Optional slides: Supplementary Fourier Transform Material

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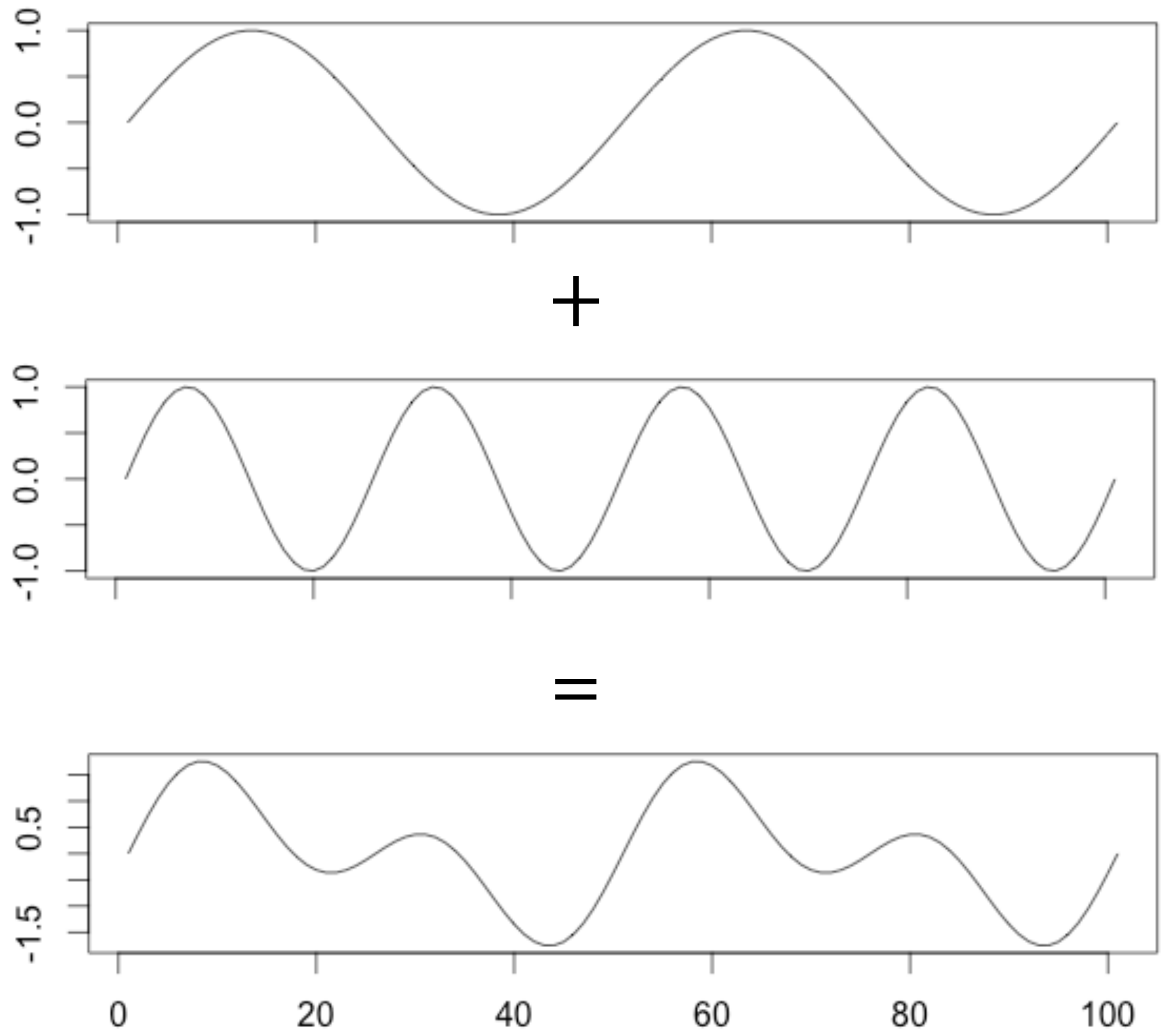
Beginning Fourier Analysis

Key principles from last term:

1) All media signals are functions

2) All functions can be expressed as sums of sinusoids

Sinusoids are “basis functions” →
You can add together carefully-chosen sinusoids to get any function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80
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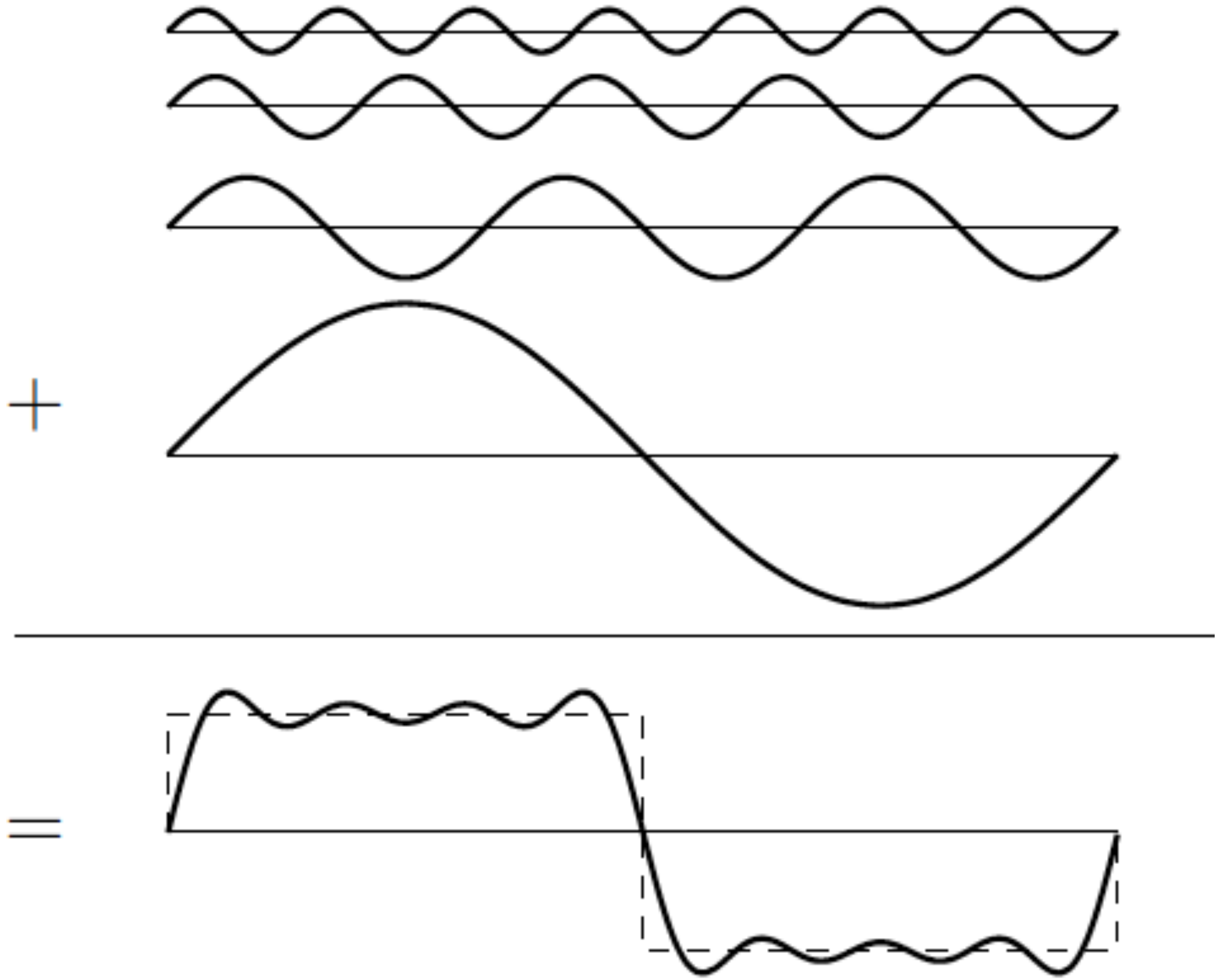
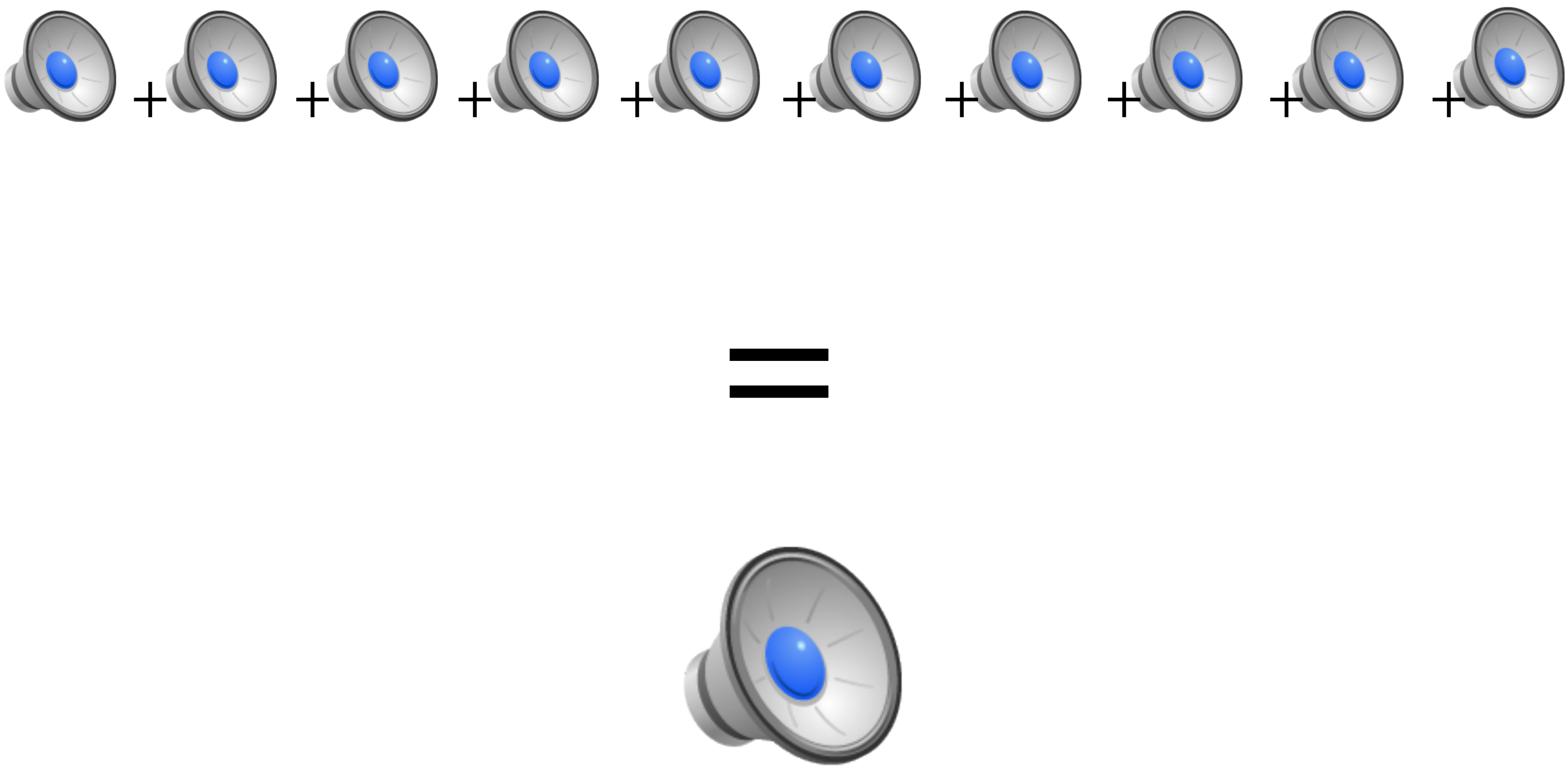
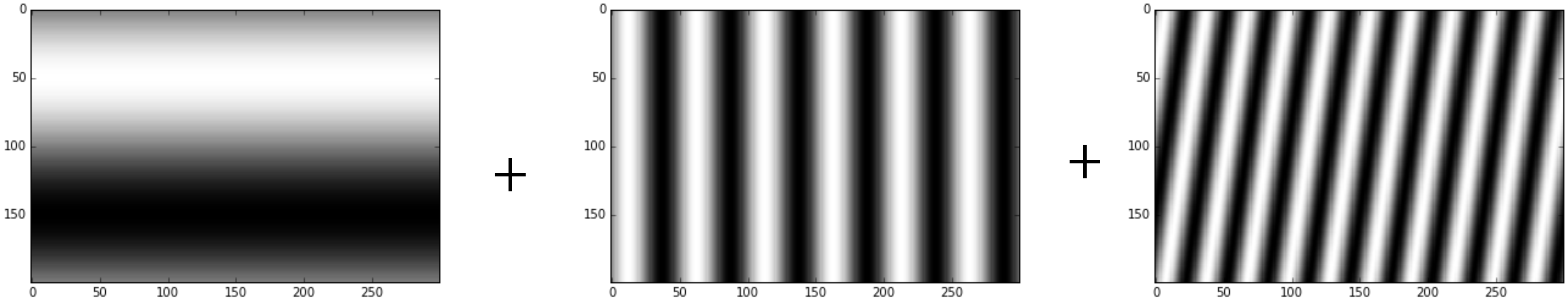
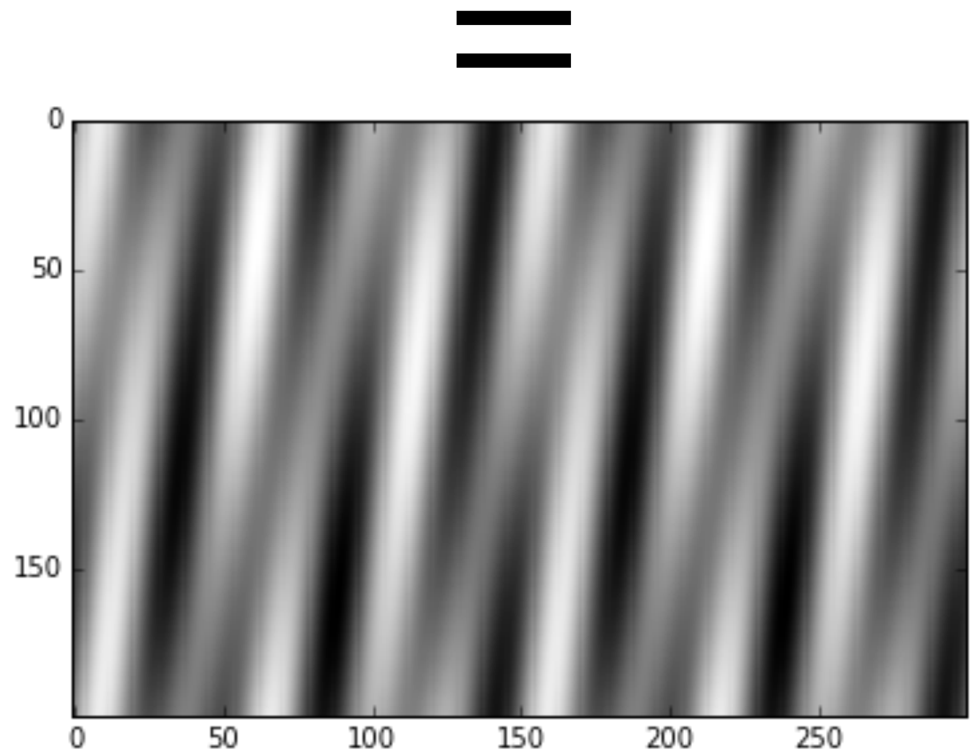





0 50 100 150
0 50 100 150 200 250
+
0 50 100 150
0 50 100 150 200 250
+
0 50 100 150
0 50 100 150 200 250


=
0 50 100 150
0 50 100 150 200 250

Fourier Theorem:

We can express *any** signal as a sum of sinusoids:

$$
\begin{aligned}
x = {} & A_1 \sin(2\pi f_1 t + \Phi_1) \\
& + A_2 \sin(2\pi f_2 t + \Phi_2) \\
& + A_3 \sin(2\pi f_3 t + \Phi_3) \\
& + \ldots
\end{aligned}
$$

**requires signal be either infinitely periodic or (more likely) finite in duration*

Fourier Analysis

Answers the question: Given a waveform, what are its sinusoidal components?

If $x(t) = A_1 \sin(2\pi f_1 t + \Phi_1)$
$+ A_2 \sin(2\pi f_2 t + \Phi_2)$
$+ A_3 \sin(2\pi f_3 t + \Phi_3)$
$+ \ldots$

What are A_k, f_k, Φ_k for all k?

spectrum of x

Why?

Audio:

- Tells us about pitch, timbre, instrumentation, mastering; speech/speaker; recording environment; ...
- Re-synthesize and process sounds (e.g. time stretch, pitch shift)
- Reason about how filters, reverb, EQ, etc. will affect a sound
- *Design* filters, reverb, EQ, etc.

Why?

Image:

- Tells us about objects, texture; text and image orientation
- Re-synthesize and process images (e.g., de-noising, blurring, sharpening)
- *Design* image effects

Other basis functions (e.g. wavelets) are great for other computer vision tasks

- E.g., find objects / faces in photos
- Dating Van Gogh's paintings (!!)

How Fourier Analysis Works

Fourier Analysis

Answers the question: Given a waveform, what are its sinusoidal components?

If $x(t) = A_1\sin(2\pi f_1 t + \Phi_1)$
$+ A_2\sin(2\pi f_2 t + \Phi_2)$
$+ A_3\sin(2\pi f_3 t + \Phi_3)$
$+ \ldots$

What are A_k, f_k, Φ_k for all k?

spectrum of x

1. Determining amplitudes: “How much” of a basis function (a specific frequency) is present in a signal?
2. Determining phase of each frequency: Reformulating phase as sine + cosine
3. Which frequencies do we really need?
4. Putting these together in a concise mathematical formula (the DFT)

1. Determining amplitude A_n for basis function at frequency f_n

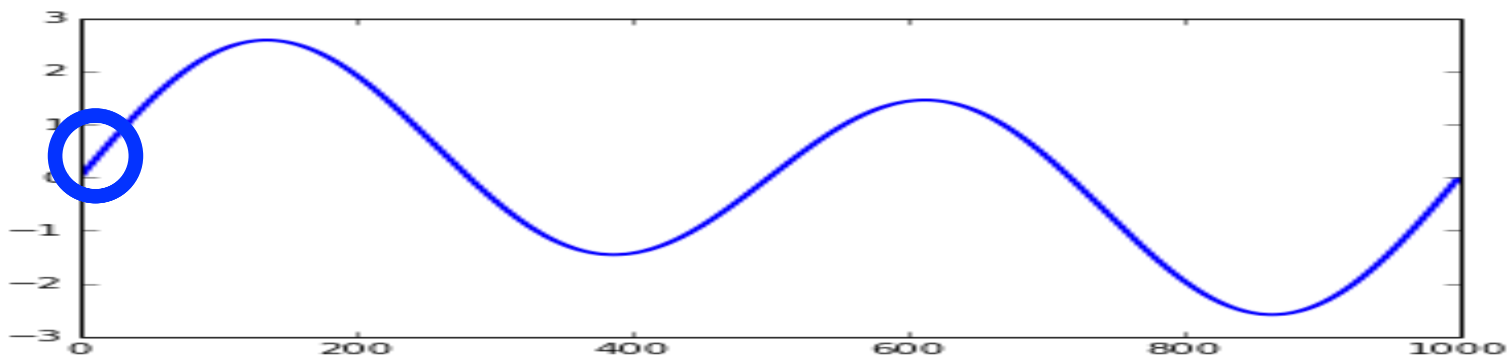
Let $x[n]$ be the n^{th} sample of x

$s_k[n]$ be the n^{th} sample of the basis sine at frequency k

and N be the length (# samples) of x

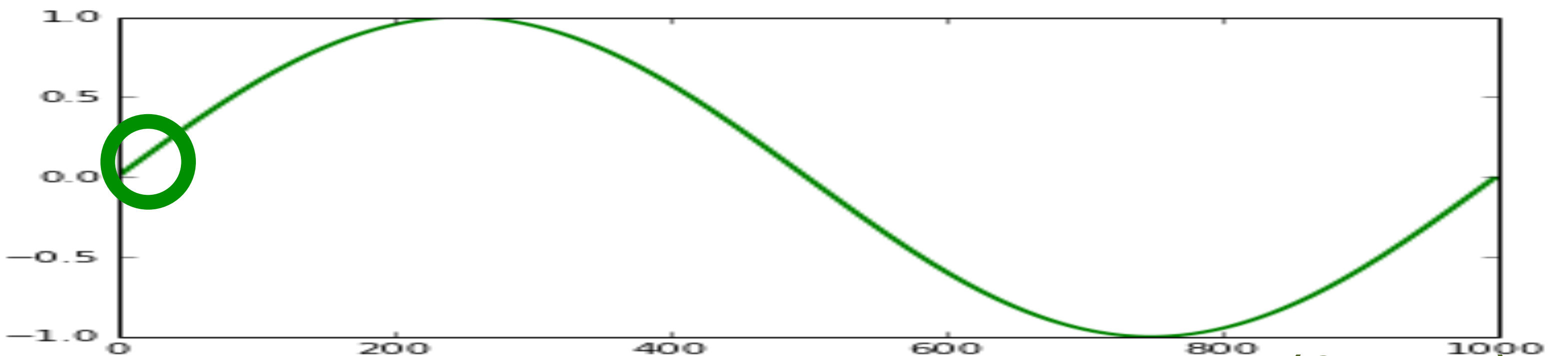
Compute:

$x[1] * s_k[1] + x[2] * s_k[2] + \ldots + x[N] * s_k[N]$



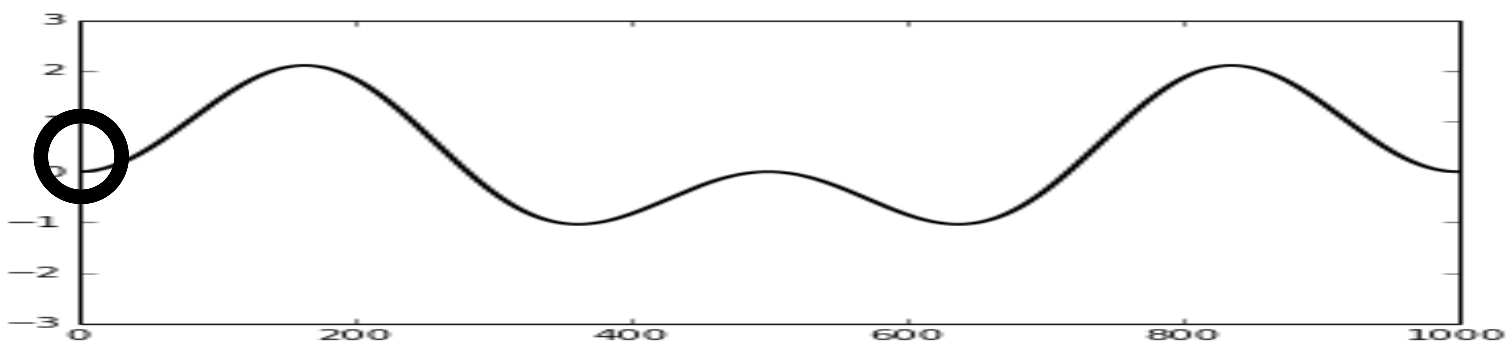
*

$x = 2 * \sin(2*pi*2*t) + .8 * \sin(2*pi*t)$

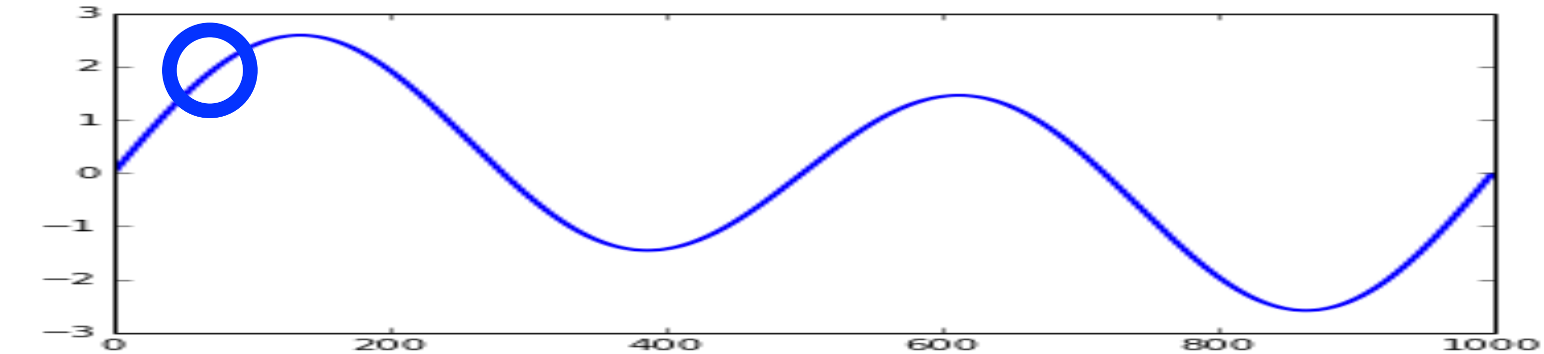


$s_1 = \sin(2*pi*t)$

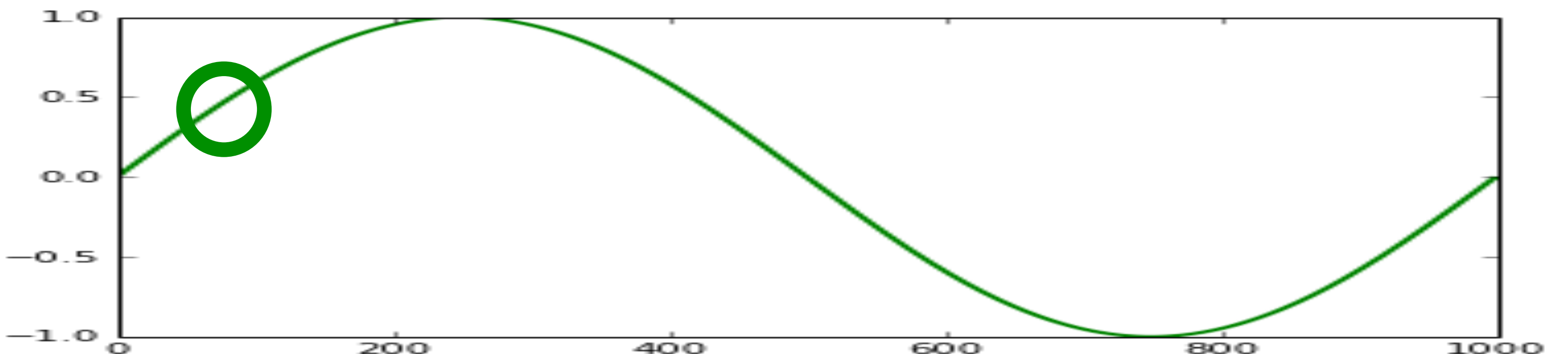
=



Point-wise product of x and s_1

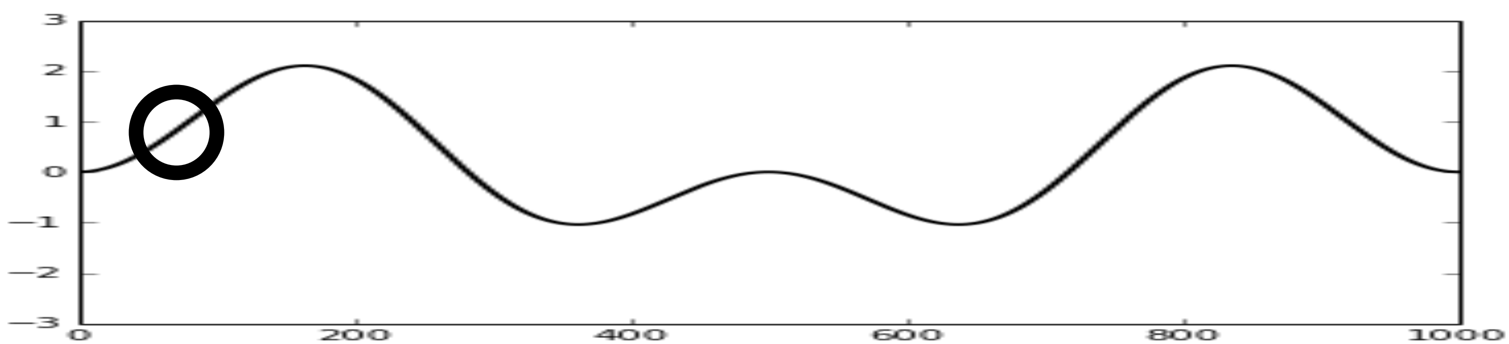


* x = 2 * sin(2*pi*2*t) + .8 * sin(2*pi*t)

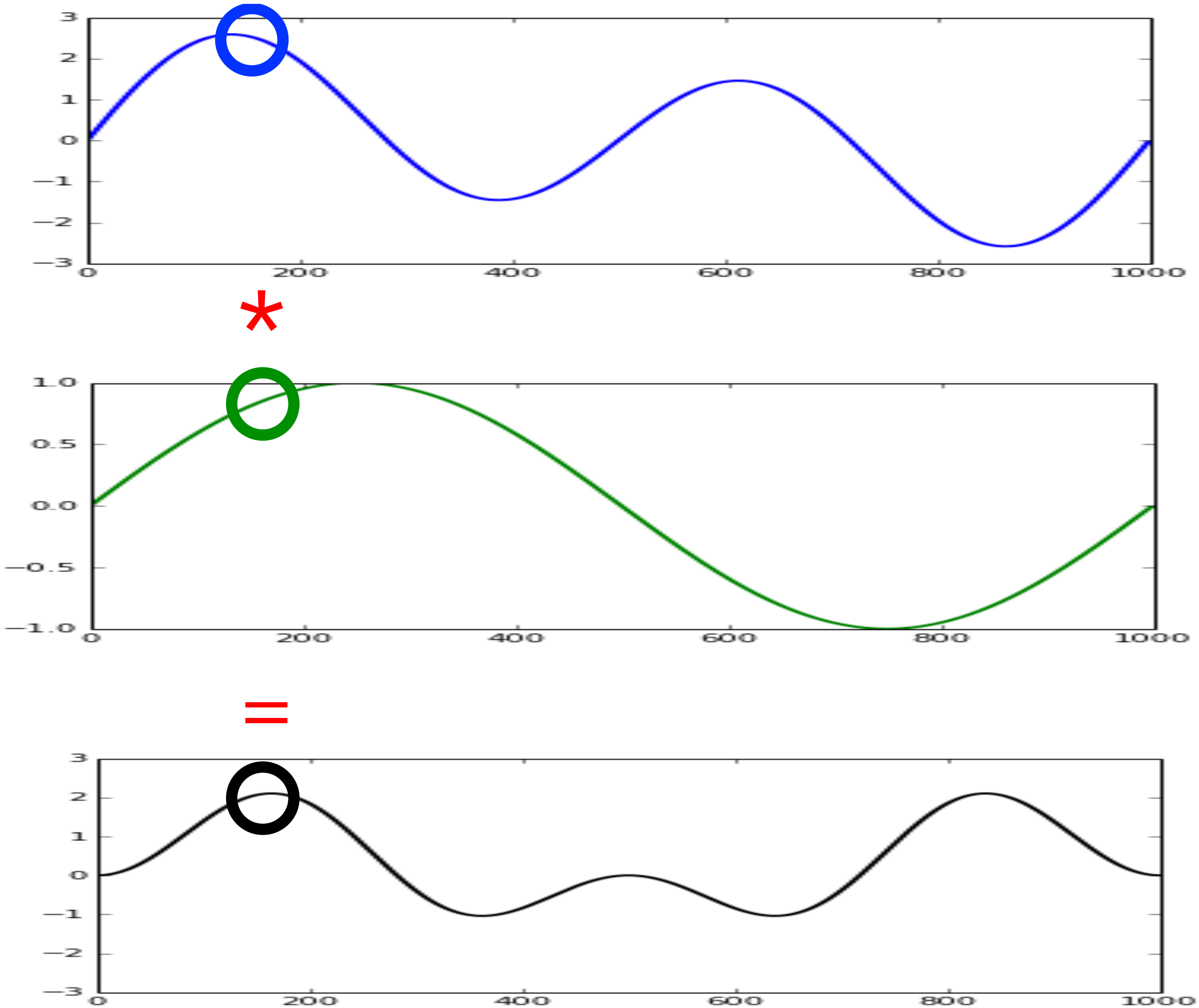


s_1 = sin(2*pi*t)

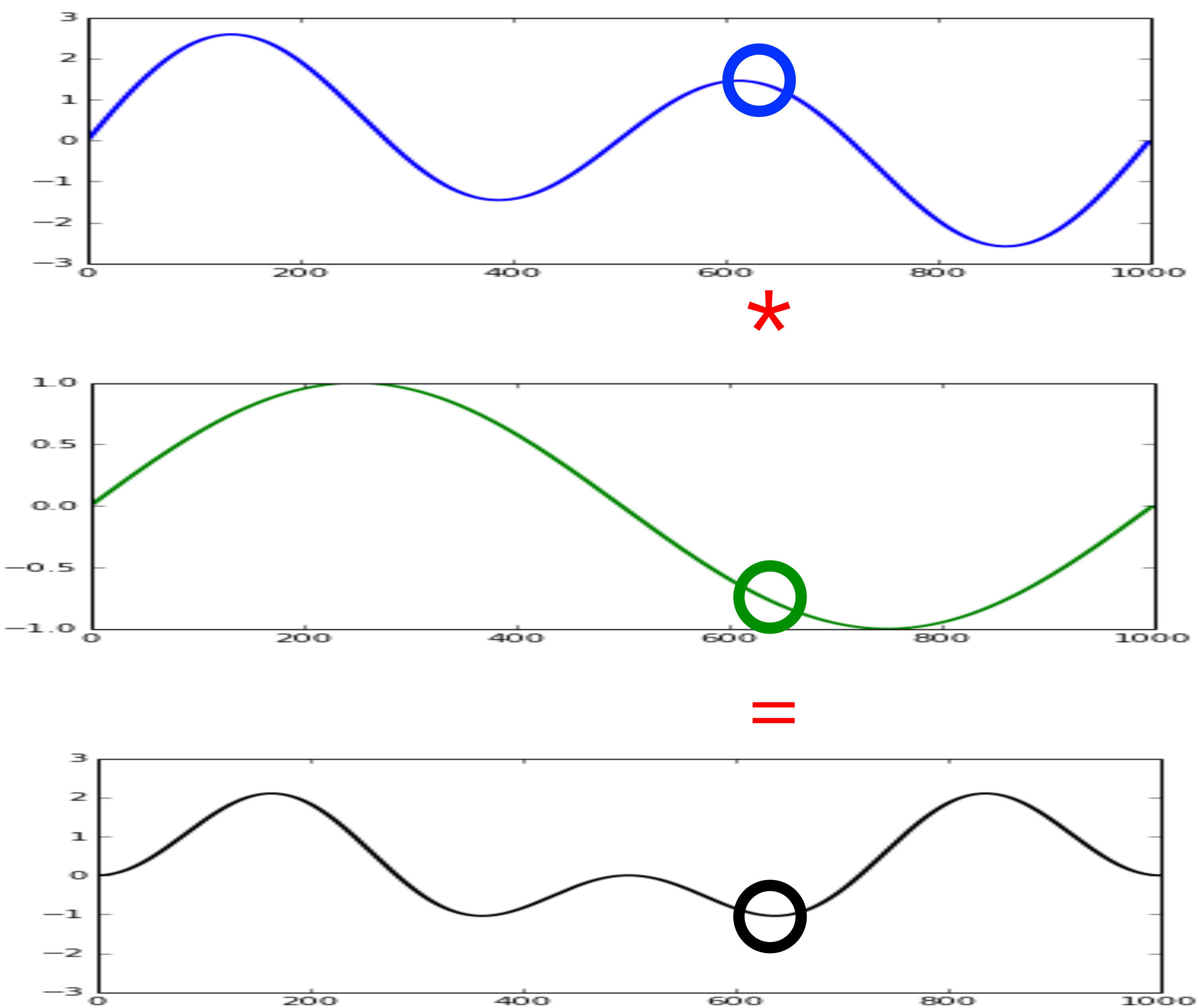
=



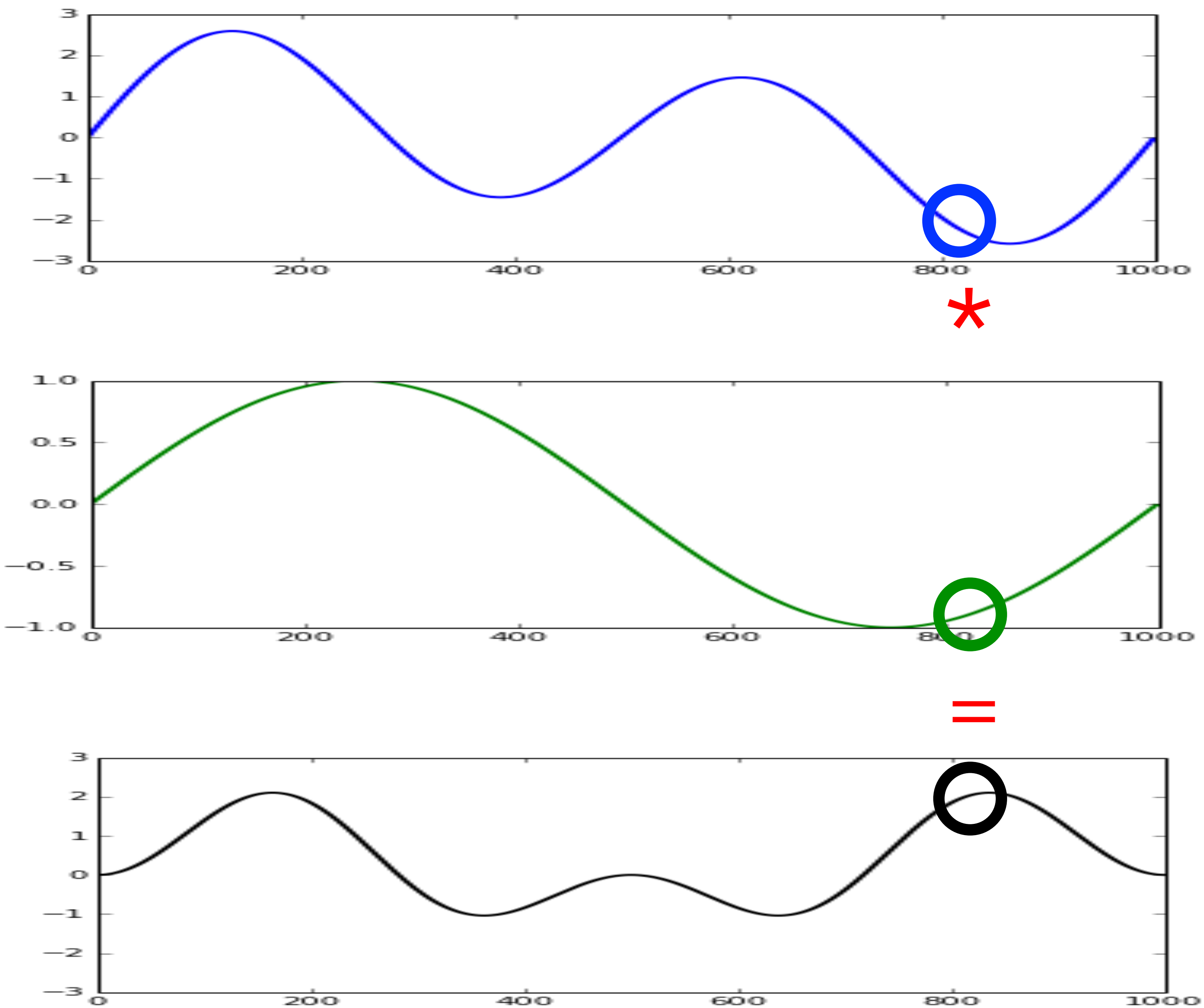
Point-wise product of x and s_1



Point-wise product of x and s_1

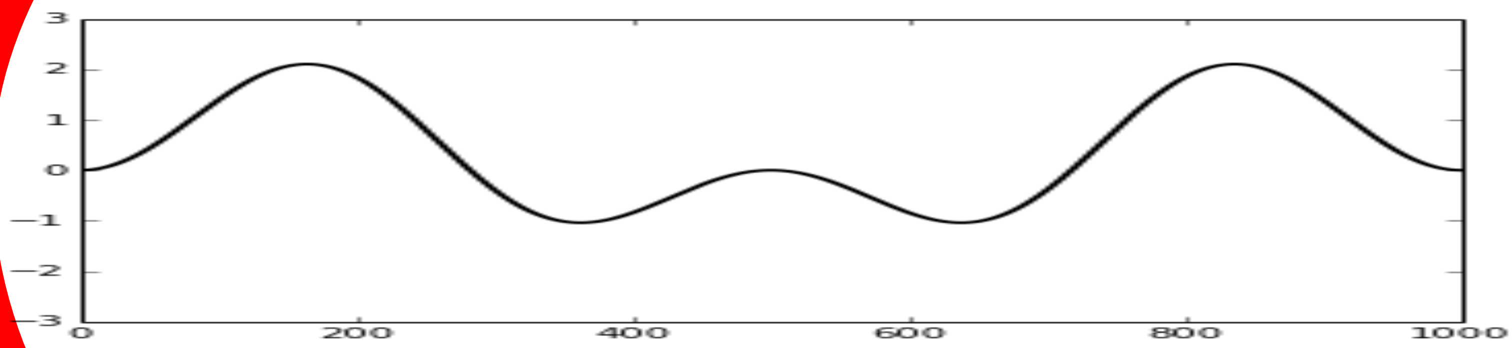


Point-wise product of x and s_1



Point-wise product of x and s_1

Finally, add up all values in this product:

sum()

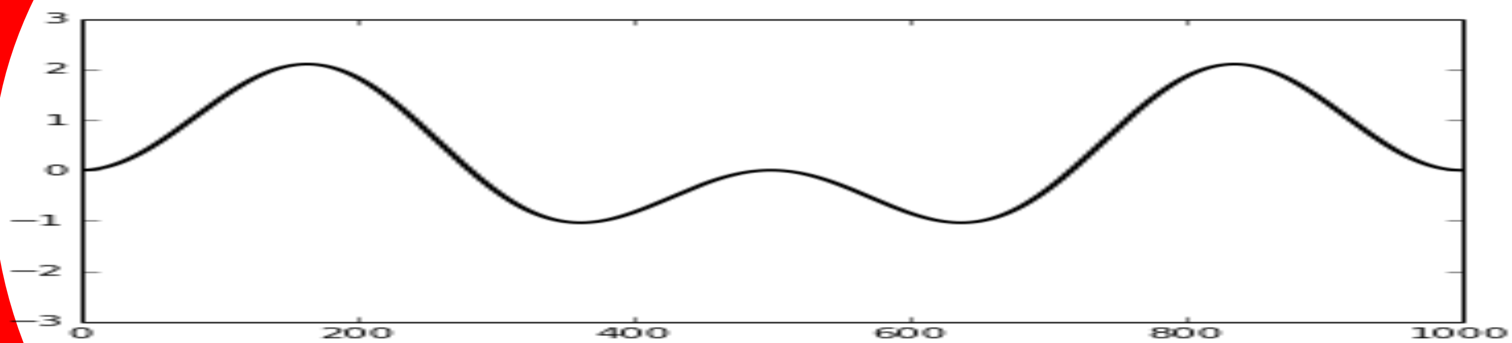
Point-wise product of x and s_1

The higher the content of this frequency f_k to the function x, the greater this sum!

If this frequency is not present in the signal, the sum will be 0!

In mathematical terminology:

sum(



)

Point-wise product of x and s_1

Compute A_{sn}, a measure of "weight" or contribution of frequency *n:*

$$A_{sk} = \sum_{n=1}^{N} x[n] \times s_k[n]$$

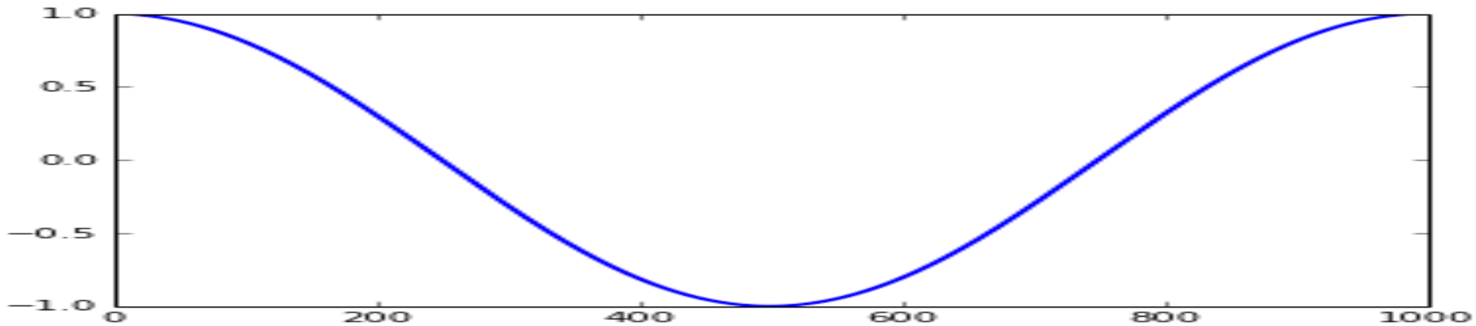
```python
t = arange(0, 1, 1./1000)
x = sin(2*pi*2*t) + .5*sin(2*pi*3*t)
s1 = sin(2*pi*1*t)
s2 = sin(2*pi*2*t)
s3 = sin(2*pi*3*t)
print sum(x * s1)
print sum(x * s2)
print sum(x * s3)
```

```
-3.0371321777e-14
500.0
250.0
```

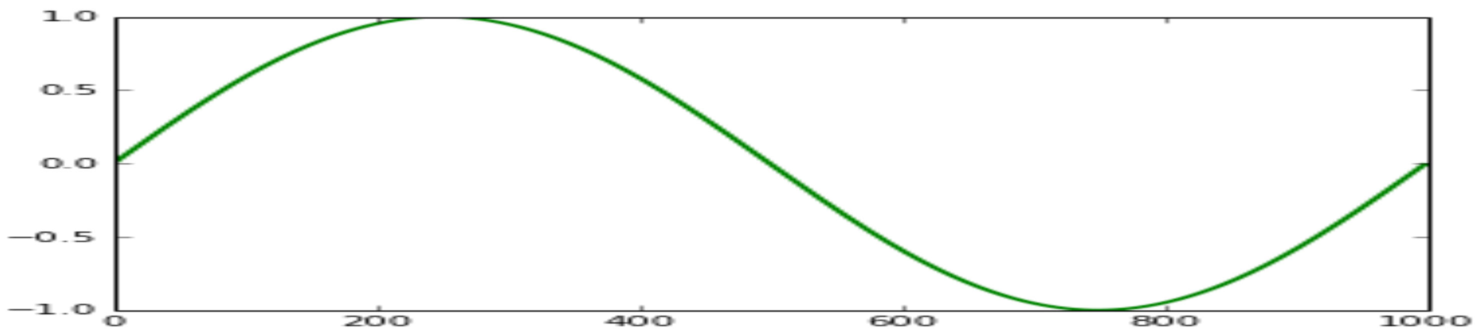
A problem:

```
t = arange(0, 1, 1./1000)
x1 = sin(2*pi*t + pi/2)
s1 = sin(2*pi*1*t)
print sum(x1 * s1)
```

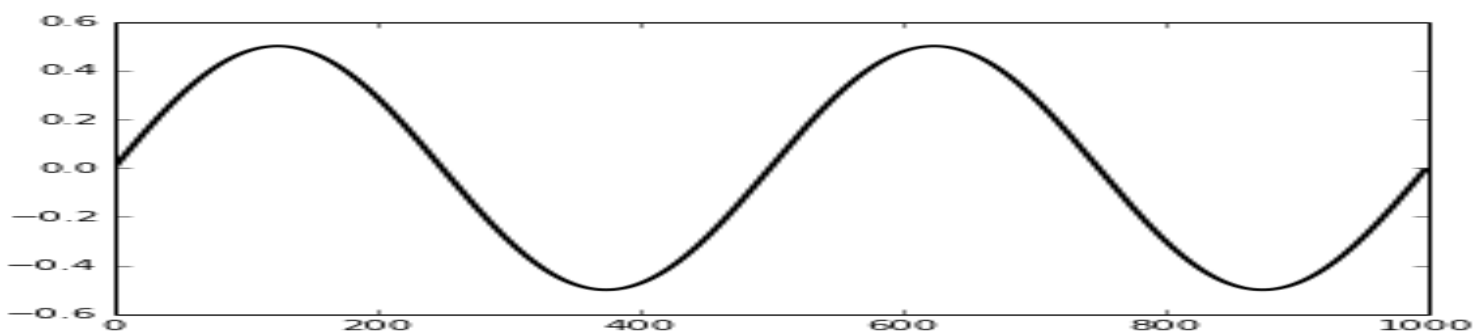
```
9.33116431345e-14
```


1.0
0.5
0.0
−0.5
−1.0
0
200
400
600
800
1000

*


1.0
0.5
0.0
−0.5
−1.0
0
200
400
600
800
1000

=


0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0
200
400
600
800
1000

Sums to 0!

Handling phase

Recognize that a sinusoid with non-zero phase is just a weighted sum of a sine and cosine at the same frequency!

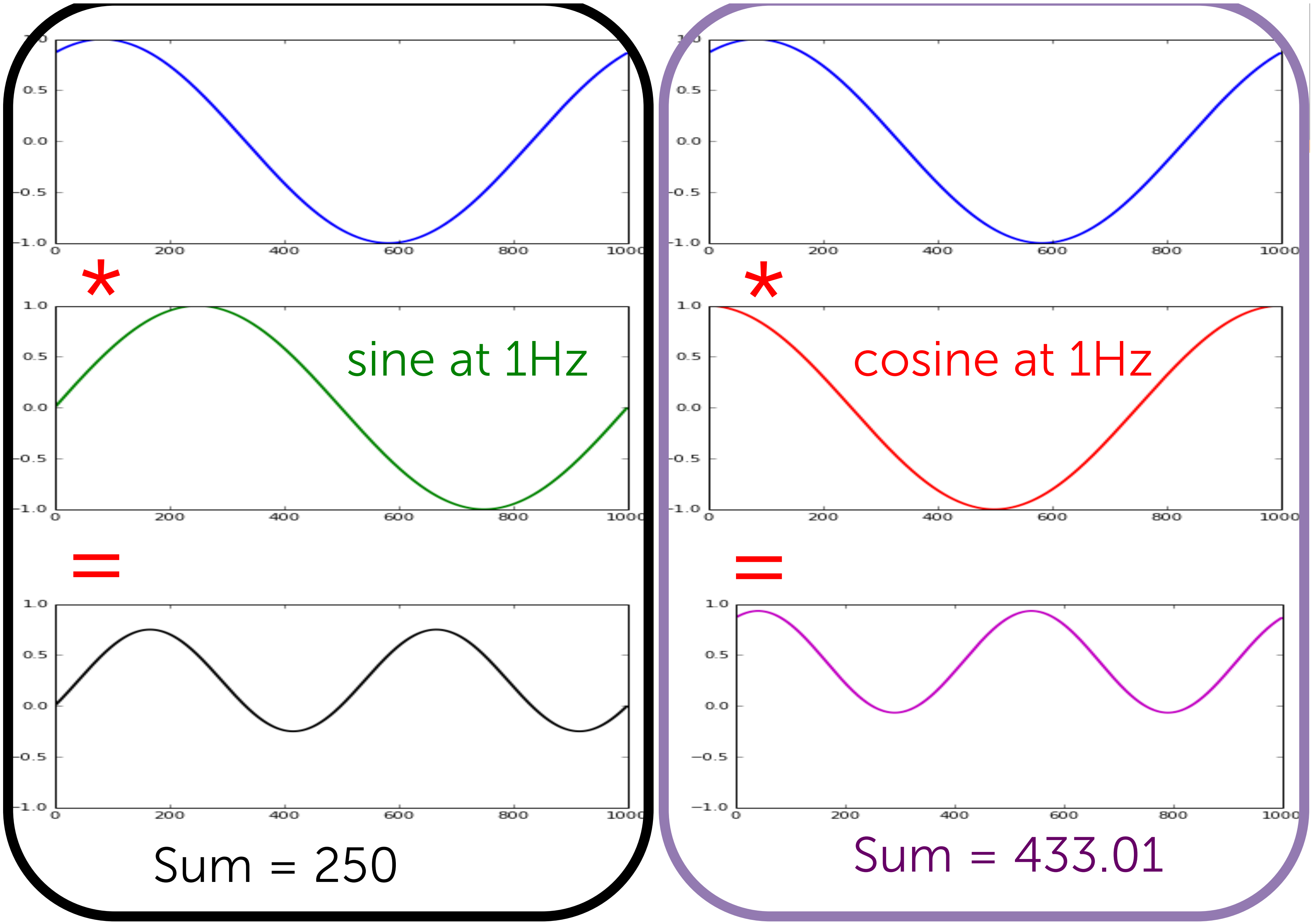
$$\sin(2\pi ft + \Phi)$$

$$=$$

$$\cos(\Phi) * \sin(2\pi ft) + \sin(\Phi) * \cos(2\pi ft)$$

Handling phase

Solution: compute summed product for both sine and cosine at the frequency f_k



A_{sk} A_{ck}

1. Determining amplitude of basis sinusoid with frequency f_k

2. Determining phase of basis sinusoid with frequency f_k

$$A_k = 2 \times \frac{\sqrt{A_{sk}^2 + A_{ck}^2}}{N}$$

$$\phi_k = \tan^{-1}\left(\frac{A_{ck}}{A_{sk}}\right)$$

Don't memorize these equations!
Most important idea: We can compute A_k and Φ_k precisely using the process of multiplying our signal with the sine and cosine, then summing.

3. Which frequencies do we really need?

If our signal is *N* samples long, we only need to look at sinusoids that oscillate:

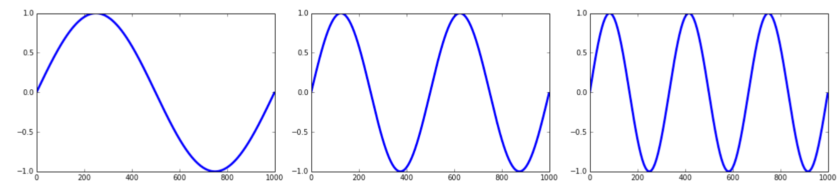
Once in *N* samples

Twice in *N samples*

3 times in *N* samples

etc.

Up to *N/2* times in N samples (Nyquist)



i.e., periods are *N*, *N*/2, *N*/3, ... 2 samples

i.e., frequencies are (1/*N*), (2/*N*), (3/*N*), ... ((N/2)/N)=1/2 oscillations per sample

As well as a constant "offset", i.e. 0 frequency

A nearly final algorithm for computing the spectrum:

For each frequency *n* in {0, 1/N, 2/N, 3/N, ..., 1/2}:

1. Compute A_{sn}, the summed product with sine at frequency *n*
2. Compute A_{cn}, the summed product with cosine at frequency *n*
3. Compute amplitude A_n and phase Φ_n from A_{sn} and A_{cn}

1. Determining amplitudes: “How much” of a basis function (a specific frequency) is present in a signal?
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Imaginary numbers

Building block: $i = \sqrt{-1}$

Examples of imaginary numbers:

*5i (same as 5*i)*

-10i

3.51*i*

Complex numbers

Have a *real* part and an *imaginary* part

E.g.:

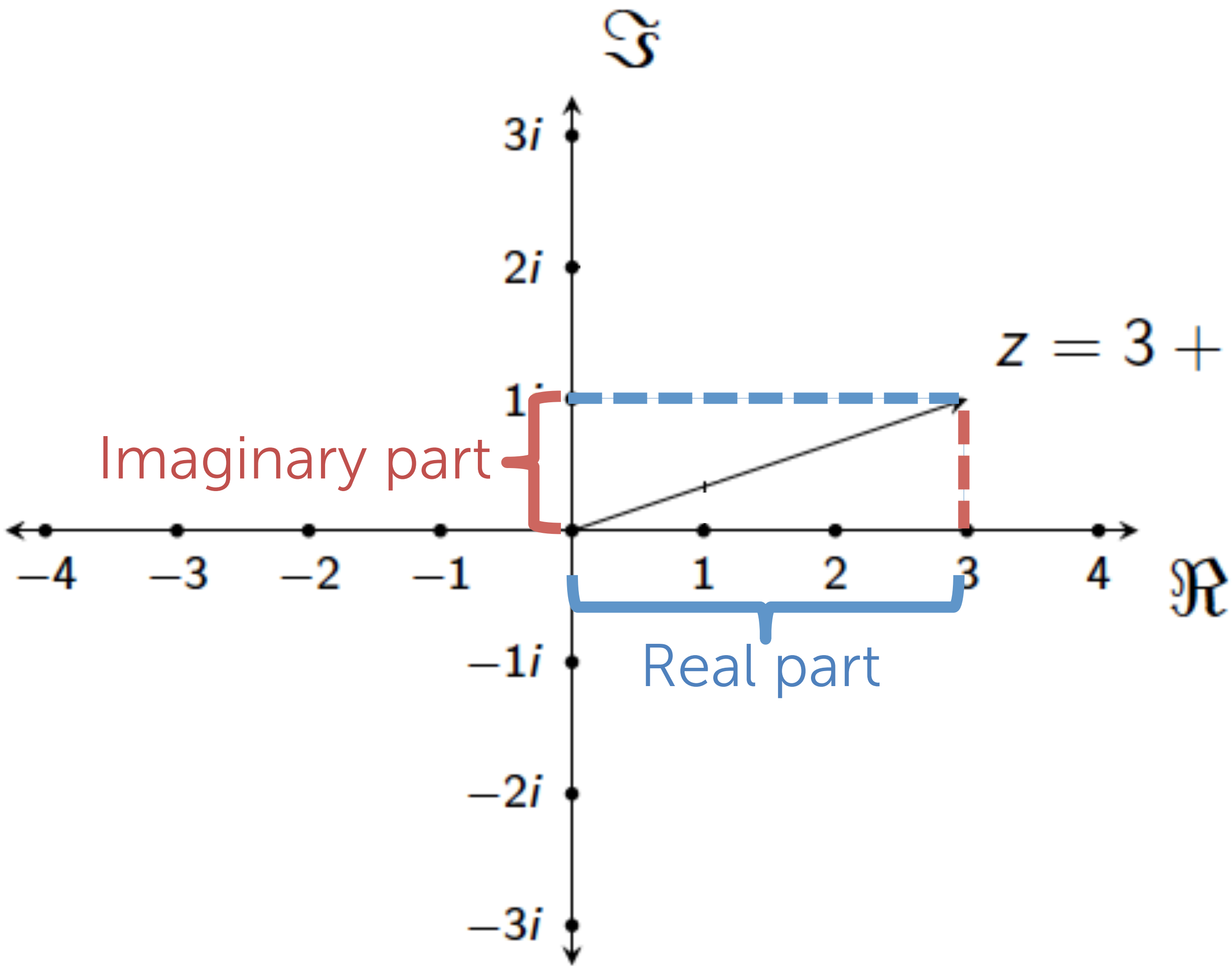
-5 + 3*i* Can't write in a simpler form!

135 - 42.5*i*

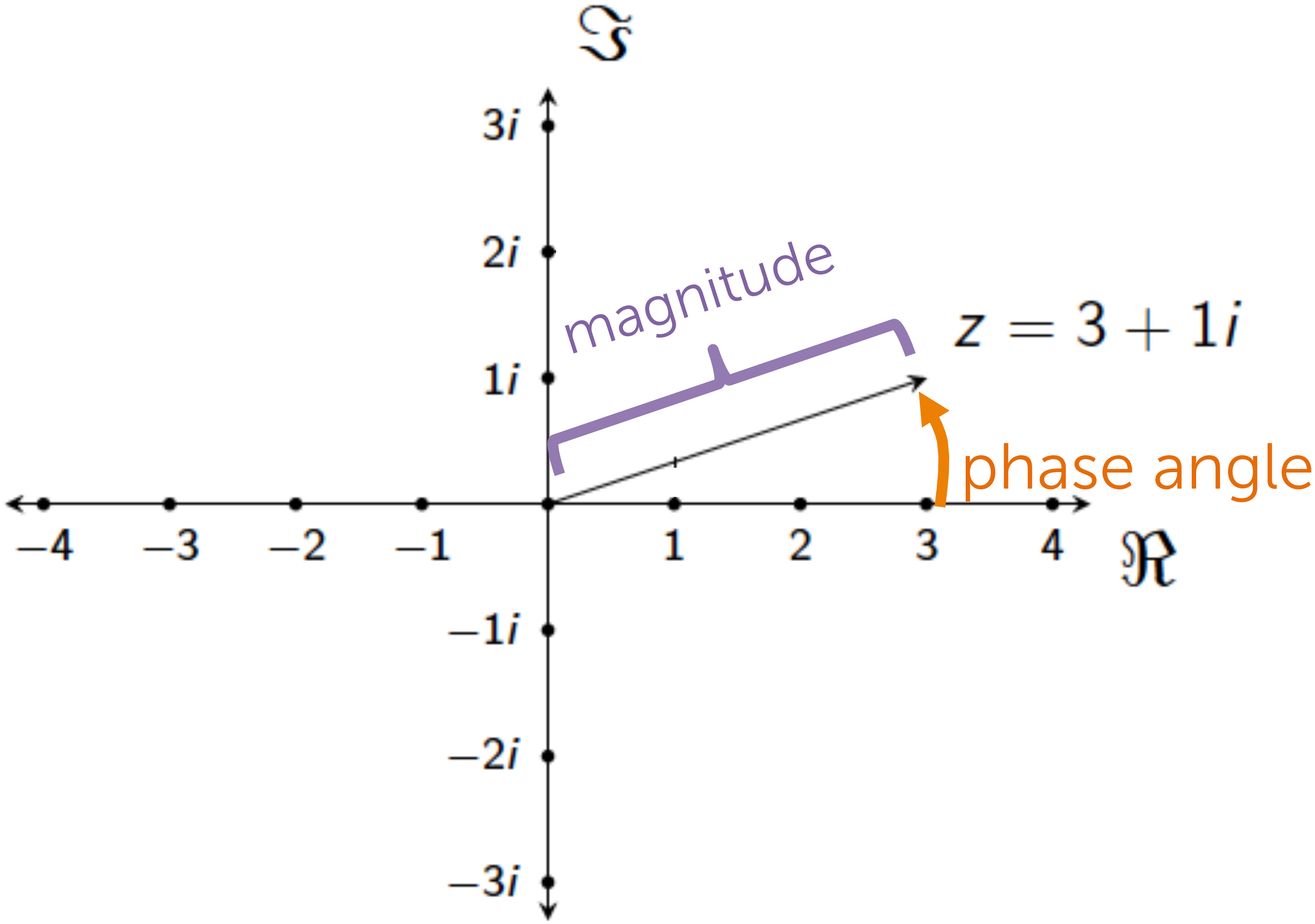
0 + 10*i* Imaginary numbers are a subset of the complex numbers

32 + 0*i* Real numbers are a subset of the complex numbers

Another way to think about complex numbers



… and yet another way to think about complex numbers



The number *e*

A mathematical constant; a real number:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$

e is approximately equal to 2.718

Euler's formula

$$e^{ix} = \cos x + i \sin x$$

for any number *x*

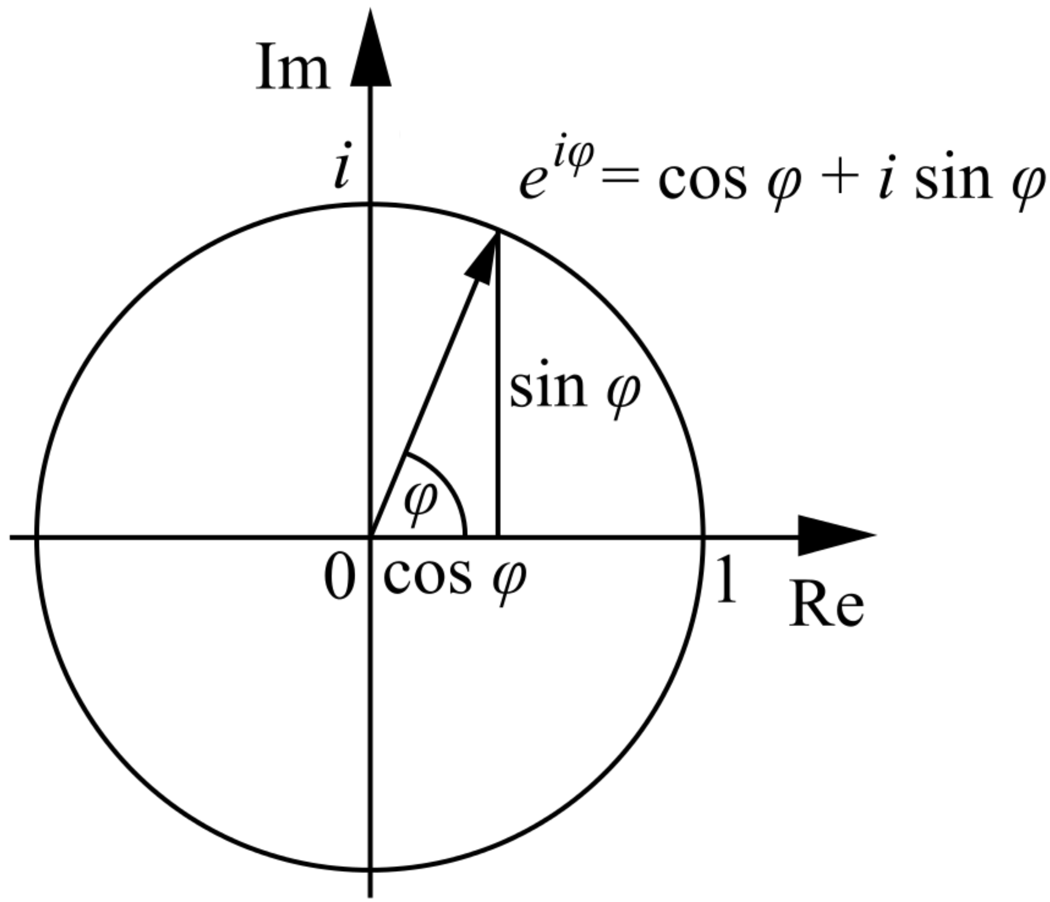
$e^{i\Phi}$ is a point in complex plane

This point lies on the unit circle: radius is 1 (magnitude of $e^{i\Phi}$ = 1)

Phase offset is $\boldsymbol{\Phi}$

Real and imaginary components can be easily derived:

- $\sin(\boldsymbol{\Phi})$ = (length of opposite side) / (length of hypotenuse)
- $\cos(\boldsymbol{\Phi})$= (length of adjacent side) / (length of hypotenuse)



Using Euler's formula for spectral analysis: The main idea

$$e^{-iq} = \cos q - i \sin q$$

Instead of multiplying each x[n] by a sine and cosine at frequency f, multiply it by $e^{-i2\pi f}$

Do this for all relevant frequencies, {0/N, 1/N, 2/N, ... }
For each given frequency, compute summed product for all sample points, n = 1 to N

The Discrete Fourier Transform

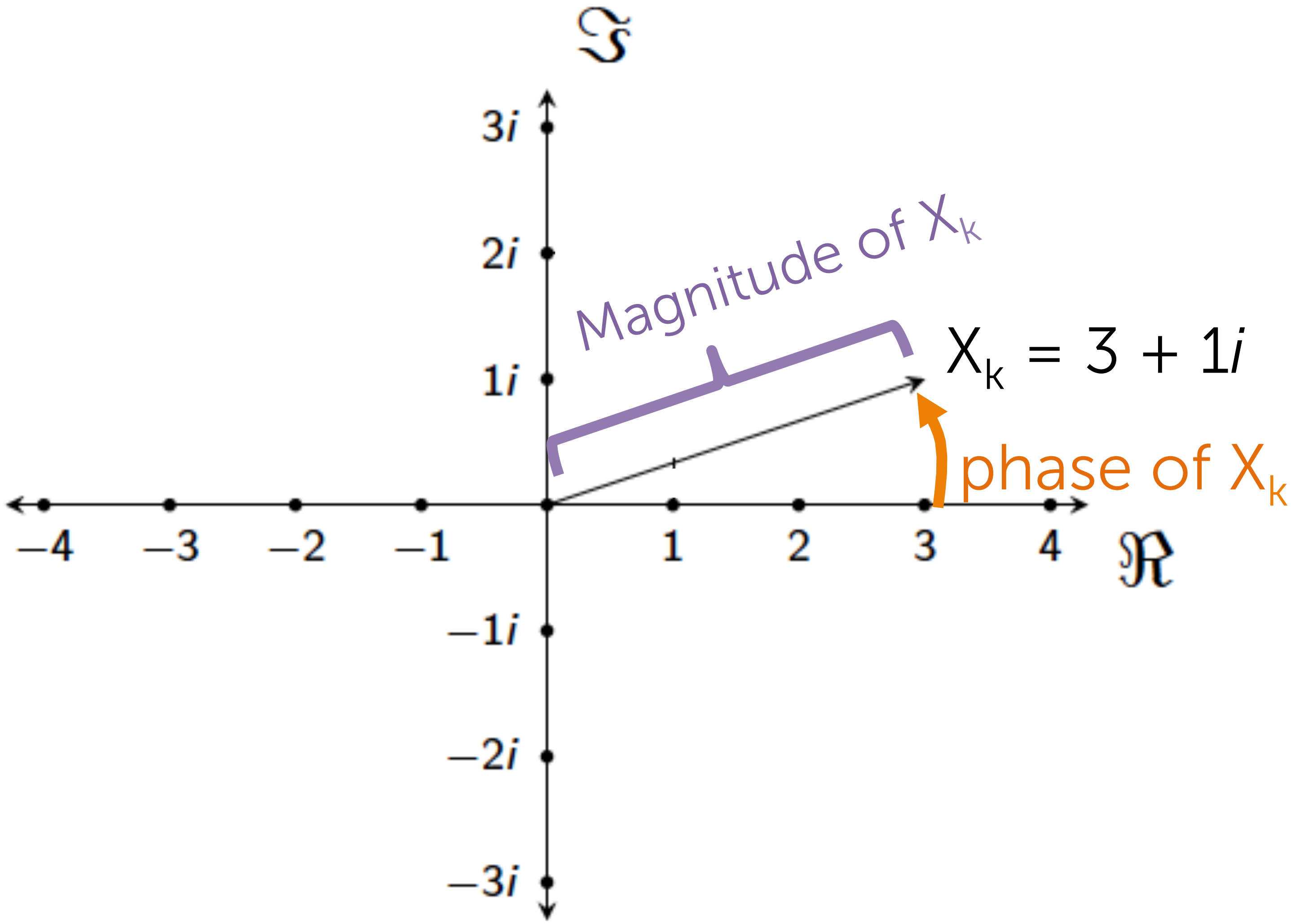
$$X_k = \sum_{n=1}^{N} x[n] \times e^{-i2\pi(k/N)n}$$

Compute X_k for k = 0, 1, 2, ...
(i.e., for freqs 0/N, 1/N, 2/N, ... osc. per sample)

Each X_k is a single complex number

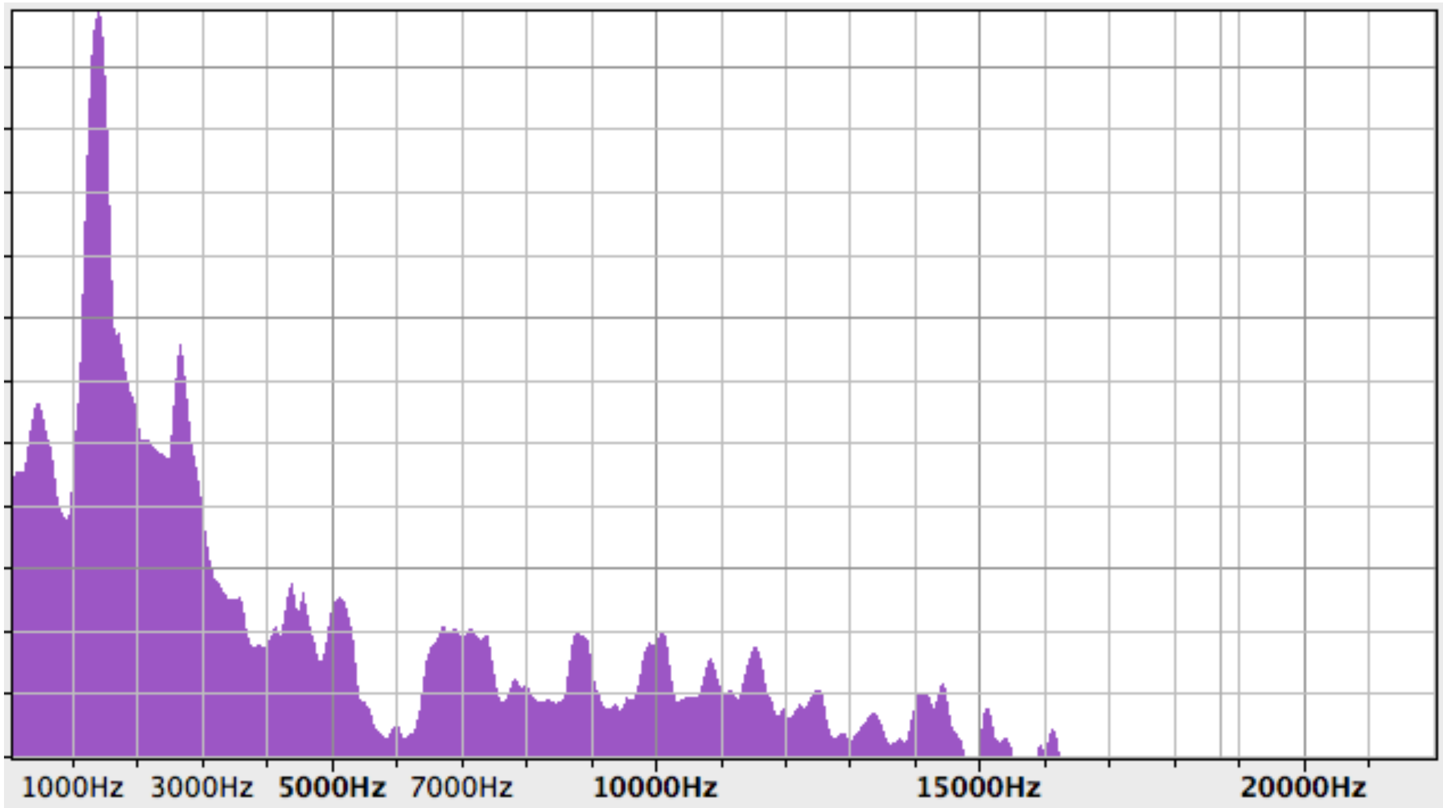
Magnitude of X_k (sometimes written $||X_k||$) is the "amount" of frequency *k* in x

Phase of X_k is the phase of frequency *k* in x



ℑ
3i
2i
1i
−1i
−2i
−3i
−4
−3
−2
−1
1
2
3
4
ℜ
Magnitude of X_k
$X_k = 3 + 1i$
phase of X_k

Plotting spectra



This is a plot of the *magnitudes* of X_k, for X_k starting at 0Hz and going up to the Nyquist rate (here 22050 Hz)

Recommended further reading

http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Book on signal processing for music and audio:
http://www.amazon.com/Digital-Signal-Processing-Primer-Applications/dp/0805316841
(available in Goldsmiths library!)