

The Calculus of Statistics

Blaise Whitesell Jeremiah Gelb

May 25, 2015

1 Probability Distribution Functions

1.1 Probability Density Functions

A probability density function (**PDF**) describes the relative likelihood $f(x)$ of possible outcomes for a continuous random variable.

- The probability of any x occurring is either positive or zero, so a PDF can never be negative.

$$f(x) \geq 0 \quad \text{for all } x$$

- The area under the curve must equal 1

$$\int_a^b f(x) \, dx = 1$$

where a and b are the lower and upper bounds, often $-\infty$ and ∞ .

1.2 Cumulative Distribution Functions

A cumulative distribution function (**CDF**) gives the probability $F(x)$ that the outcome of a continuous random variable will be less than or equal to x .

- The CDF is related to the PDF by this integral:

$$F(x) = \int_a^x f(t) \, dt \quad \text{for } a \leq x \leq b$$

- A CDF is monotonically increasing on the interval (a, b) .

$$F(a) = 0 \quad F(b) = 1$$

1.3 Using Calculus

To find the probability of an outcome in a certain range, one can integrate the PDF over that interval (c, d) contained in (a, b) .

$$\int_c^d f(x) \, dx = F(d) - F(c)$$

This gives the area under the curve, corresponding to the probability.

2 The Uniform Distribution

2.1 Definition

A uniform distribution describes a continuous random variable where every outcome on an interval (a, b) is equally likely.

$$f(x) = \kappa$$

2.2 Implications

- The distribution looks like a rectangle with length $b - a$ and height κ
- Area = $1 = \kappa(b - a)$

$$\int_a^b \frac{1}{b - a} \, dx = \frac{x}{b - a} \Big|_a^b = \frac{b}{b - a} - \frac{a}{b - a} = \frac{b - a}{b - a} = 1$$

- Therefore, we have an explicit definition for κ :

$$\kappa = \frac{1}{b - a}$$

2.3 PDF of a uniform distribution

The formal equation is piecewise. It depends only on the bounds a and b .

$$f(x) = \begin{cases} 0 & \text{if } x < a, \\ 1/(b - a) & \text{if } a \leq x \leq b, \\ 0 & \text{if } x > b. \end{cases}$$

2.4 CDF of a uniform distribution

The CDF increases linearly on the interval (a, b) .

$$f(x) = \begin{cases} 0 & \text{if } x < a, \\ (x - a)/(b - a) & \text{if } a \leq x \leq b, \\ 1 & \text{if } x > b. \end{cases}$$

Problem 1. What is the probability of losing on the first spin?

Solution. There are 200 spaces, exactly one of which corresponds to losing immediately. The continuous range of values can be modeled as a uniform distribution from 0° to 360° .

$$1/200 \times 360^\circ = 1.8^\circ$$

$$\int_0^{1.8} \frac{1}{360} d\theta = \frac{\theta}{360} \Big|_0^{1.8} = \frac{1.8}{360} - \frac{0}{360} = 0.005 \quad (1)$$

Interestingly enough, this is equivalent to $1/200$. □
