Forecasting

Forecasting without a model

Forecasting without a model: Plan

- Converting time series into cross-sectional data

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- Add lags to the y_t variable

Forecasting without a model: Plan

- Converting time series into cross-sectional data
- Add lags to the y_t variable
- Use aggregation and sliding or growing windows

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Adding predictors

There are algorithms that, based on the training sample of the dependent variable y, learning matrix of predictors X, and new predictors X_F build a forecast $\hat{y}_F$

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You can average ARIMA/ETS forecasts and forecasts from other algorithms

How to create predictors?

From one column y you can create an entire matrix of predictors X!

- Use lags y_{t-k}

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From one column y you can create an entire matrix of predictors X!

- Use lags y_{t-k}
- Use functions of lags as predictors

Using y lags

For example, let's take two lags, Ly_t and L^2y_t

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Training sample:

$$\begin{pmatrix} y_3 \\ y_4 \\ y_5 \\ \vdots \\ y_T \end{pmatrix} \quad \begin{pmatrix} y_1 & y_2 \\ y_2 & y_3 \\ y_3 & y_4 \\ \vdots & \vdots \\ y_{T-2} & y_{T-1} \end{pmatrix}$$

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Sample for prediction:

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- Aggregate function:
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Typical Predictor

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- Sliding Window: The aggregate function can be applied to, say, the previous three values y_{t-1}, y_{t-2}, y_{t-3}.
- Growing Window: The aggregate function can be applied to all previous values y_{t-1}, y_{t-2}, $\ldots$, y_1.

Using y lag functions

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$$\begin{pmatrix} y_3 \\ y_4 \\ y_5 \\ \vdots \\ y_T \end{pmatrix} \qquad \begin{pmatrix} \max\{y_1, y_2\} & \min\{y_1, y_2\} \\ \max\{y_2, y_3\} & \min\{y_1, y_2, y_3\} \\ \max\{y_3, y_4\} & \min\{y_1, \ldots, y_4\} \\ \vdots & \vdots \\ \max\{y_{T-2}, y_{T-1}\} & \min\{y_1, \ldots, y_{T-1}\} \end{pmatrix}$$

Using y lag functions

For example, let's take the maximum as a sliding window and the minimum as a growing window

Training sample:

$$
\begin{pmatrix} y_3 \\ y_4 \\ y_5 \\ \vdots \\ y_T \end{pmatrix}
\qquad
\begin{pmatrix}
\max\{y_1, y_2\} & \min\{y_1, y_2\} \\
\max\{y_2, y_3\} & \min\{y_1, y_2, y_3\} \\
\max\{y_3, y_4\} & \min\{y_1, \ldots, y_4\} \\
\vdots & \vdots \\
\max\{y_{T-2}, y_{T-1}\} & \min\{y_1, \ldots, y_{T-1}\}
\end{pmatrix}
$$

Sample for prediction:

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\begin{pmatrix} ? \end{pmatrix}
\qquad
\begin{pmatrix} \max\{y_{T-1}, y_T\} & \min\{y_1, \ldots, y_T\} \end{pmatrix}
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More predictors!

More predictors: Plan

- Trend predictors

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- Trend predictors
- Seasonal and holiday dummy

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- Cosines and sines as predictors

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Let's take t and $\sqrt{t}$ as an example

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Sample for prediction:

$$\begin{pmatrix} ? \end{pmatrix} \quad \begin{pmatrix} T+1 & \sqrt{T+1} \end{pmatrix}$$

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- For algorithms based on decision trees (random forest, gradient boosting) additional monotonic time transformations are useless
- Be aware of the possible transformation of the original variable (logarithm, Box-Cox transformation)

Seasonal and holiday dummy

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Training sample for quarterly data:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ \vdots \\ y_T \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The dummy variable trap

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Algorithms based on decision trees (random forest, gradient boosting) are resistant to the dummy variable trap

Why do we need sines and cosines?

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- Multiplying the argument by a reduces the period by a factor of a

Fourier expansion

Theorem

Any continuous and differentiable function f with period 2π can be represented as

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Practical recipe for daily data:

- Add predictors $\cos\left(\frac{2\pi}{365} \cdot t\right)$ and $\sin\left(\frac{2\pi}{365} \cdot t\right)$

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- ...

More predictors: Summary

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- Seasonality in predictors can be reflected using dummy variables or using cosine and sine functions

Predictors and $ARIMA$

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- Regression with $ARMA$ errors

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where a_t and b_t are predictors

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$ARMAX$ model

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$ARMAX$ model is not completely equivalent to regression with $ARMA$ errors, but gives approximately the same quality of predictions

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- Estimators remain consistent if trend, seasonality dummy and trigonometric predictors are added
- Not any predictor makes it possible to obtain a consistent estimator of the coefficient
- Sometimes you can get good predictions even if the assumptions are violated

$ARDL$ model

ARDL — AutoRegressive Distributed Lag model

Autoregressive model with distributed lags

The $ARDL(p, q)$ model equation

$$y_t = c + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} +$$

$$+ x_t + \alpha_1 x_{t-1} + \ldots + \alpha_q x_{t-q} + u_t$$

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- Used to find a long-term relationships between time series
- If the model assumptions aren't violated, then the OLS estimators are consistent, although they are biased
- You can add multiple predictors with different number of lags

Predictors and $ARIMA$: Summary

- For stationary data you can use regression with $ARMA$ errors or $ARMAX$ model

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- Regression with $ARMA$ errors can be constructed for the differences
- For non-stationary series it is sometimes possible to use the $ARDL$ model

Model Quality

Model Quality: Plan

- Scale-based metrics

Model Quality: Plan

- Scale-based metrics
- Percentage-based metrics

Remember the goal!

If the goal of building a model is forecasts one step ahead, then it is reasonable to compare models in predictive strength one step ahead.

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If the goal of building a model is forecasts one step ahead, then it is reasonable to compare models in predictive strength one step ahead.

If the goal is to detect the moment of model discord, then it is reasonable to look for a model that gives the minimum error when there is no discord, and the maximum error when there is discord.

Notations for brevity

For the forecast, it is important when it is built, and for how many steps ahead:

$$\hat{y}_{t+h|t}$$

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Sometimes for short:

$$\hat{y}_{t+h}$$

Problem:

$$\hat{y}_{(t+1)+2} \neq \hat{y}_{(t+2)+1}$$

Anti-quality metrics

Forecast error: $e_{t+h} = y_{t+h} - \hat{y}_{t+h}$

Anti-quality metrics

Forecast error: $e_{t+h} = y_{t+h} - \hat{y}_{t+h}$

Mean Absolute Error:

$$MAE = \frac{|e_{T+1}| + |e_{T+2}| + \ldots + |e_{T+H}|}{H}$$

Anti-quality metrics

Forecast error: $e_{t+h} = y_{t+h} - \hat{y}_{t+h}$

Mean Absolute Error:

$$MAE = \frac{|e_{T+1}| + |e_{T+2}| + \ldots + |e_{T+H}|}{H}$$

Root Mean Squared Error:

$$RMSE = \sqrt{\frac{e_{T+1}^2 + e_{T+2}^2 + \ldots + e_{T+H}^2}{H}}$$

Scaling

Convert error $e_{t+h} = y_{t+h} - \hat{y}_{t+h}$ to percentage $p_t = e_t / y_t \cdot 100$
or $p_t^s = e_t / (0.5 y_t + 0.5 \hat{y}_t) \cdot 100$

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$$sMAPE = \frac{\left|p^s_{T+1}\right| + \left|p^s_{T+2}\right| + \ldots + \left|p^s_{T+H}\right|}{H}$$

Automatically compare with naive model

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Comparing q to 1 compares our model with the naive one

Model Quality: Summary

- MAE, RMSE

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- MAPE

Model Quality: Summary

- MAE, RMSE
- MAPE
- MASE

Model Comparison

Model Comparison: Plan

- Cross-validation

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- Akaike criterion

Train and test set

Strategy:

1. Split the entire sample into training (at the beginning) and test (at the end) sets

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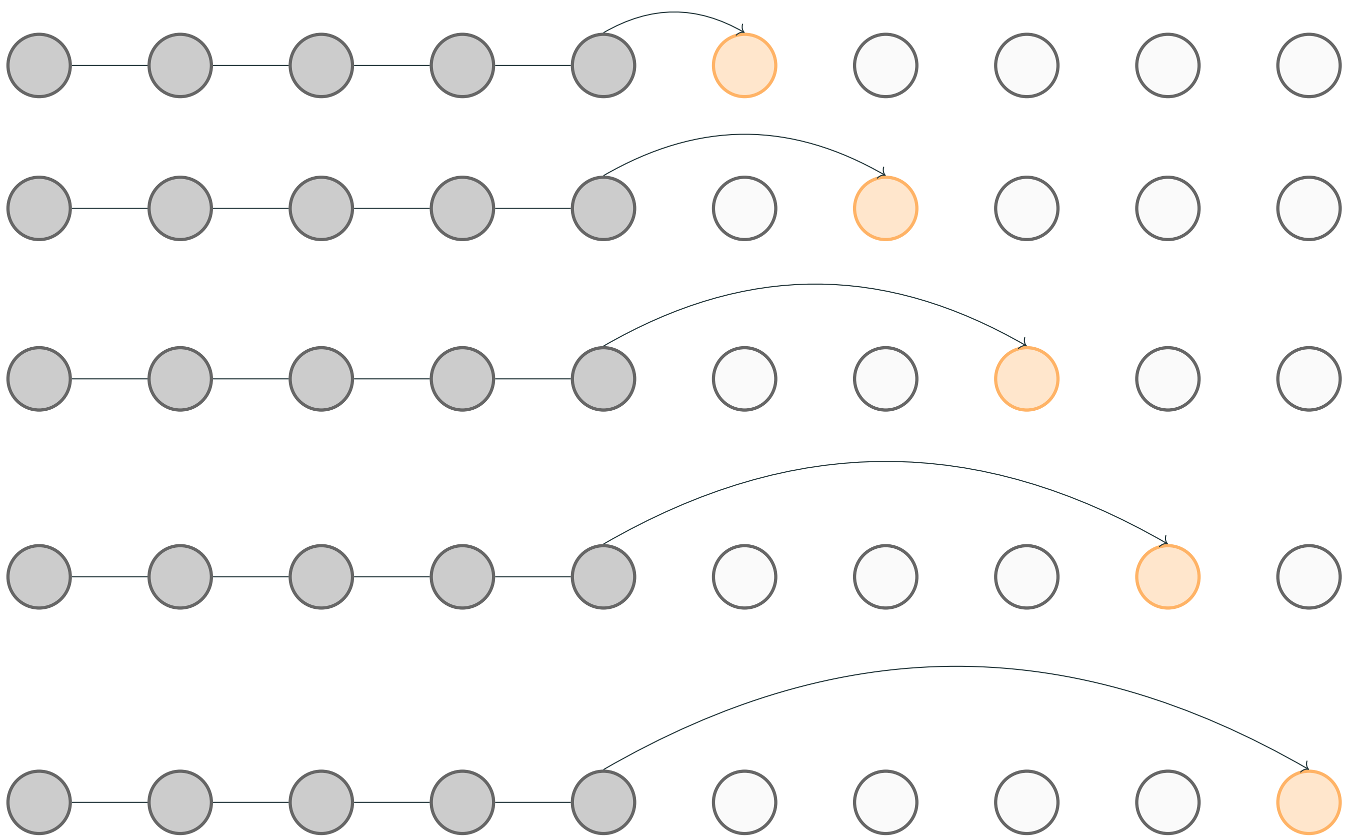
Disadvantage: forecasts have different horizons

Dividing into train and test



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6. Repeat steps 2-5

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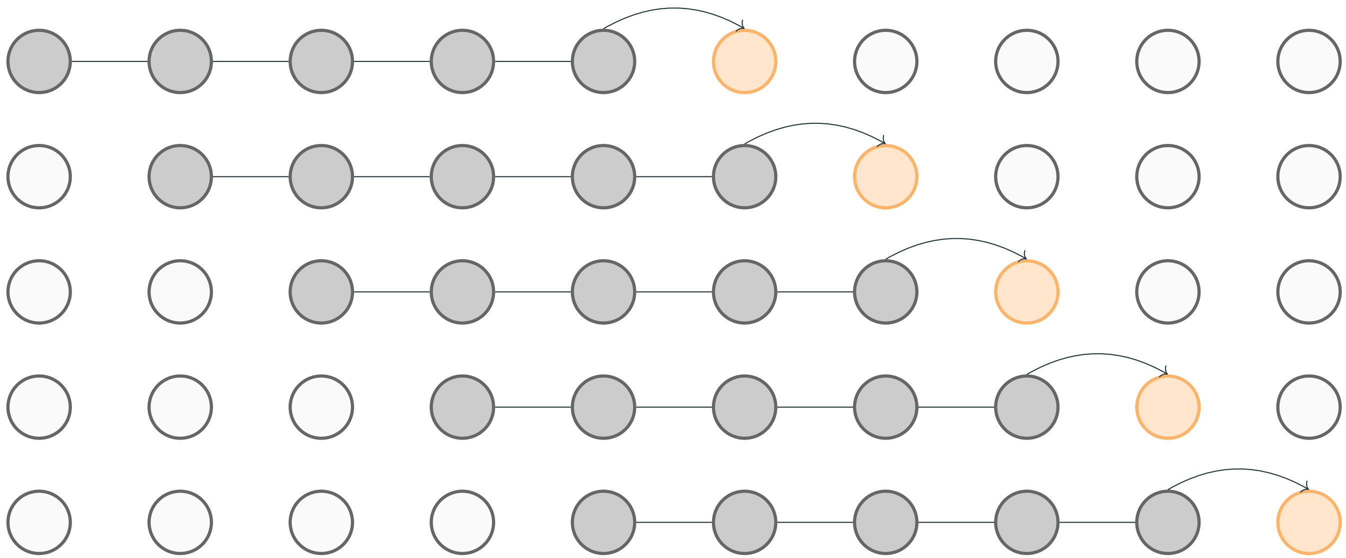
Sliding window Cross-validation

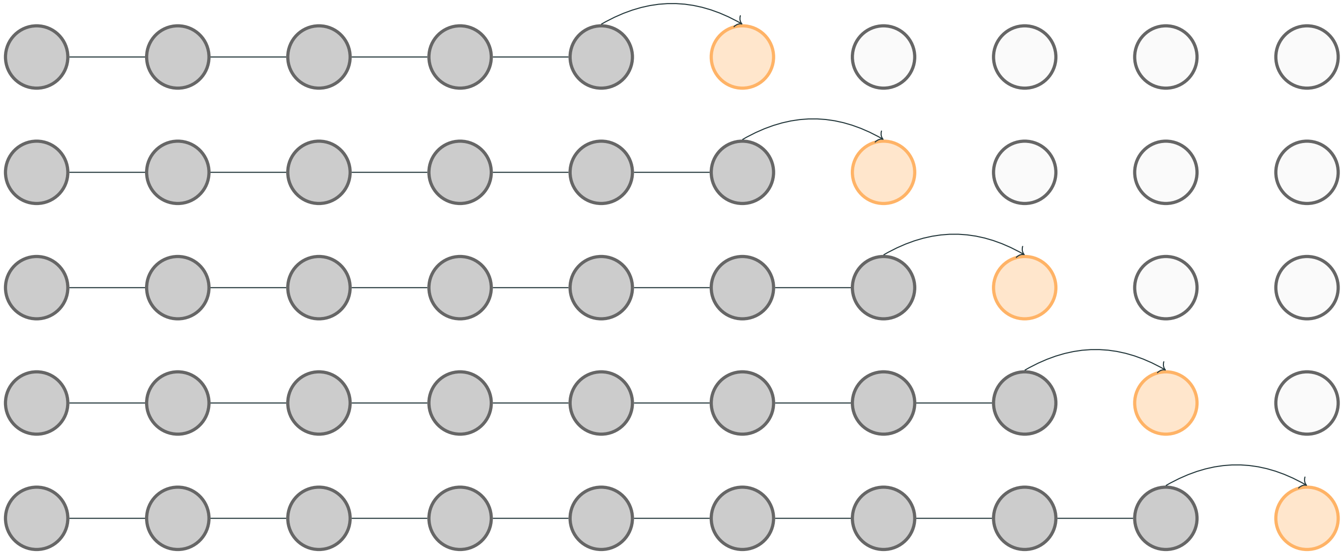


Growing window Cross-validation

Strategy:

1. Select the starting size for train sample (at the beginning)
2. Evaluate several models on the training set
3. Predict one step ahead with each model
4. Calculate prediction errors
5. Increase the training set by one observation.
6. Repeat steps 2-5
7. Compare models by MAE and choose the best one

Growing window Cross-validation



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Cross-validation can be time consuming!

Let's make cross-validation quicker!

Approximate cross-validation by one step forward based on $RMSE$ using... Akaike Information Criterion:

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Approximate cross-validation by one step forward based on $RMSE$ using... Akaike Information Criterion:

$$AIC = -2 \ln L + 2k,$$

where $\ln L$ is the logarithm of the maximum likelihood on the training set, k is the total number of model parameters

Nuances of AIC

- AIC has theoretical grounds:

$$\frac{AIC_A - AIC_B}{2} \approx KL(\text{Truth}||\text{Model A}) -$$

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- For Gaussian y_t models, the criterion approximates comparison over $RMSE$
- The models being compared must be for the same observations

Model Comparison: Summary

- Cross-validation: sliding and growing window

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- Cross-validation: sliding and growing window
- AIC is a fast and approximate analogue of cross-validation

Forecast comparison

Forecast comparison: Plan

- Diebold-Mariano test

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- Test assumptions

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DM test assumptions

Consider difference of two forecast losses:

$$d_t = e^2_{A,t} - e^2_{B,t}, \quad e_{\text{Model},t} = \hat{y}_{\text{Model},t} - y_t$$

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in particular,

$$\text{Var}(d_t) = \gamma_0$$

Test method

Under the correct $H_0 : \mu_d = 0$:

$$DM = \frac{\bar{d}}{se(\bar{d})} \to \mathcal{N}(0; 1),$$

where $se^2(\bar{d})$ is a consistent estimator for $\mathrm{Var}(\bar{d})$

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In practice, we evaluate the regression on a constant

$$\hat{d}_t = \hat{\beta}_1,$$

get $\hat{\beta}_1 = \bar{d}$ and use robust standard errors,

$$DM = \frac{\hat{\beta}_1}{se_{HAC}(\hat{\beta}_1)}.$$

How does robust estimators work?

Compare forecasts for P steps ahead,

$$\mathrm{Var}(\bar{d}) = \frac{\mathrm{Var}(d_1) + \mathrm{Var}(d_2) + \ldots + 2\,\mathrm{Cov}(d_1, d_2) + \ldots}{P^2}$$

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The naive estimator for the variance of $\widehat{\mathrm{Var}}(\bar{d})$ is

$$\frac{P\hat{\gamma}_0 + 2(P-1)\hat{\gamma}_1 + 2(P-2)\hat{\gamma}_2 + \ldots}{P^2}$$

Why compare forecasts?

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Forecast comparison: Summary

- The Diebold-Mariano test is suitable for comparing two forecasts

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- Comparing forecasts and comparing models are slightly different tasks