ARIMA processes

Stationary processes

Stationary processes: Plan

- Definition

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- Definition
- Random walk

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- Autocovariance function

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- ACF, PACF

Stationary processes

A stochastic process whose characteristics do not change over time

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Weak or wide-sense stationarity

A process (y_t) is said to be weakly stationary, if for each t and k:

$$\begin{cases} \mathbb{E}(y_t) = \mu \\ \mathrm{Cov}(y_t, y_{t+k}) = \gamma_k \end{cases}$$

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Strong or strict-sense stationarity

A process (y_t) is said to be strictly stationary, if for each k joint distribution of a r.v. $(y_t, y_{t+1}, y_{t+2}, \ldots, y_{t+k})$ does not depend on t

Stationary process: example

Independent observations

The quantities (y_t) are independent and equally distributed with finite expectation μ_y and finite variance σ_y^2

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$$\gamma_k = \mathrm{Cov}(y_t, y_{t+k}) = 0, \text{ for } k \geq 1$$

Non-Stationary Process Example

Random Walk

$$\begin{cases} y_0 = \mu \\ y_t = y_{t-1} + u_t, \text{ for } t \geq 1 \end{cases},$$

where u_t is white noise

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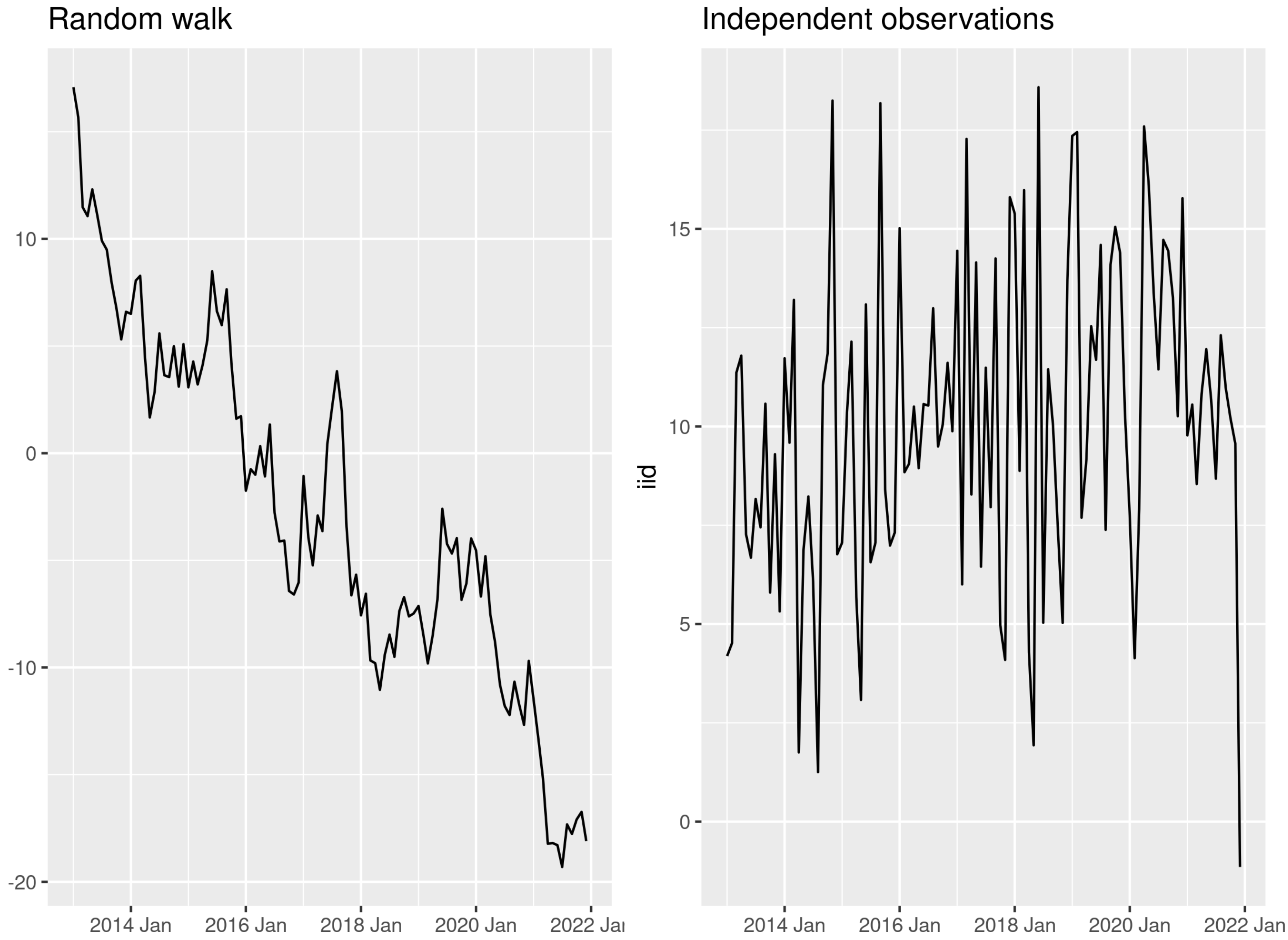
$$\gamma_k = \mathrm{Cov}(y_t, y_{t+k}) = \mathrm{Cov}(y_t, y_t + u_{t+1} + \ldots + u_{t+k}) = \mathrm{Var}(y_t)$$

Random Walk vs Random Sample



Autocovariance function

Definition

For a stationary process (y_t), the function $\gamma_k = \mathrm{Cov}(y_t, y_{t+k})$ is called autocovariance

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ACF: autocorrelation function

$$\rho_k = \mathrm{Corr}(y_t, y_{y+j}) = \frac{\mathrm{Cov}(y_t, y_{y+j})}{\sqrt{\mathrm{Var}(y_t)\,\mathrm{Var}(y_{t+k})}} =$$

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Partial Correlation

Definition

$$\mathrm{pCorr}(U, D; R_1, R_2, \ldots, R_n) = \mathrm{Corr}(U^*, D^*), \text{ where}$$
$$U^* = U - Best(U; R_1, R_2, \ldots, R_n),$$
$$D^* = D - Best(D; R_1, R_2, \ldots, R_n)$$

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The values U^* and D^* are the versions of U and D uninfluenced by the covariates $R_1, \ldots, R_n$

$$\mathrm{Cov}(U^*, R_i) = 0, \quad \mathrm{Cov}(D^*, R_i) = 0.$$

PACF

Definition

For a stationary process (y_t) the function

$$\varphi_{kk} = \mathrm{pCorr}(y_t, y_{t+k}; y_{t+1}, \ldots, y_{t+k-1})$$

is called partial autocorrelation

ACF and PACF: Intuition

For stationary process

- ACF:

$$\rho_k = \mathrm{Corr}(y_t, y_{t+k})$$

  Joint strength of relationship between y_t and y_{t+k}

- PACF:

$$\varphi_{kk} = \mathrm{pCorr}(y_t, y_{t+k}; y_{t+1}, \ldots, y_{t+k-1})$$

  Strength of relationship between y_t and y_{t+k} with the links through intermediate observations being broken

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- Autocovariance function
- Partial correlation — correlation with the effect of a set of controlling random variables removed
- In the time series, we removed the effect of intermediate observations

MA Process

MA Process: Plan

- Definition and notations with lags

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- Predictability

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Lag operator

Definition

For the process (y_t) defined at $t \in \mathbb{Z}$, lagged process Ly_t is the same sequence of values with a shifted index,

$$Ly_t = y_{t-1}$$

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$$\Delta_{12} y_t = y_t - y_{t-12} = (1 - L^{12}) y_t$$

MA process

Definition

Process (y_t), which can be represented as

$$y_t = \mu + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_q u_{t-q},$$

where $\alpha_q \neq 0$ and (u_t) is white noise, is called the $MA(q)$ process

MA — Moving Average

MA process

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MA — Moving Average

Example $MA(1)$ process:

$$y_t = 5 + u_t + 0.3 u_{t-1},$$

where (u_t) is some white noise

Notation with lags

MA with lag polynomial

Process (y_t), which can be represented as

$$y_t = \mu + P(L)u_t,$$

where $P(L)$ is a polynomial of degree q in lag L with $P(0) = 1$, and (u_t) is white noise, is called $MA(q)$ a process

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An example $MA(2)$ process:

$$y_t = 5 + (1 - 0.2L + 0.3L^2)u_t,$$

where (u_t) is white noise

ACF and Forecasts

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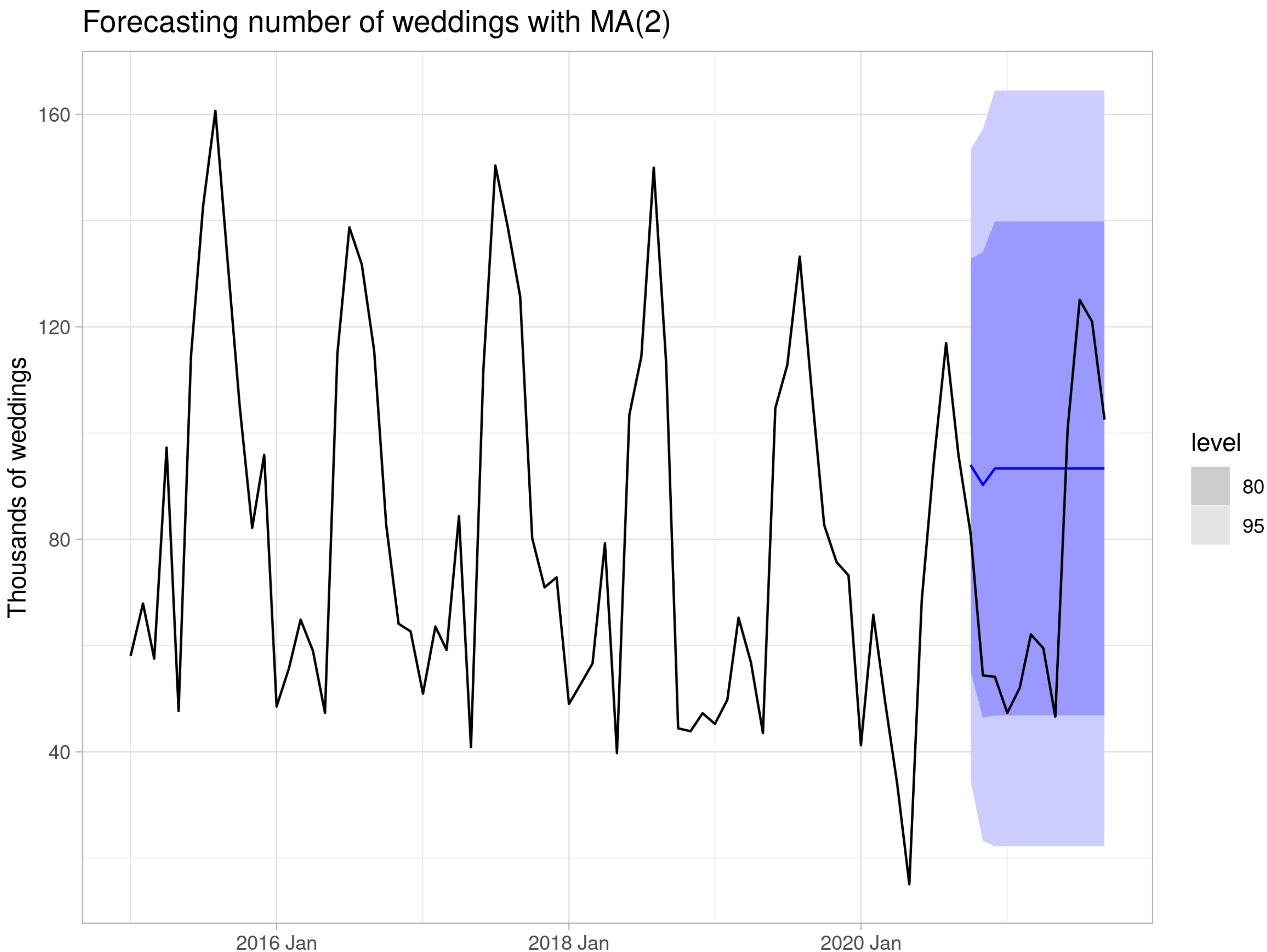
Zero $\rho_k = 0$ for $k > q$ implies independence of y_t and y_{t+k}.

Forecasts more than q steps ahead are exactly the same.

$$(y_{T+q+1} \mid \mathcal{F}_T) \sim (y_{T+q+2} \mid \mathcal{F}_T) \sim (y_{T+q+3} \mid \mathcal{F}_T) \sim \ldots$$

Predictions for $MA(2)$



$MA(\infty)$

Definition

Process (y_t), which can be represented as

$$y_t = \mu + u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \dots,$$

where (u_t) is white noise, an infinite number of $\alpha_i \neq 0$ and $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$, is called the $MA(\infty)$ process

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$MA(\infty)$:

$$y_t = 5 + u_t + 0.5u_{t-1} + 0.5^2 u_{t-2} + 0.5^3 u_{t-3} + \dots$$

$MA(\infty)$

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$MA(\infty)$:

$$y_t = 5 + u_t + 0.5 u_{t-1} + 0.5^2 u_{t-2} + 0.5^3 u_{t-3} + \dots$$

And this is not allowed:

$$y_t = 5 + u_t + \frac{1}{\sqrt{2}} u_{t-1} + \frac{1}{\sqrt{3}} u_{t-2} + \frac{1}{\sqrt{4}} u_{t-3} + \dots$$

Convergences

Theorem

If a $\sum_{i=0}^{\infty} \alpha_i^2 < \infty$ and (u_t) is a zero-mean stationary process, then the sequence of partial sums y_t^q of the form

$$y_t^q = \mu + \sum_{i=0}^{q} \alpha_i u_{t-i}$$

converges for $q \to \infty$ in mean, in probability, and in distribution

Nuance: the convergence of the weighted sum is guaranteed for the stationary (u_t)

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Bonus

…and the resulting process (y_t) is stationary

Wald's Theorem

Theorem

If (y_t) is a stationary process, then it can be represented as:

$$y_t = \sum_{i=0}^{\infty} \alpha_i u_{t-i} + r_t,$$

where

- (u_t) — white noise,
- $\sum \alpha_i^2 < \infty$,
- r_t is a linear predictable random process,
- $\operatorname{Cov}(u_t, r_t) = 0$

Predictable Process

Correct definition

A process (r_t) is called linearly predictable if

- (r_t) is stationary,
- $r_t = \beta_0 + \beta_1 r_{t-1} + \beta_2 r_{t-2} + \ldots + \beta_p r_{t-p}$

Invertibility condition

Characteristic representation

The equation $MA(q)$ of the process satisfies the invertibility condition if the characteristic polynomial $\phi(\lambda)$ has all roots $|\lambda_i| < 1$

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Lag representation

The equation $MA(q)$ of the process satisfies the invertibility condition if all roots of the lag polynomial $P(L)$ are $|\ell_i| > 1$

Example of invertible $MA(1)$ notation

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Nuance

Difference

Stationarity is a property of the (y_t) process itself.

Invertibility is a property of the equation (process notation) for (y_t).

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$MA(q)$ has a single notation when invertible

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- $MA(q)$ is a stationary process
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- Invertibility condition: roots of the characteristic polynomial $|\lambda_i| < 1$ or roots of the lag polynomial $|\ell_i| > 1$.

ARMA equation

ARMA equation: Plan

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- Non-uniqueness of solutions

About the purpose of old problems

Goal: A simple equation for a wide variety of processes

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- Non-uniqueness of equation for one process

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Let's try adding lags y_t to the equation!

New problem

$y_t - y_{t-1} = u_t - u_{t-1}$, where (u_t) is white noise

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Solutions:

- $y_t = u_t$;
- $y_t = u_t - 0.7$;

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Solutions:

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Infinite number of solutions

ARMA equation

Definition

Equation

$$y_t = c + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_q u_{t-q},$$

where (u_t) is white noise, we'll call an $ARMA$ equation

ARMA — Autoregression and Moving Average

ARMA equation

Definition

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ARMA — Autoregression and Moving Average

Definition

An equation of the form $P(L)y_t = c + Q(L)u_t$, where (u_t) is white noise, $P(L)$ and $Q(L)$ are lag polynomials with $P(0) = Q(0) = 1$, we'll call an $ARMA$ equation

ARMA equation: Summary

An equation is not a process!

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Why?

- One equation has many solutions

ARMA equation: Summary

An equation is not a process!

Why?

- One equation has many solutions
- One process can be described by several equations

ARMA process

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- Equation irreducibility

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- Solution structure

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Irreducibility of Equation

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Irreducible equation:

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- $y_0 = 2u_1$

$y_1 = 2u_1, y_2 = u_2 + u_1, y_3 = u_3 + 0.5u_2 + 0.5u_1, \ldots$

Initial Conditions

Irreducible $ARMA$ equation:

$$y_t = 0.5 y_{t-1} + u_t, \text{ where } (u_t) \text{ is white noise}$$

Let's try different initial conditions:

- $y_0 = 0$

$y_1 = u_1, y_2 = u_2 + 0.5u_1, y_3 = u_3 + 0.5u_2 + 0.25u_1, \dots$

- $y_0 = 2u_1$

$y_1 = 2u_1, y_2 = u_2 + u_1, y_3 = u_3 + 0.5u_2 + 0.5u_1, \dots$

The initial conditions also determine the past y_t!

ARMA Solutions

Theorem I

Any $ARMA$ equation with at least one y_t lag has an infinite number of solutions

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$$y_t = 0.6y_{t-1} + 0.08y_{t-2} + u_t \text{ and } y_0 = u_0, y_1 = u_0 + 4$$

And how many stationary solutions?

Correct theorem

If an $ARMA$ equation $P(L)y_t = c + Q(L)u_t$ is irreducible, then it

- has exactly one stationary solution if the lag polynomial $P(\ell)$ has all roots $|\ell_i| \neq 1$;
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AR process

Definition

$AR(p)$ process with equation

$$y_t = c + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + u_t,$$

where (u_t) is white noise and $\beta_p \neq 0$, is the solution of this equation in the form of $MA(\infty)$ with respect to (u_t)

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Definition with lags

$AR(p)$ process with equation

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where (u_t) is white noise, $P(L)$ has power p and $P(0) = 1$, is the solution of this equation in the form of $MA(\infty)$ with respect to (u_t)

Definitions by different authors

In our definition of $AR(p)$, the process is necessarily stationary.

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Some authors do not include the requirement of stationarity in the definition of $AR(p)$.

ARMA process

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$ARMA(p,q)$ process with equation

$$y_t = c + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_q u_{t-q},$$

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where (u_t) is white noise, $P(L)$ and $Q(L)$ have powers p and q, respectively, are irreducible and $P(0) = Q(0) = 1$, is the solution of this equation in the form $MA(\infty)$ with respect to (u_t)

What about non-uniqueness?

The same $ARMA(p, q)$ process (y_t) can be described by different equations!

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If the series (y_t) is an $ARMA(p,q)$ process with the equation $P(L)y_t = c + Q(L)u_t$, then this equation will be unique if the MA part satisfies the invertibility condition.

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Condition for invertibility of the $ARMA$ equation:

- the characteristic polynomial $\phi_{MA}(\lambda)$ has all roots $|\lambda_i| < 1$;
- the lag polynomial $Q(L)$ has all roots $|\ell_i| > 1$

ARMA process: Summary

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- Theoretical ACF and $PACF$ decrease exponentially
- The invertibility condition guarantees uniqueness
- An irreducible equation either has a unique stationary solution or it does not exist

ARIMA process

ARIMA process: Plan

- Stationarity of $ARMA$

ARIMA process: Plan

- Stationarity of $ARMA$
- Definition of $ARIMA$

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Nuances

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- In the canonical notation $ARMA(p, q)$ of the process $P(L)y_t = c + Q(L)u_t$ for the polynomial $P(L)$ all roots $|\ell| > 1$
- When estimating the $ARMA(p, q)$ process by the maximum likelihood method, these restrictions are imposed a priori

What to do with non-stationary processes?

Definition

The random process (y_t) is called the $ARIMA(p, 1, q)$ w.r.t. the white noise process (u_t), if (y_t) is non-stationary, but Δy_t is a stationary $ARMA(p, q)$ process w.r.t. the white noise (u_t)

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ARIMA — AutoRegressive Integrated Moving Average

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  $\ln L(y_1,\ldots,y_n \mid \theta)$ and $\ln L(y_2,\ldots,y_n \mid \theta, y_1)$ and $\ln L(y_3,\ldots,y_n \mid \theta, y_1, y_2)$ incomparable!

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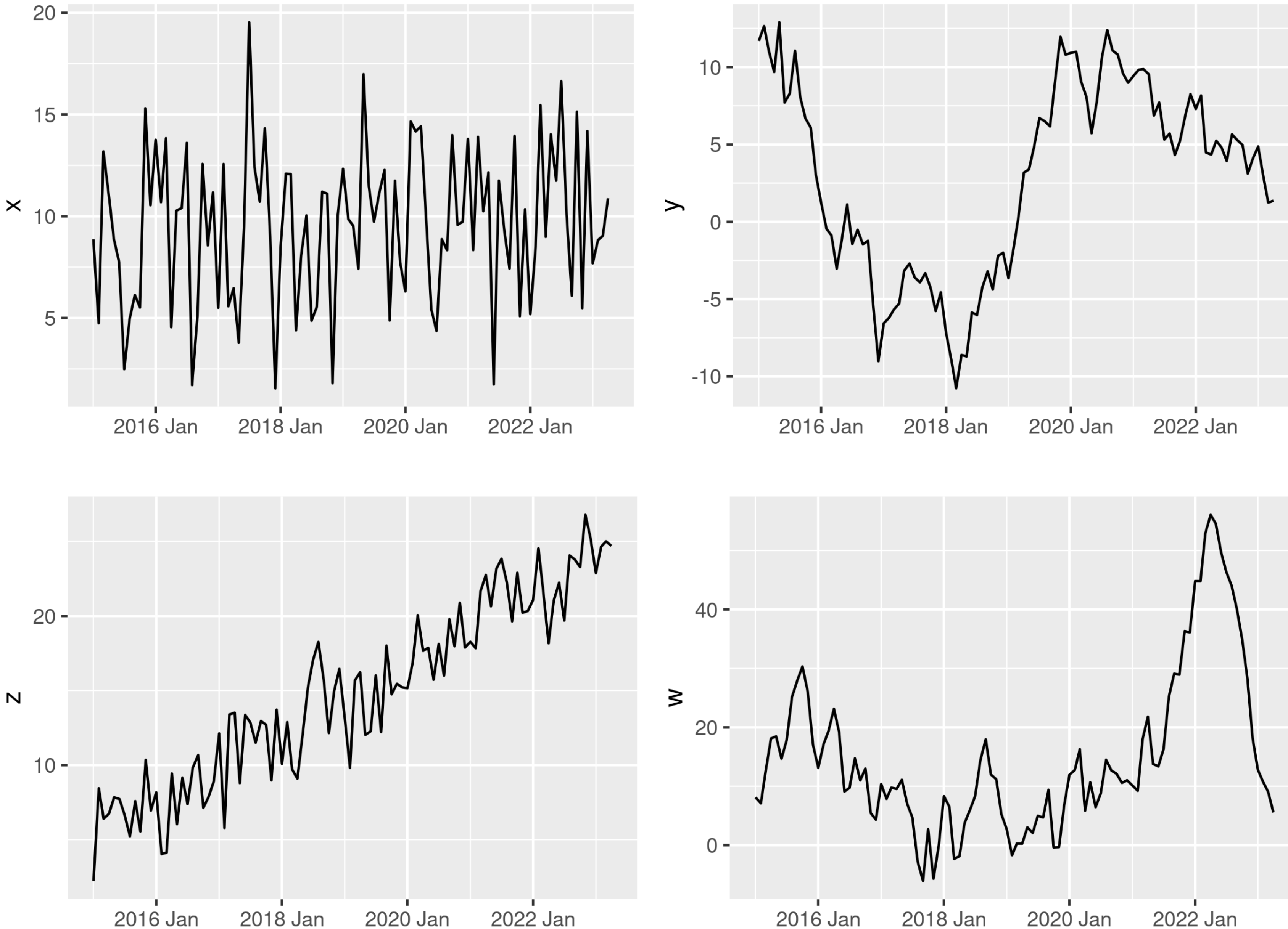
- There are unit root tests!
  ADF, KPSS, PP, …

Analysing the graphs



ARIMA: Summary

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- $ARMA$ is for stationary series
- Sometimes Δy_t or $\Delta^2 y_t$ is stationary
- Choose between $ARMA$ and $ARIMA$

SARIMA process

SARIMA process: Plan

- Seasonal $ARMA$

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SARIMA process: Plan

- Seasonal $ARMA$
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Seasonality and $ARIMA$

Using $ARMA$ and $ARIMA$ models, we can model seasonality!

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$$MA(12) : y_t = c + u_t + a_1 u_{t-1} + a_2 u_{t-2} + \ldots + a_{12} u_{t-12}.$$

$$ARIMA(12,1,0) : \Delta y_t = c + u_t + b_1 \Delta y_{t-1} + \ldots + b_{12} \Delta y_{t-12}.$$

ARMA should be economical!

Let's focus on non-zero coefficients!

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Definition

If the stationary $ARMA$ model for y_t can be written with fewer parameters as

$$P_{non}(L)P_{seas}(L^{12})y_t = c + Q_{non}(L)Q_{seas}(L^{12})u_t,$$

where the degrees of the lag polynomials are $\deg P_{non} = p$, $\deg P_{seas} = P$, $\deg Q_{non} = q$, $\deg Q_{seas} = Q$, then it is also called $SARMA(p, q)(P, Q)[12]$

Examples

- $SARMA(1, 0)(0, 2)[12]$

$$(1 - b_1 L)y_t = c + (1 + d_1 L^{12} + d_2 L^{24})u_t;$$

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$$(1 - f_1 L^{12})y_t = c + (1 + a_1 L + a_2 L^2)u_t;$$

- $SARMA(1,2)(2,1)[12]$

$$(1 - f_1 L^{12} - f_2 L^{24})(1 - b_1 L^1)y_t = c + (1 + a_1 L + a_2 L^2)(1 + d_1 L^{12})u_t$$

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By analogy with the difference $\Delta y_t = y_t - y_{t-1}$, we can consider the seasonal difference $\Delta_{12} y_t = y_t - y_{t-12}$

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d is the number of times the first difference should be taken $\Delta = 1 - L$;

D is the number of times the seasonal difference should be taken $\Delta_{12} = 1 - L^{12}$; $y_t \sim SARIMA(0, 0, 2)(1, 1, 2)[12]$ means that

$\Delta_{12} y_t \sim SARMA(0, 2)(1, 2)[12]$

How to choose?

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- You cannot use AIC!

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  And rules of thumb…

STL decomposition and the power of seasonality

Step 1. Find the STL expansion of the series (y_t)

$$y_t = trend_t + seas_t + remainder_t$$

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Step 2. Calculate the strength of seasonality

$$F_{seas} = \max\left\{1 - \frac{\mathrm{sVar}(remainder)}{\mathrm{sVar}(seas + remainder)}, 0\right\}$$

Step 3. If the strength of seasonality is above the threshold, then move to $\Delta_{12} y_t = y_t - y_{t-12}$

SARIMA: Summary

- Seasonal ARIMA is more compact

SARIMA: Summary

- Seasonal ARIMA is more compact
- The strength of seasonality from the STL expansion is used to decide if a seasonal difference $\Delta_{12} y_t$ is needed

Unit root tests: ADF test

ADF test: Plan

- Test assumptions

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- Test assumptions
- Test algorithm

ADF test: Plan

- Test assumptions
- Test algorithm
- Three variations of the test

Why do we need an stationarity tests?

We want to answer the questions:

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Name "unit root test":

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We want to answer the questions:

- Should the $ARMA$ model be used for (y_t) or for (Δy_t)?
- How to include a constant in a model?

Name "unit root test":

$$\Delta = 1 - L = P(L)$$

The equation $1 - \ell = 0$ has a root $\ell = 1$

ADF test

ADF — Augmented Dickey Fuller test

ADF test

ADF — Augmented Dickey Fuller test

Three variations of the test: without a constant, with a constant, with a trend

ADF with constant

$$\Delta y_t = c + \beta y_{t-1} + d_1 \Delta y_{t-1} + \ldots + d_p \Delta y_{t-p} + u_t,$$

ADF with constant

$$\Delta y_t = c + \beta y_{t-1} + d_1 \Delta y_{t-1} + \ldots + d_p \Delta y_{t-p} + u_t,$$

H_0: $\beta = 0$

$\Delta y_t = m + x_t$;

(x_t) is a stationary $AR(p)$ process with $\mathbb{E}(x_t) = 0$;

$y_t = y_0 + mt + \sum_{i=1}^{t} x_i$

ADF with constant

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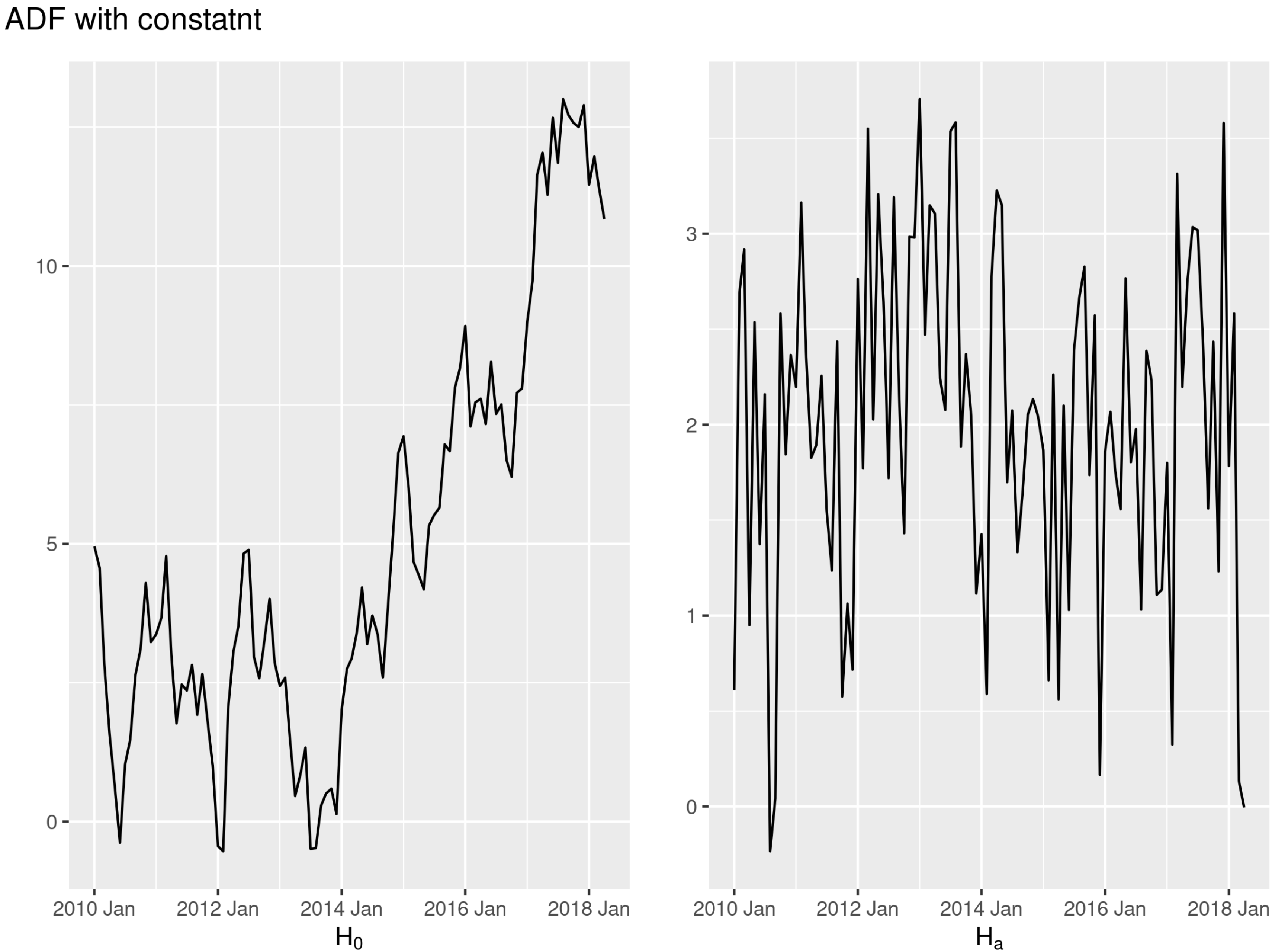
H_a: $\beta < 0$

(y_t) is a stationary $AR(p+1)$ process

ADF with constant: H_0 and H_a



ADF with constant: algorithm

Step 1. Evaluate regression

$$\widehat{\Delta y_t} = \hat{c} + \hat{\beta} y_{t-1} + \hat{d}_1 \Delta y_{t-1} + \ldots + \hat{d}_p \Delta y_{t-p}$$

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Step 3. We conclude:

If $ADF < DF^c$ then H_0 is rejected

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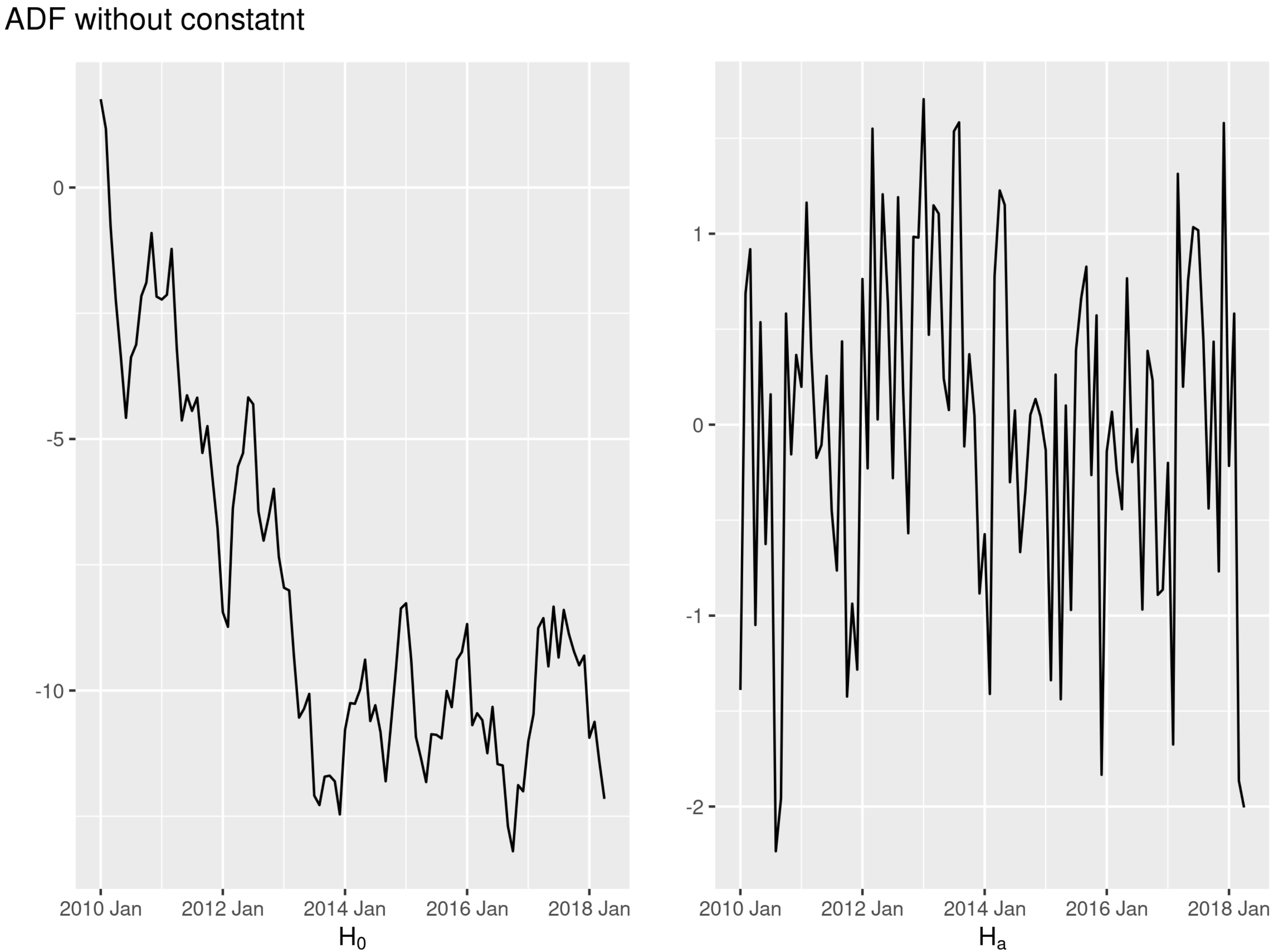
The algorithm will have regression without a constant and another distribution DF^0

ADF without constant: H_0 and H_a



ADF with trend

$$\Delta y_t = c + gt + \beta y_{t-1} + d_1 \Delta y_{t-1} + \ldots + d_p \Delta y_{t-p} + u_t,$$

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$\Delta y_t = k_1 + k_2 t + x_t$;

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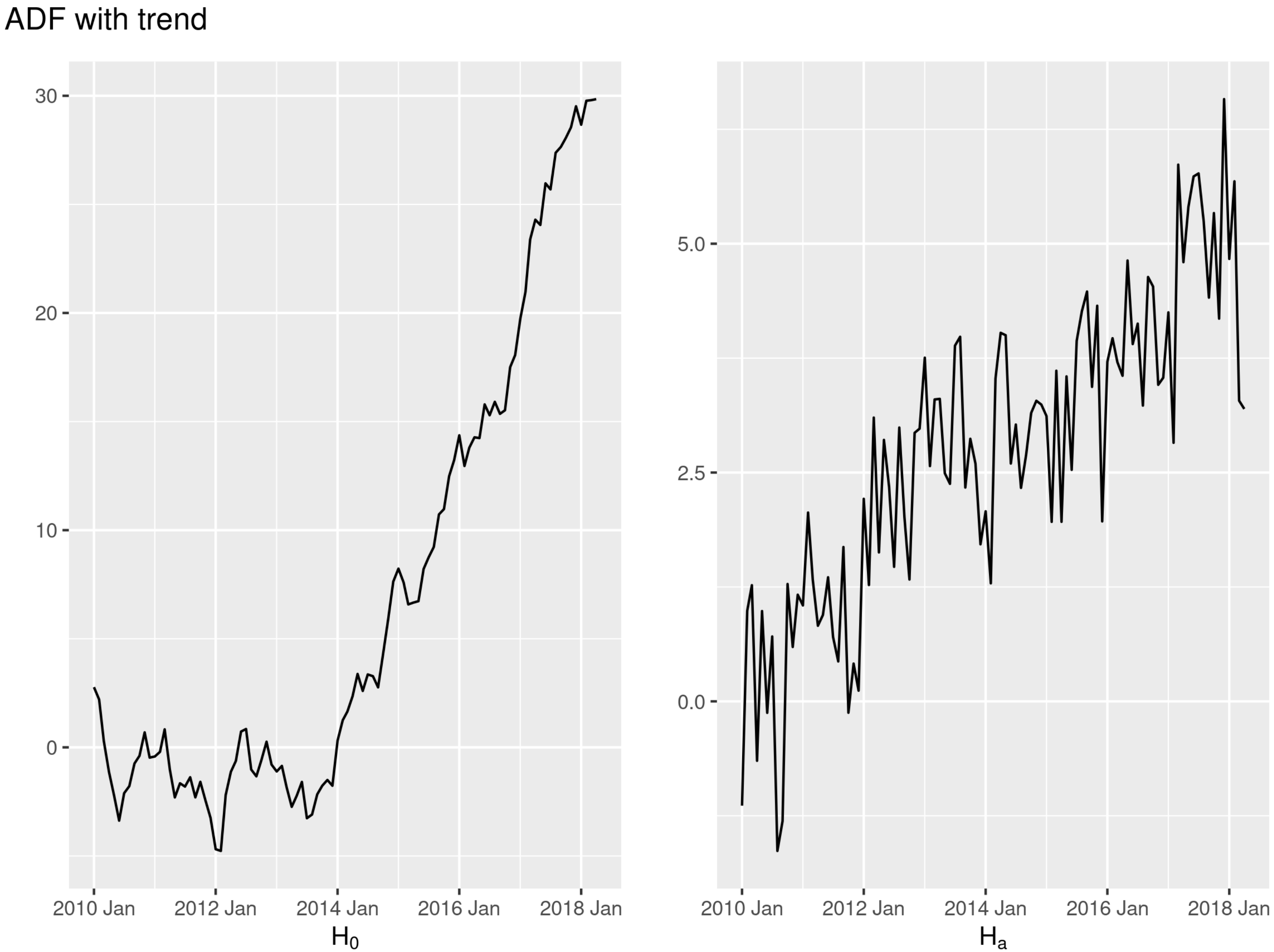

ADF test: Summary

- Applicable for making a decision about the transition to Δy_t

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- Applicable for making a decision about the transition to Δy_t
- Three variants of the ADF test with different assumptions

Unit root tests: KPSS test

KPSS test: Plan

- Long-term variance

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- Long-term variance
- Prerequisites for the test
- Two variations of the test

KPSS test

KPSS — Kwiatkowski–Phillips–Schmidt–Shin test

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Two variations of the test: with a constant, with a trend

Long-term variance

Definition

For a stationary process (y_t), the quantity λ^2 is called long-term variance if

$$\operatorname{Var}(\bar{y}) = \frac{\lambda^2}{T} + o(1/T)$$

or

$$\lim_{T \to \infty} T \operatorname{Var}(\bar{y}) = \lambda^2,$$

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Motivation

For independent observations with the constant variance

$$\mathrm{Var}(\bar{y}) = \frac{\sigma^2}{T}, \text{ where } \sigma^2 = \mathrm{Var}(y_i)$$

KPSS with constant

$$y_t = c + rw_t + x_t,$$

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H_a: $rw_t = rw_{t-1} + u_t$

$rw_0 = 0$;

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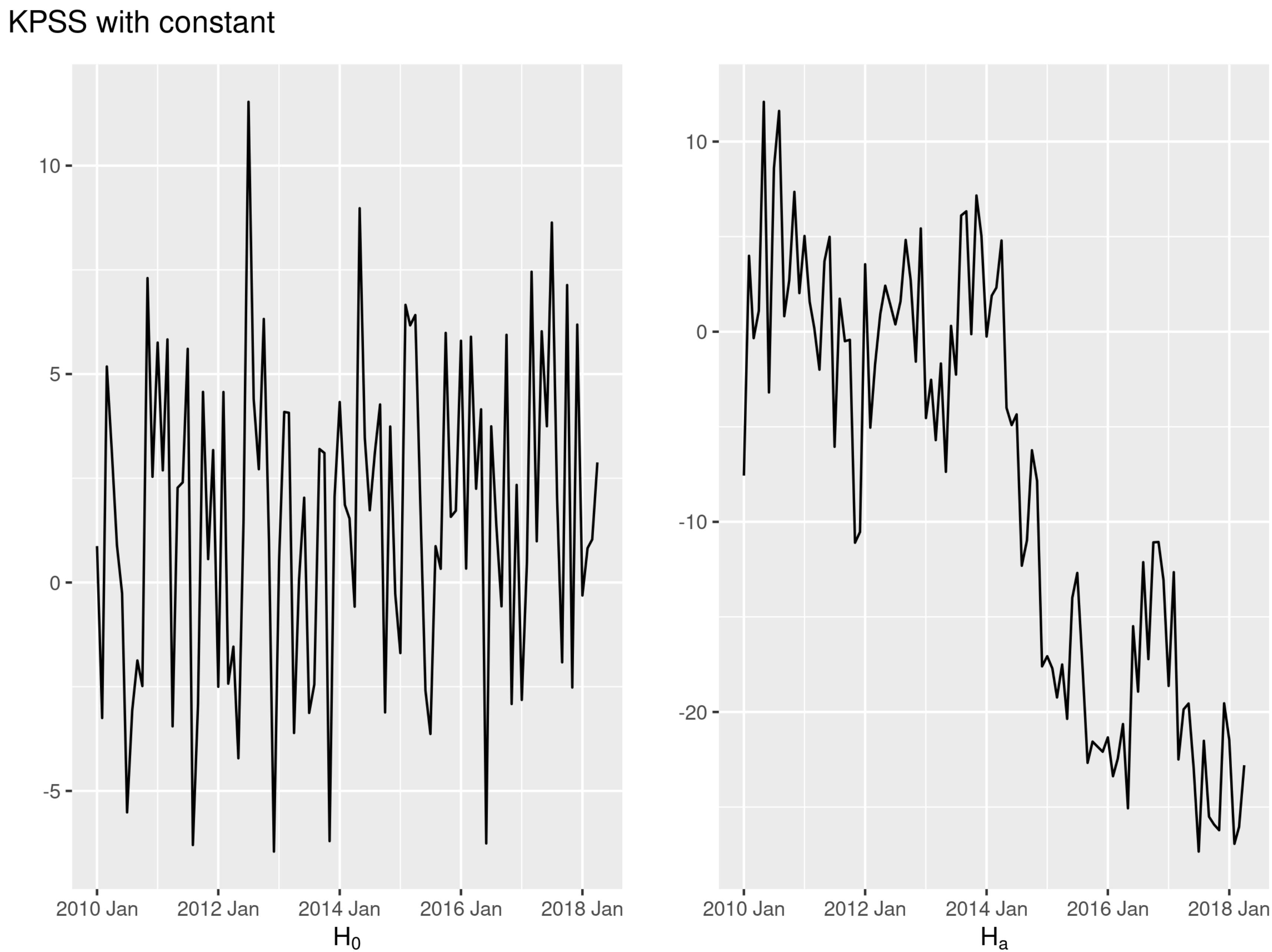
(u_t) is white noise independent of (x_t)

KPSS with constant: H_0 and H_a



KPSS with constant: algorithm

Step 1. Evaluate regression on a constant

$$\widehat{y_t} = \hat{c}$$

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Step 2. Calculate $KPSS$ statistics

$$KPSS = \frac{\sum_{t=1}^{T} S_t^2}{T^2 \hat{\lambda}^2},$$

where S_t is the accumulated sum of residuals, $S_t = \hat{u}_1 + \ldots + \hat{u}_t$,

and $\hat{\lambda}^2$ is a consistent estimator of the long-term variance.

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Under true H_0, the distribution of the $KPSS$-statistic converges to a special distribution with $KPSS^c$!

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Under true H_0, the distribution of the $KPSS$-statistic converges to a special distribution with $KPSS^c$!

Step 3. We conclude:

If $KPSS > KPSS^c$ then H_0 is rejected

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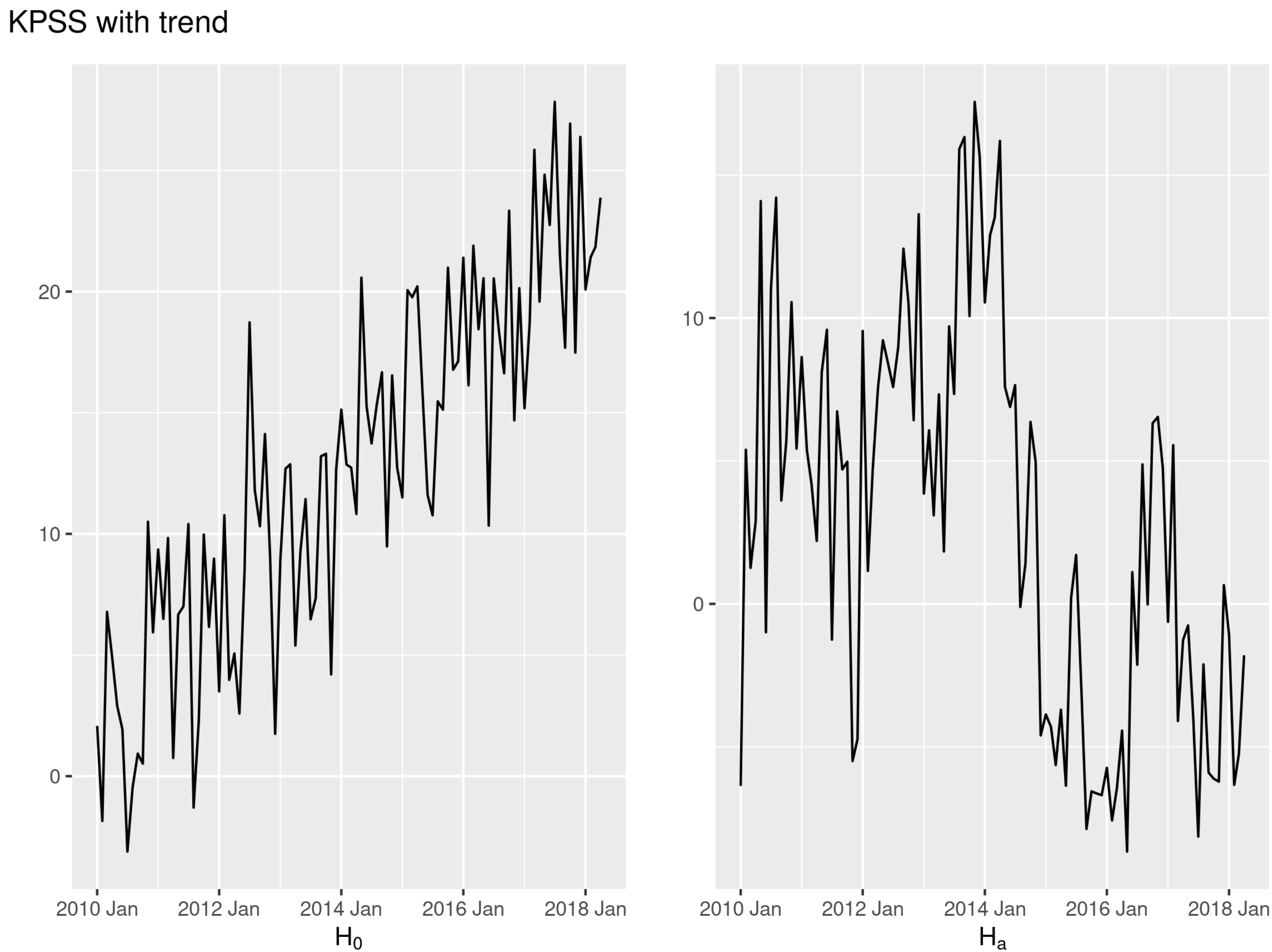
The first step of the algorithm will have a regression on a constant and a trend and the statistic under null will have another special distribution $KPSS^{ct}$

KPSS with trend: H_0 and H_a



Terminology

$$A. \quad y_t = a + bt + x_t$$

(y_t) — trend stationary (stationary around the trend)

(x_t) — a stationary process with $\mathbb{E}(x_t) = 0$

Terminology

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Recipe: Estimate regression $a + bt$ with $ARMA$ errors for (y_t).

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$$B. \quad y_t = a + \sum_{i=1}^{t} x_i \text{ or } y_t = a + bt + \sum_{i=1}^{t} x_i$$

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(y_t) — difference stationary (stationary in differences)

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Recipe: evaluate $ARMA$ for (Δy_t).

Both (y_t) are non-stationary!

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- Two versions of the KPSS test with different assumptions