A historical introduction to grey systems theory

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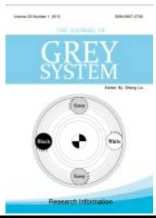

Advances in Grey Systems Research

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Abstract

The progress in grey system research is summarized. The four theories of uncertain systems – probability statistics, fuzzy mathematics, grey system, and rough set theory – are compared. The general grey numbers, the grey sequence operators and several grey system models, which are most commonly used, are introduced, such as the absolute degree of grey incidence model, the grey cluster model based on endpoint triangular whitenization functions, the grey cluster model based on center-point triangular whitenization functions, the grey prediction model of the GM(1,1) model, and the weighted multi-attribute grey target decision model. The reader will get to know a general picture of grey systems theory as a new method for studying problems with partial information known and partial information unknown, especially for uncertain systems with few data and poor information.

Key words: grey system theory; general grey numbers; grey sequence operators; grey system models; theories of uncertain systems

1. Introduction

In 1982, *Systems & Control Letters*, an international journal of North-Holland, published the first paper in grey systems theory, "The Control Problems of Grey Systems," by Julong Deng[1]. In the same year, the *Journal of Huazhong University of Science and Technology* published the first paper, also by Julong Deng, on grey systems theory in Chinese language. The publication of these papers signaled the official appearance of the cross disciplinary grey systems theory. As soon as these works appeared, they immediately caught the attention of many scholars and scientific practitioners from around the world. Numerous well-known scientists strongly supported the validity and livelihood of such research. Many young

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scholars actively participated in the investigation of grey systems theory. With great enthusiasm these young men and women carried the theoretical aspects of the theory to new heights and employed their exciting results to various fields of application. In particular, successful applications in great many fields have won the attention of the international world of science. Currently, a great number of scholars from China, the United States, England, Romania, South Africa, Germany, Japan, Australia, Canada, Poland, Spain, Cuba, Korea, Russia, Turkey, the Netherlands, Iran, and others, have been involved in the research and application of grey systems theory[2-30]. In 1989, the British journal, *The Journal of Grey System*, was launched. Currently, this publication is indexed by INSPEC (formerly Science Abstracts) of England, Mathematical Review of the United States, Science Citation Index, and other important indexing agencies from around the world. In 1997, a Chinese publication, named *Journal of Grey System*, was launched in Taiwan. Later in 2004 this publication became English. In 2011, a new journal named *Grey Systems: Theory and application,* edited by the faculty of Institute for Grey Systems Studies, Nanjing University of Aeronautics and Astronautics, was launched by Emerald in England. There are currently thousands of different professional journals in the world that have accepted and published papers in grey systems theory. As of this writing, a journal of the Association for Computing Machinery (USA), *Communications in Fuzzy Mathematics* (Taiwan), *Kybernetes: The International Journal of Systems & Cybernetics,* also sponsored by Emerald, have published more than ten special issues on grey systems theory.

Many universities around the world offer courses in grey systems theory. For instance, Nanjing University of Aeronautics and Astronautics not only offers such courses for PhD level and master level students, but also provides a service course on grey systems to all undergraduate students in different majors.

Huazhong University of Science and Technology, Nanjing University of Aeronautics and Astronautics, Wuhan University of Technology, Fuzhou University, De Montfort University, Bogazici University, University of Cape Town, Bucharest University of Economics, Kanagawa University and several universities in Taiwan recruit and produce PhD students focusing on the research in grey systems. It is estimated that thousands upon thousands of graduate students from around the world employ the thinking logic and methodology of grey systems in their research and writing of their dissertations.

Many publishers from around the world, such as Science Press, Press of National Defense Industry, Huazhong University of Science and Technology Press, Jiangsu Press of Science and Technology, Shangdong People's Press, Literature Press of Science and Technology, Taiwan Quanhua Science and Technology Books, Taiwan Guaoli Books Limited, Science and Engineering Press of Japan, the IIGSS Academic Publisher and Taylor and Francis Group (USA), Springer-Verlag (Germany) etc., have published over 100 different kinds of monographs in grey systems[31-36].

A whole array of brand new hybrid branches of study, such as grey hydrology, grey geology, grey theory and methods of breeding, grey medicine, grey systems analysis of regional economics, etc., have appeared along with the opportunity presented by grey systems theory. Agencies at national, provincial, and local governments actively sponsored research works in grey systems. Each year many theoretical and applied works on grey systems are financially supported by various foundations. It is estimated that throughout China more than 200 research outcomes of grey systems were recognized officially by national, provincial, or ministerial agencies. Since 2002, several Chinese scholars, Professor Julong Deng et al, received recognition from the World Organization of Cybernetics and Systems and IEEE Systems, Man, and Cybernetics Society.

Many important international conferences, such as the Conference on Uncertain Systems Modeling, Systems Prediction and Control, Workshops of the International Institute for General Systems Studies, Congress of World Organization of Cybernetics and Systems, IEEE conferences on Systems, Man, and Control, International Conferences of Computers and Industrial Engineering, the IEEE Conference on Networks, Sensing and Control etc., have listed grey systems theory as a special topic of interest or arranged special topic sessions for grey systems research. Grey systems theory has caught the attention of many important international conferences and become a center of discussion at many international events, which no doubt will play an important and active role for the world of systems researchers to get better acquainted with grey systems theory.

During November 18-20, 2007, the inaugural IEEE International Conference on Grey Systems and Intelligent Services (IEEE GSIS) was successfully held in Nanjing, China. Nearly 300 scholars from around the world participated in this event. The IEEE headquarter approved the organization of this event, and the National Natural Science Foundation (China), the Nanjing University of Aeronautics and Astronautics (NUAA), and the Grey Systems Society of China jointly sponsored the actual details of the conference. The Institute for Grey Systems Studies and the College of Economics and Management of the NUAA hosted the event. Scholars from China, U.S.A., England, Japan, South Africa, Russia, Turkey, Malaysia, Iran, Taiwan, Hong Kong, and other countries or geographic regions submitted 1,019 papers for possible presentation at this event. Eventually, 332 of the submissions were accepted.

As decided by the conference organizing committee, the IEEE GSIS will be organized biannually. At the end of 2007, The Technical Committee of IEEE on Grey systems was approved by System, Man, and Cybernetics Society. On the basis of such an international platform as IEEE, grey systems theory will surely be widely recognized and studied by many scholars from around the world in the years to come.

During November 10-12, 2009, and September 15-18, 2011, the second and third International Conference on Grey Systems and Intelligent Services (IEEE GSIS) were also successfully held in Nanjing. The fourth international Conference on Grey Systems and Intelligent Services (IEEE GSIS) will be held in Macao in 2013. Scholars from the U.S.A., England, South Africa, and Romania, expressed their interest of hosting such events in the future.

A computer software on most of grey systems modeling can be downloaded from the web site of the Institute for Grey System Studies of Nanjing University of Aeronautics and Astronautics (http:// igss.nuaa.edu.cn) for free.[65]

2. Comparison of Several Uncertain Systems Models

Probability and statistics, grey systems theory, rough set theory and fuzzy mathematics are the four mostly recognized research methods employed for the investigation of uncertain systems. Their research objects all contain certain kinds of uncertainty, which represents their commonality. It is exactly the differences among the uncertainties in the research objects that these four theories of uncertainty are different from each other with their respective characteristics.

Probability and statistics study the phenomena of stochastic uncertainty with emphasis placed on revealing the historical statistical laws. They investigate the chance for each possible outcome of the stochastic uncertain phenomenon to occur. Their starting point is the availability of large samples that are required to satisfy a certain typical form of distribution.

Fuzzy mathematics emphasizes on the investigation of problems with cognitive uncertainty, where the research objects possess the characteristic of clear intension and unclear extension. For instance, “young man” is a fuzzy concept, because each person knows the intension of “young man.” However, if you are going to determine the exact range within which everybody is young and outside which each person is not young, then you will find yourself in a great difficulty. That is because the concept of young man does not have a clear extension. For this kind of problem of cognitive uncertainty with clear intension and unclear extension, the situation is dealt with in fuzzy mathematics by making use of experience and the so-called membership function.[59]

The focus of grey systems theory is on uncertainty problems with small samples and poor information that are difficult for probability to handle. It explores and uncovers the realistic laws of evolution and motion of events and materials through information coverage and through the works of sequence operators. One of its characteristics is the construction of models with small amounts of data. What is clearly different to fuzzy mathematics is that grey systems theory emphasizes on the investigation of such objects that process clear extension and unclear intension. For example, by the year of 2050, China will control its total population within the range of 1.5 to 1.6 billion people. This range from 1.5 billion to 1.6 billion is a grey concept. Its extension is definite and clear. However, if one inquires further regarding exactly which specific number within the said range it will be, then he will not be able to obtain any meaningful and definite answer.

Rough set theory deals with rough non-overlapping class and a rough concept, which signifies the indiscernibility between objects. The object is approximated by both the lower approximation and the upper approximation. The redundancy can be reduced by algorithm of attribute reduction, which makes pattern discovery possible from the data which may be blurred by too much detail.[58]

Based on what is discussed above, we summarize the differences among these four most studies subject matters in table 1:

Currently, the theoretical studies of uncertain (stochastic, fuzzy, grey, rough) systems have been widely applied in all areas of natural science, social science, and engineering, including aviation, spaceflight, civil aviation, information, metallurgy, machinery, petroleum, chemical industry, electrical power, electronics, light industries, energy resources, transportation, medicine, health, agriculture, forestry, geography, hydrology, seismology, meteorology, environment protection, architecture, behavioral science, management science, law, education, military science, etc. These practical applications have brought forward definite and noticeable social and economic benefits.

Both the theoretical and applied research of uncertain (stochastic, fuzzy, grey, rough) systems has been extremely active. However, all these works have shown the emphasis on applications without much effort placed on further developing the theory and establishing innovative methods. In particular, not enough attention is given to investigate the differences and commonalities between the various available uncertain systems theories so that not enough works on melting together the traditional theories and methods with the newly appearing uncertain systems theories and methods were published. This fact more or less affected the development of uncertain (stochastic, fuzzy, grey, rough) systems theories.

As a matter of fact, each traditional or newly emerging uncertain systems theory and method complements each other and cannot be clearly separated from each other. When facing various kinds of uncertainty problems, different uncertain systems theory and method have its strengths, which complement and supplement each other instead of repelling each other. Many complex, dynamic uncertainty problems are really way beyond the scope and capacity of any single uncertain systems theory and method. It requires the researcher to combine various kinds of classical theories with the newly developed uncertain systems theories and methods. It is a must for this kind of interaction, exchange, and combination of the relevant theories and methods to occur for further, healthy development of science.

Table 1 Comparison of the different uncertainty models

Object	**Prob. Statistics**	**Fuzzy math**	**Grey systems**	**Rough sets**
Research objects	stochastics	Cognitive uncertainty	Poor information	Indiscernibility
Basic sets	Cantor sets	Fuzzy sets	Grey sets	Approximation sets
Methods	Mapping	Mapping	Information coverage	Partition
Procedures	Frequency distribution	Cut set	Sequence operator	Lower and upper approximation
Data requirement	Typical distribution	Membership Known	Any distribution	Equivalent relations
Emphasis	Intension	Extension	Intension	Intension
Objective	Historical laws	Cognitive expression	Laws of reality	Concept approximation
Characteristics	Large sample	Experience	Small sample	Information systems (tables)

3. BASICS OF GREY SYSTEMS

Many systems in studies are named after the features of the research objects, while grey systems are labeled using the color of the concerned systems. For instance, Ashby referred objects with unknown internal information as black boxes. We use “black” to indicate the unknown information, “white” for the completely known information, and “grey” for the partially known and partially unknown information. Accordingly, systems with completely known information are regarded as white, those with completely unknown information as black, and those with partially known and partially unknown information as grey. Here, the term“system” is used to indicate the studies of the structure and function of the concerned object through analyzing the existing organic connections among the object, the relevant factors, the environment, and the related changing laws.[2-3, 34-36]

Definition 1 Grey interval number

The following equation

$$\otimes \in [\underline{a}, \overline{a}], \underline{a} < \overline{a} \tag{1}$$

is called a grey interval number, or grey interval for short, where $\underline{a}$ and $\overline{a}$ are the upper and lower limits of the information separately.[32] The arithmetic of grey interval is very similar to the interval values.[37] But a grey interval is only one number which belongs to interval $[\underline{a}, \overline{a}]$, $\underline{a} < \overline{a}$. It is different from an interval.

Definition 2 General grey number

Let $g^{\pm} \in \Re$ be an unknown real number within an union set of closed or open grey intervals, shown as

$$g^{\pm} \in \bigcup_{i=1}^{n} \left[\underline{a}_i , \overline{a}_i\right] \tag{2}$$

where $i = 1, 2, \ldots, n$, n is an integer and $0 < n < \infty$, $\underline{a}_i , \overline{a}_i \in \Re$ and $\overline{a}_{i-1} \le \underline{a}_i \le \overline{a}_i \le \underline{a}_{i+1}$, for any grey interval, $\otimes_i \in [\underline{a}_i , \overline{a}_i] \subset \bigcup_{i=1}^{n} \left[\underline{a}_i , \overline{a}_i\right]$, then $g^{\pm}$ is called a general grey number. $g^{-} = \inf_{\underline{a}_i \in g^{\pm}} \underline{a}_i$ and $g^{+} = \sup_{\overline{a}_i \in g^{\pm}} \overline{a}_i$ are called the lower and upper limits of $g^{\pm}$.[37-39]

In practice, grey numbers can be represented in different forms, such as interval grey numbers, sets of discrete numbers, kernels and reduced forms, etc.[38-41]

The operation rules of general grey numbers can be defined as operation rules of real numbers, but the degree of greyness of the operation result should be considered carefully.[42]

It differs from the stochastic theory that investigates statistical laws on the basis of probabilities established on prior knowledge. The grey system theory uncovers the change laws by excavating and organizing the available raw data, representing an approach to finding data out of data, which is referred to as grey sequence generation.[32-33]

Definition 3 Sequence operator[32-33,40]

Assume that X is a sequence of numbers, and D a symbol of series operations to work on X such that after applying D, the sequence X becomes $XD = (x(1)d, x(2)d, \cdots, x(n)d)$, where D is referred to as the first order sequence operator, XD as the first order sequence worked on by the operator D.

If D_1, D_2, and D_3 are all sequence operators, D_1D_2 is referred to as a second order sequence operator, and its application on X produces a second order sequence $XD_1D_2 = (x(1)d_1d_2, x(2)d_1d_2, \cdots, x(n)d_1d_2)$. Similarly, $D_1D_2D_3$ is referred to as a third order sequence operator and $XD_1D_2D_3 = (x(1)d_1d_2d_3, x(2)d_1d_2d_3, \cdots, x(n)d_1d_2d_3)$ is a third order sequence.

The most important sequence operator is the accumulating generation operator, which is the basis to build the widely applied GM(1,1).

Definition 4 Accumulating generation operator

Assume that $X^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))$ is an original sequence, D is a sequence operator to work on $X^{(0)}$, the first order sequence $X^{(0)}D = (x^{(0)}(1)d, x^{(0)}(2)d, \ldots, x^{(0)}(n)d)$ is defined as

$$x^{(0)}(k)d = \sum_{i=1}^{k} x^{(0)}(i); k \in N_n^{+} \tag{3}$$

where N_n^{+} is a set of all integers from 1 to n, then D is called an accumulating (generation) operator.[31]

4. GREY INCIDENCE ANALYSIS and GREY CLUSTER EVALUATION MODEL

In the study of a complex system, many factors are involved, and it is the mutual interactions of these factors that determine its behavior. Grey incidence analysis[31] provides a new method to analyze which factors have primary influence, while others have less influence, on the development of the system.[41]

Several different models of grey incidence analysis have been put forwarded: Such as the model of degree of grey incidence defined by Professor Deng Julong,[31] the model of absolute degree of grey incidence,[33] the model of relative degree of grey incidence,[33] the model of synthetic degree of grey incidence,[33] the grey incidence models based on similarity and nearness,[43] and so on. The most commonly used model is the model of absolute degree of grey incidence.

Definition 5 Absolute degree of grey incidence

Let $X_i, i \in N_2^+$ be two sequences with the same length that is defined as the sum of the distances between two consecutive time moments, shown as

$$s_i = \int_1^n (X_i - x_i(1))dt \quad i \in N_2^+$$

$$s_1 - s_2 = \int_1^n \{[(X_1 - x_1(1)] - [X_2 - x_2(1)]\})dt$$

Then

$$\varepsilon_{12} = \frac{1 + |s_1| + |s_2|}{1 + |s_1| + |s_2| + |s_1 - s_2|} \tag{4}$$

is referred to as the absolute degree of grey incidence between X_1 and X_2.

The grey cluster evaluation model is mainly applicable to checking whether or not the observational objects belong to pre-determined classes such that they can be treated differently. The grey cluster model with variable weight,[32] the grey cluster model with fixed weight,[33] the grey cluster model based on endpoint triangular whitenization functions,[33] and the grey cluster model based on end-point and center-point triangular whitenization functions are the commonly used grey cluster evaluation models based on whitenization weight functions.[44,63]

Assume that *n* objects are clustered into *s* different grey classes according to *m* criteria. The observed value of object *i* in terms of criterion *j* is $x_{ij}, i \in N_n^+, j \in N_m^+$. The object *i* needs to be evaluated and diagnosed based on x_{ij}. The particular computational steps of the grey evaluation model based on endpoint triangular whitenization functions are as follows:

Step1 Based on the predetermined number *s* of grey classes for the planned evaluation, divide the individual ranges of the criteria into *s* grey classes. For example, let $[a_1, a_{s+1}]$ be the range of the values of criterion *j*. Now, divide $[a_1, a_{s+1}]$ into *s* grey classes as follows

$$[a_1, a_2], \ldots, [a_{k-1}, a_k], \ldots, [a_{s-1}, a_s], [a_s, a_{s+1}]$$

where $a_k, k \in N_{s+1}^+$, in general, can be determined based on specific requirements of a situation or relevant qualitative analysis.

Step2 Calculate the geometric midpoint between the various cells,

$\lambda_k = (a_k + a_{k+1})/2$, $k \in N_s^+$

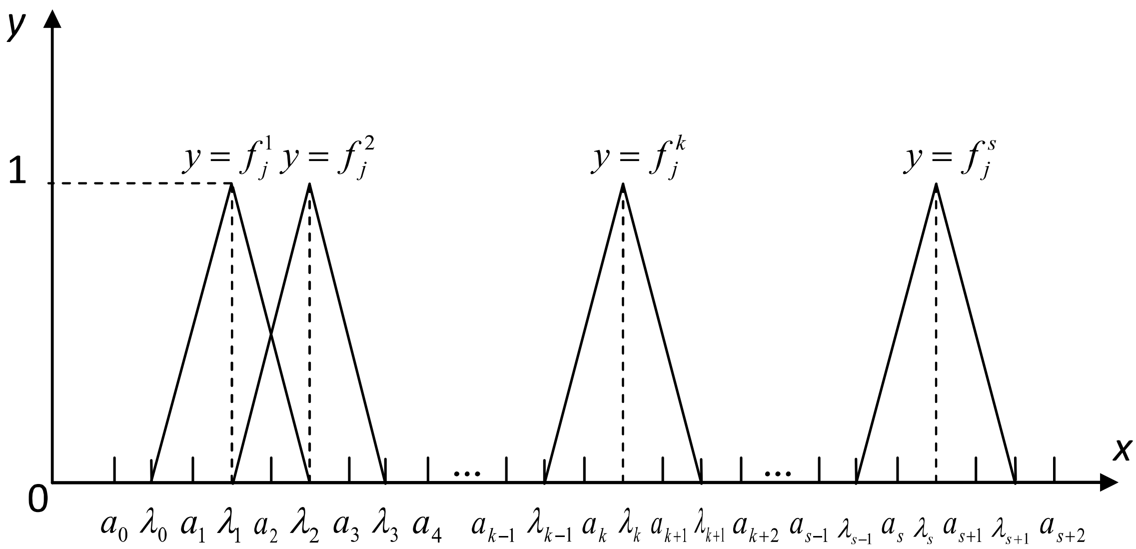

Fig. 1 General end-point triangular whitenization weight function

Step 3 Let the whitenization weight function value for λ_k to belong to the *k*th grey class being 1. When $(\lambda_k, 1)$ is connected to the starting point λ_{k-1} of the (*k*-1)th grey class and the ending point λ_{k+1} of the ($k+1$)th grey class, a triangular whitenization weight function $f_j^k(\cdot)$ is obtained in terms of criterion *j* about the *k*th grey class, $j \in N_m^+, k \in N_s^+$. For $f_j^1(\cdot)$ and $f_j^s(\cdot)$, the range of criterion *j* can be extended to the left and the right to a_0 and a_{s+2}, respectively (see fig. 1).

For an observed value *x* of criterion *j*, its degree $f_j^k(x)$ of membership in the *k*th grey class, $j \in N_m^+, k \in N_s^+$, can be computed as follows:

$$f_j^k(x) = \begin{cases} 0, & x \notin [\lambda_{k-1}, \lambda_{k+1}] \\ \dfrac{x - \lambda_{k-1}}{\lambda_k - \lambda_{k-1}}, & x \in [\lambda_{k-1}, \lambda_k) \\ \dfrac{\lambda_{k+1} - x}{\lambda_{k+1} - \lambda_k}, & x \in [\lambda_k, \lambda_{k+1}] \end{cases} \tag{5}$$

Step 4 Compute the comprehensive clustering coefficient for object *i* with respect to grey class *k*, that is

$$\sigma_i^k = \sum_{j=1}^{m} f_j^k(x_{ij}) \cdot \eta_j, i \in N_n^+, k \in N_s^+ \tag{6}$$

where $f_j^k(x_{ij})$ is the whitenization weight function of the *k*th subclass of the *j*th criterion, and η_j the weight of criterion *j* in the comprehensive clustering.

Step 5 From $\max\{\sigma_i^k\}=\sigma_i^{k^*}$, it follows that object *i* belongs to grey class k^*. When several objects belong to the same k^* grey class, the order of preference of these objects in grey class k^* can be further determined by using the magnitudes of their clustering coefficients.

The computational steps of the grey cluster model based on center-point triangular whitenization functions are mainly common but different form the following step 1.

Step 6: Based on the s grey classes required by the evaluation task, respectively, determine the centers $\lambda_1, \lambda_2, \Lambda, \lambda_{s-1}$, and λ_s of the grey class 1,2, …, *s*-1, and *s*.

The centers stand for the particular points for the observed values to belong to particular grey classes, and they could be either the centers of their individual interval or not. Also, the field of each criterion is correspondingly divided into *s* grey classes, which are represented respectively using their centers $\lambda_k, k \in N_s^+$.

The grey cluster evaluation model based on end-point triangular whitenization weight functions is applicable to the case that the boundary of each grey class is clear but the most possible point of each grey class is unknown. The grey cluster evaluation based on center-point triangular whitenization weight functions is applicable to the case that the most possible point of each grey class is known but the boundary of each grey class is unclear.

5. GREY PREDICTION MODEL

Grey prediction models can be used to predict the future behavior of system variables with small samples. Based on the practical circumstances, various grey prediction models such as GM(1,1), remnant GM(1,1), GM(0,N), and power GM(1,1) models can be used. The most commonly used model in grey system theory for prediction of sequences is GM(1,1).[32-36]

Definition 6 Model GM(1,1)

Assume that $X^{(0)}=(x^{(0)}(1), x^{(0)}(2), \Lambda, x^{(0)}(n))$ is a sequence of raw data which satisfy the law of grey exponents, $X^{(1)}$ = $X^{(1)}=(x^{(1)}(1), x^{(1)}(2), \Lambda, x^{(1)}(n)), n \ge 4$, is its accumulation generated sequence, and $Z^{(1)}=(z^{(1)}(2), z^{(1)}(3), \Lambda, z^{(1)}(n))$ is the sequence generated from $X^{(1)}$ by adjacent neighbor means, where $z^{(1)}(k)=\frac{1}{2}(x^{(1)}(k)+x^{(1)}(k-1))$, $k=2,3,...,n$. Then, $x^{(0)}(k)+az^{(1)}(k)=b$ is referred to as the model GM(1,1), where the symbol GM(1,1) stands for "the first order grey model with one variable".[32]

Assume that $X^{(0)}$ is non-negative. $\hat{a}=(a,b)^T$ is the sequence of parameters in model GM(1,1), then the least square estimate sequence of the model GM(1,1) satisfies $\hat{a}=(a,b)^T=(B^TB)^{-1}B^TY$, where

$$Y=\begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ \vdots \\ x^{(0)}(n) \end{bmatrix}, \quad B=\begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ \vdots & \vdots \\ -z^{(1)}(n) & 1 \end{bmatrix} \tag{7}$$

Example 1 Given a sequence of raw data $X^{(0)}=(x^{(0)}(1),x^{(0)}(2),x^{(0)}(3),x^{(0)}(4),x^{(0)}(5))$ = (2.874, 3.278, 3.337, 3.390, 3.679), simulate this sequence $X^{(0)}$ by GM(1,1).

Solution

$$X^{(1)}=(x^{(1)}(1),x^{(1)}(2),x^{(1)}(3),x^{(1)}(4),x^{(1)}(5)) = (2.874, 6.152, 9.489, 12.897, 16.558)$$

$$Z^{(1)}=(z^{(1)}(2),z^{(1)}(3),z^{(1)}(4),z^{(1)}(5)) = (4.513, 7.820, 11.184, 14.718)$$

Therefore

$$B=\begin{bmatrix} -z^{(1)}(2) & 1 \\ -z^{(1)}(3) & 1 \\ -z^{(1)}(4) & 1 \\ -z^{(1)}(5) & 1 \end{bmatrix}=\begin{bmatrix} -4.513 & 1 \\ -7.820 & 1 \\ -11.184 & 1 \\ -14.718 & 1 \end{bmatrix} \quad Y=\begin{bmatrix} x^{(0)}(2) \\ x^{(0)}(3) \\ x^{(0)}(4) \\ x^{(0)}(5) \end{bmatrix}=\begin{bmatrix} 3.278 \\ 3.337 \\ 3.390 \\ 3.679 \end{bmatrix}$$

$$\hat{a}=(B^TB)^{-1}\ B^TY=\begin{bmatrix} -0.03720 \\ 3.06536 \end{bmatrix}$$

From $\hat{x}^{(1)}(k)=(x^{(0)}(1)-\frac{b}{a})e^{-a(k-1)}+\frac{b}{a}$ $=85.276151e^{0.0372k}-82.402151$, the simulation values are obtained

$$\hat{X}^{(1)}=(\hat{x}^{(1)}(1),\hat{x}^{(1)}(2),\hat{x}^{(1)}(3),\hat{x}^{(1)}(4),\hat{x}^{(1)}(5)) = (2.8704, 6.1060, 9.4605, 12.9422, 16.5558)$$

$$\hat{X}^{(0)}= (2.8740, 3.2320, 3.3545, 3.4817, 3.6136)$$

The sequence of errors is

$$\varepsilon(k)=X^{(0)}(k)-\hat{X}^{(0)}(k)$$
$$=(0,\ 0.046,\ -0.0175,\ -0.0917,\ 0.0654)$$

6. GREY MODELS FOR DECISION - MAKING

Grey decision-making is about making decisions by using such decision models that involve grey elements or combine the general decision model and grey system models. The emphasis of grey decision-making is the study on the problem of choosing a specific scheme.

Definition 7 Scheme

Assume that $A=\{a_1,a_2,...a_n\}$ with $a_i, i\in N_n^+$ as the ith event, is the set of all events within the range of a study, and $B = \{b_1, b_2, \Lambda, b_m\}$ with b_j, $j \in N_m^+$ as the *j*th countermeasure, is the set of all possible countermeasures. Then the Cartesian product $S = A \times B = \{(a_i, b_j) | a_i \in A, b_j \in B\}$ is called a schemes set, where each ordered pair $s_{ij} = (a_i, b_j)$, for $a_i \in A$ and $b_j \in B$, is called a scheme[32].

For a given scheme $s_{ij} \in S$, evaluating the effects under a set of pre-determined objectives and the decision is based on the evaluation on what to take and what to let go is the decision-making.

Definition 8 Grey target

Let $d_1^{(k)}$ and $d_2^{(k)}$ be the upper and lower threshold values of the effects $u_{ij}^{(k)}$ of the scheme $s_{ij} \in S$ under objective *k*. Then $S^1 = \{r | d_1^{(k)} \le r \le d_2^{(k)}\}$ is called a one-dimensional grey target of objective *k*, and $u_{ij}^{(k)} \in [d_1^{(k)}, d_2^{(k)}]$ a satisfactory effect under objective *k*, the corresponding s_{ij} a desirable scheme with respect to objective *k*, and b_j a desirable countermeasure of event a_i with respect to objective *k*.[32]

Generally, assume that $d_1^{(1)}, d_2^{(1)}$; $d_1^{(2)}, d_2^{(2)}; \Lambda$; and $d_1^{(s)}, d_2^{(s)}$ respectively are the threshold values of effect vector u_{ij} = $(u_{ij}^{(1)}, u_{ij}^{(2)}, \Lambda, u_{ij}^{(s)}) \in S^s$ of the scheme $s_{ij} \in S$ under objectives 1, 2, …, and *s*. Then the following region of the s-dimensional Euclidean space

$$S^s = \{(r^{(1)}, r^{(2)}, \Lambda, r^{(s)}) | d_1^{(1)} \le r^{(1)} \le d_2^{(1)}, d_1^{(2)} \le r^{(2)} \le d_2^{(2)}, \Lambda, d_1^{(s)} \le r^{(s)} \le d_2^{(s)}\}$$

is called a grey target of a *s*-dimensional decision-making. Similarly, we can define satisfactory effect vector, desirable scheme and desirable countermeasures with respect to objectives 1, 2, …, and *s*.

The weighted multi-attribute grey target decision model is presented as follows:

Definition 9 Effect measure[56]

(1) Let k be a benefit type objective, that is, for k the larger the effect sample value is the better, and the decision grey target of objective k is $u_{ij}^{(k)} \in [u_{i_0 j_0}^{(k)}, \max_i \max_j \{u_{ij}^{(k)}\}]$, that is, $u_{i_0 j_0}^{(k)}$ stands for the threshold effect value of objective k. Then

$$r_{ij}^{(k)} = \frac{u_{ij}^{(k)} - u_{i_0 j_0}^{(k)}}{\max_i \max_j \{u_{ij}^{(k)}\} - u_{i_0 j_0}^{(k)}} \tag{8}$$

is referred to as the effect measure of a benefit-type objective.

(2) Let k be a cost type objective, that is, for k the smaller the effect value is the better, and the decision grey target of objective k is $u_{ij}^{(k)} \in [\min_i \min_j \{u_{ij}^{(k)}\}, u_{i_0 j_0}^{(k)}]$, that is, $u_{i_0 j_0}^{(k)}$ stands for the threshold effect value

of objective k. Then

$$r_{ij}^{(k)} = \frac{u_{i_0j_0}^{(k)} - u_{ij}^{(k)}}{u_{i_0j_0}^{(k)} - \min_i \min_j \left\{u_{ij}^{(k)}\right\}} \tag{9}$$

is referred to as the effect measure of cost type objective.

(3) Let k be a moderate-value type objective, that is, for k the closer to a moderate value A the effect value is the better, and the decision grey target of objective k is $u_{ij}^{(k)} \in [A - u_{i_0j_0}^{(k)}, A + u_{i_0j_0}^{(k)}]$, that is, both $A - u_{i_0j_0}^{(k)}$ and $A + u_{i_0j_0}^{(k)}$ are respectively the lower and upper threshold effect values of objective k. Then

(i) When $u_{ij}^{(k)} \in [A - u_{i_0j_0}^{(k)}, A]$, then

$$r_{ij}^{(k)} = \frac{u_{ij}^{(k)} - A + u_{i_0j_0}^{(k)}}{u_{i_0j_0}^{(k)}} \tag{10}$$

is referred to as the lower effect measure of moderate-value type objective.

(ii) When $u_{ij}^{(k)} \in [A, A + u_{i_0j_0}^{(k)}]$, then

$$r_{ij}^{(k)} = \frac{A + u_{i_0j_0}^{(k)} - u_{ij}^{(k)}}{u_{i_0j_0}^{(k)}} \tag{11}$$

is referred to as the upper effect measure of moderate-value type objective.

The effect measures of benefit type objectives represent the degrees of both how close the effect sample values are to the maximum sample values and how far away they are from the threshold effect values of the objectives. Similarly, the effect measures of cost type objectives represent the degrees of how close the effect sample values are to the minimum effect sample values and how far away the effect sample values are from the threshold effect values of the objectives. The lower effect measures of moderate value type objectives represent the degrees of both how close the effect sample values which are smaller than the moderate value A are to the moderate value A and how far away they are from the lower threshold effect value of the objectives, and the upper effect measures of moderate value type objectives represent the degrees of both how close the effect sample values which are larger than the moderate value A are to the moderate value A and how far away they are from the upper threshold effect value of the objectives.

For the situation of undershoot, it can also correspondingly be divided into the following four kinds:

- The effect value of a benefit-type objective is smaller than the threshold value $u_{i_0j_0}^{(k)}$, that is, $u_{ij}^{(k)} < u_{i_0j_0}^{(k)}$.
- The effect value of a cost-type objective is larger than the threshold value $u_{i_0j_0}^{(k)}$, that is, $u_{ij}^{(k)} > u_{i_0j_0}^{(k)}$.
- The effect value of a moderate-value type objective is smaller than the lower threshold effect value $A - u_{i_0j_0}^{(k)}$, that is, $u_{ij}^{(k)} < A - u_{i_0j_0}^{(k)}$.
- The effect value of a moderate-value type objective is larger than the

upper threshold effect value $A+u_{i_0j_0}^{(k)}$, that is, $u_{ij}^{(k)} > A+u_{i_0j_0}^{(k)}$.

In order to make the effect measures of each type objective satisfy the condition of normality, that is, $r_{ij}^{(k)} \in [-1,1]$. Without loss of generality, we can assume that:

(1) For a benefit type objective, $u_{ij}^{(k)} \geq -\max\limits_i \max\limits_j \left\{u_{ij}^{(k)}\right\} + 2u_{i_0j_0}^{(k)}$.

(2) For a cost type objective, $u_{ij}^{(k)} \leq -\min\limits_i \min\limits_j \left\{u_{ij}^{(k)}\right\} + 2u_{i_0j_0}^{(k)}$.

(3) For the situation that the effect value of a moderate-value type objective is smaller than the lower threshold effect value $A-u_{i_0j_0}^{(k)}$, $u_{ij}^{(k)} \geq A-2u_{i_0j_0}^{(k)}$.

(4) For the situation that the effect value of a moderate-value type objective is larger than the upper threshold effect value $A+u_{i_0j_0}^{(k)}$, $u_{ij}^{(k)} \leq A+2u_{i_0j_0}^{(k)}$.

Proposition 1 The effect measure $r_{ij}^{(k)}(i=1,2,\cdots,n; j=1,2,\cdots,m; k=1,2,\cdots,s)$, as defined in Definition 8, satisfies the following properties: (1) $r_{ij}^{(k)}$ are non-dimensional; (2) the more ideal the effect is, the larger $r_{ij}^{(k)}$ are; (3) $r_{ij}^{(k)} \in \left[-1,1\right]$. Therefore, $r_{ij}^{(k)}$ are called the uniform effect measures.

For the situation of hitting the target, $r_{ij}^{(k)} \in \left[0,1\right]$; and for the situation of missing the target, $r_{ij}^{(k)} \in \left[-1,0\right]$.[56]

Definition 9 Synthetic effect measure

Assume that η_k stands for the decision weight of objectives *k*, $k=1,2,\cdots,s$, and satisfies $\sum\limits_{k=1}^{s}\eta_k = 1$, then $\sum\limits_{k=1}^{s}\eta_k \cdot r_{ij}^{(k)}$ is called a synthetic effect measure of the scheme s_{ij}, which is still denoted as $r_{ij} = \sum\limits_{k=1}^{s}\eta_k \cdot r_{ij}^{(k)}$.

And $R=(r_{ij})_{n\times m}$ is known as the matrix of synthetic effect measures.[56]

The weighted multi-attribute grey target decision can be made by the following steps:

Step 1 Based on the set $A=\{a_1,a_2,\cdots,a_n\}$ of events and the set $B=\{b_1,b_2,\cdots,b_m\}$ of countermeasures, construct the set of schemes $S=\{s_{ij}=(a_i,b_j)\,|\,a_i \in A, b_j \in B\}$.

Step 2 Determine the decision objectives *k*, $k=1,2,\cdots,s$.

Step 3 Determine the decision weights $\eta_1,\eta_2,\cdots,\eta_s$ of the objectives.

Step 4 For each objective *k*, compute the corresponding observed effect matrix $U^{(k)}=(u_{ij}^{(k)})_{n\times m}$.

Step 5 Determine the threshold effect value of objective *k*.

Step 6 Calculate the matrix $R^{(k)} = (r_{ij}^{(k)})_{n\times m}$ of uniform effect measures of objectives k.

Step 7 From $r_{ij} = \sum_{k=1}^{s} \eta_k \cdot r_{ij}^{(k)}$, compute the matrix of synthetic effect measures $R = (r_{ij})_{n\times m}$.

Step 8 Determine the optimum scheme $s_{i_0 j_0}$.

Example 2 Select the supplier of key components of large commercial aircrafts by the weighted multi-attribute grey target decision model.

Solution The following five objectives are considered: quality, price, time of delivery, design proposal, and competitiveness.

Among these objectives, competitiveness, quality, and design proposal are qualitative. They are scored by relevant experts' evaluations satisfying that the higher the evaluation scores are the better. That is, they are benefit type objectives. Take the threshold value $u_{i_0 j_0}^{(k)} = 9, k = 1,4,5$. For the objective of cost, the lower the cost the better. Therefore, it is a cost type objective. Take the threshold value $u_{i_0 j_0}^{(2)} = 15$. The objective of "time of delivery" is one of moderate-value type. The main manufacturer desires the delivery at the end of the 16th month with an allowed two-month deviation. That is, $u_{i_0 j_0}^{(3)} = 2$, the lower threshold effect value is 16 – 2 = 14, and the upper threshold effect value is 16 + 2 = 18.

The AHP method is used to determine the weights. The detailed procedure is omitted here and table 2 shows the obtained weight for each objective:

Table 2 Evaluation system of objectives

Objective	Quality	Price	Delivery	Design	Competitiveness
Unit	Qualitative	Million US$	Month	Qualitative	Qualitative
Order	1	2	3	4	5
Weight	0.25	0.22	0.18	0.18	0.17

The effect sample vectors of each of the objectives are as follows:

$$U^{(1)} = (9.5, 9.4, 9)$$

$$U^{(2)} = (14.2, 15.1, 13.9)\ U^{(3)} = (15.5, 17.5, 19),$$

$$U^{(4)} = (9.6, 9.3, 9.4)\ U^{(5)} = (9.5, 9.7, 9.2)$$

Using the uniform effect measure, we have

$$R^{(1)} = [1, 0.8, 0]$$

$$R^{(2)} = [0.73, -0.09, 1]\quad R^{(3)} = [0.75, 0.25, -0.5]$$

$$R^{(4)} = [1, 0.5, 0.67]\quad R^{(5)} = [0.71, 1, 0.29]$$

From $r_{ij} = \sum_{k=1}^{5} \eta_k \cdot r_{ij}^{(k)}$, we compute the following vector of synthetic effect measures:

$$R=[r_{11},r_{12},r_{13}]=\left[0.8463,0.4852,0.2999\right].$$

Because $r_{11}>0,r_{12}>0,r_{13}>0$, it means that all these three suppliers have hit the target. From $\max\limits_{1\le j\le 3}\{r_{1j}\}=r_{11}=0.8463$, it follows that the main manufacturer should sign the agreement with supplier 1.

7. CONCLUSION

The general grey numbers, the grey sequence operators and several most commonly used grey system models such as the absolute degree of grey incidence model, the grey cluster model based on endpoint triangular whitenization functions, the grey cluster model based on center-point triangular whitenization functions, the grey prediction model of the model GM(1,1), and the weighted multi-attribute grey target decision model are introduced. From this paper, we can see that the research on grey methods and models remains in the ascendant. Related theories and practice achievements emerge in endless. But as a new branch of systems science, it is unavoidable that there exist immature and imperfect parts in the grey systems theory. The studies on grey analysis models, grey prediction models, grey decision-making models, grey control models and common high-dimensional model are all in progressing stage. A large number of practical and scientific problems in reality remain to be excavated, explored and solved. The testing and specific quantitative standards of grey models remain to be further studied.

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