# ARIMA processes

# Stationary processes

# Stationary processes: Plan

- Definition

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- Random walk

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- Autocovariance function

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- ACF, PACF

# Stationary processes

A stochastic process whose characteristics do not change over time

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**Weak or wide-sense stationarity**

A process $(y_t)$ is said to be weakly stationary, if for each $t$ and $k$:

$$\begin{cases} \mathbb{E}(y_t) = \mu \\ \mathrm{Cov}(y_t, y_{t+k}) = \gamma_k \end{cases}$$

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**Strong or strict-sense stationarity**

A process $(y_t)$ is said to be strictly stationary, if for each $k$ joint distribution of a r.v. $(y_t, y_{t+1}, y_{t+2}, \ldots, y_{t+k})$ does not depend on $t$

# Stationary process: example

**Independent observations**

The quantities $(y_t)$ are independent and equally distributed with finite expectation $\mu_y$ and finite variance $\sigma_y^2$

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$$\gamma_k = \mathrm{Cov}(y_t, y_{t+k}) = 0, \text{ for } k \geq 1$$

# Non-Stationary Process Example

**Random Walk**

$$\begin{cases} y_0 = \mu \\ y_t = y_{t-1} + u_t, \text{ for } t \geq 1 \end{cases},$$

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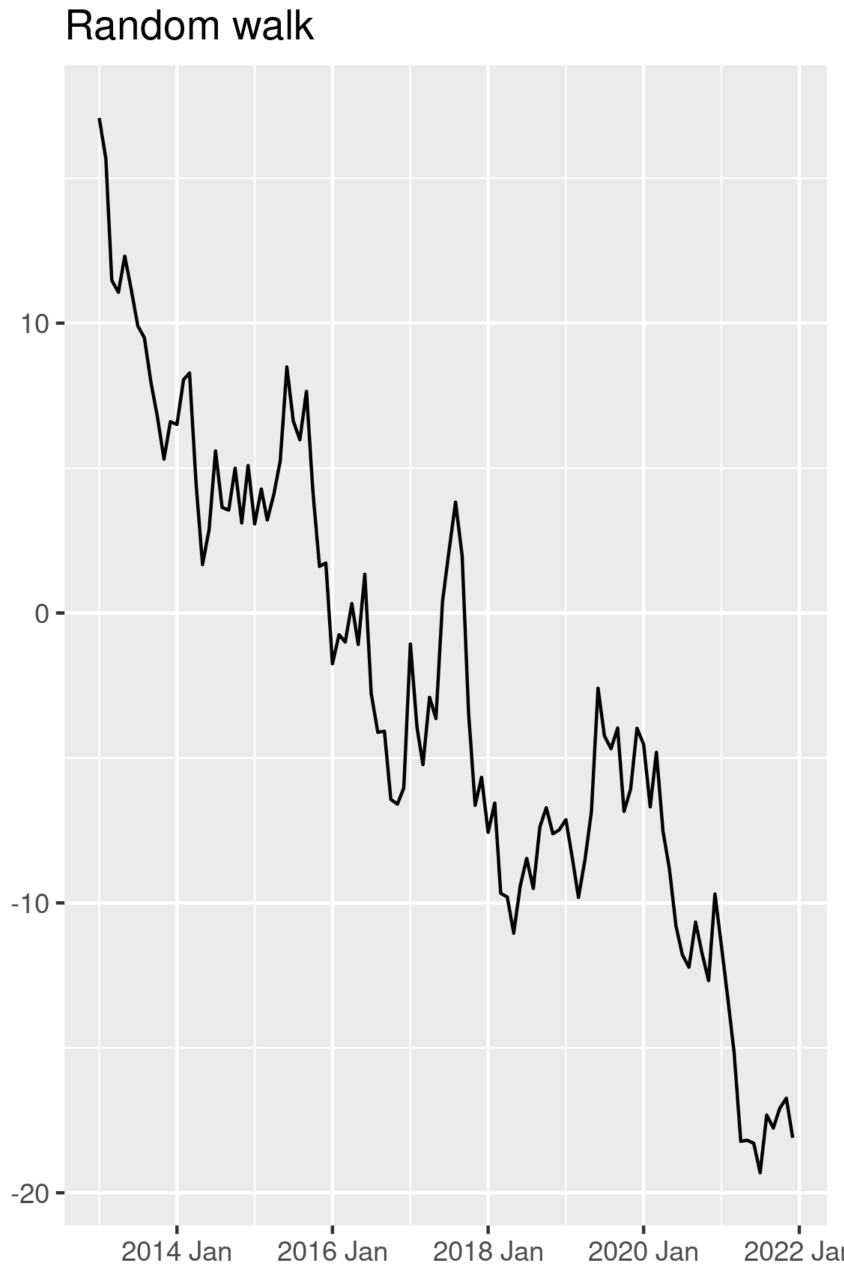
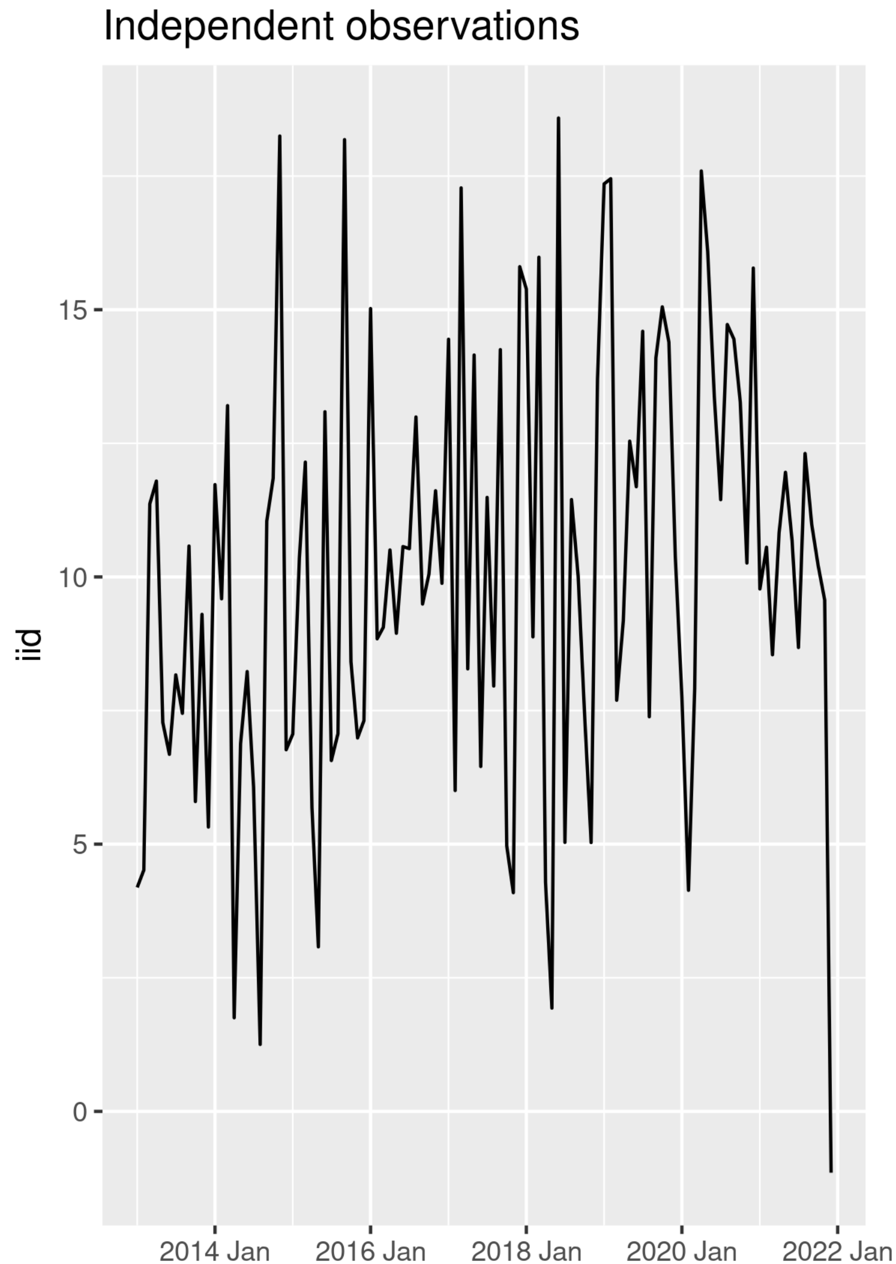
$$\gamma_k = \mathrm{Cov}(y_t, y_{t+k}) = \mathrm{Cov}(y_t, y_t + u_{t+1} + \ldots + u_{t+k}) = \mathrm{Var}(y_t)$$

# Random Walk vs Random Sample



# Autocovariance function

**Definition**

For a stationary process $(y_t)$, the function $\gamma_k = \mathrm{Cov}(y_t, y_{t+k})$ is called autocovariance

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# ACF: autocorrelation function

$$\rho_k = \mathrm{Corr}(y_t, y_{y+j}) = \frac{\mathrm{Cov}(y_t, y_{y+j})}{\sqrt{\mathrm{Var}(y_t)\,\mathrm{Var}(y_{t+k})}} =$$

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# Partial Correlation

**Definition**

$$\mathrm{pCorr}(U, D; R_1, R_2, \ldots, R_n) = \mathrm{Corr}(U^*, D^*), \text{ where}$$
$$U^* = U - Best(U; R_1, R_2, \ldots, R_n),$$
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The values $U^*$ and $D^*$ are the versions of $U$ and $D$ uninfluenced by the covariates $R_1, \ldots, R_n$

$$\mathrm{Cov}(U^*, R_i) = 0, \quad \mathrm{Cov}(D^*, R_i) = 0.$$

# PACF

**Definition**

For a stationary process $(y_t)$ the function

$$\varphi_{kk} = \mathrm{pCorr}(y_t, y_{t+k}; y_{t+1}, \ldots, y_{t+k-1})$$

is called partial autocorrelation

# ACF and PACF: Intuition

For stationary process

- ACF:

$$\rho_k = \mathrm{Corr}(y_t, y_{t+k})$$

  Joint strength of relationship between $y_t$ and $y_{t+k}$

- PACF:

$$\varphi_{kk} = \mathrm{pCorr}(y_t, y_{t+k}; y_{t+1}, \ldots, y_{t+k-1})$$

  Strength of relationship between $y_t$ and $y_{t+k}$ with the links through intermediate observations being broken

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- Partial correlation — correlation with the effect of a set of controlling random variables removed
- In the time series, we removed the effect of intermediate observations

# MA Process

# MA Process: Plan

- Definition and notations with lags

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- Stationarity

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- Definition and notations with lags
- Stationarity
- Predictability
- Reversibility

# Lag operator

**Definition**

For the process $(y_t)$ defined at $t \in \mathbb{Z}$, lagged process $Ly_t$ is the same sequence of values with a shifted index,

$$Ly_t = y_{t-1}$$

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$$\Delta_{12} y_t = y_t - y_{t-12} = (1 - L^{12}) y_t$$

# MA process

**Definition**

Process $(y_t)$, which can be represented as

$$y_t = \mu + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_q u_{t-q},$$

where $\alpha_q \neq 0$ and $(u_t)$ is white noise, is called the $MA(q)$ process

MA — Moving Average

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MA — Moving Average

Example $MA(1)$ process:

$$y_t = 5 + u_t + 0.3u_{t-1},$$

where $(u_t)$ is some white noise

# Notation with lags

**MA with lag polynomial**

Process $(y_t)$, which can be represented as

$$y_t = \mu + P(L)u_t,$$

where $P(L)$ is a polynomial of degree $q$ in lag $L$ with $P(0) = 1$, and $(u_t)$ is white noise, is called $MA(q)$ a process

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An example $MA(2)$ process:

$$y_t = 5 + (1 - 0.2L + 0.3L^2)u_t,$$

where $(u_t)$ is white noise

# ACF and Forecasts

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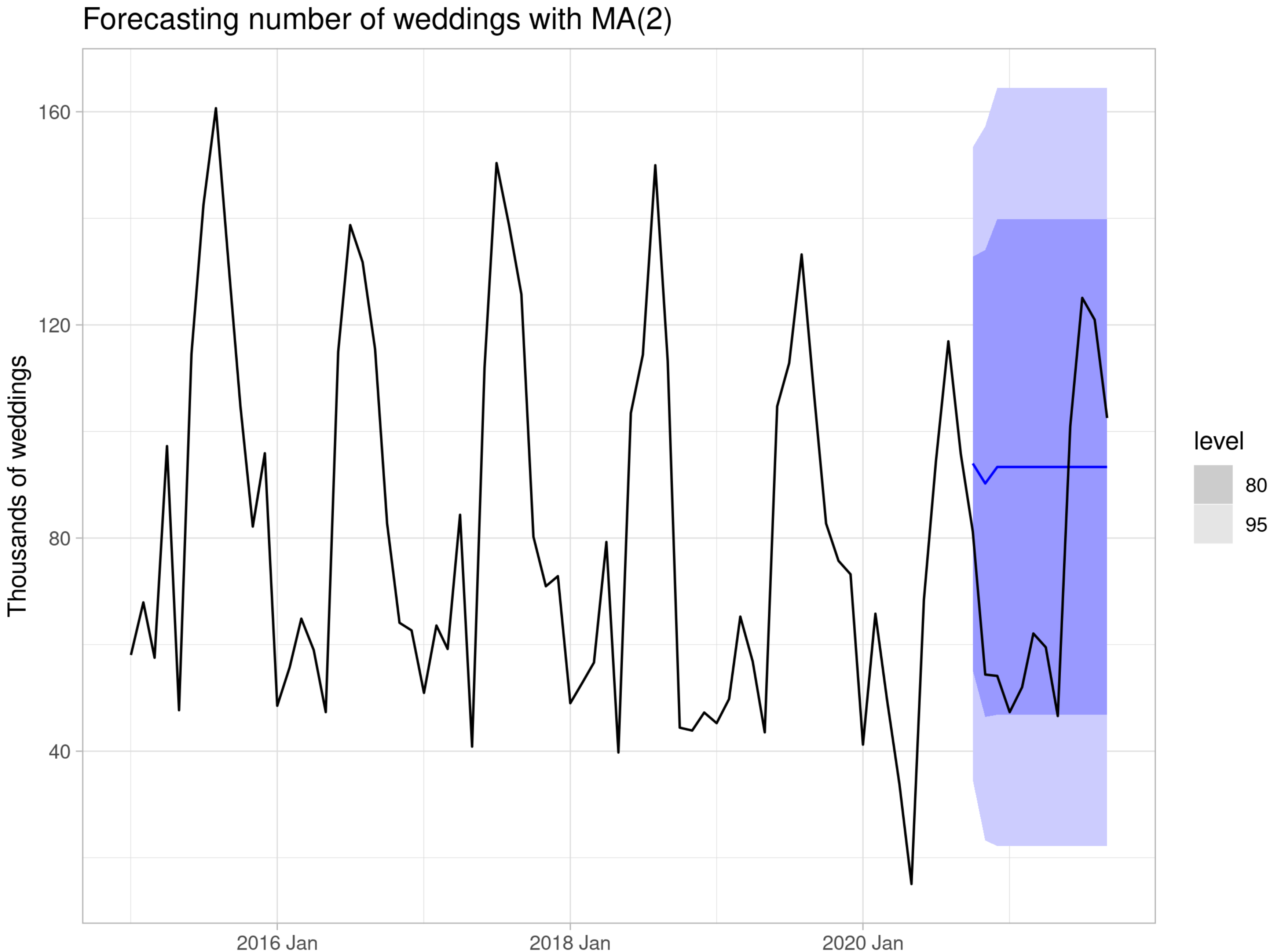
Zero $\rho_k = 0$ for $k > q$ implies independence of $y_t$ and $y_{t+k}$.

Forecasts more than $q$ steps ahead are exactly the same.

$$(y_{T+q+1} \mid \mathcal{F}_T) \sim (y_{T+q+2} \mid \mathcal{F}_T) \sim (y_{T+q+3} \mid \mathcal{F}_T) \sim \ldots$$

# Predictions for $MA(2)$



# $MA(\infty)$

**Definition**

Process $(y_t)$, which can be represented as

$$y_t = \mu + u_t + \alpha_1 u_{t-1} + \alpha_2 u_{t-2} + \ldots,$$

where $(u_t)$ is white noise, an infinite number of $\alpha_i \neq 0$ and $\sum_{i=1}^{\infty} \alpha_i^2 < \infty$, is called the $MA(\infty)$ process

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$MA(\infty)$:

$$y_t = 5 + u_t + 0.5u_{t-1} + 0.5^2 u_{t-2} + 0.5^3 u_{t-3} + \dots$$

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$MA(\infty)$:

$$y_t = 5 + u_t + 0.5 u_{t-1} + 0.5^2 u_{t-2} + 0.5^3 u_{t-3} + \dots$$

And this is not allowed:

$$y_t = 5 + u_t + \frac{1}{\sqrt{2}} u_{t-1} + \frac{1}{\sqrt{3}} u_{t-2} + \frac{1}{\sqrt{4}} u_{t-3} + \dots$$

# Convergences

**Theorem**

If a $\sum_{i=0}^{\infty} \alpha_i^2 < \infty$ and $(u_t)$ is a zero-mean stationary process, then the sequence of partial sums $y_t^q$ of the form

$$y_t^q = \mu + \sum_{i=0}^{q} \alpha_i u_{t-i}$$

converges for $q \to \infty$ in mean, in probability, and in distribution

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**Bonus**

…and the resulting process $(y_t)$ is stationary

# Wald's Theorem

**Theorem**

If $(y_t)$ is a stationary process, then it can be represented as:

$$y_t = \sum_{i=0}^{\infty} \alpha_i u_{t-i} + r_t,$$

where

- $(u_t)$ — white noise,
- $\sum \alpha_i^2 < \infty$,
- $r_t$ is a linear predictable random process,
- $\mathrm{Cov}(u_t, r_t) = 0$

# Predictable Process

**Correct definition**

A process $(r_t)$ is called linearly predictable if

- $(r_t)$ is stationary,
- $r_t = \beta_0 + \beta_1 r_{t-1} + \beta_2 r_{t-2} + \ldots + \beta_p r_{t-p}$

# Reversibility condition

**Characteristic representation**

The equation $MA(q)$ of the process satisfies the reversibility condition if the characteristic polynomial $\phi(\lambda)$ has all roots $|\lambda_i| < 1$

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**Lag representation**

The equation $MA(q)$ of the process satisfies the reversibility condition if all roots of the lag polynomial $P(L)$ are $|\ell_i| > 1$

# Example of reversible notation $MA(1)$

$$y_t = 5 + u_t + 0.5u_{t-1}, \quad \sigma_u^2 = 4$$

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$$y_t = 5 + u_t + 2u_{t-1}, \quad \sigma_u^2 = 1$$

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# Nuance

**Difference**

Stationarity is a property of the $(y_t)$ process itself.

Reversibility is a property of the equation (process notation) for $(y_t)$.

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Stationarity is a property of the $(y_t)$ process itself.

Reversibility is a property of the equation (process notation) for $(y_t)$.

$MA(q)$ has a single notation when reversible

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- $MA(q)$ is a stationary process
- $ACF$ vanishes sharply, $PACF$ tends to zero
- Reversibility condition: roots of the characteristic polynomial $|\lambda_i| < 1$ or roots of the lag polynomial $|\ell_i| > 1$.

# ARMA equation

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- Non-uniqueness of solutions

# About the purpose of old problems

Goal: A simple equation for a wide variety of processes

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Let's try adding lags $y_t$ to the equation!

# New problem

$$y_t - y_{t-1} = u_t - u_{t-1}, \text{ where } (u_t) \text{ is white noise}$$

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- $y_t = u_t$;
- $y_t = u_t - 0.7$;

# New problem

$$y_t - y_{t-1} = u_t - u_{t-1},$$ where $(u_t)$ is white noise

Solutions:

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Solutions:

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- $y_t = u_t - 0.7$;
- $y_t = u_t - 0.8$

Infinite number of solutions

# ARMA equation

**Definition**

Equation

$$y_t = c + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + u_t + \alpha_1 u_{t-1} + \ldots + \alpha_q u_{t-q},$$

where $(u_t)$ is white noise, we'll call an $ARMA$ equation

ARMA — Autoregression and Moving Average

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Equation

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where $(u_t)$ is white noise, we'll call an $ARMA$ equation

ARMA — Autoregression and Moving Average

**Definition**

An equation of the form $P(L)y_t = c + Q(L)u_t$, where $(u_t)$ is white noise, $P(L)$ and $Q(L)$ are lag polynomials with $P(0) = Q(0) = 1$, we'll call an $ARMA$ equation

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An equation is not a process!

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Why?

- One equation has many solutions

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Why?

- One equation has many solutions
- One process can be described by several equations

# ARMA process

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- Equation irreducibility

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**Definition**

$ARMA$ an equation of the form $P(L)y_t = c + Q(L)u_t$ is called irreducible, if the polynomials $P(L)$ and $Q(L)$ do not have common roots

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Irreducible equation:

$$y_t - y_{t-1} = u_t - 0.5u_{t-1} \text{ or } (1-L)y_t = (1-0.5L)u_t$$

# Initial Conditions

Irreducible $ARMA$ equation:

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Let's try different initial conditions:

- $y_0 = 0$

$y_1 = u_1, y_2 = u_2 + 0.5u_1, y_3 = u_3 + 0.5u_2 + 0.25u_1, \ldots$

# Initial Conditions

Irreducible $ARMA$ equation:

$$y_t = 0.5y_{t-1} + u_t, \text{ where } (u_t) \text{ is white noise}$$

Let's try different initial conditions:

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The initial conditions also determine the past $y_t$!

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$$y_t = 0.6y_{t-1} + 0.08y_{t-2} + u_t \text{ and } y_0 = u_0, y_1 = u_0 + 4$$

# And how many stationary solutions?

**Correct theorem**

If an $ARMA$ equation $P(L)y_t = c + Q(L)u_t$ is irreducible, then it

- has exactly one stationary solution if the lag polynomial $P(\ell)$ has all roots $|\ell_i| \neq 1$;
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# AR process

**Definition**

$AR(p)$ process with equation

$$y_t = c + \beta_1 y_{t-1} + \ldots + \beta_p y_{t-p} + u_t,$$

where $(u_t)$ is white noise and $\beta_p \neq 0$, is the solution of this equation in the form of $MA(\infty)$ with respect to $(u_t)$

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## Definition with lags

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# Definitions by different authors

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Some authors do not include the requirement of stationarity in the definition of $AR(p)$.

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# What about non-uniqueness?

The same $ARMA(p, q)$ process $(y_t)$ can be described by different equations!

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If the series $(y_t)$ is an $ARMA(p, q)$ process with the equation $P(L)y_t = c + Q(L)u_t$, then this equation will be unique if the $MA$ part satisfies the reversibility condition.

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Condition for $ARMA$ reversibility of the equation:

- the characteristic polynomial $\phi_{MA}(\lambda)$ has all roots $|\lambda_i| < 1$;
- the lag polynomial $Q(L)$ has all roots $|\ell_i| > 1$

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- Theoretical $ACF$ and $PACF$ decrease exponentially
- The reversibility condition guarantees uniqueness
- An irreducible equation either has a unique stationary solution or it does not exist

# ARIMA process

# ARIMA process: Plan

- Stationarity of $ARMA$

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- In the canonical notation $ARMA(p,q)$ of the process $P(L)y_t = c + Q(L)u_t$ for the polynomial $P(L)$ all roots $|\ell| > 1$
- When estimating the $ARMA(p,q)$ process by the maximum likelihood method, these restrictions are imposed a priori

# What to do with non-stationary processes?

**Definition**

The random process $(y_t)$ is called the $ARIMA(p, 1, q)$ w.r.t. the white noise process $(u_t)$, if $(y_t)$ is non-stationary, but $\Delta y_t$ is a stationary $ARMA(p, q)$ process w.r.t. the white noise $(u_t)$

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ARIMA — AutoRegressive Integrated Moving Average

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  $\ln L(y_1, \ldots, y_n \mid \theta)$ and $\ln L(y_2, \ldots, y_n \mid \theta, y_1)$ and $\ln L(y_3, \ldots, y_n \mid \theta, y_1, y_2)$ incomparable!

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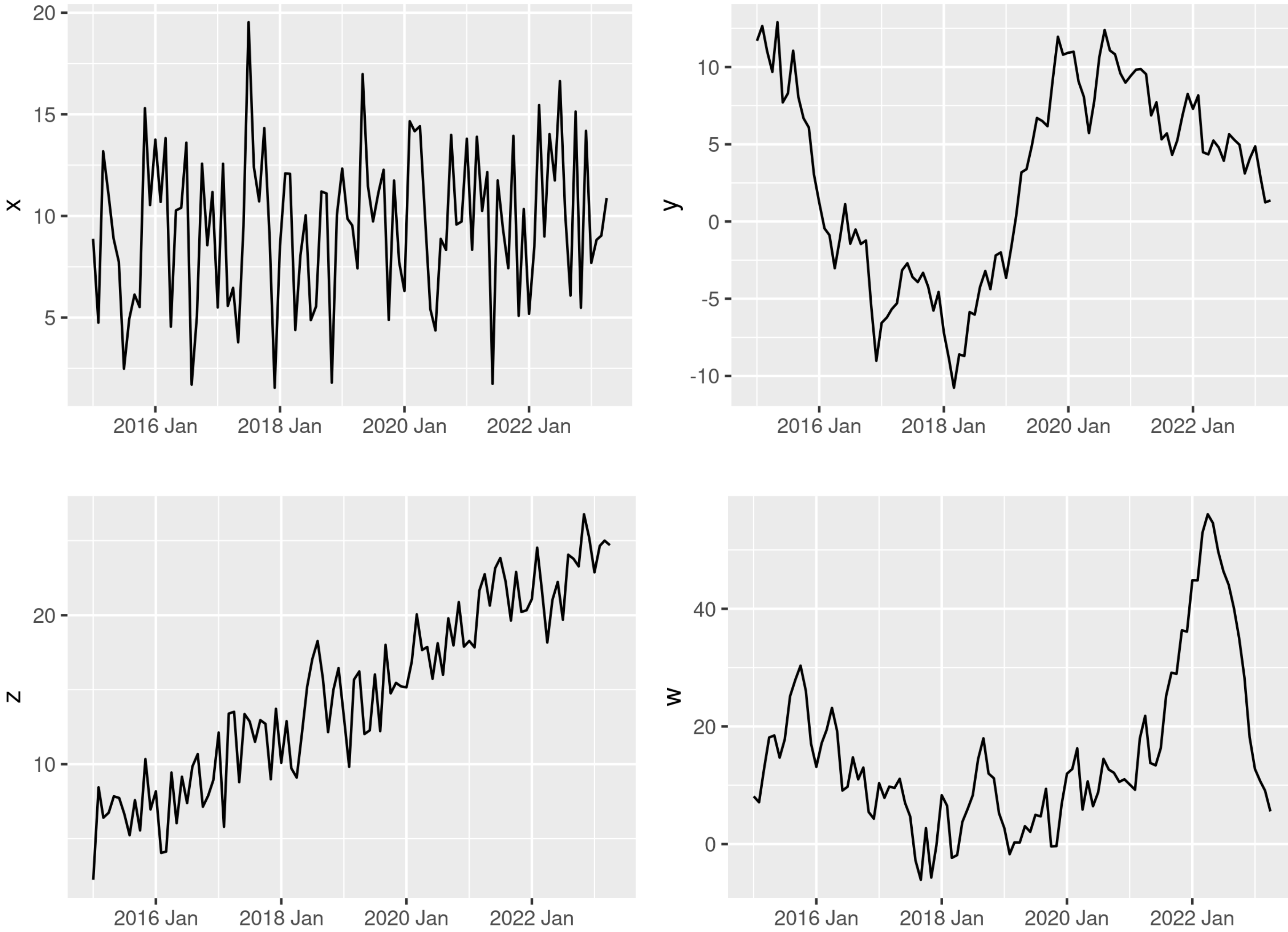
  ADF, KPSS, PP, …

# Analysing the graphs



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# SARIMA process

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- Choosing between models

# Seasonality and $ARIMA$

Using $ARMA$ and $ARIMA$ models, we can model seasonality!

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$$MA(12) : y_t = c + u_t + a_1 u_{t-1} + a_2 u_{t-2} + \ldots + a_{12} u_{t-12}.$$

$$ARIMA(12, 1, 0) : \Delta y_t = c + u_t + b_1 \Delta y_{t-1} + \ldots + b_{12} \Delta y_{t-12}.$$

# ARMA should be economical!

Let's focus on non-zero coefficients!

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**Definition**

If the stationary $ARMA$ model for $y_t$ can be written with fewer parameters as

$$P_{non}(L)P_{seas}(L^{12})y_t = c + Q_{non}(L)Q_{seas}(L^{12})u_t,$$

where the degrees of the lag polynomials are $\deg P_{non} = p$, $\deg P_{seas} = P$, $\deg Q_{non} = q$, $\deg Q_{seas} = Q$, then it is also called $SARMA(p, q)(P, Q)[12]$

# Examples

- $SARMA(1,0)(0,2)[12]$

$$(1 - b_1 L)y_t = c + (1 + d_1 L^{12} + d_2 L^{24})u_t;$$

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- $SARMA(0,2)(1,0)[12]$

$$(1-f_1L^{12})y_t = c + (1+a_1L+a_2L^2)u_t;$$

- $SARMA(1,2)(2,1)[12]$

$$(1-f_1L^{12}-f_2L^{24})(1-b_1L^1)y_t = c+(1+a_1L+a_2L^2)(1+d_1L^{12})u_t$$

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If the series $z_t = \Delta^d \Delta_{12}^D y_t$ is described by the stationary model $SARMA(p, q)(P, Q)[12]$ , then $y_t$ is said to be described by the $SARIMA(p, d, q)(P, D, Q)[12]$ model

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$d$ is the number of times the first difference should be taken $\Delta = 1 - L$;

$D$ is the number of times the seasonal difference should be taken $\Delta_{12} = 1 - L^{12}$; $y_t \sim SARIMA(0,0,2)(1,1,2)[12]$ means that

$$\Delta_{12} y_t \sim SARMA(0,2)(1,2)[12]$$

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  And rules of thumb…

# STL decomposition and the power of seasonality

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Step 2. Calculate the strength of seasonality

$$F_{seas} = \max\left\{1 - \frac{\mathrm{sVar}(remainder)}{\mathrm{sVar}(seas + remainder)}, 0\right\}$$

Step 3. If the strength of seasonality is above the threshold, then move to $\Delta_{12} y_t = y_t - y_{t-12}$

# SARIMA: Summary

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- The strength of seasonality from the STL expansion is used to decide if a seasonal difference $\Delta_{12} y_t$ is needed

# Unit root tests: ADF test

# ADF test: Plan

- Test assumptions

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- Test assumptions
- Test algorithm
- Three variations of the test

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$$\Delta = 1 - L = P(L)$$

The equation $1 - \ell = 0$ has a root $\ell = 1$

# ADF test

ADF — Augmented Dickey Fuller test

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Three variations of the test: without a constant, with a constant, with a trend

# ADF with constant

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$H_0$: $\beta = 0$

$\Delta y_t = m + x_t$;

$(x_t)$ is a stationary $AR(p)$ process with $\mathbb{E}(x_t) = 0$;

$y_t = y_0 + mt + \sum_{i=1}^{t} x_i$

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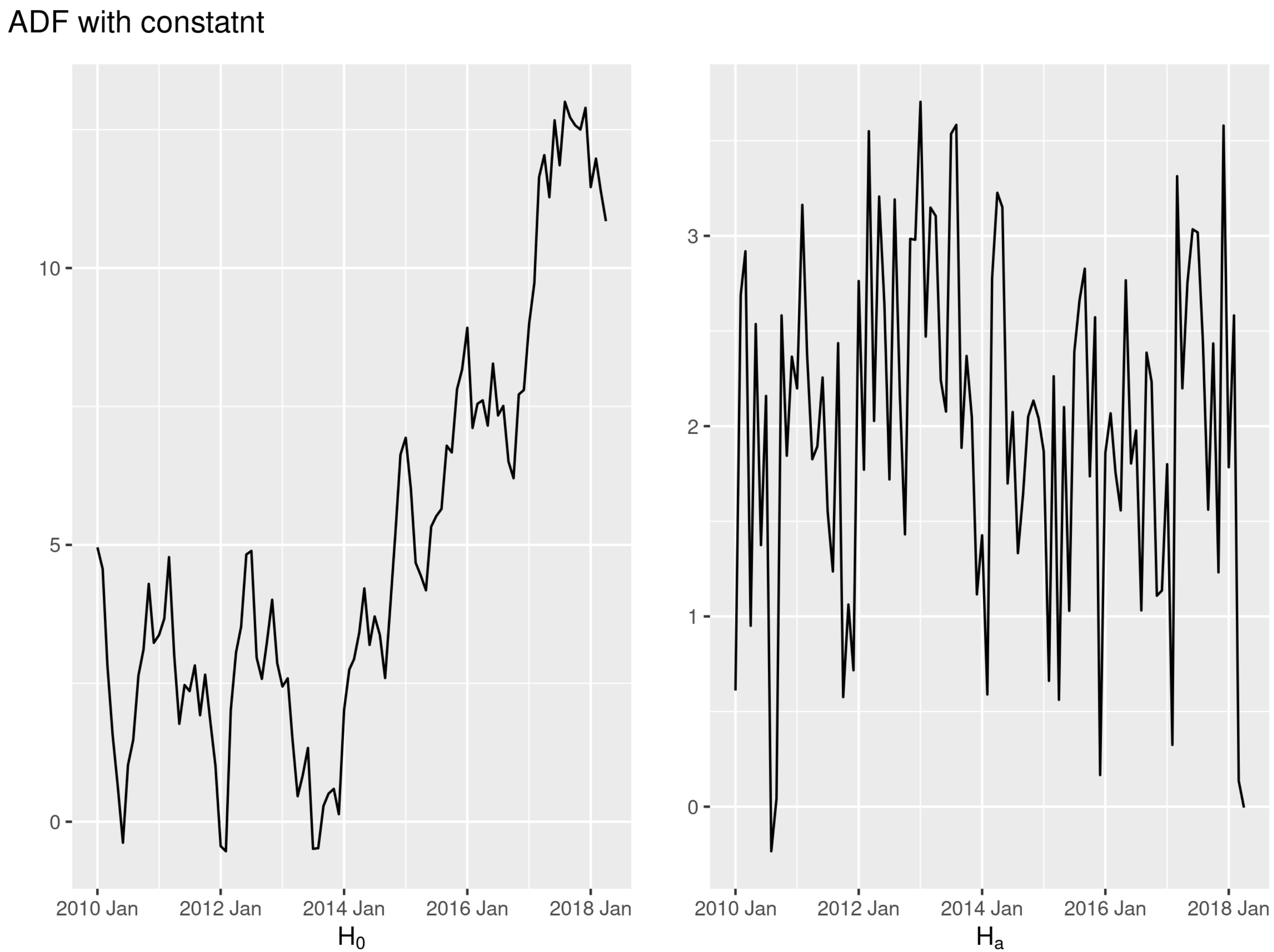
# ADF with constant: $H_0$ and $H_a$



# ADF with constant: algorithm

Step 1. Evaluate regression

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The algorithm will have regression without a constant and another distribution $DF^0$

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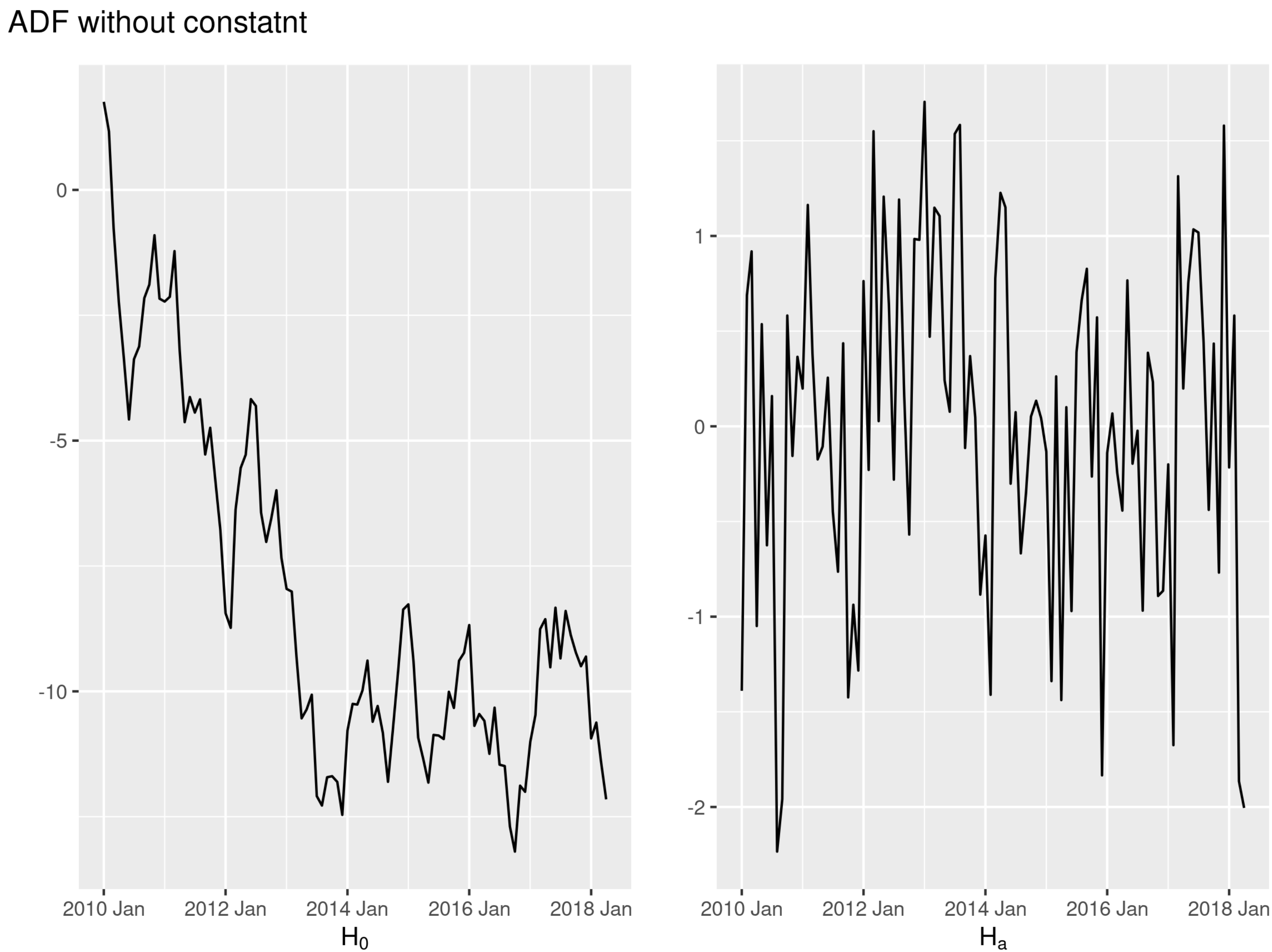

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$\Delta y_t = k_1 + k_2 t + x_t$;

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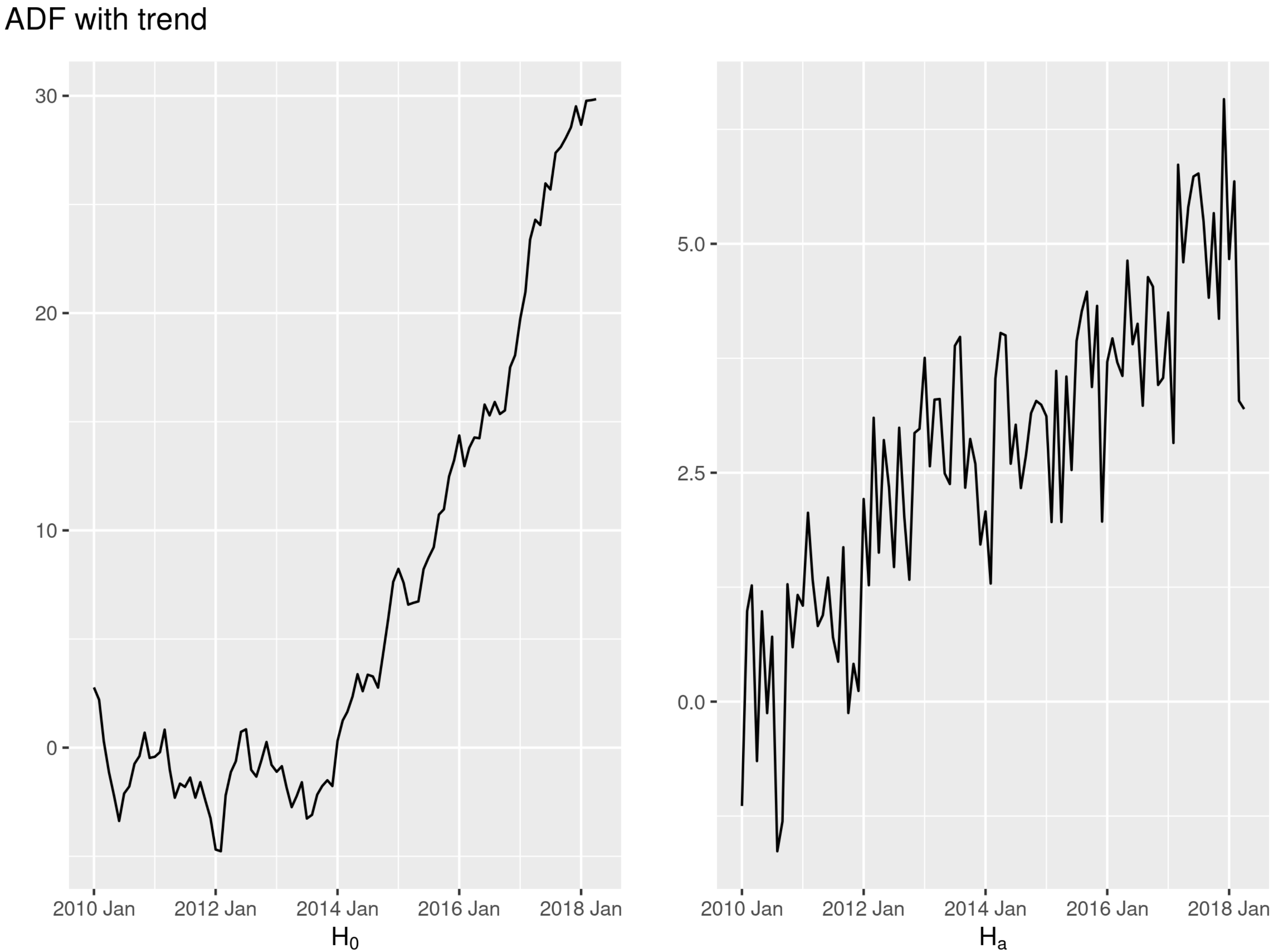

# $ADF$ test: Summary

- Applicable for making a decision about the transition to $\Delta y_t$

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- Applicable for making a decision about the transition to $\Delta y_t$
- Three variants of the ADF test with different assumptions

# Unit root tests: KPSS test

# KPSS test: Plan

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# KPSS test

KPSS — Kwiatkowski–Phillips–Schmidt–Shin test

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Two variations of the test: with a constant, with a trend

# Long-term variance

**Definition**

For a stationary process $(y_t)$, the quantity $\lambda^2$ is called long-term variance if

$$\operatorname{Var}(\bar{y}) = \frac{\lambda^2}{T} + o(1/T)$$

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$$\lim_{T \to \infty} T \operatorname{Var}(\bar{y}) = \lambda^2,$$

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**Motivation**

For independent observations with the constant variance

$$\operatorname{Var}(\bar{y}) = \frac{\sigma^2}{T}, \text{ where } \sigma^2 = \operatorname{Var}(y_i)$$

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$$y_t = c + rw_t + x_t,$$

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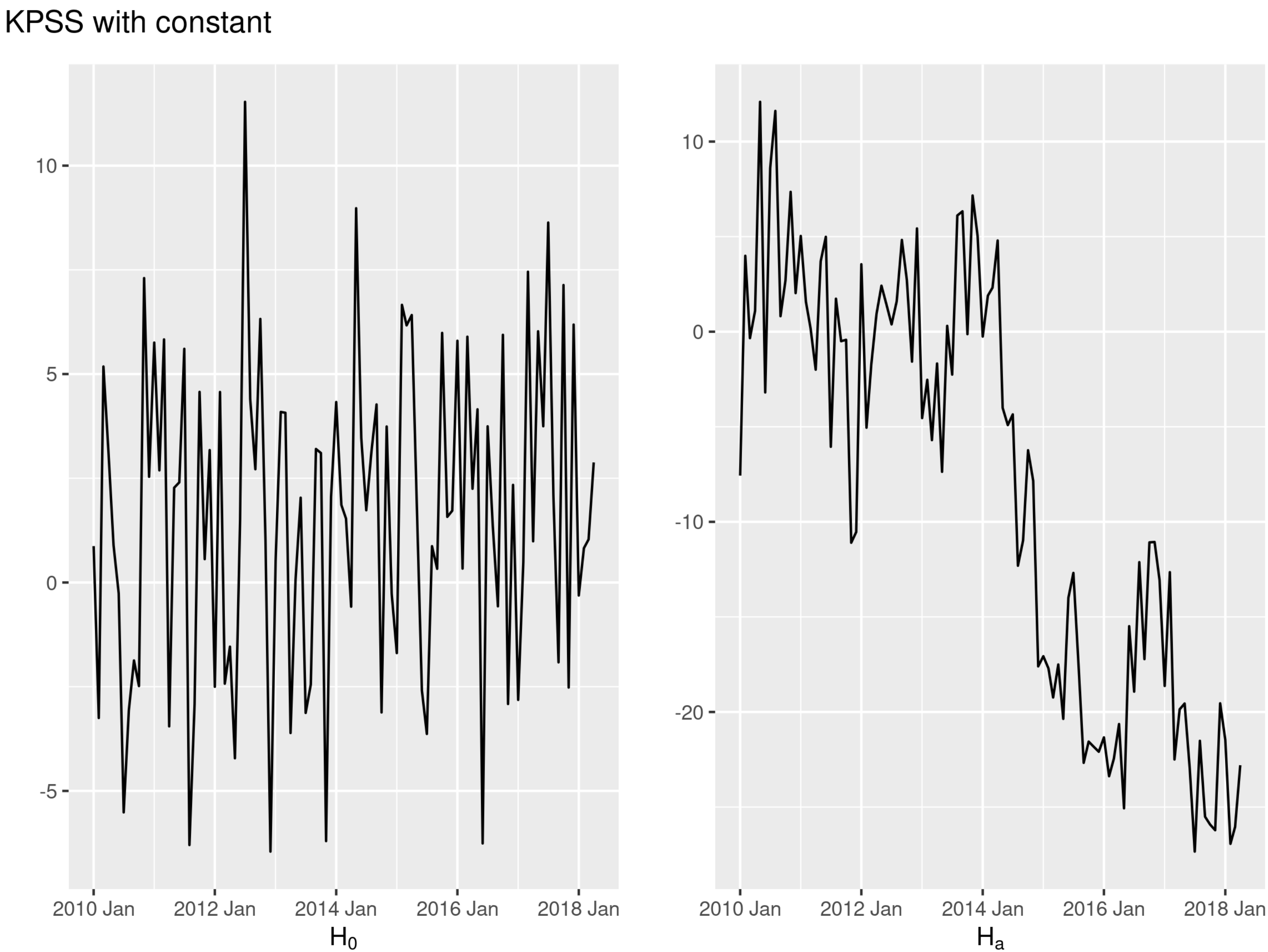
# KPSS with constant: $H_0$ and $H_a$



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Step 1. Evaluate regression on a constant

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where $S_t$ is the accumulated sum of residuals, $S_t = \hat{u}_1 + \ldots + \hat{u}_t$,

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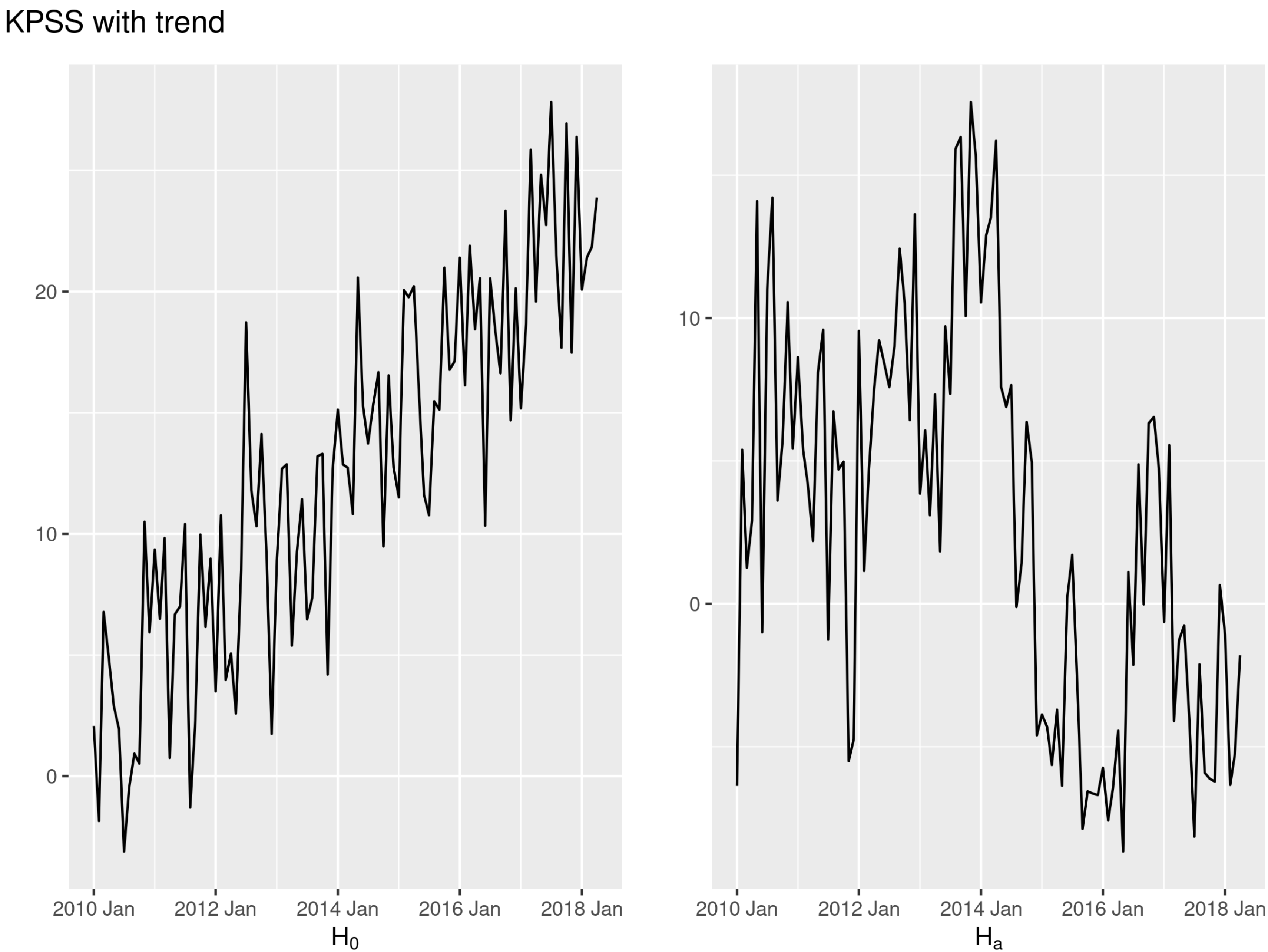
The first step of the algorithm will have a regression on a constant and a trend and the statistic under null will have another special distribution $KPSS^{ct}$

# KPSS with trend: $H_0$ and $H_a$



# Terminology

$$A. \quad y_t = a + bt + x_t$$

$(y_t)$ — trend stationary (stationary around the trend)

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$$B. \quad y_t = a + \sum_{i=1}^{t} x_i \text{ or } y_t = a + bt + \sum_{i=1}^{t} x_i$$

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Both $(y_t)$ are non-stationary!

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