Department of Mechanical Engineering, Pulchowk campus, Institute of Engineering, Tribhuvan University

ENGINEERING ECONOMICS

Time Value of Money, Interest Rate and Economic Equivalence

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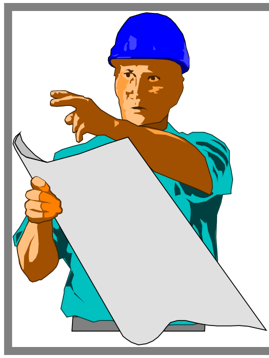
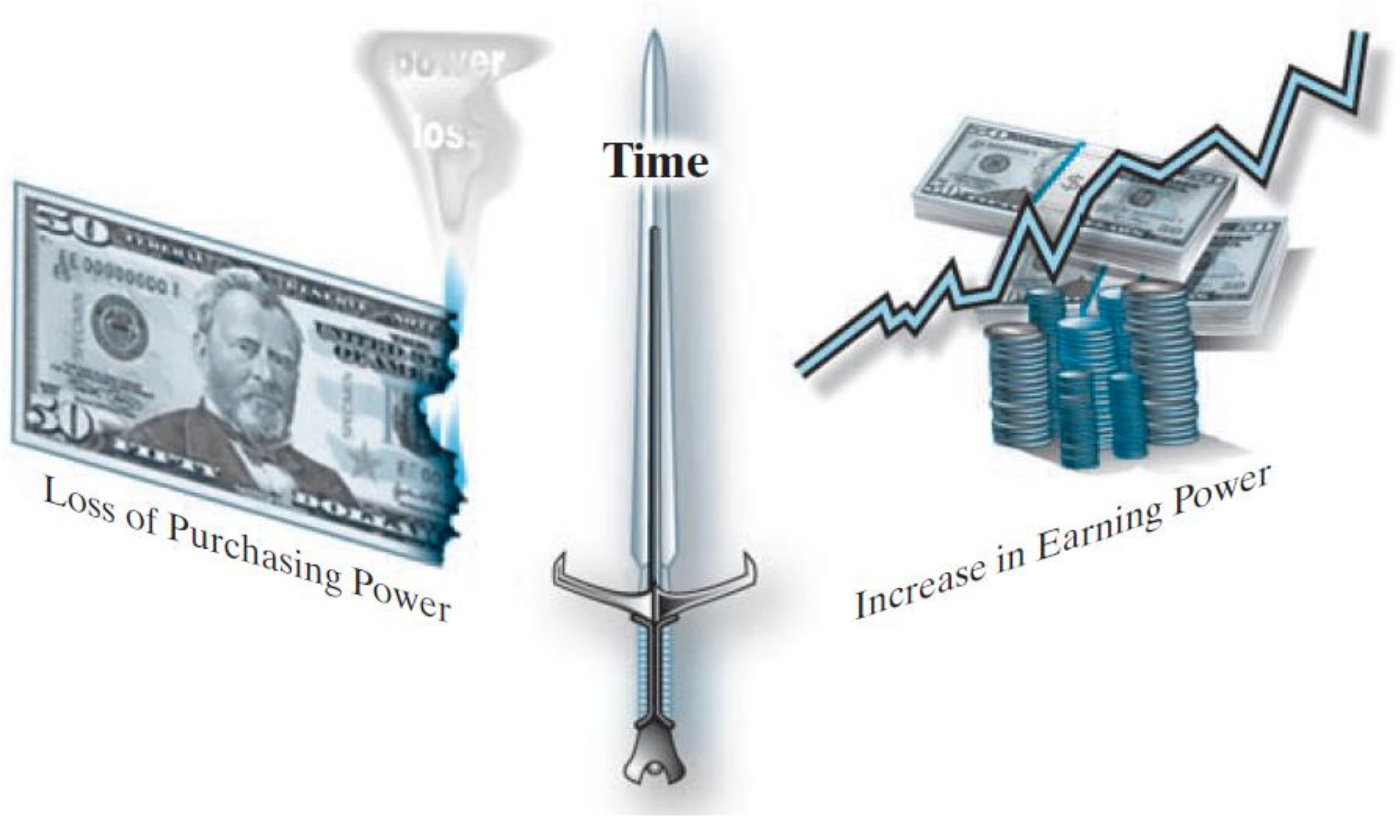
2018



Time Value of MONEY



- Money has a **time value**.
- The economic value of an amount depends on **when it is received**.
- Money has **earning power over time**.

Time Value of MONEY

A rupee **received today** has **more value** than a rupee **received at some future time**

(rate of interest = 15%)

End of year (n)	Present worth	Compound amount after n year(s)
0		100
1	86.96	100
2	75.61	100
3	65.75	100
4	57.18	100
5	49.72	100
6	43.29	100
7	37.59	100
8	32.69	100
9	28.43	100
10	24.72	100

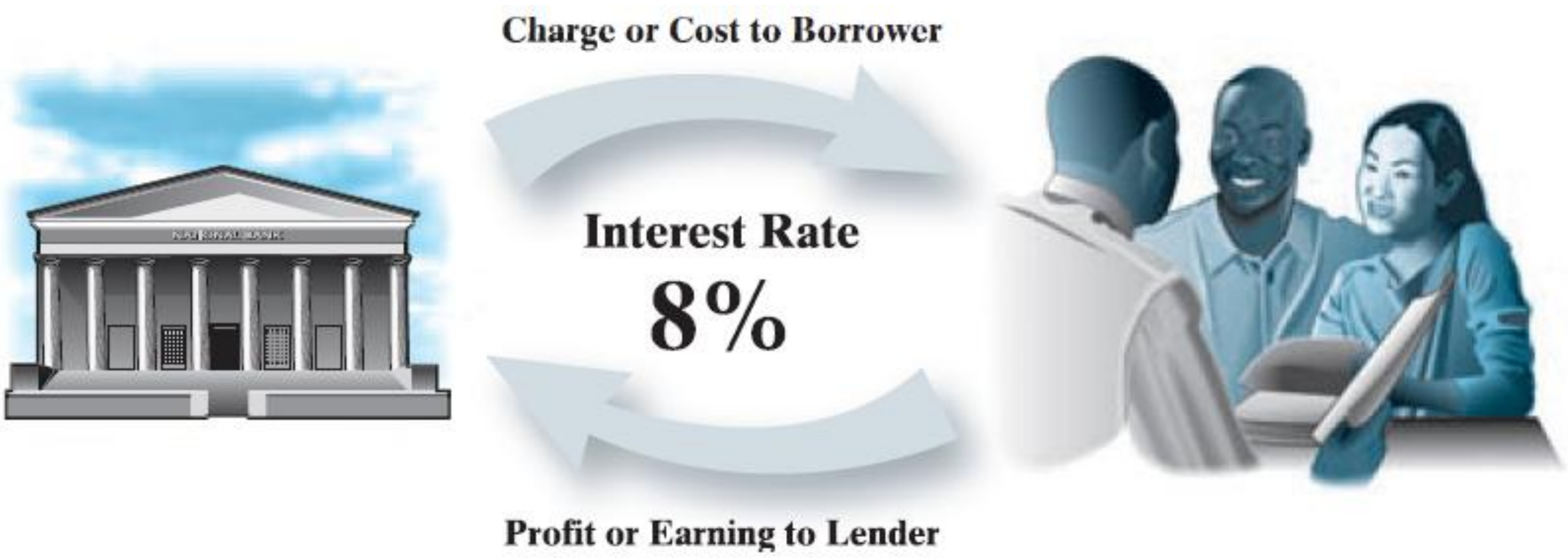
Time Value of MONEY

- **Gains achieved** or **losses** incurred **by delaying consumption**

	Account Value	Cost of Refrigerator
Case 1: Inflation exceeds earning power	$N = 0$ \$100 $N = 1$ \$106 (earning rate = 6%)	$N = 0$ \$100 $N = 1$ \$108 (inflation rate = 8%)
Case 2: Earning power exceeds inflation	$N = 0$ \$100 $N = 1$ \$106 (earning rate = 6%)	$N = 0$ \$100 $N = 1$ \$104 (inflation rate = 4%)

Cost of money => Interest

The cost of money is measured by an ***interest rate***, a percentage that is periodically applied and added to an amount of money over a specified length of time



Financial terms

- **Principal:** initial amount of money in transactions of debt or investment
- **Interest rate:** measures the cost of money, expressed as percentage per period of time
- **Interest period:** a period of time when interest is frequently calculated (usually one year)
- **Number of periods:** specified length of time when transaction is done
- **Future amount:** amount of money resulted from cumulative effects of interest rate over a number of several interest periods

Some Notations

A_n = Discrete payment or receipt at the end of interest period

i = interest rate per interest period

N = number of interest periods

P = A sum of money at a time chosen for analysis, or called present value (PV) or present worth

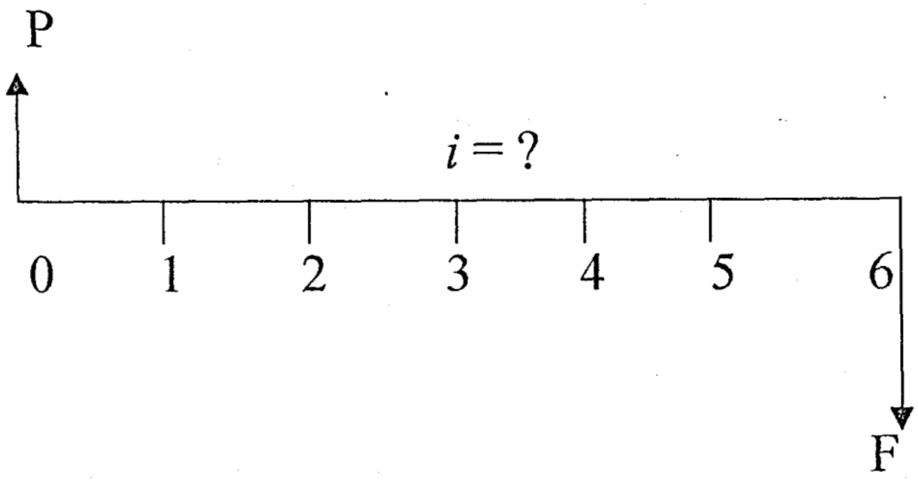
F = A future sum of money at the end of analysis period, or Fn ,at the end of some interest period

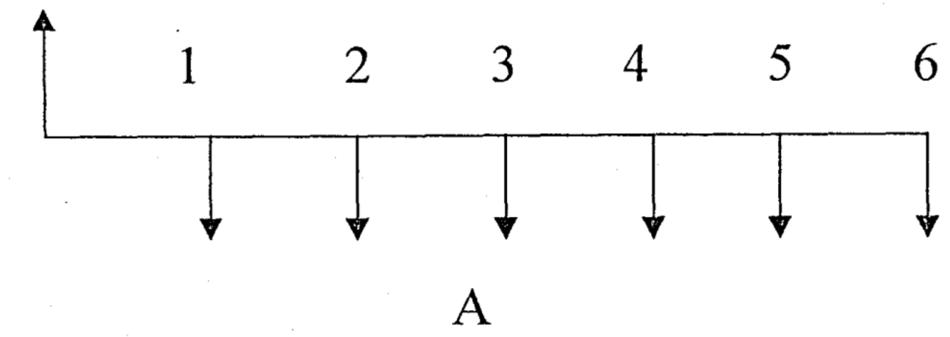
A = An end- of -period payment or receipt in a uniform series

V_n = An equivalent sum of money at the end of specified period, considering time value of money; $V_0 = P$; $V_n = F$

Cash flow Diagrams

It is a **convenient way of representing problems** involving time value of money in graphic forms.





The cash flow diagram **shows net flows at the end of interest periods**.

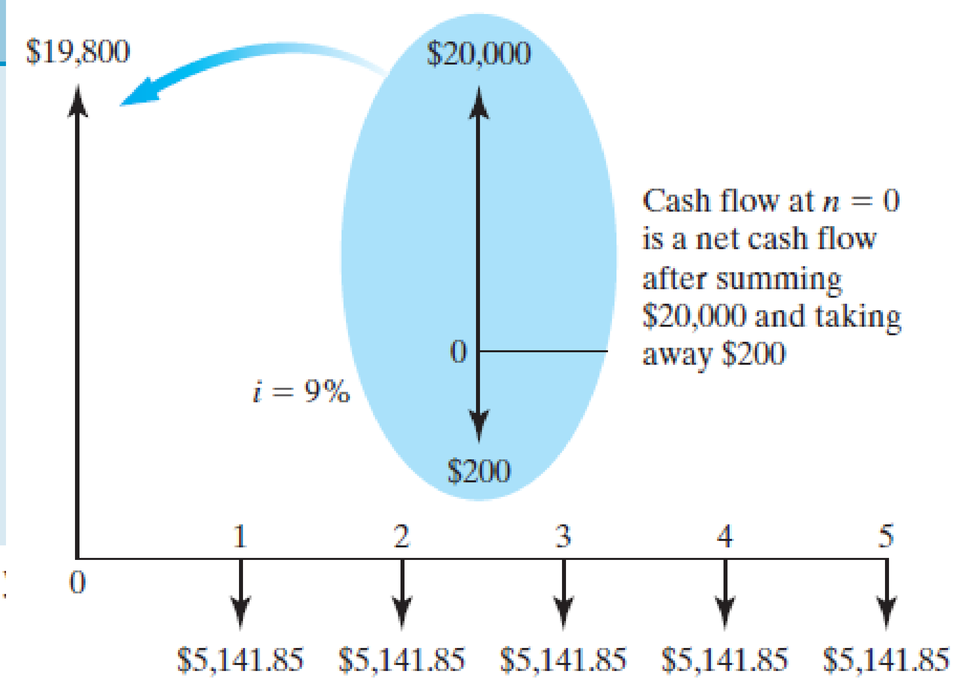
Cash flow Diagrams

Repayment Plans for Example Given in Text (for N = 5 years, and i = 9%)

End of Year	Receipts	Payments
		Plan 1
Year 0	\$20,000.00	\$ 200.00
Year 1		5,141.85
Year 2		5,141.85
Year 3		5,141.85
Year 4		5,141.85
Year 5		5,141.85

P = \$20,000, A = \$5,141.85, F = \$30,772.48

Note: You actually borrow \$19,800 with the origination fee of \$200, but



Simple Interest

Under simple interest, the **interest earned during each interest period** will **not earn an additional interest on interest amount** in the remaining periods.

$$I = (\,i\,P\,)\,N$$

$$F = P + I = P\,(1 + i\,N)$$

Compound Interest

Under this interest, the interest earned in each period is based on the total amount owed at the end of the previous period.

This means **interest will be earned on the interest charged in the previous period** as well.

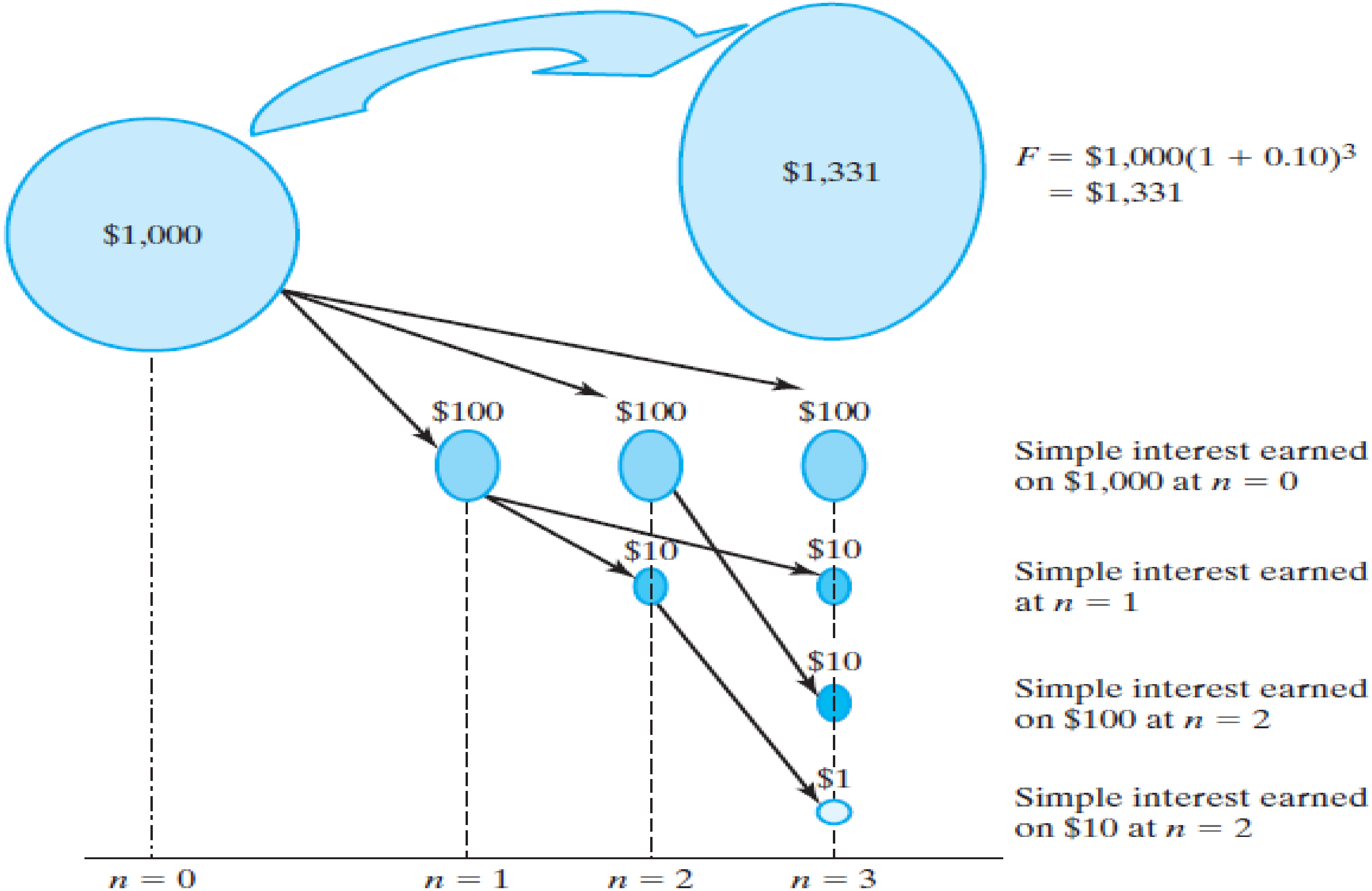
At the end of first year, $\mathbf{F = P + iP = P(1+i)}$

At the end of 2nd year, $\mathbf{F = P\,(1+i) + i\,[\,P\,(1+i)\,]}$

$$\mathbf{= P\,(1+i)^2}$$

A the end of n periods, $\mathbf{F = P\,(1+i)^n}$

Relationship between simple interest and compound interest



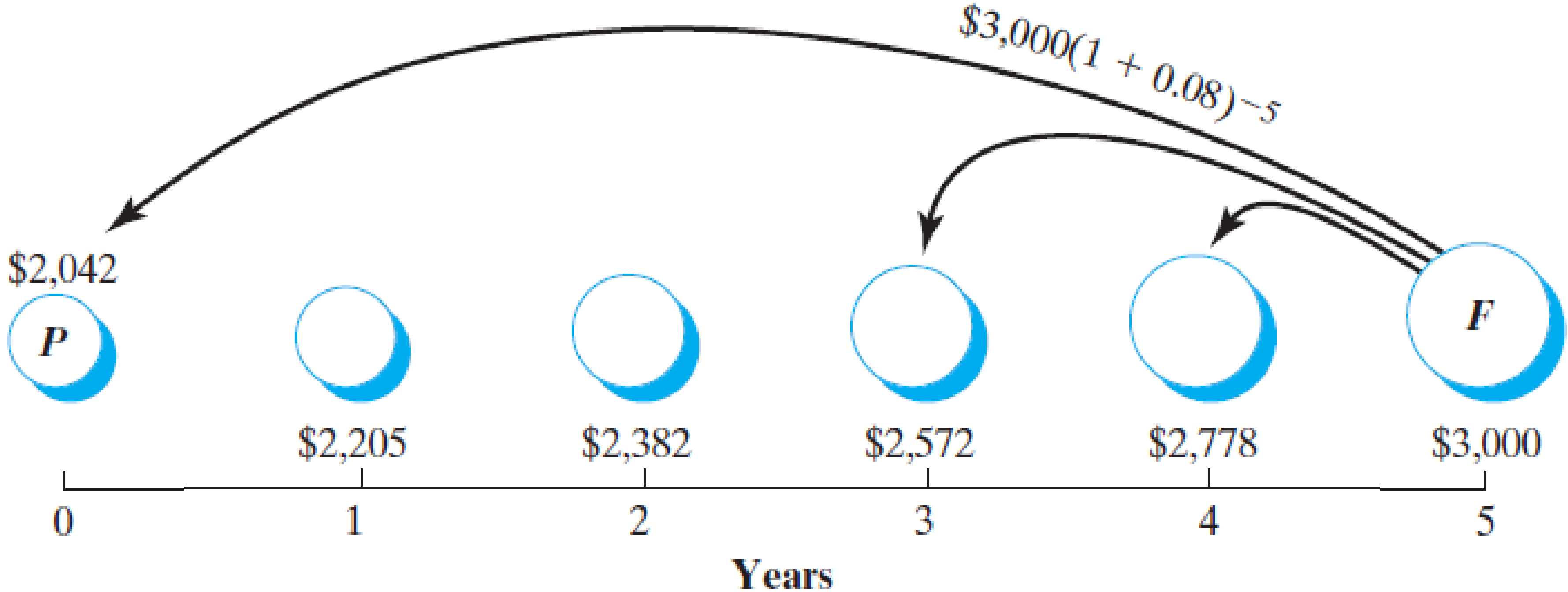
$$\Delta I = P[(1+i)^N - 1] - (iP)N$$

$$= P[(1+i)^N - (1+iN)].$$

Economic Equivalence

Economic equivalence exists **when cash flows have the same economic effect** and thus **could be traded for one another** in the financial market.

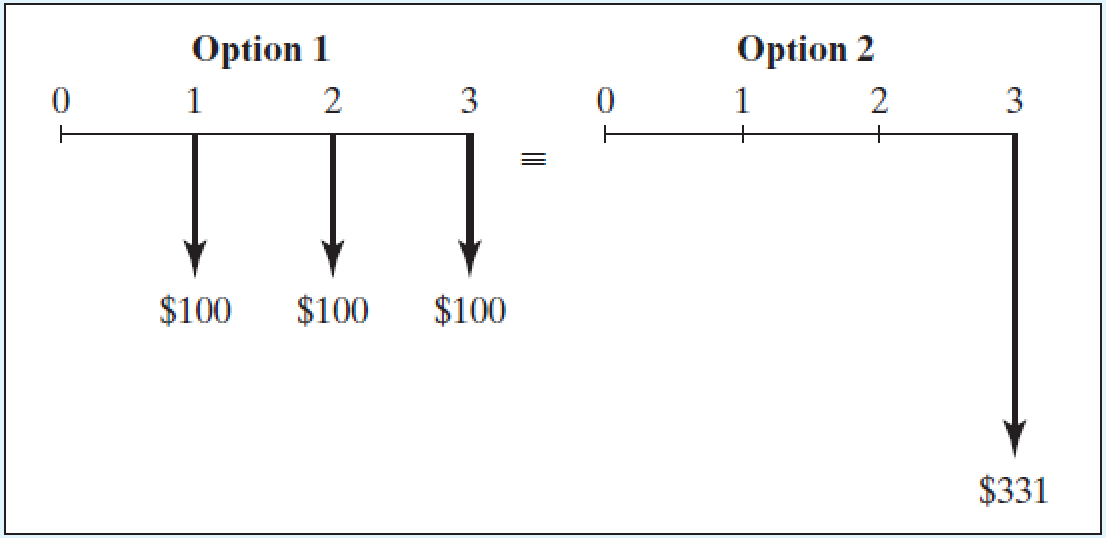
Suppose you are **offered the alternative of** receiving **either Rs 3,000 at the end of 5 years** or **Rs *P* today**. There is no question that the Rs 3,000 will be paid in full (no risk). Having no current need for the money, you would deposit the ***P*** rupees in an account that **pays 8% interest**. What value of ***P*** would make you indifferent in your choice between ***P*** rupees today and the promise of Rs 3,000 at the end of 5 years from Now?



You borrowed $ 1,000 from a bank for 3 years at 10% annual interest.

The bank offered you two options
(a) repaying the loan all at once at the end of 3 years, or
(b) repaying the interest charges for each year at the end of that year.

Options	Year 1	Year 2	Year 3	Total[1]
Option 1: End-of-year repayment of interest, principal repaid at end of loan	$100	$100	$1100	$1300
Option 2: One end-of-loan repayment of both principal and interest	0	0	1331	1331



$$F_3 \text{ for } \$100 \text{ at } n = 1 : \$100(1 + .10)^{3-1} = \$121;$$

$$F_3 \text{ for } \$100 \text{ at } n = 2 : \$100(1 + .10)^{3-2} = \$110;$$

$$F_3 \text{ for } \$100 \text{ at } n = 3 : \$100(1 + .10)^{3-3} = \underline{\$100};$$

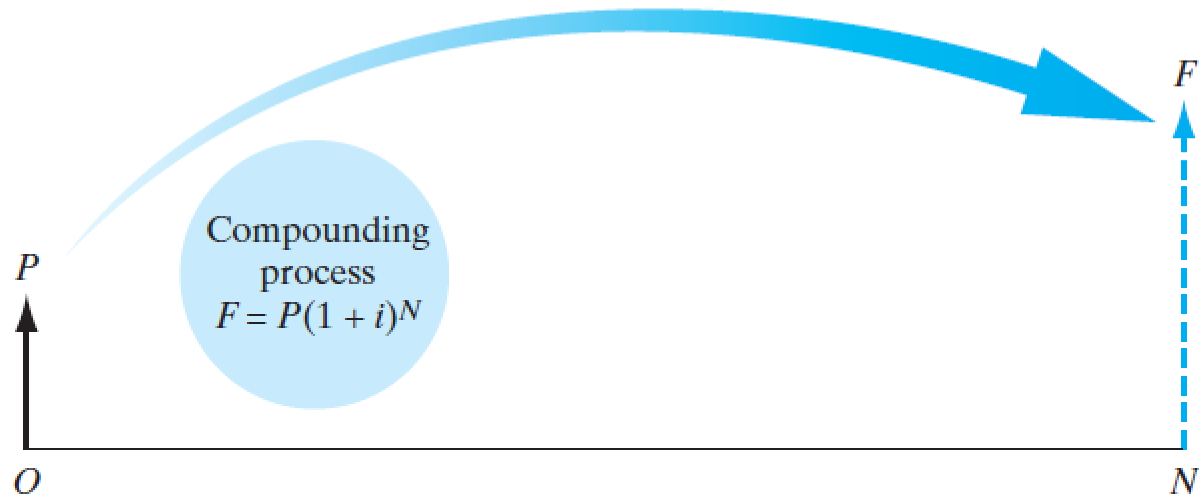
$$\text{Total} = \$331.$$

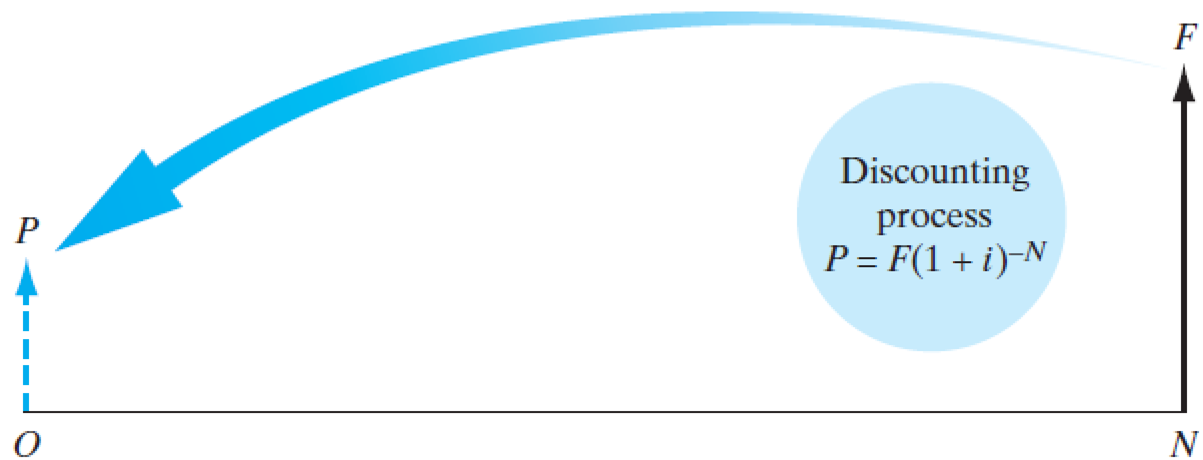
Since **two interest payments** are **equivalent**, the bank would be **economically indifferent to a choice** between the two plans

Five Types of Cash flows

- Single cash flow
- Uniform Cash flows
- Linear Gradient Series
- Geometric Gradient Series
- Irregular series

Single cash flow





Single cash flow

$F = P(1+i)^N = P\ (F/P, i, N)$

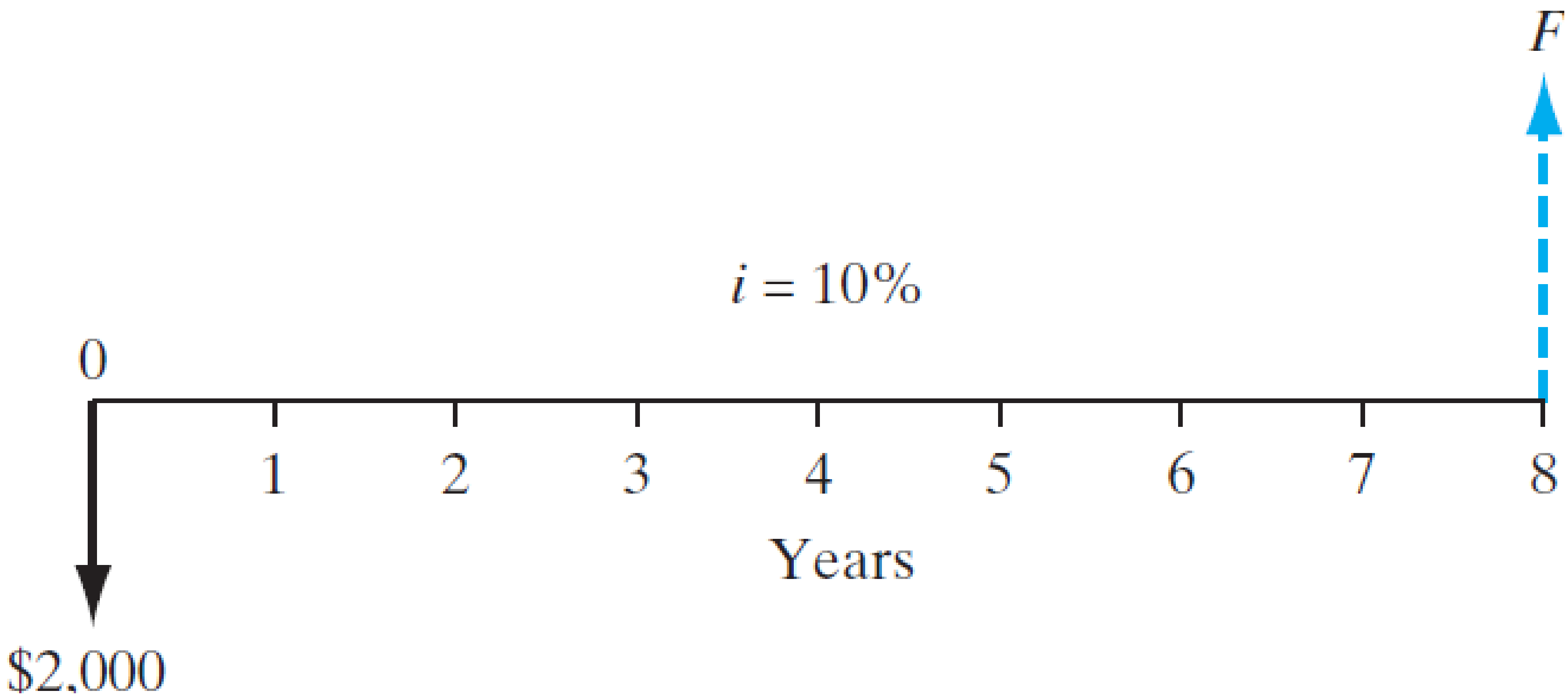
where, $(1+i)^N$ is called ***single payment compound amount factor***.

$P = F(1+i)^{-N} = F\ (P/F, i, N)$

where, $(1+i)^{-N}$ is called ***single payment present value (worth) factor***.

Single cash flow (Example)

If you had **$2,000 now and invested** it **at 10%,** how much would it be **worth in eight years?**



Single cash flow (Example)

SOLUTION

Given: $P = \$2{,}000$, $i = 10\%$ per year, and $N = 8$ years.
Find: F.

We can solve this problem in any of three ways:

1. **Using a calculator.** You can simply use a calculator to evaluate the $(1 + i)^N$ term (financial calculators are preprogrammed to solve most future-value problems):

$$F = \$2{,}000(1 + 0.10)^8$$
$$= \$4{,}287.18.$$

2. **Using compound-interest tables.** The interest tables can be used to locate the compound-amount factor for $i = 10\%$ and $N = 8$. The number you get can be substituted into the equation. Compound-interest tables are included as Appendix A of this book. From the tables, we obtain

$$F = \$2{,}000(F/P, 10\%, 8) = \$2{,}000(2.1436) = \$4{,}287.20.$$

Single cash flow (Example)

APPENDIX A Interest Factors for Discrete Compounding

10.0%

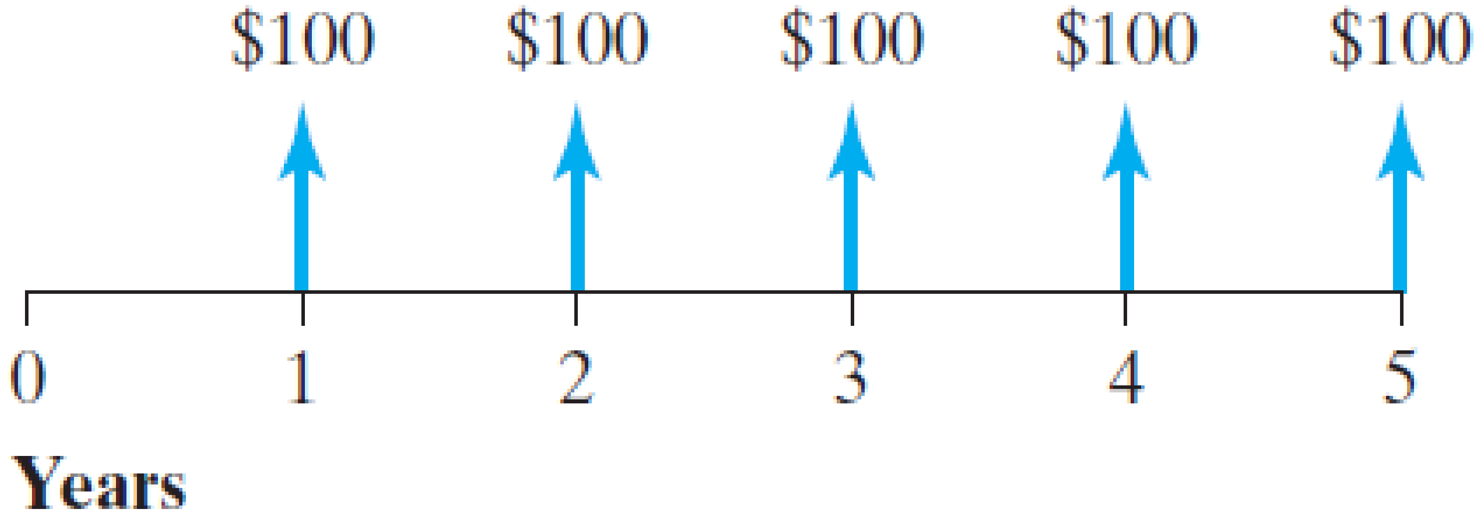
	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.1000	0.9091	1.0000	1.0000	0.9091	1.1000	0.0000	0.0000	1
2	1.2100	0.8264	2.1000	0.4762	1.7355	0.5762	0.4762	0.8264	2
3	1.3310	0.7513	3.3100	0.3021	2.4869	0.4021	0.9366	2.3291	3
4	1.4641	0.6830	4.6410	0.2155	3.1699	0.3155	1.3812	4.3781	4
5	1.6105	0.6209	6.1051	0.1638	3.7908	0.2638	1.8101	6.8618	5
6	1.7716	0.5645	7.7156	0.1296	4.3553	0.2296	2.2236	9.6842	6
7	1.9487	0.5132	9.4872	0.1054	4.8684	0.2054	2.6216	12.7631	7
8	2.1436	0.4665	11.4359	0.0874	5.3349	0.1874	3.0045	16.0287	8
9	2.3579	0.4241	13.5795	0.0736	5.7590	0.1736	3.3724	19.4215	9
10	2.5937	0.3855	15.9374	0.0627	6.1446	0.1627	3.7255	22.8913	10
11	2.8531	0.3505	18.5312	0.0540	6.4951	0.1540	4.0641	26.3963	11
12	3.1384	0.3186	21.3843	0.0468	6.8137	0.1468	4.3884	29.9012	12
13	3.4523	0.2897	24.5227	0.0408	7.1034	0.1408	4.6988	33.3772	13

Single cash flow (Example)

3. Using Excel. Many financial software programs for solving compound-interest problems are available for use with personal computers. Excel provides financial functions to evaluate various interest formulas, where the future-worth calculation looks like the following:

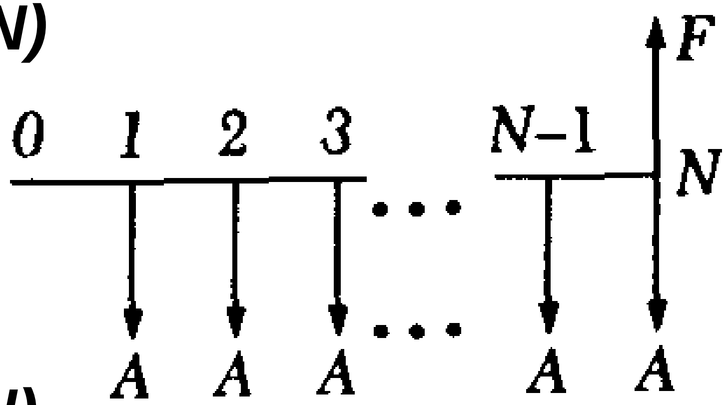
$$=FV(10\%,8,0,-2000)$$

Uniform Cash flows



Uniform Cash flows

$$F = A\left[\frac{(1+i)^N - 1}{i}\right] = \boldsymbol{A(F/A,\, i,\, N)}$$



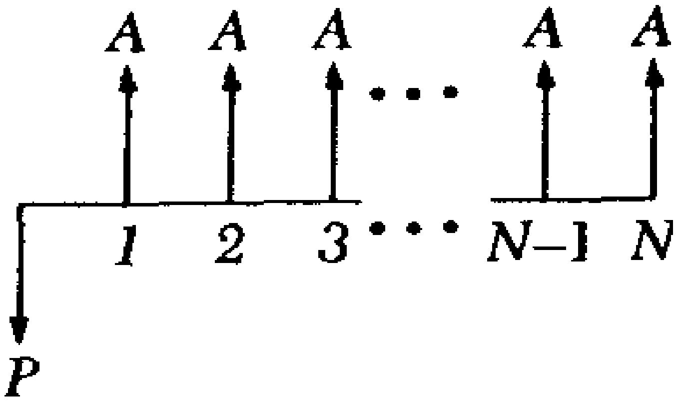
$$A = F\left[\frac{i}{(1+i)^N - 1}\right] = \boldsymbol{F(A/P,\, i,\, N)}$$

where $[(1+i)^N - 1] / i$ is called *equal payment series compound amount factor* or *uniform series* ***compound amount factor***.

where $i/ [(1+i)^N - 1]$ is called *equal payment series* ***sinking - fund factor***.

Uniform Cash flows

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A\ (P/A,\ i,\ N)$$

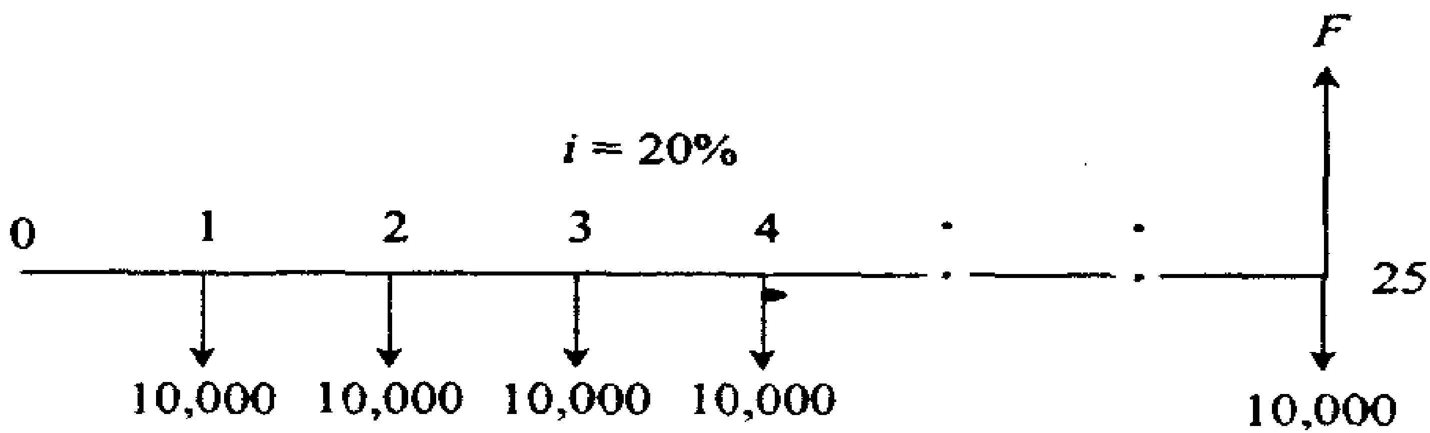


$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right] = P\ (A/P,\ i,\ N)$$

where $[(1+i)^N - 1] / i(1+i)^N$ is called *equal payment series* ***present value (worth) factor***.

where $i(1+i)^N / [(1+i)^N - 1]$ is called *equal payment series* ***capital recovery factor*** or ***annuity factor***.

A person now 35 years old. he plan to invest an equal sum of Rs. 10,000 at the end of every year for next 25 years starting from the end of the next year. The bank gives 20% interest rate compounded annually.
Find maturity value when he is 60 years old.



$$F = A\,\frac{(1+i)^n - 1}{i}$$

uniform series compound amount factor

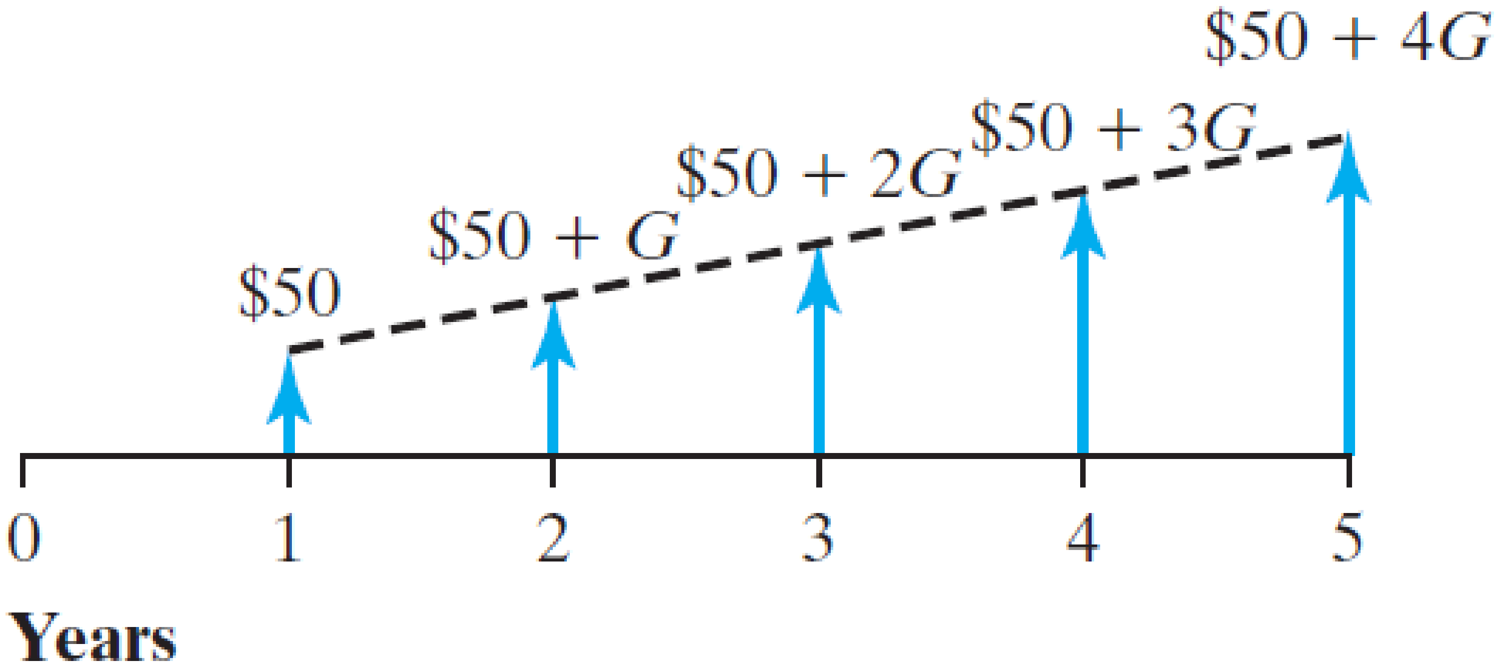
$$= A(F/A,\ i,\ n)$$

$$= 10{,}000(F/A,\ 20\%,\ 25)$$

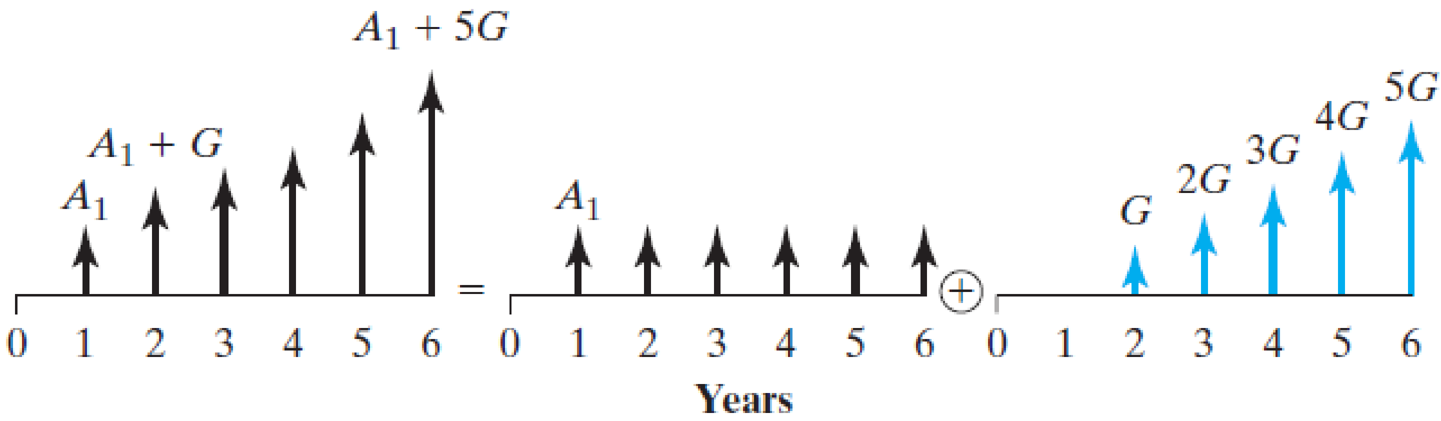
$$= 10{,}000 \times 471.981$$

$$= \text{Rs. } 47{,}19{,}810$$

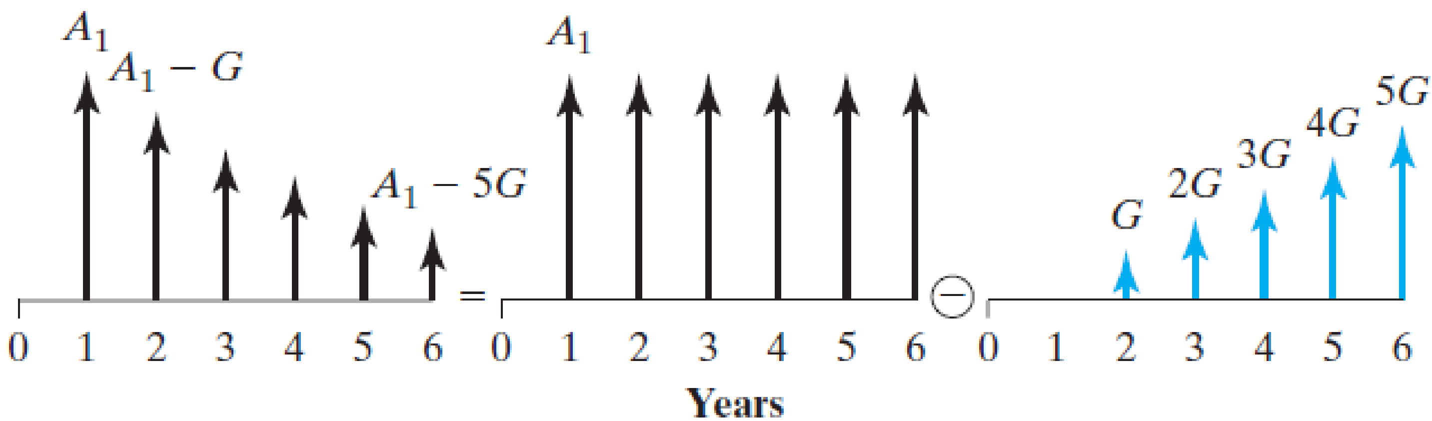
Linear Gradient Series



Linear Gradient Series



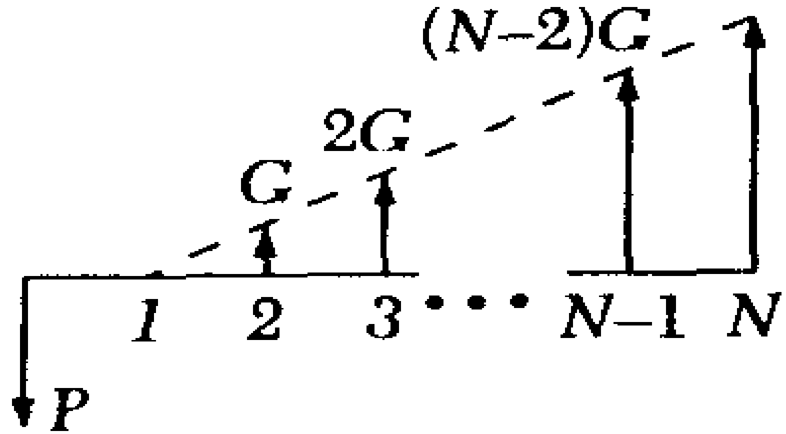
(a) Increasing gradient series



(b) Decreasing gradient series

Linear Gradient Series

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$$



Gradient series is to be **taken as composite series**, or as a **set of two cash flows** – one as **uniform series** and **another as gradient series**.

$P = P_1 + P_2$

$P_2 = G\,[(1+i)^N - iN - 1] / i^2(1+i)^N = G(P/G,i,N)$

where (P/G, i, N) is called *gradient series present value (worth) factor*.

Linear Gradient series

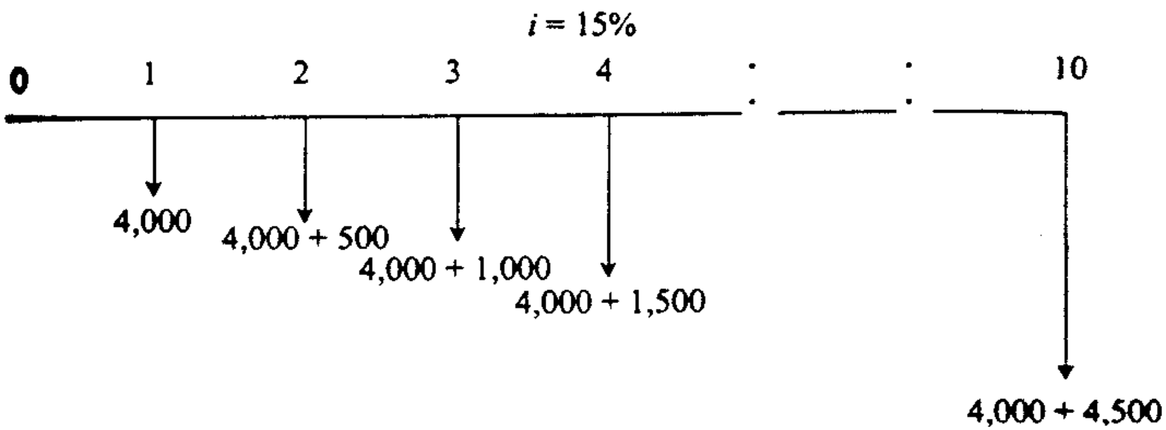
$$A = G\,[(1+i)^N - iN - 1] \,/\, i[(1+i)^N - 1]$$

$$= G(A/G, i, N)$$

where (A/G, i, N) is called *Gradient to equal payment series conversion factor*.

A person has **10 years of service**. He would like to deposit **20% of his salary**, which *is **Rs. 4,000**, at* the end of the first year, and thereafter **he wishes to deposit** *the* amount with an **annual increase of Rs. 500** for the **next 9 years** with an **interest rate of 15%**.

Find the **total amount at the end of the 10th year** of the above series?



$$A = A1 + G\,\frac{(1+i)^n - in - 1}{i(1+i)^n - i}$$

$$= A1 + G(A/G, i, n)$$

$$= 4{,}000 + 500(A/G, 15\%, 10)$$

$$= 4{,}000 + 500 \times 3.3832$$

$$= \text{Rs. } 5{,}691.60$$

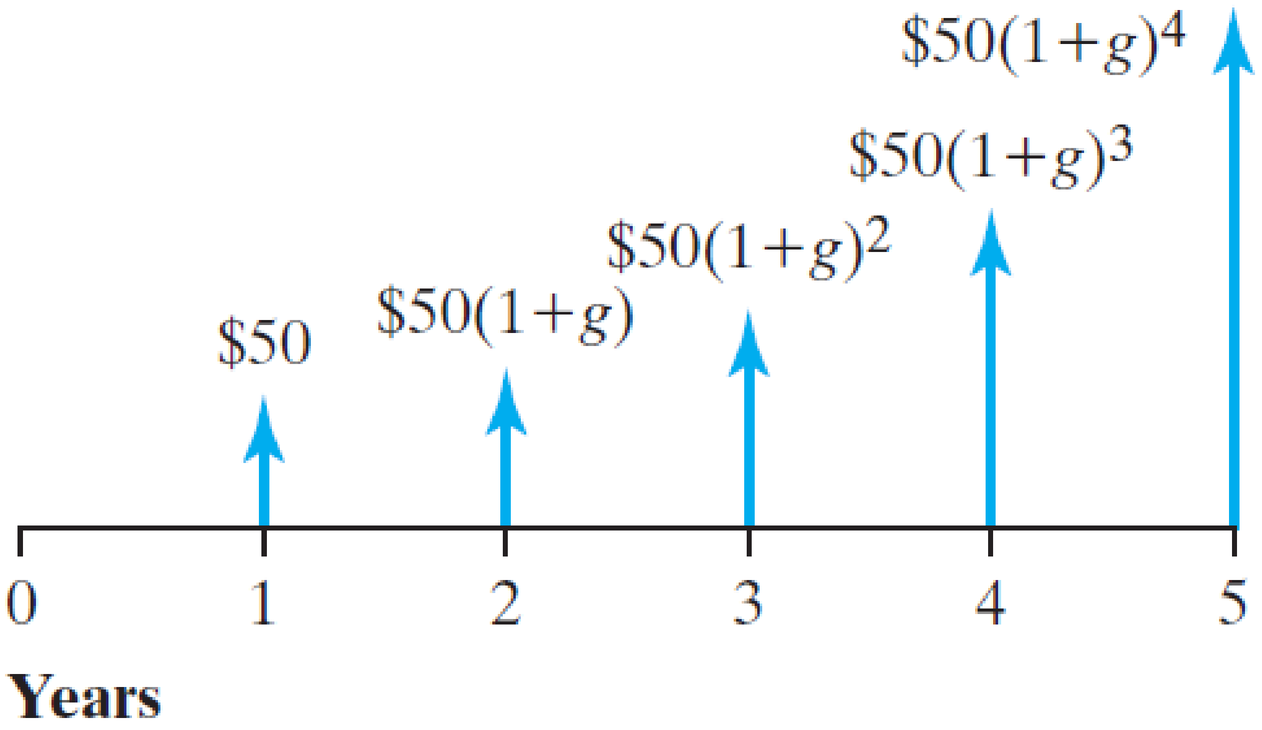
$$F = A(F/A, i, n)$$

$$= A(F/A, 15\%, 10)$$

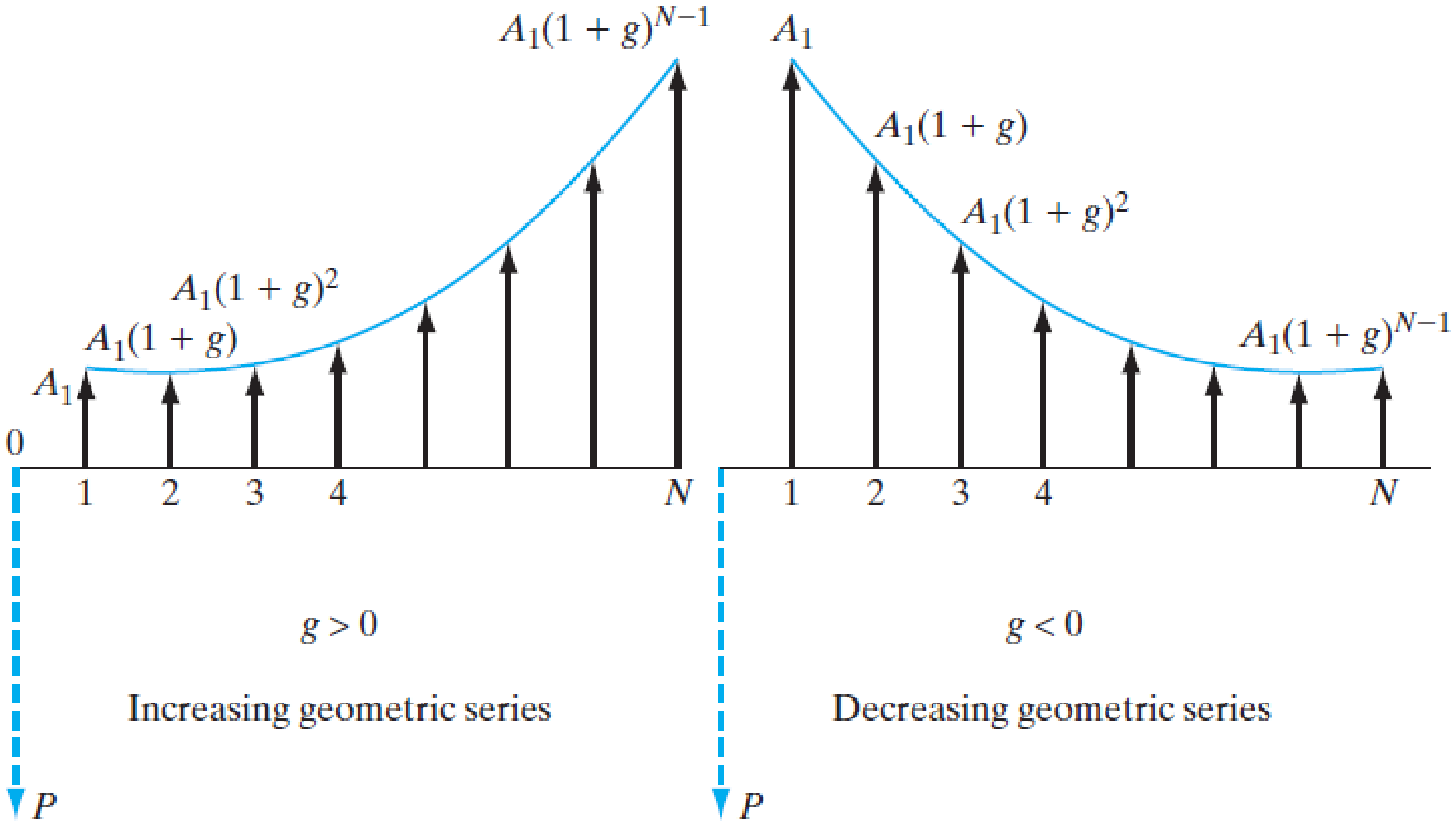
$$= 5{,}691.60(20.304)$$

$$= \text{Rs. } 1{,}15{,}562.25$$

Geometric Gradient Series



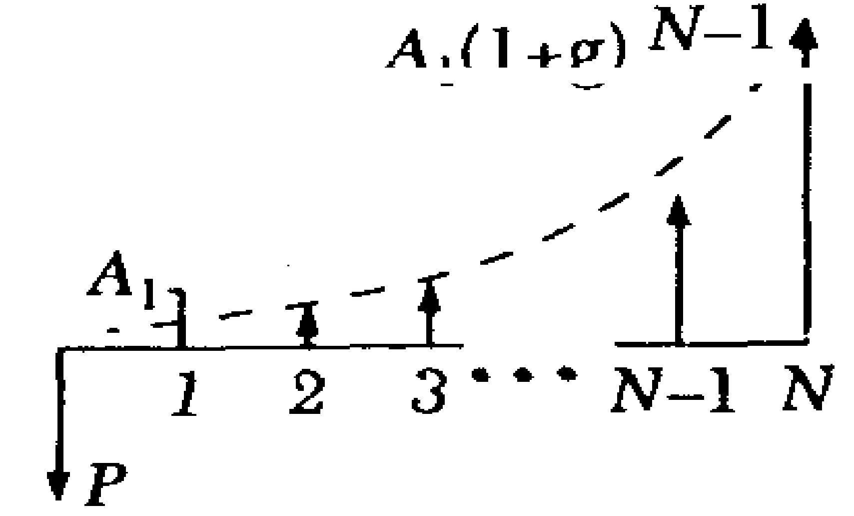
Geometric Gradient Series



Geometric Gradient Series

$$P = \begin{cases} A_1\left[\dfrac{1-(1+g)^N(1+i)^{-N}}{i-g}\right] \\ \dfrac{NA_1}{1+i} \quad (\text{if } i = g) \end{cases}$$

Geometric gradient
Present worth factor
(P/A$_1$, g, i, N)



Geometric Gradient Series

$$F = A_1[(1+i)^N - (1+g)^N] / (i - g) \qquad \text{if } i \neq g$$

$$= N A_1 / (1+i)^{N-1} \qquad \text{if } i = g$$

$$= A_1(F/A_1, g, i, N)$$

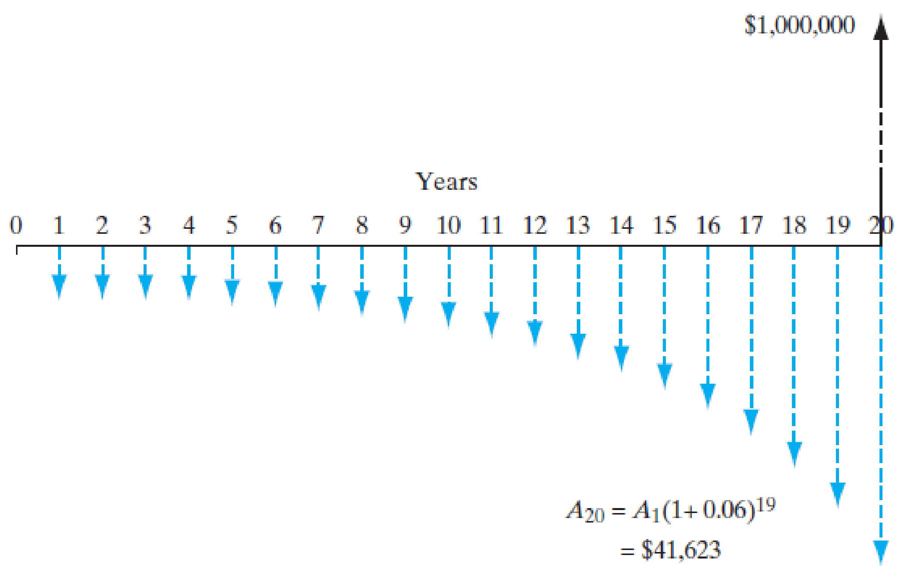
where $(F/A_1, g, i, N)$ is called *future value (worth) equivalent of geometric gradient series*.

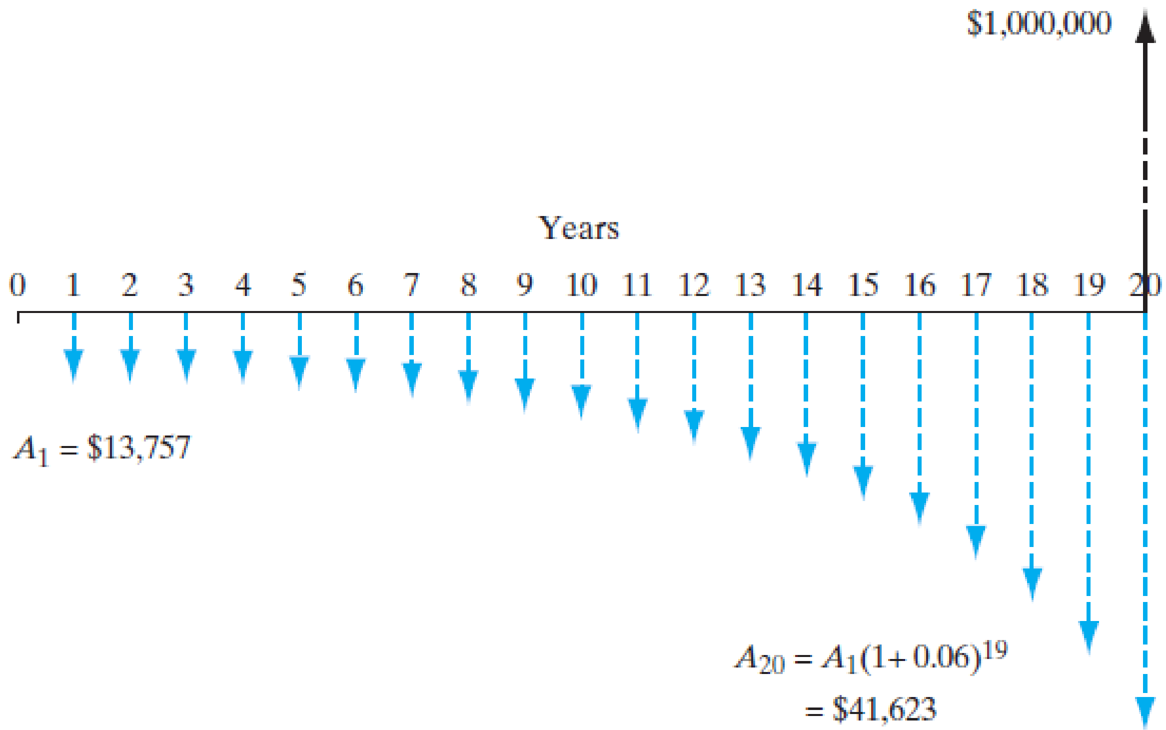
Jimmy Carpenter, a self-employed individual, is opening a retirement account at a bank. His goal is to accumulate \$1,000,000 in the account by the time he retires from work in 20 years' time. A local bank is willing to open a retirement account that pays 8% interest compounded annually throughout the 20 years. Jimmy expects that his annual income will increase 6% yearly during his working career. He wishes to start with a deposit at the end of year 1 (A_1) and increase the deposit at a rate of 6% each year thereafter. What should be the size of his first deposit (A_1)? The first deposit will occur at the end of year 1, and subsequent deposits will be made at the end of each year. The last deposit will be made at the end of year 20.

SOLUTION

Given: $F = \$1{,}000{,}000$, $g = 6\%$ per year, $i = 8\%$ per year, and $N = 20$ years.

Find: A_1 as in Figure 3.35.





Given: $F = \$1{,}000{,}000$, $g = 6\%$ per year, $i = 8\%$ per year, and $N = 20$ years.
Find: A_1 as in Figure 3.35.

We have

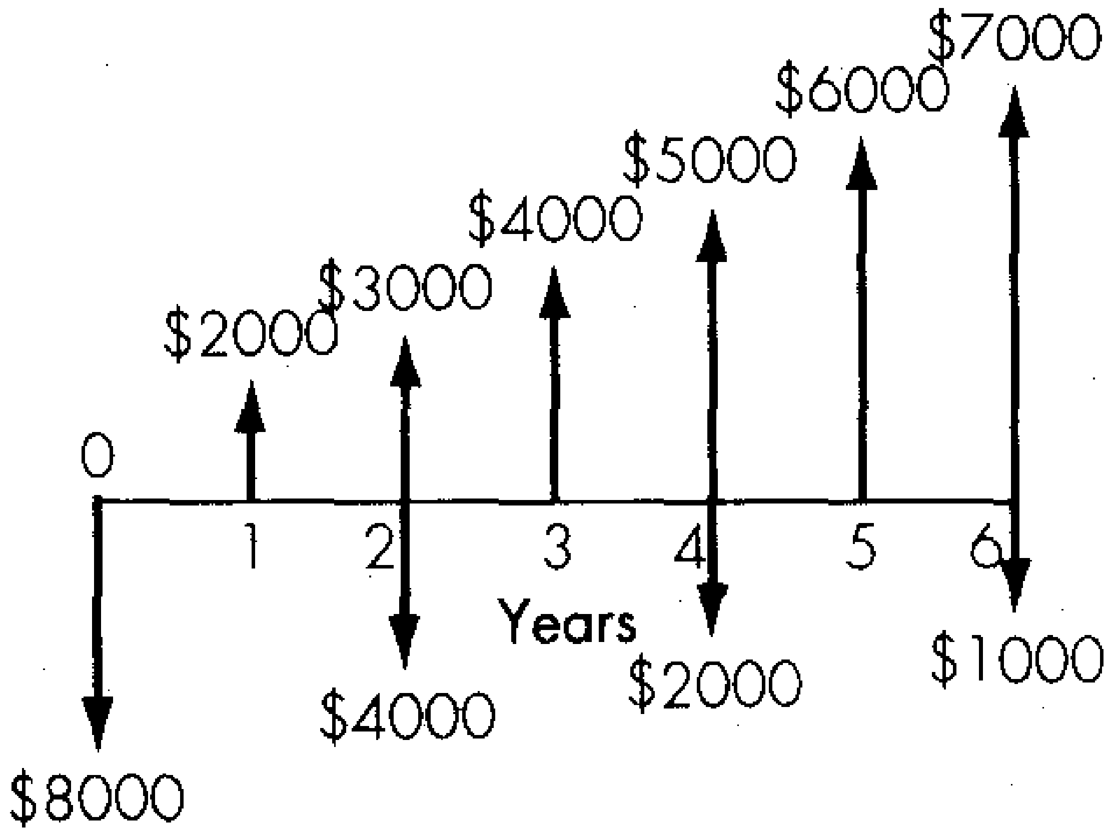
$$F = A_1(P/A_1, 6\%, 8\%, 20)\,(F/P, 8\%, 20)$$

$$= A_1(72.6911).$$

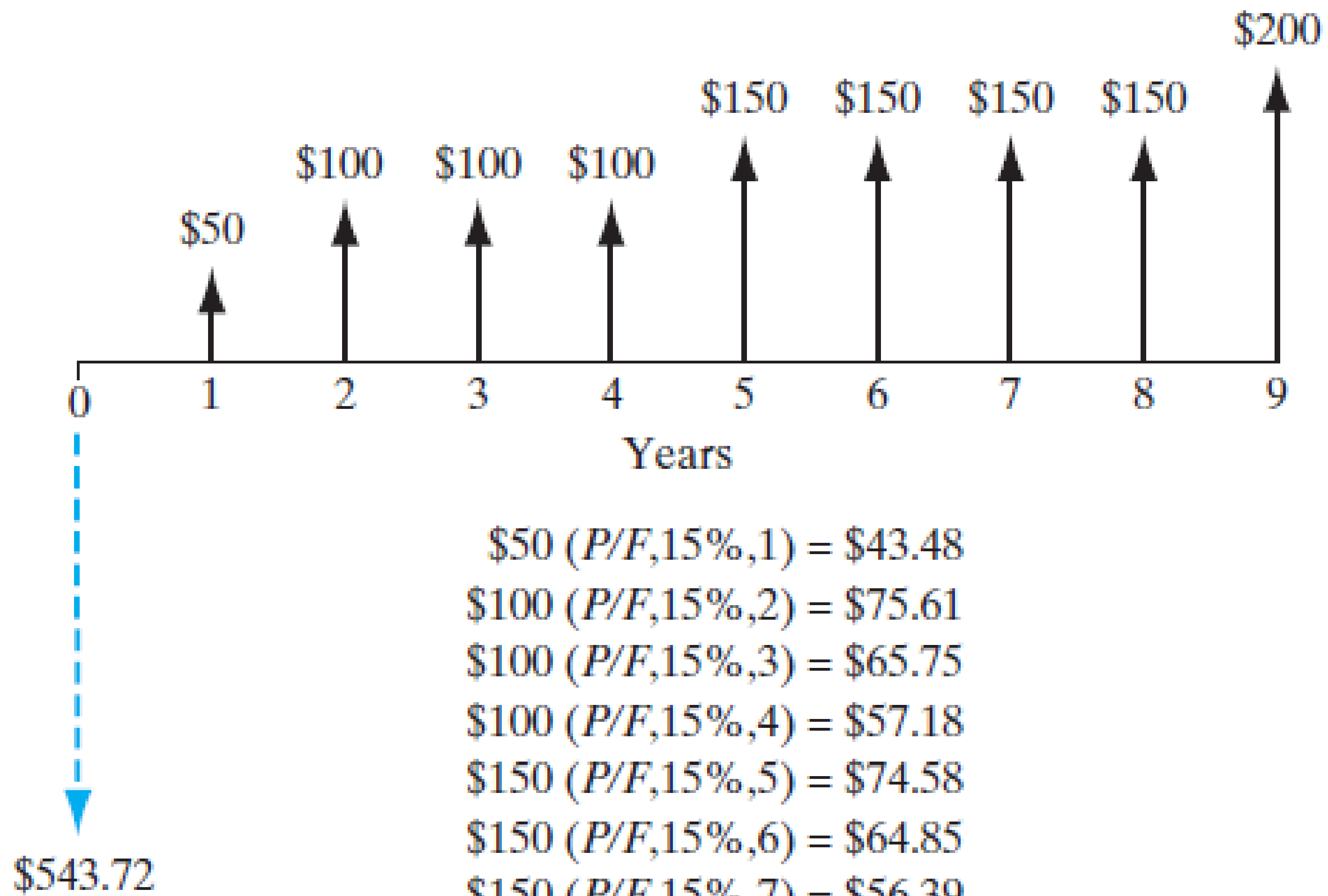
Solving for A_1 yields

$$A_1 = \$1{,}000{,}000/72.6911 = \$13{,}757.$$

Irregular series



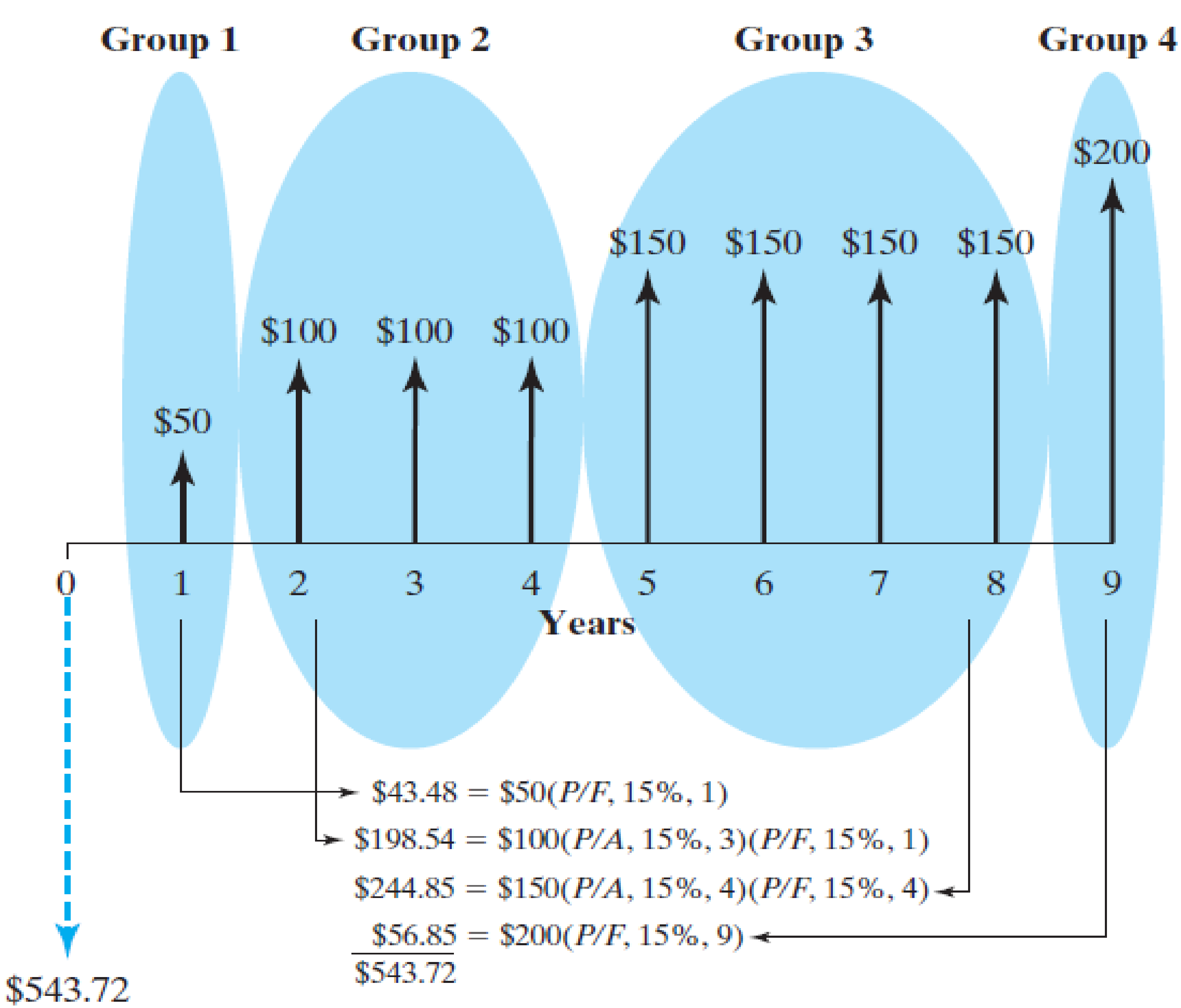
Irregular series



$50 $(P/F,15\%,1)$ = $43.48
$100 $(P/F,15\%,2)$ = $75.61
$100 $(P/F,15\%,3)$ = $65.75
$100 $(P/F,15\%,4)$ = $57.18
$150 $(P/F,15\%,5)$ = $74.58
$150 $(P/F,15\%,6)$ = $64.85
$150 $(P/F,15\%,7)$ = $56.39
$150 $(P/F,15\%,8)$ = $49.04
$200 $(P/F,15\%,9)$ = $57.78

$543.72

Irregular series



Assignment L4

3.3, 3.5, 3.6, 3.17, 3.21, 3.30, 3.38, 3.45, 3.51, 3.68

Practice

If you desire to withdraw the following amounts over the next five years from a savings account that earns 8% interest compounded annually, how much do you need to deposit now?

N	**Amount**
2	$32,000
3	43,000
4	46,000
5	28,000

8.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0800	0.9259	1.0000	1.0000	0.9259	1.0800	0.0000	0.0000	1
2	1.1664	0.8573	2.0800	0.4808	1.7833	0.5608	0.4808	0.8573	2
3	1.2597	0.7938	3.2464	0.3080	2.5771	0.3880	0.9487	2.4450	3
4	1.3605	0.7350	4.5061	0.2219	3.3121	0.3019	1.4040	4.6501	4
5	1.4693	0.6806	5.8666	0.1705	3.9927	0.2505	1.8465	7.3724	5
6	1.5869	0.6302	7.3359	0.1363	4.6229	0.2163	2.2763	10.5233	6
7	1.7138	0.5835	8.9228	0.1121	5.2064	0.1921	2.6937	14.0242	7
8	1.8509	0.5403	10.6366	0.0940	5.7466	0.1740	3.0985	17.8061	8
9	1.9990	0.5002	12.4876	0.0801	6.2469	0.1601	3.4910	21.8081	9
10	2.1589	0.4632	14.4866	0.0690	6.7101	0.1490	3.8713	25.9768	10
11	2.3316	0.4289	16.6455	0.0601	7.1390	0.1401	4.2395	30.2657	11
12	2.5182	0.3971	18.9771	0.0527	7.5361	0.1327	4.5957	34.6339	12
13	2.7196	0.3677	21.4953	0.0465	7.9038	0.1265	4.9402	39.0463	13
14	2.9372	0.3405	24.2149	0.0413	8.2442	0.1213	5.2731	43.4723	14

Practice

An individual deposits an annual bonus into a savings account that pays 8% interest compounded annually. The size of the bonus increases by $2,000 each year, and the initial bonus amount was $5,000. Determine how much will be in the account immediately after the fifth deposit.

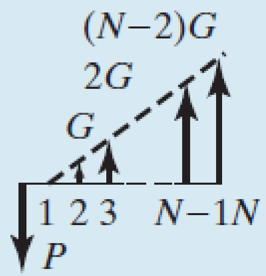
Linear gradient

Present worth $(P/G, i, N)$

$$P = G\left[\frac{(1+i)^N - iN - 1}{i^2(1+i)^N}\right]$$

Conversion factor $(A/G, i, N)$

$$A = G\left[\frac{(1+i)^N - iN - 1}{i[(1+i)^N - 1]}\right]$$



8.0%

	Single Payment		Equal Payment Series				Gradient Series		
N	Compound Amount Factor (F/P,i,N)	Present Worth Factor (P/F,i,N)	Compound Amount Factor (F/A,i,N)	Sinking Fund Factor (A/F,i,N)	Present Worth Factor (P/A,i,N)	Capital Recovery Factor (A/P,i,N)	Gradient Uniform Series (A/G,i,N)	Gradient Present Worth (P/G,i,N)	N
1	1.0800	0.9259	1.0000	1.0000	0.9259	1.0800	0.0000	0.0000	1
2	1.1664	0.8573	2.0800	0.4808	1.7833	0.5608	0.4808	0.8573	2
3	1.2597	0.7938	3.2464	0.3080	2.5771	0.3880	0.9487	2.4450	3
4	1.3605	0.7350	4.5061	0.2219	3.3121	0.3019	1.4040	4.6501	4
5	1.4693	0.6806	5.8666	0.1705	3.9927	0.2505	1.8465	7.3724	5
6	1.5869	0.6302	7.3359	0.1363	4.6229	0.2163	2.2763	10.5233	6
7	1.7138	0.5835	8.9228	0.1121	5.2064	0.1921	2.6937	14.0242	7
8	1.8509	0.5403	10.6366	0.0940	5.7466	0.1740	3.0985	17.8061	8
9	1.9990	0.5002	12.4876	0.0801	6.2469	0.1601	3.4910	21.8081	9
10	2.1589	0.4632	14.4866	0.0690	6.7101	0.1490	3.8713	25.9768	10
11	2.3316	0.4289	16.6455	0.0601	7.1390	0.1401	4.2395	30.2657	11
12	2.5182	0.3971	18.9771	0.0527	7.5361	0.1327	4.5957	34.6339	12
13	2.7196	0.3677	21.4953	0.0465	7.9038	0.1265	4.9402	39.0463	13
14	2.9372	0.3405	24.2149	0.0413	8.2442	0.1213	5.2731	43.4723	14