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On the Conservation of Rothalpy in Turbomachines

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ABSTRACT

The conditions under which rothalpy is conserved are investigated by means of the energy and moment-of-momentum equations for unsteady flow of a viscous, compressible fluid. Differential and integral equations are given for the total enthalpy and rothalpy in both stationary and rotating coordinates. From the equations in rotating coordinates it is shown that rothalpy may change due to: (1) pressure fluctuations caused by flow unsteadiness in the rotating frame; (2) angular acceleration of the rotor; (3) work done by viscous stresses on the relative flow in the rotating frame; (4) work done by body forces on the relative flow; (5) changes in entropy due to viscous dissipation and heat transfer. Conclusions of this investigation are compared with those of previous authors, some of whom have stated that rothalpy is conserved even in viscous flows. A modified Euler's turbomachine equation which includes viscous effects is derived and errors in previous derivations noted.

NOMENCLATURE

A	entrance or exit area of control volume
a	acceleration
$\vec{e}$	unit vector
$\vec{F}$	force per unit volume
$\vec{f}$	body force per unit mass
h	enthalpy
I	rothalpy
$\vec{M}$	moment acting on fluid in control volume
$\dot{m}$	mass flow rate
$\vec{n}$	unit normal vector
P	power
p	static pressure
p^*	rotary stagnation pressure, Eq. (23)
$\vec{q}$	heat flux vector
r	radius in cylindrical coordinates
S	surface area
s	entropy
T	absolute temperature
$\vec{U}$	blade velocity
$\vec{V}$	absolute velocity vector
$\vec{W}$	relative velocity vector
$\mathcal{V}$	volume
z	distance along axis of rotation
θ	angle in cylindrical coordinates
ρ	density
$\overleftrightarrow{\tau}$	viscous stress tensor
Φ	rate of viscous dissipation per unit volume
ω	angular velocity of rotor

Subscripts

c	centrifugal
f	frictional
in	into control volume
out	out of control volume
z	component along axis of rotation
0	total or stagnation quantity
θ	component in tangential direction

Superscripts

$'$	relative to rotor-fixed coordinates

INTRODUCTION

It is well known that for steady adiabatic flow in stationary passages with negligible change in potential energy, the total or stagnation enthalpy

$$h_0 = h + \frac{V^2}{2} \tag{1}$$

is conserved. For steady, isentropic flow in rotating passages and negligible change in potential energy it is the rothalpy (Wu, 1953)

$$I = h + \frac{1}{2}(W^2 - U^2) \tag{2}$$

which is conserved. In Eq. (2) $\vec{W}$ is the velocity of the fluid relative to the frame rotating with the blade, and $\vec{U}$ is the blade velocity. Since the absolute velocity of the flow is

$$\vec{V} = \vec{U} + \vec{W} \tag{3}$$

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and $\vec{U} = \omega r \vec{e}_\theta$, the rothalpy can also be written

$$I = \left(h + \frac{1}{2}V^2\right) - \vec{U}\cdot\vec{V} \; = \; h_0 - \omega r V_\theta \tag{4}$$

The turbomachinery textbooks of Vavra (1960), Dixon (1978), Cumpsty (1989), and Whitfield and Baines (1990) prove that rothalpy is conserved in adiabatic steady flows. The proofs rely on a simplified form of the energy equation which does not include the work done on the relative flow by the viscous stresses. Since it is not always recognized that this work has been neglected, the books differ on whether rothalpy is conserved for viscous flows in impellers.

According to Cumpsty (1989), "In a moving passage the rothalpy is therefore constant provided:

(a) the flow is steady in the rotating frame;

(b) no work is done on the flow in the rotating frame (for example by friction from the casing);

(c) there is no heat flow to or from the flow."

This statement appears in Chapter 1 of Cumpsty's book, following the usual proof that rothalpy is constant in steady adiabatic flow. Although it is a clear and (as will be shown) accurate summary of the conditions under which rothalpy is conserved, it is contradicted later in the book (Chapter 6), where the equation $I = \text{const}$ appears without comment among the equations governing the flow of a viscous compressible fluid relative to the impeller of a centrifugal compressor.

Dixon (1978) claims that the rothalpy "... is constant for an adiabatic irreversible flow process, relative to a rotating component." According to Vavra (1960) the rothalpy is constant along a relative streamline in adiabatic steady flows "... even though entropy changes occur because of frictional effects." Whitfield and Baines (1990) state that "... for any adiabatic flow in a turbomachine, I is constant, regardless of external work transfers or radius changes." Although the last authors had made it clear that their derivation was for steady adiabatic flow, they did not mention friction.

On the other hand, a few theoretical analyses have shown that rothalpy is not conserved when work is done on the relative flow by the viscous stresses. Csanady (1964) derived an integral (control volume) equation for the rothalpy which included the viscous power dissipation[1]. Sehra (1979) derived a differential equation for the rate of change of the mean (pitchwise averaged) rothalpy along the mean streamline. Sehra's result is quite complicated (see Sehra and Kerrebrock, 1980), but it clearly shows that changes in the mean rothalpy are due to both molecular viscosity and the apparent stresses due to velocity fluctuations relative to the blade.

Urbach (1990) derived differential equations for the total enthalpy and rothalpy in the rotor-fixed frame. By integrating his equation for the total enthalpy he obtained a result which he called the "Euler Turbomachinery Equation with Viscous Correction." Urbach's equation differs from Csanady's in that there are two terms instead of one representing the useful mechanical power. Each of these terms is the product of a mass flow rate in the radial or axial direction and the corresponding change of UV_θ in that direction. This discrepancy was not mentioned by Urbach, although he did cite Csanady's book[2]. Urbach did obtain the correct expression for the viscous correction term, which Vavra (1960) had omitted altogether in his form of Euler's turbine equation. Vavra argued that his equation nevertheless accounted for friction.

1 Csanady's derivation contains an incorrect assumption, but as shown in Appendix A, this assumption can be removed without affecting the result.

2 It is shown in Appendix B that had the integration been performed correctly, Urbach's result would have agreed with both Csanady's and that obtained in the present paper.

Computations of viscous flow in centrifugal impellers have been carried out by numerous authors. The results reported in a series of papers by J. Moore, J.G. Moore, and R.H. Timmins (1980, 1984, 1985) are of particular interest because they include rothalpy production at the wall of the stationary shroud. Work done by the viscous stress at the shroud wall was found to increase the rothalpy by about 1 or 2 percent of the total work input, depending on the impeller design and operating conditions.

The present investigation began as an attempt on the author's part to resolve the contradictions in the literature regarding conservation of rothalpy in viscous flow. The equation for the rate of change of rothalpy following the motion of a fluid particle relative to the impeller is derived by transforming the differential equations for the total enthalpy and moment of momentum of a viscous, compressible, heat-conducting fluid into coordinates rotating with the impeller. This approach is similar to that used previously by Urbach (1990), whose work was unknown to the author at the time this investigation was started. Urbach assumed a flow which was adiabatic and steady in coordinates fixed to an impeller rotating with constant angular velocity. The present analysis includes heat transfer and allows unsteadiness due to either velocity fluctuations within the rotor passage or angular acceleration of the rotor. It also provides a simple and rigorous derivation of the integral form of the rothalpy equation, resolving inconsistencies in prior analyses.

ANALYSIS

1. Equations for total enthalpy

The total enthalpy h_0 changes according to the following equation for a viscous, compressible fluid with heat conduction (Liepmann and Roshko, 1957):

$$\rho\frac{Dh_0}{Dt} \; = \; \frac{\partial p}{\partial t} + \nabla\cdot(\vec{\vec{\tau}}\cdot\vec{V}) + \rho\vec{f}\cdot\vec{V} - \nabla\cdot\vec{q} \tag{5}$$

If Eq. (5) is integrated over a region moving with the fluid and use is made of Reynolds' transport theorem (Aris, 1962) to transform to a fixed control volume, the result is

$$\frac{\partial}{\partial t}\int \rho h_0 d\mathcal{V} + \int \rho h_0 \vec{V}\cdot\vec{n}dS \; = \; \int \frac{\partial p}{\partial t} d\mathcal{V} + \int \vec{n}\cdot\vec{\vec{\tau}}\cdot\vec{V}dS$$
$$+ \int \rho\vec{f}\cdot\vec{V}d\mathcal{V} - \int \vec{q}\cdot\vec{n}dS \tag{6}$$

The viscous stress-resultant $\vec{n}\cdot\vec{\vec{\tau}}$ does work only on moving boundaries, so that the second integral on the right of Eq. (6) is nonzero only on those boundaries. For a flow passage with all its boundaries stationary this integral vanishes completely. If the boundaries are also adiabatic and the flow steady the only remaining term is the second on the left, which yields the familiar result that the net rate of efflux of total enthalpy from the control volume is zero.

Equation (5) is now transformed from a fixed system of cylindrical coordinates (r, θ, z) to a system (r', θ', z') rotating about the z-axis by means of the relations

$$\left.\begin{aligned} r' &= r \\ \theta' &= \theta - \int \omega dt \\ z' &= z \\ t' &= t \end{aligned}\right\} \tag{7}$$

It follows that

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial r'} \;,\quad \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta'} \;,\quad \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} \tag{8a}$$

or

$$\nabla = \nabla' \tag{8b}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \omega\frac{\partial}{\partial \theta'} \tag{8c}$$

The substantial derivative of any scalar function $f(r,\theta,z,t)$ is therefore

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{V}\cdot\nabla f$$

$$= \frac{\partial f}{\partial t'} - \omega\frac{\partial f}{\partial \theta'} + (\vec{U}+\vec{W})\cdot\nabla' f$$

$$= \frac{\partial f}{\partial t'} + \vec{W}\cdot\nabla' f \equiv \frac{Df}{Dt'} \qquad (9)$$

where D/Dt' means the substantial derivative in the rotating frame[3]. Letting $f = h_0$, combining Eqs. (5) and (9), and noting that the viscous stresses do no work of deformation in solid-body rotation (i.e., $\vec{\vec{\tau}}:\nabla\vec{U} = 0$), one obtains

$$\rho\frac{Dh_0}{Dt} = \frac{\partial p}{\partial t'} + \omega r F_\theta + \nabla\cdot(\vec{\vec{\tau}}\cdot\vec{W}) + \rho\vec{f}\cdot\vec{W} - \nabla\cdot\vec{q} \qquad (10)$$

where

$$F_\theta \equiv -\frac{1}{r}\frac{\partial p}{\partial \theta} + (\nabla\cdot\vec{\vec{\tau}})_\theta + \rho f_\theta \qquad (11)$$

is the tangential component of the net force per unit volume of fluid. The second term on the right of Eq. (10) is the moment of this force about the z-axis times the angular velocity ω of the rotor. That is, it is the power per unit volume which must be supplied by the surrounding fluid and ultimately the blades just to rotate a fluid particle with angular velocity ω against the tangential pressure, friction, and body forces acting on it. The third term on the right of Eq. (10) is the rate at which work is done on the relative motion of the fluid in the rotating frame by viscous stresses. The fourth term on the right is the rate at which work is done on the relative motion by the body force, and the fifth the net rate of heat addition.

Integrating Eq. (10) over a region moving with the fluid and using Reynolds' transport theorem to transform to a control volume which rotates with the impeller leads to the result

$$\frac{\partial}{\partial t'}\int \rho h_0 d\forall' + \int \rho h_0 \vec{W}\cdot\vec{n}\, dS' = \int\left(\frac{\partial p}{\partial t'} + \omega r F_\theta + \rho\vec{f}\cdot\vec{W}\right)d\forall'$$

$$+ \int \vec{n}\cdot(\vec{\vec{\tau}}\cdot\vec{W} - \vec{q})dS' \qquad (12)$$

On the rotating surfaces $\vec{W} = 0$, so the viscous stress-resultant $\vec{n}\cdot\vec{\vec{\tau}}$ does no work on these surfaces. Only the stresses on the fixed surfaces, which appear to rotate relative to the rotor frame, do work on the relative flow and change the total enthalpy.

Another form of the differential equation for the total enthalpy can be obtained by splitting the third term on the right of Eq. (10) as follows:

$$\nabla\cdot(\vec{\vec{\tau}}\cdot\vec{W}) = (\nabla\cdot\vec{\vec{\tau}})\cdot\vec{W} + \vec{\vec{\tau}}:\nabla\vec{W} \qquad (13)$$

The first term on the right of Eq. (13) is the rate at which the net viscous force per unit volume $\nabla\cdot\vec{\vec{\tau}}$ does work on the relative flow, while the second term is the rate of viscous dissipation,

$$\Phi \equiv \vec{\vec{\tau}}:\nabla\vec{V} = \vec{\vec{\tau}}:\nabla\vec{W} \qquad (14)$$

3 Equation (9) holds only for scalar functions. For vector functions the substantial derivatives in the fixed and rotating frames will be different because the unit vectors change direction. Because of Eqs. (8b) and (9) primes on ∇ and D/Dt are unnecessary and are dropped in subsequent equations, but by Eq.(8c) the prime is necessary to distinguish the partial time derivative $\partial/\partial t'$ in the rotating frame from the derivative $\partial/\partial t$ in the fixed frame.

which is never negative. Since the rate of increase of entropy due to viscous dissipation and heat transfer is given by the equation (Liepmann and Roshko, 1957)

$$\rho T\frac{Ds}{Dt} = \Phi - \nabla\cdot\vec{q} \qquad (15)$$

an alternate form of Eq. (10) is

$$\rho\frac{Dh_0}{Dt} = \frac{\partial p}{\partial t'} + \omega r F_\theta + (\nabla\cdot\vec{\vec{\tau}} + \rho\vec{f})\cdot\vec{W} + \rho T\frac{Ds}{Dt} \qquad (16)$$

2. Moment-of-momentum equation

The differential and integral forms of the moment-of-momentum equation are derived in numerous references, e.g., Vavra (1960). All that is needed here is the z-component of this equation, which is readily obtained by multiplying the θ-component of the linear momentum equation, $F_\theta = \rho a_\theta$, by r. The tangential acceleration a_θ in cylindrical coordinates is

$$a_\theta = \left[\frac{\partial}{\partial t} + V_r\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \frac{V_\theta}{r}\frac{\partial}{\partial\theta} + V_z\frac{\partial}{\partial z}\right]V_\theta$$

$$= \left(\frac{\partial}{\partial t'} + \vec{W}\cdot\nabla + \frac{1}{r}V_r\right)V_\theta$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial t'} + \vec{W}\cdot\nabla\right)(rV_\theta) = \frac{1}{r}\frac{D}{Dt}(rV_\theta) \qquad (17)$$

Hence the moment-of-momentum equation is

$$rF_\theta = \rho\frac{D}{Dt}(rV_\theta) \qquad (18)$$

An even simpler derivation of this familiar result can be found in Cumpsty (1989). From Eq. (11) it is obvious that F_θ includes all the forces on the fluid particle, namely pressure, viscous and body forces. Integration of Eq. (18) over the rotating control volume surrounding the fluid in the impeller gives what is perhaps the most familiar form of this equation, namely, the Euler turbomachinery equation[4]:

$$M_z \equiv \int rF_\theta d\forall' = \frac{\partial}{\partial t'}\int \rho r V_\theta d\forall' + \int \rho r V_\theta \vec{W}\cdot\vec{n}\, dS' \qquad (19)$$

3. Differential equation for rothalpy

The following two forms of the differential equation for the rothalpy are readily obtained from Eqs. (4), (9), (10), (13)-(15) and (18):

$$\rho\frac{DI}{Dt} = \frac{\partial p}{\partial t'} - \rho r V_\theta\frac{d\omega}{dt} + \nabla\cdot(\vec{\vec{\tau}}\cdot\vec{W}) + \rho\vec{f}\cdot\vec{W} - \nabla\cdot\vec{q} \qquad (20a)$$

$$\rho\frac{DI}{Dt} = \frac{\partial p}{\partial t'} - \rho r V_\theta\frac{d\omega}{dt} + (\nabla\cdot\vec{\vec{\tau}} + \rho\vec{f})\cdot\vec{W} + \rho T\frac{Ds}{Dt} \qquad (20b)$$

4 This term is unfortunately used for both the moment-of-momentum equation and the result of combining that equation with a simplified version of the energy equation which neglects unsteady and viscous effects. The latter usage is historically inaccurate, because Euler's mémoire on reaction turbines (1754), in which the torque was calculated from the rate of change of moment of momentum (see also Rouse and Ince, 1957), preceded the first law of thermodynamics by about 100 years. Moreover, as Cumpsty (1989) points out, "the Euler equation is valid no matter how the momentum of momentum is produced, and even viscous drag on the blades can produce a positive work input." Calling the combined equation, which lacks the term for the work done by friction, the "Euler turbomachinery equation" may be responsible for some of the confusion whether friction affects rothalpy.

From Eq. (20b) it is clear that changes in rothalpy of a fluid particle are caused by:

(1) pressure fluctuations due to flow unsteadiness in the rotating frame;

(2) angular acceleration of the rotor;

(3) work done by the net viscous force per unit volume $\nabla\cdot\vec{\tau}$ on the relative flow in the rotor;

(4) work done by body forces on the relative flow;

(5) changes in entropy due to viscous dissipation and heat transfer.

Therefore the rothalpy of a fluid particle is conserved provided that the three conditions (a)-(c) of Cumpsty (1989) quoted in the Introduction are satisfied, and in addition, provided that the body force does no work on the relative motion in the rotating frame. The last condition, although not mentioned by Cumpsty, is not very important in most turbomachinery applications. If necessary, the work done by a conservative body force could always be accounted for by simply adding its potential energy to the total enthalpy or rothalpy on the left of Eqs. (10) and (20).

An equation similar to Eq. (20b) was derived by Wu (1952) for an inviscid flow without body forces or rotor angular acceleration (cf. his Eq. (20)). Equation (20b) includes the effects neglected by Wu, of which the most important is probably friction.

4. Rothalpy equation for a control volume

Integration of Eq. (20a) over a control volume rotating with the impeller leads to

$$\frac{\partial}{\partial t'}\int\rho I\,d\forall' + \int\rho I\vec{W}\cdot\vec{n}\,dS' = \int\left(\frac{\partial p}{\partial t'} - \rho r V_\theta\frac{d\omega}{dt} + \rho\vec{f}\cdot\vec{W}\right)d\forall' + \int\vec{n}\cdot(\vec{\tau}\cdot\vec{W} - \vec{q})\,dS' \qquad (21)$$

As mentioned above in connection with Eq. (12), the viscous stresses on the rotating surfaces do no work on the relative flow, so they affect neither the total enthalpy nor rothalpy. For an unshrouded rotor this means that only the casing friction and possibly the viscous stresses on the control surfaces at the rotor entrance and exit do work on the relative flow. On the casing $\vec{W} = -\vec{U}$, so if the stress resultant $\vec{n}\cdot\vec{\tau}$ exerted on the fluid by the casing is opposite to the blade velocity the rothalpy would increase from inlet to outlet, as Moore et al (1980, 1984, 1985) calculated. For a shrouded rotor no work is done by the viscous stresses on any of the internal rotor passages.

5. Alternate form of rothalpy differential equation

From the definition of I in Eq. (2) and the thermodynamic relation $dh = Tds + (1/\rho)dp$ it follows that

$$\rho\frac{DI}{Dt} = \rho T\frac{Ds}{Dt} + \frac{Dp}{Dt} + \frac{1}{2}\rho\frac{D}{Dt}(W^2 - U^2)$$

$$= \rho T\frac{Ds}{Dt} + \frac{Dp^*}{Dt} + \frac{1}{2}(U^2 - W^2)\frac{D\rho}{Dt} \qquad (22)$$

where

$$p^* = p + \frac{1}{2}\rho(W^2 - U^2) \qquad (23)$$

is the rotary stagnation pressure introduced by Hawthorne (1974). It is interchangeable with the rothalpy in incompressible, isentropic flows, for then $p^* = \rho I$.

Equating the expressions for $\rho DI/Dt$ on the rights of Eqs. (20b) and (22) gives

$$\frac{Dp^*}{Dt} = \frac{\partial p}{\partial t'} - \rho r V_\theta\frac{d\omega}{dt} + \frac{1}{2}(W^2 - U^2)\frac{D\rho}{Dt} + (\nabla\cdot\vec{\tau} + \rho\vec{f})\cdot\vec{W} \qquad (24)$$

Note that the entropy production terms do not appear in this version of the equation. For steady incompressible flow only the net viscous and body forces affect p^*. If these forces were absent p^* would be conserved.

SPECIAL CASES

1. Unsteady and steady isentropic flow

For an inviscid flow with no body forces or heat transfer Eqs. (10) and (20) for the total enthalpy and rothalpy reduce to

$$\rho\frac{Dh_0}{Dt} = \frac{\partial p}{\partial t'} - \omega\frac{\partial p}{\partial\theta} = \frac{\partial p}{\partial t} \qquad (25)$$

$$\rho\frac{DI}{Dt} = \frac{\partial p}{\partial t'} - \rho r V_\theta\frac{d\omega}{dt} \qquad (26)$$

Equation (25) shows that unsteadiness is necessary in order for an adiabatic turbomachine to do useful work on an inviscid flow (Dean, 1959). But this conclusion requires only unsteadiness in the laboratory frame. Assuming that the flow in the rotating frame is steady does not remove the possibility that the machine can do work, because the blade-to-blade pressure variation in the rotating frame does work at a fixed point in the laboratory frame. This comes from the term $\omega r F_\theta = -\omega(\partial p/\partial\theta)$. A nice physical explanation of this is given by Dean.

Equation (26) shows how unsteadiness affects the rothalpy. If the flow is steady in the rotating frame, $\partial p/\partial t' = d\omega/dt = 0$, and Eqs. (25) and (26) become

$$\rho\vec{W}\cdot\nabla h_0 = \omega r F_\theta \qquad (27)$$

$$\rho\vec{W}\cdot\nabla I = 0 \qquad (28)$$

In words, the rate of change of total enthalpy of a fluid particle as it moves along its relative streamline in the rotating frame is equal to the rate at which the net tangential pressure force per unit volume does work, and the rothalpy is constant along a streamline of the relative flow. Contrary to Vavra's (1960) statement quoted in the Introduction, rothalpy would not be constant along the streamline in the presence of friction.

2. Steady viscous flow in rotor

If the flow is steady in coordinates rotating with a constant angular velocity ω, and there is no heat transfer nor body forces, Eq. (21) reduces to

$$\int\rho I\vec{W}\cdot\vec{n}\,dS' = \int\vec{n}\cdot\vec{\tau}\cdot\vec{W}\,dS' \qquad (29)$$

The integral on the left is nonzero only on the portions of the control surface which form the entrance and exit sections. The integral on the right is nonzero only on the stationary surfaces, such as the casing of the turbomachine, and on the entrance and exit sections. Thus Eq. (29) can be written

$$\dot{m}(\bar{I}_{out} - \bar{I}_{in}) = P_f \qquad (30)$$

where $\dot{m}$ is the mass flow rate through the rotor,

$$\bar{I} \equiv \frac{1}{\dot{m}}\int_A\rho I\vec{W}\cdot\vec{n}\,dS' \qquad (31)$$

is the mass-averaged rothalpy over the entrance or exit area A, and

$$P_f = \int\vec{n}\cdot\vec{\tau}\cdot\vec{W}\,dS' \qquad (32)$$

is the power loss due to viscous friction. Although the integral in Eq. (32) is over both the casing and the entrance and exit sections, usually the viscous stresses on the latter sections could be neglected and only the friction from the casing considered. In some cases it may be possible to make simple engineering estimates of P_f and determine the change in average rothalpy from Eq. (30).

Setting $I = h_0 - \omega r V_\theta$ in Eq. (30), one obtains a more familiar-looking result

$$(\bar{h}_0)_{out} - (\bar{h}_0)_{in} = \omega\left[(\overline{rV}_\theta)_{out} - (\overline{rV}_\theta)_{in}\right] + (P_f/\dot{m}) \qquad (33)$$

Different versions of this equation have appeared in the literature. Equation 11(57) of Vavra (1960) is Eq. (33) without the frictional loss term. Vavra argued that his equation would be valid even if friction were present, because the effect of friction would be to increase the moment of momentum at exit above what would be obtained without friction. That seems a tacit acknowledgment of the missing frictional term.

A result equivalent to Eqs. (29) and (32) was derived by Csanady (1964), who erroneously assumed constant radial mass flow rate. As shown in Appendix A, his result can be obtained without that assumption.

Urbach (1990) derived a different version of this equation which has separate terms for the change in moment of momentum in the axial and radial directions. It is also based on the faulty assumption of constant axial and radial mass flow rates. In Appendix B the present results are compared with those of Urbach, and it is shown that Eq. (33) is the correct result.

CONCLUDING REMARKS

It has been shown that rothalpy is conserved only under five rather restrictive conditions. Two of these conditions are that the flow is steady in the rotor frame and that the rotor angular velocity is constant. Both pressure fluctuations in the relative flow and angular acceleration of the rotor can change the rothalpy. Rotor acceleration will not cause a significant change as long as the angular velocity variation is small during the residence time of a fluid particle in the rotor. The third and fourth conditions for constancy of rothalpy are that the net viscous and body forces do no work on the relative flow. The effect of body forces is usually of little consequence. The last condition is that the flow is isentropic, i.e., there is no heat transfer or viscous dissipation. The effect of heat transfer is probably negligible in most cases, although it might be important in the hot rotating sections of gas turbines or in interstage passages in cooled multistage centrifugal compressors.

Unsteadiness of the flow in the rotating passages and friction on the non-rotating surfaces will probably be the major factors affecting the rothalpy. It should be pointed out that even when the flow is adiabatic and steady in rotor-fixed coordinates, the rothalpy changes because the work of the viscous stresses affects the moment of momentum and total enthalpy differently. The reason for this is that the total enthalpy can be affected by nonisentropic processes in the rotor whereas the change in moment of momentum is the same whether the mechanical work is done reversibly or irreversibly by the torque.

Finally, although rothalpy is in general not conserved in viscous flows, it has often been assumed constant in turbomachine through-flow calculations which incorporate loss models (Bosman and Marsh, 1974; Hirsch and Warzee, 1977). Such an assumption cannot be rigorously justified, but it does greatly simplify the calculations, in that it allows the differential energy equation to be replaced by the algebraic equation $I = \text{const.}$ along streamlines. The cited analyses utilize a "consistent" loss model, in which the net viscous force $\nabla\cdot\overleftrightarrow{\tau}$ is replaced by a "drag" force $\vec{D}$, so that for steady flow without external body forces the assumption $DI/Dt = 0$ reduces Eq. (20b) to $\rho T Ds/Dt = -\vec{D}\cdot\vec{W}$. From Eq. (20a) it can be seen that this requires $\nabla\cdot(\vec{q} - \overleftrightarrow{\tau}\cdot\vec{W}) = 0$, i.e., all the work done by the viscous stress is carried away by conduction. This is the special condition postulated by Denton (1986) to show that his viscous loss model implies conservation of rothalpy. Earlier, Hirsch and Warzee (1977) had concluded that the simple loss model conserves rothalpy, but without mentioning the heat transfer implications. Unfortunately, the simple loss models do not correctly account for entropy generation by viscous dissipation, so the fact that they conserve rothalpy is merely fortuitous.

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APPENDIX A: CORRECTION OF CSANADY'S DERIVATION

Csanady (1964) derived an integral equation for the rothalpy by writing a steady-state power balance in the rotating frame as

$$P_c + P_f = \int\left(h + \frac{W^2}{2} + \phi\right)dG \qquad (A1)$$

where P_f is the power developed by fluid friction, ϕ ($=gz$ in his notation) is the potential energy, $dG = \rho\vec{W}\cdot\vec{n}dS'$ is the differential mass flow rate, and P_c is the power input due to the centrifugal force,

$$P_c = \int\rho\omega^2 r W_r d\mathcal{V}' \qquad (A2)$$

Csanady incorrectly assumed that the radial mass flow rate $\int\rho W_r dS'$ was constant in order to reduce Eq. (A2) to the desired form. His result can be obtained without this assumption as follows:

Since $W_r = Dr/Dt = \vec{W}\cdot\nabla r$, Eq.(A2) can be written

$$P_c = \int\rho\omega^2\vec{W}\cdot\nabla(r^2/2)d\mathcal{V}'$$

$$= \frac{1}{2}\int\omega^2[\nabla\cdot(\rho\vec{W}r^2) - r^2\nabla\cdot(\rho\vec{W})]d\mathcal{V}' \qquad (A3)$$

But for steady flow in the rotor the continuity equation is $\nabla\cdot(\rho\vec{W}) = 0$, so the second term in the integrand on the right of Eq. (A3) vanishes. Since $\omega = const.$, applying Gauss's theorem to the remaining integral gives

$$P_c = \int\frac{\omega^2 r^2}{2}\rho\vec{W}\cdot\vec{n}dS' = \int\frac{U^2}{2}dG \qquad (A4)$$

Combining Eqs. (A1) and (A4) gives Csanady's result,

$$P_f = \int(I + \phi)dG \qquad (A5)$$

When there are no body forces this becomes the same as Eq. (29) of the present paper.

APPENDIX B: COMPARISON WITH RESULTS OF URBACH

Urbach's report (1990) covers much of the same ground as the present paper, and, as will be shown in this Appendix, the differential equations derived here for the substantial rates of change of total enthalpy and rothalpy in the rotating frame reduce to Urbach's equations when one imposes the additional restrictions of steady relative flow, no angular acceleration of the rotor, and no body forces nor heat transfer. It is also shown here that Urbach's differential equation for total enthalpy, when properly integrated over the rotor control volume, leads to Eq. (33) of this paper rather than Urbach's result.

Urbach's Eq. (22) is the differential equation for the rothalpy. In the present notation it reads

$$\rho\frac{DI}{Dt} = \nabla\cdot(\vec{\vec{\tau}}\cdot\vec{W}) \qquad (B1)$$

Equation (20) of this paper reduces to the same result under the conditions cited in the previous paragraph.

Urbach's Eq. (26) for the total enthalpy may be written

$$\rho\frac{Dh_0}{Dt} = \vec{U}\cdot(-\nabla p + \nabla\cdot\vec{\vec{\tau}}) + \nabla\cdot(\vec{\vec{\tau}}\cdot\vec{W}) \qquad (B2)$$

Equation (10) of this paper reduces to the same result, because when there are no body forces the scalar product of $\vec{U}$ with the momentum equation gives

$$\vec{U}\cdot(-\nabla p + \nabla\cdot\vec{\vec{\tau}}) = \rho\vec{U}\cdot\frac{D\vec{V}}{Dt} = \omega r F_\theta \qquad (B3)$$

Thus the differential equations for total enthalpy and rothalpy derived in this paper are in full agreement with the corresponding equations of Urbach (1990) under the stated conditions. The derivations differ mainly in the handling of the first term in the right of Eq. (B2). Here it was replaced by $\omega r F_\theta$, which by Eq. (18) is $\rho\omega$ times the rate of change of moment of momentum following the fluid particle. Urbach replaced it by $\rho\vec{U}\cdot D\vec{V}/Dt$, expressing $D\vec{V}/Dt$ as the sum of the relative acceleration in the rotating frame and the Coriolis and centripetal accelerations. Since $\partial\vec{W}/\partial t' = 0$ in the present case,

$$\frac{D\vec{V}}{Dt} = (\vec{W}\cdot\nabla)\vec{W} + 2\vec{\omega}\times\vec{W} - \nabla\left(\frac{U^2}{2}\right) \qquad (B4)$$

Urbach's Eq. (25) for the total enthalpy follows from Eqs. (B2) - (B4):

$$\rho\frac{Dh_0}{Dt} = \rho\vec{U}\cdot[(\vec{W}\cdot\nabla)\vec{W} + 2\vec{\omega}\times\vec{W}] + \nabla\cdot(\vec{\vec{\tau}}\cdot\vec{W}) \qquad (B5)$$

Although this equation is correct for steady flow in the rotor, the complexity of the first two terms on the right somewhat obscures their physical meanings, and it is easy to make errors when integrating these terms over the rotor control volume, as Urbach did. In integrating the first term on the right, which can be written

$$\rho\vec{U}\cdot(\vec{W}\cdot\nabla)\vec{W} = \rho U\left(W_r\frac{\partial W_\theta}{\partial r} + \frac{W_\theta}{r}\frac{\partial W_\theta}{\partial\theta} + W_z\frac{\partial W_\theta}{\partial z} + \frac{W_r W_\theta}{r}\right) \qquad (B6)$$

Urbach split the moment-of-momentum transport terms into separate radial and axial components and assumed constant axial and radial mass flow rates. This is obviously untrue in a mixed-flow impeller, because the flow enters in a nearly axial direction and leaves in a direction which is neither radial nor axial.

The integration can be performed without these assumptions, however, by recognizing that Eq. (B6) can be rewritten as

$$\rho\vec{U}\cdot(\vec{W}\cdot\nabla)\vec{W} = \rho\omega\vec{W}\cdot\nabla(rW_\theta) = \nabla\cdot(\rho\vec{W}UW_\theta) \qquad (B7)$$

where the steady-state continuity equation $\nabla\cdot(\rho\vec{W}) = 0$ was used to obtain the last equality.

The second term on the right of Eq. (B5) is just twice the integrand in Eq. (A2), so that

$$\rho\vec{U}\cdot(2\vec{\omega}\times\vec{W}) = 2\rho\omega^2 rW_r = \rho\vec{W}\cdot\nabla(U^2) \qquad (B8)$$

Therefore, substituting Eqs. (B7) and (B8) into Eq. (B5), integrating over the rotating control volume and using Gauss's and Reynolds' theorems, one obtains

$$\int\rho I\vec{W}\cdot\vec{n}\,dS' = \int\vec{n}\cdot\vec{\vec{\tau}}\cdot\vec{W}\,dS' \qquad (B9)$$

This is precisely the result derived in the body of this paper (cf. Eq. (29)). Thus Eq. (33) of this paper, which follows from it, is the correct form of the so-called "Euler turbomachinery equation with viscous correction," rather than Urbach's Eq. (40).