Trend-seasonal decomposition and exponential smoothing models

Data and Tasks

Data and Tasks: Plan

- Time series is a data type

Data and Tasks: Plan

- Time series is a data type
- Tasks for one row

Data and Tasks: Plan

- Time series is a data type
- Tasks for one row
- Tasks for multiple rows

What is a time series?

Time series

A sequence of observations ordered in time

$$0, 0, 5, 7, 102, 53, 23$$

What is a time series?

Time series

A sequence of observations ordered in time

$$0, 0, 5, 7, 102, 53, 23$$

Time series

A sequence of random variables ordered in time

$$y_1, y_2, y_3, y_4, \ldots, y_T$$

Tasks for one series

- Predict the following values

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- Predict the following values
- Restore missing values in the middle of a series

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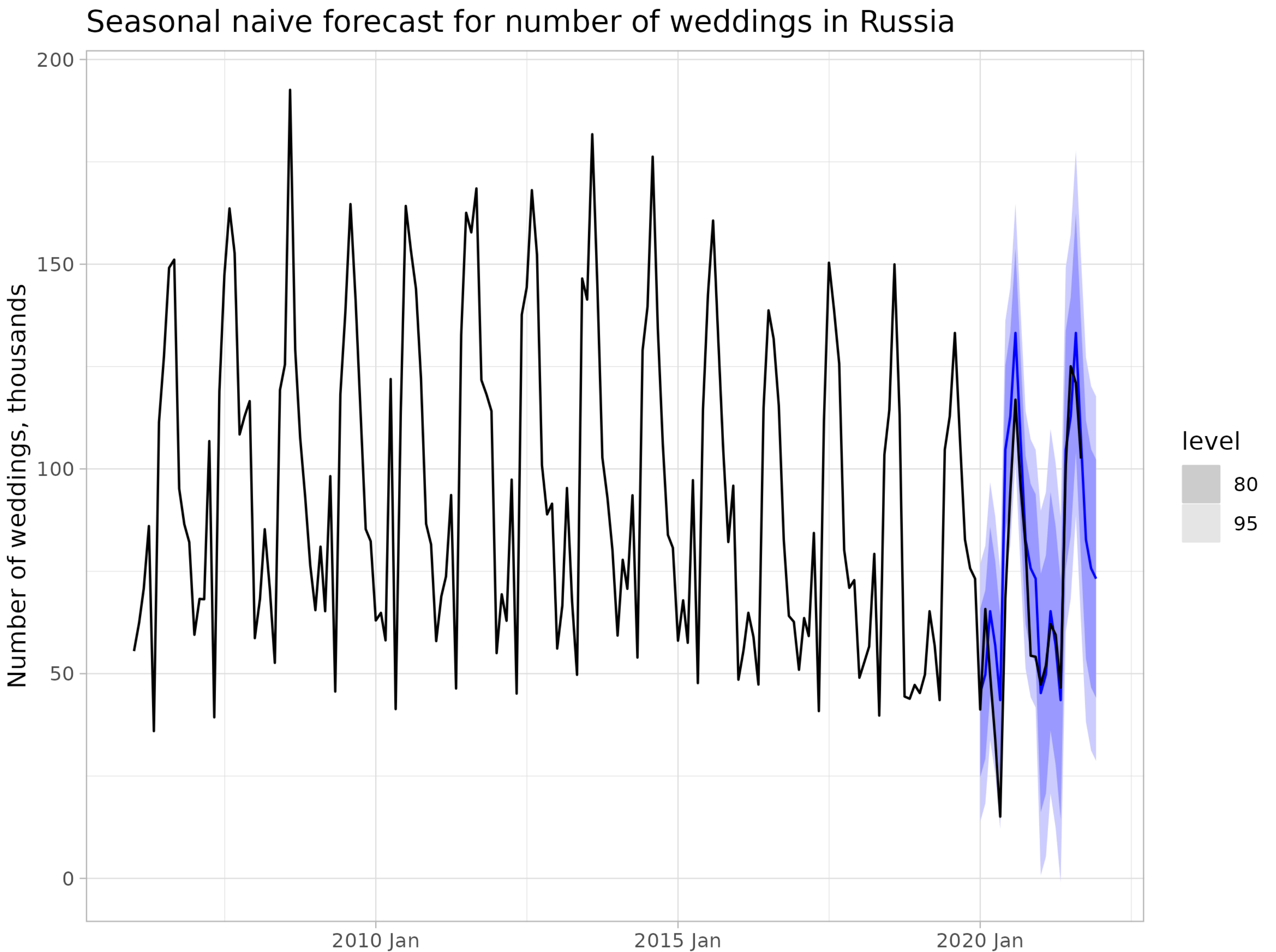
Forecasting



Tasks for multiple series

- Use additional series when studying the target series

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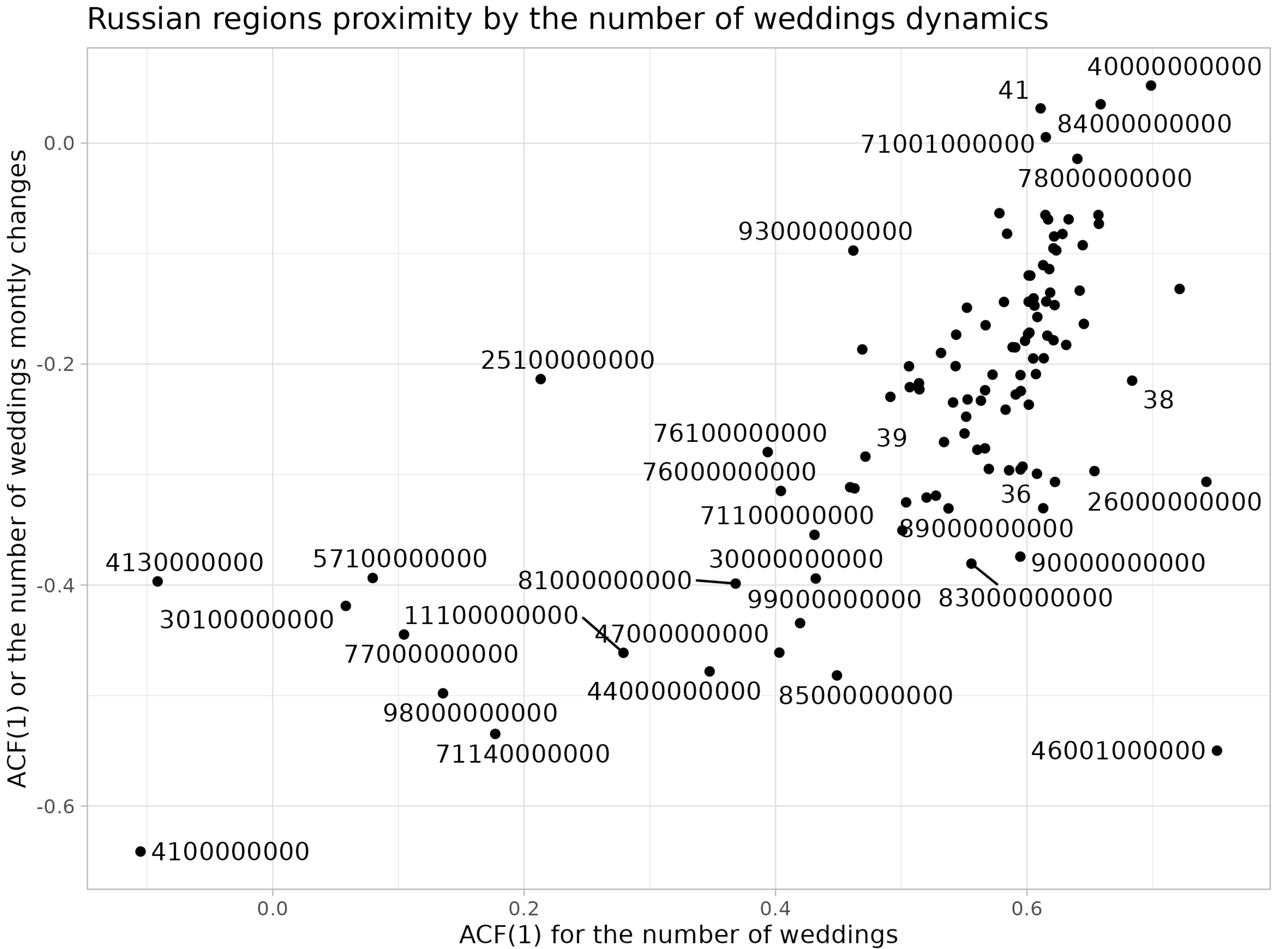
Measuring series proximity



Models and algorithms

Models

- Explicit assumptions about the values $y_1, y_2, \dots, y_T$

ETS, ARIMA, ORBIT, PROPHET, …

Models and algorithms

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- Explicit assumptions about the values $y_1, y_2, \ldots, y_T$
- Estimation method: maximum likelihood, Bayesian approach

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Models

- Explicit assumptions about the values $y_1, y_2, \ldots, y_T$
- Estimation method: maximum likelihood, Bayesian approach
- Point and interval forecasts, hypothesis testing

ETS, ARIMA, ORBIT, PROPHET, ...

Models and algorithms

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- Fuzzy assumptions about the values $y_1, y_2, \ldots, y_T$

STL, gradient boosting, random forest, …

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- Fuzzy assumptions about the values $y_1, y_2, \ldots, y_T$
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Models and algorithms

Algorithms

- Fuzzy assumptions about the values $y_1, y_2, \ldots, y_T$
- A special instruction for actions
- Point estimates without confidence intervals

STL, gradient boosting, random forest, ...

Course Focus

Forecasting one-dimensional series using models

Series Components

Series Components: Plan

- Trend, cyclicity and seasonality

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- Additive and multiplicative expansion

Series Components: Plan

- Trend, cyclicity and seasonality
- Additive and multiplicative expansion
- A formal definition?

Looking for components

Additive series expansion:

$$y_t = trend_t + seas_t + remainder_t$$

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Trend — smoothly changing component of the series

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Seasonal component — a component with a clear frequency and stable intensity

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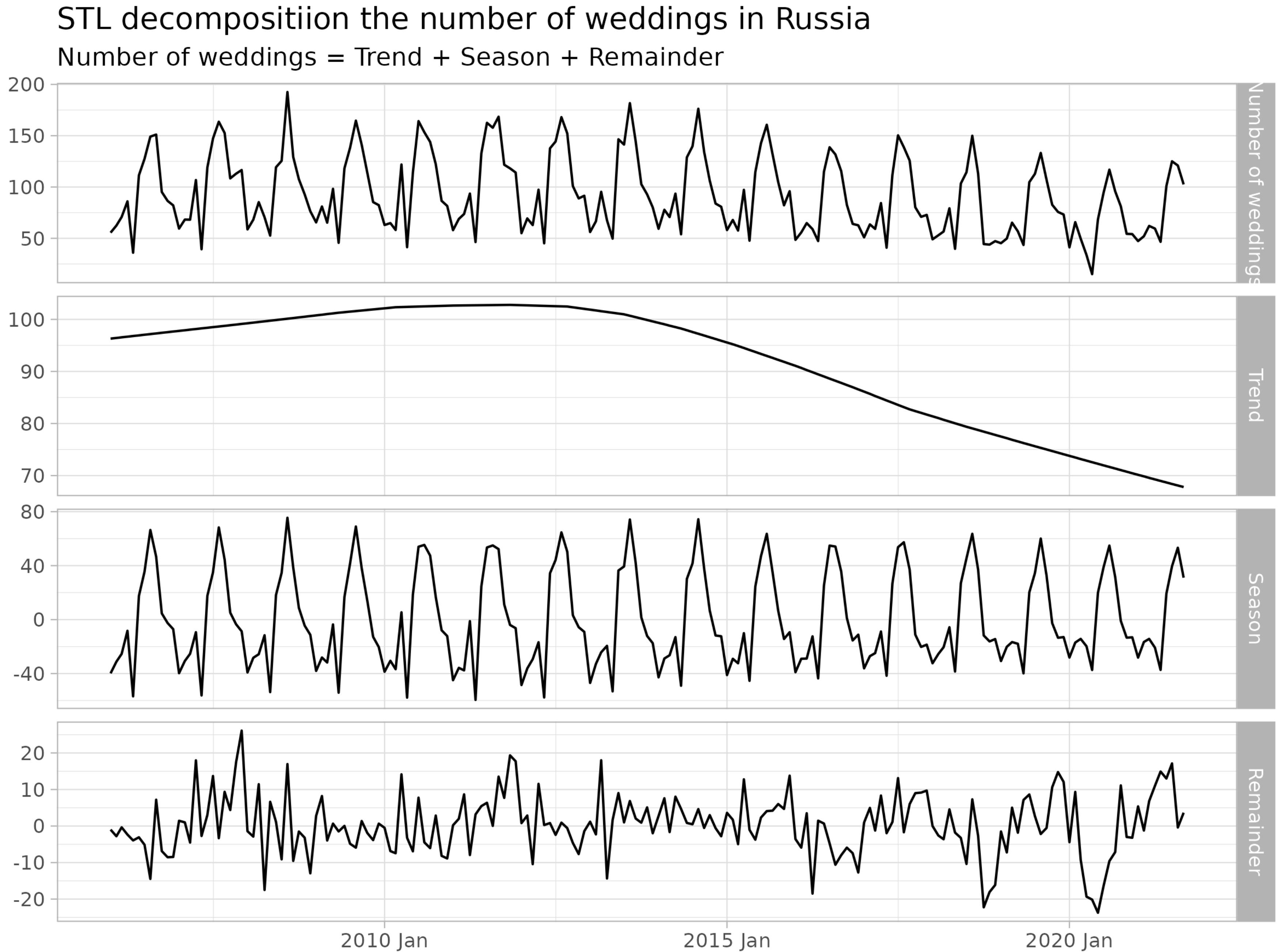
Trend — smoothly changing component of the series

Seasonal component — a component with a clear frequency and stable intensity

Random component (remainder) — everything else

Trend, seasonality and residual



Strict definition?

There will be no single strict definition for the components!

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Some models and algorithms formally define these components

Cyclical component

Sometimes the series can be decomposed further

$$y_t = trend_t + cycle_t + seas_t + remainder_t$$

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Cyclical component — component with floating frequency and unstable intensity

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Cyclical component — component with floating frequency and unstable intensity

Trend (in the narrow sense) — a smoothly changing monotonous component of a series

Additive and multiplicative expansion

Additive series decomposition:

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Multiplicative series decomposition:

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Additive and multiplicative expansion

Additive series decomposition:

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Multiplicative series decomposition:

$$y_t = trend_t \cdot seas_t \cdot remainder_t$$

Let's transform one into another:

$$\ln y_t = \ln trend_t + \ln seas_t + \ln remainder_t$$

Box-Cox Transformation

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(Generalized) Box-Cox transformation:

$$bc_\lambda(y_t) = \begin{cases} \ln y_t, \text{ if } \lambda = 0, \\ \operatorname{sign}(y_t)(|y_t|^\lambda - 1)/\lambda, \text{ if } \lambda \neq 0 \end{cases}$$

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How to select the parameter λ?

- Some models contain it inside and estimate λ within themselves
- You can choose λ by yourself to stabilize the amplitude of oscillations of the series

What to choose?

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It is important to understand the purpose of constructing the decomposition

Why expand?

- Interesting by itself

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- For predicting a series using component prediction

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- To classify the new series into one of the given classes
- To identify unknown clusters in series

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- The seasonal component has clear periodicity and stable amplitude
- The exact formalization of the components depends on the model

Naive Models

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White noise

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Time series u_t is white noise if:

- $\mathbb{E}(u_t) = 0$;
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- An integral part of all models; most often, white noise is not modelled explicitly
- Often independence and normality are assumed
  ARCH, GARCH volatility models are based on the fact that u_t and u_s can be dependent!

Independent observations

Model

$$y_t = \mu + u_t,$$

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Interval forecast h steps ahead:

$$[\bar{y} - 1.96\hat{\sigma}; \bar{y} + 1.96\hat{\sigma}]$$

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Naive model

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Interval forecast h steps ahead:

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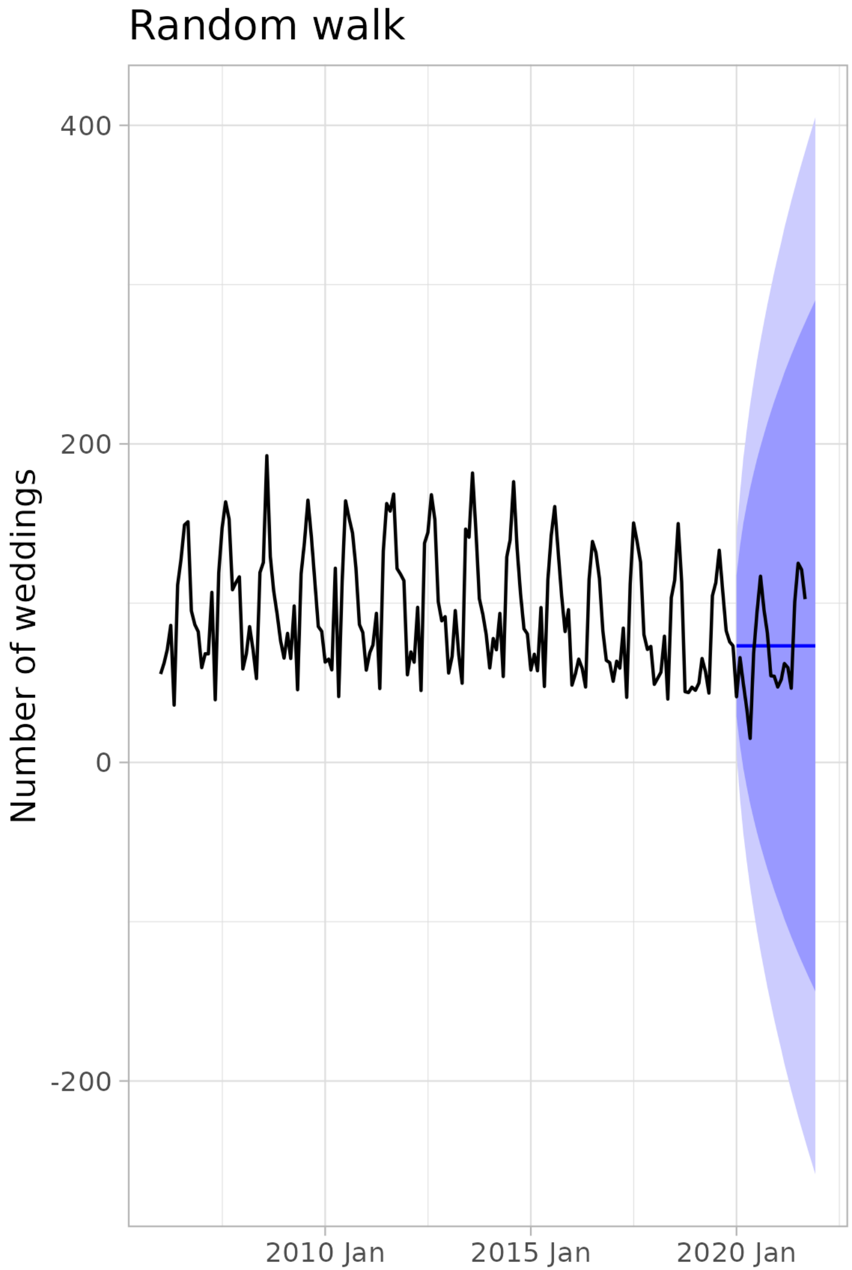
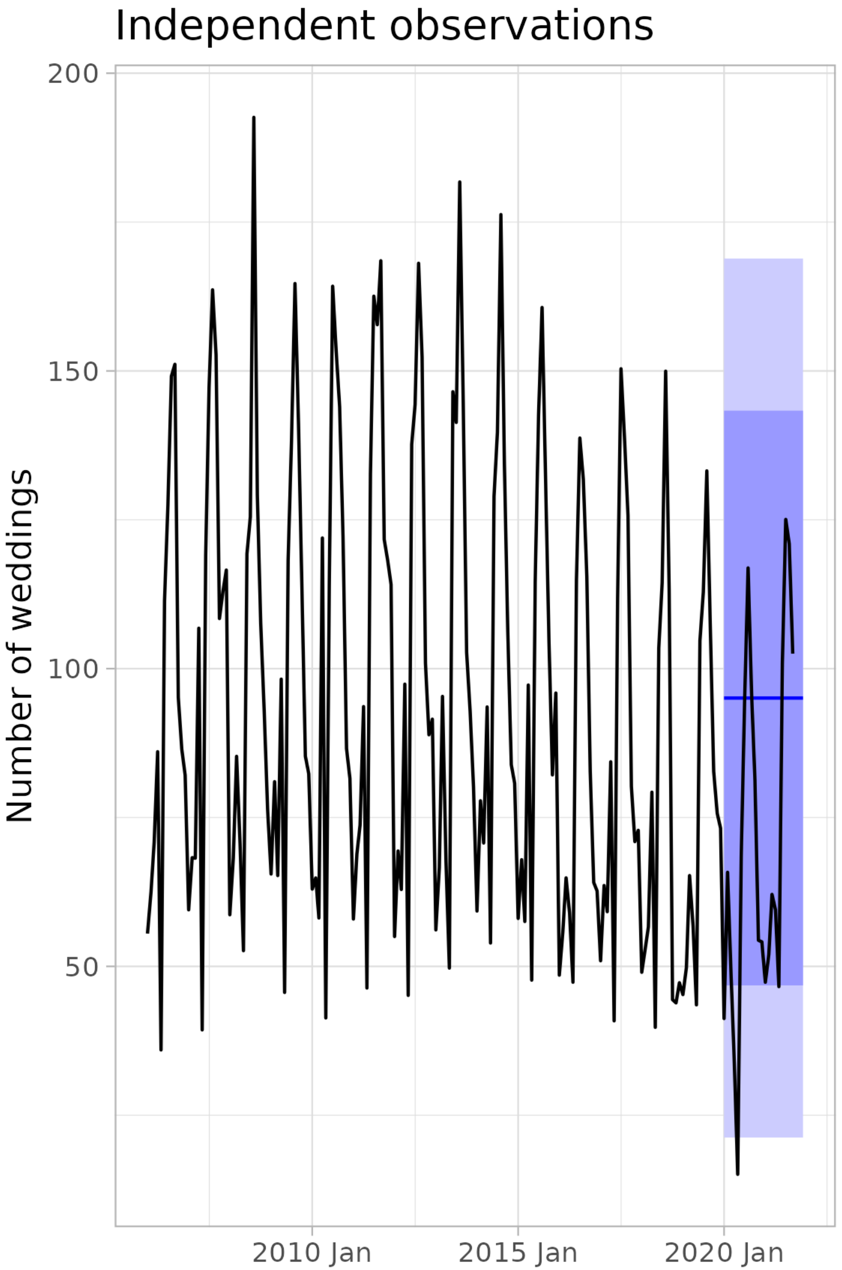
First predictions!





Seasonal random walk

Seasonal naive model

$$y_t = y_{t-12} + u_t,$$

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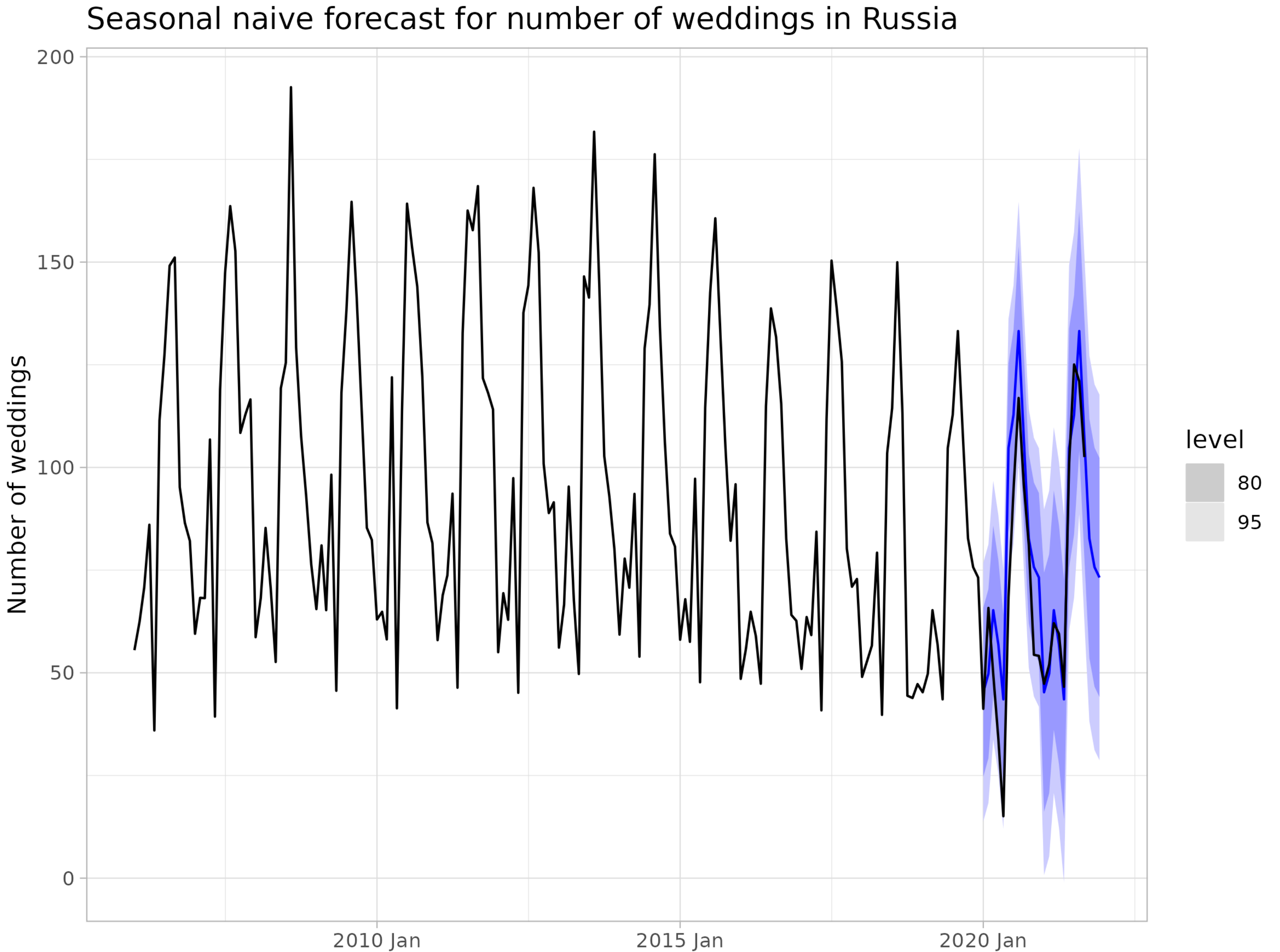
Interval forecast for h seasons ahead:

$$[y_T - 1.96\hat{\sigma}\sqrt{h}; y_T + 1.96\hat{\sigma}\sqrt{h}]$$

Not bad already!



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- Ideas for complex model:
  the stationary series models are similar to the independent observations model;
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- Benchmark for comparison:
  when evaluating a complex model, it is very important to have a base of comparison
- Averaging with other models' forecasts:
  you can average forecasts of a complex model and a naive seasonal one!

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STL algorithm

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- Local regression

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STL

STL — Seasonal Trend decompositon with LOESS

STL — seasonality and trend decomposition using LOESS

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LOESS — LOcal regrESSion

LOESS — local linear regression

STL as a black box

Input:

Row Y_t

Algorithm parameters $n_p, n_i, n_o, n_l, n_s, n_t$

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Decomposition $Y_t = T_t + S_t + R_t$

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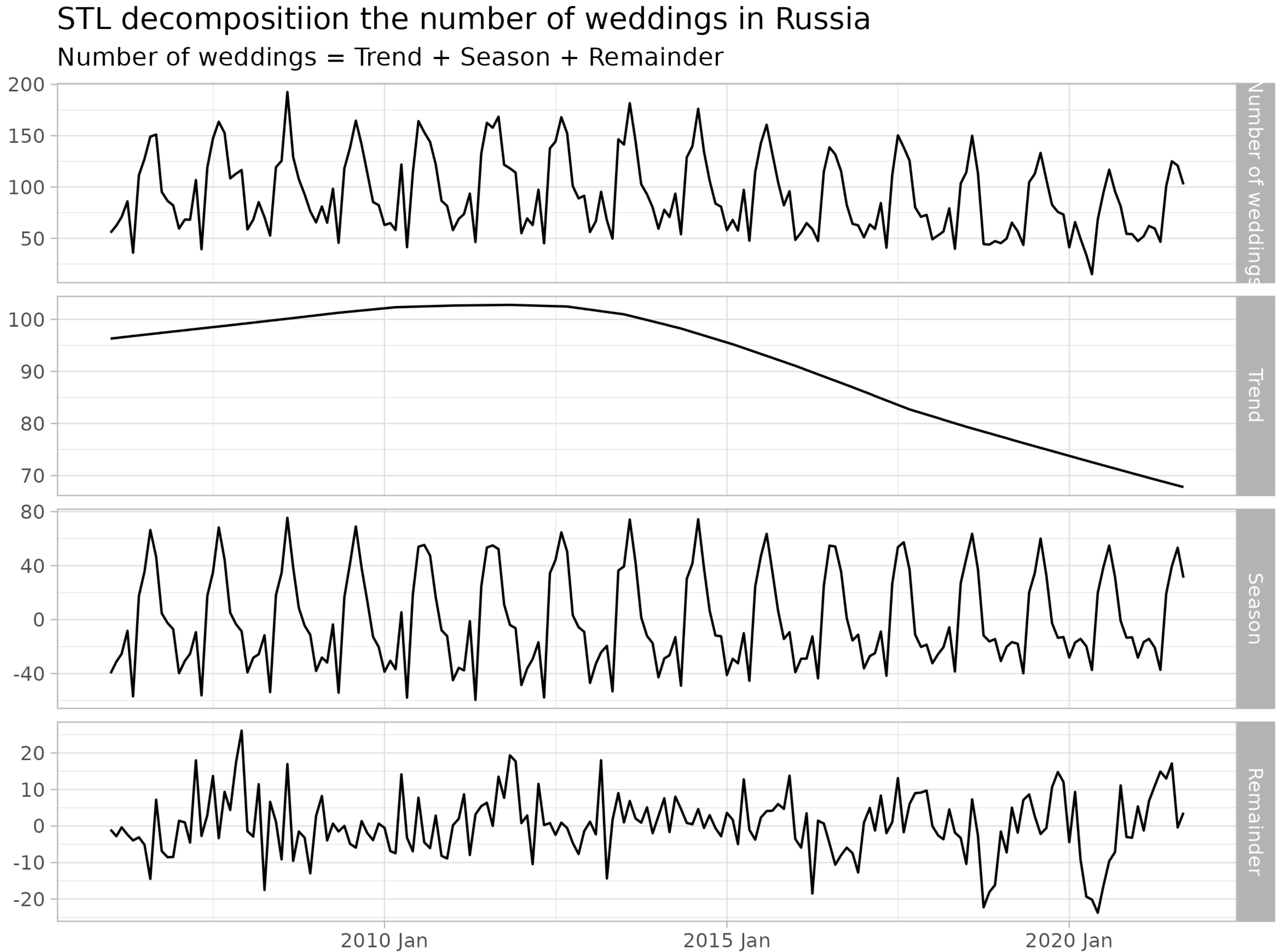
Black box set up

STL: result



STL decompositiion the number of weddings in Russia
Number of weddings = Trend + Season + Remainder
200
150
100
50
Number of weddings
100
90
80
70
Trend
80
40
0
-40
Season
20
10
0
-10
-20
Remainder
2010 Jan
2015 Jan
2020 Jan

LOESS

- We want to build a forecast for the point x
- Find local estimates $\hat{\beta}_1(x)$, $\hat{\beta}_2(x)$

$$\min \sum_i K_h(x_i - x)(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i)^2$$

- Predicting:

$$\hat{y} = \hat{\beta}_1(x) + \hat{\beta}_2(x)x.$$

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- The function $K_h(x_i - x)$ decreases with increasing distance $|x_i - x|$;
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For example, h is the number of points x_i next to x that we take into account

Nuances of local regression

- Select degrees of the polynomial

$$\min \sum_i K_h(x_i - x)(y_i - \hat{\beta}_1 - \hat{\beta}_2 x_i - \hat{\beta}_3 x_i^2)^2$$

- Select kernel function

$$K_h(d) = \frac{1}{\sqrt{2\pi}h} \exp\left(-d^2/2h^2\right)$$

- Select window width h

STL algorithm

Purpose: decomposition of $Y_t = T_t + S_t + R_t$

The algorithm contains two loops: outer and inner loop

1. Initialize $T_t = 0$, $R_t = 0$

   Outer loop:

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   on the first pass, $\rho_t = 1$ for each observation;

   on subsequent passes, ρ_t depends negatively on the new value of R_t

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3. Update the current decomposition $Y_t = T_t + S_t + R_t$ taking into account new weights ρ_t

STL: inner loop

Goal: update the decomposition $Y_t = T_t + S_t + R_t$.

1. Remove the previously calculated trend from the series:

$$Y_t^{det} = Y_t - T_t.$$

STL: inner loop

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$$Y_t^{det} = Y_t - T_t.$$

2. Divide the detrended series into 12 series (one for each season)
3. Smooth each of the series individually with LOESS:

$$C^{jan} = LOESS_\rho(Y_{jan}^{det}), C^{feb} = LOESS_\rho(Y_{feb}^{det}), \ldots$$

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4. Extract the low-frequency component (double moving average + LOESS):

$$L_t = LOESS(MA(MA(C_t)))$$

STL: inner loop

1-3. Remove the previously calculated trend from the series, break it down into 12 series and smooth each of them with LOESS.

4. Extract the low-frequency component L_t.

5. Get new seasonal component by removing the low-frequency component:

$$S_t^{new} = C_t - L_t$$

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$$S_t^{new} = C_t - L_t$$

6. Get new trend component by removing new seasonal component from the original series and smoothing with LOESS:

$$T_t^{new} = LOESS_\rho(Y_t - S_t^{new})$$

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  the larger the number n_o, the weaker the impact of outliers;
  $n_o = 1$ is often sufficient
- n_i — number of passes of the inner loop:
  $n_i = 2$ is often enough to achieve convergence.

STL smoothing parameters

- n_l — low pass filter smoothing strength

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- n_s — seasonal contract smoothing strength

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- n_t — smoothing strength when highlighting a trend at the last step

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  What to configure?

1. Be sure to specify the periodicity n_p
2. Maybe play around with n_s

STL algorithm: Summary

- LOESS — local regression

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- LOESS — local regression
- STL is a well-proven algorithm without an underlying model
- If you wish, you can play around with the smoothing parameters

Series Characteristics

Series Characteristics: Plan

- Sample autocorrelation

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- Sample autocorrelation
- Sample partial autocorrelation

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Tasks for multiple series

- Classify the new series into one of the existing classes
- Understand which series are close to each other
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How to solve?

1. Generate features for each series
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Classify using random forest;

Measure distance using the Mahalanobis metric;

Cluster using hierarchical clustering

Creating features

Two sets of features:

- Sample ACF (autocorrelation function, AutoCorrelation Function)

Creating features

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From one series we get:

$ACF_1, ACF_2, ACF_3, \ldots$

$PACF_1, PACF_2, PACF_3, \ldots$

ACF

Sample ACF

Let's evaluate a set of paired regressions:

$$\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 y_{t-1}, \quad ACF_1 = \hat{\beta}_2;$$

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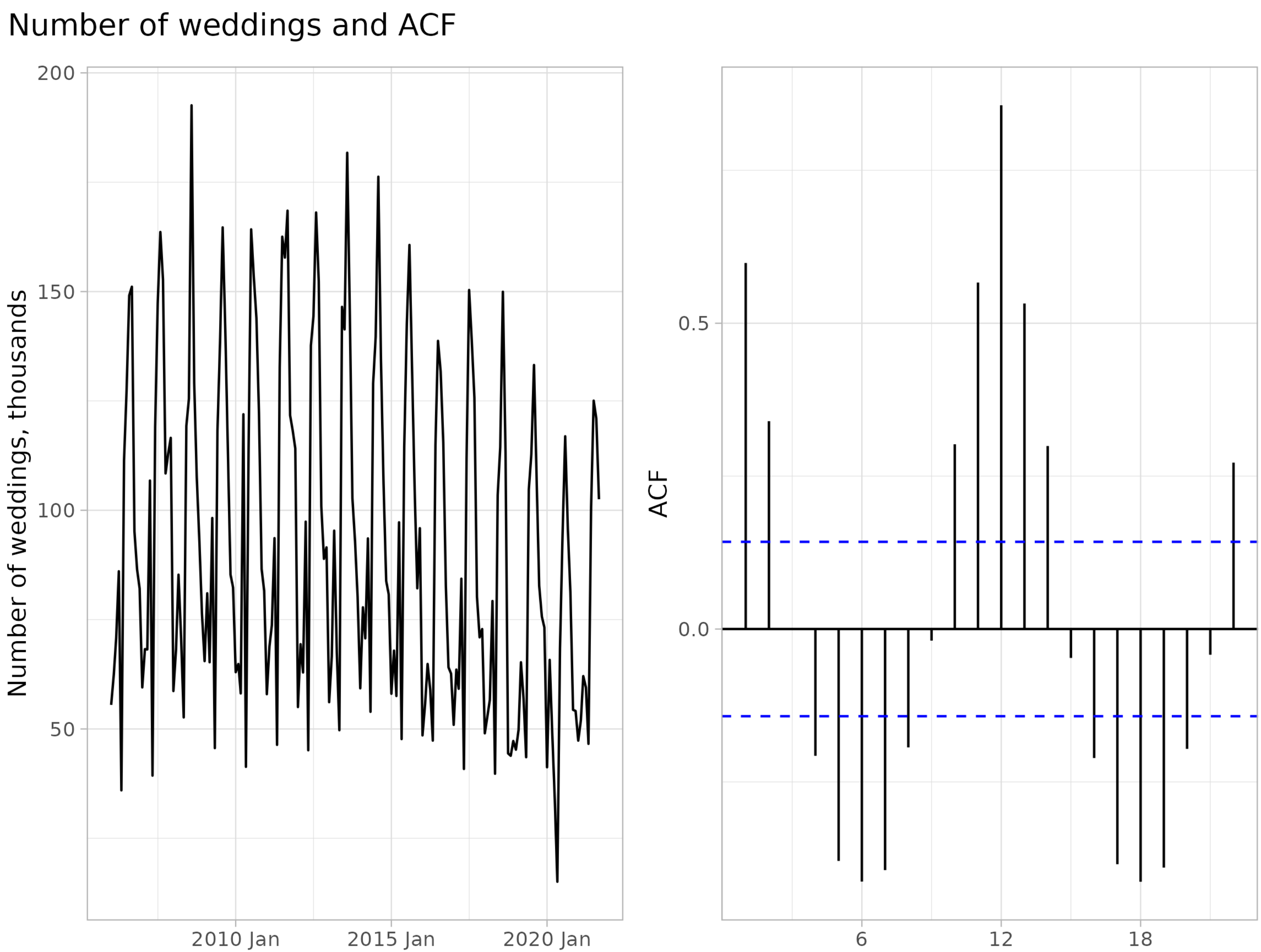
Meaning ACF_2: How many units is y_t above average on average if y_{t-2} is one unit above average.

Series and its ACF



Why is ACF a correlation?

Classic definition

Sample ACF

ACF_k — sample correlation between series y_t and series y_{t-k}

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PACF

Sample PACF

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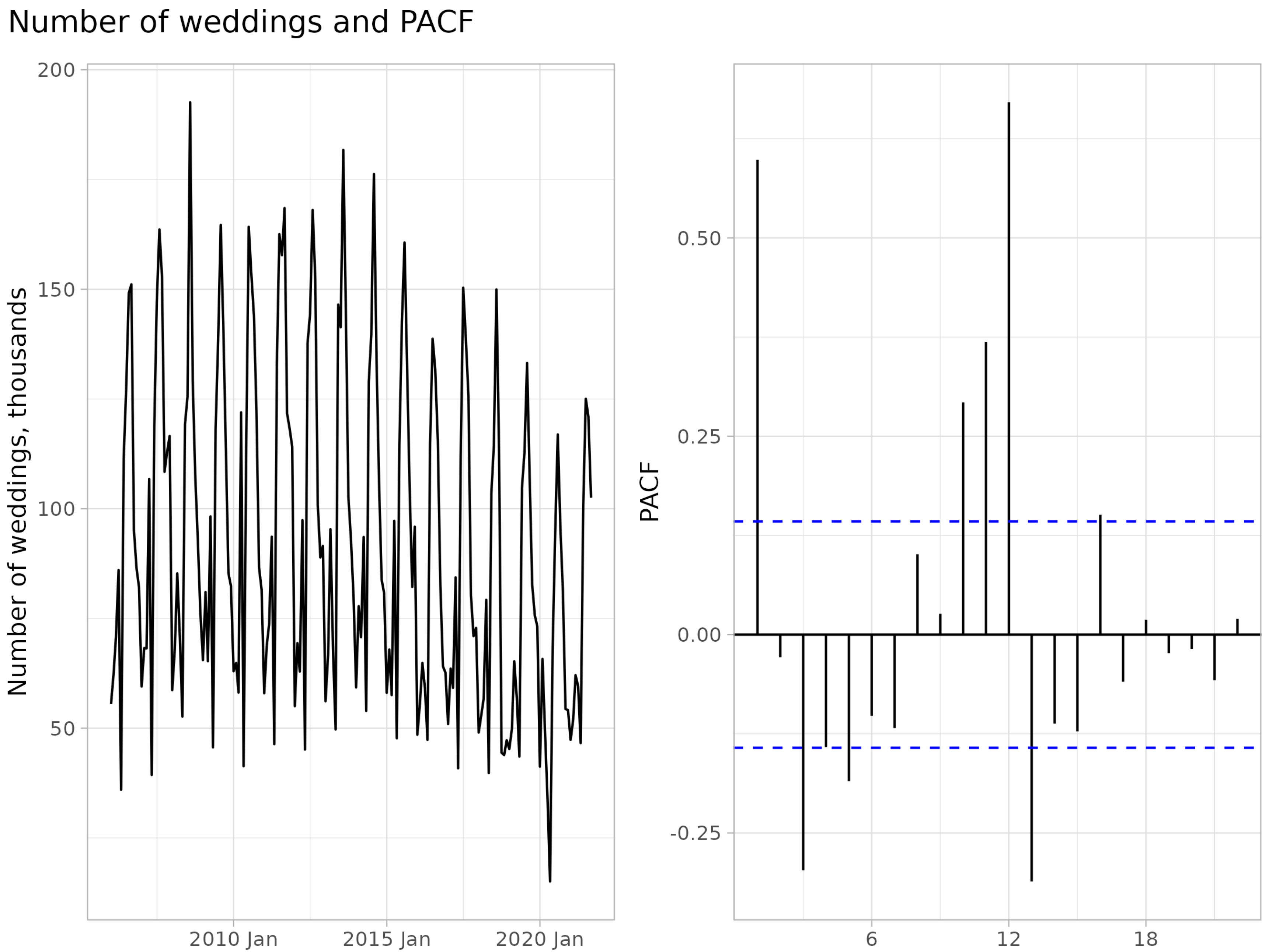
Meaning $PACF_2$: how many units is y_t above average on average if y_{t-2} is one unit above average, and y_{t-1} is at the middle level

Series and its PACF



Why is PACF a correlation?

Classic definition

Custom PACF

$PACF_4$ — sample correlation between a_t residuals and b_t residuals:

a_t — regression residuals

$$y_t \mid 1, y_{t-1}, y_{t-2}, y_{t-3};$$

b_t — regression residuals

$$y_{t-4} \mid 1, y_{t-1}, y_{t-2}, y_{t-3}$$

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Classic definition

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The difference between the definitions of small

STL features

Output:

$$y_t = T_t + S_t + R_t$$

STL features

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Let's measure:

- Strength of F_{trend} trend
- Strength of F_{seas} seasonality

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We got the decomposition:

$$y_t = trend_t + seas_t + remainder_t.$$

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For an ideal decomposition with uncorrelated components:

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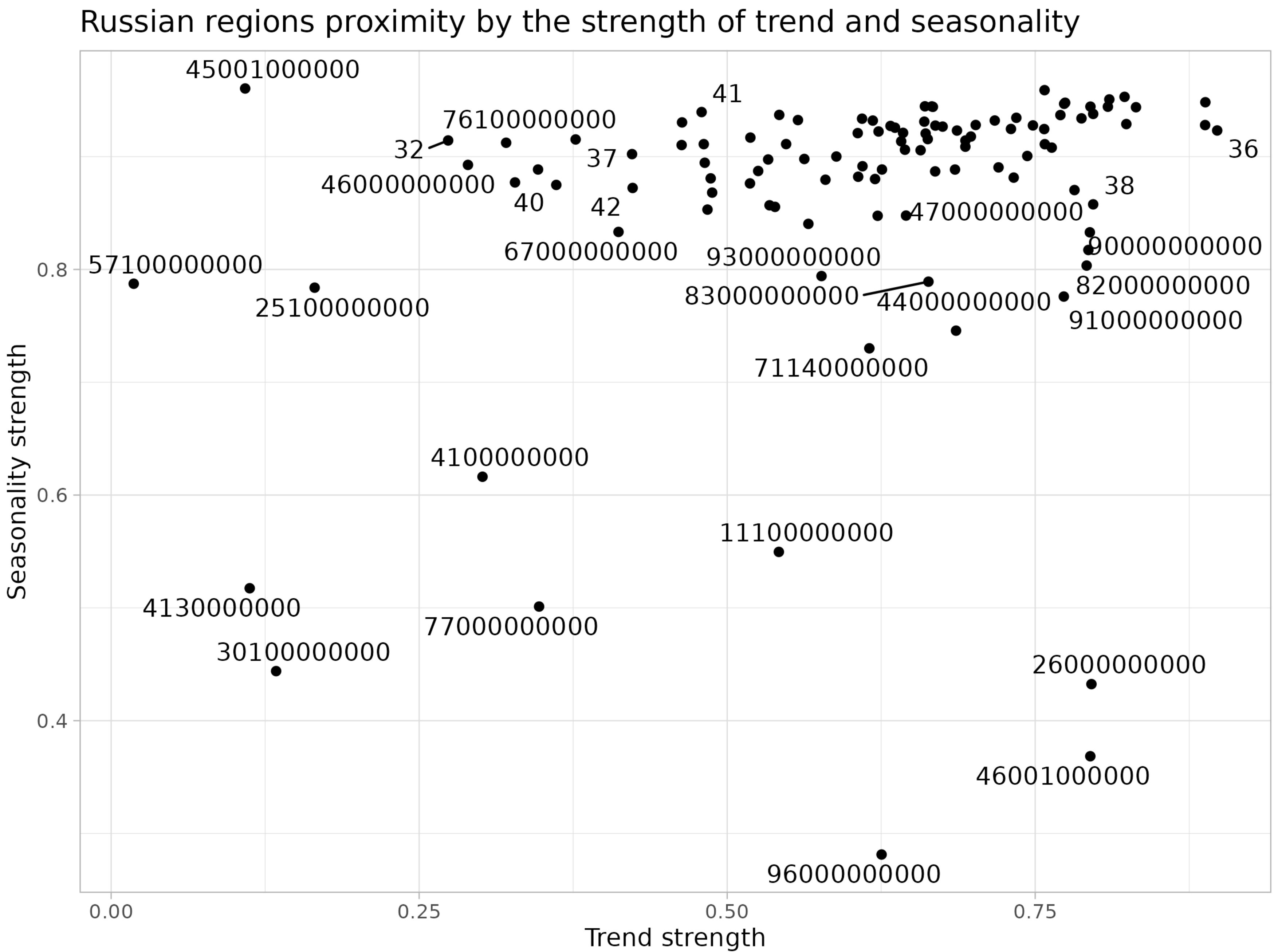

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- ACF — coefficients in paired regressions or correlations
- PACF — coefficients in multiple regressions or correlations
- STL allows you to measure strength of trend and seasonality in comparison to the residual component

ETS Model (Part I)

ETS Model: Plan

- ETS as a model

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- Formulas for predictions

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- Idea of damped trend

How many ETS models in total?

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Formally: 30 options

Historical names

ETS(ANN) — simple exponential smoothing

ETS(AAA) — an additive Holt-Winters method

ETS(AAM) — the multiplicative Holt-Winters method

ETS(AAdM) — Holt-Winters method with a fading trend

ETS(ANN) terminology

y_t — the observed series;

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Parameters: α, σ^2, ℓ_0

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Substitute $\alpha = 1$:

$$y_t = \ell_t = \ell_{t-1} + u_t$$

Estimation

Maximum likelihood method is used for estimation

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Main idea: decompose the likelihood into a sum

$$\ln L(y \mid \theta) = \ln L(y_1 \mid \theta) + \ln L(y_2 \mid y_1, \theta) + \ldots + \\ + \ln L(y_T \mid y_{T-1}, \ldots, y_1, \theta),$$

where $\theta = (\alpha, \ell_0, \sigma^2)$

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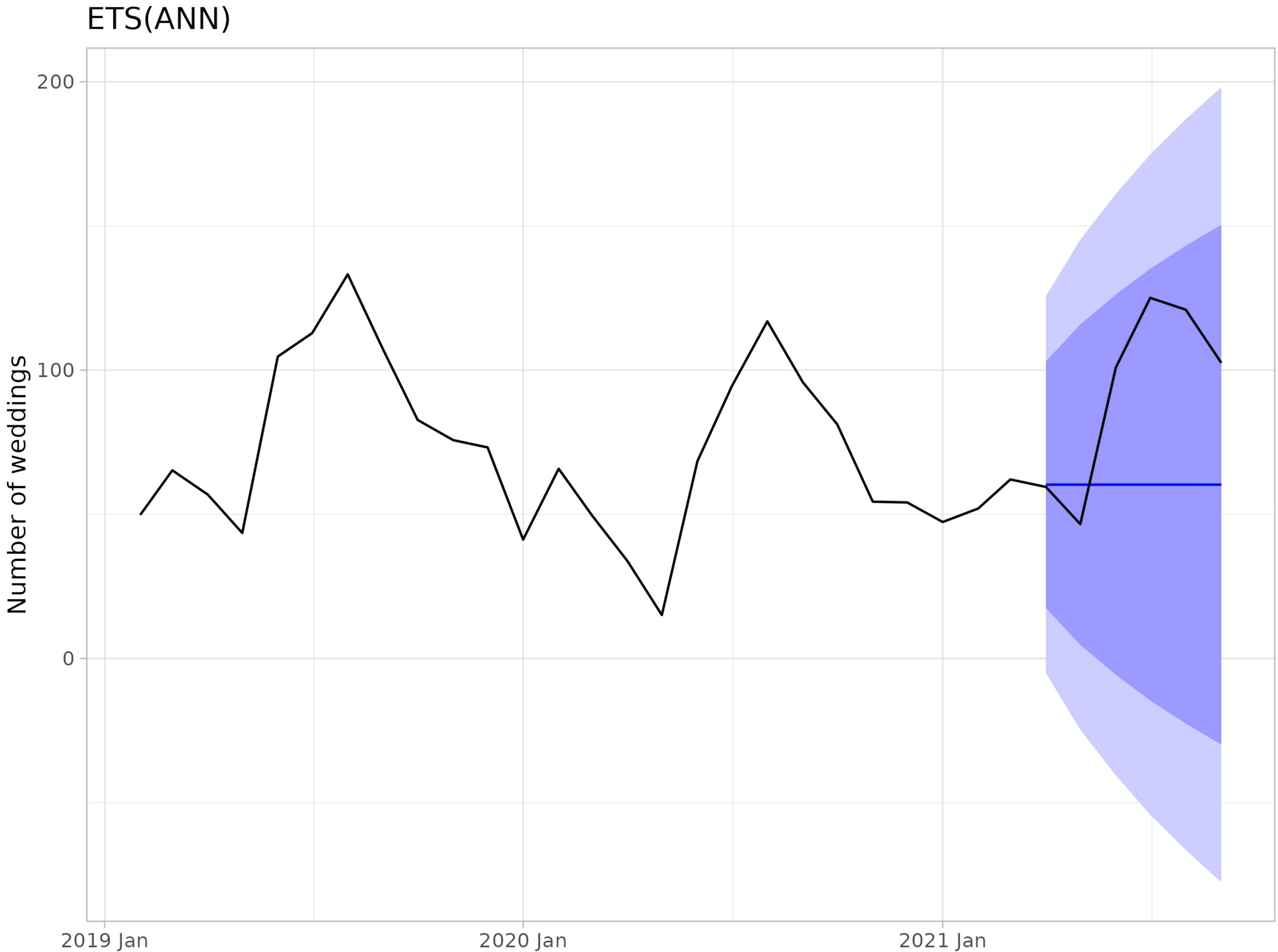
where $\theta = (\alpha, \ell_0, \sigma^2)$

Unfortunately, there are no explicit formulas for the estimators

Forecasting



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$$(y_{T+2} \mid \mathcal{F}_T) \sim \mathcal{N}(\ell_T; \sigma^2(\alpha^2 + 1))$$

Predictive intervals

From distribution law

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$$[\hat{\ell}_T - 1.96\hat{\sigma}\sqrt{\hat{\alpha}^2 + 1}; \hat{\ell}_T + 1.96\hat{\sigma}\sqrt{\hat{\alpha}^2 + 1}].$$

What was discovered in the 1950s?

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$$\min_{\alpha} \sum (y_t - \hat{\ell}_t)^2$$

Adding trend!

y_t — the observed series;

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ETS(AAN):

A — additive error;

A — additive trend;

N — no seasonality

ETS(AAN): equations

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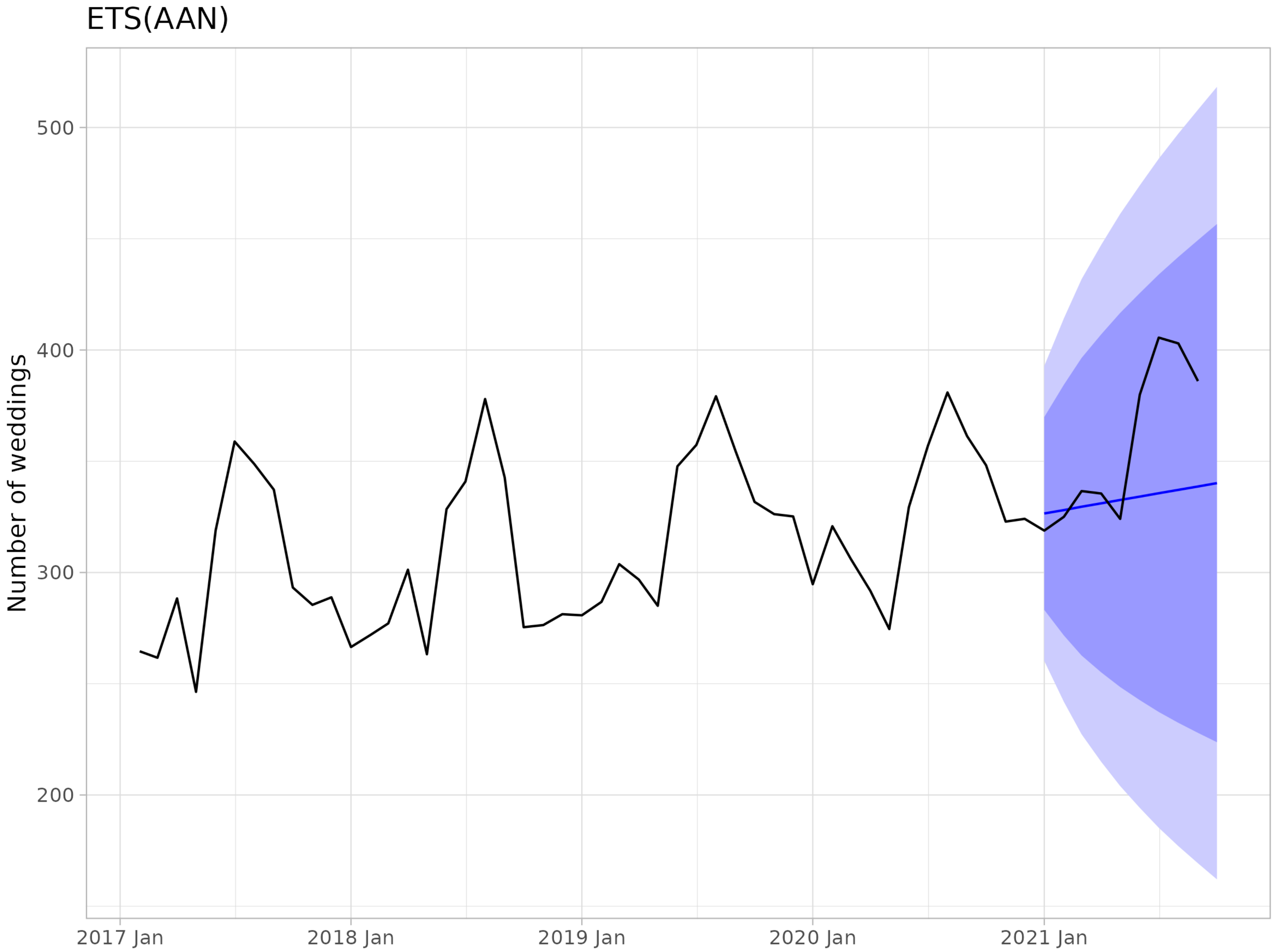
Parameters: α, β, σ^2, ℓ_0, b_0

ETS(AAN): Forecasting



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Forecast 2 steps ahead

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$$y_{T+2} = \ell_{T+1} + b_{T+1} + u_{T+2} = (\ell_T + b_T + \alpha u_{T+1}) + \\ + (b_T + \beta u_{T+1}) + u_{T+2}$$

$$(y_{T+2} \mid \mathcal{F}_T) \sim \mathcal{N}(\ell_T + 2b_T; \sigma^2((\alpha+\beta)^2 + 1))$$

Problem with trend in ETS(AAN)

In the ETS(AAN) model growth rate of the ℓ_t trend is defined by the formula

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Consequently,

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Long-term forecast of a positive indicator at $b_T < 0$ will become negative

Contradiction

In short-term we expect a change in the indicator:

we want a trend in the model.

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we want a trend in the model.

In long-term negative values are impossible:

we don't want a trend in the model.

Solution: damped or fading trend.

Extra parameters are expensive!

We want richer trend dynamics — we need additional parameters.

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Additional parameters — risk overfitting of the model, wider confidence intervals for the remaining parameters.

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We want richer trend dynamics — we need additional parameters.

Additional parameters — risk overfitting of the model, wider confidence intervals for the remaining parameters.

Let's solve the problem with only one new parameter!

Damped trend

We introduce the trend damping parameter $\phi \in (0; 1)$ into the slope equation:

$$b_t = \phi b_{t-1} + \beta u_t, \text{ starts at } b_0$$

Damped trend

We introduce the trend damping parameter $\phi \in (0;1)$ into the slope equation:

$$b_t = \phi b_{t-1} + \beta u_t, \text{ starts at } b_0$$

And for the rest of the equations:

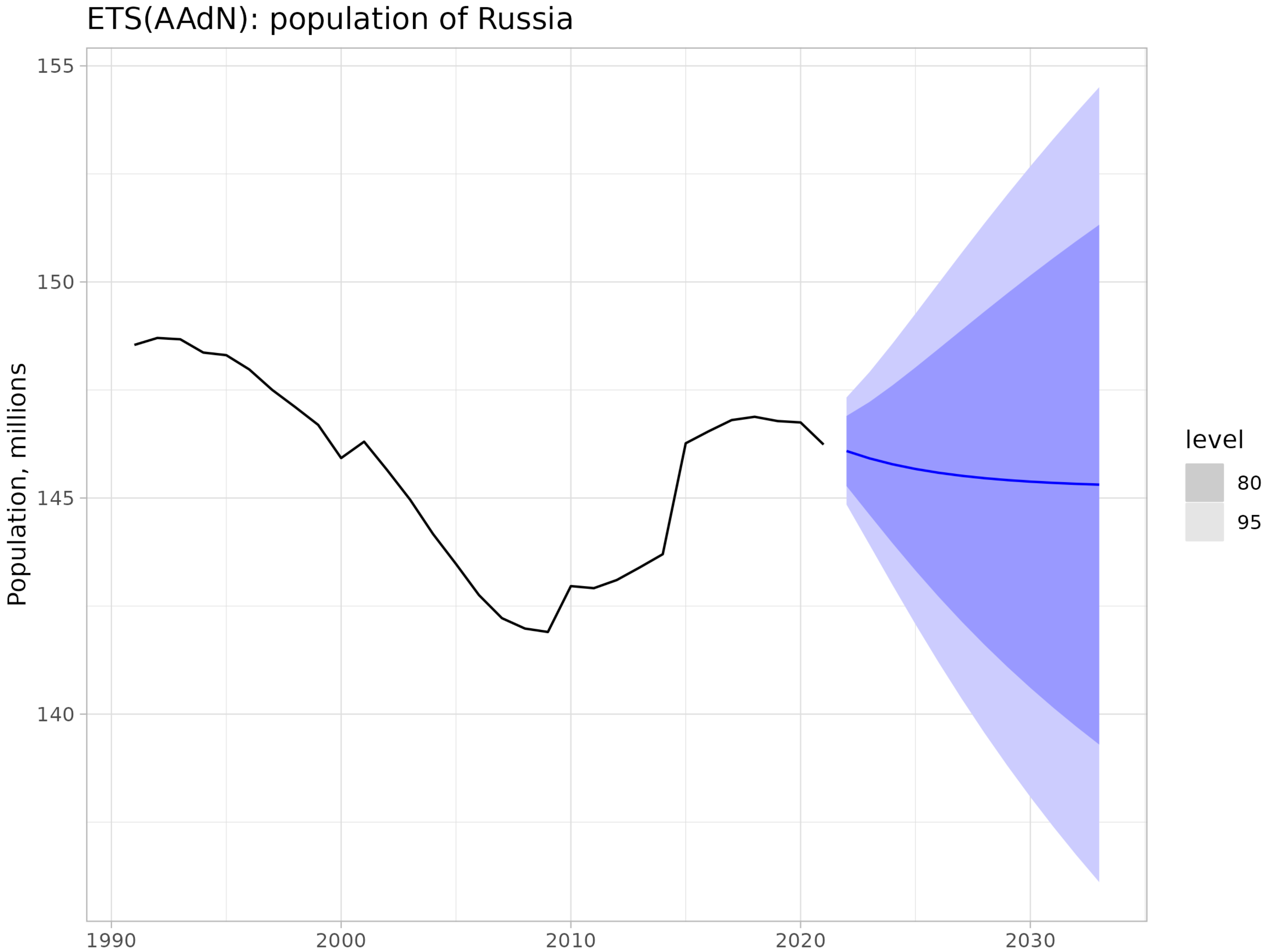
$$\begin{cases} y_t = \ell_{t-1} + \phi b_{t-1} + u_t; \\ \ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha u_t, \text{ starts at } \ell_0 \end{cases}$$

General form of ETS(AAdN)

$$\begin{cases} y_t = \ell_{t-1} + \phi b_{t-1} + u_t; \\ \ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha u_t, \text{ starts at } \ell_0; \\ b_t = \phi b_{t-1} + \beta u_t, \text{ starts at } b_0; \\ u_t \sim \mathcal{N}(0; \sigma^2) \text{ and are independent} \end{cases}$$

Parameters: α, σ^2, ℓ_0, b_0, β, ϕ

ETS(AAdN): Forecasting



ETS Model: Summary

- Formulas for exponential smoothing have been around for a long time

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ETS Model: Summary

- Formulas for exponential smoothing have been around for a long time
- ETS — a wide class of modern models
- The slope of the trend line can change
- Damped trend: on a small forecasting horizon there is a trend, on a large horizon — none

ETS Model (Part II)

ETS Model: Plan

- Adding seasonality

ETS Model: Plan

- Adding seasonality
- Formulas for predictions

ETS Model: Plan

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- Decomposition into components

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Adding seasonality!

y_t — the observed series;

ℓ_t — trend, cleaned series;

b_t — current growth rate of the cleaned series;

s_t — seasonal component;

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ETS(AAA):

A — additive error;

A — additive trend;

A — additive seasonality

ETS(AAA): equations

$$\begin{cases} y_t = \ell_{t-1} + b_{t-1} + s_{t-12} + u_t; \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t, \text{ starts at } \ell_0; \\ u_t \sim \mathcal{N}(0; \sigma^2) \text{ and are independent} \\ b_t = b_{t-1} + \beta u_t, \text{ starts at } b_0; \\ s_t = s_{t-12} + \gamma u_t; \text{ starts at} s_0, s_{-1}, \ldots, s_{-11} \end{cases}$$

Parameters: α, β, γ, σ^2, ℓ_0, b_0, s_0, s_{-1}, ..., s_{-11}

Restriction: $s_0 + s_{-1} + \ldots + s_{-11} = 0$

ETS(AAA): equations

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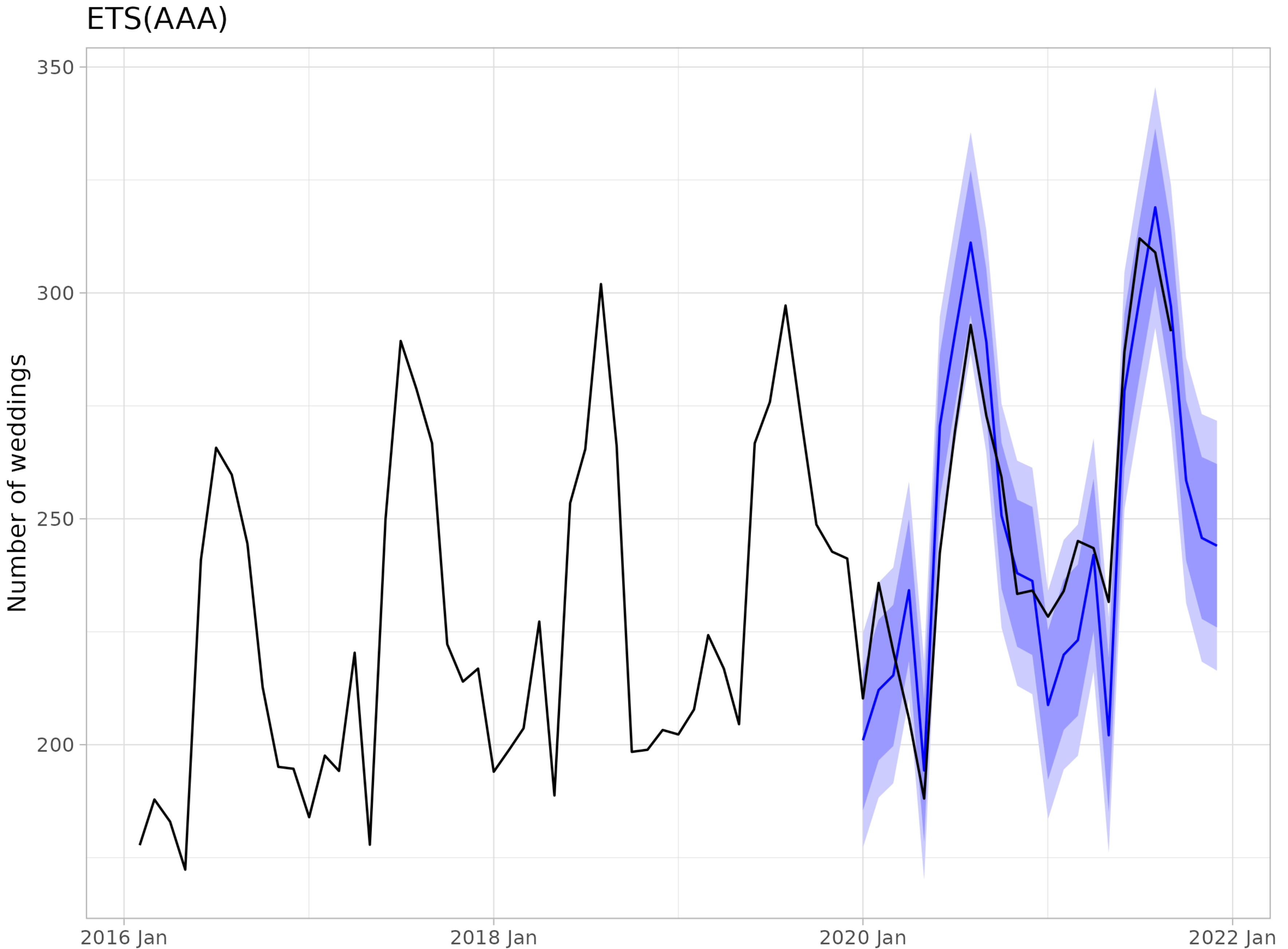
How many independent parameters are we estimating?

Correct answer: 17

ETS(AAA): Forecasting



Forecast 1 step ahead

$$
\begin{cases}
y_t = \ell_{t-1} + b_{t-1} + s_{t-12} + u_t; \\
\ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t, \text{ starts at } \ell_0; \\
u_t \sim \mathcal{N}(0; \sigma^2) \text{ and are independent.} \\
b_t = b_{t-1} + \beta u_t, \text{ starts at } b_0; \\
s_t = s_{t-12} + \gamma u_t
\end{cases}
$$

$$
y_{T+1} = \ell_T + b_T + s_{T-11} + u_{T+1}
$$

$$
(y_{T+1} \mid \mathcal{F}_T) \sim \mathcal{N}(\ell_T + b_T + s_{T-11}; \sigma^2)
$$

Forecast 2 steps ahead

$$\begin{cases} y_t = \ell_{t-1} + b_{t-1} + s_{t-12} + u_t; \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t, \text{ starts at } \ell_0; \\ u_t \sim \mathcal{N}(0; \sigma^2) \text{ and are independent} \\ b_t = b_{t-1} + \beta u_t, \text{ starts at } b_0; \\ s_t = s_{t-12} + \gamma u_t \end{cases}$$

$$y_{T+2} = \ell_{T+1} + b_{T+1} + s_{T-10} + u_{T+2} = (\ell_T + b_T + \alpha u_{T+1}) + \\ + (b_T + \beta u_{T+1}) + s_{T-10} + u_{T+2}$$

$$(y_{T+2} \mid \mathcal{F}_T) \sim \mathcal{N}(\ell_T + 2b_T + s_{T-10}; \sigma^2((\alpha + \beta)^2 + 1))$$

Decomposition for free!

Consider the output of ETS(AAA):

Parameter estimates: $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\sigma}^2$, $\hat{\ell}_0$, $\hat{b}_0$, $\hat{s}_0$, $\hat{s}_{-1}$, ..., $\hat{s}_{-11}$.

Constraints: $\hat{s}_0 + \hat{s}_{-1} + \ldots + \hat{s}_{-11} = 0$.

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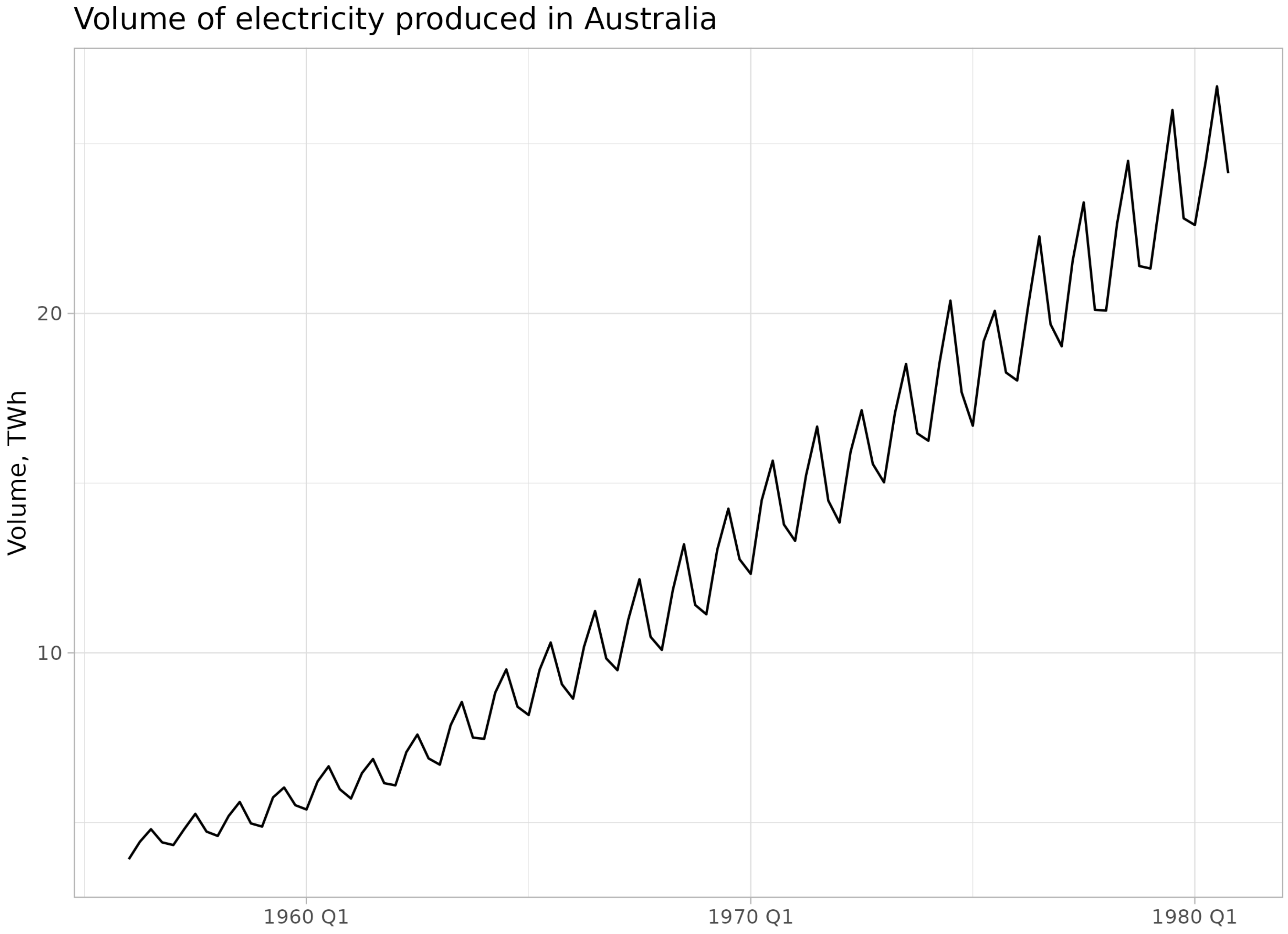
Constraints: $\hat{s}_0 + \hat{s}_{-1} + \ldots + \hat{s}_{-11} = 0$.

Estimated component values: $\hat{\ell}_t$, $\hat{b}_t$, $\hat{s}_t$.

We automatically get decomposition: $y_t = \hat{\ell}_t + \hat{s}_t + remainder_t$.

Oscillation amplitude can vary



Various oscillation amplitude

Possible solutions:

- Switch to logarithms, $y_t \to \ln y_t$

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ETS(MNM): equations

ETS(MNM) for monthly data:

$$\begin{cases} y_t = \ell_{t-1} \cdot s_{t-12} \cdot (1 + u_t); \\ \ell_t = \ell_{t-1} \cdot (1 + \alpha u_t), \text{ starts at } \ell_0; \\ s_t = s_{t-12} \cdot (1 + \gamma u_t), \text{ starts at } s_0, \ldots, s_{-11}; \\ u_t \sim \mathcal{N}(0; \sigma^2) \text{ and are independent} \end{cases}$$

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ETS(ANA):

$$\begin{cases} y_t = \ell_{t-1} + s_{t-12} + u_t; \\ \ell_t = \ell_{t-1} + \alpha u_t, \text{ starts at } \ell_0; \\ s_t = s_{t-12} + \gamma u_t, \text{ starts at } s_0, \ldots, s_{-11}; \end{cases}$$

Units

Series y_t, ℓ_t — initial units.

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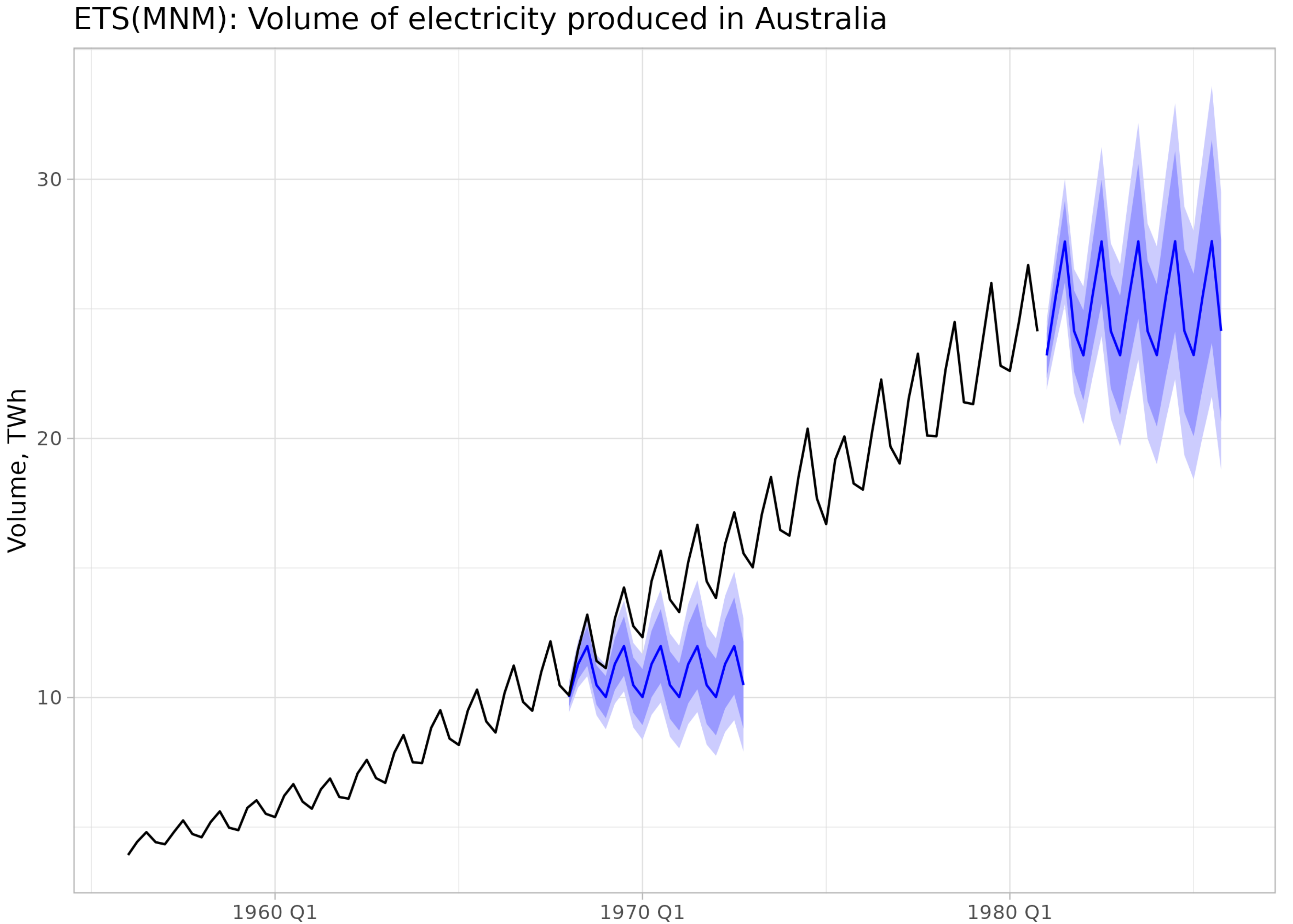
Series y_t, ℓ_t — initial units.

The s_t series is measured relative to one, for example, $s_t = 0.9$ — 10% below the trend.

The u_t series is measured relative to zero, for example, $u_t = -0.1$ — a 10% drop.

ETS(MNM): Forecasting



Which option to choose?

Different amplitude of fluctuations: indication of multiplicative models.

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Automatic selection based on the AIC criterion works.

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You can get the ETS(AAdA) model with seasonality.

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Different amplitude of fluctuations: indication of multiplicative models.

Automatic selection based on the AIC criterion works.

You can get the ETS(AAdA) model with seasonality.

Some of the multiplicative models can be numerically unstable or not implemented in the software.

ETS Model: Summary

- The slope of the trend and seasonality may change

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ETS Model: Summary

- The slope of the trend and seasonality may change
- Automatic decomposition into components
- Multiplicative models take into account changing oscillation amplitudes
- A lot of possible combinations

Theta method

Theta method: Plan

- An unexpected leader

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Theta method: Plan

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- Special case of ETS

Theta method

Appeared in 2000 and became a sensation at the competition M3 for predicting series.

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Works for non-seasonal series.

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Initially suggested without a statistical model.

Author's version

1. Expand the series into two theta lines ($\theta = 0$, $\theta = 2$)

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You can pre-delete seasonality and add it back in the end

What is a theta line?

Zero theta line — regression on time:

$$\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 t$$

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Theta line for arbitrary theta:

$$\Delta^2 y_t^{new} = \theta \Delta^2 y_t$$

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- The zero theta line catches the long-term trend of the series
- Theta line ($\theta = 2$) catches the short-term trend: acceleration of theta line is θ times stronger than of the initial series
- Averaging reduces the variance of predictions

How is the theta line selected?

We take $\theta = 2$:

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The new series y_t^{new} is completely determined by y_1^{new}, y_2^{new}

We solve the optimization problem:

$$\sum_{t=1}^{T} (y_t - y_t^{new})^2 \to \min$$

Statistical Model

Formal model appeared in 2003:

$$\begin{cases} y_t = \ell_t + b + u_t; \\ \ell_t = \ell_{t-1} + b + \alpha u_t; \\ \ell_1 = y_1 \end{cases}$$

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Or:

$$\Delta y_t = b + (\alpha - 1) u_{t-1} + u_t$$

Theta method — ETS variant

A special case of a more general model — ETS(AAN):

$$\begin{cases} y_t = \ell_{t-1} + b_{t-1} + u_t; \\ \ell_t = \ell_{t-1} + b_{t-1} + \alpha u_t, \text{ starts at } \ell_1; \\ b_t = b_{t-1} + \beta u_t, \text{ starts at } b_0; \\ u_t \sim \mathcal{N}(0; \sigma^2) \text{ and are independent.} \end{cases}$$

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Nuances of initialization are possible.

Theta method: Summary

- Works well for non-seasonal data

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- Works well for non-seasonal data
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