The Calculus of Statistics

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1 Probability Distribution Functions

1.1 Probability Density Functions

A probability density function (**PDF**) describes the relative likelihood $f(x)$ of possible outcomes for a continuous random variable.

- The probability of any x occurring is either positive or zero, so a PDF can never be negative.

$$f(x) \geq 0 \quad \text{for all } x$$

- The area under the curve must equal 1

$$\int_a^b f(x)\,\mathrm{d}x = 1$$

where a and b are the lower and upper bounds, often $-\infty$ and ∞.

1.2 Cumulative Distribution Functions

A cumulative distribution function (**CDF**) gives the probability $F(x)$ that the outcome of a continuous random variable will be less than or equal to x.

- The CDF is related to the PDF by this integral:

$$F(x) = \int_a^x f(t)\,\mathrm{d}t \quad \text{for } a \leq x \leq b$$

- A CDF is monotonically increasing on the interval (a, b).

$$F(a) = 0 \qquad F(b) = 1$$

1.3 Using Calculus

To find the probability of an outcome in a certain range, one can integrate the PDF over that interval (c, d) contained in (a, b).

$$\int_c^d f(x)\,\mathrm{d}x \qquad = F(d) - F(c)$$

This gives the area under the curve, corresponding to the probability.

2 The Uniform Distribution

2.1 Definition

A uniform distribution describes a continuous random variable where every outcome on an interval (a, b) is equally likely.

$$f(x) = \kappa$$

2.2 Implications

- The distribution looks like a rectangle with length $b - a$ and height κ
- Area $= 1 = \kappa(b - a)$

$$\int_a^b \frac{1}{b-a}\,\mathrm{d}x = \left.\frac{x}{b-a}\right|_a^b = \frac{b}{b-a} - \frac{a}{b-a} = \frac{b-a}{b-a} = 1$$

- Therefore, we have an explicit definition for κ:

$$\kappa = \frac{1}{b-a}$$

2.3 PDF of a uniform distribution

The formal equation is piecewise. It depends only on the bounds a and b.

$$f(x) = \begin{cases} 0 & \text{if } x < a, \\ 1/(b-a) & \text{if } a \leq x \leq b, \\ 0 & \text{if } x > b. \end{cases}$$

2.4 CDF of a uniform distribution

The CDF increases linearly on the interval (a, b).

$$f(x) = \begin{cases} 0 & \text{if } x < a, \\ (x-a)/(b-a) & \text{if } a \leq x \leq b, \\ 1 & \text{if } x > b. \end{cases}$$

Problem 1 (Spin to Win)**.** *What is the probability of losing on the first spin?*

Solution. There are 200 spaces, exactly one of which corresponds to losing immediately. The range of values can be modeled as a uniform distribution from $0°$ to $360°$.

$$1/200 \times 360° = 1.8°$$

$$\int_0^{1.8} \frac{1}{360}\,\mathrm{d}\theta = \left.\frac{\theta}{360}\right|_0^{1.8} = \frac{1.8}{360} - \frac{0}{360} = 0.005 \tag{1}$$

Interestingly enough, this is equivalent to $1/200$. □