Online Graph Coloring: Baselines, Degree Heuristics, and Empirical Results

COMP 6651 Project

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1 Problem Description

Graph coloring represents a fundamental area of study in graph theory, originating from the famous Four Color Problem first posed in 1852 [1]. In online coloring, an algorithm must assign colors to graph vertices as they appear sequentially, without the ability to preview future vertices or modify previously assigned colors [3]. Research has shown that any such online approach may need at minimum $(2n/(\log n)^2)\chi(G)$ colors under worst-case conditions [4].

The FirstFit algorithm stands out as perhaps the most straightforward online graph coloring method [3]. This greedy approach works by establishing a predetermined ordering of available colors, then assigning each new vertex the lowest-ranked color that preserves proper coloring constraints. Researchers have studied this algorithm extensively across various types of graphs [3, 5, 6]. A different notable method, CBIP (Coloring Based on Interval Partitioning), was developed specifically for bipartite graphs: as each vertex $v = \sigma(i)$ is presented, the algorithm identifies the complete connected component containing v within the currently known portion of the graph [7].

While FirstFit and CBIP provide foundational strategies, recent studies have explored enhancements through degree-based heuristics. For instance, reordering vertices based on their degrees before coloring can lead to improved performance [5]. Additionally, batch processing techniques, where vertices are grouped and reordered within batches, have shown promise in reducing the number of colors used [2].

The purpose of this project is to implement, test, and empirically compare simple online baselines and degree-based heuristics under reproducible conditions. We sweep graph size n and density p, and analyze how ordering vertices by degree and per-batch degree reordering affect the efficiency of greedy coloring.

2 Implementation Details

Graph Data Structure

We use a lightweight, mutable, undirected graph. Vertices are positive integers; adjacency maps vertex IDs to sets of neighbors. Self-loops are ignored, and edges are stored symmetrically.

Core Graph Functions
Data structures: adjacency map adj : $V \to 2^V$ (sets of neighbours).

INIT_GRAPH(n):
  define adj(i) $\leftarrow \emptyset$ for $i \in [1..n]$;
  return adj

ENSURE_VERTEX(adj, v):
  if $v \notin$ adj **then**
  adj(v) $\leftarrow \emptyset$

ADD_EDGE(adj, u, v):
  if $u = v$ **then return**;
  ENSURE_VERTEX on u, v;
  adj(u) $\leftarrow$ adj(u) $\cup \{v\}$;
  adj(v) $\leftarrow$ adj(v) $\cup \{u\}$

NEIGHBOURS(adj, v):
  return adj(v) $\quad \triangleright$ empty if unseen

DEGREE(adj, v):
  return |adj(v)|

CONNECTED_COMPONENT(adj, s, Allowed):
  if $s \notin$ Allowed **then return** $\emptyset$
  $\mathcal{C} \leftarrow \{s\}$; $Q \leftarrow [s]$
  while Q not empty **do**
    $x \leftarrow$ pop_front(Q)
    for $y \in$ adj(x) **do**
      if ($y \in$ Allowed) $\wedge$ ($y \notin \mathcal{C}$) **then**
        $\mathcal{C} \leftarrow \mathcal{C} \cup \{y\}$; push_back($Q, y$)
  return $\mathcal{C}$

BIPARTITION_COMPONENT(adj, $\mathcal{C}$):
  color $\leftarrow \emptyset$; $A \leftarrow \emptyset$; $B \leftarrow \emptyset$
  for each $v \in \mathcal{C}$ **do**
    if $v \in$ color **then continue**
    color $\leftarrow \emptyset$; $A \leftarrow \emptyset$; $B \leftarrow \emptyset$
    $Q \leftarrow [v]$; color(v) $\leftarrow 0$; $A \leftarrow A \cup \{v\}$
    while Q not empty **do**
      $x \leftarrow$ pop_front(Q)
      for $y \in$ adj(x) $\cap\, \mathcal{C}$ **do**
        if $y \notin$ color **then**
          color(y) $\leftarrow 1 -$ color(x)
          (A if color(y) $= 0$ else B) $\leftarrow \cdot \cup \{y\}$
          push_back(Q, y)
  return (A, B)

Generator

We construct k-colorable graphs reproducibly by partitioning vertices into k independent sets and adding cross edges with probability p.

Generate(n, k, p)
// Output: graph $G = (V, E)$ and online order σ
 $S[1..k] \leftarrow k$ empty sets ▷ partition of V
 for $i = 1..k$ **do** $S[i] \leftarrow S[i] \cup \{i\}$ ▷ seed non-empty
 for $v = k + 1..n$ **do** $S[\text{rand}(1..k)] \leftarrow S[\cdot] \cup \{v\}$
 $G \leftarrow$ **INIT_GRAPH**(n)
 for $i = 1..k$ **do**
  for each $v \in S[i]$ **do**
   for $j = 1..k,\ j \neq i$ **do**
    choose $u \in S[j]$ uniformly; **ADD_EDGE**(G, v, u) ▷ mandatory cross edge
    for each $u_2 \in S[j] \setminus \{u\}$ **do**
     if rand() $< p$ **then** **ADD_EDGE**(G, v, u_2) ▷ random cross edges
 define $\sigma([1..n]) \leftarrow V$ and uniformly randomize order ▷ online presentation order
 return (G, σ)

Algorithms

We retain four algorithms; pseudocode mirrors the Python implementations.

- **FirstFit (online)**: Greedy assignment of the smallest permissible color by arrival order.
- **CBIP (online, bipartite-specific)**: Uses component bipartition; reported for $k = 2$.
- **HeuristicDegree (semi-online reorder)**: Reorders the arrival sequence by non-increasing degree, then applies FirstFit.
- **BatchDegree (semi-online)**: Processes batches of size t, reorders within batch by degree, colors via FirstFit.

FirstFit(G)
// Input: $G = (V, E)$ — Current online graph
 define $c(V)$ ▷ stores coloring of V
 initialize $c(V)$ to *nil* ▷ no colors set yet
 define $\sigma([1..n]) \leftarrow V$ ▷ presentation order of vertices, $\sigma : [1..|V|] \rightarrow V$
 uniformly randomize order of σ ▷ randomize online order of vertices
 $i \leftarrow 1$
 define vertex v, set of vertices N
 while $i \leq |V|$ **do**
  $v \leftarrow \sigma(i)$
  $N \leftarrow$ all neighbours of v in $V \cap \sigma[1..i-1]$ ▷ include only current revealed vertices
  $c(v) \leftarrow \min(\mathbb{N} \setminus c(N))$ ▷ $c(N)$ the set of colors of vertices in N
  $i \leftarrow i + 1$
 return c

HeuristicDegree(G)
// Input: $G = (V, E)$ — Current offline reorder followed by online greedy
compute $d(v)$ for all $v \in V$ ▷ static degree in G
define $\sigma([1..n]) \leftarrow V$
sort σ by $d(v)$ in non-increasing order ▷ highest degree first
return FirstFit(G) on σ

CBIP(G)
// Input: $G = (V, E)$ — Online bipartite coloring (reported for $k = 2$)
initialize $c(V)$ to nil; $R \leftarrow \emptyset$ ▷ R: revealed vertices
define $\sigma([1..n]) \leftarrow V$
for $i = 1$ to $|V|$ **do**
$v \leftarrow \sigma(i)$; $R \leftarrow R \cup \{v\}$
$\mathcal{C} \leftarrow$ ConnectedComponent(G, v, R)
$(A, B) \leftarrow$ Bipartition$(G, \mathcal{C})$
if $(v \notin A)$ **and** $(v \in B)$ **then** swap(A, B)
$U \leftarrow \{c(u) : u \in (B \cap R)\}$ ▷ colors used on the opposite side
$c(v) \leftarrow \min(\mathbb{N} \setminus U)$
return c

BatchDegree(G, t)
// Input: $G = (V, E)$ — Semi-online batches of size t with degree reorder
initialize $c(V)$ to nil; $R \leftarrow \emptyset$; compute $d(v)$ for all v
define $\sigma([1..n]) \leftarrow V$
for $s \in \{1, 1 + t, 1 + 2t, \ldots\}$ **do**
$B \leftarrow \sigma[s : s + t]$ ▷ current batch
reorder B by $d(v)$ in non-increasing order
for each $v \in B$ **do**
$N \leftarrow$ neighbours of v in R
$c(v) \leftarrow \min(\mathbb{N} \setminus c(N))$
$R \leftarrow R \cup \{v\}$
return c

3 Implementation correctness

- **Software correctness**: Unit tests validate that self-loops are ignored, parallel edges are not duplicated (set-based adjacency), adjacency is symmetric, and EDGES I/O is consistent (each undirected edge written once, tolerant reader).
- **Algorithm correctness (examples)**: FirstFit yields proper colorings on random k-colorable graphs and uses 3 colors on a triangle; CBIP colors bipartite instances properly (e.g., 4-cycle uses at most 2 colors). We manually verified outputs on small graphs with known chromatic numbers, and used boundary testing to ensure robustness.

4 Experimental Setup

Our experimental framework is implemented in Python 3.13.7 and organized as follows:

- **Parameter grid**: Iterate over $n \in \{50, 100, 150, 200, 300, 400, 500, 600, 700, 800, 900, 1000\}$, $k \in \{2, 3, 4\}$, $p \in \{0.05, 0.1, 0.2, 0.3\}$; for each triple generate $N = 100$ independent instances.
- **Deterministic artifacts**: For every instance we persist two plain-text files: an `xxx.edges` file (one undirected edge $u\ v$ per line, $u < v$) and an `xxx.order` file (vertex arrival permutation σ).
- **Reproducibility**: Algorithms never sample randomness internally when consuming a stored instance; they read the pre-generated edge list and vertex order, ensuring identical outcomes across runs/machines.
- **Graph generation** (`generator/generate.py`): Implements the partition-based k-colorable construction plus probabilistic cross edges with probability p; writes artifacts only if absent (idempotent cache behavior).
- **Core structure** (`graph.py`): Provides adjacency maintenance, component discovery, and bipartition logic used by CBIP; omits multi-edges and self-loops.
- **Algorithms** (`algorithms/*.py`): Each file exposes a pure function mapping (G, σ) (and batch size where relevant) to a coloring dictionary. No global state, facilitating multiprocessing.
- **Simulation driver** (`simulations.py`): Orchestrates sweeps over the parameter grid, loads cached instances, dispatches algorithm runs, aggregates per-instance color counts, and computes competitive ratios.
- **Metrics** (`metrics.py`): Defines helper routines (e.g., competitive ratio, summary statistics: mean, standard deviation) separated for clarity and testability.
- **Parallelism**: Uses Python `multiprocessing` (process pool) for independent instances within the same (n, k, p) slice when algorithms are stateless. BatchDegree may be run in the main process to avoid serialization of internal lambdas.
- **Progress aggregation**: Intermediate results accumulated into in-memory structures, then flushed periodically to CSVs under `outputs/` (one per algorithm variant plus a consolidated enriched summary).
- **Failure tolerance**: Instance-level exceptions (should none occur in normal operation) are logged and skipped without aborting the entire sweep.
- **Main entry** (`main.py`): Provides CLI handling (e.g., selecting algorithms, enabling parallel mode, specifying output paths) and invokes simulation routines; supports partial grids for quick exploratory runs.
- **Test coverage** (`tests/test_sanity.py`): Validates structural invariants (symmetry, loop suppression) and correctness of edge writing/reading to guarantee reliability of persisted dataset before large simulations run.
- **Post-processing** (notebooks / `plot_results.ipynb`): Loads summary CSVs, produces fit tables, residual plots, and family figures; writes PNG outputs consumed by LaTeX report.

5 Results

We report two simulation tracks. Each uses fixed instance artifacts (edge list and arrival order) for strict reproducibility and repeats $N = 100$ times per (n, k, p).

5.1 Simulation I: FirstFit and CBIP

Parameters and setup

- **Order**: Use the persisted online order σ per instance; no reshuffling at evaluation time.
- **FirstFit**: No tunable parameter; assigns smallest feasible colour on arrival.
- **CBIP**: Reported only for $k = 2$ (bipartite inputs). Computes the revealed component and a bipartition on-the-fly; colours from the opposite side's palette.
- **Repetitions**: $N = 100$ per (n, k, p); performance summarised by mean and standard deviation of the competitive ratio.

Table 1: FirstFit: results across $k \in \{2, 3, 4\}$ with density p.

Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit	2	0.05	50	100	2.04	0.31
FirstFit	2	0.05	100	100	2.52	0.43
FirstFit	2	0.05	150	100	2.8	0.53
FirstFit	2	0.05	200	100	3.25	0.54
FirstFit	2	0.05	300	100	3.7	0.77
FirstFit	2	0.05	400	100	4.16	0.69
FirstFit	2	0.05	500	100	4.26	0.95
FirstFit	2	0.05	600	100	4.58	0.93
FirstFit	2	0.05	700	100	4.7	1.15
FirstFit	2	0.05	800	100	4.96	1.22
FirstFit	2	0.05	900	100	5.29	1.23
FirstFit	2	0.05	1000	100	5.15	1.23
FirstFit	3	0.05	50	100	1.87	0.19
FirstFit	3	0.05	100	100	2.36	0.19
FirstFit	3	0.05	150	100	2.73	0.24
FirstFit	3	0.05	200	100	3.1	0.2
FirstFit	3	0.05	300	100	3.78	0.23
FirstFit	3	0.05	400	100	4.31	0.25
FirstFit	3	0.05	500	100	4.82	0.36
FirstFit	3	0.05	600	100	5.37	0.31
FirstFit	3	0.05	700	100	5.72	0.4
FirstFit	3	0.05	800	100	6.22	0.4
FirstFit	3	0.05	900	100	6.61	0.43
FirstFit	3	0.05	1000	100	6.94	0.54
FirstFit	4	0.05	50	100	1.63	0.15

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit	4	0.05	100	100	2.04	0.14
FirstFit	4	0.05	150	100	2.37	0.13
FirstFit	4	0.05	200	100	2.68	0.16
FirstFit	4	0.05	300	100	3.23	0.15
FirstFit	4	0.05	400	100	3.79	0.15
FirstFit	4	0.05	500	100	4.24	0.17
FirstFit	4	0.05	600	100	4.71	0.18
FirstFit	4	0.05	700	100	5.12	0.17
FirstFit	4	0.05	800	100	5.57	0.16
FirstFit	4	0.05	900	100	5.95	0.19
FirstFit	4	0.05	1000	100	6.36	0.2
FirstFit	2	0.1	50	100	2.11	0.55
FirstFit	2	0.1	100	100	2.51	0.8
FirstFit	2	0.1	150	100	2.91	0.96
FirstFit	2	0.1	200	100	2.89	1.05
FirstFit	2	0.1	300	100	3.14	1.11
FirstFit	2	0.1	400	100	3.37	1.41
FirstFit	2	0.1	500	100	3.66	1.55
FirstFit	2	0.1	600	100	3.57	1.54
FirstFit	2	0.1	700	100	3.37	1.68
FirstFit	2	0.1	800	100	3.88	1.8
FirstFit	2	0.1	900	100	4.02	1.87
FirstFit	2	0.1	1000	100	3.6	1.71
FirstFit	3	0.1	50	100	2.08	0.29
FirstFit	3	0.1	100	100	2.86	0.29
FirstFit	3	0.1	150	100	3.22	0.45
FirstFit	3	0.1	200	100	3.77	0.48
FirstFit	3	0.1	300	100	4.54	0.65
FirstFit	3	0.1	400	100	5.01	0.79
FirstFit	3	0.1	500	100	5.67	0.89
FirstFit	3	0.1	600	100	5.9	1.01
FirstFit	3	0.1	700	100	6.58	1.13
FirstFit	3	0.1	800	100	6.61	1.28
FirstFit	3	0.1	900	100	7.15	1.59
FirstFit	3	0.1	1000	100	7.43	1.39
FirstFit	4	0.1	50	100	1.88	0.15
FirstFit	4	0.1	100	100	2.47	0.19
FirstFit	4	0.1	150	100	3.03	0.21
FirstFit	4	0.1	200	100	3.52	0.24
FirstFit	4	0.1	300	100	4.35	0.32
FirstFit	4	0.1	400	100	5.13	0.32
FirstFit	4	0.1	500	100	5.78	0.42
FirstFit	4	0.1	600	100	6.43	0.36
FirstFit	4	0.1	700	100	6.98	0.62
FirstFit	4	0.1	800	100	7.5	0.52

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit	4	0.1	900	100	7.95	0.7
FirstFit	4	0.1	1000	100	8.47	0.8
FirstFit	2	0.2	50	100	1.72	0.7
FirstFit	2	0.2	100	100	1.98	0.86
FirstFit	2	0.2	150	100	1.95	0.83
FirstFit	2	0.2	200	100	2.12	1.1
FirstFit	2	0.2	300	100	2	0.99
FirstFit	2	0.2	400	100	1.97	1
FirstFit	2	0.2	500	100	2.14	1.08
FirstFit	2	0.2	600	100	2.12	1.07
FirstFit	2	0.2	700	100	2.19	1.06
FirstFit	2	0.2	800	100	2.23	1.26
FirstFit	2	0.2	900	100	2.05	1.19
FirstFit	2	0.2	1000	100	1.97	1.02
FirstFit	3	0.2	50	100	2.16	0.52
FirstFit	3	0.2	100	100	2.69	0.81
FirstFit	3	0.2	150	100	2.98	1.07
FirstFit	3	0.2	200	100	3.15	1.24
FirstFit	3	0.2	300	100	3.45	1.25
FirstFit	3	0.2	400	100	3.73	1.6
FirstFit	3	0.2	500	100	3.87	1.89
FirstFit	3	0.2	600	100	3.97	1.88
FirstFit	3	0.2	700	100	3.72	1.97
FirstFit	3	0.2	800	100	3.73	1.76
FirstFit	3	0.2	900	100	4.02	1.9
FirstFit	3	0.2	1000	100	3.69	1.9
FirstFit	4	0.2	50	100	2.17	0.32
FirstFit	4	0.2	100	100	2.71	0.53
FirstFit	4	0.2	150	100	3.38	0.67
FirstFit	4	0.2	200	100	3.73	1.02
FirstFit	4	0.2	300	100	4.59	1.14
FirstFit	4	0.2	400	100	4.76	1.47
FirstFit	4	0.2	500	100	5.46	1.62
FirstFit	4	0.2	600	100	5.88	1.76
FirstFit	4	0.2	700	100	5.89	1.8
FirstFit	4	0.2	800	100	5.86	1.93
FirstFit	4	0.2	900	100	5.83	2.5
FirstFit	4	0.2	1000	100	5.9	2.25
FirstFit	2	0.3	50	100	1.43	0.49
FirstFit	2	0.3	100	100	1.61	0.75
FirstFit	2	0.3	150	100	1.46	0.57
FirstFit	2	0.3	200	100	1.44	0.59
FirstFit	2	0.3	300	100	1.49	0.57
FirstFit	2	0.3	400	100	1.6	0.68
FirstFit	2	0.3	500	100	1.47	0.58

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit	2	0.3	600	100	1.47	0.49
FirstFit	2	0.3	700	100	1.52	0.55
FirstFit	2	0.3	800	100	1.53	0.68
FirstFit	2	0.3	900	100	1.51	0.64
FirstFit	2	0.3	1000	100	1.64	0.74
FirstFit	3	0.3	50	100	1.81	0.63
FirstFit	3	0.3	100	100	2.07	0.94
FirstFit	3	0.3	150	100	2.08	1.06
FirstFit	3	0.3	200	100	2.21	1.07
FirstFit	3	0.3	300	100	2.17	0.98
FirstFit	3	0.3	400	100	2.26	1.05
FirstFit	3	0.3	500	100	2.32	1.09
FirstFit	3	0.3	600	100	2.06	0.97
FirstFit	3	0.3	700	100	2.17	1.02
FirstFit	3	0.3	800	100	2.15	1.02
FirstFit	3	0.3	900	100	2.2	0.97
FirstFit	3	0.3	1000	100	2.35	1.13
FirstFit	4	0.3	50	100	2.01	0.45
FirstFit	4	0.3	100	100	2.4	0.75
FirstFit	4	0.3	150	100	2.56	0.98
FirstFit	4	0.3	200	100	2.82	1.13
FirstFit	4	0.3	300	100	2.8	1.1
FirstFit	4	0.3	400	100	3.05	1.24
FirstFit	4	0.3	500	100	2.8	1.19
FirstFit	4	0.3	600	100	2.96	1.26
FirstFit	4	0.3	700	100	3.11	1.64
FirstFit	4	0.3	800	100	3.02	1.27
FirstFit	4	0.3	900	100	2.98	1.28
FirstFit	4	0.3	1000	100	3.08	1.47

Table 2: CBIP: results for bipartite inputs ($k = 2$) with density p.

Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
CBIP	2	0.05	50	100	1.91	0.26
CBIP	2	0.05	100	100	1.99	0.27
CBIP	2	0.05	150	100	2.01	0.27
CBIP	2	0.05	200	100	2.05	0.21
CBIP	2	0.05	300	100	2.06	0.28
CBIP	2	0.05	400	100	2.05	0.23
CBIP	2	0.05	500	100	2.04	0.27
CBIP	2	0.05	600	100	2.05	0.25
CBIP	2	0.05	700	100	2.07	0.22

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
CBIP	2	0.05	800	100	2.07	0.25
CBIP	2	0.05	900	100	2.06	0.25
CBIP	2	0.05	1000	100	2.05	0.21
CBIP	2	0.1	50	100	1.76	0.33
CBIP	2	0.1	100	100	1.78	0.32
CBIP	2	0.1	150	100	1.85	0.29
CBIP	2	0.1	200	100	1.79	0.3
CBIP	2	0.1	300	100	1.88	0.29
CBIP	2	0.1	400	100	1.87	0.35
CBIP	2	0.1	500	100	1.86	0.3
CBIP	2	0.1	600	100	1.87	0.28
CBIP	2	0.1	700	100	1.83	0.31
CBIP	2	0.1	800	100	1.88	0.32
CBIP	2	0.1	900	100	1.84	0.3
CBIP	2	0.1	1000	100	1.86	0.29
CBIP	2	0.2	50	100	1.47	0.38
CBIP	2	0.2	100	100	1.56	0.38
CBIP	2	0.2	150	100	1.56	0.36
CBIP	2	0.2	200	100	1.56	0.39
CBIP	2	0.2	300	100	1.55	0.39
CBIP	2	0.2	400	100	1.55	0.37
CBIP	2	0.2	500	100	1.59	0.35
CBIP	2	0.2	600	100	1.57	0.35
CBIP	2	0.2	700	100	1.64	0.35
CBIP	2	0.2	800	100	1.61	0.34
CBIP	2	0.2	900	100	1.54	0.36
CBIP	2	0.2	1000	100	1.55	0.37
CBIP	2	0.3	50	100	1.35	0.36
CBIP	2	0.3	100	100	1.4	0.41
CBIP	2	0.3	150	100	1.33	0.35
CBIP	2	0.3	200	100	1.3	0.33
CBIP	2	0.3	300	100	1.38	0.32
CBIP	2	0.3	400	100	1.4	0.34
CBIP	2	0.3	500	100	1.33	0.37
CBIP	2	0.3	600	100	1.36	0.33
CBIP	2	0.3	700	100	1.41	0.38
CBIP	2	0.3	800	100	1.32	0.34
CBIP	2	0.3	900	100	1.34	0.34
CBIP	2	0.3	1000	100	1.41	0.4

5.2 Simulation II: HeuristicDegree and BatchDegree

Parameterisation and heuristic choices

- **HeuristicDegree**: Offline reorder by non-increasing static degree, then FirstFit. No extra parameter beyond using the same fixed σ as baseline to define V.

- **BatchDegree**: Semi-online with batch size t; within each batch, reorder by static degree and colour via FirstFit. We report $t \in \{10, 50\}$ as representative settings, chosen for a good speed/quality trade-off observed in pilot runs.
- **Alternatives explored (not selected)**: We evaluated several on-arrival colouring heuristics: (i) *Randomised colour* tie-breaking, (ii) *MRU* preference (biasing towards most recently used colour), and (iii) *Least-used* global colour selection. Across (n, p), these yielded competitive ratios that were comparable to or worse than FirstFit and significantly worse than BatchDegree, especially at higher densities. Consequently, we focus on BatchDegree for clearer improvements.
- **Impact of t**: Larger t generally reduces colours by enabling more effective local ordering, with diminishing returns beyond $t \approx 50$ at our scales.

Table 3: HeuristicDegree: degree-sorted FirstFit across $k \in \{2, 3, 4\}$ with density p.

Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit(HeuristicDegree)	2	0.05	50	100	1.37	0.42
FirstFit(HeuristicDegree)	2	0.05	100	100	1.68	0.64
FirstFit(HeuristicDegree)	2	0.05	150	100	1.96	0.79
FirstFit(HeuristicDegree)	2	0.05	200	100	2.14	0.9
FirstFit(HeuristicDegree)	2	0.05	300	100	2.56	1.03
FirstFit(HeuristicDegree)	2	0.05	400	100	2.7	1.17
FirstFit(HeuristicDegree)	2	0.05	500	100	2.54	1.33
FirstFit(HeuristicDegree)	2	0.05	600	100	2.95	1.44
FirstFit(HeuristicDegree)	2	0.05	700	100	3.02	1.53
FirstFit(HeuristicDegree)	2	0.05	800	100	3.14	1.48
FirstFit(HeuristicDegree)	2	0.05	900	100	3.51	1.59
FirstFit(HeuristicDegree)	2	0.05	1000	100	3.39	1.7
FirstFit(HeuristicDegree)	3	0.05	50	100	1.59	0.19
FirstFit(HeuristicDegree)	3	0.05	100	100	1.99	0.24
FirstFit(HeuristicDegree)	3	0.05	150	100	2.37	0.24
FirstFit(HeuristicDegree)	3	0.05	200	100	2.69	0.36
FirstFit(HeuristicDegree)	3	0.05	300	100	3.21	0.32
FirstFit(HeuristicDegree)	3	0.05	400	100	3.66	0.46
FirstFit(HeuristicDegree)	3	0.05	500	100	4.01	0.51
FirstFit(HeuristicDegree)	3	0.05	600	100	4.54	0.57
FirstFit(HeuristicDegree)	3	0.05	700	100	4.75	0.66
FirstFit(HeuristicDegree)	3	0.05	800	100	5.02	0.73
FirstFit(HeuristicDegree)	3	0.05	900	100	5.35	0.85
FirstFit(HeuristicDegree)	3	0.05	1000	100	5.66	1
FirstFit(HeuristicDegree)	4	0.05	50	100	1.46	0.12
FirstFit(HeuristicDegree)	4	0.05	100	100	1.81	0.12
FirstFit(HeuristicDegree)	4	0.05	150	100	2.11	0.14
FirstFit(HeuristicDegree)	4	0.05	200	100	2.4	0.14

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit(HeuristicDegree)	4	0.05	300	100	2.91	0.14
FirstFit(HeuristicDegree)	4	0.05	400	100	3.34	0.2
FirstFit(HeuristicDegree)	4	0.05	500	100	3.8	0.18
FirstFit(HeuristicDegree)	4	0.05	600	100	4.19	0.21
FirstFit(HeuristicDegree)	4	0.05	700	100	4.65	0.19
FirstFit(HeuristicDegree)	4	0.05	800	100	4.97	0.26
FirstFit(HeuristicDegree)	4	0.05	900	100	5.31	0.23
FirstFit(HeuristicDegree)	4	0.05	1000	100	5.7	0.28
FirstFit(HeuristicDegree)	2	0.1	50	100	1.35	0.51
FirstFit(HeuristicDegree)	2	0.1	100	100	1.44	0.64
FirstFit(HeuristicDegree)	2	0.1	150	100	1.54	0.81
FirstFit(HeuristicDegree)	2	0.1	200	100	1.58	0.93
FirstFit(HeuristicDegree)	2	0.1	300	100	1.68	1.02
FirstFit(HeuristicDegree)	2	0.1	400	100	1.74	1.09
FirstFit(HeuristicDegree)	2	0.1	500	100	1.86	1.27
FirstFit(HeuristicDegree)	2	0.1	600	100	1.82	1.22
FirstFit(HeuristicDegree)	2	0.1	700	100	1.84	1.32
FirstFit(HeuristicDegree)	2	0.1	800	100	1.81	1.34
FirstFit(HeuristicDegree)	2	0.1	900	100	1.92	1.39
FirstFit(HeuristicDegree)	2	0.1	1000	100	1.87	1.44
FirstFit(HeuristicDegree)	3	0.1	50	100	1.68	0.4
FirstFit(HeuristicDegree)	3	0.1	100	100	2.13	0.57
FirstFit(HeuristicDegree)	3	0.1	150	100	2.49	0.71
FirstFit(HeuristicDegree)	3	0.1	200	100	2.56	0.91
FirstFit(HeuristicDegree)	3	0.1	300	100	3.15	1.25
FirstFit(HeuristicDegree)	3	0.1	400	100	3.63	1.24
FirstFit(HeuristicDegree)	3	0.1	500	100	3.74	1.46
FirstFit(HeuristicDegree)	3	0.1	600	100	3.82	1.64
FirstFit(HeuristicDegree)	3	0.1	700	100	4.42	1.77
FirstFit(HeuristicDegree)	3	0.1	800	100	4.51	1.87
FirstFit(HeuristicDegree)	3	0.1	900	100	4.42	2.12
FirstFit(HeuristicDegree)	3	0.1	1000	100	5.01	1.89
FirstFit(HeuristicDegree)	4	0.1	50	100	1.65	0.19
FirstFit(HeuristicDegree)	4	0.1	100	100	2.11	0.27
FirstFit(HeuristicDegree)	4	0.1	150	100	2.66	0.24
FirstFit(HeuristicDegree)	4	0.1	200	100	2.98	0.38
FirstFit(HeuristicDegree)	4	0.1	300	100	3.68	0.53
FirstFit(HeuristicDegree)	4	0.1	400	100	4.25	0.58
FirstFit(HeuristicDegree)	4	0.1	500	100	4.8	0.65
FirstFit(HeuristicDegree)	4	0.1	600	100	5.09	1.08
FirstFit(HeuristicDegree)	4	0.1	700	100	5.75	0.92
FirstFit(HeuristicDegree)	4	0.1	800	100	6.03	0.91
FirstFit(HeuristicDegree)	4	0.1	900	100	6.37	1.22
FirstFit(HeuristicDegree)	4	0.1	1000	100	6.95	1.05
FirstFit(HeuristicDegree)	2	0.2	50	100	1.07	0.21

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit(HeuristicDegree)	2	0.2	100	100	1.11	0.28
FirstFit(HeuristicDegree)	2	0.2	150	100	1.14	0.37
FirstFit(HeuristicDegree)	2	0.2	200	100	1.12	0.35
FirstFit(HeuristicDegree)	2	0.2	300	100	1.2	0.51
FirstFit(HeuristicDegree)	2	0.2	400	100	1.15	0.36
FirstFit(HeuristicDegree)	2	0.2	500	100	1.2	0.49
FirstFit(HeuristicDegree)	2	0.2	600	100	1.2	0.57
FirstFit(HeuristicDegree)	2	0.2	700	100	1.21	0.46
FirstFit(HeuristicDegree)	2	0.2	800	100	1.21	0.4
FirstFit(HeuristicDegree)	2	0.2	900	100	1.18	0.43
FirstFit(HeuristicDegree)	2	0.2	1000	100	1.36	0.65
FirstFit(HeuristicDegree)	3	0.2	50	100	1.38	0.52
FirstFit(HeuristicDegree)	3	0.2	100	100	1.52	0.65
FirstFit(HeuristicDegree)	3	0.2	150	100	1.67	0.76
FirstFit(HeuristicDegree)	3	0.2	200	100	1.53	0.78
FirstFit(HeuristicDegree)	3	0.2	300	100	1.6	0.8
FirstFit(HeuristicDegree)	3	0.2	400	100	1.73	0.99
FirstFit(HeuristicDegree)	3	0.2	500	100	1.88	1.05
FirstFit(HeuristicDegree)	3	0.2	600	100	1.8	1.06
FirstFit(HeuristicDegree)	3	0.2	700	100	1.7	0.97
FirstFit(HeuristicDegree)	3	0.2	800	100	1.82	1.04
FirstFit(HeuristicDegree)	3	0.2	900	100	1.94	1.34
FirstFit(HeuristicDegree)	3	0.2	1000	100	1.99	1.28
FirstFit(HeuristicDegree)	4	0.2	50	100	1.62	0.4
FirstFit(HeuristicDegree)	4	0.2	100	100	2.01	0.65
FirstFit(HeuristicDegree)	4	0.2	150	100	2.26	0.84
FirstFit(HeuristicDegree)	4	0.2	200	100	2.47	1.06
FirstFit(HeuristicDegree)	4	0.2	300	100	2.73	1.19
FirstFit(HeuristicDegree)	4	0.2	400	100	2.84	1.35
FirstFit(HeuristicDegree)	4	0.2	500	100	3.3	1.68
FirstFit(HeuristicDegree)	4	0.2	600	100	2.89	1.7
FirstFit(HeuristicDegree)	4	0.2	700	100	3.3	1.7
FirstFit(HeuristicDegree)	4	0.2	800	100	3.59	1.9
FirstFit(HeuristicDegree)	4	0.2	900	100	3.56	2.06
FirstFit(HeuristicDegree)	4	0.2	1000	100	3.41	2.03
FirstFit(HeuristicDegree)	2	0.3	50	100	1.03	0.13
FirstFit(HeuristicDegree)	2	0.3	100	100	1.02	0.11
FirstFit(HeuristicDegree)	2	0.3	150	100	1.08	0.2
FirstFit(HeuristicDegree)	2	0.3	200	100	1.04	0.17
FirstFit(HeuristicDegree)	2	0.3	300	100	1.08	0.2
FirstFit(HeuristicDegree)	2	0.3	400	100	1.1	0.3
FirstFit(HeuristicDegree)	2	0.3	500	100	1.05	0.14
FirstFit(HeuristicDegree)	2	0.3	600	100	1.08	0.25
FirstFit(HeuristicDegree)	2	0.3	700	100	1.05	0.17
FirstFit(HeuristicDegree)	2	0.3	800	100	1.12	0.3

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Algorithm	k	p	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFit(HeuristicDegree)	2	0.3	900	100	1.08	0.24
FirstFit(HeuristicDegree)	2	0.3	1000	100	1.05	0.22
FirstFit(HeuristicDegree)	3	0.3	50	100	1.12	0.29
FirstFit(HeuristicDegree)	3	0.3	100	100	1.23	0.32
FirstFit(HeuristicDegree)	3	0.3	150	100	1.24	0.46
FirstFit(HeuristicDegree)	3	0.3	200	100	1.18	0.31
FirstFit(HeuristicDegree)	3	0.3	300	100	1.2	0.34
FirstFit(HeuristicDegree)	3	0.3	400	100	1.21	0.33
FirstFit(HeuristicDegree)	3	0.3	500	100	1.27	0.43
FirstFit(HeuristicDegree)	3	0.3	600	100	1.23	0.42
FirstFit(HeuristicDegree)	3	0.3	700	100	1.19	0.27
FirstFit(HeuristicDegree)	3	0.3	800	100	1.2	0.34
FirstFit(HeuristicDegree)	3	0.3	900	100	1.31	0.45
FirstFit(HeuristicDegree)	3	0.3	1000	100	1.22	0.33
FirstFit(HeuristicDegree)	4	0.3	50	100	1.38	0.38
FirstFit(HeuristicDegree)	4	0.3	100	100	1.38	0.53
FirstFit(HeuristicDegree)	4	0.3	150	100	1.44	0.57
FirstFit(HeuristicDegree)	4	0.3	200	100	1.48	0.6
FirstFit(HeuristicDegree)	4	0.3	300	100	1.55	0.6
FirstFit(HeuristicDegree)	4	0.3	400	100	1.52	0.58
FirstFit(HeuristicDegree)	4	0.3	500	100	1.5	0.64
FirstFit(HeuristicDegree)	4	0.3	600	100	1.49	0.57
FirstFit(HeuristicDegree)	4	0.3	700	100	1.68	0.82
FirstFit(HeuristicDegree)	4	0.3	800	100	1.59	0.61
FirstFit(HeuristicDegree)	4	0.3	900	100	1.55	0.53
FirstFit(HeuristicDegree)	4	0.3	1000	100	1.65	0.7

Table 4: BatchDegree: degree-ordered batches with explicit batch size and density p.

Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=10)	2	0.05	10.0	50	100	1.94	0.33
FirstFitBatchDegree(size=10)	2	0.05	10.0	100	100	2.47	0.47
FirstFitBatchDegree(size=10)	2	0.05	10.0	150	100	2.83	0.53
FirstFitBatchDegree(size=10)	2	0.05	10.0	200	100	3.25	0.61
FirstFitBatchDegree(size=10)	2	0.05	10.0	300	100	3.6	0.8
FirstFitBatchDegree(size=10)	2	0.05	10.0	400	100	4.13	0.89
FirstFitBatchDegree(size=10)	2	0.05	10.0	500	100	4.4	0.87
FirstFitBatchDegree(size=10)	2	0.05	10.0	600	100	4.54	0.97
FirstFitBatchDegree(size=10)	2	0.05	10.0	700	100	4.8	1.14
FirstFitBatchDegree(size=10)	2	0.05	10.0	800	100	5.1	1.08
FirstFitBatchDegree(size=10)	2	0.05	10.0	900	100	4.95	1.21
FirstFitBatchDegree(size=10)	2	0.05	10.0	1000	100	5.08	1.32
FirstFitBatchDegree(size=10)	3	0.05	10.0	50	100	1.82	0.18

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Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=10)	3	0.05	10.0	100	100	2.28	0.17
FirstFitBatchDegree(size=10)	3	0.05	10.0	150	100	2.74	0.2
FirstFitBatchDegree(size=10)	3	0.05	10.0	200	100	3.1	0.2
FirstFitBatchDegree(size=10)	3	0.05	10.0	300	100	3.76	0.25
FirstFitBatchDegree(size=10)	3	0.05	10.0	400	100	4.38	0.25
FirstFitBatchDegree(size=10)	3	0.05	10.0	500	100	4.84	0.3
FirstFitBatchDegree(size=10)	3	0.05	10.0	600	100	5.33	0.3
FirstFitBatchDegree(size=10)	3	0.05	10.0	700	100	5.76	0.4
FirstFitBatchDegree(size=10)	3	0.05	10.0	800	100	6.27	0.36
FirstFitBatchDegree(size=10)	3	0.05	10.0	900	100	6.57	0.52
FirstFitBatchDegree(size=10)	3	0.05	10.0	1000	100	6.98	0.56
FirstFitBatchDegree(size=10)	4	0.05	10.0	50	100	1.61	0.15
FirstFitBatchDegree(size=10)	4	0.05	10.0	100	100	2.01	0.12
FirstFitBatchDegree(size=10)	4	0.05	10.0	150	100	2.35	0.14
FirstFitBatchDegree(size=10)	4	0.05	10.0	200	100	2.68	0.13
FirstFitBatchDegree(size=10)	4	0.05	10.0	300	100	3.23	0.14
FirstFitBatchDegree(size=10)	4	0.05	10.0	400	100	3.76	0.15
FirstFitBatchDegree(size=10)	4	0.05	10.0	500	100	4.23	0.17
FirstFitBatchDegree(size=10)	4	0.05	10.0	600	100	4.69	0.17
FirstFitBatchDegree(size=10)	4	0.05	10.0	700	100	5.16	0.14
FirstFitBatchDegree(size=10)	4	0.05	10.0	800	100	5.54	0.22
FirstFitBatchDegree(size=10)	4	0.05	10.0	900	100	6	0.17
FirstFitBatchDegree(size=10)	4	0.05	10.0	1000	100	6.37	0.18
FirstFitBatchDegree(size=50)	2	0.05	50.0	50	100	1.37	0.42
FirstFitBatchDegree(size=50)	2	0.05	50.0	100	100	2.14	0.58
FirstFitBatchDegree(size=50)	2	0.05	50.0	150	100	2.52	0.79
FirstFitBatchDegree(size=50)	2	0.05	50.0	200	100	2.99	0.78
FirstFitBatchDegree(size=50)	2	0.05	50.0	300	100	3.31	0.9
FirstFitBatchDegree(size=50)	2	0.05	50.0	400	100	3.78	0.99
FirstFitBatchDegree(size=50)	2	0.05	50.0	500	100	3.95	1.35
FirstFitBatchDegree(size=50)	2	0.05	50.0	600	100	4.19	1.23
FirstFitBatchDegree(size=50)	2	0.05	50.0	700	100	4.3	1.49
FirstFitBatchDegree(size=50)	2	0.05	50.0	800	100	4.67	1.39
FirstFitBatchDegree(size=50)	2	0.05	50.0	900	100	4.89	1.38
FirstFitBatchDegree(size=50)	2	0.05	50.0	1000	100	4.7	1.35
FirstFitBatchDegree(size=50)	3	0.05	50.0	50	100	1.59	0.19
FirstFitBatchDegree(size=50)	3	0.05	50.0	100	100	2.16	0.22
FirstFitBatchDegree(size=50)	3	0.05	50.0	150	100	2.59	0.23
FirstFitBatchDegree(size=50)	3	0.05	50.0	200	100	3.03	0.25
FirstFitBatchDegree(size=50)	3	0.05	50.0	300	100	3.71	0.26
FirstFitBatchDegree(size=50)	3	0.05	50.0	400	100	4.28	0.28
FirstFitBatchDegree(size=50)	3	0.05	50.0	500	100	4.79	0.3
FirstFitBatchDegree(size=50)	3	0.05	50.0	600	100	5.21	0.4
FirstFitBatchDegree(size=50)	3	0.05	50.0	700	100	5.76	0.35
FirstFitBatchDegree(size=50)	3	0.05	50.0	800	100	6.11	0.5

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Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=50)	3	0.05	50.0	900	100	6.46	0.45
FirstFitBatchDegree(size=50)	3	0.05	50.0	1000	100	6.98	0.53
FirstFitBatchDegree(size=50)	4	0.05	50.0	50	100	1.46	0.12
FirstFitBatchDegree(size=50)	4	0.05	50.0	100	100	1.89	0.14
FirstFitBatchDegree(size=50)	4	0.05	50.0	150	100	2.29	0.14
FirstFitBatchDegree(size=50)	4	0.05	50.0	200	100	2.6	0.14
FirstFitBatchDegree(size=50)	4	0.05	50.0	300	100	3.2	0.14
FirstFitBatchDegree(size=50)	4	0.05	50.0	400	100	3.72	0.16
FirstFitBatchDegree(size=50)	4	0.05	50.0	500	100	4.18	0.17
FirstFitBatchDegree(size=50)	4	0.05	50.0	600	100	4.66	0.15
FirstFitBatchDegree(size=50)	4	0.05	50.0	700	100	5.11	0.16
FirstFitBatchDegree(size=50)	4	0.05	50.0	800	100	5.51	0.18
FirstFitBatchDegree(size=50)	4	0.05	50.0	900	100	5.91	0.19
FirstFitBatchDegree(size=50)	4	0.05	50.0	1000	100	6.31	0.19
FirstFitBatchDegree(size=10)	2	0.1	10.0	50	100	1.93	0.55
FirstFitBatchDegree(size=10)	2	0.1	10.0	100	100	2.5	0.74
FirstFitBatchDegree(size=10)	2	0.1	10.0	150	100	2.63	0.99
FirstFitBatchDegree(size=10)	2	0.1	10.0	200	100	2.87	1.04
FirstFitBatchDegree(size=10)	2	0.1	10.0	300	100	3.06	1.22
FirstFitBatchDegree(size=10)	2	0.1	10.0	400	100	3.3	1.41
FirstFitBatchDegree(size=10)	2	0.1	10.0	500	100	3.49	1.61
FirstFitBatchDegree(size=10)	2	0.1	10.0	600	100	3.08	1.43
FirstFitBatchDegree(size=10)	2	0.1	10.0	700	100	3.35	1.7
FirstFitBatchDegree(size=10)	2	0.1	10.0	800	100	3.87	1.71
FirstFitBatchDegree(size=10)	2	0.1	10.0	900	100	3.88	1.93
FirstFitBatchDegree(size=10)	2	0.1	10.0	1000	100	3.6	1.82
FirstFitBatchDegree(size=10)	3	0.1	10.0	50	100	2.09	0.24
FirstFitBatchDegree(size=10)	3	0.1	10.0	100	100	2.75	0.32
FirstFitBatchDegree(size=10)	3	0.1	10.0	150	100	3.27	0.47
FirstFitBatchDegree(size=10)	3	0.1	10.0	200	100	3.78	0.52
FirstFitBatchDegree(size=10)	3	0.1	10.0	300	100	4.34	0.72
FirstFitBatchDegree(size=10)	3	0.1	10.0	400	100	5	0.77
FirstFitBatchDegree(size=10)	3	0.1	10.0	500	100	5.56	1.06
FirstFitBatchDegree(size=10)	3	0.1	10.0	600	100	6.07	1.01
FirstFitBatchDegree(size=10)	3	0.1	10.0	700	100	6.46	1.13
FirstFitBatchDegree(size=10)	3	0.1	10.0	800	100	6.57	1.35
FirstFitBatchDegree(size=10)	3	0.1	10.0	900	100	7.12	1.41
FirstFitBatchDegree(size=10)	3	0.1	10.0	1000	100	7.32	1.5
FirstFitBatchDegree(size=10)	4	0.1	10.0	50	100	1.84	0.16
FirstFitBatchDegree(size=10)	4	0.1	10.0	100	100	2.47	0.19
FirstFitBatchDegree(size=10)	4	0.1	10.0	150	100	3.01	0.2
FirstFitBatchDegree(size=10)	4	0.1	10.0	200	100	3.51	0.22
FirstFitBatchDegree(size=10)	4	0.1	10.0	300	100	4.37	0.28
FirstFitBatchDegree(size=10)	4	0.1	10.0	400	100	5.16	0.3
FirstFitBatchDegree(size=10)	4	0.1	10.0	500	100	5.75	0.43

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Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=10)	4	0.1	10.0	600	100	6.34	0.49
FirstFitBatchDegree(size=10)	4	0.1	10.0	700	100	6.84	0.54
FirstFitBatchDegree(size=10)	4	0.1	10.0	800	100	7.45	0.78
FirstFitBatchDegree(size=10)	4	0.1	10.0	900	100	8.01	0.7
FirstFitBatchDegree(size=10)	4	0.1	10.0	1000	100	8.48	0.91
FirstFitBatchDegree(size=50)	2	0.1	50.0	50	100	1.35	0.51
FirstFitBatchDegree(size=50)	2	0.1	50.0	100	100	1.79	0.85
FirstFitBatchDegree(size=50)	2	0.1	50.0	150	100	1.96	0.95
FirstFitBatchDegree(size=50)	2	0.1	50.0	200	100	2.06	1.12
FirstFitBatchDegree(size=50)	2	0.1	50.0	300	100	2.5	1.31
FirstFitBatchDegree(size=50)	2	0.1	50.0	400	100	2.59	1.45
FirstFitBatchDegree(size=50)	2	0.1	50.0	500	100	2.64	1.53
FirstFitBatchDegree(size=50)	2	0.1	50.0	600	100	2.7	1.56
FirstFitBatchDegree(size=50)	2	0.1	50.0	700	100	3.18	1.76
FirstFitBatchDegree(size=50)	2	0.1	50.0	800	100	2.93	1.67
FirstFitBatchDegree(size=50)	2	0.1	50.0	900	100	2.95	1.75
FirstFitBatchDegree(size=50)	2	0.1	50.0	1000	100	3.23	1.93
FirstFitBatchDegree(size=50)	3	0.1	50.0	50	100	1.68	0.4
FirstFitBatchDegree(size=50)	3	0.1	50.0	100	100	2.48	0.47
FirstFitBatchDegree(size=50)	3	0.1	50.0	150	100	3	0.52
FirstFitBatchDegree(size=50)	3	0.1	50.0	200	100	3.53	0.7
FirstFitBatchDegree(size=50)	3	0.1	50.0	300	100	4.42	0.66
FirstFitBatchDegree(size=50)	3	0.1	50.0	400	100	4.78	1.14
FirstFitBatchDegree(size=50)	3	0.1	50.0	500	100	5.33	1.01
FirstFitBatchDegree(size=50)	3	0.1	50.0	600	100	5.59	1.44
FirstFitBatchDegree(size=50)	3	0.1	50.0	700	100	6.04	1.33
FirstFitBatchDegree(size=50)	3	0.1	50.0	800	100	6.49	1.27
FirstFitBatchDegree(size=50)	3	0.1	50.0	900	100	6.69	1.67
FirstFitBatchDegree(size=50)	3	0.1	50.0	1000	100	7.08	1.57
FirstFitBatchDegree(size=50)	4	0.1	50.0	50	100	1.65	0.19
FirstFitBatchDegree(size=50)	4	0.1	50.0	100	100	2.32	0.19
FirstFitBatchDegree(size=50)	4	0.1	50.0	150	100	2.88	0.19
FirstFitBatchDegree(size=50)	4	0.1	50.0	200	100	3.41	0.19
FirstFitBatchDegree(size=50)	4	0.1	50.0	300	100	4.31	0.25
FirstFitBatchDegree(size=50)	4	0.1	50.0	400	100	5.02	0.32
FirstFitBatchDegree(size=50)	4	0.1	50.0	500	100	5.74	0.42
FirstFitBatchDegree(size=50)	4	0.1	50.0	600	100	6.42	0.47
FirstFitBatchDegree(size=50)	4	0.1	50.0	700	100	6.89	0.54
FirstFitBatchDegree(size=50)	4	0.1	50.0	800	100	7.4	0.64
FirstFitBatchDegree(size=50)	4	0.1	50.0	900	100	8.04	0.64
FirstFitBatchDegree(size=50)	4	0.1	50.0	1000	100	8.32	0.84
FirstFitBatchDegree(size=10)	2	0.2	10.0	50	100	1.52	0.58
FirstFitBatchDegree(size=10)	2	0.2	10.0	100	100	1.79	0.85
FirstFitBatchDegree(size=10)	2	0.2	10.0	150	100	1.75	0.9
FirstFitBatchDegree(size=10)	2	0.2	10.0	200	100	1.78	0.75

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Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=10)	2	0.2	10.0	300	100	1.79	0.87
FirstFitBatchDegree(size=10)	2	0.2	10.0	400	100	1.82	0.86
FirstFitBatchDegree(size=10)	2	0.2	10.0	500	100	1.63	0.76
FirstFitBatchDegree(size=10)	2	0.2	10.0	600	100	1.77	0.9
FirstFitBatchDegree(size=10)	2	0.2	10.0	700	100	2	1.08
FirstFitBatchDegree(size=10)	2	0.2	10.0	800	100	1.89	1.03
FirstFitBatchDegree(size=10)	2	0.2	10.0	900	100	1.7	0.8
FirstFitBatchDegree(size=10)	2	0.2	10.0	1000	100	1.93	1.25
FirstFitBatchDegree(size=10)	3	0.2	10.0	50	100	2.07	0.58
FirstFitBatchDegree(size=10)	3	0.2	10.0	100	100	2.59	0.83
FirstFitBatchDegree(size=10)	3	0.2	10.0	150	100	2.88	0.94
FirstFitBatchDegree(size=10)	3	0.2	10.0	200	100	3.14	1.26
FirstFitBatchDegree(size=10)	3	0.2	10.0	300	100	3.24	1.46
FirstFitBatchDegree(size=10)	3	0.2	10.0	400	100	3.43	1.56
FirstFitBatchDegree(size=10)	3	0.2	10.0	500	100	3.74	1.71
FirstFitBatchDegree(size=10)	3	0.2	10.0	600	100	3.83	1.92
FirstFitBatchDegree(size=10)	3	0.2	10.0	700	100	3.4	1.68
FirstFitBatchDegree(size=10)	3	0.2	10.0	800	100	3.62	1.77
FirstFitBatchDegree(size=10)	3	0.2	10.0	900	100	3.82	1.92
FirstFitBatchDegree(size=10)	3	0.2	10.0	1000	100	3.98	2.25
FirstFitBatchDegree(size=10)	4	0.2	10.0	50	100	2.07	0.34
FirstFitBatchDegree(size=10)	4	0.2	10.0	100	100	2.74	0.49
FirstFitBatchDegree(size=10)	4	0.2	10.0	150	100	3.42	0.65
FirstFitBatchDegree(size=10)	4	0.2	10.0	200	100	3.72	0.86
FirstFitBatchDegree(size=10)	4	0.2	10.0	300	100	4.64	1.04
FirstFitBatchDegree(size=10)	4	0.2	10.0	400	100	4.68	1.51
FirstFitBatchDegree(size=10)	4	0.2	10.0	500	100	5.11	1.64
FirstFitBatchDegree(size=10)	4	0.2	10.0	600	100	5.25	1.84
FirstFitBatchDegree(size=10)	4	0.2	10.0	700	100	5.52	1.84
FirstFitBatchDegree(size=10)	4	0.2	10.0	800	100	6.32	2.05
FirstFitBatchDegree(size=10)	4	0.2	10.0	900	100	6.45	2.03
FirstFitBatchDegree(size=10)	4	0.2	10.0	1000	100	5.81	2.23
FirstFitBatchDegree(size=50)	2	0.2	50.0	50	100	1.07	0.21
FirstFitBatchDegree(size=50)	2	0.2	50.0	100	100	1.17	0.39
FirstFitBatchDegree(size=50)	2	0.2	50.0	150	100	1.34	0.69
FirstFitBatchDegree(size=50)	2	0.2	50.0	200	100	1.3	0.62
FirstFitBatchDegree(size=50)	2	0.2	50.0	300	100	1.45	0.68
FirstFitBatchDegree(size=50)	2	0.2	50.0	400	100	1.44	0.85
FirstFitBatchDegree(size=50)	2	0.2	50.0	500	100	1.6	0.94
FirstFitBatchDegree(size=50)	2	0.2	50.0	600	100	1.4	0.69
FirstFitBatchDegree(size=50)	2	0.2	50.0	700	100	1.45	0.73
FirstFitBatchDegree(size=50)	2	0.2	50.0	800	100	1.39	0.75
FirstFitBatchDegree(size=50)	2	0.2	50.0	900	100	1.65	0.98
FirstFitBatchDegree(size=50)	2	0.2	50.0	1000	100	1.48	0.74
FirstFitBatchDegree(size=50)	3	0.2	50.0	50	100	1.38	0.52

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Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=50)	3	0.2	50.0	100	100	1.81	0.8
FirstFitBatchDegree(size=50)	3	0.2	50.0	150	100	2.04	0.95
FirstFitBatchDegree(size=50)	3	0.2	50.0	200	100	2.31	1.17
FirstFitBatchDegree(size=50)	3	0.2	50.0	300	100	2.61	1.57
FirstFitBatchDegree(size=50)	3	0.2	50.0	400	100	2.59	1.41
FirstFitBatchDegree(size=50)	3	0.2	50.0	500	100	2.91	1.76
FirstFitBatchDegree(size=50)	3	0.2	50.0	600	100	3.1	1.7
FirstFitBatchDegree(size=50)	3	0.2	50.0	700	100	2.8	1.67
FirstFitBatchDegree(size=50)	3	0.2	50.0	800	100	3.33	2.12
FirstFitBatchDegree(size=50)	3	0.2	50.0	900	100	3.27	1.87
FirstFitBatchDegree(size=50)	3	0.2	50.0	1000	100	2.94	1.79
FirstFitBatchDegree(size=50)	4	0.2	50.0	50	100	1.62	0.4
FirstFitBatchDegree(size=50)	4	0.2	50.0	100	100	2.49	0.57
FirstFitBatchDegree(size=50)	4	0.2	50.0	150	100	3.15	0.8
FirstFitBatchDegree(size=50)	4	0.2	50.0	200	100	3.32	1.13
FirstFitBatchDegree(size=50)	4	0.2	50.0	300	100	4.06	1.16
FirstFitBatchDegree(size=50)	4	0.2	50.0	400	100	4.32	1.54
FirstFitBatchDegree(size=50)	4	0.2	50.0	500	100	4.95	1.93
FirstFitBatchDegree(size=50)	4	0.2	50.0	600	100	5.34	2.01
FirstFitBatchDegree(size=50)	4	0.2	50.0	700	100	5.32	2.01
FirstFitBatchDegree(size=50)	4	0.2	50.0	800	100	5.65	2.33
FirstFitBatchDegree(size=50)	4	0.2	50.0	900	100	5.9	2.33
FirstFitBatchDegree(size=50)	4	0.2	50.0	1000	100	6.41	2.35
FirstFitBatchDegree(size=10)	2	0.3	10.0	50	100	1.31	0.45
FirstFitBatchDegree(size=10)	2	0.3	10.0	100	100	1.26	0.43
FirstFitBatchDegree(size=10)	2	0.3	10.0	150	100	1.39	0.61
FirstFitBatchDegree(size=10)	2	0.3	10.0	200	100	1.41	0.69
FirstFitBatchDegree(size=10)	2	0.3	10.0	300	100	1.35	0.52
FirstFitBatchDegree(size=10)	2	0.3	10.0	400	100	1.2	0.34
FirstFitBatchDegree(size=10)	2	0.3	10.0	500	100	1.34	0.59
FirstFitBatchDegree(size=10)	2	0.3	10.0	600	100	1.33	0.54
FirstFitBatchDegree(size=10)	2	0.3	10.0	700	100	1.3	0.48
FirstFitBatchDegree(size=10)	2	0.3	10.0	800	100	1.3	0.52
FirstFitBatchDegree(size=10)	2	0.3	10.0	900	100	1.4	0.63
FirstFitBatchDegree(size=10)	2	0.3	10.0	1000	100	1.41	0.53
FirstFitBatchDegree(size=10)	3	0.3	10.0	50	100	1.63	0.56
FirstFitBatchDegree(size=10)	3	0.3	10.0	100	100	1.84	0.75
FirstFitBatchDegree(size=10)	3	0.3	10.0	150	100	1.99	0.94
FirstFitBatchDegree(size=10)	3	0.3	10.0	200	100	2.02	0.99
FirstFitBatchDegree(size=10)	3	0.3	10.0	300	100	1.9	0.76
FirstFitBatchDegree(size=10)	3	0.3	10.0	400	100	2.02	0.98
FirstFitBatchDegree(size=10)	3	0.3	10.0	500	100	2.03	0.89
FirstFitBatchDegree(size=10)	3	0.3	10.0	600	100	1.94	0.88
FirstFitBatchDegree(size=10)	3	0.3	10.0	700	100	2.09	1.01
FirstFitBatchDegree(size=10)	3	0.3	10.0	800	100	1.9	0.93

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Algorithm	k	p	**BatchSize**	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=10)	3	0.3	10.0	900	100	2.17	1.16
FirstFitBatchDegree(size=10)	3	0.3	10.0	1000	100	2.1	0.94
FirstFitBatchDegree(size=10)	4	0.3	10.0	50	100	1.85	0.49
FirstFitBatchDegree(size=10)	4	0.3	10.0	100	100	2.34	0.74
FirstFitBatchDegree(size=10)	4	0.3	10.0	150	100	2.55	0.97
FirstFitBatchDegree(size=10)	4	0.3	10.0	200	100	2.59	1.14
FirstFitBatchDegree(size=10)	4	0.3	10.0	300	100	2.79	1.1
FirstFitBatchDegree(size=10)	4	0.3	10.0	400	100	2.65	1.06
FirstFitBatchDegree(size=10)	4	0.3	10.0	500	100	2.93	1.33
FirstFitBatchDegree(size=10)	4	0.3	10.0	600	100	2.86	1.23
FirstFitBatchDegree(size=10)	4	0.3	10.0	700	100	2.92	1.38
FirstFitBatchDegree(size=10)	4	0.3	10.0	800	100	3.09	1.53
FirstFitBatchDegree(size=10)	4	0.3	10.0	900	100	3.03	1.52
FirstFitBatchDegree(size=10)	4	0.3	10.0	1000	100	2.98	1.34
FirstFitBatchDegree(size=50)	2	0.3	50.0	50	100	1.03	0.13
FirstFitBatchDegree(size=50)	2	0.3	50.0	100	100	1.04	0.14
FirstFitBatchDegree(size=50)	2	0.3	50.0	150	100	1.1	0.39
FirstFitBatchDegree(size=50)	2	0.3	50.0	200	100	1.13	0.31
FirstFitBatchDegree(size=50)	2	0.3	50.0	300	100	1.15	0.36
FirstFitBatchDegree(size=50)	2	0.3	50.0	400	100	1.14	0.42
FirstFitBatchDegree(size=50)	2	0.3	50.0	500	100	1.17	0.45
FirstFitBatchDegree(size=50)	2	0.3	50.0	600	100	1.17	0.39
FirstFitBatchDegree(size=50)	2	0.3	50.0	700	100	1.1	0.25
FirstFitBatchDegree(size=50)	2	0.3	50.0	800	100	1.14	0.33
FirstFitBatchDegree(size=50)	2	0.3	50.0	900	100	1.16	0.34
FirstFitBatchDegree(size=50)	2	0.3	50.0	1000	100	1.16	0.31
FirstFitBatchDegree(size=50)	3	0.3	50.0	50	100	1.12	0.29
FirstFitBatchDegree(size=50)	3	0.3	50.0	100	100	1.36	0.46
FirstFitBatchDegree(size=50)	3	0.3	50.0	150	100	1.43	0.61
FirstFitBatchDegree(size=50)	3	0.3	50.0	200	100	1.31	0.44
FirstFitBatchDegree(size=50)	3	0.3	50.0	300	100	1.46	0.68
FirstFitBatchDegree(size=50)	3	0.3	50.0	400	100	1.56	0.78
FirstFitBatchDegree(size=50)	3	0.3	50.0	500	100	1.63	0.8
FirstFitBatchDegree(size=50)	3	0.3	50.0	600	100	1.61	0.79
FirstFitBatchDegree(size=50)	3	0.3	50.0	700	100	1.64	0.93
FirstFitBatchDegree(size=50)	3	0.3	50.0	800	100	1.51	0.63
FirstFitBatchDegree(size=50)	3	0.3	50.0	900	100	1.61	0.67
FirstFitBatchDegree(size=50)	3	0.3	50.0	1000	100	1.58	0.81
FirstFitBatchDegree(size=50)	4	0.3	50.0	50	100	1.38	0.38
FirstFitBatchDegree(size=50)	4	0.3	50.0	100	100	1.66	0.69
FirstFitBatchDegree(size=50)	4	0.3	50.0	150	100	1.95	0.8
FirstFitBatchDegree(size=50)	4	0.3	50.0	200	100	2.16	1.02
FirstFitBatchDegree(size=50)	4	0.3	50.0	300	100	2.21	1.11
FirstFitBatchDegree(size=50)	4	0.3	50.0	400	100	2.17	1.09
FirstFitBatchDegree(size=50)	4	0.3	50.0	500	100	2.33	1.2

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Algorithm	k	p	BatchSize	n	N	$\overline{\rho(Alg)}$	$SD(\rho(Alg))$
FirstFitBatchDegree(size=50)	4	0.3	50.0	600	100	2.61	1.36
FirstFitBatchDegree(size=50)	4	0.3	50.0	700	100	2.2	1.18
FirstFitBatchDegree(size=50)	4	0.3	50.0	800	100	2.09	1.07
FirstFitBatchDegree(size=50)	4	0.3	50.0	900	100	2.34	1.18
FirstFitBatchDegree(size=50)	4	0.3	50.0	1000	100	2.71	1.53

6 Analysis

We now analyse outcomes for the two simulation tracks. Each subsection first presents raw behavior (grouped figures) followed by model-fitting interpretation. Competitive ratio ρ = used colors/k closer to 1 indicates tighter adherence to the target palette.

6.1 Simulation I (FirstFit and CBIP)

Grouped FirstFit performance A composite grid (Figure 1) summarises $k \in \{2, 3, 4\}$ across densities $p \in \{0.05, 0.1, 0.2, 0.3\}$.

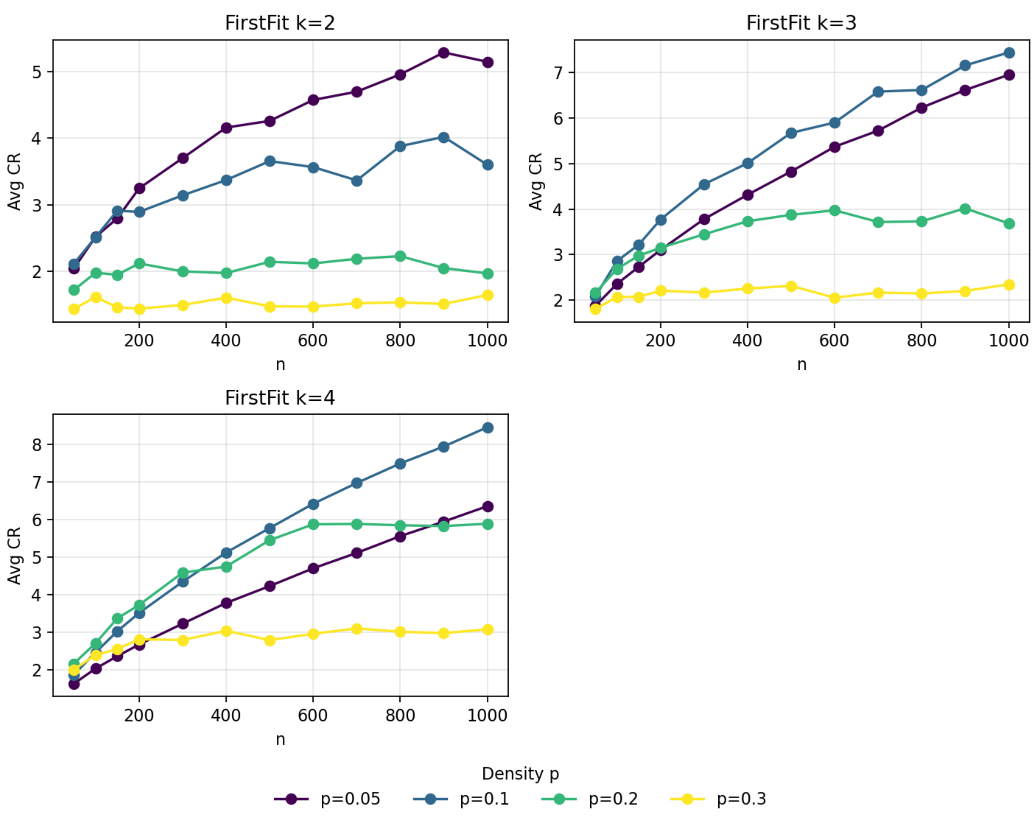


Figure 1: FirstFit competitive ratio vs n for $k = 2, 3, 4$ across densities p.

FirstFit grid: In this dataset, several curves show that higher density p corresponds to *lower* competitive ratios over large ranges of n, and lines can cross. This non-monotonic effect suggests that denser instances sometimes create more regular neighbourhoods where FirstFit reuses colours efficiently, whereas sparser instances trigger extra colours earlier due to irregular conflicts. The $k = 2$ panels stay near $\rho \approx 1$ (consistent with near-bipartite behaviour). For $k = 3$ and $k = 4$, variability increases and the ordering by p is not stable across all n; both increasing and decreasing segments with p appear. Overall, the impact of p on FirstFit is **non-monotonic** in our results, with several cases where larger p yields lower ρ.

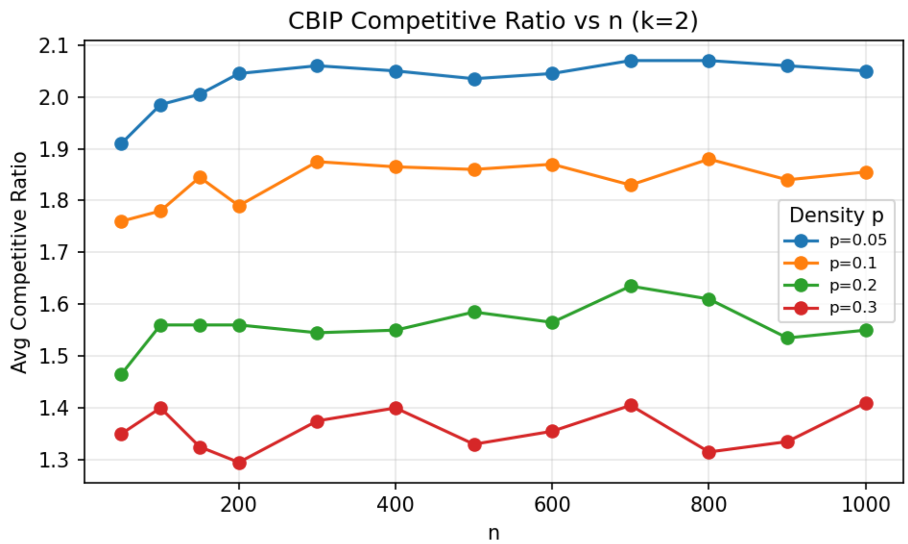


Figure 2: CBIP competitive ratio vs n for $k = 2$ across densities p (nearly flat).

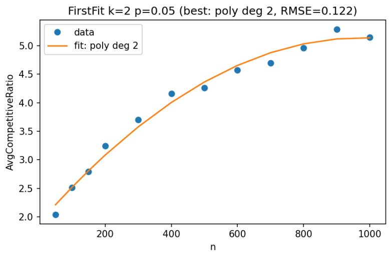
CBIP ($k = 2$): CBIP traces almost constant $\rho \approx 1$ across n and all p tested. Its BFS bipartition guarantees minimal colors required for bipartite substructure revealed so far. Variance is negligible compared to FirstFit.

Why is k restricted to 2 in CBIP?

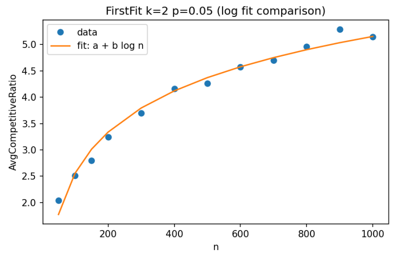
Bipartition offers a linear-time certificate and immediate 2-coloring. Extending CBIP beyond bipartite graphs would require online k-coloring (NP-complete for $k \geq 3$), removing tractable structural guarantees and making per-arrival component re-coloring infeasible at our scales. Hence we restrict CBIP reporting to $k = 2$.

Model fitting To characterise growth, we fitted $\log n$, $\sqrt{n}$, and quadratic models to mean ρ vs n per (k, p). For FirstFit, $\log n$ often suffices at low p (sublinear growth of conflicts), while higher p shifts preference toward $\sqrt{n}$ or quadratic where acceleration of conflicts appears. CBIP exhibits near-constant behavior, with $a + b \log n$ degenerating to an almost flat line and lowest RMSE.

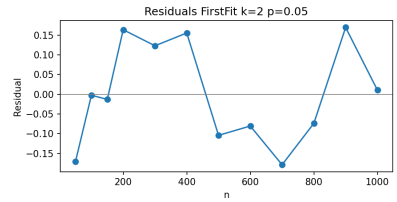
Best-fit summaries



(a) Best fit (poly deg 2)



(b) Log fit comparison



(c) Residuals (best model)

Figure 3: FirstFit $k = 2, p = 0.05$: mild sublinear growth. Quadratic (degree-2) model selected (lowest RMSE); $\log(n)$ fit shows similar gentle curvature with slightly higher error.

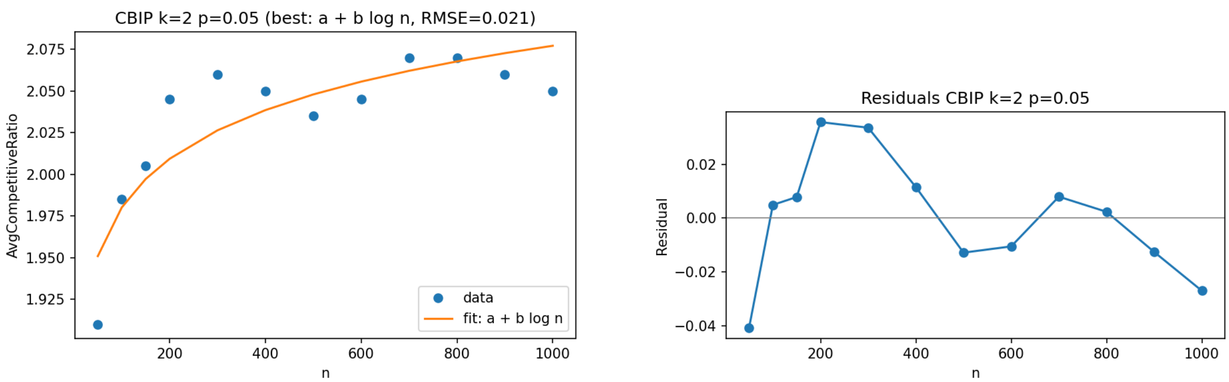
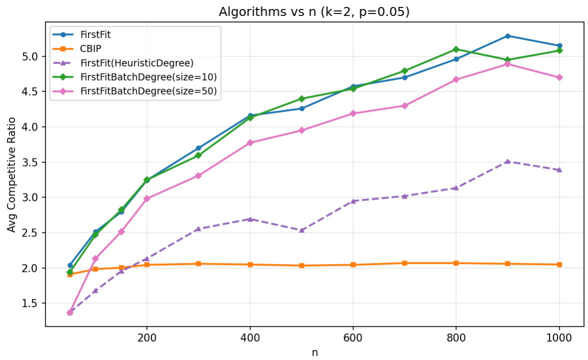


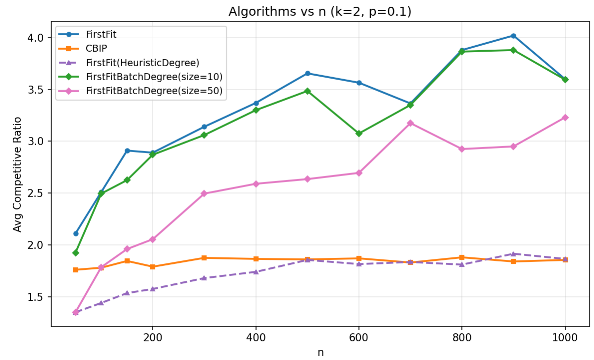
Figure 4: CBIP $k = 2, p = 0.05$: near-flat residuals confirm structural optimality.

6.2 Simulation II (HeuristicDegree and BatchDegree)

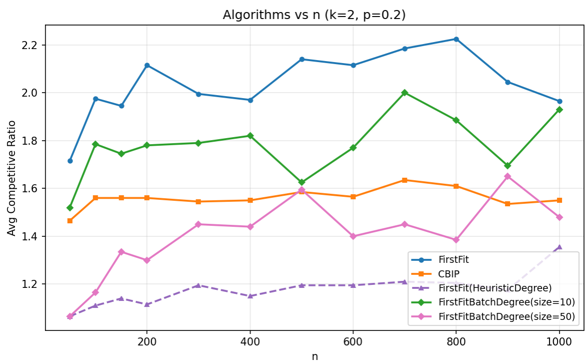
Grouped comparison figures by density For each k, we present a 2 x 2 grid of $p \in \{0.05, 0.1, 0.2, 0.3\}$; each subplot compares FirstFit, HeuristicDegree, and BatchDegree variants over n.



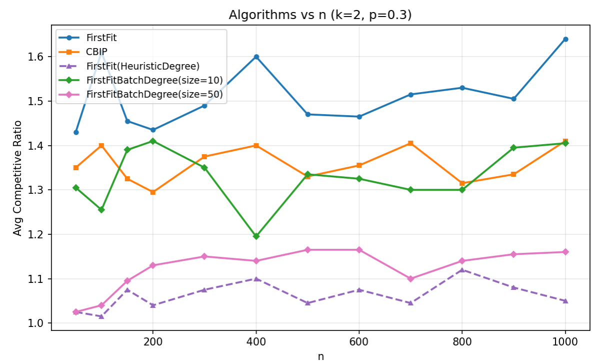
(a) $k = 2,\ p = 0.05$



(b) $k = 2,\ p = 0.1$



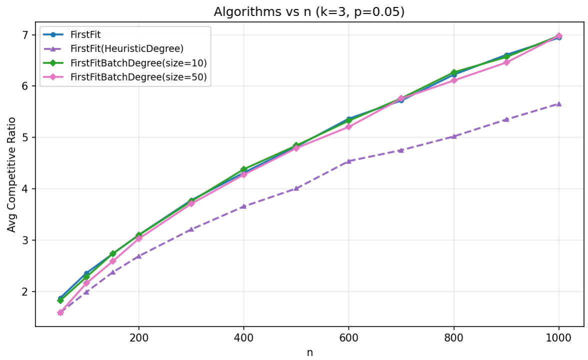
(c) $k = 2,\ p = 0.2$



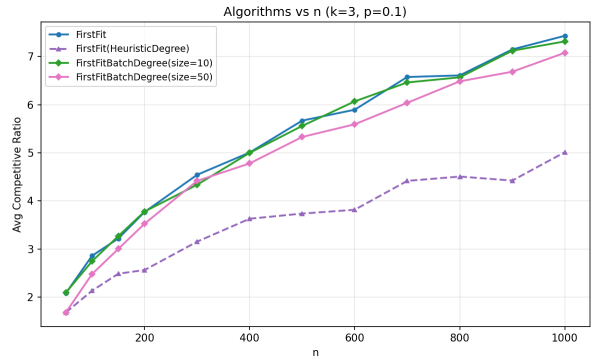
(d) $k = 2,\ p = 0.3$

Figure 5: Simulation II grouped comparison for $k = 2$: FirstFit vs HeuristicDegree vs BatchDegree across densities.

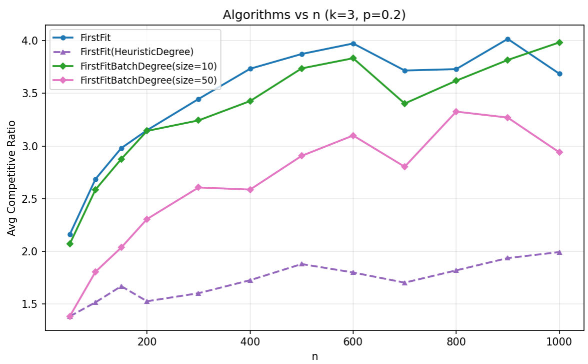
Analysis ($k = 2$): HeuristicDegree reduces ρ relative to FirstFit across all densities and remains the strongest variant. **However, BatchDegree still improves upon raw FirstFit at higher densities when the batch size is larger.** For example, at $p = 0.2$ and $n = 1000$ FirstFit attains $\rho \approx 1.97$ while BatchDegree($t = 50$) lowers this to ≈ 1.48 (25% reduction); at $p = 0.3$ and $n = 1000$ FirstFit is ≈ 1.64 versus ≈ 1.16 for BatchDegree($t = 50$) (29% reduction). Smaller batches ($t = 10$) yield weaker or inconsistent gains. Thus the ordering hierarchy is HeuristicDegree $<$ BatchDegree($t = 50$) $<$ FirstFit (in ρ) for dense settings, while BatchDegree may revert closer to FirstFit at lower p.



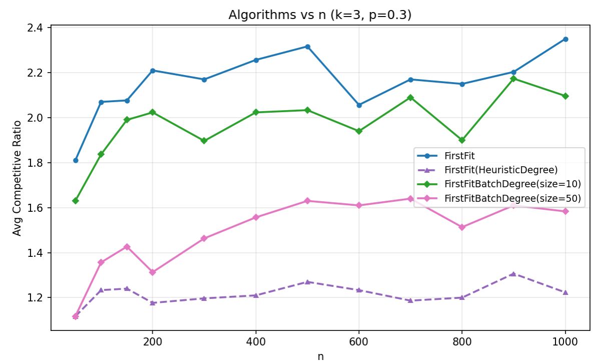
(a) $k = 3, \ p = 0.05$



(b) $k = 3, \ p = 0.1$



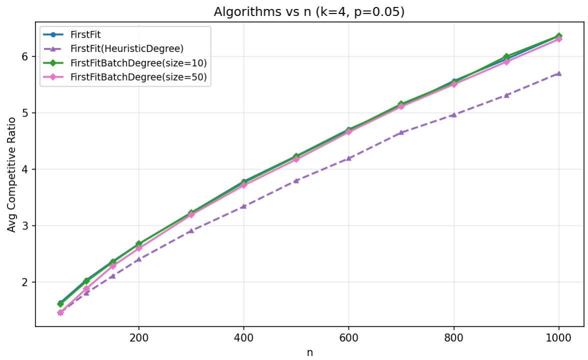
(c) $k = 3, \ p = 0.2$



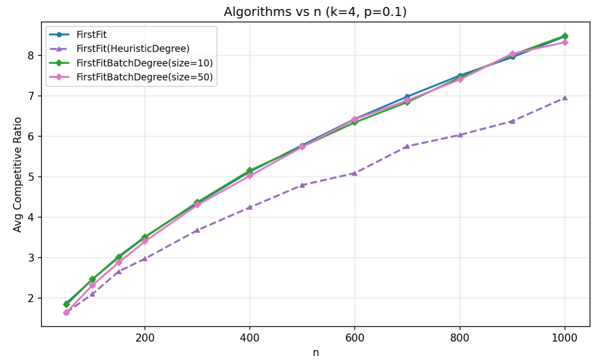
(d) $k = 3, \ p = 0.3$

Figure 6: Simulation II grouped comparison for $k = 3$: FirstFit vs HeuristicDegree vs BatchDegree across densities.

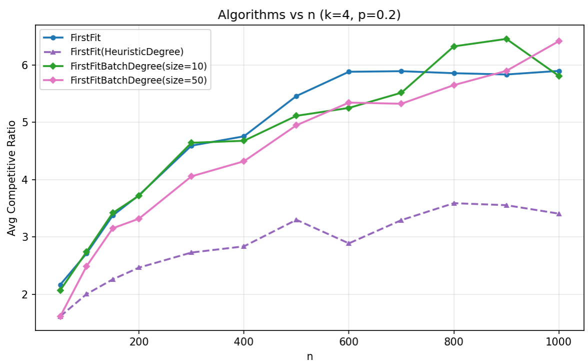
Analysis ($k = 3$): HeuristicDegree remains the strongest performer, but larger batches again provide **meaningful improvement over FirstFit in dense regimes**. At $p = 0.2$, $n = 1000$ FirstFit reaches $\rho \approx 3.69$ while BatchDegree($t = 50$) reduces this to ≈ 2.94 (20% reduction); at $p = 0.3$, $n = 1000$ FirstFit is ≈ 2.35 versus ≈ 1.58 for BatchDegree($t = 50$) (33% reduction). BatchDegree($t = 10$) offers smaller or unstable gains. Ordering relationship for dense cases: HeuristicDegree $<$ BatchDegree($t = 50$) $<$ FirstFit. For sparser graphs the BatchDegree curves can drift upward toward FirstFit, reflecting diminished benefit when local batches see fewer high-degree conflicts to exploit.



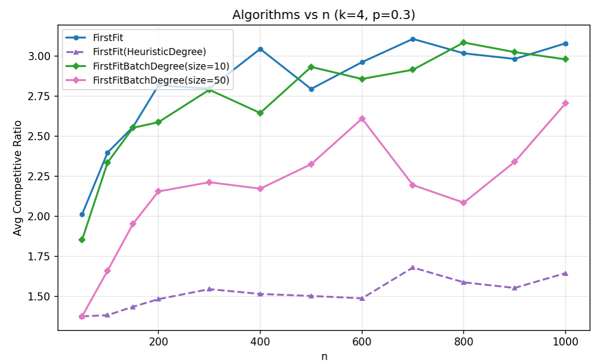
(a) $k = 4,\ p = 0.05$



(b) $k = 4,\ p = 0.1$



(c) $k = 4,\ p = 0.2$



(d) $k = 4,\ p = 0.3$

Figure 7: Simulation II grouped comparison for $k = 4$: FirstFit vs HeuristicDegree vs BatchDegree across densities.

Analysis ($k = 4$): HeuristicDegree again delivers the lowest ratios. BatchDegree($t = 50$) shows **modest** improvement over FirstFit at the highest density ($p = 0.3$, $n = 1000$: ρ drops from ≈ 3.08 to ≈ 2.71, 12% reduction) while at $p = 0.2$ improvements do not materialise (BatchDegree slightly worse than FirstFit for large n). This indicates diminishing returns of limited-window reordering as k grows: global sorting preserves advantages, whereas intra-batch ordering may not sufficiently coordinate color reuse across partitions.

7 Conclusion

This project implemented and rigorously evaluated online graph coloring baselines (FirstFit, CBIP) and degree-based ordering heuristics (HeuristicDegree, BatchDegree) over reproducible, cached datasets spanning n, k, and p. We standardised algorithm descriptions, ensured correctness via unit tests, and produced a coherent analysis pipeline (figures, fits, and summaries).

Key findings:

- **FirstFit behavior**: Competitive ratio trends with density p are non-monotonic; higher p can yield lower ratios via more regular neighbourhoods enabling color reuse. Growth with n is generally mild to sublinear.
- **CBIP (k=2)**: Near-optimal ($\rho \approx 1$) and highly stable across n and p, confirming bipartite structural leverage.

- **Ordering pays off (nuanced)**: HeuristicDegree is consistently best. BatchDegree with larger batches ($t = 50$) still achieves substantial reductions vs raw FirstFit in dense settings (e.g., $\approx 25\%$ for $k = 2$, $p = 0.2$; $\approx 33\%$ for $k = 3$, $p = 0.3$; $\approx 12\%$ for $k = 4$, $p = 0.3$), though it does not surpass the global ordering and may revert toward FirstFit in sparser cases.

- **Model fits**: Simple forms (log, sqrt, quadratic) capture mild growth; quadratic often attains the lowest error while log remains a close, interpretable alternative.

Overall, our efforts produced a clean, reproducible framework and empirical evidence that strategic ordering, particularly in batched form, meaningfully improves online coloring quality, while CBIP remains the gold standard for bipartite inputs.

8 Team Work Distribution

List tasks per member (generator, algorithms, experiments, analysis, writing, QA).

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