THEORY OF MACHINES AND MECHANISMS II

Mechanical IV/I

Chapter 6

Vibration of Single Degree of Freedom Systems

Text/Reference Books

- Theory of Vibrations with Applications: **W. T. Thomson**
- Mechanical Vibrations: **S. S. Rao**
- Fundamentals of Mechanical Vibrations: **G. H. Kelly**
- Introductory Course on Theory and Practice of Mechanical Vibrations: **J. S. Rao, K. Gupta**
- Engineering Vibration: **D. J. Inman**
- Mechanical Vibrations: **Tse, Morse, Hinkle**
- An Introduction to Mechanical Vibrations: **R. F. Steidel**

Text/Reference Books

- Fundamentals of Vibration: **L. Meirovitch**
- Mechanical Vibrations: **J P Den Hartog**
- Vibration Problems in Engineering: **S. Timoshenko**
- Engineering Vibration Analysis with Application to Control Systems: **C. F. Beards**
- Mechanical Vibrations: **V. P. Singh**
- Mechanical Vibrations: **G. K. Grover**

Revision

Second order linear differential equation

- Particular and complementary solution
- Initial value problem

Second order partial differential equation

Matrix

- Matrix multiplication, Inverse
- Eigenvalues and Eigenvectors

Strength of Materials

- End conditions for beams
- Beam deflection

6.1 Introduction

- Any motion that repeats itself after an interval of time is called **vibration** or **oscillation**.
- The study of vibration is concerned with the oscillatory motions of bodies and the forces associated with them.
- All bodies possessing mass and elasticity are capable of vibration.

Causes and Effects of Vibration

Causes: Unbalanced force, dry friction, external excitation, earthquakes, winds, etc.

Bad Effects: Excessive stresses, undesirable noise, looseness of parts, and partial or complete failure of parts

Good Effects: Hearing, breathing, musical instruments, vibrating screens, shakers, etc.

Basic Terms

Periodic Motion

A motion which repeats itself after equal intervals of time.

Time Period

Time taken to complete one cycle.

Frequency

Number of cycles per unit time.

Simple Harmonic Motion

A periodic motion of a particle whose acceleration is always directed towards the mean position and is proportional to its distance form the mean position.

Amplitude

The maximum displacement of a vibrating body from the mean position.

Free Vibration

The vibration of a system because of its own elastic properties.

Natural Frequency

Frequency of free vibration of the system.

Forced Vibration

The vibration which the system executes under an external force. The frequency of vibration in this case is the same as that of excitation.

Resonance

The vibration of the system when the frequency of the external force is equal to the natural frequency of the system. The amplitude of vibration at resonance frequency becomes excessive.

Damping

Resistance to the motion of the vibrating body.

Degree of Freedom

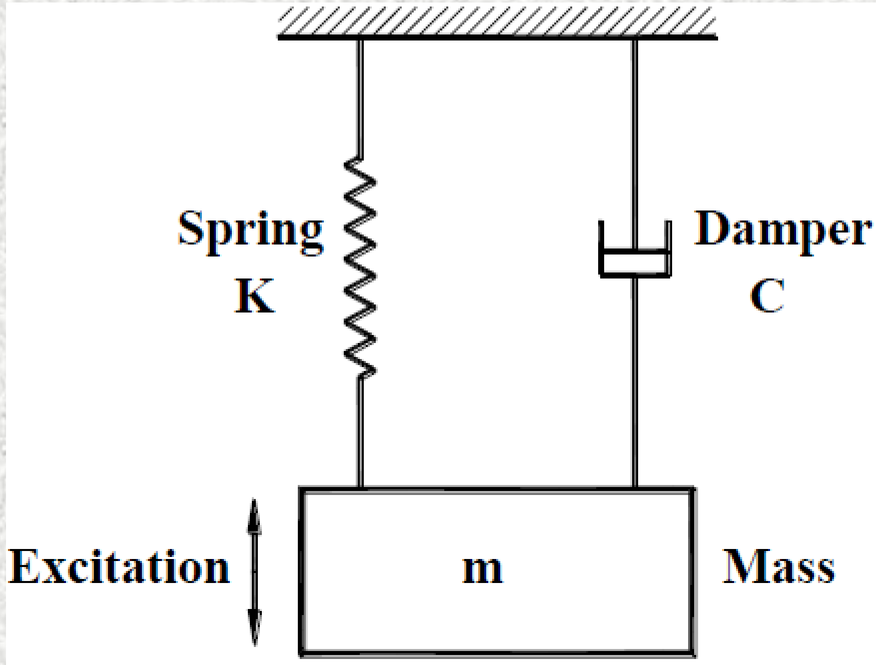
The minimum number of independent coordinates required to specify the motion of a system at any instant is known as degree of freedom of the system.

Phase Difference

It is the angle between two rotating vectors representing simple harmonic motions of the same frequency.

6.2 Elements of a Vibratory System

The elements of a vibratory system under idealized conditions are: the mass, the spring, the damper, and the excitation.



Mass

The mass *m* is assumed to be rigid body. Energy is stored in the mass in the form of kinetic energy.

Spring

The spring *k* possesses elasticity and is assumed to be of negligible mass. Energy is stored in the spring in the form of potential energy.

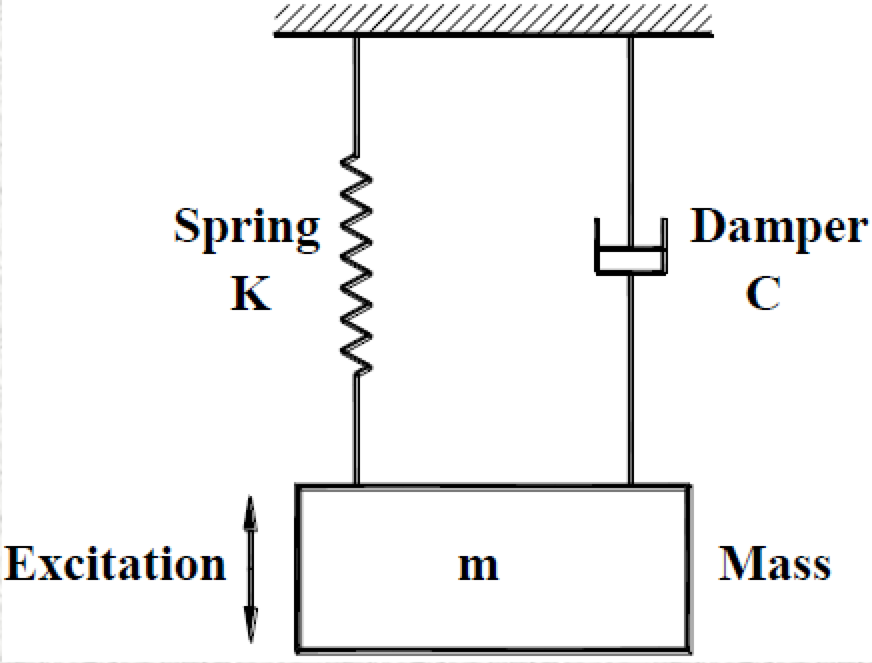
Spring is characterized by its stiffness; i.e., $k = \dfrac{dF}{dx}$

Damper

The damper C has neither mass nor elasticity. Energy is dissipated in the damper in the form of heat.

The damper is characterized by its damping constant or damping coefficient, i.e.,

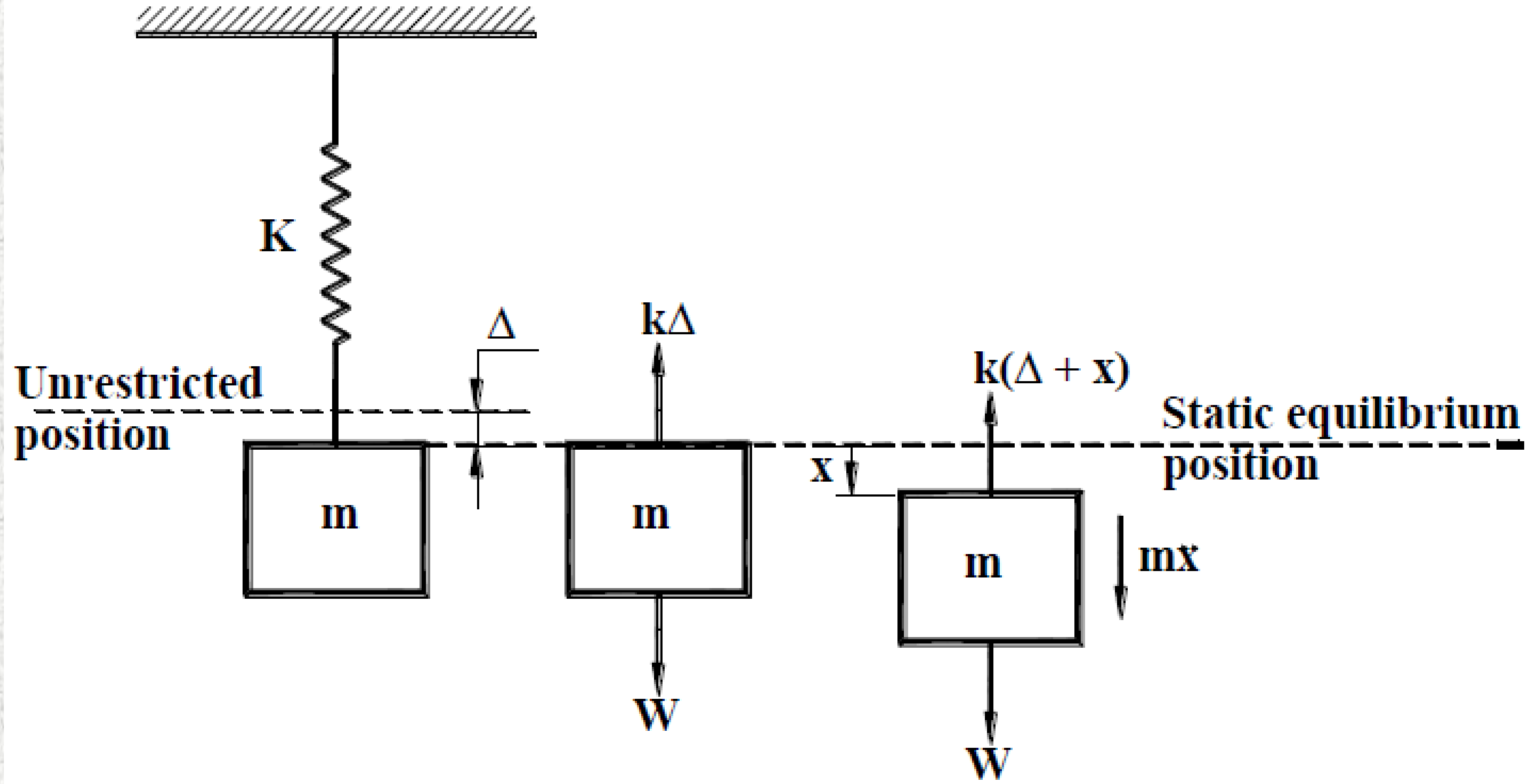
$$c = \frac{dF}{dv}$$



Excitation

Energy enters the system with the application of external force known as excitation.

6.3 Undamped Free Vibrations of SDOF Systems



In equilibrium position, the gravitational pull *W*, is balanced by a force of spring, such that

$$W = k\Delta \qquad \text{.......... (1)}$$

where Δ is the static deflection of the spring.

If the mass is displaced from its equilibrium position by a distance *x* and then released, so after time *t*,

$$m\ddot{x} = \sum F = W - k(\Delta + x) \qquad \text{.......... (2)}$$

Since W = kΔ, we obtain

$$m\ddot{x} = -k\,x \qquad \text{.......... (3)}$$

which can be rewritten in standard differential equation form as

$$m\ddot{x} + k\,x = 0 \qquad \text{.......... (4)}$$

Equation (4) is a homogeneous second order linear differential equation and is also recognized as equation for simple harmonic motion. The general solution for the equation is

$$x = A \sin \omega_n t + B \cos \omega_n t \quad \ldots\ldots\ldots\ (5)$$

where A and B are constants which can be found by considering the initial conditions, and ω_n is the circular natural frequency of the motion.

Differentiating Equation (5) twice, we get

$$\dot{x} = \omega_n (A \cos \omega_n t - B \sin \omega_n t) \quad \ldots\ldots\ldots\ (6)$$

$$\ddot{x} = -\omega_n^2 (A \sin \omega_n t + B \cos \omega_n t) \quad \ldots\ldots\ldots\ (7)$$

Substituting Equations (5) and (7) in Equation (4), we get

$$-m\omega_n^2 (A \sin \omega_n t + B \cos \omega_n t) + k (A \sin \omega_n t + B \cos \omega_n t) = 0$$

$$(-m\omega_n^2 + k)(A \sin \omega_n t + B \cos \omega_n t) = 0 \quad \ldots\ldots\ldots\ (8)$$

Since $(A \sin \omega_n t + B \cos \omega_n t) \neq 0$

$$-m\omega_n^2 + k = 0$$

$$\therefore \quad \omega_n = \sqrt{\frac{k}{m}} \quad \text{..........} \; (9)$$

The natural frequency , in *Hz*, is given by

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{..........} \; (10)$$

The period of the natural vibration is

$$T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} \quad \text{..........} \; (11)$$

These equations can also be expressed in terms of static displacement Δ as

$$f_n = \frac{1}{2\pi}\sqrt{\frac{g}{\Delta}}$$

$$T = 2\pi\sqrt{\frac{\Delta}{g}}$$

Solution with initial conditions

Assume that at $t = 0$, $x = x(0)$ and $\dot{x} = \dot{x}(0)$

Substituting $t = 0$ and $x = x(0)$ in Equation (5),

$$x(0) = 0 + B \quad \therefore \quad B = x(0) \quad \ldots\ldots\ldots\ldots (12)$$

Again substituting $t = 0$ and $\dot{x} = \dot{x}(0)$ in Equation (6),

$$\dot{x}(0) = \omega_n A - 0 \qquad \therefore \quad A = \frac{\dot{x}(0)}{\omega_n} \quad \ldots\ldots\ldots\ldots (13)$$

Hence, general equation of motion of a simple spring mass system is given by

$$x = \frac{\dot{x}(0)}{\omega_n} \sin \omega_n t + x(0) \cos \omega_n t \quad \ldots\ldots\ldots\ldots (14)$$

Equation (14) can also be expressed in other forms as

$$x = A_1 \sin(\omega_n t + \phi_1) \quad \ldots\ldots\ldots\ldots (15)$$

$$\text{where } A_1 = \sqrt{\left(\frac{\dot{x}(0)}{\omega_n}\right)^2 + \left(x(0)\right)^2}, \text{ and } \phi_1 = \tan^{-1}\left(\frac{x(0)}{\dot{x}(0)/\omega_n}\right)$$

or

$$x = A_2 \cos(\omega_n t + \phi_2) \quad \ldots\ldots\ldots\ldots (16)$$

$$\text{where } A_2 = \sqrt{\left(\frac{\dot{x}(0)}{\omega_n}\right)^2 + \left(x(0)\right)^2}, \text{ and } \phi_2 = \tan^{-1}\left(\frac{\dot{x}(0)/\omega_n}{x(0)}\right)$$

Example 6.1

A steel wire ($E = 1.96 \times 10^{11} N/m^2$) is of 2 mm diameter and is 30 m long. It is fixed at the upper end and carries a mass *m* at its lower end. Find *m* so that the frequency of longitudinal vibration is 4 Hz.

32.49 kg

Example 6.2

A helical spring of stiffness *k* is cut into two halves and a mass *m* is connected to the two halves as shown in **Figure E6.2a**. The natural time period of this system is found to be 0.5 s. If an identical spring is cut so that one part is one-fourth and the other part three fourths of the original length, and the mass *m* is connected to the two parts as shown in **Figure E6.2b**, what would be the natural period of the system?

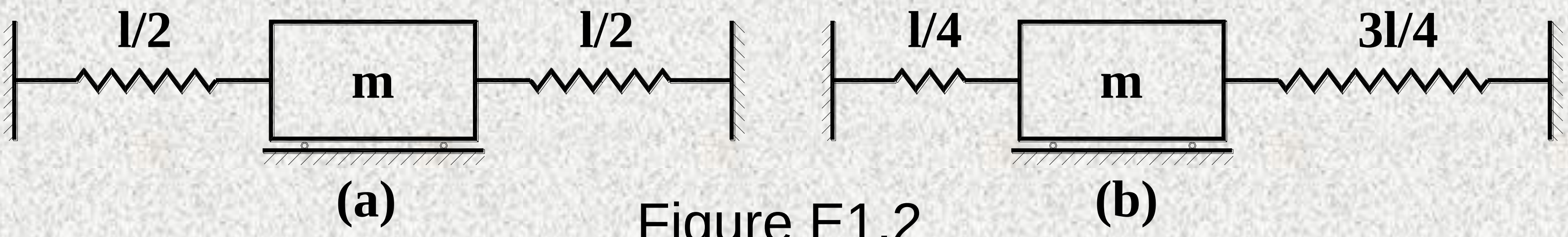


Figure E1.2

0.433 s

Example 6.3

What is the natural frequency of the 200 kg block of **Figure E6.3**?

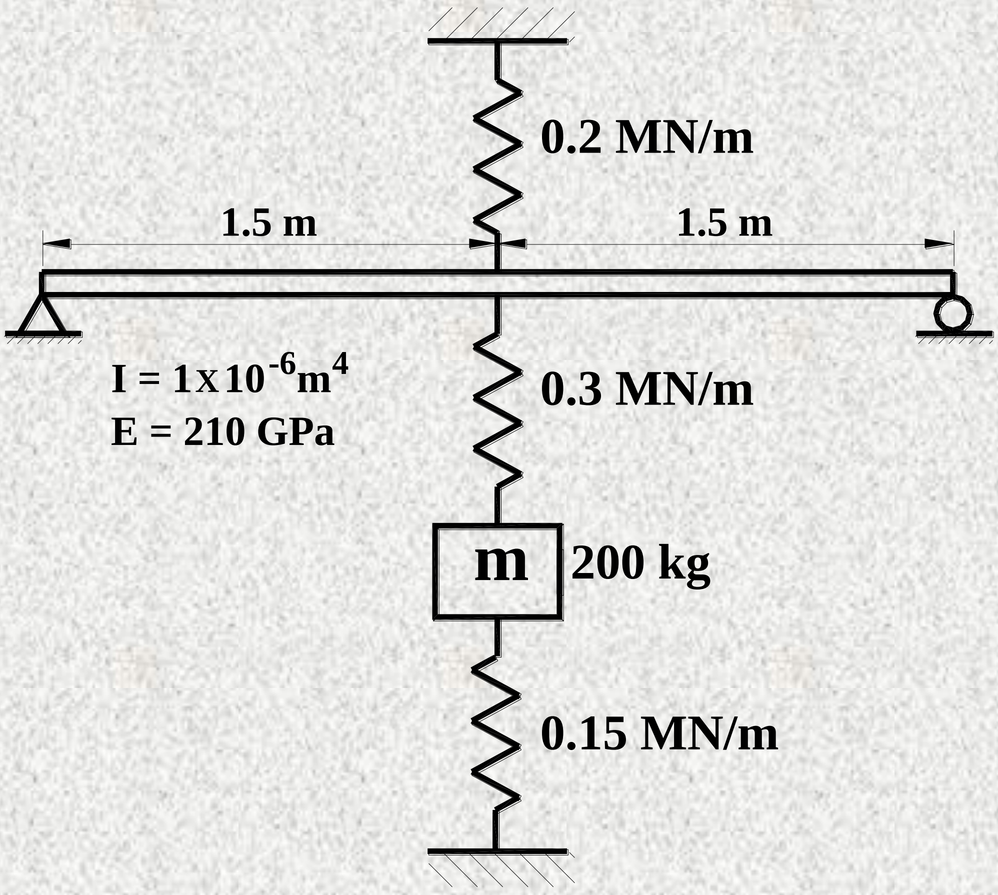


Figure E1.3

41.6 rad/s

Example 6.4

Find the spring stiffness of the system shown in **Figure E6.4**, at the point of force application. The mass less rigid bar 1-2 is free to have rectangular and angular motions.

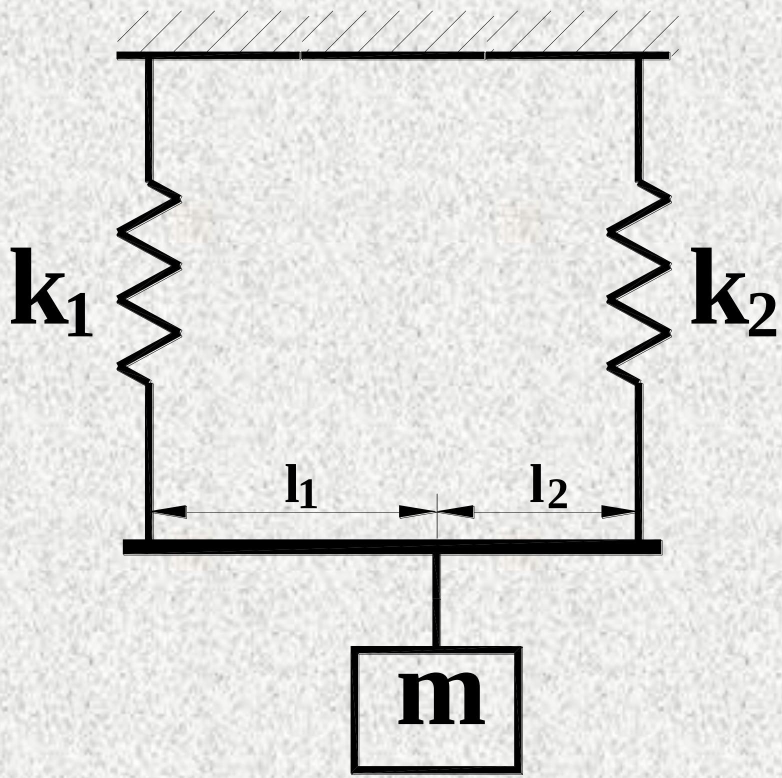


Figure E6.4

$$\frac{(l_1 + l_2)^2 k_1 k_2}{k_1 l_1{}^2 + k_2 l_2{}^2}$$

Example 6.5

In **Figure E6.5**, the mass m is suspended by means of spring k_2 from the end of a rigid mass less beam which is of the length l and attached to the frame at its left end. It is also supported in a horizontal position by a spring k_1 attached to the frame as shown. What is the natural frequency f of the system?

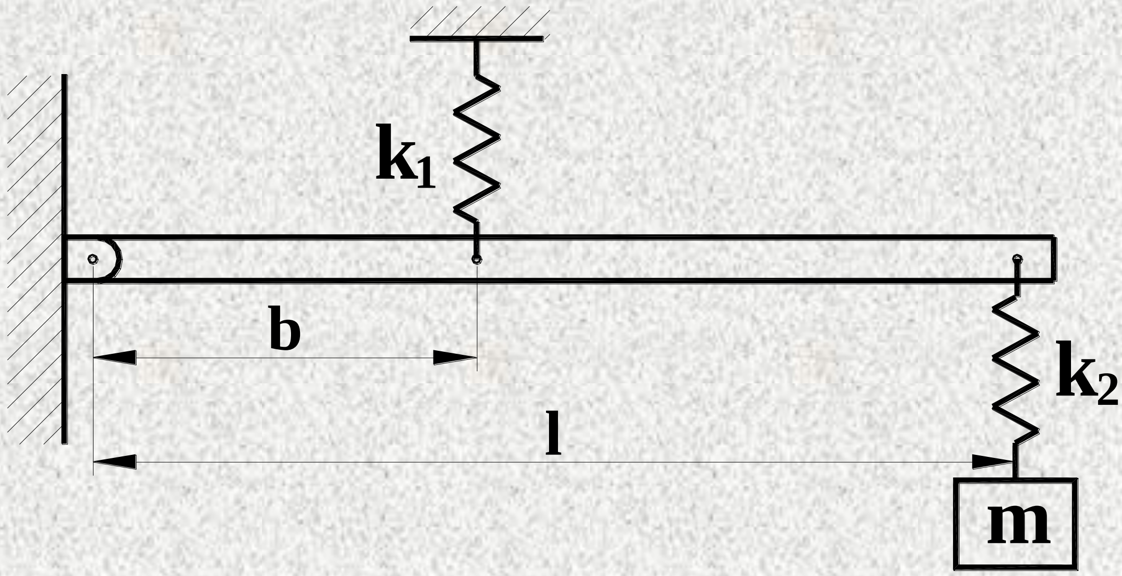


Figure E6.5

$$\left[\frac{1}{2\pi}\sqrt{\frac{k_1 k_2}{m\left\{k_1+(l/b)^2 k_2\right\}}}\right]$$