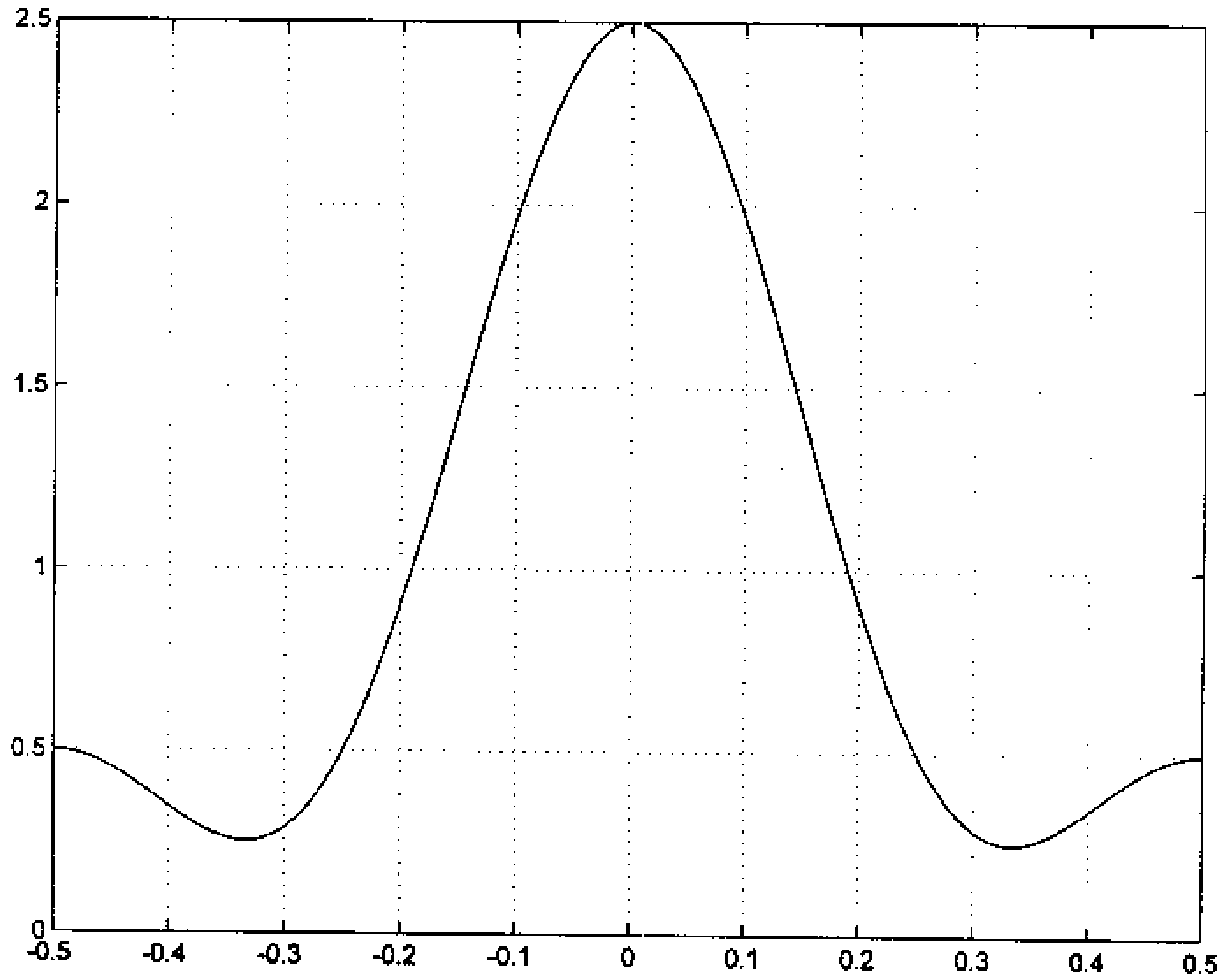



```
% probprob17_35.m
%
clear all
a=-0.75;
varu1=1-a^2;
k=[-10:10]';
r=(varu1/(1-a^2))*a.^abs(k);
figure
stem(k,r)
grid
Nfft=1024;
f=[0:Nfft-1]'/Nfft-0.5;
P=varu1./(abs(fftshift(fft([1 -a]',Nfft)))).^2;
figure
plot(f,P)
grid
```

36) $$P_x(f) = \sum_{k=-\infty}^{\infty} r_x(k)\, e^{-j2\pi f k}$$

$$= 1 + \frac{1}{2} e^{j2\pi f} + \frac{1}{2} e^{-j2\pi f} + \frac{1}{4} e^{j4\pi f} + \frac{1}{4} e^{-j4\pi f}$$

$$= 1 + \cos 2\pi f + \frac{1}{2}\cos 4\pi f$$




```
% probprob17_36.m
%
clear all
f=[-0.5:0.001:0.5]';
P=1+cos(2*pi*f)+0.5*cos(4*pi*f);
figure
plot(f,P)
grid
```

37) No since $r_x(k) = 1 \quad k=0$

$\frac{1}{2} \quad k = \pm 1$

38) $P_x(f) = (1 + e^{-j2\pi f} + \frac{1}{2} e^{-j4\pi f})$

$\cdot (1 + e^{j2\pi f} + \frac{1}{2} e^{j4\pi f})$

$= 1 + e^{j2\pi f} + \frac{e^{j4\pi f}}{2} + e^{-j2\pi f} + 1 + \frac{e^{j2\pi f}}{2}$

$+ \frac{e^{-j4\pi f}}{2} + \frac{e^{-j2\pi f}}{2} + \frac{1}{4}$

$= \frac{1}{2} e^{j4\pi f} + \frac{3}{2} e^{j2\pi f} + 9/4 + \frac{3}{2} e^{-j2\pi f} + \frac{1}{2} e^{-j4\pi f}$

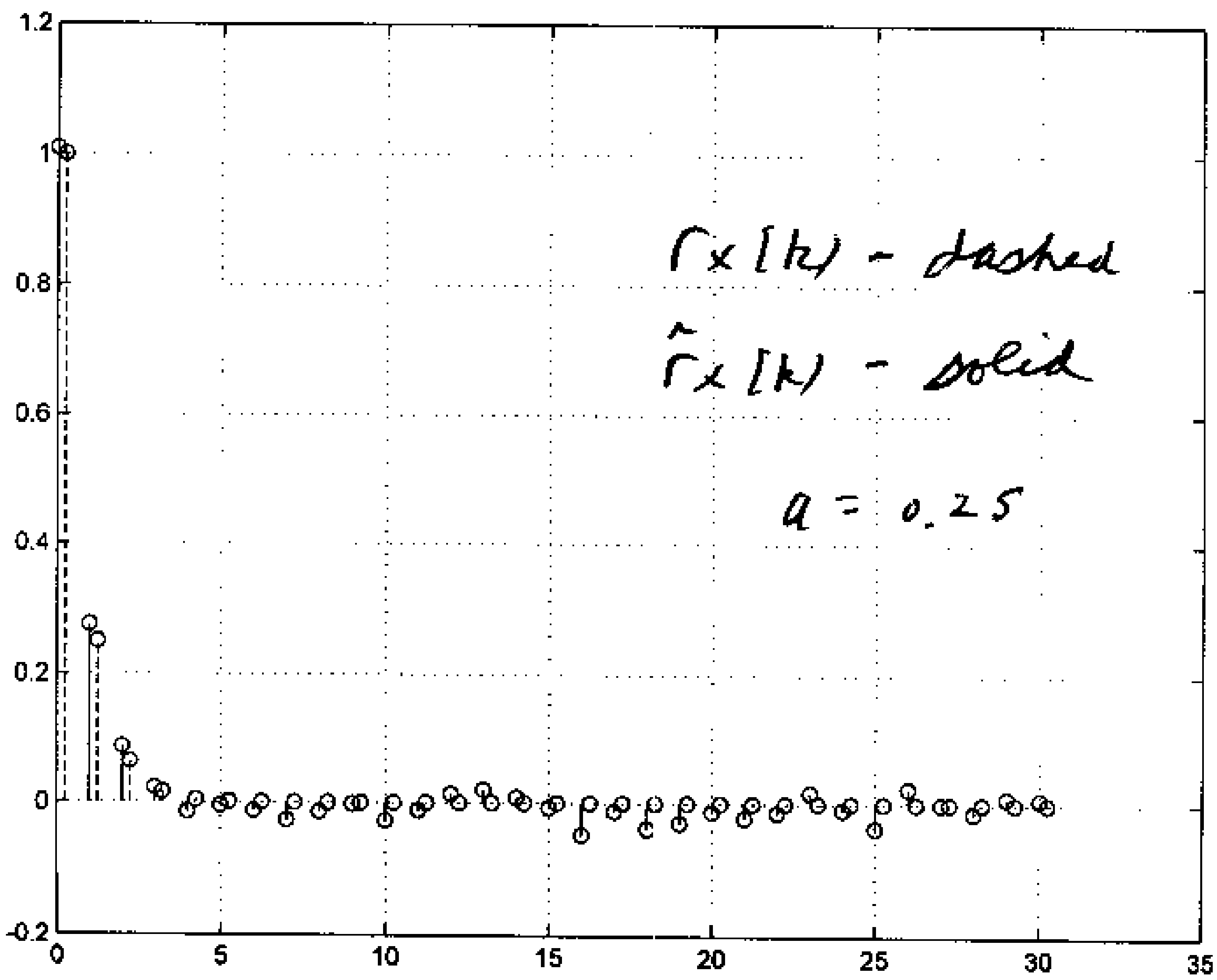
$r_x(k) = 9/4 \quad k=0$

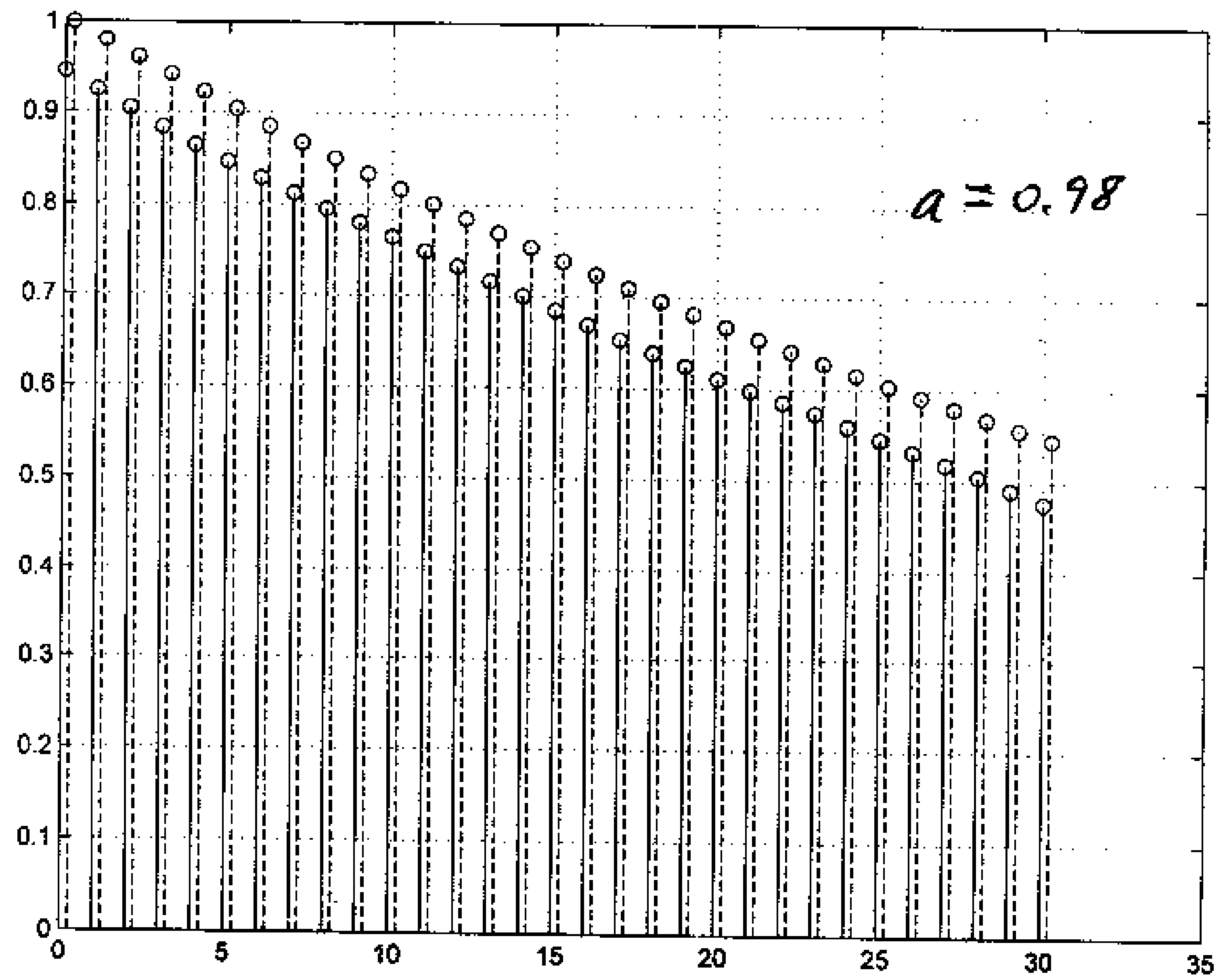
$3/2 \quad k = \pm 1$

$1/2 \quad k = \pm 2$

$0 \quad$ otherwise

39)






```
% probprob17_39.m
%
clear all
randn('state',0)
n=[0:1999]';N=length(n);
a1=0.25;a2=0.98;
varu1=1-a1^2;varu2=1-a2^2;
varx1=varu1/(1-a1^2);varx2=varu2/(1-a2^2);
x1(1,1)=sqrt(varx1)*randn(1,1);
x2(1,1)=sqrt(varx2)*randn(1,1);
for i=2:N
   x1(i,1)=a1*x1(i-1)+sqrt(varu1)*randn(1,1);
   x2(i,1)=a2*x2(i-1)+sqrt(varu2)*randn(1,1);
end
r1true=(varu1/(1-a1^2))*a1.^[0:30]';
r2true=(varu2/(1-a2^2))*a2.^[0:30]';
for k=0:30
   r1est(k+1,1)=(1/(N-k))*sum(x1(1:N-k).*x1(1+k:N));
   r2est(k+1,1)=(1/(N-k))*sum(x2(1:N-k).*x2(1+k:N));
end
figure
stem([0:30]',r1est,'-')
hold on
stem([0:30]'+0.25,r1true,'--')
grid
hold off
figure
stem([0:30]',r2est,'-')
hold on
stem([0:30]'+0.25,r2true,'--')
grid
hold off
```

40) $P_X(-f) = P_X(f)$ and $P_X(f)$ is real for any a, b. Need only $P_X(f) \geq 0$

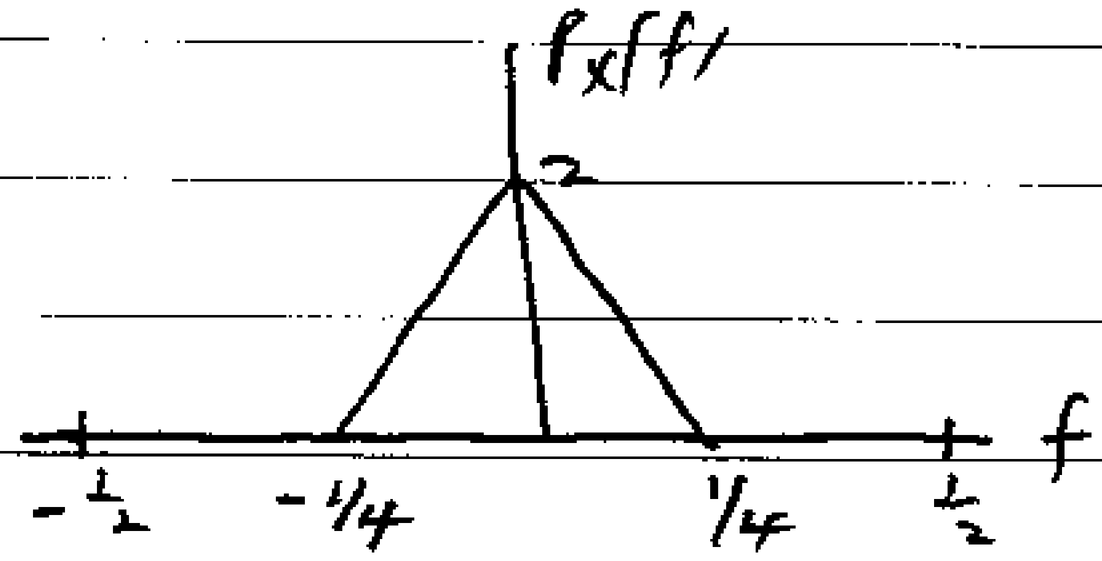
$P_X(0) \geq 0$, $P_X(\frac{1}{2}) \geq 0$ since $P_X(f)$ either increases or decreases for $0 < f < \frac{1}{2}$.

$P_X(0) = a + b \geq 0$

$P_X(\frac{1}{2}) = a - b \geq 0$

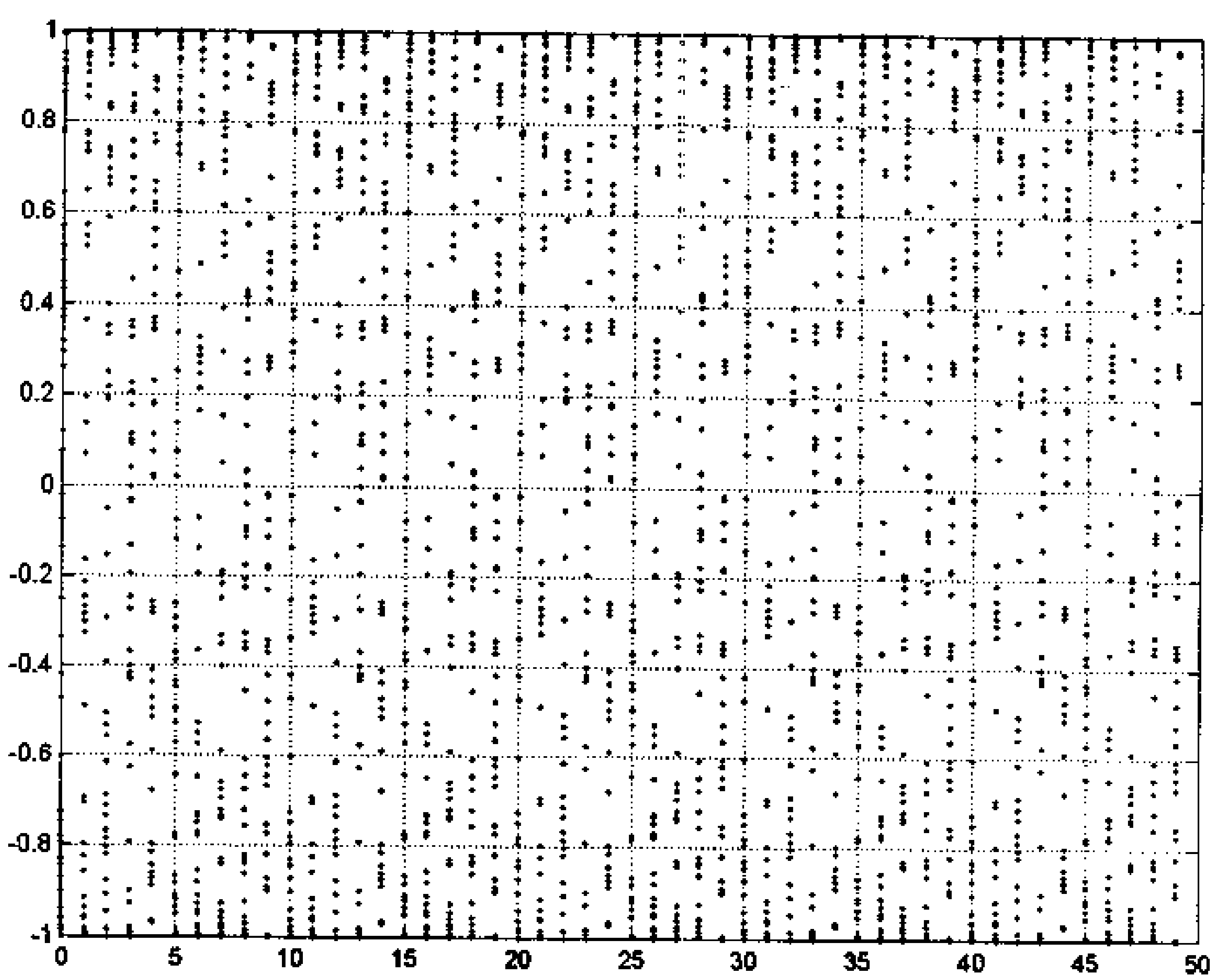
$\Rightarrow a \geq -b, \; a \geq b \Rightarrow a \geq |b|$ and $a > 0$

41)



Total average power $= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_X(f)\,df =$ area of triangle $= \frac{1}{2}(\frac{1}{2})(2) = \frac{1}{2}$

42)



Compare to Fig 16.12 ⇒ more probability mass near ±1.

Theoretically $E[x[n]] = 0$

$$r_x[2] = \frac{1}{2}\cos(2\pi(0.1)2)$$
$$= 0.1545$$

```
Ex10 =

  -0.0105
```

← $\hat{E}[x[10]]$

```
Ex12 =

   0.0117
```

← $\hat{E}[x[12]]$

```
r1012 =

   0.1501
```

← $\hat{E}[x[10]x[12]]$

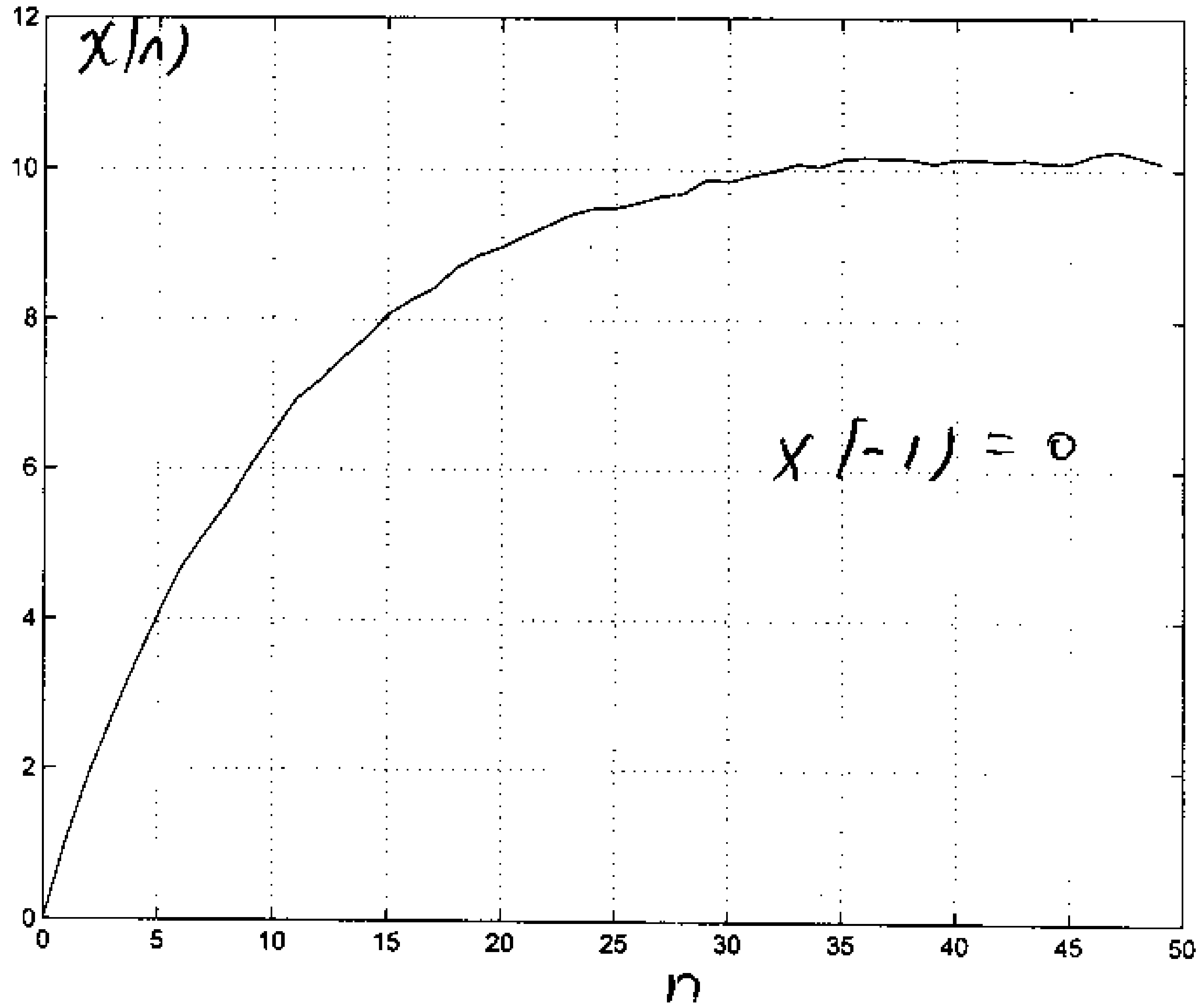
```
r1214 =

   0.1533
```

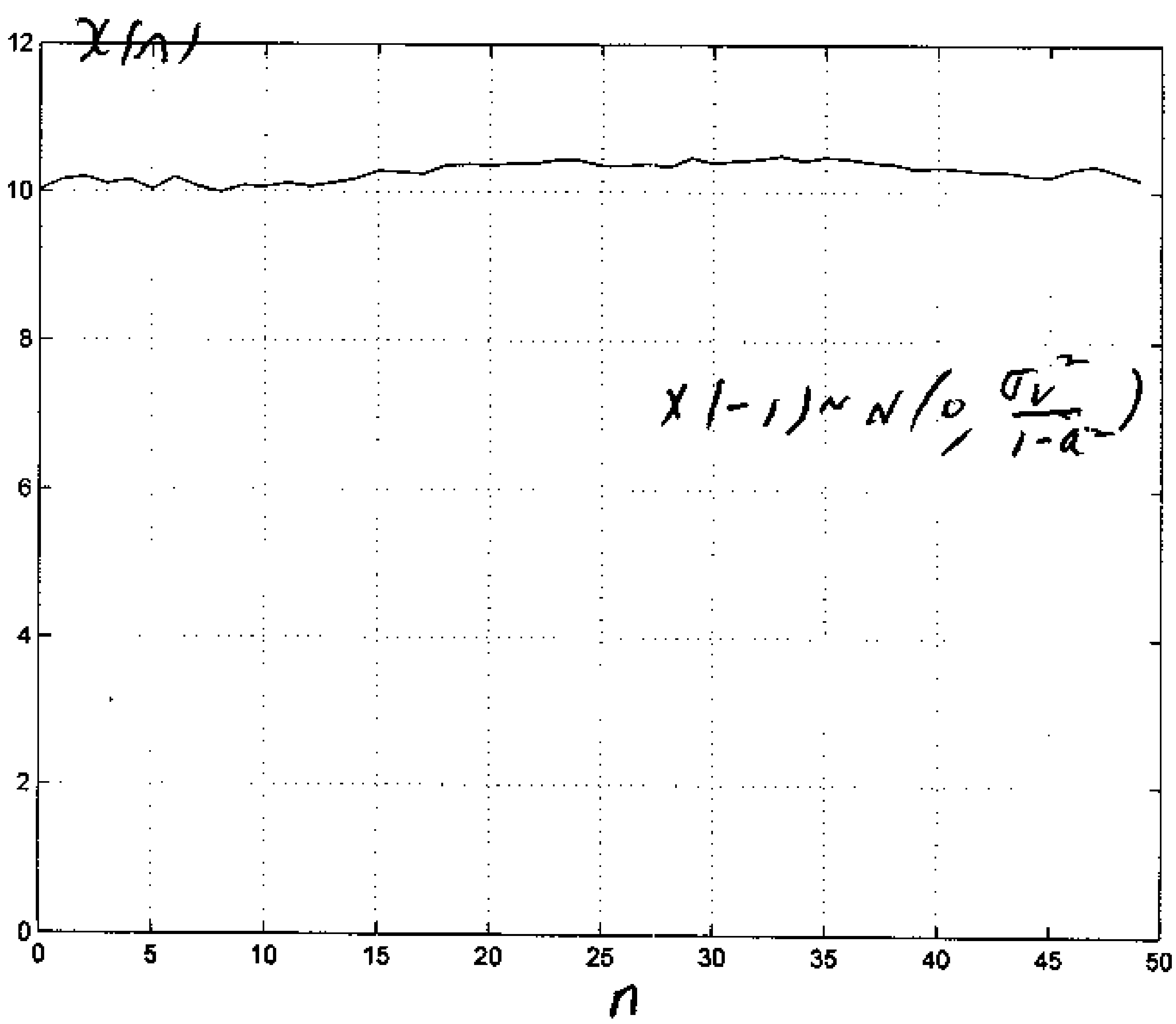
← $\hat{E}[x[12]x[14]]$

```
r2true =

   0.1545
```

```
% probprobl7_42.m
%
clear all
rand('state',0)
N=50;
n=[0:N-1]';
for i=1:1000
   x(:,i)=cos(2*pi*0.1*n+2*pi*rand(1,1));
end
figure
for i=1:50
   plot(n,x(:,i),'.')
 hold on
end
grid
hold off
Ex10=mean(x(10,:))
Ex12=mean(x(12,:))
r1012=mean(x(10,:).*x(12,:))
r1214=mean(x(12,:).*x(14,:))
r2true=0.5*cos(2*pi*0.1*2)
```

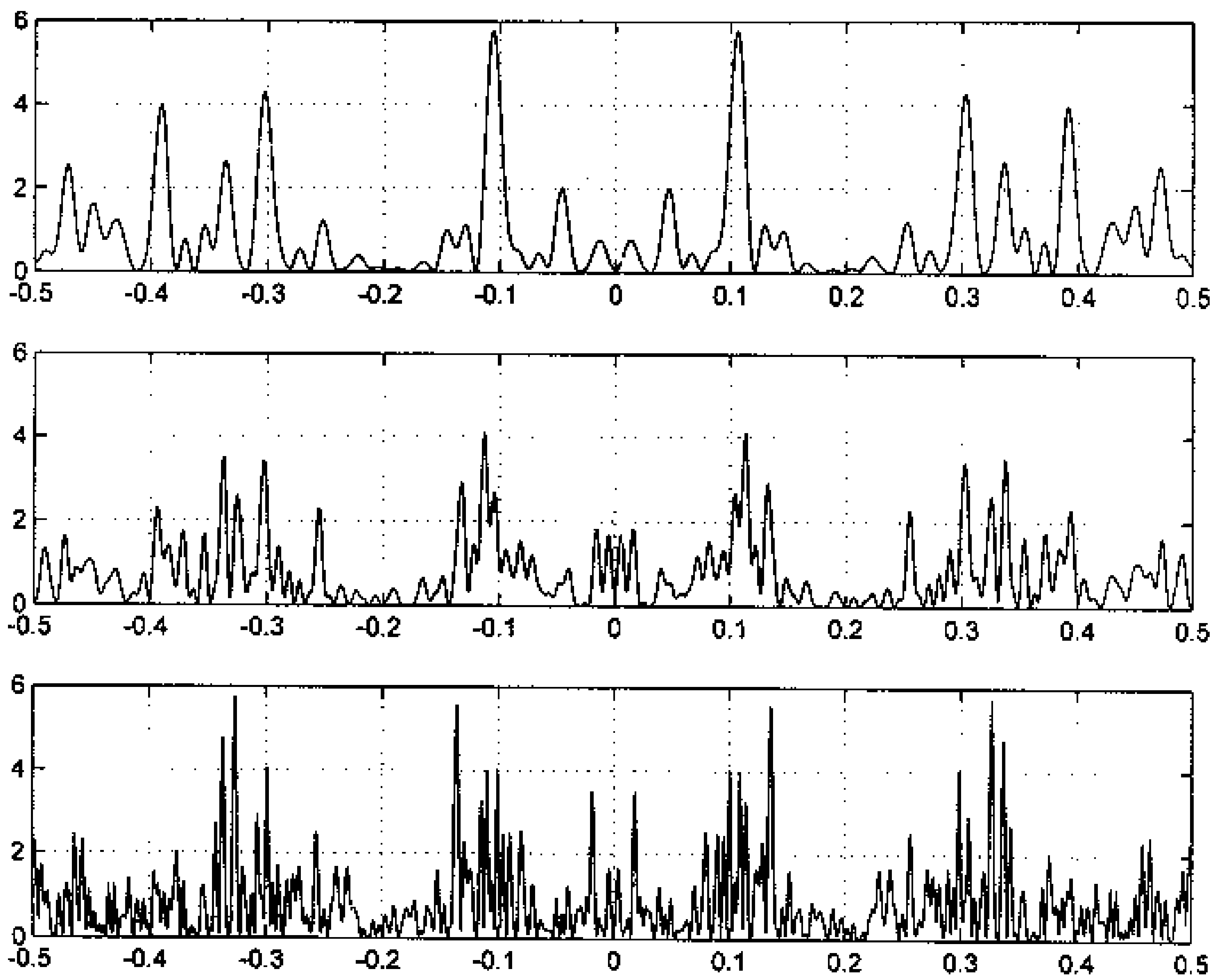
43) $r_x(0) = \frac{\sigma_v^2}{1-a^2} = \frac{1}{1-0.95^2} = 10.25$





```
% probprob17_43.m
%
clear all
randn('state',0)
N=50;M=10000;
a=0.95;varu=1;
x1=zeros(N,M);x2=x1;
for m=1:M
   for n=1:N
      if n==1
         x1(1,m)=0;
         x2(1,m)=sqrt(varu/(1-a^2))*randn(1,1);
         else
      u=randn(1,1);
      x1(n,m)=a*x1(n-1,m)+sqrt(varu)*u;
      x2(n,m)=a*x2(n-1,m)+sqrt(varu)*u;
   end
end
end
   for n=1:N
      varest1(n,1)=(x1(n,:)*x1(n,:)')/M;
      varest2(n,1)=(x2(n,:)*x2(n,:)')/M;
   end
   figure
   plot([0:N-1]',varest1)
   grid
   axis([0 N 0 12])
   figure
   plot([0:N-1]',varest2)
   grid
    axis([0 N 0 12])
```

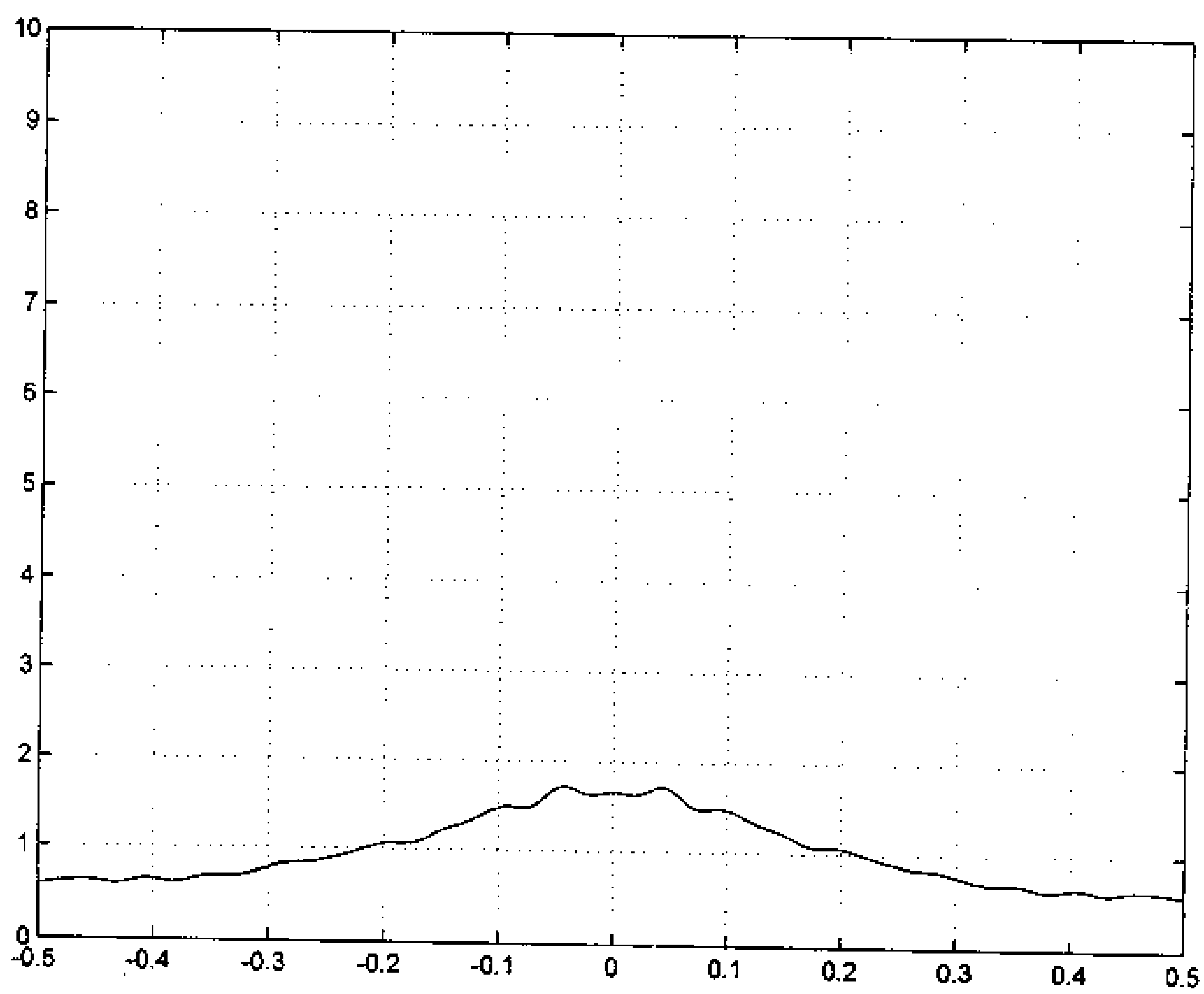
44)



True PSD is $P_X(f) = \sigma_x^2 = 1$
No need to average "down the ensemble" - see next problem.

```
% probprob17_44.m
%

clear all
randn('state',0)
x=randn(256,1);
Nfft=1024;
f=[0:Nfft-1]'/Nfft-0.5;
P1=(1/64)*abs(fftshift(fft(x(1:64),Nfft))).^2;
P2=(1/128)*abs(fftshift(fft(x(1:128),Nfft))).^2;
P3=(1/256)*abs(fftshift(fft(x(1:256),Nfft))).^2;
figure
subplot(3,1,1)
plot(f,P1)
axis([-0.5 0.5 0 6])
grid
subplot(3,1,2)
plot(f,P2)
grid
subplot(3,1,3)
plot(f,P3)
grid
```

45)



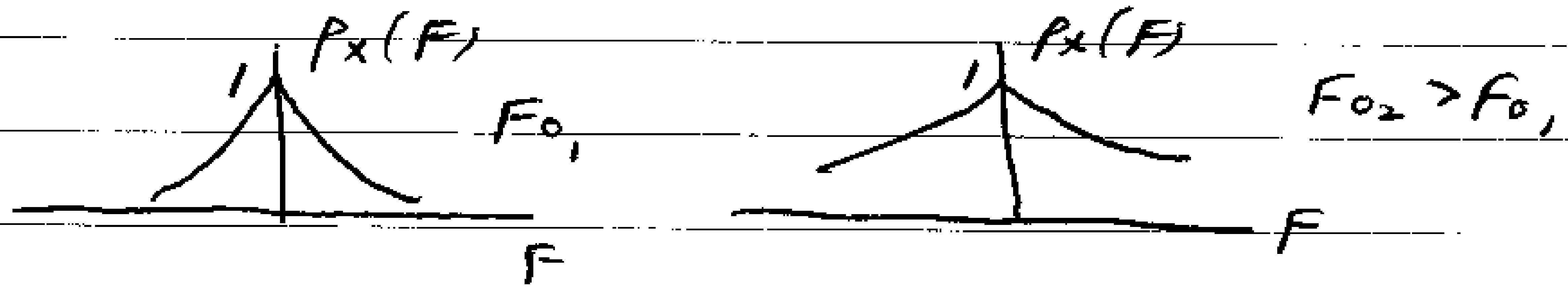
```
% probprob17_45.m
%
clear all
randn('state',0)
L=31;I=1000;
n=[0:I*L-1]';
N=length(n);
a=0.25;
varu=1-a^2;
varx=varu/(1-a^2);
x(1,1)=sqrt(varx)*randn(1,1);
for i=2:N
   x(i,1)=a*x(i-1)+sqrt(varu)*randn(1,1);
end
Nfft=1024;
Pav=zeros(Nfft,1);
f=[0:Nfft-1]'/Nfft-0.5;
for i=0:I-1
   nstart=1+i*L;nend=L+i*L;
   y=x(nstart:nend);
Pav=Pav+(1/(I*L))*abs(fftshift(fft(y,Nfft))).^2;
end
plot(f,Pav)
grid
axis([-0.5 0.5 0 10])
```

46) $$\mu_X(t) = E[\cos(2\pi F_0 t + \Theta)]$$
$$= \int_0^{2\pi} \cos(2\pi F_0 t + \theta) \frac{1}{2\pi} d\theta$$
$$= 0$$

$$r_X(\tau) = E[X(t)X(t+\tau)]$$
$$= \int_0^{2\pi} \cos(2\pi F_0 t + \theta)\cos(2\pi F_0(t+\tau)+\theta)\frac{d\theta}{2\pi}$$
$$= \int_0^{2\pi} \frac{1}{2}\cos(2\pi F_0\tau) + \frac{1}{2}\cos(4\pi F_0 t + 2\pi F_0\tau + 2\theta)\frac{d\theta}{2\pi}$$
$$= \int_0^{2\pi} \frac{1}{2}\cos(2\pi F_0\tau)\frac{1}{2\pi}d\theta = \frac{1}{2}\cos(2\pi F_0\tau)$$

47) Total average power $= 2\int_{10}^{100} e^{-F}\,dF$
$$= 2\left(-e^{-F}\right)\Big|_{10}^{100} = 2\left(-e^{-100} + e^{-10}\right)$$
$$= 2\left(e^{-10} - e^{-100}\right)$$

48) As F_0 increases, $P_x(F)$ will widen



As $F_0 \to \infty$, $P_x(F)$ becomes flat with frequency and $r_x(\tau) \to \delta(\tau)$

49) $P_x(F) = \int_{-\infty}^{\infty} r_x(\tau)\, e^{-j2\pi F\tau}\, d\tau$

$$= \int_{-\infty}^{\infty} r_x(\tau) \cos 2\pi F\tau \, d\tau$$

$$- j\int_{-\infty}^{\infty} \underbrace{r_x(\tau)}_{\text{even function}} \underbrace{\sin 2\pi F\tau}_{\text{odd function}}\, d\tau$$

Second integral must be zero since even x odd = odd

Similarly for (17.51)

50) $E[x(t)] = \frac{1}{T}\int_{t-T}^{t} \underbrace{E[v(\zeta)]}_{=0}\, d\zeta = 0$

$$E[x^2(t)] = E\left[\frac{1}{T^2}\int\int v(\zeta_1)\, v(\zeta_2)\, d\zeta_1\, d\zeta_2\right]$$

$$= \frac{1}{T^2}\int\int \underbrace{E[v(\zeta_1)\, v(\zeta_2)]}_{N_0/2\ \delta(\zeta_2-\zeta_1)}\, d\zeta_1\, d\zeta_2$$

$$= \frac{1}{T^2}\int_{t-T}^{t} N_0/2 \, d\mathfrak{z}_1$$

$$= \frac{N_0}{2T^2}T = \frac{N_0}{2T} < \infty$$

$$\text{var}(X(t)) = \frac{N_0}{2T} \Rightarrow$$ finite average total power

5.1) $r_X(\tau) = N_0W \dfrac{\sin 2\pi W\tau}{2\pi W\tau} + \mu^2$

$$r_X[k] = E[X[n]X[n+k]] = E[X(n\Delta_t)X((n+k)\Delta_t)]$$

$$= r_X(k\Delta_t) = r_X(k/4W)$$

$$= N_0W \frac{\sin 2\pi Wk/4W}{2\pi Wk/4W} + \mu^2$$

$$= N_0W \frac{\sin \pi/2\, k}{\pi/2\, k} + \mu^2$$

$$\text{var}(\hat{\mu}_N) = \frac{1}{N}\sum_{k=-(N-1)}^{N-1} (1-|k|/N)\, N_0W \frac{\sin \pi/2\,k}{\pi/2\,k} \quad \left(\begin{matrix}\text{for}\\ N=20\end{matrix}\right)$$

If we had sampled at the Nyquist rate,
$r_X[k] = N_0W\,\delta[k] + \mu^2$ and

$$\text{var}_{Ny}(\hat{\mu}_N) = N_0W/N/2 \quad \text{(half as many samples now)}$$

$$\Rightarrow \frac{\text{var}(\hat{\mu}_N)}{N_0W/N/2} = \frac{1}{2}\sum_{k=-(N-1)}^{N-1}(1-|k|/N)\frac{\sin \pi/2\,k}{\pi/2\,k}$$

For $N=20$

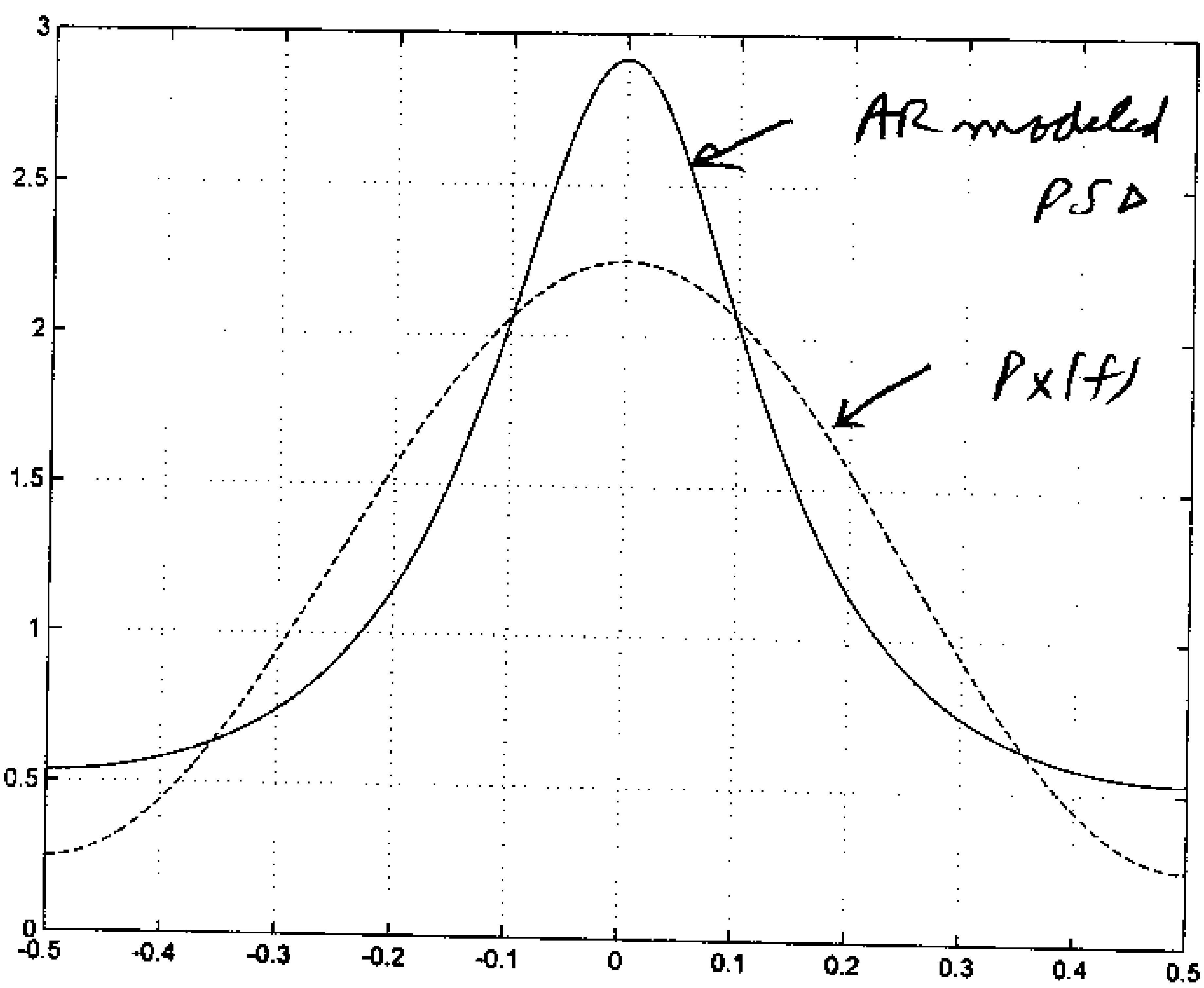
$$\frac{\mathrm{var}(\hat{\mu}_N)}{N\sigma W/N/2} = \frac{1}{2} \sum_{k=-19}^{19} \left(1 - \frac{|k|}{20}\right) \frac{\sin \pi/2\, k}{\pi/2\, k}$$

$$= \frac{1}{2} \left[1 + 2 \sum_{k=1}^{19} \left(1 - \frac{k}{20}\right) \frac{\sin \pi/2\, k}{\pi/2\, k} \right)$$

$$\frac{\mathrm{var}(\hat{\mu}_N)}{\mathrm{var}_{Ny}(\hat{\mu}_N)} = 0.9841$$

$\Rightarrow$ nearly the same almost no advantage in sampling faster.

5.21



$$P_x(f) = (1+\tfrac{1}{2}e^{-j2\pi f})(1+\tfrac{1}{2}e^{j2\pi f})$$
$$= 5/4 + \tfrac{1}{2}e^{-j2\pi f} + \tfrac{1}{2}e^{j2\pi f}$$

$$\Rightarrow \quad r_x[0] = 5/4$$
$$r_x[1] = 1/2$$

$$a = r_x[1]/r_x[0] = 2/5 \quad \text{from (17.57)}$$

$$\sigma_u^2 = r_x[0](1-a^2)$$
$$= 5/4\,(1-(2/5)^2) = 5/4\,(21/25)$$
$$= 21/20$$

```
% probprob17_52.m
%
clear all
f=[-0.5:0.001:0.5]';
P=abs(1+0.5*exp(-j*2*pi*f)).^2;
PAR=(21/20)./abs(1-(2/5)*exp(-j*2*pi*f)).^2;
figure
plot(f,P,'--',f,PAR,'-')
grid
```

Chapter 18

1) $P_X(f) = |H(f)|^2 \sigma_v^2$

$= |1 - e^{-j2\pi f} - e^{-j4\pi f}|^2$

Let $z = e^{j2\pi f}$

$P_X(f) = (1 - z^{-1} - z^{-2})(1 - z - z^2)$

$= 1 - z - z^2 - z^{-1} + 1 + z - z^{-2} + z^{-1} + 1$

$= 3 - z^2 - z^{-2}$

$P_X(f) = 3 - e^{j4\pi f} - e^{-j4\pi f}$

$= 3 - 2\cos 4\pi f$

$r_X[k] = \mathcal{F}^{-1}\{P_X(f)\}$

$$= \begin{cases} 3 & k = 0 \\ -1 & k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$

2) $\mu_X = \sum_{k=-\infty}^{\infty} h[k]\mu_V = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k 2$

$= \frac{2}{1-\frac{1}{2}} = 4$

3) Not WSS $\Rightarrow$ PSD can't be defined
(note: filter is not LSI)

4)

want $H(0.25) = 0$

$1 - b_1 e^{-j2\pi f_0} - b_2 e^{-j4\pi f_0} = 0 \qquad f_0 = 0.25$

$1 - b_1 e^{-j\pi/2} - b_2 e^{-j\pi} = 0$

$1 - b_1(-j) - b_2(-1) = 0$

$\Rightarrow$ let $b_1 = 0 \quad b_2 = -1$

$$\mathcal{H}(z) = 1 + z^{-2}$$

5) $H(f) = \mathcal{H}(e^{j2\pi f})$

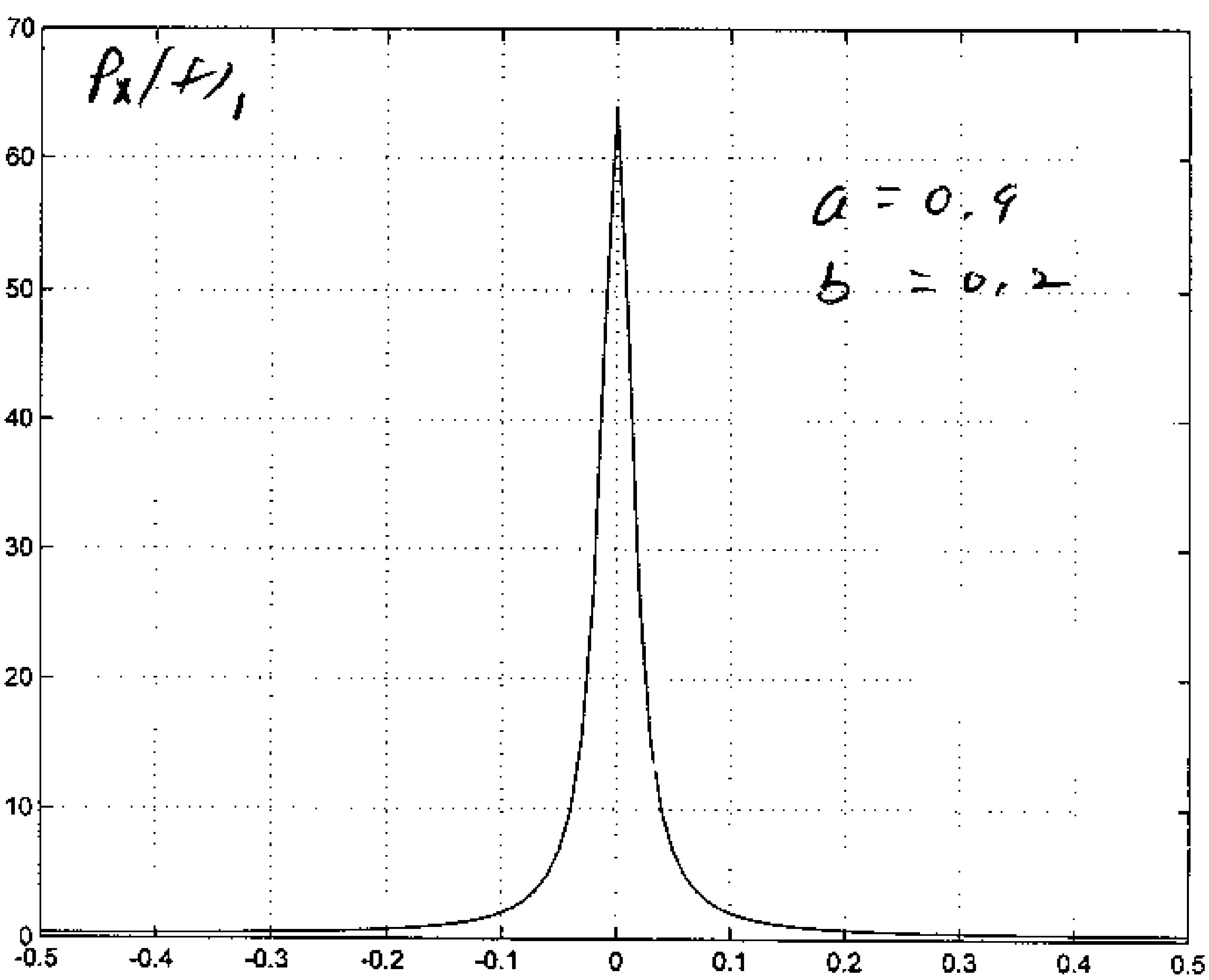
$$X(z) = a z^{-1} X(z) + U(z) - b z^{-1} U(z)$$

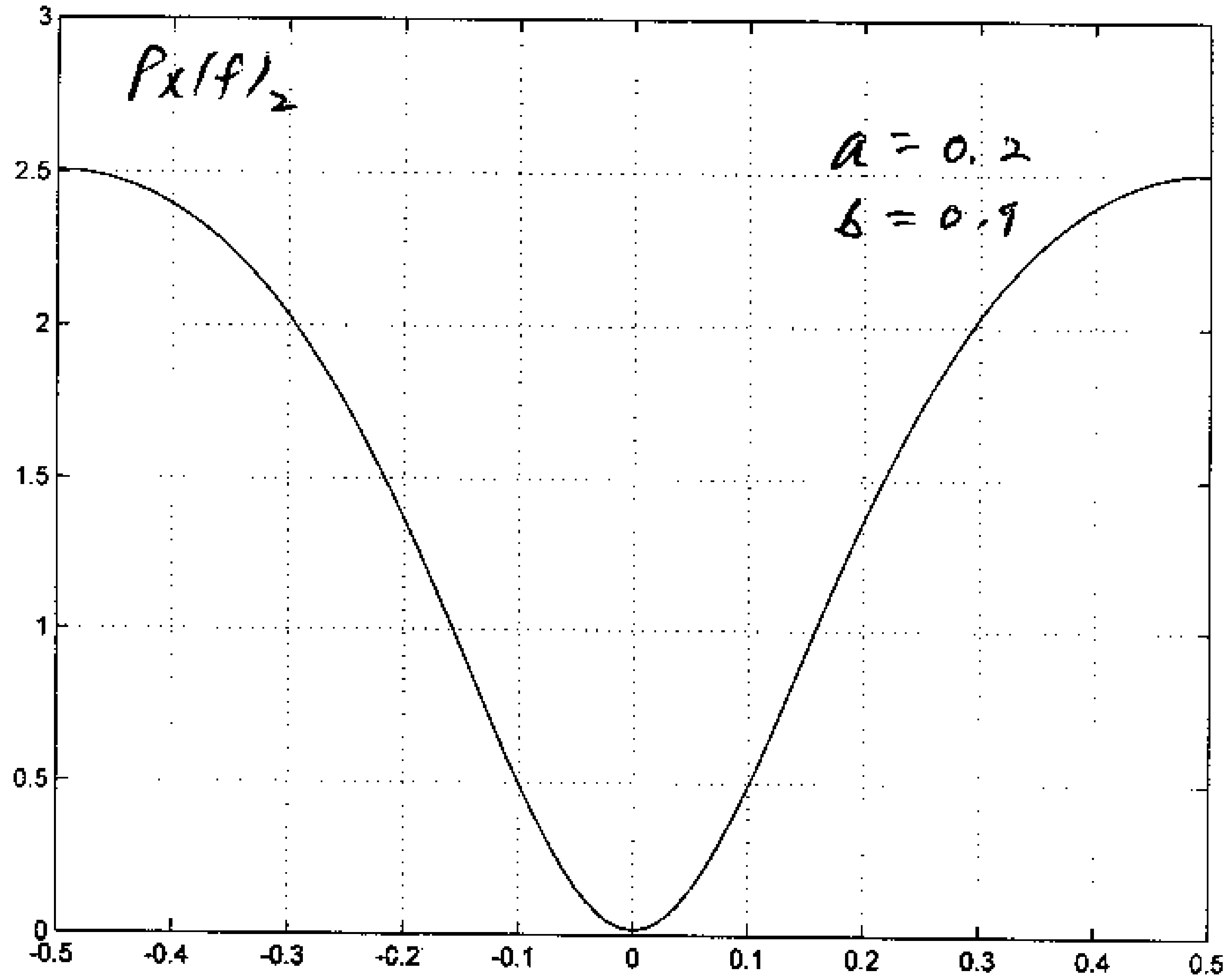
$$X(z)(1 - a z^{-1}) = U(z)(1 - b z^{-1})$$

$$\mathcal{H}(z) = \frac{1 - b z^{-1}}{1 - a z^{-1}}$$

$$P_X(f) = |H(f)|^2 \sigma_u^2 = |H(f)|^2$$

$$= \left| \frac{1 - b e^{-j2\pi f}}{1 - a e^{-j2\pi f}} \right|^2$$





$P_X(f)_2 = 1/P_X(f)_1 \Rightarrow P_X(f)_1$ is lowpass PSD and $P_X(f)_2$ is a highpass PSD

```
% probprob18_5.m
%
clear all
a=0.9;b=0.2;
f=[-0.5:0.01:0.5]';
B=1-b*exp(-j*2*pi*f);
A=1-a*exp(-j*2*pi*f);
P=abs(B./A).^2;
figure
plot(f,P)
grid
a=0.2;b=0.9;
f=[-0.5:0.01:0.5]';
B=1-b*exp(-j*2*pi*f);
A=1-a*exp(-j*2*pi*f);
P=abs(B./A).^2;
figure
plot(f,P)
grid
```

6) From previous problem

$$P_x(f) = \left| \frac{1 - b e^{-j2\pi f}}{1 - a e^{-j2\pi f}} \right|^2$$

Here $a = b = 0.5 \Rightarrow P_x(f) = 1$ or

$H(f) = 1 \Rightarrow x[n] = v[n] =$ white noise

7) $P_x(f) = |H(f)|^2 P_v(f)$

$$= |1 - e^{-j2\pi f}|^2 (1 - \cos 2\pi f)$$

$$= (1 - e^{-j2\pi f})(1 - e^{j2\pi f})(1 - \cos 2\pi f)$$

$$= (2 - 2\cos 2\pi f)(1 - \cos 2\pi f)$$

$$= 2(1 - \cos 2\pi f)^2$$

$$= 2(1 - 2\cos 2\pi f + \cos^2 2\pi f)$$

$$= 2(1 - 2\cos 2\pi f + \tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi f)$$

$$= 3 - 4\cos 2\pi f + \cos 4\pi f$$

$$P_x(f) = \sum_{k=-\infty}^{\infty} r_x[k] \cos 2\pi f k$$

$$\Rightarrow \quad r_x[k] = \begin{cases} 3 & k = 0 \\ -2 & k = 1 \\ \tfrac{1}{2} & k = 2 \\ 0 & k \geq 3 \end{cases}$$

8) $\mathcal{F}\left\{ \sigma_v^2 \sum_{m=-\infty}^{\infty} h[m] h[m+k] \right\} =$

$$\sigma_v^2 \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[m] h[m+k] e^{-j2\pi f k}$$

Let $l = m + k \Rightarrow k = l - m$

$$= \sigma_v^2 \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} h[m] h[l] e^{-j2\pi f(l-m)}$$

$$= \sigma_v^2 \underbrace{\sum_{l=-\infty}^{\infty} h[l] e^{-j2\pi f l}}_{H(f)} \underbrace{\sum_{m=-\infty}^{\infty} h[m] e^{j2\pi f m}}_{H^*(f)}$$

$$= \sigma_v^2 |H(f)|^2$$

9) $$r_x[k] = \sigma_v^2 \sum_{m=-\infty}^{\infty} h[m]h[m+k]$$

$$= \sigma_v^2 \Big(h[0]h[k] + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} h[m]h[m+k] \Big)$$

$$r_x[0] = \sigma_v^2 \Big(h^2[0] + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} h^2[m] \Big)$$

$$\geq \sigma_v^2 h^2[0] = \sigma_v^2$$

Output power $= r_x[0] \geq \sigma_v^2 =$ input power

No

10) $$P_v(f) = |H(f)|^2 P_x(f)$$

$$\sigma_v^2 = |H(f)|^2 \frac{1}{|1 - \frac{1}{2} e^{-j2\pi f}|^2}$$

$$4 = \frac{|H(f)|^2}{|1 - \frac{1}{2} e^{-j2\pi f}|^2}$$

$$|H(f)| = 2 |1 - \tfrac{1}{2} e^{-j2\pi f}|$$

$$\Rightarrow H(f) = 2(1 - \tfrac{1}{2} e^{-j2\pi f}) = 2 - e^{-j2\pi f}$$

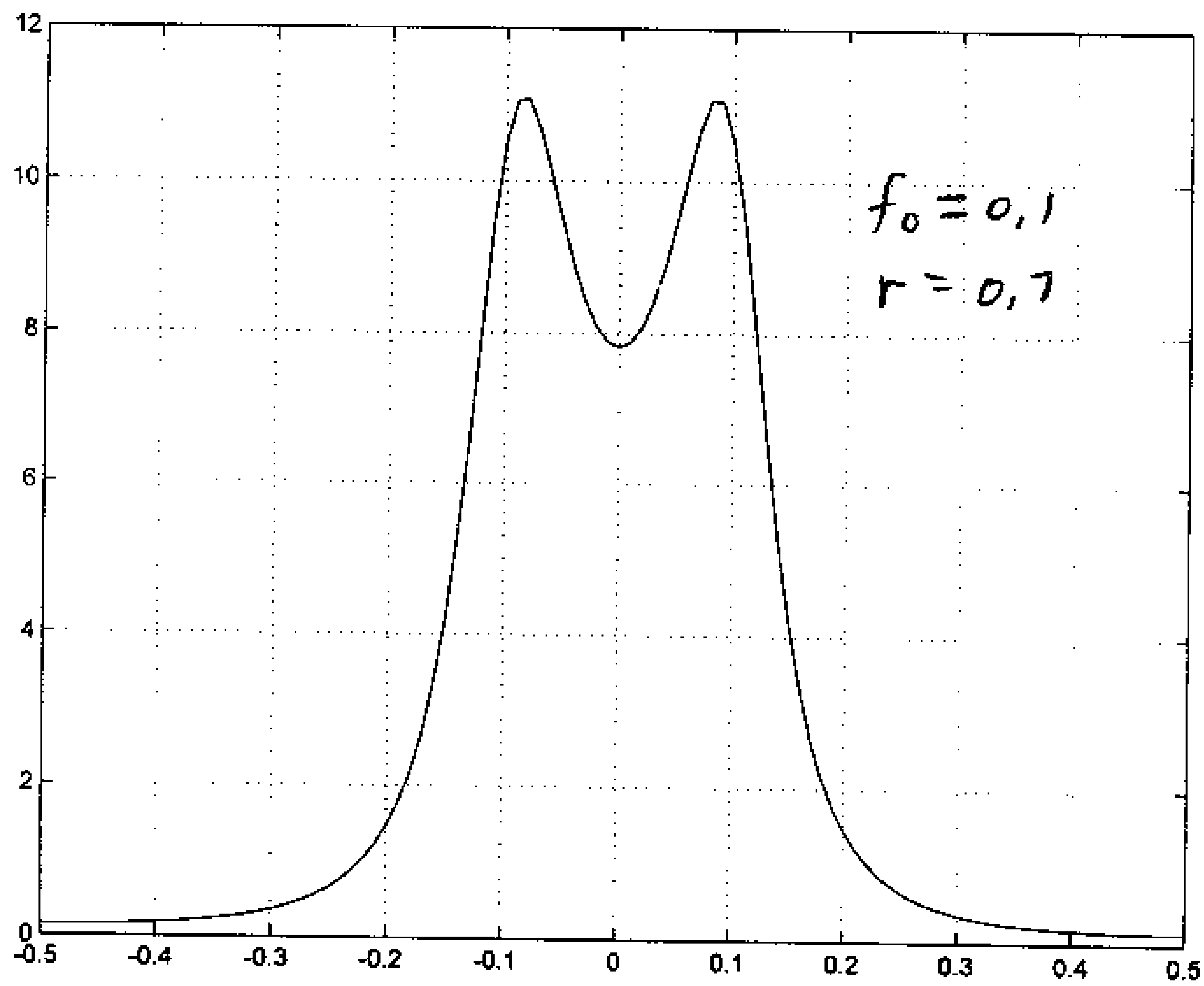
$$h(n) = 2 \qquad n=0$$
$$\qquad -1 \qquad n=1$$

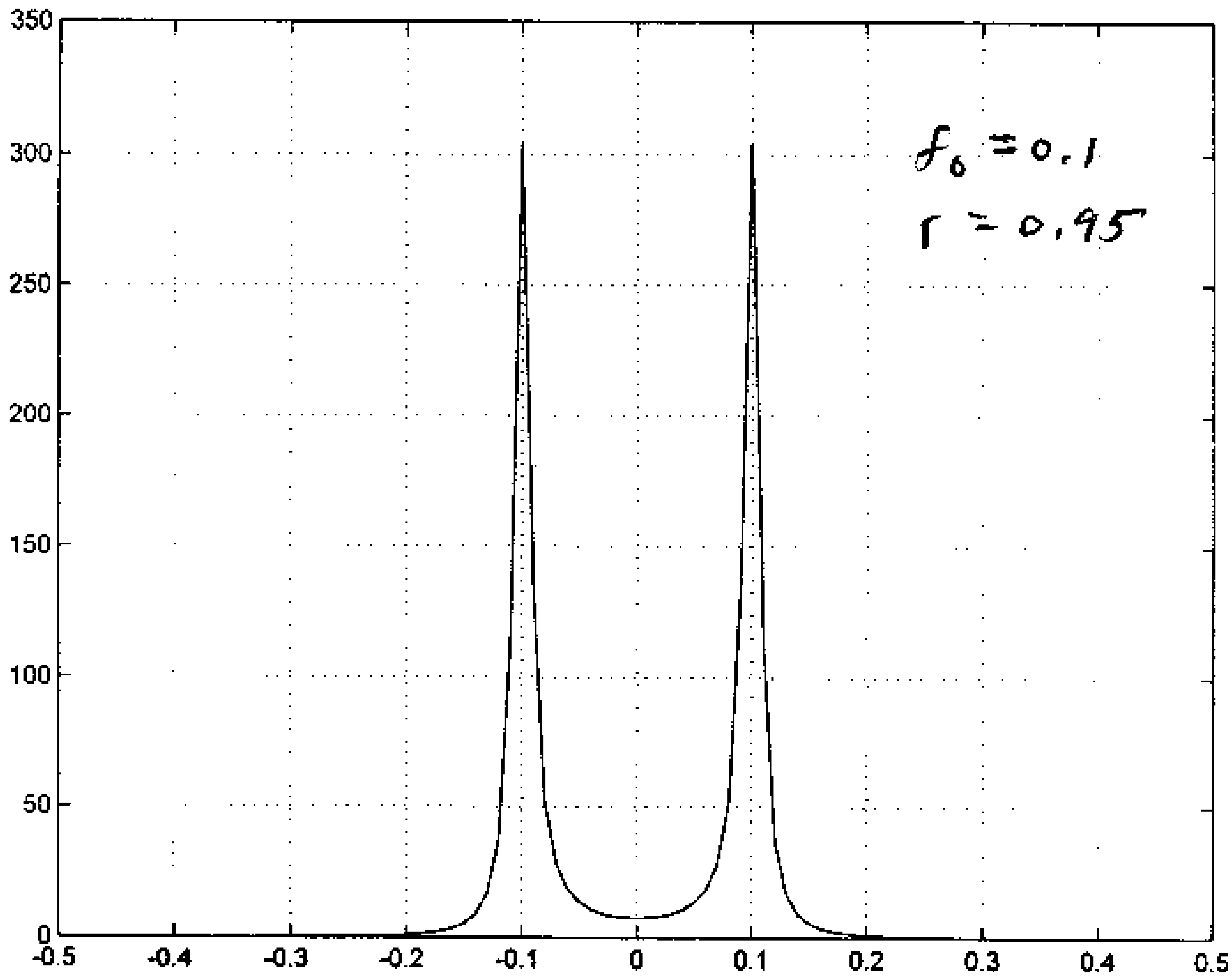
$$v(n) = 2x(n) - x(n-1)$$

(1) $$H(f) = (1 - 2r\cos 2\pi f_0 e^{-j2\pi f} + r^2 e^{-j4\pi f})^{-1}$$

$$P_x(f) = |H(f)|^2 \sigma_v^2 = |H(f)|^2$$

$$= \frac{1}{|1 - 2r\cos 2\pi f_0 e^{-j2\pi f} + r^2 e^{-j4\pi f}|^2}$$





The filter has $H(z) = \dfrac{1}{1 - a_1 z^{-1} - a_2 z^{-2}}$

where $a_1 = 2r\cos 2\pi f_0$ $a_2 = -r^2$. The poles are at

$$z^2 - a_1 z - a_2 = 0 \quad \text{or}$$

$$z^2 - 2r\cos 2\pi f_0 + r^2 = 0$$

$$(z - re^{j2\pi f_0})(z - re^{-j2\pi f_0}) = 0$$

$\Rightarrow$ Poles at $re^{\pm j2\pi f_0}$

As $r \to 1$, poles approach unit circle of z-plane so that $1 - a_1 z^{-1} - a_2 z^{-2} \to 0$ for $z = e^{\pm j2\pi f_0}$

and $H(z)\big|_{z = e^{\pm j2\pi f}} \to \infty$

```
% probprob18_11.m
%
clear all
f0=0.1;r=0.7;
f=[-0.5:0.01:0.5]';
A=1-2*r*cos(2*pi*f0)*exp(-j*2*pi*f)+r^2*exp(-j*4*pi*f);
P=1./abs(A).^2;
figure
plot(f,P)
grid
f0=0.1;r=0.95;
f=[-0.5:0.01:0.5]';
A=1-2*r*cos(2*pi*f0)*exp(-j*2*pi*f)+r^2*exp(-j*4*pi*f);
P=1./abs(A).^2;
figure
plot(f,P)
grid
```

12)
$$H_{opt}(f) = \frac{P_s(f)}{P_s(f)+\sigma_W^2}$$

$$= 0 \quad \text{for } |f| > B$$

Filter does not allow through frequency components consisting of noise only

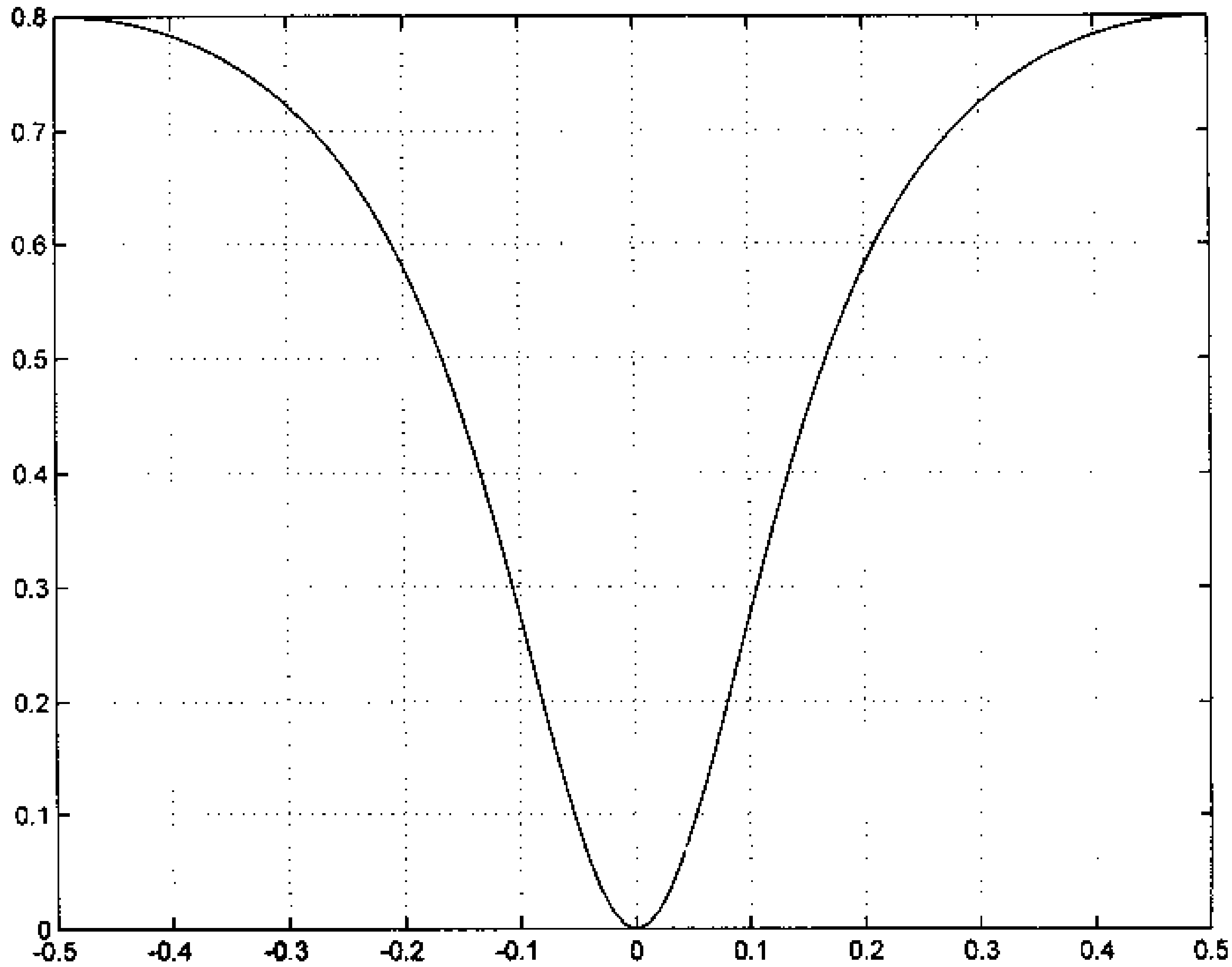
13)
$$H_{opt}(f) = \frac{P_s(f)}{P_s(f)+\sigma_W^2} = \frac{2-2\cos 2\pi f}{3-2\cos 2\pi f}$$

$$\mathrm{mse}_{min} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_W(f)P_s(f)}{P_s(f)+P_W(f)}\,df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{P_s(f)}{P_s(f)+1}\,df$$

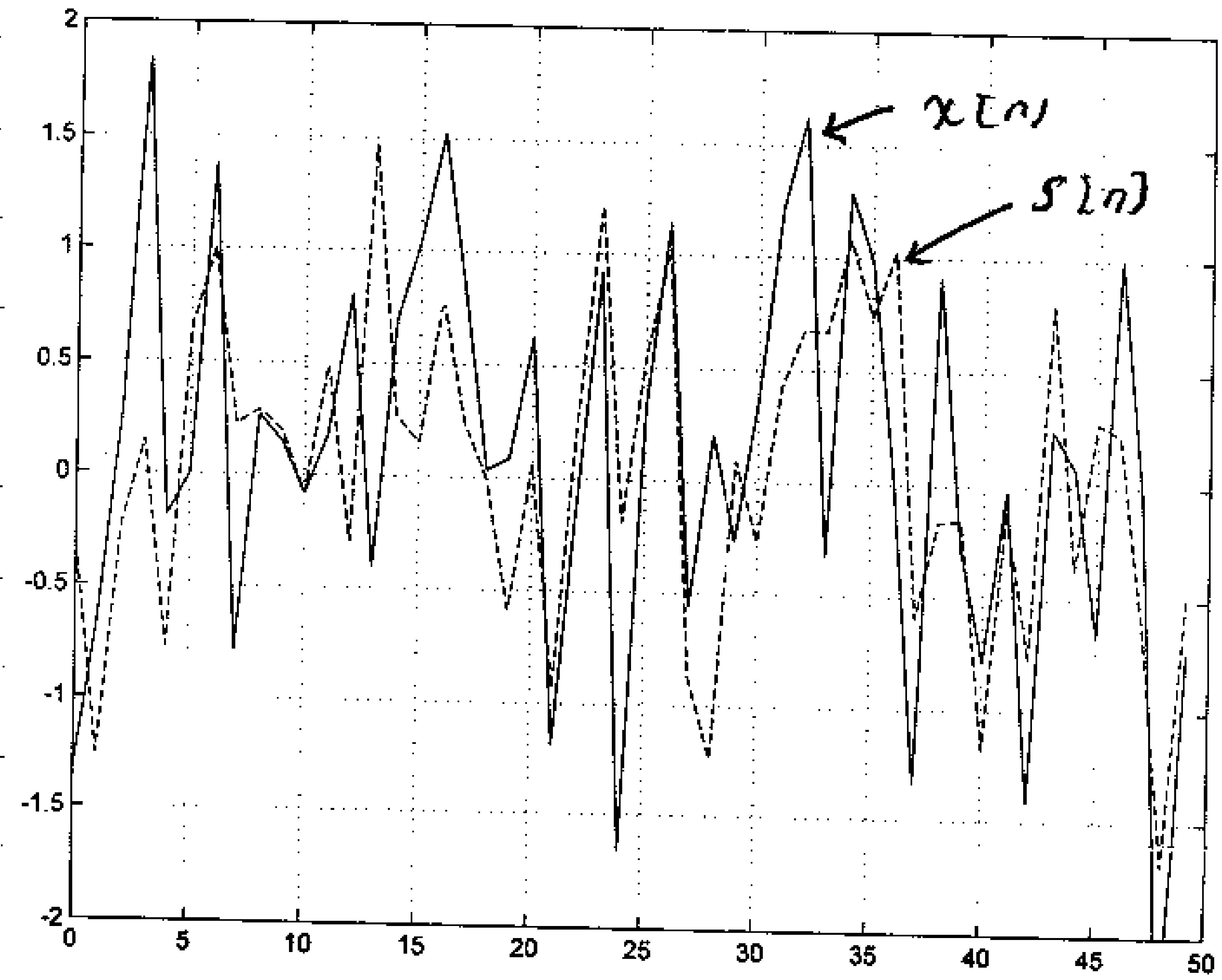
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2-2\cos 2\pi f}{3-2\cos 2\pi f}\,df$$

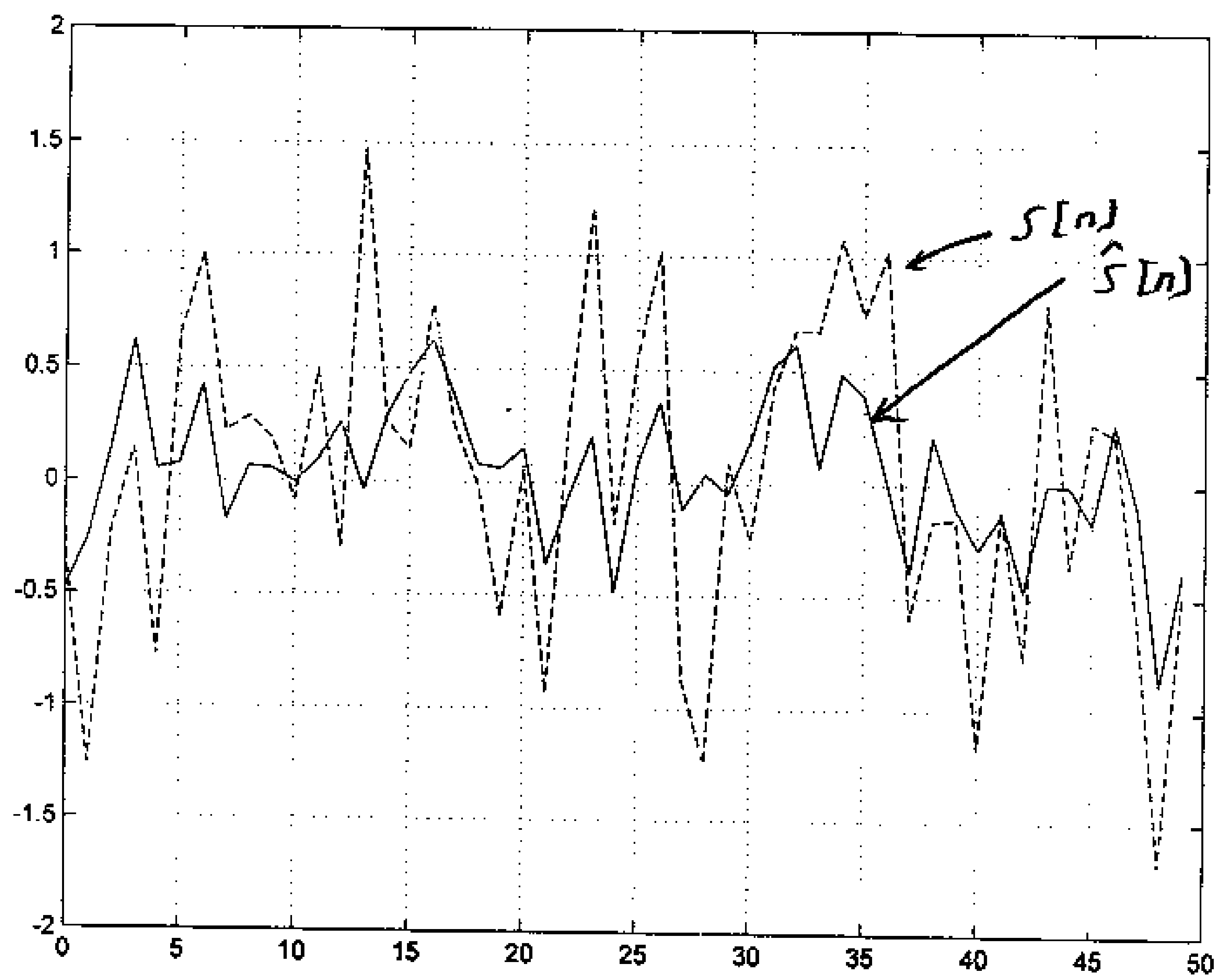
$$\approx 0.5552$$




```
% probprob18_13.m
%
clear all
f=[-0.5:0.01:0.5]';
Ps=2-2*cos(2*pi*f);
H=Ps./(Ps+1);
figure
plot(f,H)
grid
msemin=mean(H)
```

14)


x[n]
s[n]


s[n]
$\hat{s}$[n]

```
% probprob18_14.m
%
%  This program implements a Wiener smoother for extracting
%  an AR(1) signal in white Gaussian noise
clear all
randn('state',0)
a=0.25;varu=0.5;vars=varu/(1-a^2);varw=1;N=50; %set up parameters
for n=0:N-1
   nn=n+1;
   if n==0 % use Gaussian random processes
      s(nn,1)=sqrt(vars)*randn(1,1); % initialize first sample
                                     % to avoid transient
   else
      s(nn,1)=a*s(nn-1)+sqrt(varu)*randn(1,1);
   end
end
x=s+sqrt(varw)*randn(N,1); %add white Gaussian noise
Nfft=1024;
Ps=varu./(abs(1-a*exp(-j*2*pi*[0:Nfft-1]'/Nfft)).^2);
% filtering of real AR(1) signal in WGN
Hf=Ps./(Ps+varw); % form Wiener smoother
sestf=Hf.*fft(x,Nfft);  % signal estimate in frequency domain
sest=real(ifft(sestf,Nfft)); % inverse Fourier transform
nn=[0:N-1]';
figure
plot(nn,s,'--',nn,x,'-')
grid
axis([0 50 -2 2])
figure
plot(nn,s,'--',nn,sest(1:N),'-')
grid
axis([0 50 -2 2])
```

15) $z P_x(z) = \mathcal{H}(z) P_x(z)$

$\Rightarrow \mathcal{H}(z) = z$ or $h[k] = \delta[k+1]$

noncausal filter. This is actually the correct result if (18.28) were to be satisfied for $-\infty < l < \infty$.

16) $E[(x[n_0+1] - a\, x[n_0])\, x[n_0-k]]$

$= E[u[n_0+1]\, x[n_0-k]]$

$= 0 \qquad k \geq 0$

Since $E[v[n_0+1]\,x[n_0-k]]$

$$= E\left[v[n_0+1]\sum_{\ell=0}^{\infty} a^{\ell} v[n_0-k-\ell]\right]$$

$$= \sum_{\ell=0}^{\infty} a^{\ell}\, \underbrace{E[v[n_0+1]\,v[n_0-k-\ell]]}_{\sigma_v^2\,\delta[k+\ell+1]}$$

but $\ell \ge 0$ $k \ge 0 \Rightarrow k+\ell \ge 0$ and thus $\delta[k+\ell+1] = 0$ for all $k \ge 0$.

17) $mse_{MIN} = E[(x[n_0+1] - a\,x[n_0])^2]$

$$= E[v^2[n_0+1]] = \sigma_v^2$$

but $r_x[0] = \sigma_v^2/1-a^2$

$$\Rightarrow mse_{MIN} = r_x[0](1-a^2)$$

$a = 0.98$ easier to predict

18) $$\rho_{x[n_0],x[n_0+1]} = \frac{\mathrm{cov}(x[n_0],\,x[n_0+1])}{\sqrt{\mathrm{var}(x[n_0])\,\mathrm{var}(x[n_0+1])}}$$

$$= \frac{E[x[n_0]\,x[n_0+1]]}{\sqrt{r_x[0]\,r_x[0]}}$$

$$= \frac{r_x[1]}{r_x[0]}$$

$$r_x[k] = \frac{\sigma_v^2}{1-a^2}\,a^{|k|} \Rightarrow \frac{r_x[1]}{r_x[0]} = a$$

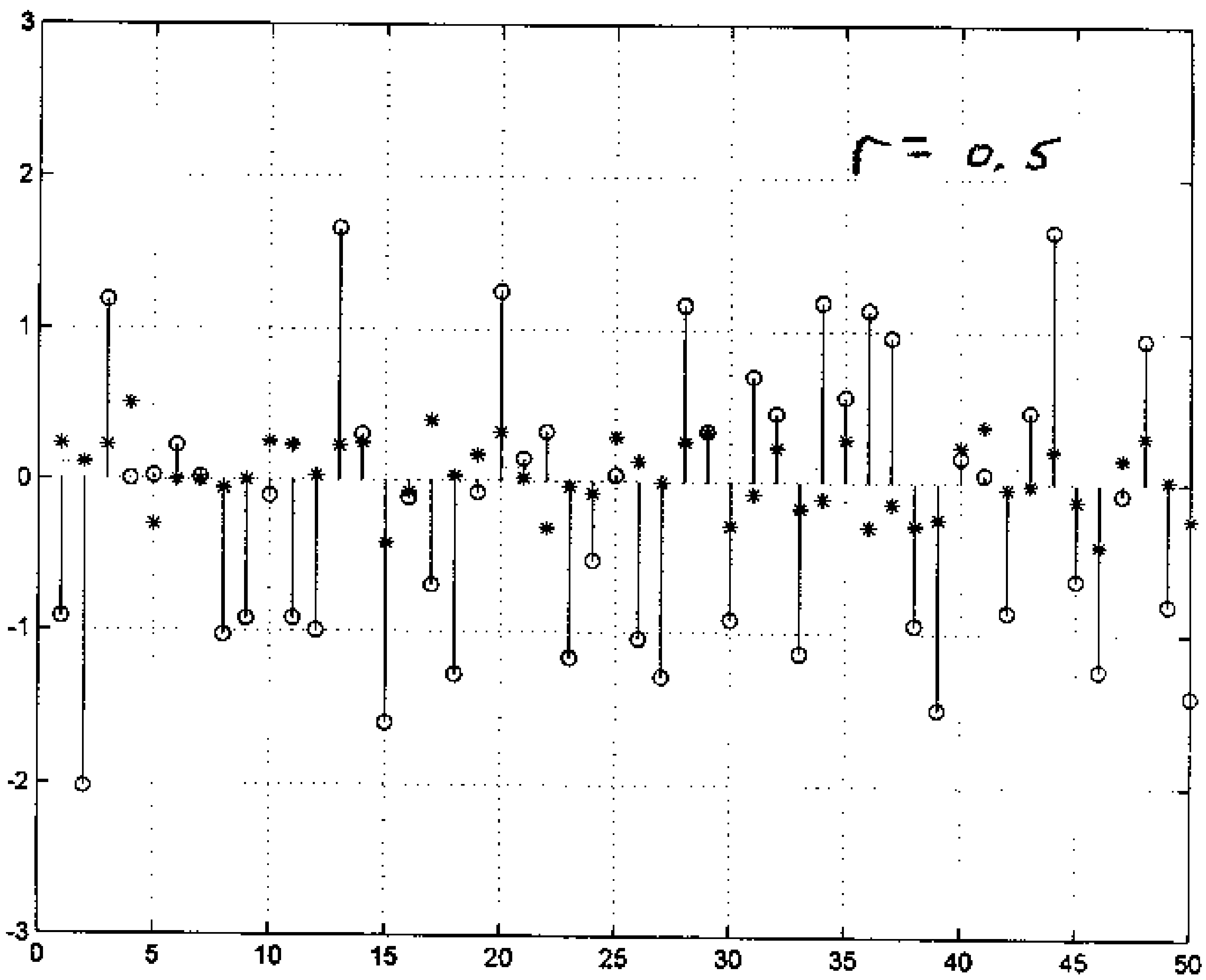
$$mse_{MIN} = r_x[0]\left(1-\rho^2_{x[n_0],x[n_0+1]}\right)$$

If $\rho^2 \to \pm 1$, $mse_{MIN} \to 0$

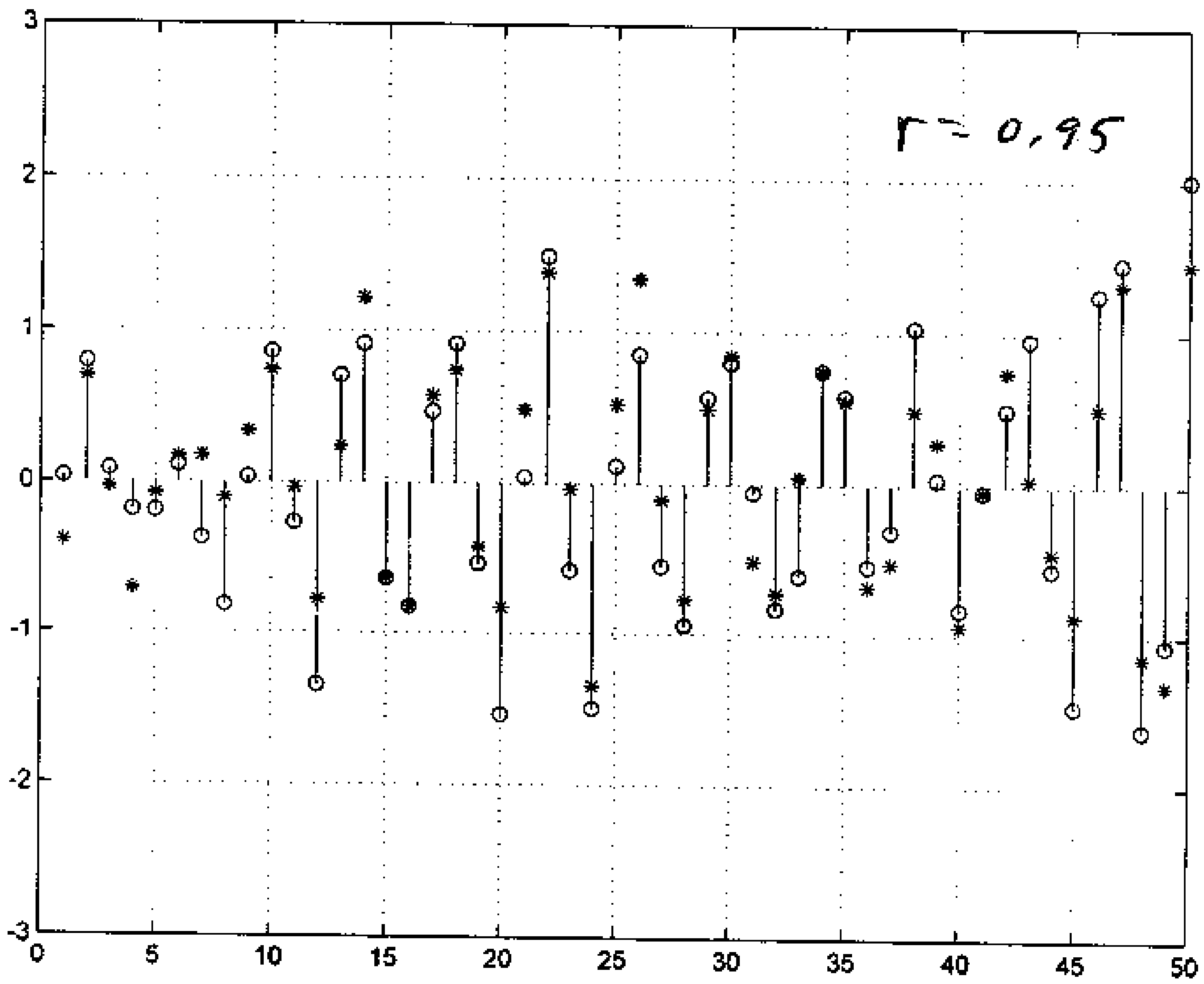
$\rho = 0$ $\quad mse_{MIN} = r_x[0]$ since $\hat{x}[n_0+1] = 0$

19) $\hat{x}[n_0+1] = a[1]\,x[n_0] + a[2]\,x[n_0-1]$

$= -r^2 x[n_0-1]$



$$\hat{mse}_{MIN} = \underbrace{\frac{1}{N}\sum_{n=2}^{N-1}\left[x[n]-(-r^2 x[n-2])\right]^2}_{\text{estimate of } mse_{MIN}} = 0.8007$$



$\hat{mse}_{MIN} = 0.1485$

Easier to predict x(n) as r → 1.

```
% probprob18_19.m
%
clear all
randn('state',0)
r=0.5;varu=1-r^4;
for n=1:150
   if n==1
      x(n,1)=sqrt(varu)*randn(1,1);
   elseif n==2
      x(n,1)=sqrt(varu)*randn(1,1);
   else
      x(n,1)=-r^2*x(n-2,1)+sqrt(varu)*randn(1,1);
      xpred(n,1)=-r^2*x(n-2,1);
   end
end
msemin1=mean((x(101:150)-xpred(101:150)).^2)
figure
stem([1:50]',x(101:150))
hold on
plot([1:50]',xpred(101:150),'*')
axis([0 50 -3 3])
grid
hold off
r=0.95;varu=1-r^4;
```

```
for n=1:150
   if n==1
      x(n,1)=sqrt(varu)*randn(1,1);
   elseif n==2
      x(n,1)=sqrt(varu)*randn(1,1);
   else
      x(n,1)=-r^2*x(n-2,1)+sqrt(varu)*randn(1,1);
      xpred(n,1)=-r^2*x(n-2,1);
   end
end
msemin2=mean((x(101:150)-xpred(101:150)).^2)
figure
stem([1:50]',x(101:150))
hold on
plot([1:50]',xpred(101:150),'*')
axis([0 50 -3 3])
grid
hold off
```

20)
$$\hat{x}[n_0+1] = \sum_{k=0}^{M-1} h[k]\, x[n_0-k]$$

$$E\left[\left(x[n_0+1] - \sum_{k=0}^{M-1} h[k]\, x[n_0-k]\right) x[n_0-l]\right] = 0 \quad \text{for } l = 0, 1, \ldots, M-1$$

by orthogonality principle

$$E[x[n_0+1]\, x[n_0-l]] = \sum_{k=0}^{M-1} h[k]\, E[x[n_0-k]\, x[n_0-l]]$$

$$r_x[l+1] = \sum_{k=0}^{M-1} h[k]\, r_x[l-k] \quad l = 0, 1, \ldots, M-1$$

$$mse_{MIN} = E\left[\left(x[n_0+1] - \sum_{k=0}^{M-1} h_{OPT}[k]\, x[n_0-k]\right)^2\right]$$

$$= E\left[\left(x[n_0+1] - \sum_{k=0}^{M-1} h_{OPT}[k]\, x[n_0-k]\right)\left(x[n_0+1]\right)\right]$$

$$- E\left[\underbrace{\left(x[n_0+1] - \sum_{k=0}^{M-1} h[k]\, x[n_0-k]\right)}_{\epsilon[n_0+1]} \sum_{l=0}^{M-1} h_{OPT}[l]\, x[n_0-l]\right]$$

second term is zero since it is equal to

$$\sum_{\ell=0}^{M-1} h_{opt}[\ell]\, \underbrace{E\{\varepsilon[n_0+1]\, x[n_0-\ell]\}}_{=\, 0 \text{ by orthogonality principle}}$$

$$mse_{min} = E\{x^2[n_0+1]\} - \sum_{k=0}^{M-1} h_{opt}[k] \cdot E\{x[n_0+1]x[n_0-k]\}$$

$$= r_x[0] - \sum_{k=0}^{M-1} h_{opt}[k]\, r_x[k+1]$$

21) $M=1 \Rightarrow r_x[1] = h[0]\, r_x[0]$

$\Rightarrow h_{opt}[0] = r_x[1]/r_x[0]$

But if $x[n_0+1] \to Y$

$x[n_0] \to X$

$\hat{Y} = \rho_{x,y} X$ for x, y zero mean

and $\rho_{x,y} = \dfrac{E(x[n_0]\, x[n_0+1])}{\sqrt{E\{x^2[n_0]\}\, E\{x^2[n_0+1]\}}}$

$$= \frac{r_x[1]}{r_x[0]} = h_{opt}[0]$$

22) $\begin{bmatrix} 1+b^2 & -b \\ -b & 1+b^2 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} -b \\ 0 \end{bmatrix}$

$$\begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \frac{1}{(1+b^2)^2 - b^2} \begin{bmatrix} 1+b^2 & b \\ b & 1+b^2 \end{bmatrix} \begin{bmatrix} -b \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} -b-b^3 \\ -b^2 \end{bmatrix}}{1+b^2+b^4}$$

$$mse_{min} = r_x[0] - h_{opt}[0]\, r_x[1] - h_{opt}[1]\, r_x[2]$$

$$= 1+b^2 - \left(\frac{-b-b^3}{1+b^2+b^4}\right)(-b) - \left(\frac{-b^2}{1+b^2+b^4}\right)(0)$$

$$= 1+b^2 - \frac{b^2(1+b^2)}{1+b^2+b^4}$$

$$= 1 + \frac{b^2+b^4+b^6-b^2-b^4}{1+b^2+b^4}$$

$$= 1 + \frac{b^6}{1+b^2+b^4} > 1$$

for infinite length predictor $mse_{min} = \sigma_v^2 = 1$

23) $\underline{R}_x = \sigma_x^2 \underline{I} \Rightarrow \underline{R}_x^{-1} = 1/\sigma_x^2\, \underline{I}$

$$\begin{bmatrix} h[0] \\ \vdots \\ h[M-1] \end{bmatrix} = \frac{1}{\sigma_x^2} \underline{I} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \underline{0}$$

$\Rightarrow \hat{x}[n_0+1] = 0$ for any M

Can't predict white noise since there is no correlation between samples

24) Here $b = \frac{1}{2}$

$$\hat{x}[n_0+1] = h[0]x[n_0] + h[1]x[n_0-1]$$

$$h[0] = \frac{-\frac{1}{2}-(\frac{1}{2})^3}{1+(\frac{1}{2})^2+(\frac{1}{2})^4} = -10/21$$

$$h[1] = \frac{-(\frac{1}{2})^2}{1+(\frac{1}{2})^2+(\frac{1}{2})^4} = -4/21$$

$$\hat{x}[n_0+1] = -\frac{10}{21}x[n_0] - \frac{4}{21}x[n_0-1]$$

$$mse_{MIN} = 1 + \frac{(\frac{1}{2})^6}{1+(\frac{1}{2})^2+(\frac{1}{2})^4} = 85/84$$

$$= 1.0119$$

From simulation $\hat{mse}_{MIN} = 1.0117$

```
% probprob18_24.m
%
clear all
randn('state',0)
b=0.5;
N=10000;
u=randn(N,1);
for n=2:N
   x(n,1)=u(n)-b*u(n-1);
end
for n=3:N
   xpred(n,1)=-(10/21)*x(n-1)-(4/21)*x(n-2);
   error(n,1)=x(n)-xpred(n);
end
mse=error(3:N)'*error(3:N)/(N-2)
```

25)

$$\begin{bmatrix} 1 & \cos 2\pi f_0 \\ \cos 2\pi f_0 & 1 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} \cos 2\pi f_0 \\ \cos 4\pi f_0 \end{bmatrix}$$

$$\begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \frac{\begin{bmatrix} 1 & -\cos 2\pi f_0 \\ -\cos 2\pi f_0 & 1 \end{bmatrix}}{1-\cos^2 2\pi f_0} \begin{bmatrix} \cos 2\pi f_0 \\ \cos 4\pi f_0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} \cos 2\pi f_0 - \cos 2\pi f_0 \cos 4\pi f_0 \\ \cos 4\pi f_0 - \cos^2 2\pi f_0 \end{bmatrix}}{1-\cos^2 2\pi f_0}$$

Let $\theta = 2\pi f_0$

$$\begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \frac{\begin{bmatrix} \cos\theta - \cos\theta\,(2\cos^2\theta - 1) \\ 2\cos^2\theta - 1 - \cos^2\theta \end{bmatrix}}{1-\cos^2\theta}$$

$$= \frac{\begin{bmatrix} \cos\theta\,(1 - 2\cos^2\theta + 1) \\ \cos^2\theta - 1 \end{bmatrix}}{1-\cos^2\theta}$$

$$= \begin{bmatrix} 2\cos\theta \\ -1 \end{bmatrix}$$

$$\hat{x}[n_0+1] = 2\cos 2\pi f_0\, x[n_0] - x[n_0-1]$$

$$mse_{MIN} = r_x[0] - h[0]\, r_x[1] - h[1]\, r_x[2]$$
$$= 1 - 2\cos\theta\cos\theta - (-1)\cos 2\theta$$
$$= 1 - 2\cos^2\theta + (2\cos^2\theta - 1)$$
$$= 0$$

26) By orthogonality principle

$$E\left(\left(x[n_0+L] - \sum_{k=0}^{\infty} h[k]\, x[n_0-k]\right) x[n_0-l]\right) = 0 \qquad l \geq 0$$

$$E\left(x[n_0+L]\, x[n_0-l]\right) = \sum_{k=0}^{\infty} h[k]\, E\left(x[n_0-k]\, x[n_0-l]\right)$$

$$r_x[l+L] = \sum_{k=0}^{\infty} h[k]\, r_x[l-k] \qquad l \geq 0$$

$$mse_{MIN} = E\left(\left(x[n_0+L] - \sum_{k=0}^{\infty} h_{OPT}[k]\, x[n_0-k]\right) x[n_0+L]\right)$$

second term $= 0$ (see Problem 18.21)

$$= r_x[0] - \sum_{k=0}^{\infty} h_{OPT}[k]\, r_x[k+L]$$

$$\hat{x}[n_0+L] = \sum_{k=0}^{\infty} h[k]\, x[n_0-k]$$

Assume that $h[k] = 0$ $k = 1, 2, \ldots$ and we show that all the equations can be satisfied.

$$\frac{\sigma_u^2}{1-a^2}\, a^{|l+L|} = h[0]\, \frac{\sigma_u^2}{1-a^2}\, a^{|l|} \qquad l \geq 0$$

But $l \geq 0$, $l + L \geq 0$ $\Rightarrow$

$$a^{l+L} = h(0)\, a^{l} \Rightarrow h_{OPT}(0) = a^{L} \text{ and}$$

$\hat{x}(n_0+1) = a^{l} x(n_0)$ is optimal predictor

$$MSE_{MIN} = r_x(0) - h_{OPT}(0)\, r_x[L]$$
$$= r_x(0) - a^{L} r_x(0)\, a^{L}$$
$$= r_x(0)\,(1 - a^{2L})$$

As L increases prediction becomes poorer since correlation between $x(n_0)$ and $x[n_0+L] \to 0$ $\quad (r_x[L] = r_x(0)\, a^{L})$

27) $\hat{x}(n_0) = h(1)\, x(n_0-1) + h[-1)\, x(n_0+1)$

By orthogonality principle

$$E\{(x(n_0) - h(-1)\, x(n_0+1) - h[1)\, x(n_0-1))\, x(n_0-l)\} = 0$$
$$l = -1, 1$$

$$r_x(l) = h[-1]\, r_x(l+1) + h[1]\, r_x(l-1)$$

$$\begin{bmatrix} r_x(0) & r_x[2) \\ r_x(2) & r_x(0) \end{bmatrix} \begin{bmatrix} h[-1) \\ h(1) \end{bmatrix} = \begin{bmatrix} r_x(-1) \\ r_x(1) \end{bmatrix}$$

$$r_x(k) = r_x(0)\, a^{|k|}$$

$$\begin{bmatrix} r_x[0) & r_x(0)\, a^2 \\ r_x(0)\, a^2 & r_x(0) \end{bmatrix} \begin{bmatrix} h(-1) \\ h(1) \end{bmatrix} = \begin{bmatrix} r_x(0)\, a \\ r_x(0)\, a \end{bmatrix}$$

$$\begin{bmatrix} 1 & a^2 \\ a^2 & 1 \end{bmatrix} \begin{bmatrix} h[-1] \\ h[1] \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix}$$

$$\begin{bmatrix} h[-1] \\ h[1] \end{bmatrix} = \frac{\begin{bmatrix} 1 & -a^2 \\ -a^2 & 1 \end{bmatrix}}{1-a^4} \begin{bmatrix} a \\ a \end{bmatrix}$$

$$= \frac{\begin{pmatrix} a-a^3 \\ a-a^3 \end{pmatrix}}{1-a^4} = \begin{bmatrix} \frac{a(1-a^2)}{1-a^4} \\ \frac{a(1-a^2)}{1-a^4} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{a}{1+a^2} \\ \frac{a}{1+a^2} \end{pmatrix}$$

$$\Rightarrow \quad \hat{x}[n_0] = \frac{a}{1+a^2}\left(x[n_0+1] + x[n_0-1] \right)$$

If $a \to 1$ samples are averaged

$a \to 0$ $\hat{x}[n_0] \to 0$ since samples are uncorrelated

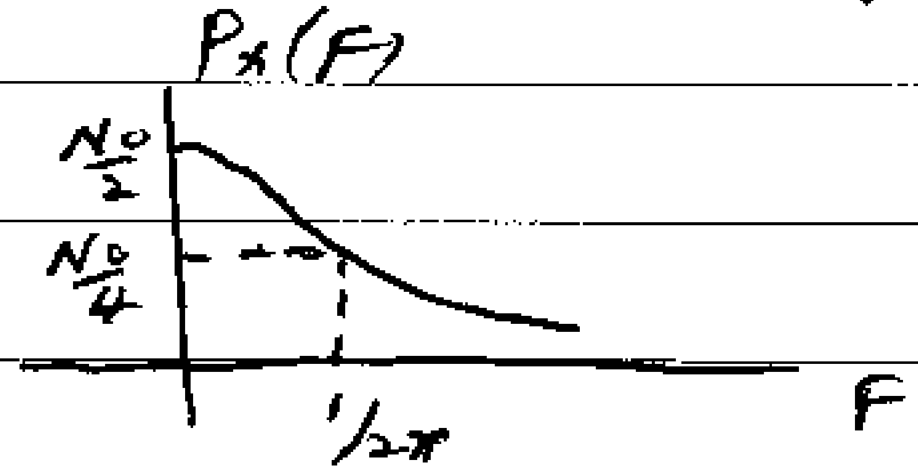
28) $P_x(F) = |H(F)|^2 P_w(F)$

$$H(F) = \mathcal{F}\{ e^{-t} u_s(t) \} = \int_0^\infty e^{-t} e^{-j2\pi F t}\, dt$$

$$= \left. \frac{e^{-(1+j2\pi F)t}}{-(1+j2\pi F)} \right|_0^\infty$$

$$= \frac{1}{1+j2\pi F}$$

$$P_x(F) = \frac{N_0/2}{|1+j2\pi F|^2} = \frac{N_0/2}{1+(2\pi F)^2}$$



29) $P_x(F) = |H(F)|^2 N_0/2$

$$H(F) = \int_0^T 1\, e^{-j2\pi F t}\, dt$$

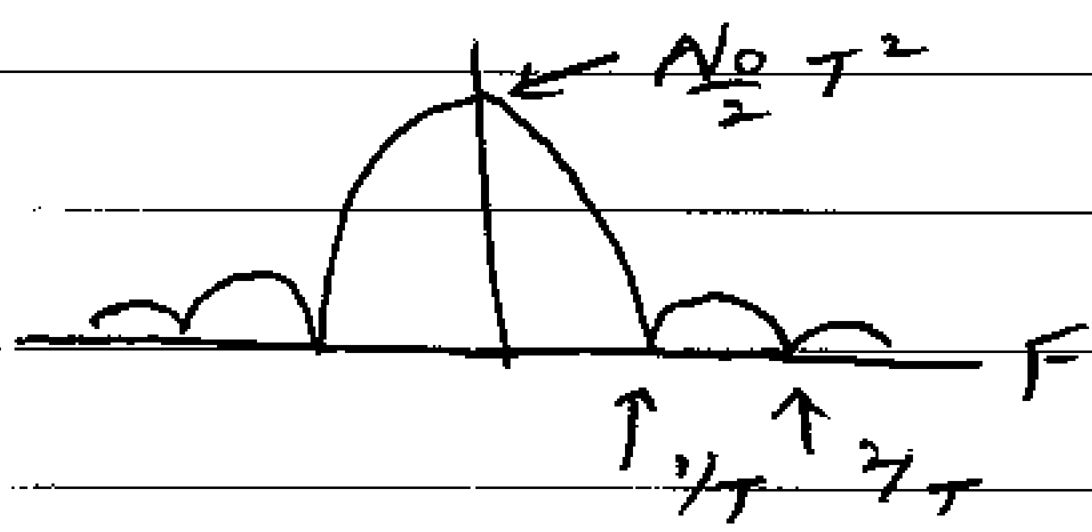
$$= \frac{e^{-j2\pi F t}}{-j2\pi F}\Big|_0^T = \frac{e^{-j2\pi F T}-1}{-j2\pi F}$$

$$= \frac{e^{-j\pi F T}\left(e^{-j\pi F T} - e^{j\pi F T}\right)}{-j2\pi F}$$

$$= e^{-j\pi F T}\frac{(-2j\sin\pi F T)}{-j2\pi F}$$

$$= e^{-j\pi F T}\, T\, \frac{\sin\pi F T}{\pi F T}$$

$$P_x(F) = \frac{N_0}{2}T^2\left(\frac{\sin\pi F T}{\pi F T}\right)^2$$



30) $$P_x(F) = \underbrace{|H(F)|^2}_{=1} P_v(F)$$

$$= P_v(F)$$

PSD unchanged, only phases are affected by filter, which do not enter into PSD calculation since the PSD measures only power

31) $$r_x(\tau) = h(-\tau) \star h(\tau) \star r_v(\tau)$$

$$= \frac{N_0}{2} h(-\tau) \star h(\tau)$$

since

$$x(t) \star \delta(t) = x(t)$$

$$r_x(\tau) = \int_{-\infty}^{\infty} \frac{N_0}{2} h(-u) h(\tau-u)\, du$$

Let $t = -u$

$$r_x(\tau) = \frac{N_0}{2} \int_{-\infty}^{\infty} h(t) h(t+\tau)\, dt$$

32) $$h(t) = \frac{1}{RC} e^{-t/RC} u_s(t)$$ (from Fourier transform Tables)

From previous problem

$$r_x(0) = \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(t)\, dt$$

$$= \frac{N_0}{2} \int_{0}^{\infty} \left(\frac{1}{RC} e^{-t/RC}\right)^2 dt$$

$$= \frac{N_0}{2}\left(\frac{1}{RC}\right)^2 \left. \frac{e^{-2t/RC}}{-2/RC} \right|_0^\infty$$

$$= \frac{N_0}{2}\left(\frac{1}{RC}\right)^2 \frac{RC}{2} = \frac{N_0}{4RC} < \infty$$

33) $E[z(t)] = E[x(t)] + E[y(t)] = 0$

$E[z(t)z(t+\tau)] = E[(x(t)+y(t))(x(t+\tau) + y(t+\tau))]$

$= E[x(t)x(t+\tau)] + E[y(t)y(t+\tau)]$

since $x(t)$ and $y(t)$ are uncorrelated

$$\Rightarrow \left.\begin{array}{l} \mu_z(t) = \mu = 0 \\ r_z(\tau) = r_x(\tau) + r_y(\tau) \end{array}\right\} \Rightarrow \text{WSS}$$

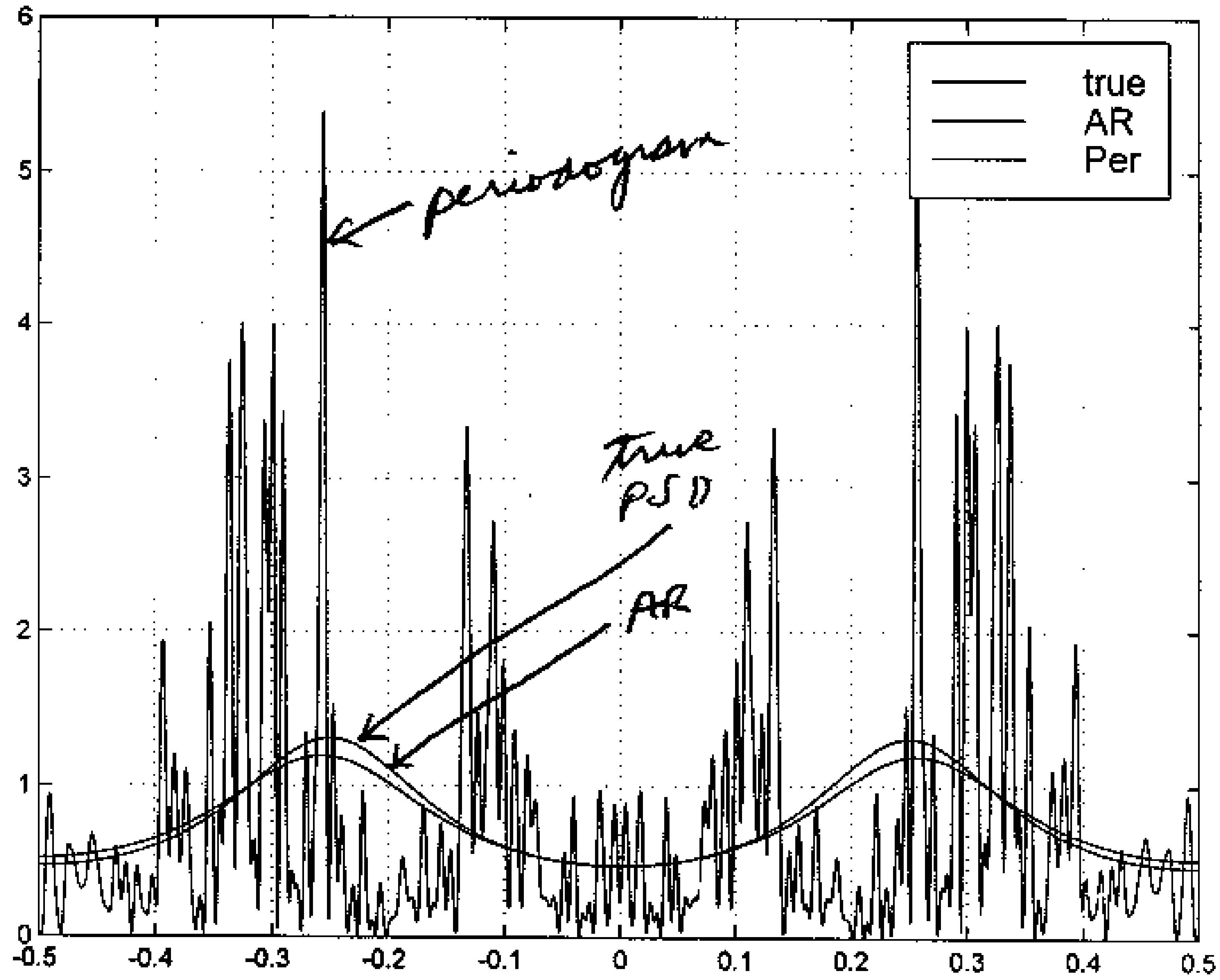
Also, $P_z(F) = P_x(F) + P_y(F)$

34) $\hat{a}[1] = -0.0294$

$\hat{a}[2] = -0.2150$

True PSD is $P_x(f) = \dfrac{\sigma_u^2}{|1 - a[1]e^{-j2\pi f} - a[2]e^{-j4\pi f}|^2}$

with $a[1] = 0$, $a[2] = -0.25$, $\sigma_u^2 = 15/16$




```
% probprob18_34.m

clear all

randn('state',0)
rad=0.5;varu=1-rad^4;
for n=1:150
   if n==1
      xseg(n,1)=sqrt(varu)*randn(1,1);
   elseif n==2
      xseg(n,1)=sqrt(varu)*randn(1,1);
   else
      xseg(n,1)=-rad^2*xseg(n-2,1)+sqrt(varu)*randn(1,1);
   end
end
```

to next page

```
N=length(xseg);
Nfft=1024; % set up FFT length for Fourier transforms
freq=[0:Nfft-1]'/Nfft-0.5; % PSD frequency points to be plotted
P_per=(1/N)*abs(fftshift(fft(xseg,Nfft))).^2; % compute periodogram
p=2;
for k=1:p+1 % estimate ACS for k=0,1,...,p (MATLAB indexes must start at 1)
   rX(k,1)=(1/N)*sum(xseg(1:N-k+1).*xseg(k:N));
end
r=rX(2:p+1); % fill in right-hand-side vector
for i=1:p % fill in autocorrelation matrix
   for j=1:p
      R(i,j)=rX(abs(i-j)+1);
   end
end
a=inv(R)*r % solve linear equations to find AR filter parameters
varu=rX(1)-a'*r; % find excitation noise variance
den=abs(fftshift(fft([1;-a],Nfft))).^2; % compute denominator of AR PSD
P_AR=varu./den; % compute AR PSD
dentrue=abs(fftshift(fft([1 0 rad^2]',Nfft))).^2;
P_true=varu./dentrue;

plot(freq,P_true,freq,P_AR,freq,P_per)
legend('true','AR','Per')
grid
```

Chapter 19

1) Each random process is WSS since

$$\mu_X[n] = 0$$
$$r_X[k] = \sigma_v^2 \delta[k]$$
$$\mu_Y[n] = 0$$
$$r_Y[k] = E[(-1)^n v[n] (-1)^{n+k} v[n+k]]$$
$$= (-1)^{2n+k} r_v[k]$$
$$= (-1)^k \sigma_v^2 \delta[k] = \sigma_v^2 \delta[k]$$

but

$$E[x[n] y[n+k]] = E[v[n] (-1)^{n+k} v[n+k]]$$
$$= (-1)^{n+k} \sigma_v^2 \delta[k]$$
$$= 0 \qquad k \neq 0$$
$$\sigma_v^2 (-1)^n \qquad k = 0$$

depends on n $\Rightarrow$ not jointly WSS

2)
$$E[x[n] y[n+k]] = E[(a_1 v_1[n] + a_2 v_2[n])(b_1 v_1[n+k] + b_2 v_2[n+k])]$$

$$= a_1 a_2 r_{v_1}[k] + a_1 b_2 r_{v_1 v_2}[k] + a_2 b_1 r_{v_2 v_1}[k]$$
$$+ a_2 b_2 r_{v_2}[k]$$

doesn't depend on n. Also easy to show $x[n]$, $y[n]$ are individually WSS $\Rightarrow$ jointly WSS

$$r_{XY}[k] = \left(\tfrac{1}{2}\right)^{|k-1|}$$

3)


Not symmetric about $k=0$ } different
$r_{x,y}[0]$ not maximum }
$r_{x,y}[k] \to 0$ as $|k| \to \infty$ same

4) $$r_{x,y}[k] = E[x[n](x[n+k]+w[n+k])]$$
$$= E[x[n]x[n+k]] + E[x[n]w[n+k]]$$
$$= r_x[k] + r_{x,w}[k]$$

$$\Rightarrow P_{x,y}(f) = P_x(f) + P_{x,w}(f)$$

5) $$r_{x,y}[k] = E[x[n]y[n+k]]$$
$$= E[x[n]x[n+k]w[n+k]]$$
$$= E[x[n]x[n+k]]\underbrace{E[w[n+k]]}_{=0} \text{ (ind.)}$$
$$= 0$$

Effect of $w[n]$ is to "modulate" $x[n]$ with $+1$ or -1, which causes $x[n]$ and $y[n]$ to be uncorrelated.

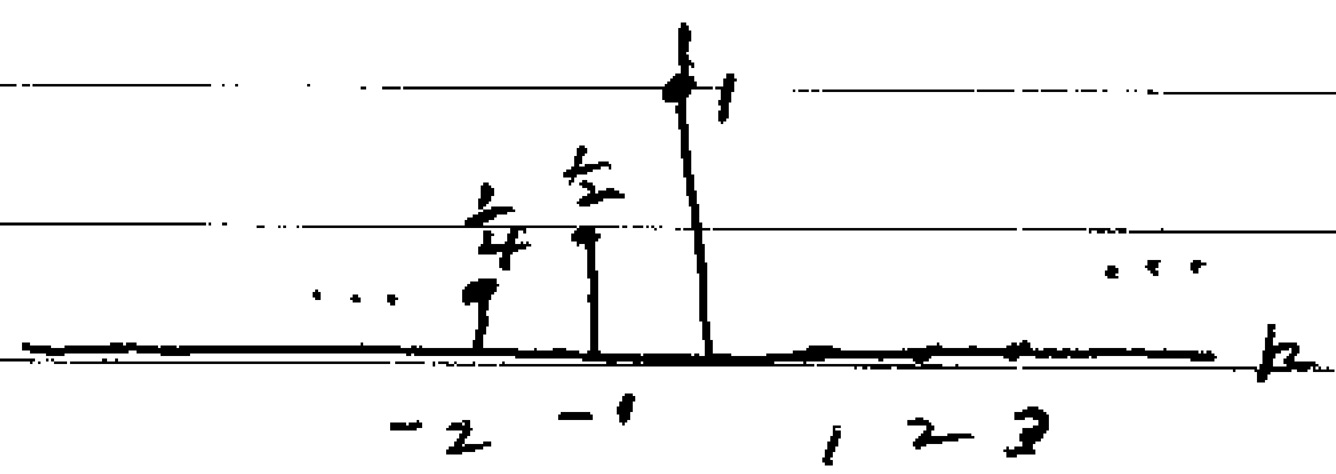
6) $$r_{x,u}[k] = E[x[n]u[n+k]]$$
$$= E[x[n](x[n+k]-ax[n+k-1])]$$
$$= r_x[k] - a r_x[k-1]$$
$$= \frac{\sigma_u^2}{1-a^2}\left(a^{|k|} - a\,a^{|k-1|}\right)$$
$$= 0 \qquad k>0$$

For $k \le 0$

$$r_{x,v}[k] = \frac{\sigma_v^2}{1-a^2}\left(a^{-k} - a\,a^{1-k}\right)$$

$$= \frac{\sigma_v^2}{1-a^2}\,a^{-k}(1-a^2) = \sigma_v^2\, a^{-k}$$

$$= \left(\tfrac{1}{2}\right)^{-k}$$



7) $$|r_{x,y}[k]| \le \sqrt{r_x[0]\, r_y[0]}$$
$$= \sqrt{4(3)} = 2\sqrt{3}$$

8) $$P_{x,y}(f) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_n \sum_m E\left[x[n]y[m]\right] e^{j2\pi f n} e^{-j2\pi f m}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_n \sum_m \underbrace{r_{x,y}[m-n]\, e^{-j2\pi f(m-n)}}_{g[m-n]}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{k=-2M}^{2M} (2M+1-|k|)\, r_{x,y}[k]\, e^{-j2\pi f k}$$

$$= \lim_{M\to\infty} \sum_{k=-2M}^{M} \left(1 - \frac{|k|}{2M+1}\right) r_{x,y}[k]\, e^{-j2\pi f k}$$

$$= \sum_{k=-\infty}^{\infty} r_{x,y}[k]\, e^{-j2\pi f k}$$ since

as $M \to \infty$ $\left(1 - \frac{|k|}{2M+1}\right) \to 1$ for all finite k

9) $r_{X,Y}[k] = E[\cos(2\pi f_0 n + \Theta_1)\cos(2\pi f_0(n+k) + \underbrace{\Theta_2}_{=\Theta_1})]$

$$= r_X[k] = \frac{1}{2}\cos 2\pi f_0 k$$

$$P_{X,Y}(f) = P_X(f) = \frac{1}{4}\delta(f+f_0) + \frac{1}{4}\delta(f-f_0)$$

If Θ_1, Θ_2 are independent $r_{X,Y}[k] = 0$
$\Rightarrow P_{X,Y}(f) = 0$. In this case the Fourier components at $f = f_0$ are out of phase on the average and thus the crosscorrelation or $E[X^*_{2M+1}(f)Y_{2M+1}(f)] = 0$.

10) $P_{X,Y}(f) = 1 + 2e^{-j2\pi f}$

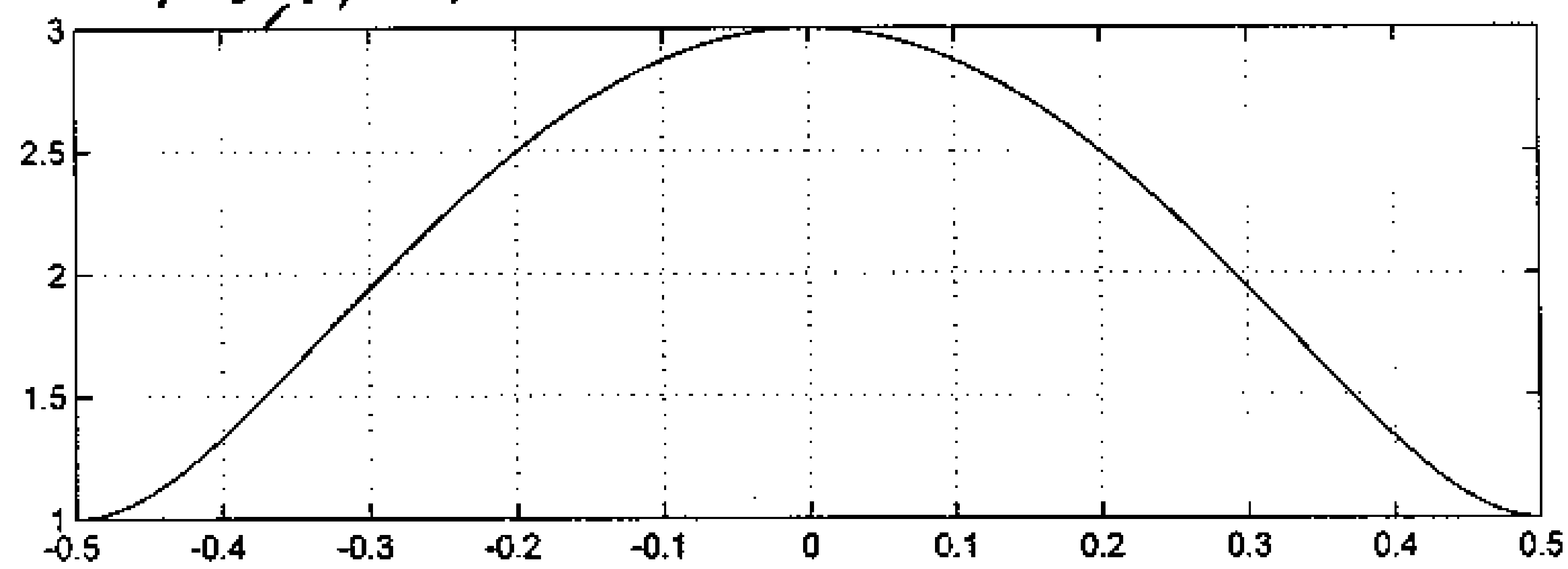
$$= 1 + 2\cos 2\pi f - 2j\sin 2\pi f$$

$$|P_{X,Y}(f)| = \sqrt{(1+2\cos 2\pi f)^2 + (2\sin 2\pi f)^2}$$
$$= \sqrt{5 + 4\cos 2\pi f}$$

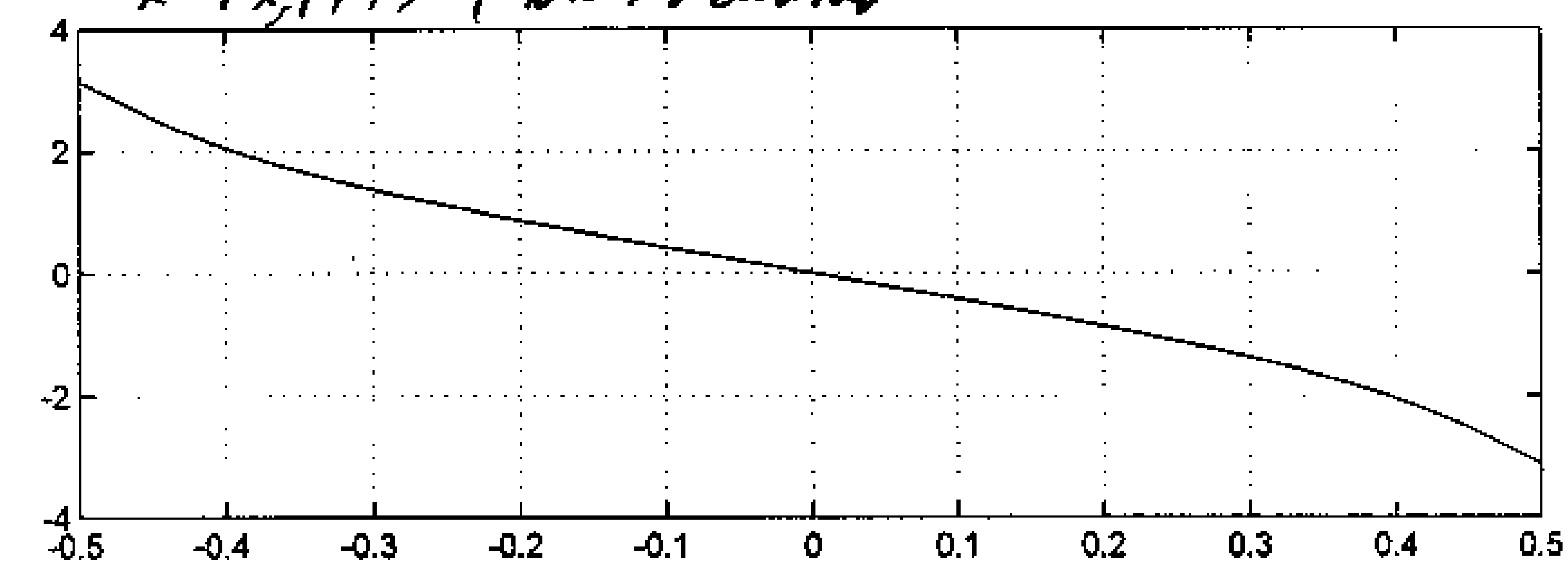
$$\angle P_{X,Y}(f) = \arctan \frac{-2\sin 2\pi f}{1+2\cos 2\pi f}$$

```
%probprob19_10.m
%
clear all
f=[-0.5:0.01:0.5]';
Pxymag=sqrt(5+4*cos(2*pi*f));
Pxyphase=atan2(-2*sin(2*pi*f),1+2*cos(2*pi*f));
subplot(2,1,1)
plot(f,Pxymag)
grid
subplot(2,1,2)
plot(f,Pxyphase)
grid
```

$|P_{X,Y}(f)|$



$\measuredangle P_{X,Y}(f)$ (in radians)



11) $r_{X,Y}[k] = E[X[n]Y[n+k]]$

$$= E[U[n](U[n+k] - bU[n+k-1])]$$

$$= r_U[k] - b\, r_U[k-1]$$

$$P_{X,Y}(f) = P_U(f) - b\, e^{-j2\pi f} P_U(f)$$

$$= 1 - b\, e^{-j2\pi f} \quad (P_U(f) = 1)$$

As $b \to 0$, $P_{X,Y}(f) \to 1 = P_X(f)$ since $Y[n] \to X[n]$.

12) $r_Z[k] = E[(X[n]-Y[n])(X[n+k]-Y[n+k])]$

$$= r_X[k] - r_{X,Y}[k] - r_{Y,X}[k] + r_Y[k]$$

$$P_Z(f) = P_X(f) - P_{X,Y}(f) - P_{Y,X}(f) + P_Y(f)$$

13) $$P_{x,y}(f) = 1 - b e^{-j2\pi f}$$
$$P_x(f) = P_u(f) = 1$$
$$P_y(f) = |1 - b e^{-j2\pi f}|^2 P_u(f)$$
$$= |1 - b e^{-j2\pi f}|^2$$

$$\gamma_{x,y}(f) = \frac{1 - b e^{-j2\pi f}}{\sqrt{1 \cdot |1 - b e^{-j2\pi f}|^2}}$$

$$= \frac{|1 - b e^{-j2\pi f}| \, e^{j \measuredangle (1 - b e^{-j2\pi f})}}{|1 - b e^{-j2\pi f}|}$$

$$= e^{j \measuredangle (1 - b e^{-j2\pi f})}$$

As $b \to 0$, $\gamma_{x,y}(f) \to e^{j \measuredangle 1} = e^{j \cdot 0} = 1$

14) $$P_{x,y}(f) = 1 - e^{-j2\pi f}$$

at $f = 0$, $P_{x,y}(f) = 0 \Rightarrow \gamma_{x,y}(f) = 0$

$$\left.\begin{aligned} x[n] &= u[n] \\ y[n] &= u[n] - u[n-1] \end{aligned}\right\} \text{ from Prob. 19.11 with } b = 1$$

at $f = 0$, the Fourier component of $y[n]$ will be zero since if $u[n] = \mu$ (a constant), $y[n] = \mu - \mu = 0$.

15) $$r_y[k] = E[y[n]y[n+k]]$$
$$= E[(-x[n])(-x[n+k])]$$
$$= r_x[k]$$

$$r_{x,y}[k] = E[x[n]y[n+k]]$$
$$= E[x[n](-x[n+k])]$$
$$= -r_x[k]$$
$$\Rightarrow \quad P_{x,y}(f) = -P_x(f)$$

$$\gamma_{x,y}(f) = \frac{P_{x,y}(f)}{\sqrt{P_x(f)P_y(f)}} = \frac{-P_x(f)}{\sqrt{P_x^2(f)}}$$

$$= -1 \qquad \text{since } P_x(f) \geq 0$$

$y[n]$ is perfectly predictable since $|\gamma_{x,y}(f)| = 1$
Use $\hat{y}[n_0] = -x[n_0]$

16) First principal minor $= P_x(f) \geq 0$
Second principal minor $= P_x(f)P_y(f) - P_{x,y}(f)P_{y,x}(f)$

$$= P_x(f)P_y(f) - P_{x,y}(f)P^*_{x,y}(f) \quad \text{from p 19.9}$$

$$= P_x(f)P_y(f)\left[1 - \underbrace{\frac{|P_{x,y}(f)|^2}{P_x(f)P_y(f)}}_{|\gamma_{x,y}(f)|^2 \leq 1}\right] \geq 0$$

17) $z[n] = x[n] + y[n]$
$$r_z[k] = r_x[k] + r_y[k]$$
$$= 2\delta[k] + \left(\tfrac{1}{2}\right)^{|k|}$$

$$P_z(f) = 2 + \mathcal{F}\{(\tfrac{1}{2})^{|k|}\}$$

But $y(n)$ is AR process with $\sigma_v^2 = 1 - a^2$ and $a = \frac{1}{2}$ $\Rightarrow$

$$\mathcal{F}\{(\tfrac{1}{2})^{|k|}\} = \frac{1-(\frac{1}{2})^2}{|1 - \frac{1}{2}e^{-j2\pi f}|^2}$$

$$P_z(f) = 2 + \frac{3/4}{1 + \frac{1}{4} - \cos 2\pi f}$$

$$= 2 + \frac{3/4}{5/4 - \cos 2\pi f}$$

$$= \frac{13/4 - 2\cos 2\pi f}{5/4 - \cos 2\pi f}$$

18) $\quad r_{x,y}(l) = \sum_{k=-\infty}^{\infty} h(k)\, r_x(l-k) \qquad -\infty < l < \infty$

$$\Rightarrow \quad P_{x,y}(f) = H(f)\, P_x(f)$$

$$H_{opt}(f) = P_{x,y}(f) / P_x(f)$$

If $y(n) = s(n)$, $\quad x(n) = s(n) + w(n)$

where $r_{s,w}(k) = 0$ all k

$$P_x(f) = P_s(f) + P_w(f)$$

$$P_{x,y}(f) = \mathcal{F}\{r_{xy}(k)\}$$ uncorrelated

$$r_{x,y}(k) = E\{(s(n) + w(n))\, s(n+k)\}$$

$$= E\{s(n)\, s(n+k)\} = r_s(k)$$

$$\Rightarrow \quad P_{x,y}(f) = P_s(f)$$

$$H_{opt}(f) = P_s(f) / (P_s(f) + P_w(f))$$

19) $r_{x,y}[k] = \delta[k] + 2\delta[k-1]$

$P_{x,y}(f) = 1 + 2e^{-j2\pi f}$

$P_x(f) = P_u(f) = 1$

$H_{opt}(f) = P_{x,y}(f) / P_x(f)$ from prob 22.18

$= 1 + 2e^{-j2\pi f}$

20) $P_{x,y}(f) = H(f) P_x(f)$ from (19.28)

$$\gamma_{x,y}(f) = \frac{H(f) P_x(f)}{\sqrt{P_x(f)\, |H(f)|^2 P_x(f)}}$$

$$= \frac{H(f)}{|H(f)|} \quad (P_x(f) \geq 0)$$

$\Rightarrow |\gamma_{x,y}(f)| = 1$ (for $H(f) \neq 0$)

21)

$$x[n] \rightarrow \underbrace{\boxed{H(f)} \xrightarrow{y[n]} \boxed{\frac{1}{H(f)}}}_{=1} \rightarrow x[n]$$

22)
$$\hat{X}(f_0) = \sum_{k=-M}^{M} x[k] e^{-j2\pi f_0 k}$$

$$= \sum_{k=-M}^{M} x[k] h[-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[-k] \quad (h[k] = 0 \text{ for } |k| > M)$$

$$= \sum_{k=-\infty}^{\infty} x[k] h[n-k] \Big|_{n=0}$$

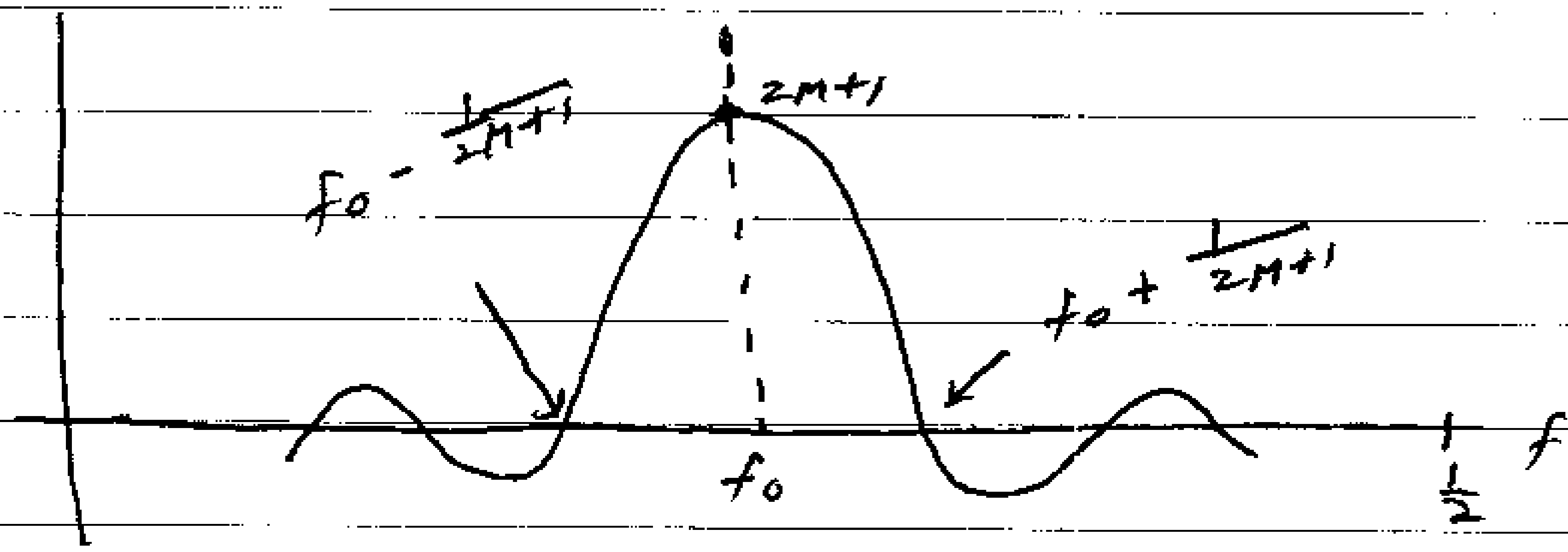
$$H(f) = \sum_{k=-M}^{M} e^{j2\pi f_0 k} e^{-j2\pi f k}$$

$$= \sum_{k=-M}^{M} e^{jk\theta} \qquad \theta = 2\pi(f_0 - f)$$

$$= \frac{\sin[(2M+1)\theta/2]}{\sin \theta/2}$$

$$= \frac{\sin((2M+1)\pi(f_0-f))}{\sin(\pi(f_0-f))}$$

$$= \frac{\sin[(2M+1)\pi(f-f_0))}{\sin(\pi(f-f_0))}$$



as $M \to \infty$, this is narrowband filter centered about f_0.

23) $r_{x,y}(\tau) = E[X(t)Y(t+\tau)]$

$$= E\left[X(t)\,\frac{1}{T}\int_{t+\tau-T}^{t+\tau} X(\xi)\,d\xi\right]$$

$$= \frac{1}{T}\int_{t+\tau-T}^{t+\tau} E[X(t)X(\xi)]\,d\xi$$

$$= \frac{1}{T}\int_{t+\tau-T}^{t+\tau} \frac{N_0}{2}\,\delta(\xi-t)\,d\xi$$

For this to be nonzero the impulse at $\zeta = t$ must be in $[t+\tau-T,\ t+\tau]$ or

$$t+\tau-T \le t \le t+\tau$$

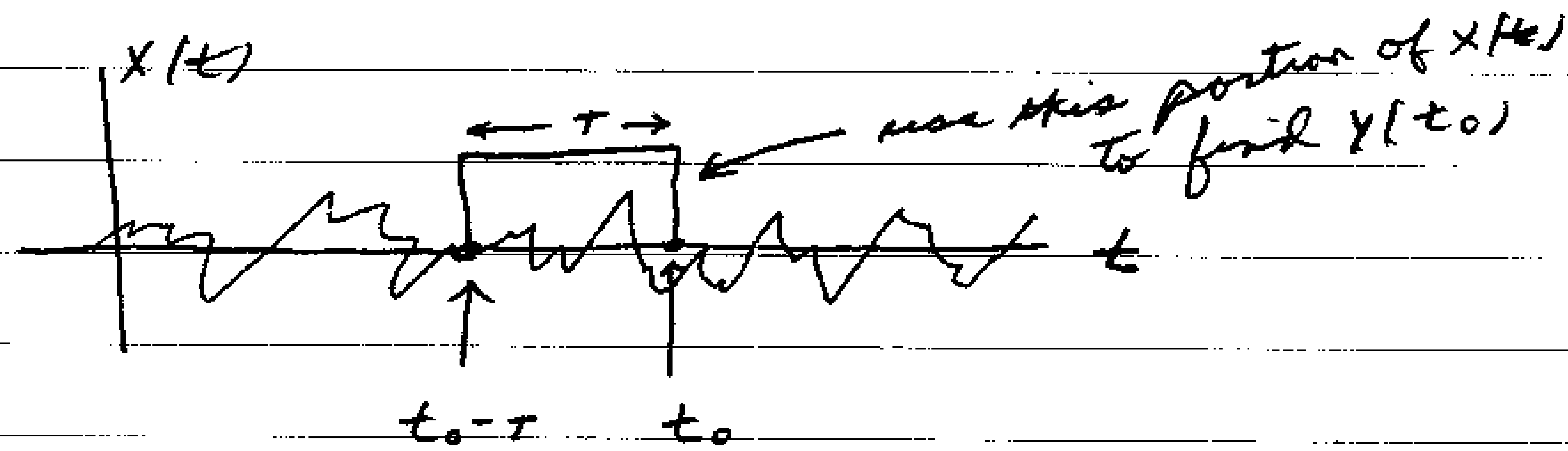
$$\tau - T \le 0 \le \tau$$

$$\Rightarrow \quad \tau - T \le 0 \quad \text{or} \quad \tau \le T$$

$$0 \le \tau \quad \text{or} \quad \tau \ge 0$$

$\Rightarrow \quad 0 \le \tau \le T$ Then

$$r_{x,y}(\tau) = \begin{cases} \frac{1}{T}\frac{N_0}{2} & 0 \le \tau \le T \\ 0 & \text{otherwise} \end{cases}$$



$$r_{x,y}(\tau) = E[x(t_0)\,y(t_0+\tau)]$$

$$= E[x(t_0-\tau)\,y(t_0)]$$

↑ based on $x(t)$ for $t_0 - T \le t \le t_0$

If $\tau = -1$ for example

$r_{x,y}(\tau) = E[x(t_0+1)\,y(t_0)] = 0$ since $y(t_0)$ composed of $x(t)$ prior to $t = t_0+1$ (and all $x(t)$'s are uncorrelated).

24) $r_{x,y}(\tau) = h(\tau) \star r_x(\tau)$
$$= e^{-\tau} u(\tau) \star N_0/2\, \delta(\tau)$$

But $x(\tau) \star \delta(\tau) = x(\tau)$

$$\Rightarrow \quad r_{x,y}(\tau) = \frac{N_0}{2} e^{-\tau} u(\tau)$$

25) Yes, if $H_1(F) = H_2(F)$

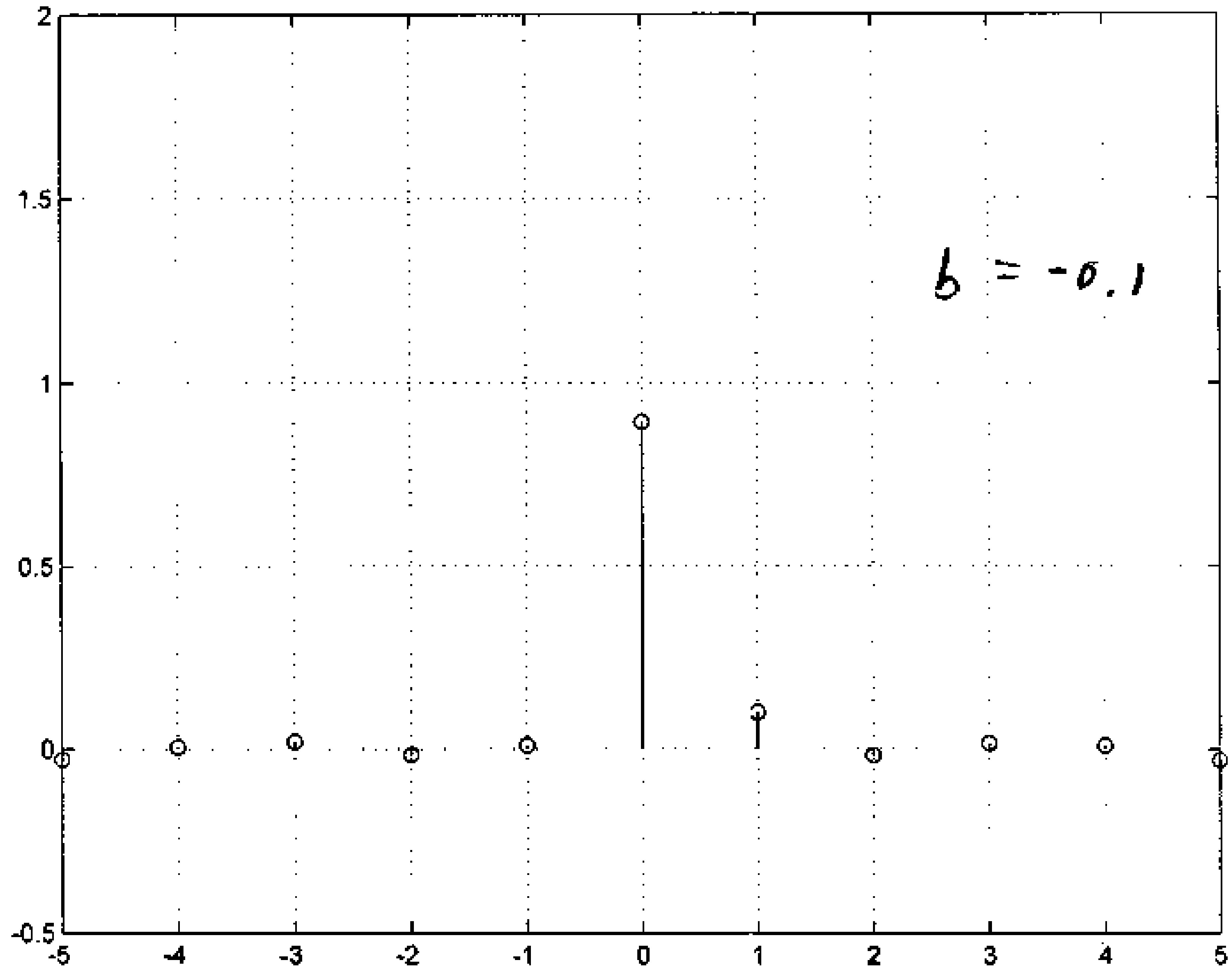
$$P_{x,y}(F) = |H_1(F)|^2 P_v(F)$$

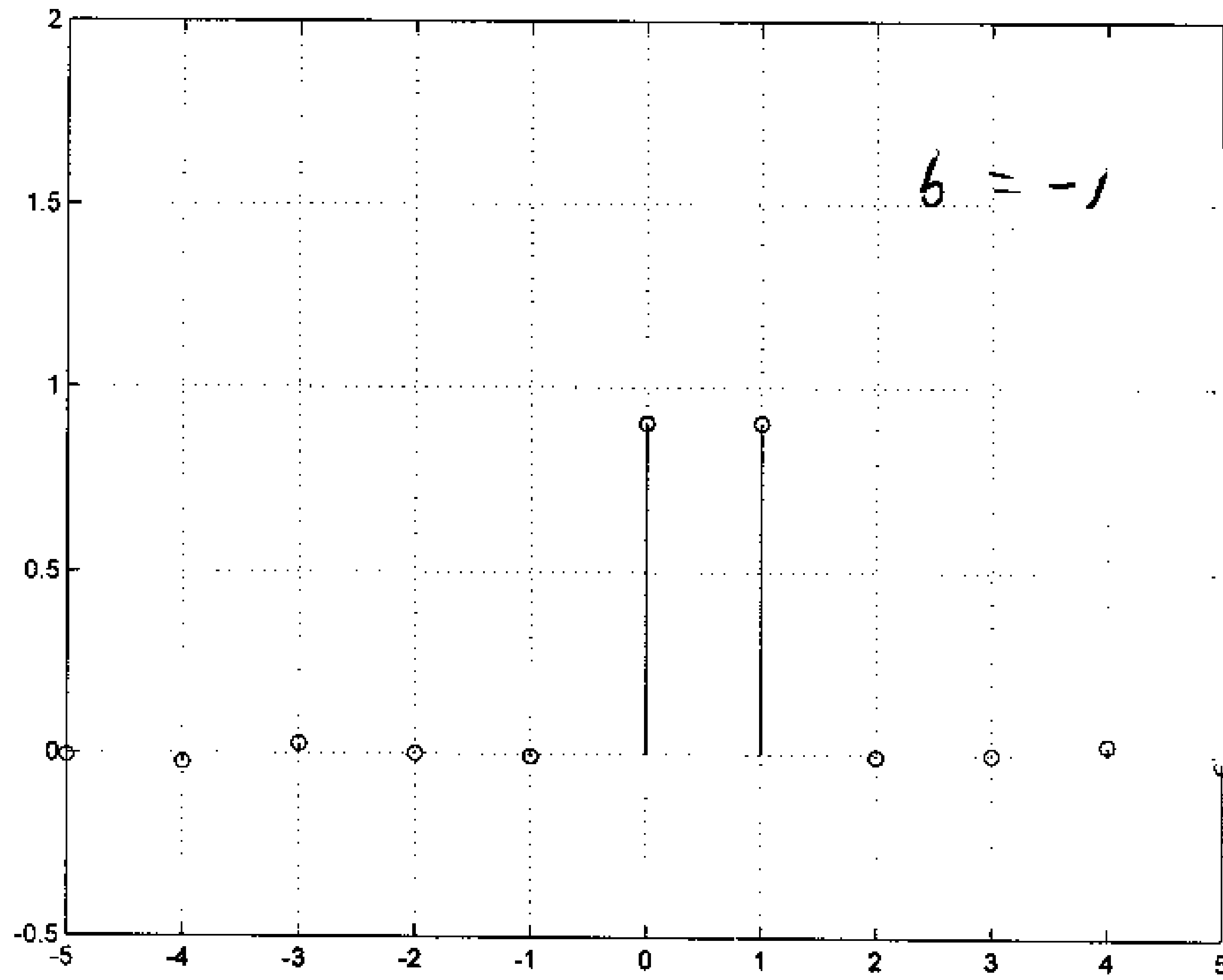
In this case, $y(t) = x(t)$.

Let $H_1(F) = H_2(F) = e^{-j2\pi F \tau_0}$
This is just a delay of τ_0 sec through each filter and thus $y(t) = x(t) = v(t-\tau_0)$.

26) $r_{x,y}[k] = r_v[k] - b\, r_v[k-1]$ from Prob 19.11
$$= \delta[k] - b\, \delta[k-1]$$
$$= \begin{cases} 1 & k = 0 \\ -b & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

For b close to zero $r_{x,y}[k] \simeq \delta[k]$ since then $x[n] = y[n] = v[n]$ and
$$r_{x,y}[k] = r_x[k] = \delta[k]$$





rxy =

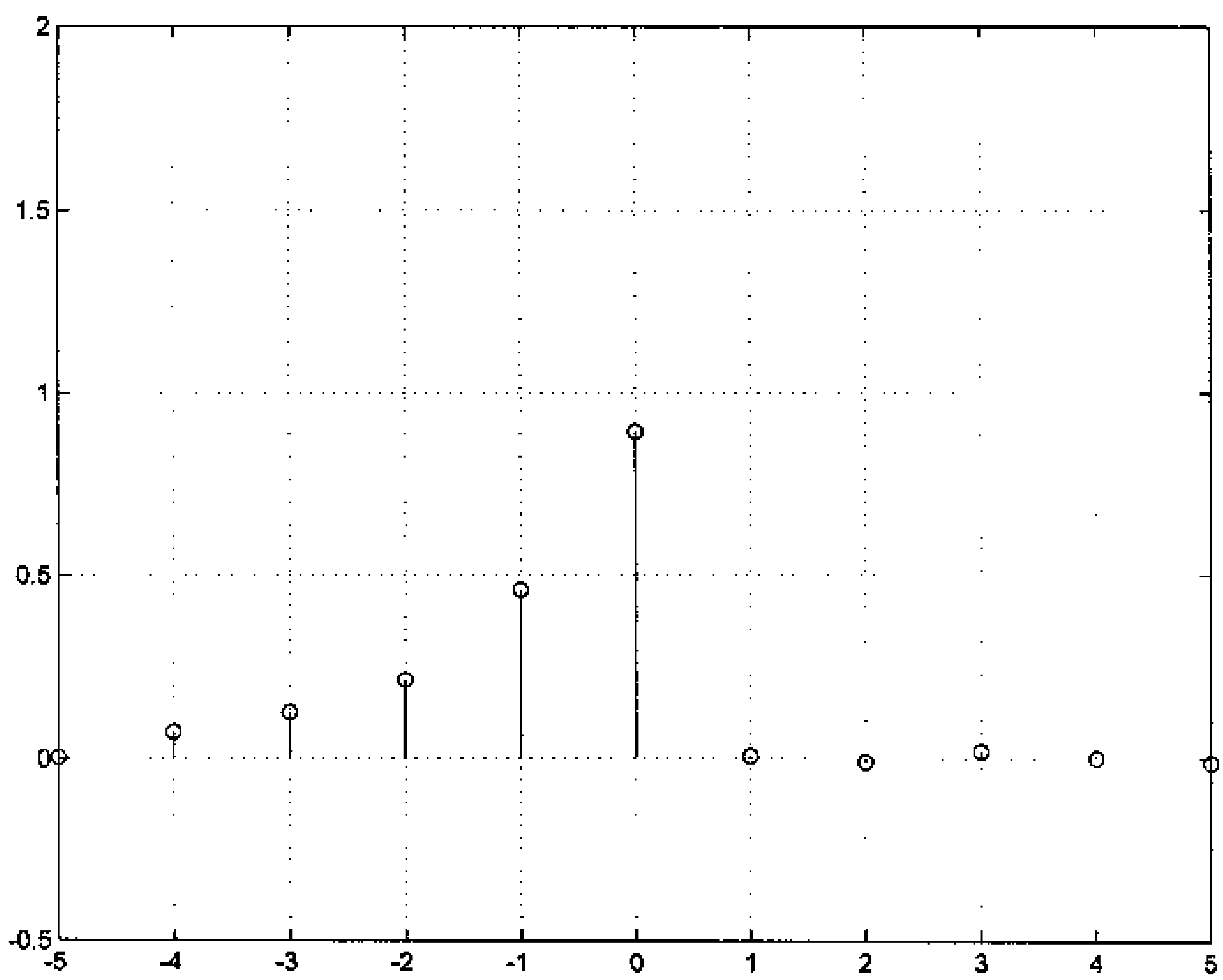
k=-5 -0.0077
-0.0242
0.0259
0.0004
-0.0062
0.9034
0.9031
-0.0064
-0.0007
0.0267
k=5 -0.0238

```
% probprob19_26.m
%
clear all
randn('state',0)
b=-1;
N=1000;M=5;
u=randn(N+1,1);
x=u(2:N+1,1);
y=u(2:N+1,1)-b*u(1:N,1);
for k=0:M
   rxypos(k+1,1)=(1/(N-k))*sum(x(1:N-k).*y(1+k:N));
end
for k=1:M
   rxyneg(k+1,1)=(1/(N-k))*sum(x(k+1:N).*y(1:N-k));
end
rxy=[flipud(rxyneg(2:M+1,1));rxypos];

figure
stem([-M:M]',rxy)
grid
axis([-5 5 -0.5 2])
```

27) $r_{x,v}(k) = \left(\frac{1}{2}\right)^{-k} \quad k \le 0$

$\qquad\qquad 0 \quad k > 0$

from Prob 19.6



```
% probprob19_27.m
%
clear all
randn('state',0)
N=1000;;M=5;
a=0.5;varu=1;
xx(1,1)=sqrt(varu/(1-a^2))*randn(1,1);
for n=2:1001
   uu(n,1)=sqrt(varu)*randn(1,1);
   xx(n,1)=a*xx(n-1,1)+uu(n);
end
x=xx(2:1001,1);u=uu(2:1001,1);
for k=0:M
   rxupos(k+1,1)=(1/(N-k))*sum(x(1:N-k).*u(1+k:N));
end
for k=1:M
   rxuneg(k+1,1)=(1/(N-k))*sum(x(k+1:N).*u(1:N-k));
end
rxu=[flipud(rxuneg(2:M+1,1));rxupos];

figure
stem([-M:M]',rxu)
grid
axis([-5 5 -0.5 2])
```

28) $r_{x_1 x_2}(\tau) = E[x_1(t)x_2(t+\tau)]$

$$= E[U(t-t_0)U(t+\tau)]$$

$$= r_U(\tau + t_0)$$

But $r_U(\tau) = \int_{-W}^{W} P_U(F) e^{j2\pi F\tau} dF$

$$= \int_{-W}^{W} \frac{N_0}{2} e^{j2\pi F\tau} dF$$

$$= \frac{N_0}{2} \int_{-W}^{W} \cos 2\pi F\tau \, dF$$

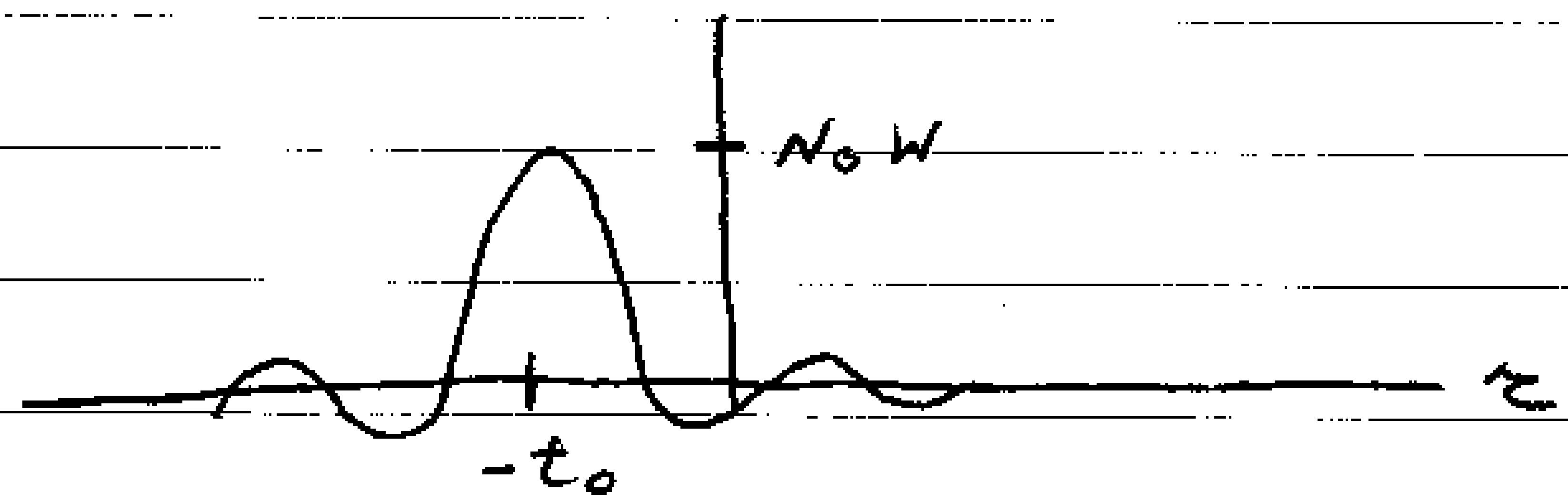
$$= \frac{N_0}{2} \left. \frac{\sin 2\pi F\tau}{2\pi\tau} \right|_{-W}^{W}$$

$$= \frac{N_0}{2} \left[\frac{\sin 2\pi W\tau}{2\pi\tau} - \frac{\sin 2\pi(-W)\tau}{2\pi\tau} \right]$$

$$= \frac{N_0}{2} \; \frac{2 \sin 2\pi W \tau}{2\pi\tau}$$

$$= N_0 W \; \frac{\sin 2\pi W \tau}{2\pi W \tau}$$

$$\Gamma_{x_1, x_2}(\tau) = N_0 W \; \frac{\sin(2\pi W(\tau + t_0))}{2\pi W(\tau + t_0)}$$



Find peak of estimated CCF, call it $\hat{\tau}$.

Then

$$\hat{\tau} = -\hat{t}_0 = -d/c \cos\hat{\theta}$$

$$\Rightarrow \quad \hat{\theta} = \arccos(-c\hat{\tau}/d)$$

Chapter 20

1) Since samples are independent

$$P[X[0]>0, \ldots, X[4]>0] = \prod_{n=0}^{4} P[X[n]>0] = Q^5(0)$$

$$= \left(\tfrac{1}{2}\right)^5 = \tfrac{1}{32}$$

Same for second set since also IID.

2) $X[n] = \frac{1}{2}(U[n] + U[n-1])$

Since $r_X[k] = 0$ for $k > 1$

$\Rightarrow$ $X[0]$, $X[3]$ are uncorrelated and since $U[n]$ is Gaussian, $X[n]$ is Gaussian random process $\Rightarrow$ $X[0]$, $X[3]$ are ind.

$$P[X[0]>0, X[3]>0] = P[X$$

$$= P[X[0]>0]^2$$

Since $X[n] \sim N(0, \frac{1}{4}(1+1)) = N(0, \frac{1}{2})$

$$P[X[0]>0, X[3]>0] = Q\left(\frac{0-0}{\sqrt{1/2}}\right)^2 = Q(0)^2$$

$$= \left(\tfrac{1}{2}\right)^2 = 1/4$$

3) $X[n] = \sum_{i=0}^{n} U(i) \sim N(0, (n+1)\sigma_U^2)$

Here $\sigma_U^2 = 2$

$$P[-3 \le X[n] \le 3] = Q\left(\frac{-3}{\sqrt{2(n+1)}}\right) - Q\left(\frac{3}{\sqrt{2(n+1)}}\right)$$

$$= 1 - 2Q\left(\frac{3}{\sqrt{2(n+1)}}\right) = 1 - 2Q\left(\frac{3}{\sqrt{12}}\right)$$

$$= 1 - 2\Phi(\sqrt{3}/2) \doteq 0.6135$$

4) $$x[n] = \sum_{i=0}^{n} u[i]$$

$$y[n] = x[n] - x[n-1] = \sum_{i=0}^{n} u[i] - \sum_{i=0}^{n-1} u[i]$$

$= u[n]$ which is WGN

5) $$\underbrace{\begin{bmatrix} y[0] \\ y[1] \end{bmatrix}}_{\underline{y}} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{\underline{G}} \underbrace{\begin{bmatrix} x[-1] \\ x[0] \\ x[1] \end{bmatrix}}_{\underline{x}}$$ (or use results in Example 20.4)

$$\underline{y} \sim N(\underline{G}\underline{\mu}_x, \underline{G}\underline{C}\underline{G}^T)$$

$$\underline{\mu}_x = \underline{0}, \quad \underline{C} = \sigma_x^2 \underline{I} \Rightarrow \underline{y} \sim N(\underline{0}, \sigma_x^2 \underline{G}\underline{G}^T)$$

$$\underline{G}\underline{G}^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Not independent since $\underline{C}_y = \underline{G}\underline{C}\underline{G}^T = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is not diagonal.

6) Same as previous problem except

$$\underline{C}_y = \underline{G}\underline{C}\underline{G}^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} r_x[0] & r_x[1] & r_x[2] \\ r_x[1] & r_x[0] & r_x[1] \\ r_x[2] & r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} -r_x[0] + r_x[1] & r_x[2] - r_x[1] \\ r_x[0] - r_x[1] & r_x[1] - r_x[0] \\ r_x[1] - r_x[2] & r_x[0] - r_x[1] \end{bmatrix}$$

$$= \begin{bmatrix} 2(r_x[0] - r_x[1]) & 2r_x[1] - r_x[0] - r_x[2] \\ 2r_x[1] - r_x[0] - r_x[2] & 2(r_x[0] - r_x[1]) \end{bmatrix}$$

But $r_x[k] = \dfrac{\sigma_u^2}{1-a^2} a^{|k|} = \dfrac{a^{|k|}}{1-a^2}$

$$\underline{C}_y = \frac{1}{1-a^2} \begin{bmatrix} 2(1-a) & 2a-1-a^2 \\ 2a-1-a^2 & 2(1-a) \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{1+a} & \dfrac{-(a^2-2a+1)}{1-a^2} \\ & \dfrac{2}{1+a} \end{bmatrix}$$

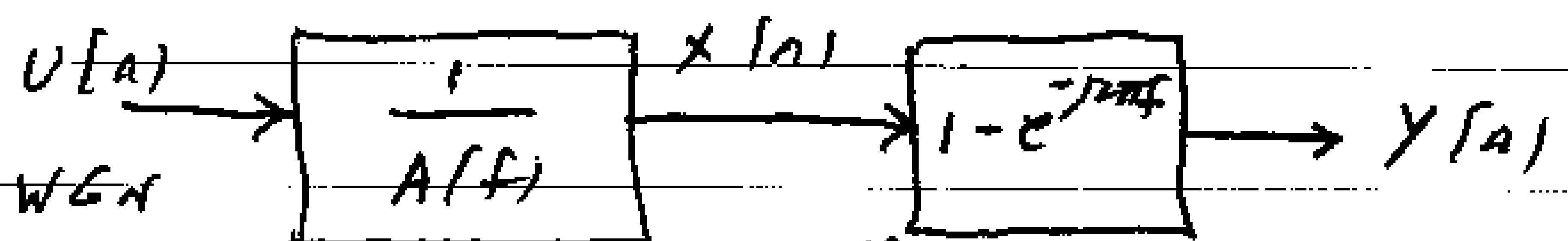
$$= \begin{bmatrix} \dfrac{2}{1+a} & \dfrac{-(a-1)^2}{1-a^2} \\ & \dfrac{2}{1+a} \end{bmatrix} = \begin{bmatrix} \dfrac{2}{1+a} & -\dfrac{1-a}{1+a} \\ & \dfrac{2}{1+a} \end{bmatrix}$$

as $a \to 1$, $\underline{C}_y \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} y[0] \\ y[1] \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$\Rightarrow$ $y[0] \sim N(0,1)$

$y[1] \sim N(0,1)$

and independent



$A(f) = 1 - a e^{-j2\pi f}$ difference

As $a \to 1$, $y[n] \to u[n] = WGN$

7) $y[n] = \sum_{i=0}^{3} h[i]x[n-i]$

$= h[0]x[n] + h[1]x[n-1] + h[2]x[n-2] + h[3]x[n-3]$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \end{bmatrix} = \underbrace{\begin{bmatrix} h[3] & h[2] & h[1] & h[0] & 0 & 0 \\ 0 & h[3] & h[2] & h[1] & h[0] & 0 \\ 0 & 0 & h[3] & h[2] & h[1] & h[0] \end{bmatrix}}_{3\times 6 \;=\; \underline{G}} \begin{bmatrix} x[-3] \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \end{bmatrix}$$

8) $P_x(f) = \sigma_x^2$

$P_y(f) = |H(f)|^2 \sigma_x^2$

$= |e^{j2\pi f} - e^{-j2\pi f}|^2 \sigma_x^2$

$= (e^{j2\pi f} - e^{-j2\pi f})(e^{-j2\pi f} - e^{j2\pi f})\sigma_x^2$

$= (2 - e^{j4\pi f} - e^{-j4\pi f})\sigma_x^2$

$\Rightarrow r_y[k] = 2\sigma_x^2 \delta[k] - \sigma_x^2 \delta[k-2] - \sigma_x^2 \delta[k+2]$

and since $x[n]$ is WGN $\Rightarrow \mu_x = 0$ and hence $\mu_y = 0$. Also,

$$\underline{C}_y = \begin{bmatrix} r_y[0] & r_y[1] & \cdots & r_y[N-1] \\ r_y[1] & r_y[0] & \cdots & r_y[N-2] \\ & & \ddots & \\ & & & r_y[0] \end{bmatrix}$$

Hence, $\underline{y} \sim N(\underline{0}, \underline{C}_y)$

where $\underline{C}_y = \sigma_x^2 \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 2 & 0 & -1 & \cdots & 0 \\ 0 & 0 & \cdots & & 0 & -1 \;\; 0 & 2 \end{bmatrix}$

Yes samples that are not separated by 2 time units are independent, for example, $y(0)$ and $y(1)$ are independent.

9) $\phi_{\underline{z}}(\underline{\omega}) = E_{\underline{z}}(e^{j\underline{\omega}^T\underline{z}})$ (definition)

$= E_{\underline{x},\underline{y}}(e^{j\underline{\omega}^T(\underline{x}+\underline{y})})$

$= E_{\underline{x},\underline{y}}(e^{j\underline{\omega}^T\underline{x}}\, e^{j\underline{\omega}^T\underline{y}})$

$= E_{\underline{x}}(e^{j\underline{\omega}^T\underline{x}})\, E_{\underline{y}}(e^{j\underline{\omega}^T\underline{y}})$ (independence)

$= \phi_{\underline{x}}(\underline{\omega})\, \phi_{\underline{y}}(\underline{\omega})$

$= \left(e^{j\underline{\omega}^T\underline{\mu}_x - \frac{1}{2}\underline{\omega}^T\underline{C}_x\underline{\omega}}\right) \cdot \left(e^{j\underline{\omega}^T\underline{\mu}_y - \frac{1}{2}\underline{\omega}^T\underline{C}_y\underline{\omega}}\right)$

$= e^{j\underline{\omega}^T\underbrace{(\underline{\mu}_x+\underline{\mu}_y)}_{\underline{\mu}_z} - \frac{1}{2}\underline{\omega}^T\underbrace{(\underline{C}_x+\underline{C}_y)}_{\underline{C}_z}\underline{\omega}}$

$\Rightarrow \underline{z} \sim N(\underline{\mu}_x + \underline{\mu}_y,\ \underline{C}_x + \underline{C}_y)$

$\Rightarrow$ $z[n]$ is Gaussian random process by definition

10) $$E[z[n]] = E[x[n]y[n]] = E[x[n]]E[y[n]] = \mu_x \mu_y$$

$$E[z[n]z[n+k]] = E(x[n]y[n]x[n+k]y[n+k])$$
$$= E[x[n]x[n+k]y[n]y[n+k]]$$
$$= E[x[n]x[n+k]]\,E[y[n]y[n+k]]$$

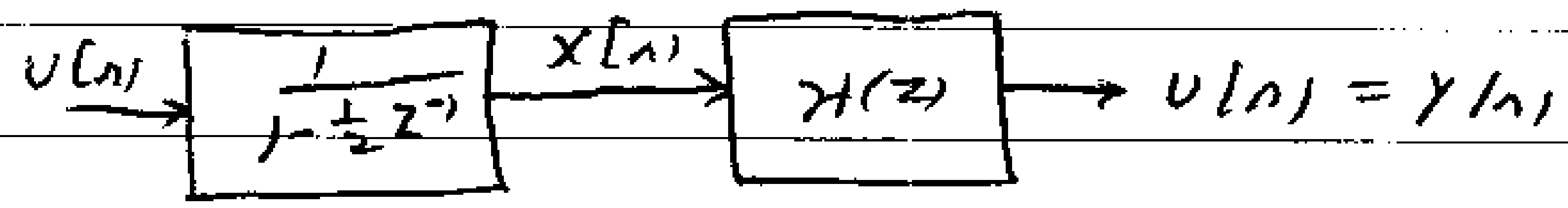
due to independence

$$= r_x[k]\,r_y[k] \qquad x[n], y[n] \text{ are WSS}$$

$\Rightarrow$ $z[n]$ is WSS with $\mu_z = \mu_x \mu_y$

$$P_z(f) = P_x(f) \star P_y(f)$$

11)



$y[n]$ is WGN with $\sigma_y^2 = \sigma_u^2 = 1$

$z = y^2[0] + y^2[1] \sim \chi_2^2$ See Sect. 10.5.6

$$p_Z(z) = \tfrac{1}{2} e^{-\frac{1}{2}z} u(z)$$

$$P[y^2[0] + y^2[1] > 1] = \int_1^\infty \tfrac{1}{2} e^{-\frac{1}{2}z}\,dz$$

$$= -e^{-\frac{1}{2}z}\Big|_1^\infty = e^{-\frac{1}{2}}$$

12) $$\frac{\partial \phi_x(\mathbf{w})}{\partial w_k} = e^{-\frac{1}{2}\mathbf{w}^T\mathbf{C}\mathbf{w}} \frac{\partial\, \mathbf{w}^T\mathbf{C}\mathbf{w}}{\partial w_k}\left(-\tfrac{1}{2}\right)$$

$$= -\tfrac{1}{2}\phi_x(\mathbf{w}) \frac{\partial}{\partial w_k} \sum_i \sum_j w_i w_j E[x_i x_j]$$

$$\frac{\partial}{\partial \omega_k} \sum_i \sum_j \omega_i \omega_j E\{x_i x_j\} = \sum_i \sum_j (\omega_i \delta_{jk} + \delta_{ik} \omega_j) E\{x_i x_j\}$$

$$= \sum_i \omega_i E\{x_i x_k\} + \sum_j \omega_j E\{x_k x_j\}$$

$$= \sum_j \omega_j \cdot E\{x_j x_k\} + \sum_j \omega_j \cdot E\{x_k x_j\}$$

$$= 2 L_k$$

$$\Rightarrow \frac{\partial \phi_{\underline{x}}}{\partial \omega_k} = -\phi_{\underline{x}}(\underline{\omega}) L_k$$

$$\frac{\partial L_i}{\partial \omega_k} = \frac{\partial}{\partial \omega_k} \sum_j \omega_j E\{x_i x_j\}$$

$$= \sum_j \delta_{jk} E\{x_i x_j\} = E\{x_i x_k\}$$

Now $\dfrac{\partial \phi_{\underline{x}}(\underline{\omega})}{\partial \omega_1} = -\phi_{\underline{x}}(\underline{\omega}) L_1$

$$\frac{\partial^2 \phi_x(\underline{\omega})}{\partial \omega_1 \partial \omega_2} = -\phi_x(\underline{\omega}) \frac{\partial L_1}{\partial \omega_2} - \frac{\partial \phi_x}{\partial \omega_2} L_1$$

$$= -\phi_x(\underline{\omega}) E\{x_1 x_2\} - [-\phi_x(\underline{\omega}) L_2] L_1$$

$$= -\phi_x(\underline{\omega}) E\{x_1 x_2\} + \phi_x(\underline{\omega}) L_1 L_2$$

$$\frac{\partial^3 \phi_{\underline{x}}(\underline{\omega})}{\partial \omega_1 \partial \omega_2 \partial \omega_3} = +\phi_{\underline{x}}(\underline{\omega}) L_3 E\{x_1 x_2\} + (-\phi_{\underline{x}}(\underline{\omega}) L_3) L_1 L_2$$

$$+ \phi_x(\underline{\omega}) \frac{\partial L_1}{\partial \omega_3} L_2 + \phi_x(\underline{\omega}) L_1 \frac{\partial L_2}{\partial \omega_3}$$

$$= \phi_{\underline{x}}(\underline{\omega}) \{ L_3 E\{x_1 x_2\} - L_1 L_2 L_3 + E\{x_1 x_3\} L_2 + L_1 E\{x_2 x_3\} \}$$

Since $h_i|_{\underline{\omega}=\underline{0}} = 0$, we have

$$\frac{\partial^4 \phi_{\underline{x}}(\underline{\omega})}{\partial\omega_1\partial\omega_2\partial\omega_3\partial\omega_4}\bigg|_{\underline{\omega}=\underline{0}} = \phi_{\underline{x}}(\underline{0})\Big[E(X_3X_4)E(X_1X_2) - 0$$

$$+ E(X_1X_3)E(X_2X_4) + E(X_1X_4) \cdot E(X_2X_3)\Big]$$

$$= E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

(3) $$E(\hat{r}_x(0)) = \frac{1}{N}\sum_{n=0}^{N-1}\underbrace{E(x^2(n))}_{\sigma_x^2} = \sigma_x^2$$

$$E(\hat{r}_x^2(0)) = E\left(\frac{1}{N^2}\sum_m\sum_n x^2(m)x^2(n)\right)$$

$$= \frac{1}{N^2}\sum_m\sum_n \underbrace{E(x^2(m)x^2(n))}_{r_x^2(0) + r_x^2(n-m) + r_x^2(m-n)}$$

$$= \frac{1}{N^2}\sum_m\sum_n\left[(\sigma_x^2)^2 + 2(\sigma_x^2\delta(n-m))^2\right]$$

$$= (\sigma_x^2)^2 + 2(\sigma_x^2)^2\frac{1}{N^2}\underbrace{\sum_m\sum_n \delta(n-m)}_{N}$$

$$= (\sigma_x^2)^2 + \frac{2(\sigma_x^2)^2}{N}$$

$$\mathrm{var}(\hat{r}_x(0)) = E(\hat{r}_x^2(0)) - E^2(\hat{r}_x(0))$$

$$= \frac{2(\sigma_x^2)^2}{N}$$

14) $x[n] = u[n] + u[n-1]$ with $\sigma_u^2 = 1$ has

ACS $r_x[k] = \begin{cases} 2 & k=0 \\ 1 & k=\pm 1 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow E[x[0]x[1]x[2]x[3]] = r_x^2[1] + r_x^2[2] + r_x[3]r_x[1]$$

from (20.13)

$$= 1 + 0 + 0 = 1$$

15) From Example 20.5

$$P_Y(f) = r_x^2[0]\,\delta(f) + 2\,P_X(f) \star P_X(f)$$

But $P_X(f) = 2 \Rightarrow r_x[0] = 2$

$$P_Y(f) = 4\,\delta(f) + 2\,\mathcal{F}\{r_x^2[k]\}$$

Since $r_x[k] = 2\,\delta[k]$, $r_x^2[k] = 4\,\delta[k]$

$$P_Y(f) = 4\,\delta(f) + 8$$

16) $\bar{X}(t)$ is a Gaussian random process since any set of samples are a linear transformation of samples of $Z(t)$. As $\Delta t \to 0$ $W \to \infty$ and $Z(t)$ becomes WGN with PSD $P_Z(F) = N_0/2$ or $Z(t) \to U(t)$. Also,

$$\sum_{n=0}^{[t/\Delta t]} Z(n\Delta t)\Delta t \to \int_0^t Z(\xi)\,d\xi$$

$$= \int_0^t U(\xi)\,d\xi = X(t)$$

17) $$\hat{A} = \frac{1}{T}\int_0^T [A + v(\zeta)]\, d\zeta$$

$$= A + \frac{1}{T}\underbrace{\int_0^T v(\zeta)\, d\zeta}_{\substack{\text{sample of Wiener} \\ \text{random process at} \\ t=T \Rightarrow \text{Gaussian} \\ \text{random variable}}}$$

$\hat{A}$ is a Gaussian random variable with $E(\hat{A}) = A$ and

$$\mathrm{var}(\hat{A}) = \mathrm{var}\left[\frac{1}{T}\int_0^T v(\zeta)\, d\zeta\right]$$

$$= E\left[\left(\frac{1}{T}\int_0^T v(\zeta)\, d\zeta\right)^2\right]$$

since $E\left[\frac{1}{T}\int_0^T v(\zeta)\, d\zeta\right] = 0$

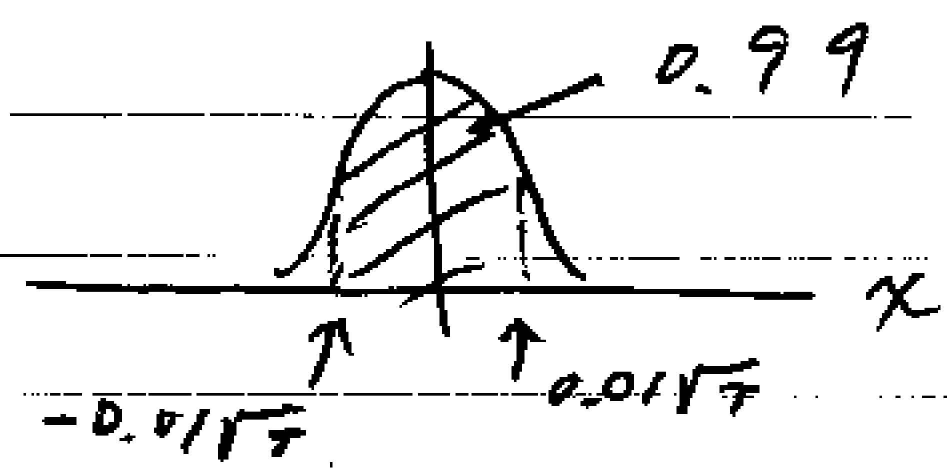
$$\mathrm{var}(\hat{A}) = \frac{1}{T^2}\int_0^T\!\!\int_0^T \underbrace{E[v(\zeta_1)v(\zeta_2)]}_{\delta(\zeta_2-\zeta_1)}\, d\zeta_1\, d\zeta_2$$

$$= \frac{1}{T^2}\int_0^T 1 \cdot d\zeta_2 = 1/T$$

$$\Rightarrow \hat{A} \sim N(A, 1/T)$$

$$P(|\hat{A} - A| \le 0.01) = 0.99$$

$$P\left[\left|\underbrace{\frac{\hat{A}-A}{\sqrt{1/T}}}_{N(0,1)}\right| \le 0.01\sqrt{T}\right] = 0.99$$



$$1 - 2Q(0.01\sqrt{T}) = 0.99$$
$$Q(0.01\sqrt{T}) = 0.005$$

Using Qinv.m $\Rightarrow$ $Q^{-1}(0.005) = 2.5758$

$$\Rightarrow \quad 0.01\sqrt{T} = 2.5758$$
$$T = 66,347$$

18) The X_i's are independent since each one is composed of nonoverlapping $V(t)$'s which themselves are independent ($V(t)$ is WGN). Also X_i's are Gaussian since each one is an increment of $X(t)$, which is Gaussian, for example, $X(t_2) - X(t_1) = \int_{t_1}^{t_2} V(\xi)\,d\xi = X_2$

$$E(X_i) = E\left(\int_{t_{i-1}}^{t_i} V(\xi)\,d\xi\right) = 0$$

$$\text{var}(X_i) = E(X_i^2) = \int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{t_i} \underbrace{E(V(\xi_1)V(\xi_2))}_{\frac{N_0}{2}\delta(\xi_2-\xi_1)}\,d\xi_1\,d\xi_2$$

$$= \frac{N_0}{2}\int_{t_{i-1}}^{t_i} d\xi_2$$

$$= \frac{N_0}{2}(t_i - t_{i-1}) = \frac{N_0}{2}\Delta t$$

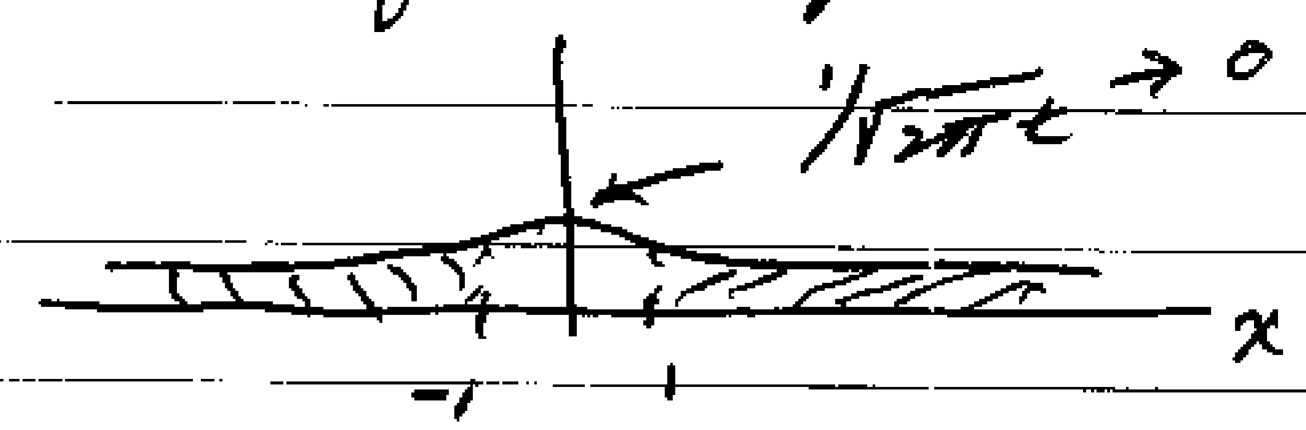
$\Rightarrow$ X_i's are IID with $X_i \sim N(0, \frac{N_0}{2}\Delta t)$

$$\Rightarrow X(t_n) = \sum_{i=1}^{n} X_i$$

19) $X(t) \sim N(0, t)$

$$P(|X(t)| > 1) = 2P(X(t) > 1)$$
$$= 2Q(1/\sqrt{t})$$

as $t \to \infty$, $P(|X(t)| > 1) \to 1$
since $\mathrm{var}(X(t)) \to \infty \Rightarrow$ probability density function spreads out



20) $2X(t)\cos 2\pi F_0 t =$

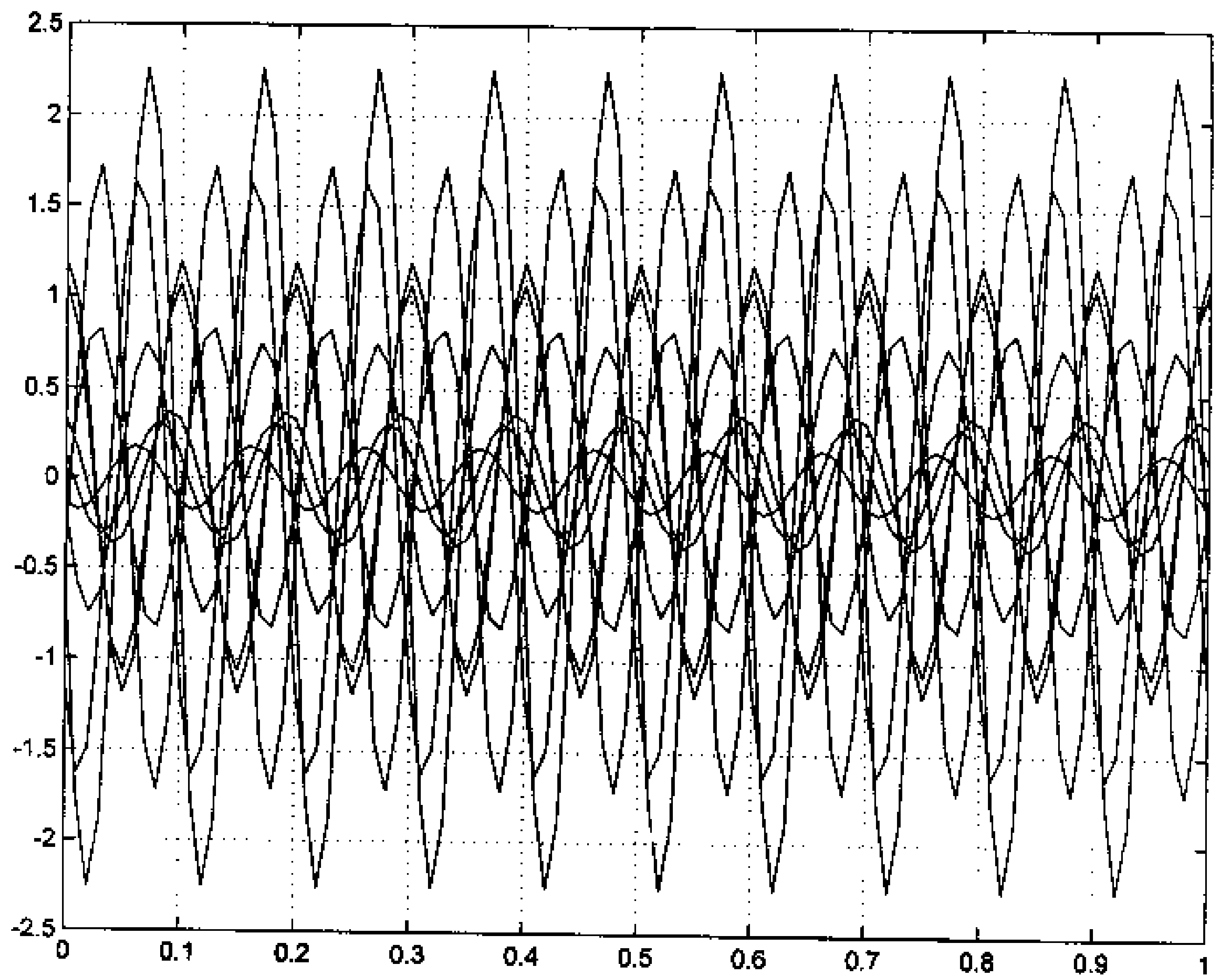
$$2(U\cos 2\pi F_0 t - V\sin 2\pi F_0 t)\cos 2\pi F_0 t$$
$$= 2U\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi F_0 t\right)$$
$$-2V\left(\tfrac{1}{2}\sin 4\pi F_0 t\right)$$
$$= U + \underbrace{U\cos 4\pi F_0 t - V\sin 4\pi F_0 t}_{\text{filtered out}}$$

$$-2X(t)\sin 2\pi F_0 t = -2(U\cos 2\pi F_0 t - V\sin 2\pi F_0 t) \cdot \sin 2\pi F_0 t$$
$$= -2U\left(\tfrac{1}{2}\sin 4\pi F_0 t\right) + 2V\left(\tfrac{1}{2} - \tfrac{1}{2}\cos 4\pi F_0 t\right)$$

$$= V - \underbrace{U\sin 4\pi F_0 t - V\cos 4\pi F_0 t}_{\text{filtered out}}$$

Need $W/2 < 2F_0$ to filter out sinusoidal signals.

21) Use $U \sim N(0, \sigma^2) = N(0,1)$
$V \sim N(0, \sigma^2) = N(0,1)$ } independent




```
% probprob20_21.m
%
clear all
randn('state',0)
del=0.01;
t=[0:del:1]';
plot(t,randn(1,1)*cos(2*pi*10*t)-randn(1,1)*sin(2*pi*10*t));
hold on
for m=2:10
   plot(t,randn(1,1)*cos(2*pi*10*t)-randn(1,1)*sin(2*pi*10*t));
end
grid
```

22) $y(t) = (U_1+U_2)\cos 2\pi F_0 t - (V_1+V_2)\sin 2\pi F_0 t$

$r_x(0) = \sigma^2 = 1 \Rightarrow U_i \sim N(0,1)$
$V_i \sim N(0,1)$

and all U_i's, V_i's are independent

$$\underbrace{\begin{bmatrix} y(0) \\ y(\frac{1}{4}) \end{bmatrix}}_{\underline{y}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1/\sqrt{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{\underline{G}} \underbrace{\begin{bmatrix} U_1 + U_2 \\ V_1 + V_2 \end{bmatrix}}_{\underline{W} \sim N(\underline{0},\, 2\underline{I})}$$

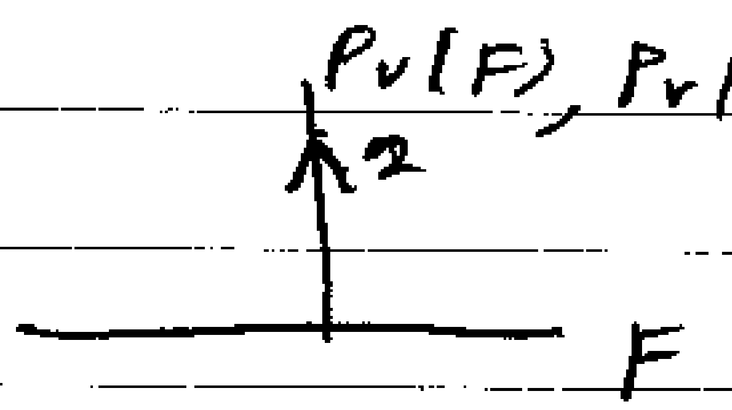
$$\underline{y} \sim N(\underline{0}, \underline{C}_y)$$

$$\underline{C}_y = \underline{G}\,\underline{C}_W\,\underline{G}^T = 2\,\underline{G}\,\underline{G}^T$$

$$= 2 \begin{bmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix}$$

23) $$P_x(F) = \underbrace{\tfrac{1}{2} P_U(F+F_0)}_{\delta(F+10)} + \underbrace{\tfrac{1}{2} P_V(F-F_0)}_{\delta(F-10)}$$

$$\Rightarrow \quad P_U(F) = P_V(F) = 2\,\delta(F)$$



24) Same as Problem 20

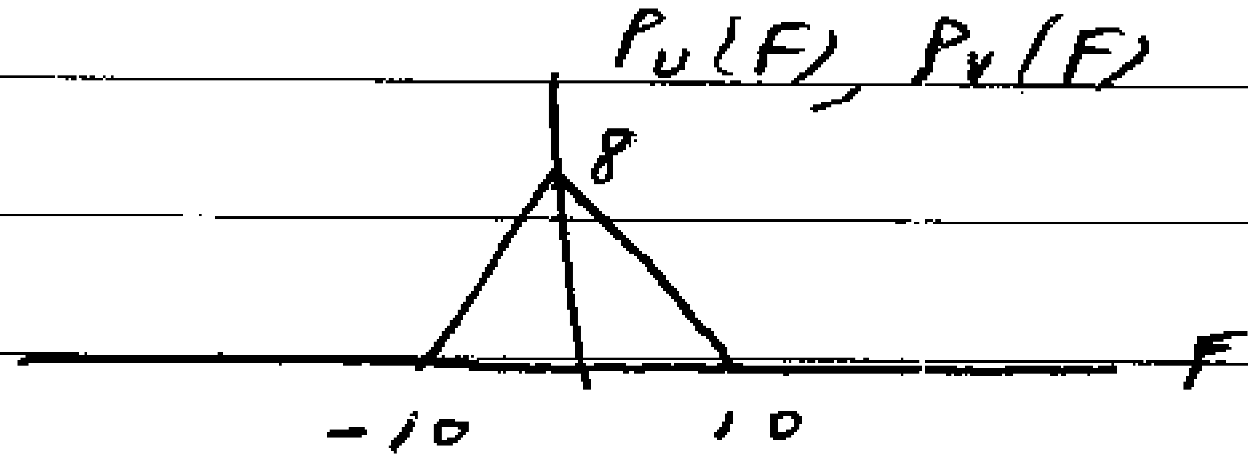
$$2x(t)\cos 2\pi F_0 t = U(t) + \left[U(t)\cos 4\pi F_0 t - V(t)\sin 4\pi F_0 t\right]$$

But $U(t)\cos 4\pi F_0 t - V(t)\sin 4\pi F_0 t$ is a bandpass random process centered at $2F_0$ and whose lower bandedge is at $2F_0 - W/2$.

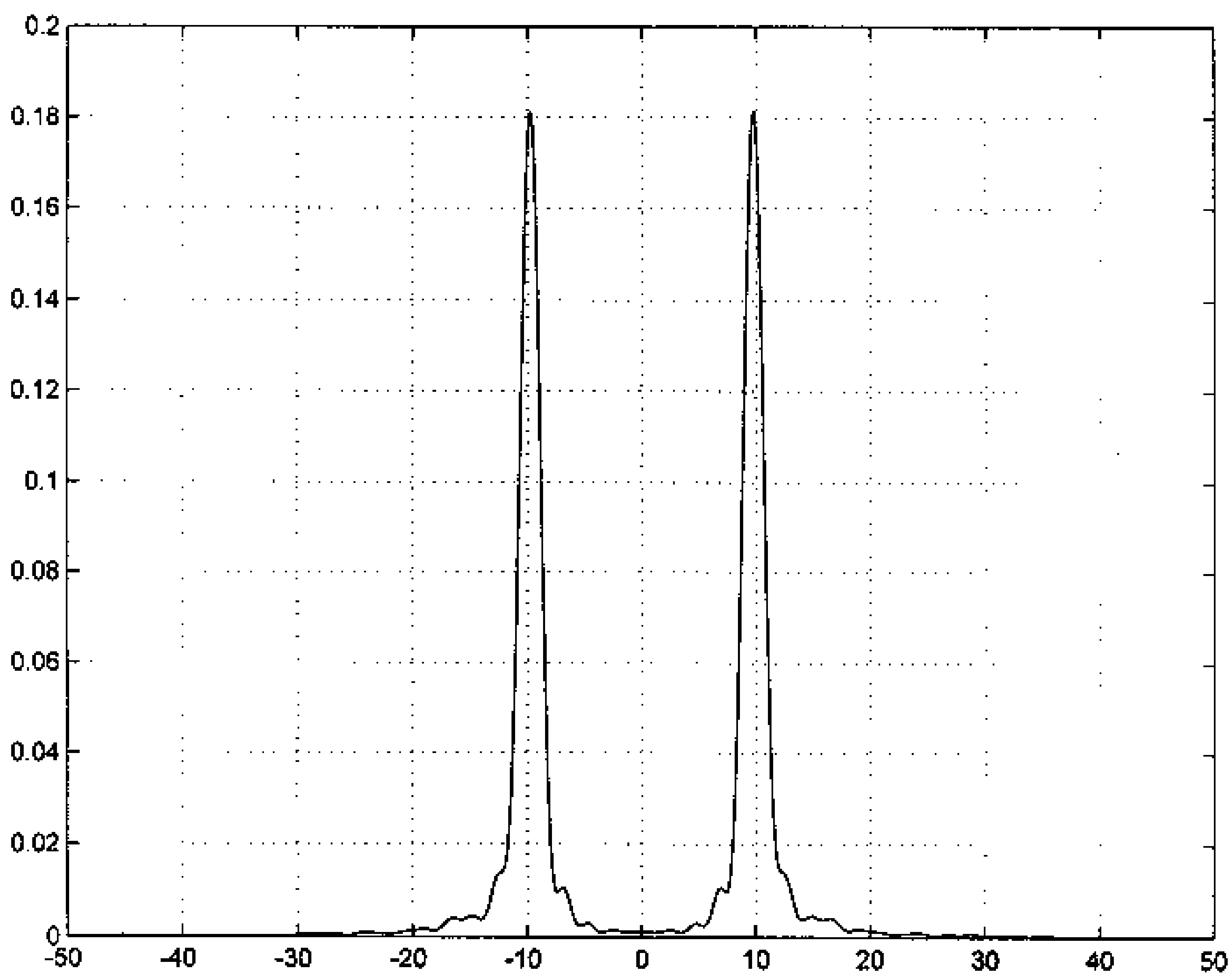
as long as $2F_0 - W/2 > W/2$ or $F_0 > W/2$
it will be filtered out

Same for $-2X(t)\sin 2\pi F_0 t$

25) $P_X(F) = \frac{1}{2} P_V(F+F_0) + \frac{1}{2} P_V(F-F_0)$



26)



Yes, the process is Gaussian with mean zero and PSD shown above. Hence the multivariate Gaussian PDF describes the distribution of any set of samples. Also, we can represent $X(t)$ using the bandpass description.

```
% probprob20_26.m
%
clear all
rand('state',0)
t=[0:0.01:0.99]';
F0=10;
s=cos(2*pi*F0*t);
ss=[s;zeros(1000-length(s),1)];
tt=[0:0.01:9.99]';
x=zeros(1000,1);
for i=1:100
   tau=round(10*i+10*(rand(1,1)-0.5));
   x=x+rand(1,1)*shift(ss,tau);
end
Fs=100;
L=50;I=20;
n=[0:I*L-1]';
Nfft=1024;
Pav=zeros(Nfft,1);
f=[0:Nfft-1]'/Nfft-0.5;
F=f*Fs;
for i=0:I-1
   nstart=1+i*L;nend=L+i*L;
   y=x(nstart:nend);
Pav=Pav+(1/(I*L))*abs(fftshift(fft(y,Nfft))).^2;
end
Pest=Pav/Fs;
plot(F,Pest)
grid
axis([-50 50 0 0.2])
```

27) From Figure 20.12 $N_0/2 = 4$, $W/2 = 10$

$$P[A(t_0) > \gamma] = e^{-\frac{1}{2}\gamma^2/\sigma^2}$$

But $r_U(0) = r_V(0) = N_0 W$ from (20.23)

$$\Rightarrow E[U^2(t)] = E[V^2(t)] = \sigma^2 = N_0 W$$

or $\sigma^2 = (8)(20) = 160$

$\gamma = 10$

$$P(A(t_0) \le 10) = 1 - e^{-\frac{1}{2}\gamma^2/\sigma^2}$$
$$= 1 - e^{-\frac{1}{2}(100)/160}$$
$$= 1 - e^{-5/16} = 0.2683$$

Result doesn't depend on t_0

28) $$H_1(f) = \tfrac{1}{2} e^{j2\pi f} + \tfrac{1}{2} e^{-j2\pi f}$$
$$= \cos 2\pi f$$
$$H_2(f) = \tfrac{1}{2} + \tfrac{1}{2} e^{-j4\pi f}$$
$$= \left(\tfrac{1}{2} e^{+j2\pi f} + \tfrac{1}{2} e^{-j2\pi f}\right) e^{-j2\pi f}$$
$$= \cos 2\pi f \; e^{-j2\pi f}$$
$$P_1(f) = |H_1(f)|^2 \sigma_v^2 = \cos^2 2\pi f$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi f$$
$$P_2(f) = |H_2(f)|^2 \sigma_v^2 = \cos^2 2\pi f$$
$$= \tfrac{1}{2} + \tfrac{1}{2}\cos 4\pi f$$

29) $$P_x(f) = |H(f)|^2 \sigma_v^2 = |1 - \tfrac{1}{2} e^{-j2\pi f}|^2$$

Let $H(f) = 1 - \frac{1}{2} e^{-j2\pi f}$ $\quad \sigma_v^2 = 1$

$$\Rightarrow \quad h(n) = \begin{cases} 1 & n = 0 \\ -\tfrac{1}{2} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(n) = v(n) - \tfrac{1}{2} v(n-1)$$

where $v(n)$ is WGN with $\sigma_v^2 = 1$

30) $P_x(f) = 2 - 2\cos 2\pi f$

$= 2 - 2\left[\frac{1}{2}\left(e^{j2\pi f} + e^{-j2\pi f}\right)\right]$

$= 2 - e^{j2\pi f} - e^{-j2\pi f}$

Let $z = e^{j2\pi f}$

$P_x(z) = 2 - z - z^{-1}$

$= z^{-1}(2z - z^2 - 1)$

$= -z^{-1}(z^2 - 2z + 1)$

$= -z^{-1}(z-1)(z-1)$

$= (-1 + z^{-1})(z-1)$

$= (1 - z^{-1})(1 - z)$

or $P_x(f) = \underbrace{(1 - e^{-j2\pi f})}_{H(f)}(1 - e^{j2\pi f})\underbrace{\sigma_v^2}_{=1}$

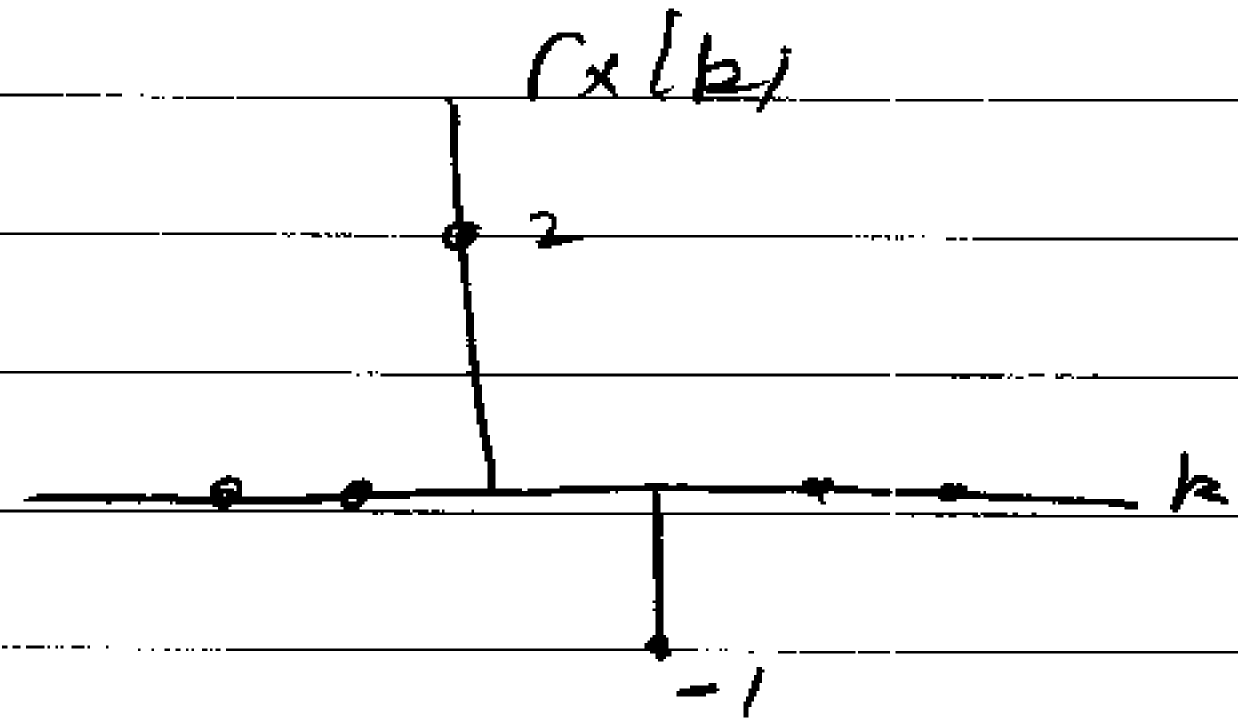
$h[n] = 1 \quad n = 0$
$\quad\;\; -1 \quad n = 1$
$\quad\;\; 0 \quad$ otherwise

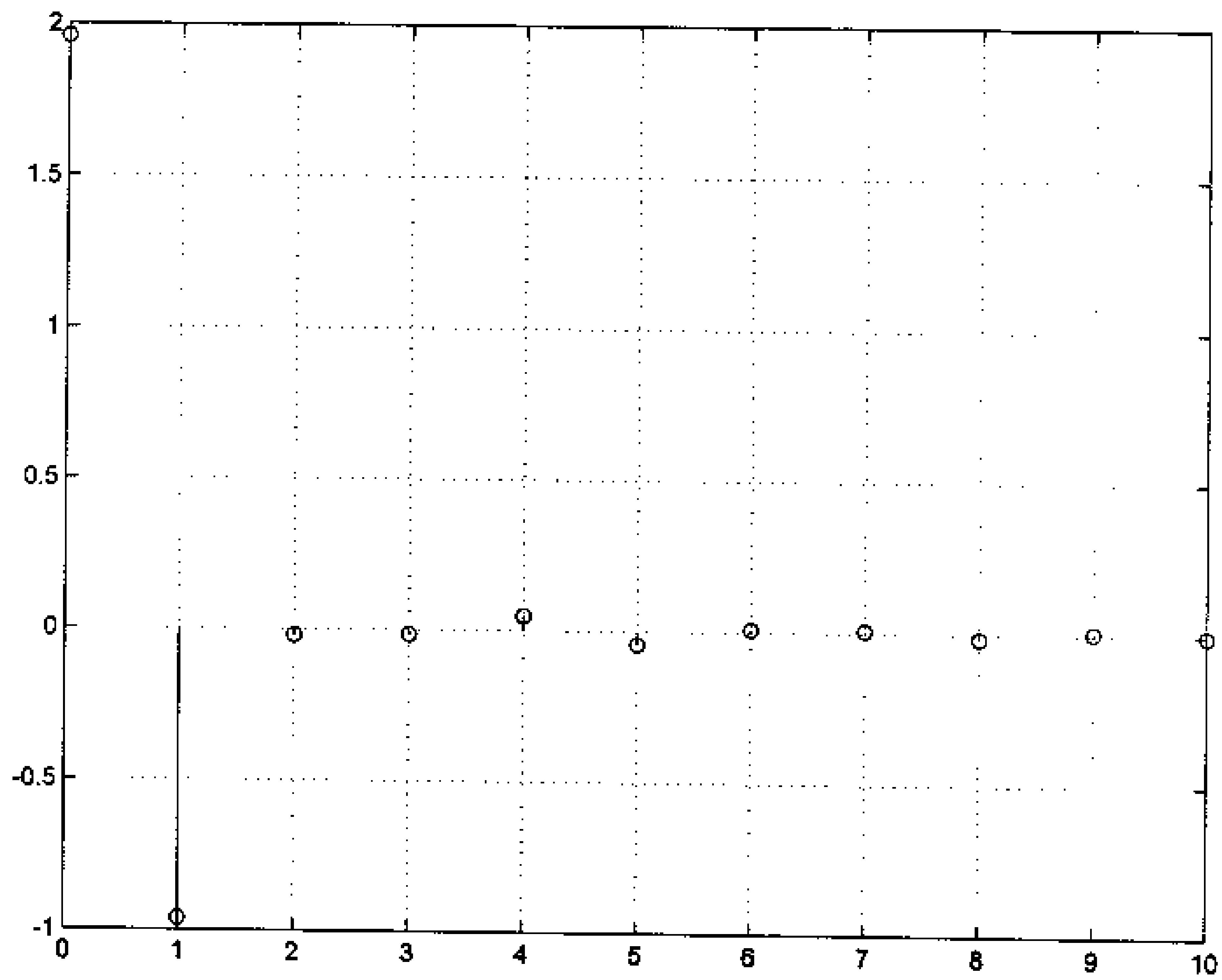
Use $x[n] = v[n] - v[n-1]$ $\quad v[n]$ is WGN with $\sigma_v^2 = 1$

31) $r_x[k] = E[(v[n] - v[n-1])(v[n+k] - v[n+k-1])]$

$= r_v[k] - r_v[k-1] - r_v[k+1] + r_v[k]$

$= 2\delta[k] - \delta[k-1] - \delta[k+1]$






```
rx =

k=0    1.9591
      -0.9614
      -0.0195
      -0.0154
       0.0483
      -0.0423
       0.0104
       0.0095
      -0.0142
       0.0062
k=10  -0.0023
```

```
% probprob20_31.m
%
clear all
randn('state',0)
u=randn(10001,1);
x=u(2:10001)-u(1:10000);
for k=0:10
   rx(k+1,1)=(sum(x(1:10000-k).*x(k+1:10000)))/(10000-k);
end
stem([0:10]',rx)
grid
```

Chapter 21

1) $P[N(t)=k] = P[N(t-\Delta t)=k-1,\ N(t)-N(t-\Delta t)=1]$
$+P[N(t-\Delta t)=k,\ N(t)-N(t-\Delta t)=0]$
$= P[N(t-\Delta t)=k-1]\,P[N(t)-N(t-\Delta t)=1]$
$+P[N(t-\Delta t)=k]\,P[N(t)-N(t-\Delta t)=0]$
independence of increments
$= P[N(t-\Delta t)=k-1]\,P[N(t+\Delta t)-N(t)=1]$
$+P[N(t-\Delta t)=k]\,P[N(t+\Delta t)-N(t)=0]$
stationarity of increments

$$P_k(t) = P_{k-1}(t-\Delta t)\lambda\Delta t + P_k(t-\Delta t)(1-\lambda\Delta t)$$

$$\frac{P_k(t)-P_k(t-\Delta t)}{\Delta t} = P_{k-1}(t-\Delta t)\lambda - P_k(t-\Delta t)\lambda$$

as $\Delta t \to 0 \Rightarrow$

$$\frac{dP_k(t)}{dt} = -P_k(t)\lambda + P_{k-1}(t)\lambda$$

or
$$\frac{dP_k(t)}{dt} + \lambda P_k(t) = \lambda P_{k-1}(t)$$

2) $P_k(0^+) = P[N(0^+)=k] = 0$ since
$N(0)=0$
Taking Laplace transforms

$$s\mathcal{P}_k(s) - \cancelto{0}{P_k(0^+)} + \lambda\mathcal{P}_k(s) = \lambda\mathcal{P}_{k-1}(s)$$

$$\Rightarrow \mathcal{P}_k(s) = \frac{\lambda}{\lambda+s}\mathcal{P}_{k-1}(s)$$

or $P_1(s) = \frac{\lambda}{\lambda+s} P_0(s)$

$P_2(s) = \frac{\lambda}{\lambda+s} P_1(s) = \left(\frac{\lambda}{\lambda+s}\right)^2 P_0(s)$

or $P_k(s) = \left(\frac{\lambda}{\lambda+s}\right)^k P_0(s)$

But $P_0(s) = \mathcal{L}\{p_0(t)\}$

$= \mathcal{L}\{e^{-\lambda t} u(t)\}$

$= \frac{1}{s+\lambda}$

$P_k(s) = \frac{\lambda^k}{(s+\lambda)^{k+1}}$

From Tables $\mathcal{L}^{-1}\left\{\frac{1}{(s+\lambda)^n}\right\} = \frac{t^{n-1} e^{-\lambda t}}{(n-1)!}$

$\Rightarrow \quad p_k(t) = \lambda^k \left(\frac{t^k e^{-\lambda t}}{k!}\right)$

$= e^{-\lambda t} \frac{(\lambda t)^k}{k!} \qquad k = 0, 1, \ldots$

3) Same as in $[0, 5]$ (stationarity)

$p\{N(5) = 6\} = e^{-1(5)} \frac{(1(5))^6}{6!}$

$= 0.1462$

$E\{N(t)\} = \lambda t = 5$

4) Because of independence and stationarity of increments

$$P = p^5(\text{2 arrivals})$$

$$= p^5[N(1)=2] = \left[e^{-2(1)}\frac{[(2)(1)]^2}{2!}\right]^5$$

$$= (2e^{-2})^5 = 32e^{-10} = 0.0015$$

5) $P(N(t+\Delta t)-N(t)=1) = \lambda\Delta t$ Axiom 3

$\to 0$ as $\Delta t \to 0$

6) $P[N(60)>12] = 1-P(N(60)\le 12)$

$$= 1-\sum_{k=0}^{12} e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

$$= 1-\sum_{k=0}^{12} e^{-\frac{1}{5}(60)}\frac{(\frac{1}{5}(60))^k}{k!}$$

$$= 1-e^{-12}\sum_{k=0}^{12}\frac{(12)^k}{k!}$$

$$= 0.4240$$

7) $\lambda = \dfrac{E[N(t)]}{t}$

$\lambda = 2 \Rightarrow \hat{\lambda} = 1.9629$

$\lambda = 5 \Rightarrow \hat{\lambda} = 4.9072$

```
%probprob21_7.m
%
clear all
rand('state',0)
lambda=2; % set arrival rate
   for i=1:10000
         z(i,1)=(1/lambda)*log(1/(1-rand(1,1))); %generate interarrival time
   if i==1 % generate arrival time
      t(i,1)=z(i);
      else
         t(i,1)=t(i-1,1)+z(i,1);
      end
         end
   lambdaest=10000/t(10000)
```

8) $$P[T > t] = P[\min(T_1^{(1)}, T_1^{(2)}) > t]$$
$$= P[T_1^{(1)} > t, T_1^{(2)} > t]$$
$$= P[T_1^{(1)} > t]\, P[T_1^{(2)} > t] \quad \text{(ind.)}$$
$$= P^2[T_1 > t]$$
$$= \left(\int_t^\infty \lambda e^{-\lambda u}\, du\right)^2$$
$$= \left[-e^{-\lambda u}\Big|_t^\infty\right]^2$$
$$= e^{-2\lambda t} = 1 - F_T(t)$$
↑ CDF

$$p_T(t) = \frac{d}{dt}\left(1 - e^{-2\lambda t}\right)$$
$$= 2\lambda e^{-2\lambda t}$$

$$\Rightarrow E[T] = \frac{1}{2\lambda} = \frac{1}{2} E[T_1^{(1)}]$$

Waiting time for first arrival is halved.

9) Let $N_s(t) = N_1(t) + N_2(t)$

$N_s(0) = N_1(0) + N_2(0) = 0$ axiom 1

$N_s(t_4) - N_s(t_3) = (N_1(t_4) - N_1(t_3))$
$+ (N_2(t_4) - N_2(t_3))$
$= X_1 + X_2$

$N_s(t_2) - N_s(t_1) = (N_1(t_2) - N_1(t_1))$
$+ (N_2(t_2) - N_2(t_1))$
$= X_3 + X_4$

X_1 X_2 X_3 X_4 are independent since $\{X_1, X_3\}$ are ind. of $\{X_2 X_4\}$ due to ind. of Poisson processes. Also X_1, X_3 are ind due to ind. increments as are X_2 X_4.
$\Rightarrow$ $X_1 + X_2$ ind of $X_3 + X_4$. Similarly, increments of $N_s(t)$ are stationary.
$\Rightarrow$ axiom 2

$N_1(t) \sim Poiss(\lambda_1 t)$
$N_2(t) \sim Poiss(\lambda_2 t)$ } ind.

$\Rightarrow$ from characteristic function

$\phi_{N_s}(\omega) = \phi_{N_1}(\omega)\phi_{N_2}(\omega)$

$= e^{\lambda_1 t(e^{j\omega}-1)} e^{\lambda_2 t(e^{j\omega}-1)}$

$= e^{(\lambda_1+\lambda_2)t(e^{j\omega}-1)}$

$\Rightarrow \quad N_s(t) \sim \text{Poiss}((\lambda_1+\lambda_2)t)$

$P[N_s(t+\Delta t) - N_s(t) = 1] =$

$P[N_s(\Delta t) - N_s(0) = 1]$ (stat.)

$= P[N_s(\Delta t) = 1]$

$$= e^{-(\lambda_1+\lambda_2)\Delta t} \frac{(\lambda_1+\lambda_2)\Delta t}{1!}$$

as $\Delta t \to 0$

$\rightarrow (1-(\lambda_1+\lambda_2)\Delta t)((\lambda_1+\lambda_2)\Delta t)$

$\rightarrow (\lambda_1+\lambda_2)\Delta t$ Axiom 3

Can show $\dfrac{P[N_s(\Delta t) = k]}{\Delta t} \to 0 \qquad k \geq 2$

10) $E[N(t_2) - N(t_1)] = \lambda t_2 - \lambda t_1 = \lambda(t_2 - t_1)$

$\text{var}(N(t_2) - N(t_1)) = \text{var}(N(t_2 - t_1))$
since increments are stationary and $N(0) = 0$

$= \lambda(t_2 - t_1)$ for Poisson random variable

11) From Problem 21.9

$\lambda_s = 3\lambda$

$$P[N(t) = k] = e^{-3\lambda t} \frac{(3\lambda t)^k}{k!}$$

12) $P[X > k_1 + k_2 \mid X > k_1] =$

$$\frac{P[X > k_1, X > k_1 + k_2]}{P[X > k_1]}$$

$$= \frac{P(X > k_1 + k_2)}{P(X > k_1)}$$

But $P(X > k) = \sum_{i=k+1}^{\infty} (1-p)^{i-1} p$

$$= p \frac{(1-p)^k}{1-(1-p)} = (1-p)^k$$

$$\Rightarrow \quad = \frac{(1-p)^{k_1+k_2}}{(1-p)^{k_1}} = (1-p)^{k_2}$$

$$P[X > k_1 + k_2 \mid X > k_1] = (1-p)^{k_2}$$

$$P(X\Delta t > (k_1+k_2)\Delta t \mid X\Delta t > k_1 \Delta t) = (1-p)^{k_2}$$

$$P[\underbrace{X\Delta t}_{\text{first arrival in seconds} = Z_1} > z_1 + z_2 \mid X\Delta t > z_1) = (1-\lambda\Delta t)^{z_2/\Delta t}$$

$$P[Z_1 > z_1 + z_2 \mid Z_1 > z_1) \rightarrow (e^{-\lambda})^{z_2} = e^{-\lambda z_2}$$

$$= P[Z_1 > z_2)$$

13) Due to memoryless property

$$P[\text{wait} < 1 \text{ minute}) = \int_0^{60} \lambda e^{-\lambda z} dz$$

where $\lambda = 1/60$

$$= -e^{-\lambda z}\Big|_0^{60} = 1 - e^{-60\lambda}$$
$$= 1 - e^{-1} = 0.6321$$

14) Want $E[T_{10^6}]$

$$E[T_{10^6}] = k/\lambda = \frac{10^6}{1000} = 1000 \text{ sec.}$$

15) $y = x_1 + x_2$

$$p_Y(y) = \int_{-\infty}^{\infty} p_{X_1}(u)\, p_{X_2}(y-u)\, du$$
$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda u} u_s(u)\, \lambda e^{-\lambda(y-u)} u_s(y-u)\, du$$

where $u_s(\,)$ is unit step

$$= \int_0^y \lambda^2 e^{-\lambda y}\, du = \lambda^2 e^{-\lambda y} y \qquad y > 0$$

$$p_Y(y) = \begin{cases} \lambda^2 y e^{-\lambda y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

See (2.1.8)

16) $P[t - \Delta t \leq T_k \leq t] = P[(k-1)$ arrivals in $[0, t-\Delta t]$ and 1 arrival in $(t-\Delta t, t]]$

$= P[(k-1)$ arrivals in $[0, t-\Delta t)]$
$\cdot P[1$ arrival in $(t-\Delta t, t]]$ ind. increments

$$= e^{-\lambda(t-\Delta t)} \frac{[\lambda(t-\Delta t)]^{k-1}}{(k-1)!} \lambda \Delta t$$

$$\frac{P[\,t-\Delta t \le T_k \le t\,]}{\Delta t} = e^{-\lambda(t-\Delta t)} \frac{[\lambda(t-\Delta t)]^{k-1}}{(k-1)!}\lambda$$

$$\rightarrow \; e^{-\lambda t} \frac{(\lambda t)^{k-1}}{k!}\lambda = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t}$$

17) $E(T_{1000}) = \dfrac{1000}{\lambda} = \dfrac{1000}{100/60} = 600$ sec

$\Rightarrow$ 10 minutes

18) $E_{X,Y,Z}(g(X,Y,Z)) = \iiint g(x,y,z)\, p_{X,Y,Z}(x,y,z)\, dx\, dy\, dz$

$= \iiint g(x,y,z)\, p_{X,Y|Z}(x,y|z)\, p_Z(z)\, dx\, dy\, dz$

$= \int \underbrace{\left[\iint g(x,y,z)\, p_{X,Y|Z}(x,y|z)\, dx\, dy\right]}_{E_{X,Y|Z}[g(X,Y,Z)|z]} p_Z(z)\, dz$

$= E_Z\left[E_{X,Y|Z}[g(X,Y,Z)|z]\right]$

19) Probability is equivalent since $N(0) = 0$ and $N(t_3) = k_3 \Leftrightarrow N(t_3) - N(t_2) = k_3 - k_2$ assuming $N(t_2) = k_2$, which is given.

$P[N(t_3)-N(t_2)=k_3-k_2 \mid N(t_2)=k_2,\ N(t_1)-N(0)=k_1]$

But $N(t_3)-N(t_2)$ is ind of $N(t_1)-N(0)$ since these are increments for nonoverlapping intervals.

$$\Rightarrow = P[N(t_3)-N(t_2)=k_3-k_2 \mid N(t_2)=k_2]$$

$$= P[N(t_3)=k_3 \mid N(t_2)=k_2]$$

20) $$P[T_2 \leq 1] = \int_0^1 \frac{\lambda^2}{1!} t e^{-\lambda t} dt$$

$$= \lambda^2 \left[-\frac{e^{-\lambda t}}{\lambda^2} - \frac{t e^{-\lambda t}}{\lambda} \right] \Big|_0^1$$

$$= \lambda^2 \left[-\frac{e^{-\lambda}}{\lambda^2} - \frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2} \right]$$

$$= 1 - e^{-\lambda} - \lambda e^{-\lambda}$$

$$= 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} = 0.2642$$

$$\hat{P}[T_2 \leq 1] = 0.2622$$

```
%probprob21_20.m
%


clear all
rand('state',0)
lambda=1; % set arrival rate
T=1; % set time interval in seconds
P=0;nreal=10000;
for m=1:nreal
   clear z t
   for i=1:1000
         z(i,1)=(1/lambda)*log(1/(1-rand(1,1))); %generate interarrival time
   if i==1 % generate arrival time
      t(i,1)=z(i);
      else
         t(i,1)=t(i-1,1)+z(i,1);
      end
      if t(i)>T % test to see if desired time interval has elapsed
         break
      end
   end
   M=length(t)-1; % number of arrivals in interval [0,T]
   arrivals=t(1:M); % arrival times in interval [0,T]
   if length(arrivals)>=2
      P=P+1/nreal;
   end
end
P
```

21) Rate is $\lambda/2$ - See Example 21.6 and assume $p = \frac{1}{2}$ which models "dropped arrivals" $\Rightarrow \lambda' = \lambda p = \frac{1}{2}\lambda$. The individual random processes are still Poisson and are independent of each other - not easy to prove!

22) $\text{var}(X(t_0)) = E[X^2(t_0)] - \underbrace{E^2[X(t_0)]}_{(\lambda t_0 E[U_i])^2}$

$$E[X^2(t_0)] = \frac{1}{j^2} \left. \frac{d^2\phi(\omega)}{d\omega^2} \right|_{\omega=0}$$

$$= \frac{1}{j^2} \frac{d^2}{dw^2} \left[e^{\lambda t_0 (\phi_{v_1}(w) - 1)} \right] \Big|_{w=0}$$

$$= \frac{1}{j^2} \frac{d}{dw} \left(\lambda t_0 \phi'_{v_1}(w) e^{\lambda t_0 (\phi_{v_1}(w) - 1)} \right) \Big|_{w=0}$$

$$= \frac{1}{j^2} \Big[(\lambda t_0 \phi'_{v_1}(w))^2 e^{\lambda t_0 (\phi_{v_1}(w) - 1)} + \lambda t_0 \phi''_{v_1}(w) e^{\lambda t_0 (\phi_{v_1}(w) - 1)} \Big] \Big|_{w=0}$$

$$= \frac{1}{j^2} \left[(\lambda t_0 \phi'_{v_1}(0))^2 + \lambda t_0 \phi''_{v_1}(0) \right] \quad (\phi_{v_1}(0) = 1)$$

$$\mathrm{var}(x(t_0)) = -(\lambda t_0 \phi'_{v_1}(0))^2 - \lambda t_0 \phi''_{v_1}(0) - (\lambda t_0 E(v_1))^2$$

$$= -(\lambda t_0 j E[v_1])^2 - \lambda t_0 j^2 E[v_1^2] - (\lambda t_0 E[v_1])^2$$

$$= \lambda t_0 E[v_1^2]$$

23.) $E[x(t_0)] = \lambda t_0 E(v_1)$

$$= \lambda t_0 (1 \cdot p + (-1)(1-p))$$

$$= \lambda t_0 (2p - 1)$$

24) From simulation average points per game = 28.2404

```
%probprob21_24.m
%

clear all
rand('state',0)
lambda=1/60; % set arrival rate
T=2400; % set time interval in seconds
nreal=10000;
for m=1:nreal
   points(m,1)=0;
   clear z t
   for i=1:1000

         z(i,1)=(1/lambda)*log(1/(1-rand(1,1))); %generate interarrival time
   if i==1 % generate arrival time
      t(i,1)=z(i);
      else
         t(i,1)=t(i-1,1)+z(i,1);
      end
      if t(i)>T % test to see if desired time interval has elapsed
         break
      end

   if rand(1,1)<0.6
      shot=2;
   else
      shot=3;
   end

   if shot==2&rand(1,1)<0.5
      points(m,1)=points(m,1)+2;
   elseif shot==3&rand(1,1)<0.3
      points(m,1)=points(m,1)+3;
   end

      end
end
mean(points)
```

25) $E[X(t_0)] = E_{N(t_0)}\left[E_{V_1,\ldots,V_k|N(t_0)}\left[\sum_{i=1}^{k} V_i \,|\, N(t_0)=k \right]\right]$

$= E_{N(t_0)}\left[E_{V_1,\ldots,V_k}\left[\sum_{i=1}^{k} V_i \right]\right]$

since V_i's are ind. of $N(t_0)$.

$$= E_{N(t_0)}\left(\sum_{i=1}^{k} E[v_i]\right)$$

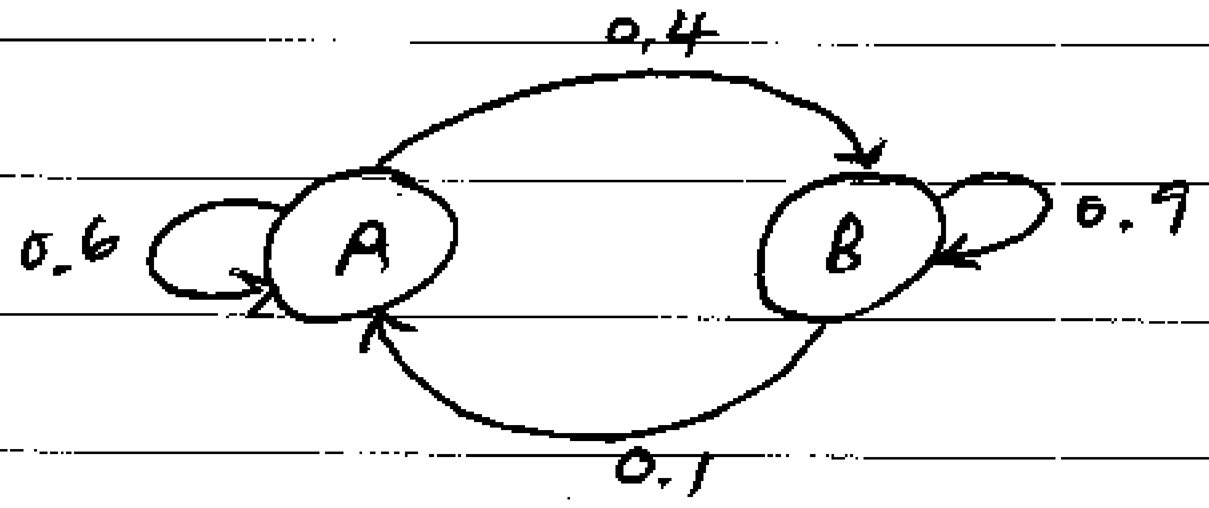
$$= E_{N(t_0)}\left(k E[v_1]\right)$$ v_i's have same mean

$$= E_{N(t_0)}[k]\, E[v_1]$$

$$= \lambda t_0 E[v_1]$$

Chapter 22

1)



$$\underline{P} = \begin{array}{c|cc} & A & B \\ A & 0.6 & 0.4 \\ B & 0.1 & 0.9 \end{array}$$

2) $P[x[0]=0, x[1]=1, x[2]=0, x[3]=1, x[4]=1]$

$= P[x[0]=0]\, P[x[1]=1 \mid x[0]=0]\, P[x[2]=0 \mid x[1]=1]$

$\cdot\, P[x[3]=1 \mid x[2]=0]\, P[x[4]=1 \mid x[3]=1]$

$= (\frac{1}{2})(\frac{1}{4})(\frac{1}{2})(\frac{1}{4})(\frac{1}{2}) = 1/128$

3) $P = P[x[0]=0]\, P_{01} P_{10} P_{01} P_{11} P_{11}$

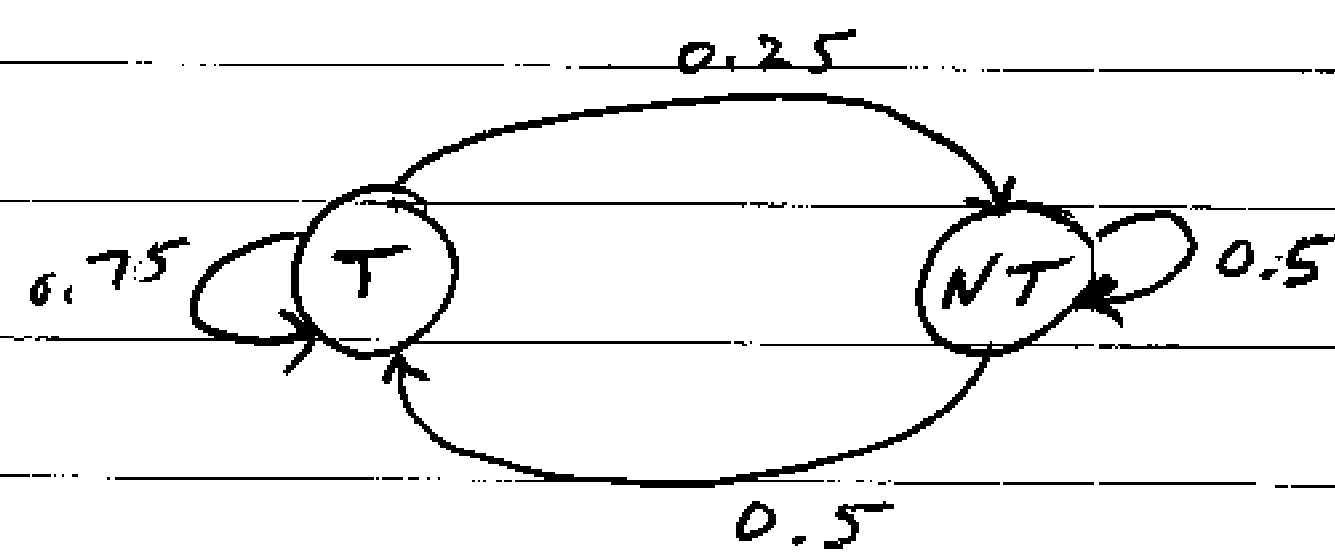
$= p_0[0]\, \alpha \beta \alpha (1-\beta)^2 = p_0[0]\, \alpha^2 \beta (1-\beta)^2$

$P = P[x[0]=1]\, P_{11} P_{10} P_{01} P_{11} P_{11}$

$= p_1[0]\, (1-\beta)\beta\alpha(1-\beta)^2 = p_0[0]\, \alpha\beta(1-\beta)^3$

initial state is different

4)



5) $P(Y[2]=1 \mid Y[1]=1, Y[0]=0) =$

$$\frac{P(Y[0]=0, Y[1]=1, Y[2]=1)}{P(Y[0]=0, Y[1]=1)}$$

$$= \frac{P(X[0]=0, X[1]=1, X[2]=0)}{P(X[0]=0, X[1]=1)}$$

$$= \frac{P(X[0]=0)P(X[1]=1)P(X[2]=0)}{P(X[0]=0)P(X[1]=1)}$$

$$= P(X[2]=0) = 1-p$$

$P(Y[2]=1 \mid Y[1]=1) =$

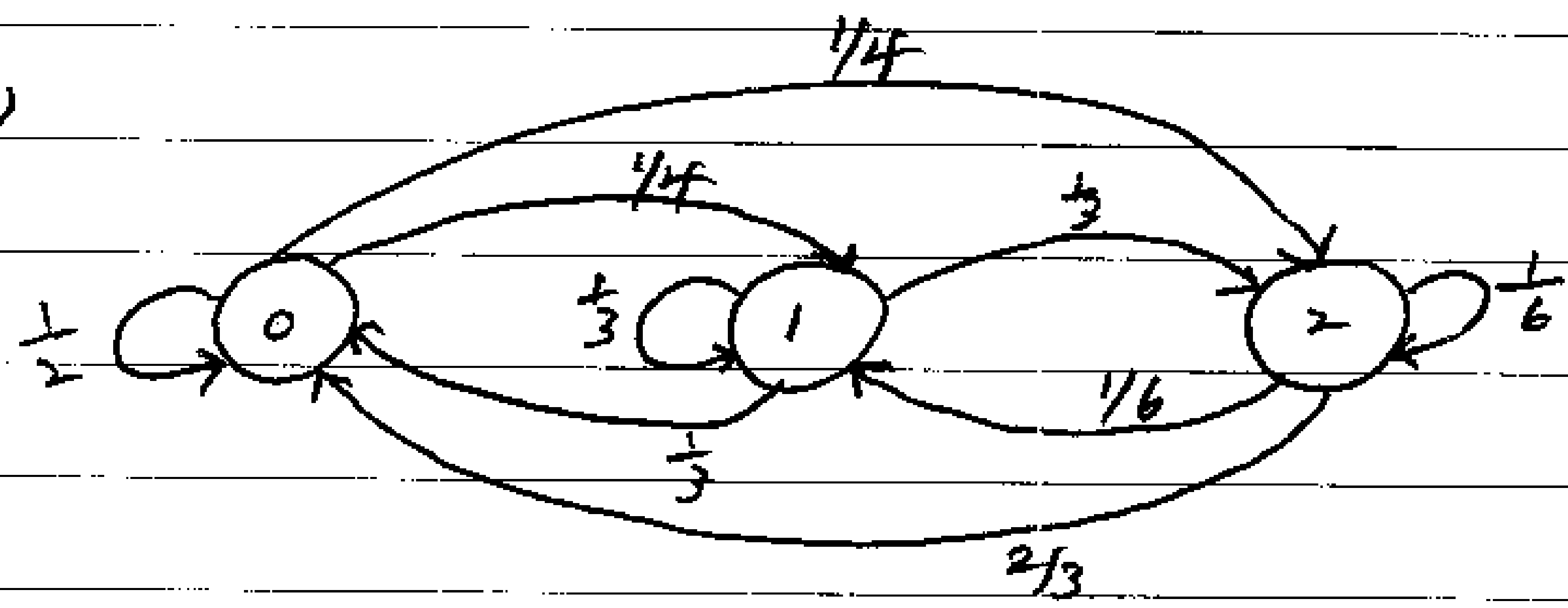
$$\frac{P(Y[1]=1, Y[2]=1)}{P(Y[1]=1)}$$

$$= \frac{P[X[0]=1, X[1]=0, X[2]=1) + P(X[0]=0, X[1]=1, X[2]=0)}{P[X[0]=0, X[1]=1) + P[X[0]=1, X[1]=0)}$$

$$= \frac{p^2(1-p) + p(1-p)^2}{p(1-p) + p(1-p)}$$

$$= \frac{p + (1-p)}{2} = \frac{1}{2} \neq 1-p \text{ in general}$$

6)



7) Outcomes are 1, 2, 3, 4, 5, 6
$\Rightarrow$ states are 1, 2, 3, 4, 5, 6

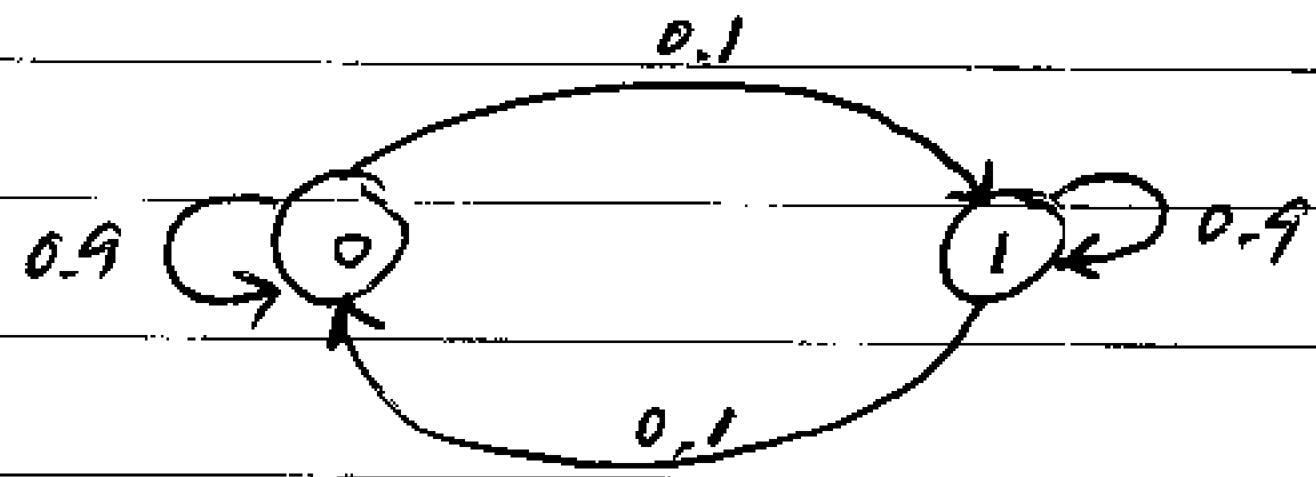
Note that $P(X[n] = j \mid X[n-1] = i) = 0$ for $j < i$ since maximum can't decrease with more tosses

$$\begin{array}{cc|cccccc|} & j = & 1 & 2 & 3 & 4 & 5 & 6 \\ & 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & 2 & 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ i = & 3 & 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ & 4 & 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ & 5 & 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ & 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

8)

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & p & 0 & 1-p \\ 1 & 1-p & 0 & p & 0 \\ 2 & 0 & 1-p & 0 & p \\ 3 & p & 0 & 1-p & 0 \end{array}$$

9) $P_e = P[\,1 \text{ decoded} \mid 0 \text{ sent}\,]$



$\underline{P} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}$ $\quad \underline{p}^T[5] = \underline{p}^T[0]\,\underline{P}^5$

$\underline{p}^T[0] = [\,p_0[0] \;\; p_1[0]\,] = [\,1 \;\; 0\,]$ since "0" is sent

$$\underline{p}^T[5] = [\,1 \;\; 0\,] \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}^5 = [\,\underset{\substack{\uparrow \\ p_0[5]}}{0.6638} \;\; \underset{\substack{\uparrow \\ p_1[5]}}{0.3362}\,]$$

$P_e = p_1[5] = 0.3362$

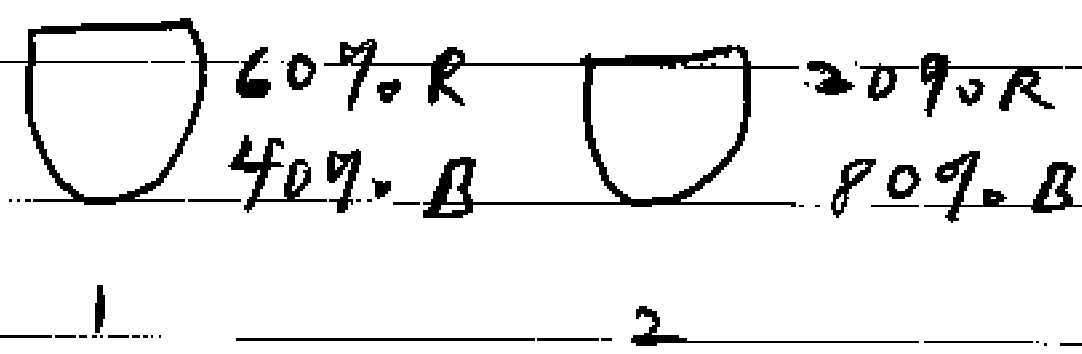
10) Need $(1-\alpha-\beta)^n \simeq 0$

$$\left(1-\tfrac{1}{2}\right)^n = 10^{-6}$$

$$n = \frac{\ln 10^{-6}}{\ln \frac{1}{2}} = 19.93$$

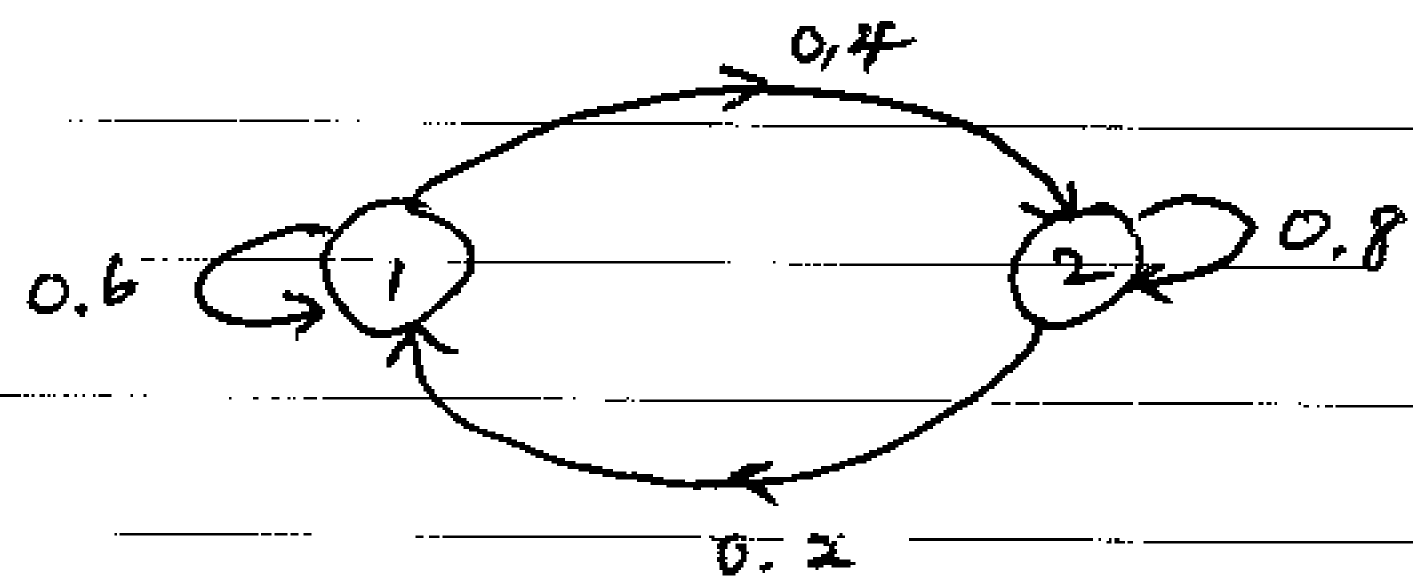
$\Rightarrow \quad n = 20$

11)



red chosen $\Rightarrow$ use urn 1

black chosen $\Rightarrow$ use urn 2



Here the states are the urns and the outcomes are $\{1, 2\}$. Thus,

$$p(\text{red}) = p(\text{red} \mid \text{urn } 1)\, p(\text{urn } 1) + p(\text{red} \mid \text{urn } 2)\, p(\text{urn } 2)$$

$$= 0.6\,\pi_1 + 0.2\,\pi_2$$

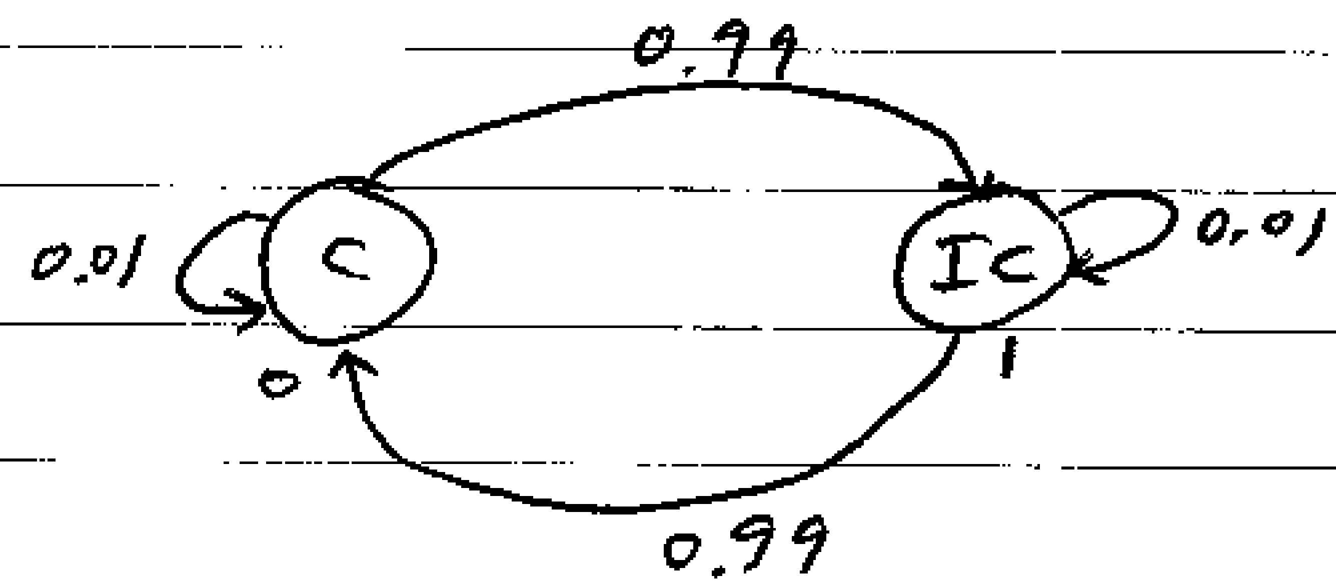
$$\underline{P} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\Rightarrow \quad \pi_1 = \frac{\beta}{\alpha+\beta} = \frac{0.2}{0.6} = \frac{1}{3}$$

$$\pi_2 = 2/3$$

$$P(\text{red}) = 0.6\left(\tfrac{1}{3}\right) + 0.2\left(\tfrac{2}{3}\right) = \frac{1}{3}$$

12)



$\alpha = 0.99 \quad \beta = 0.99$

$$\pi_C = \beta/\alpha+\beta = \frac{0.99}{0.99+0.99} = \frac{1}{2}$$

$$\pi_{IC} = \frac{1}{2}$$

13) $$P^2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} > 0$$

$\Rightarrow$ P^n converges.

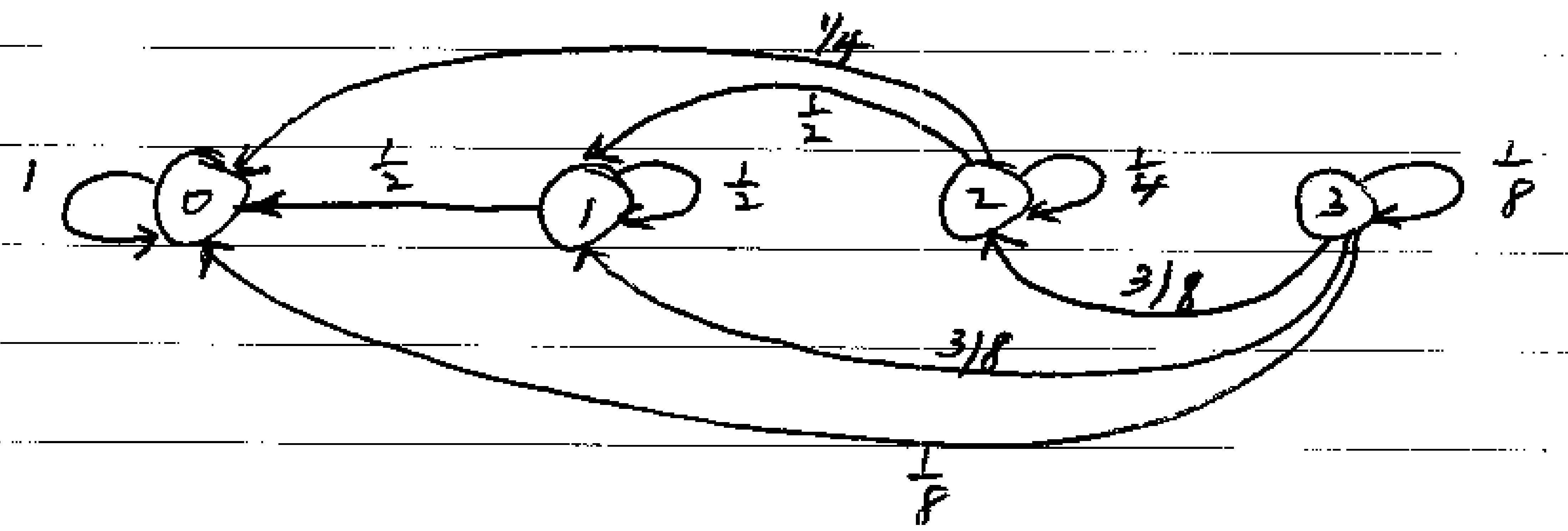
14) $$P^\infty = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & & \end{bmatrix}$$

$\Rightarrow \pi^T = [\frac{1}{3} \;\; \frac{2}{3} \;\; 0 \;\; 0]$

```
% probprob22_14.m
%
P=[0.5 0.5 0 0;0.25 0.75 0 0;0.25 0.25 0.25 0.25;0.25 0.25 0.25 0.25];
for i=1:30
   P^i
   pause
end
```

15) Let states be number of working bulbs.

$\Rightarrow$ 0, 1, 2, 3



$$\underline{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

Want π_0. $\underline{P}^{\infty} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \underline{\pi}^T = [1\ 0\ 0\ 0]$ and $\pi_0 = 1$

Probability of being in state 0 = 1 $\Rightarrow$ all light bulbs will eventually fail.

```
% probprob22_15.m
%
P=[1 0 0 0;0.5 0.5 0 0;0.25 0.5 0.25 0;1/8 3/8 3/8 1/8];
for i=1:30
   P^i
end
```

16) $\pi_0 = \dfrac{\beta}{\alpha+\beta} = \dfrac{\frac{1}{4}}{\frac{2}{3}+\frac{1}{4}} = \dfrac{3}{11}$

$\pi_1 = \dfrac{\alpha}{\alpha+\beta} = \dfrac{\frac{2}{3}}{\frac{2}{3}+\frac{1}{4}} = \dfrac{8}{11}$

17) $\underline{P}^n \rightarrow [\underline{v}_1 \ \underline{v}_2 \ \underline{v}_3] \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \\ \underline{w}_3^T \end{bmatrix}}_{\begin{bmatrix} \underline{w}_1^T \\ \underline{0}^T \\ \underline{0}^T \end{bmatrix}}$

$= \underline{v}_1 \underline{w}_1^T$

$\underline{P}^\infty \underline{1} = \underline{1}$ (row elements sum to 1)

$\underline{v}_1 \underbrace{\underline{w}_1^T \underline{1}}_{1/c} = \underline{1}$

$\underline{v}_1 = c\,\underline{1}$

$\underline{\pi}^T \underline{P}^\infty = \underline{\pi}^T$ repeated application of $\underline{\pi}^T \underline{P} = \underline{\pi}^T$ from

$\underbrace{\underline{\pi}^T \underline{v}_1}_{1/d} \underline{w}_1^T = \underline{\pi}^T$

$\Rightarrow \underline{w}_1 = d\underline{\pi}$

$\underline{w}_1^T \underline{v}_1 = (d\underline{\pi})^T c\,\underline{1} = 1 \Rightarrow cd = 1$

since $\underline{\pi}^T \underline{1} = 1$

18) $\underline{P}^{100} = \begin{bmatrix} 0.2165 & 0.4021 & 0.3814 \\ 0.2165 & 0.4021 & 0.3814 \\ 0.2165 & 0.4021 & 0.3814 \end{bmatrix}$

yes, for $\underline{P}^{\infty}$ all rows are the same.

19) $\underline{\pi} = (\underline{I} - \underline{P}^T + \underline{1}\underline{1}^T)^{-1}\underline{1}$

$= [0.2165 \quad 0.4021 \quad 0.3814]^T$

```
% probprob22_19.m
%
clear all
P=[0.1 0.4 0.5;0.2 0.5 0.3;0.3 0.3 0.4];
piprob=inv(eye(3)-P'+ones(3,1)*ones(1,3))*ones(3,1)
```

20) It means that the n-step transition probability matrix must be that given as $n \to \infty$ or since $\underline{P}^2 = \underline{P} \Rightarrow \underline{P}(\underline{P}^2) = \underline{P}^3 = \underline{P}$ etc.

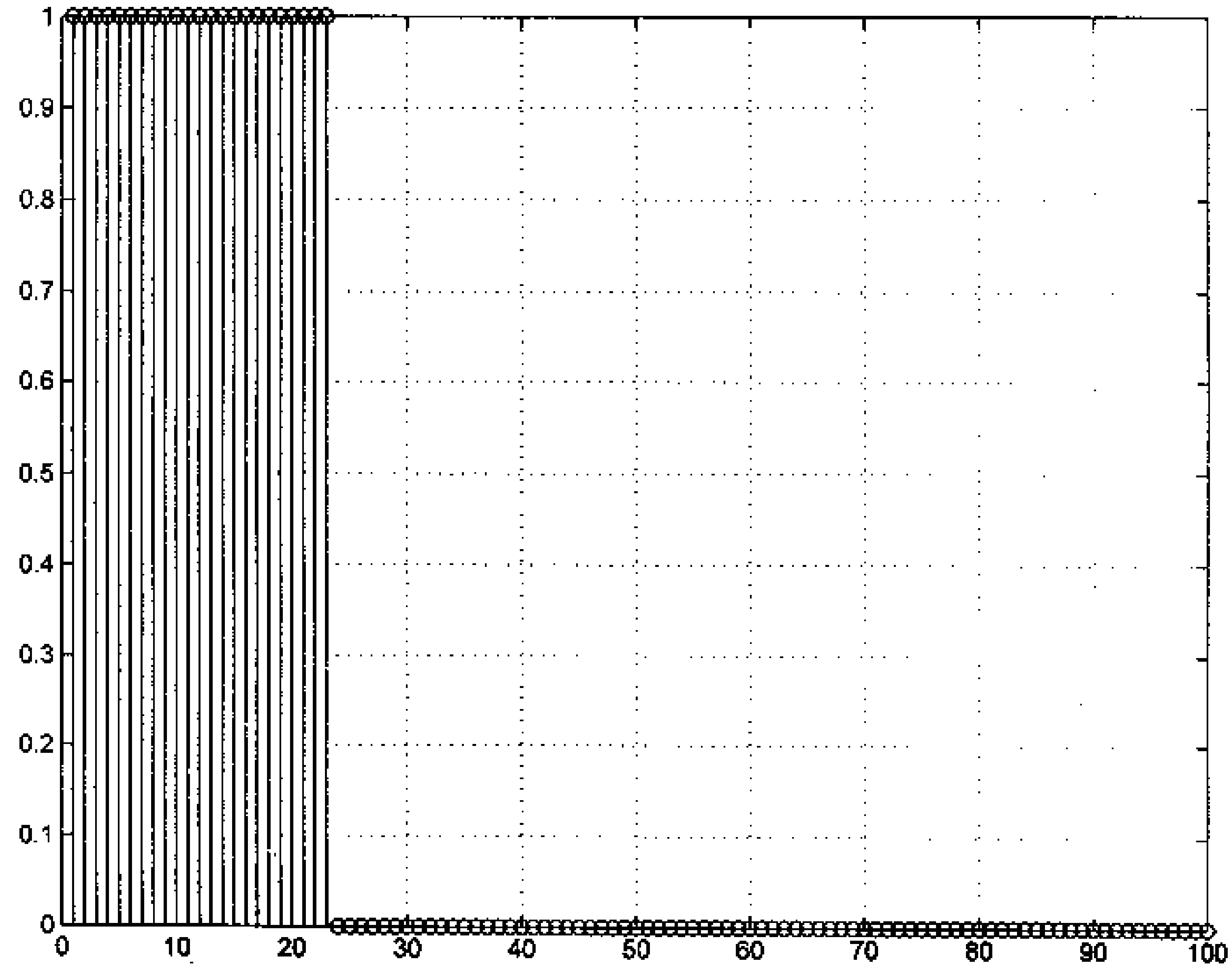
Also $\underline{\pi}^T = [0.2 \quad 0.1 \quad 0.7]$

21) $\pi_0 = \frac{\beta}{\alpha+\beta} = \frac{0.01}{0.01+0.01} = \frac{1}{2}$

$\pi_1 = \frac{1}{2}$

Same probabilities as fair loaded coin but realization will be very different. Expect to see long runs of 0's and 1's.

22)



$\hat{\pi}_1 = 0.5722$

```
% probprob22_22.m
%
clear all
rand('state',0)
ntrials=10000;
x=zeros(ntrials,1);
x(1)=floor(rand(1,1)+0.5);
for i=2:ntrials
   if x(i-1)==0
      if rand(1,1)<=0.01
         x(i)=1;
      end
   else
      x(i)=floor(rand(1,1)+0.99);
   end
end
stem([1:100]',x(1:100))
grid
mean(x)
```

23)
$$\begin{array}{ccc} & 0 & 1 \\ \text{late } 0 & & \\ \text{on-time } 1 & & \end{array} \begin{bmatrix} 0.2 & 0.8 \\ 0.4 & 0.6 \end{bmatrix}$$

$p_0[0] = 0.1$
↑ not needed here

$$\pi_0 = \frac{\beta}{\alpha+\beta} = \frac{0.4}{0.8+0.4} = \frac{1}{3}$$

24) $\underline{\pi}^T = [0.6964 \quad 0.1786 \quad 0.1250]$

$P[\text{rain}] = 0.6964 > \frac{1}{3}$
This is because $P_{00} = 6/8$ or if it is raining then the probability of rain the next day is higher than in Example 22.8. Same for other states as well.

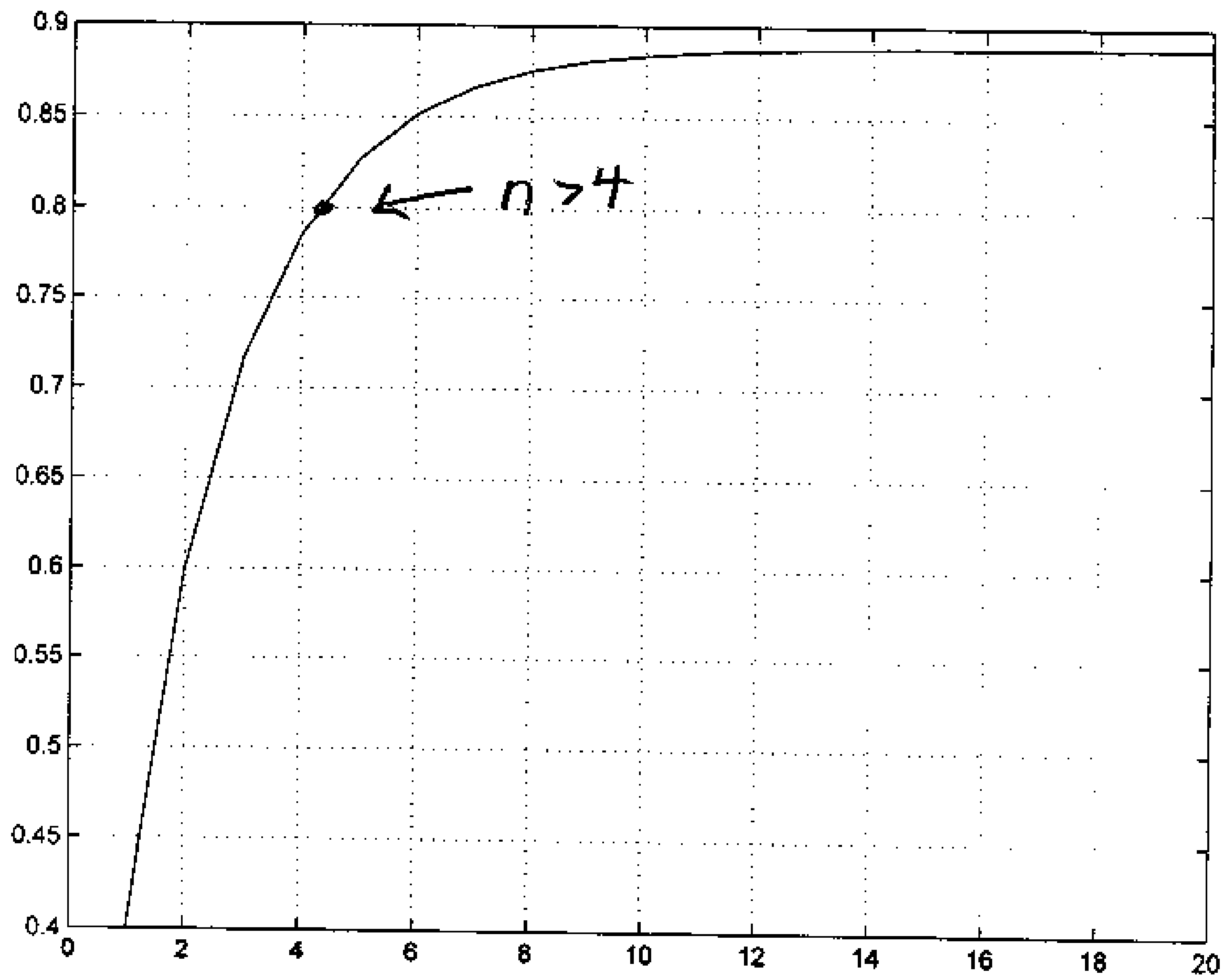
```
%probprob22_24.m
%
P=[6/8 1/8 1/8;5/8 2/8 1/8;4/8 3/8 1/8];
p=inv(eye(3)-P'+ones(3,1)*ones(1,3))*ones(3,1)
```

25) If for example there are currently 2 machines operating, then the next day there cannot be 3 machines operating $\Rightarrow P_{23} = 0$

$$\underline{p}^T[n] = \underline{p}^T[0]\,\underline{P}^n$$

when $\underline{p}^T[0] = [0\ 0\ 0\ 1]$ all machines initially working want n for which $[\underline{p}^T[n]]_0 > 0.8$. or

$p_0[n] > 0.8$.



$\Rightarrow$ $n = 5$

```
% probprob22_25.m
%
clear all
P=[1 0 0 0;0.5 0.5 0 0;0.1 0.3 0.6 0;0.4 0.3 0.1 0.1];
for n=1:20
   p=[0 0 0 1]*P^n;
   p_0(n,1)=p(1,1);
end
plot([1:20]',p_0)
grid
```

2 6) $P[k) = \binom{N}{k}\left(\frac{1}{2}\right)^N$ for N fish left in pond

$N = 4$ $P(k) = \frac{1}{16}, \frac{4}{16}, \frac{6}{16}, \frac{4}{16}, \frac{1}{16}$ $k = 0, 1, 2, 3, 4$

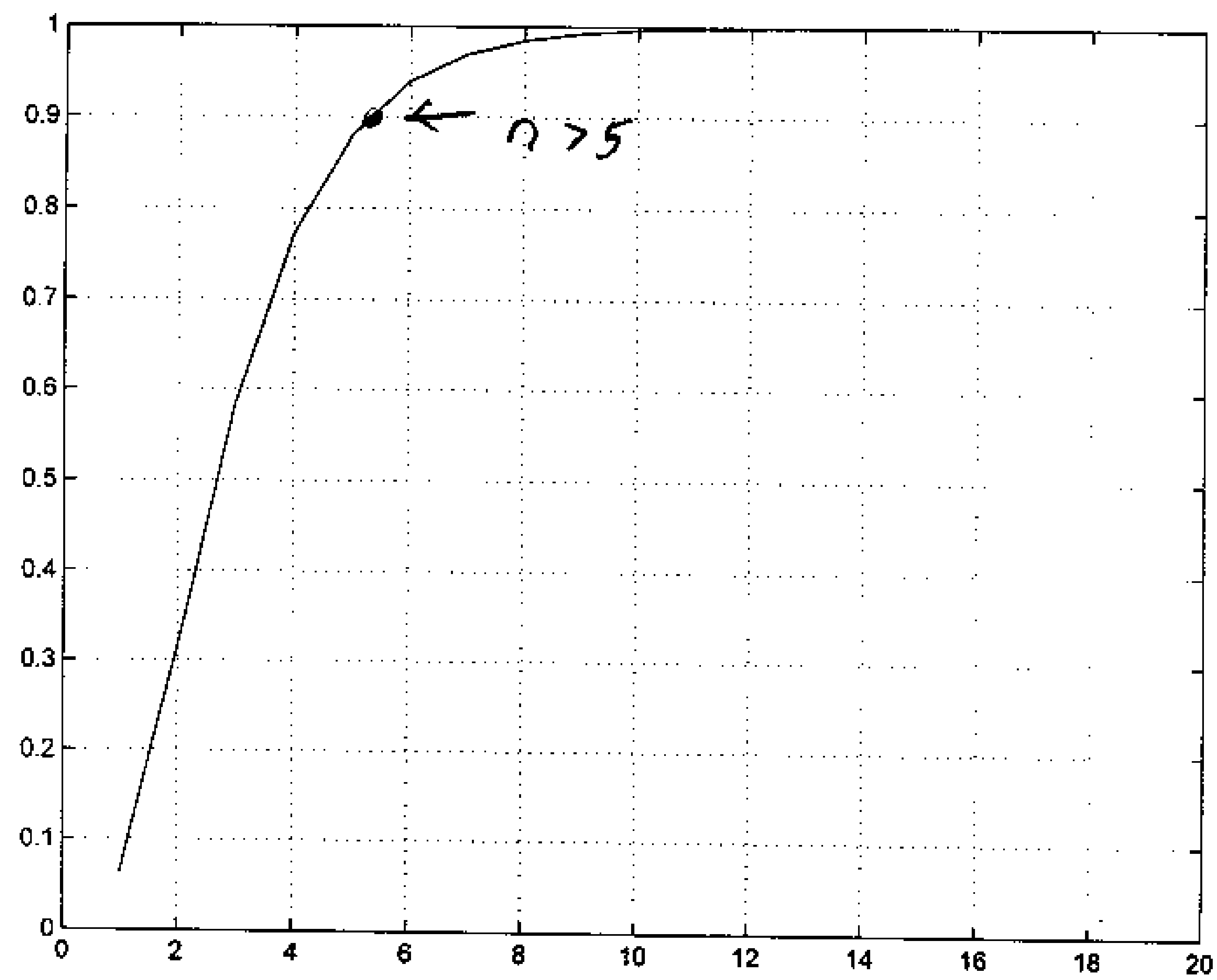
$N = 3$ $P(k) = \frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ $k = 0, 1, 2, 3$

$N = 2$ $P(k) = \frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ $k = 0, 1, 2$

$N = 1$ $P(k) = \frac{1}{2}, \frac{1}{2}$ $k = 0, 1$

$$\underline{P} = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & 3/8 & 3/8 & 1/8 & 0 \\ 1/16 & 4/16 & 6/16 & 4/16 & 1/16 \end{bmatrix}$$

want $q_0[n] > 0.9$



$\Rightarrow$ $n = 6$

```
% probprob22_26.m
%
clear all
P=[1 0 0 0 0;0.5 0.5 0 0 0;1/4 1/2 1/4 0 0;1/8 3/8 3/8 1/8 0;1/16 4/16 6/16 4/16 1/16];
for n=1:20
   p=[0 0 0 0 1]*P^n;
   p_0(n,1)=p(1,1);
end
plot([1:20]',p_0)
grid
```

27) $\underline{P}^T(\underline{P}^T\underline{1}) = \underline{P}^{T^2}\underline{1}$

but $\underline{P}^T\underline{1} = \underline{1} \Rightarrow \underline{P}^T(\underline{P}^T\underline{1}) = \underline{P}^T\underline{1}$

$\Rightarrow \underline{P}^{T^2} = \underline{P}^T$ etc

or $\underline{P}^{\infty^T}\underline{1} = \underline{1}$

but from problem 22.17 $\underline{P}^\infty = \underline{1}\underline{\pi}^T$

$$\Rightarrow \underline{P}^{\infty^T}\underline{1} = (\underline{1}\underline{\pi}^T)^T\underline{1} = \underline{1}$$

$$\underline{\pi}\underbrace{\underline{1}^T\underline{1}}_{K} = \underline{1}$$

$$\Rightarrow \underline{\pi} = \frac{1}{K}\underline{1}$$

28) $\frac{1}{N}\sum_{n=0}^{N-1} x(n) = \hat{\pi}_1 = 0.3240$

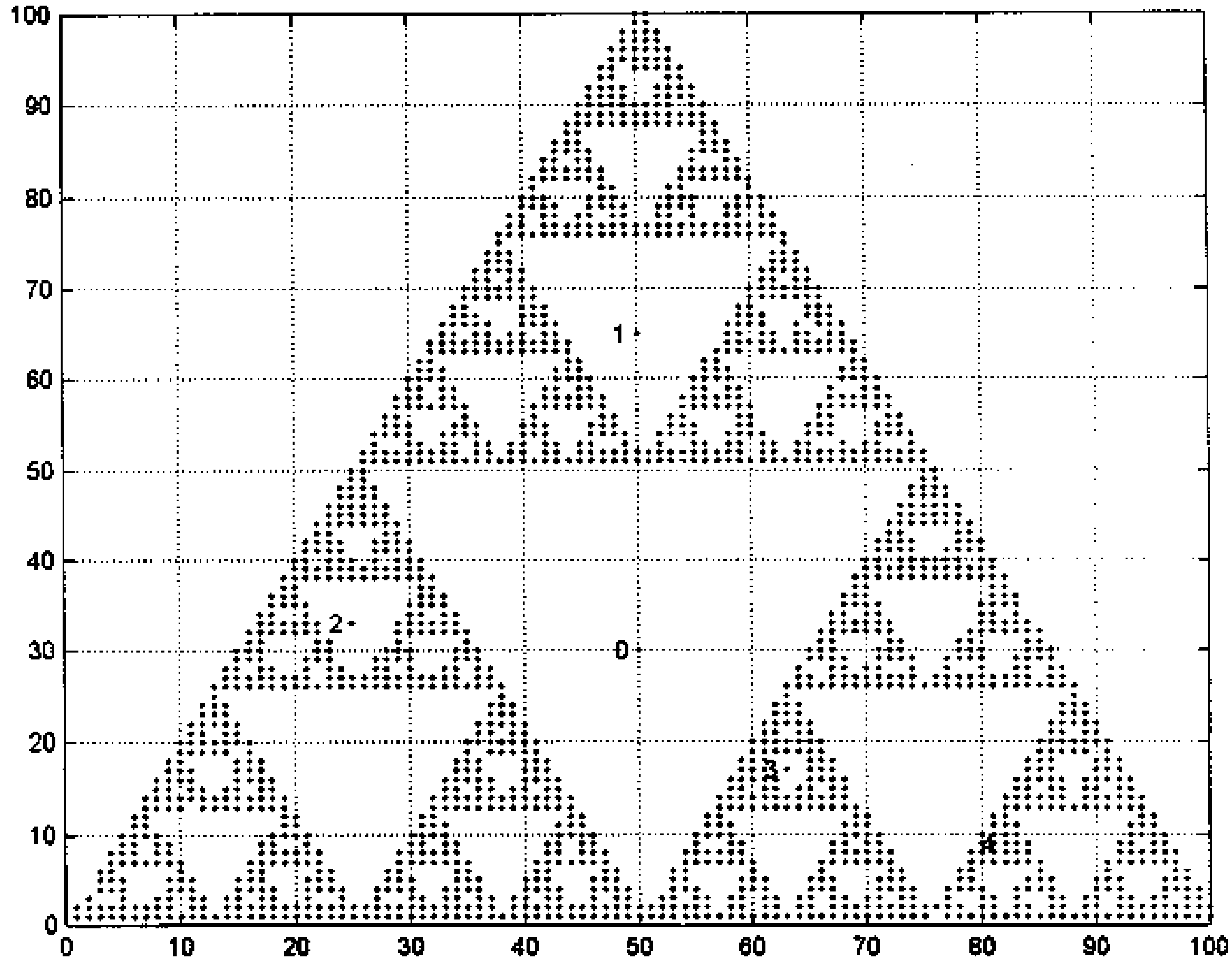
but $\pi_1 = \frac{1}{3}$

```
% probprob22_28.m
%
clear all
rand('state',0)
p=0.25;nholes=1000;
x=zeros(nholes,1);
x(1)=floor(rand(1,1)+0.5);
for i=2:nholes
   if x(i-1)==0
      if rand(1,1)>1-p
         x(i)=1;
      end
   else
      x(i)=floor(rand(1,1)+0.5);
   end
end
mean(x)
```

29) $\hat{E}[T_R] = 3.0929$

```
% probprob22_29.m
%
clear all
rand('state',0)
p=0.25;nholes=1000;
x=zeros(nholes,1);
x(1)=floor(rand(1,1)+0.5);
for i=2:nholes
   if x(i-1)==0
      if rand(1,1)>1-p
         x(i)=1;
      end
   else
      x(i)=floor(rand(1,1)+0.5);
   end
end
ind=find(x==1);% assumes a one for x(1)
for i=2:length(ind)
   holes(i,1)=ind(i)-ind(i-1);
end
mean(holes(2:length(ind)))
```

30)



Same steady-state figure. Only difference is in some initial values.

```
% probprob22_30.m
%
clear all
rand('state',0)
r(:,1)=[0 0]';
r(:,2)=[100 0]';
r(:,3)=[50 100]';
x0=[50 30]'; % set initial state
plot(x0(1),x0(2),'.') % plot state outcome as point
text(x0(1)-2,x0(2),'0')
axis([0 100 0 100])
hold on
xn_1=x0;
for n=1:10000 % generate states
   j=floor(3*rand(1,1)+1); % choose at random one of three reference points
   xn=round(0.5*(r(:,j)+xn_1)); % generate new state
   plot(xn(1),xn(2),'.') % plot state outcome as point
   if n<5
      text(xn(1)-2,xn(2),num2str(n))
   end

   xn_1=xn; % make current state the previous one for next transition
end
grid
hold off
```