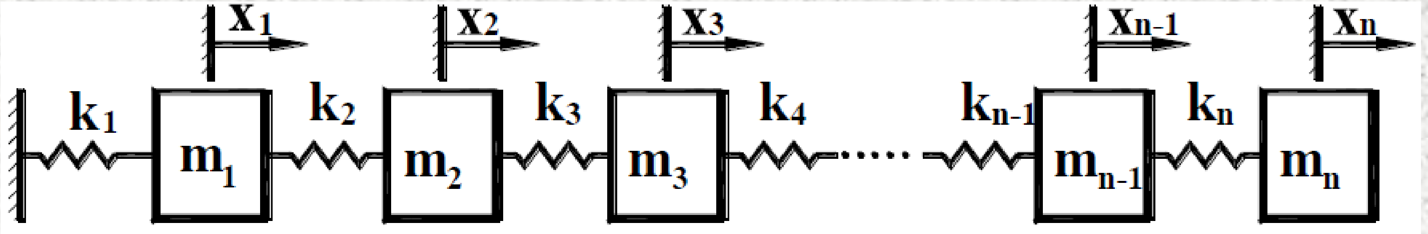
THEORY OF MACHINES AND MECHANISMS II

Mechanical IV/I

Chapter 8

Vibration of Multi Degree of Freedom Systems

8.1 Equations of Motion in Matrix Form



$$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2)$$

$$m_2\ddot{x}_2 = k_2(x_1 - x_2) - k_3(x_2 - x_3)$$

$$m_3\ddot{x}_3 = k_3(x_2 - x_3) - k_4(x_3 - x_4)$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$m_n\ddot{x}n = k_n(x_{n-1} - x_n)$$

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0$$

$$m_3\ddot{x}_3 - k_3x_2 + (k_3 + k_4)x_3 - k_4x_4 = 0$$

$$\cdots \quad \cdots \quad \cdots$$

$$m_n\ddot{x}_n - k_nx_{n-1} + k_nx_n = 0$$

The above equations are the required equations of motion which can also be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 & .. & 0 \\ 0 & m_2 & 0 & .. & 0 \\ 0 & 0 & m_3 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ .. \\ \ddot{x}_n \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 & .. & 0 \\ -k_2 & k_2+k_3 & -k_3 & .. & 0 \\ 0 & -k_3 & k_3+k_4 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & k_n \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

where

[M] is a square matrix of n^{th} order and with only diagonal elements in this case;

[K] is a square stiffness matrix also of n^{th} order, this matrix being a symmetrical one;

{x} is a column matrix of *n* elements corresponding to the dynamic displacements of the respective *n* masses.

8.2 Flexibility and Stiffness Matrix

The equations of motion of several degrees of freedom can be expressed in terms of flexibility influence coefficients.

The flexibility influence coefficients a_{ij} is defines as the displacement at point *i* due to a unit force applied at point *j*.

With forces f_1, f_2, and f_3 acting at stations *1*, *2*, and *3*, the principle of superposition can be applied to determine the displacements in terms of flexibility influence coefficients

$$
\begin{aligned}
x_1 &= a_{11} f_1 + a_{12} f_2 + a_{13} f_3 \\
x_2 &= a_{21} f_1 + a_{22} f_2 + a_{23} f_3 \\
x_3 &= a_{31} f_1 + a_{32} f_2 + a_{33} f_3
\end{aligned}
$$

In matrix notation, the equation is

$$\{x\} = [A]\{f\}$$

where

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is the flexibility matrix

Similarly the stiffness coefficient k_{ij} is defines as force or moment required at the point i when a unit displacement (rectilinear or angular) is at point j, with all other coordinates fixed, i.e., zero displacements of all other points.

$$f_1 = k_{11} x_1 + k_{12} x_2 + k_{13} x_3$$

$$f_2 = k_{21} x_1 + k_{22} x_2 + k_{23} x_3$$

$$f_3 = k_{31} x_1 + k_{32} x_2 + k_{33} x_3$$

In matrix notation, the equation is

$$\{f\} = [K]\ \{x\}$$

where

$$[K] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

is the stiffness matrix.

$$\{x\} = [A]\{f\}$$

If the above equation is premultiplied by the inverse of the flexibility matrix $[A]^{-1}$, we obtain the equation

$$[A]^{-1}\{x\} = \{f\} = [K]\{x\}$$

$$[A]^{-1} = [K] \quad \text{or,} \quad [K]^{-1} = [A]$$

We thus find that the flexibility matrix and the stiffness matrix are inverse of each other.

Reciprocity Theorem

The reciprocity theorem states that the deflection at any point in the system due to unit load acting at any other point of the same system is equal to the deflection at the second point due to unit load acting on the first point.

$$a_{ij} = a_{ji};\ k_{ij} = k_{ji}$$

8.3 Natural Frequencies and Mode shapes Eigenvalues and Eigenvectors

For the undamped system of several degrees of freedom, the equations of motion expressed in matrix form is

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

If we premultiply the above equation by $[M]^{-1}$, we obtain the following terms

$$[M]^{-1}[M] = [I], \text{ a unit matrix}$$
$$[M]^{-1}[K] = [D], \text{ a dynamic matrix}$$

Thus, the above matrix equation becomes

$$[I]\{\ddot{x}\} + [D]\{x\} = \{0\}$$

Assuming harmonic motion, i.e., $\ddot{x} = -\lambda X$, where $\lambda = \omega^2$

$$[D - \lambda I]\{X\} = \{0\}$$

From the above equation, we from the determinant

$$|D - \lambda I| = 0$$

which is the characteristic equation of the system. The roots λ_i of the characteristic equation are called eigenvalues and the natural frequencies of the system are determined from the by the relationship

$$\lambda_i = \omega_i^2$$

$$[D - \lambda I]\{X\} = \{0\}$$

By substituting λ_i into the matrix equation, we obtain the corresponding mode shape X_i which is called the eigenvector. Thus for an *n*-degrees of freedom system, there will be *n* eigenvalues and *n* eigenvectors.

8.4 Orthogonal Properties of the Eigenvectors

The normal modes, or the eigenvectors of the system, can be shown to be orthogonal with respect to the mass and stiffness matrices.

Let the equation for the i^{th} mode be

$$[K]\{X\}_i = \lambda_i [M]\{X\}_i \quad \ldots\ldots\ldots\ldots (1)$$

Premultiply by the transpose of mode j

$$\{X\}'_j [K]\{X\}_i = \lambda_i \{X\}'_j [M]\{X\}_i \quad \ldots\ldots\ldots\ldots (2)$$

Next, start with the equation for the j^{th} mode and premultiply by $\{X\}'_i$ to obtain

$$\{X\}'_i [K]\{X\}_j = \lambda_j \{X\}'_i [M]\{X\}_j \quad \ldots\ldots\ldots\ldots (3)$$

Since [K] and [M] are symmetric matrices, the following relationships hold

$$\begin{aligned} \{X\}'_j [M]\{X\}_i &= \{X\}'_i [M]\{X\}_j \\ \{X\}'_j [K]\{X\}_i &= \{X\}'_i [K]\{X\}_j \end{aligned} \quad \ldots\ldots\ldots\ldots (4)$$

Subtracting Equation (3) from Equation (2), we obtain

$$0 = (\lambda_i - \lambda_j)\ \{X\}'_i\ [M]\ \{X\}_j \quad \ldots\ldots\ldots\ (5)$$

If $\lambda_i \neq \lambda_j$, the above equation requires that

$$\{X\}'_i\ [M]\ \{X\}_j = 0 \quad \ldots\ldots\ldots\ (6)$$

It is also evident form Equation (2) or Equation (3) that as a consequence of Equation (6)

$$\{X\}'_i\ [K]\ \{X\}_j = 0 \quad \ldots\ldots\ldots\ (7)$$

Equations (6) and (7) define the orthogonal character of the normal modes.

If $i = j$, Equation (5) is satisfied for any finite value of the products given by Equations (6) or (7). We therefore let

$$\begin{aligned} \{X\}'_i\ [M]\ \{X\}_i &= [M]_i \\ \{X\}'_i\ [K]\ \{X\}_i &= [K]_i \end{aligned} \quad \ldots\ldots\ldots\ (8)$$

These are called the generalized mass and the generalized stiffness respectively.

Example 8.1

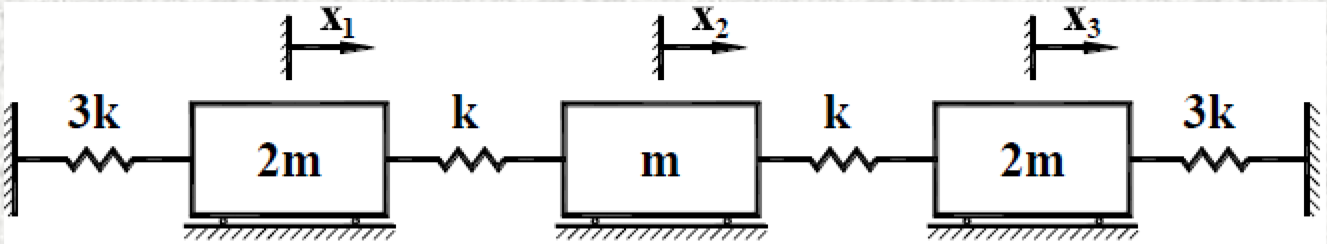
For the system shown in **Figure E8.1**

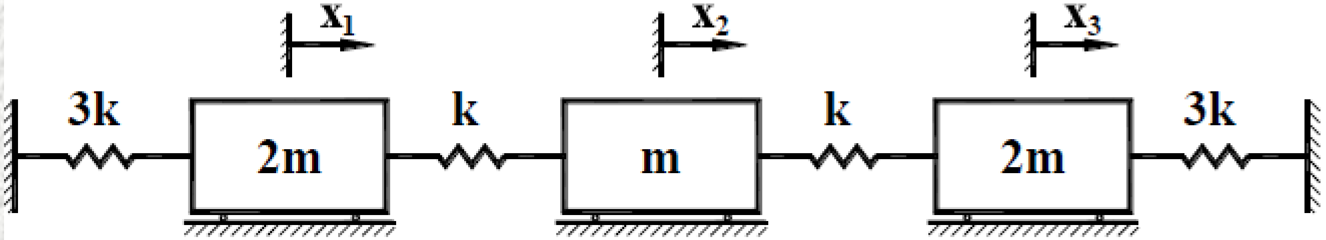
a) Write three differential equations of motion by Newton's second law of motion and put these in matrix form.

b) Determine the flexibility matrix and write the differential equation of motion in matrix form in terms of flexibility matrix.

c) Determine the stiffness matrix and write the differential equation of motion in matrix form in terms of stiffness matrix.

d) Show that the product of stiffness matrix and flexibility matrix is a unit matrix.



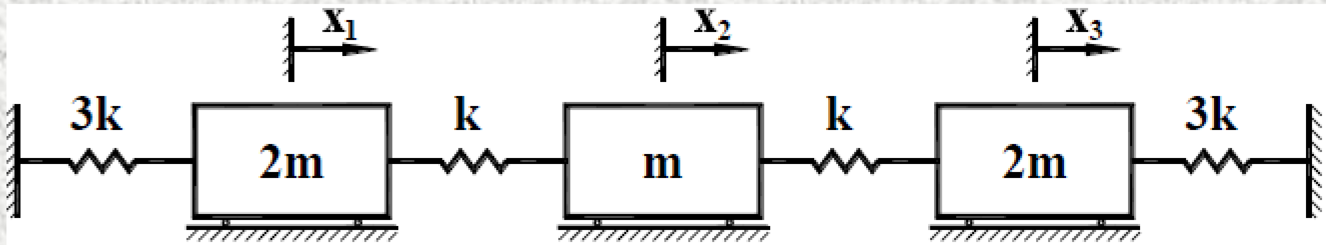


$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 4k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 4k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$[A] = \begin{bmatrix} \frac{7}{24k} & \frac{1}{6k} & \frac{1}{24k} \\ \frac{1}{6k} & \frac{2}{3k} & \frac{1}{6k} \\ \frac{1}{24k} & \frac{1}{6k} & \frac{7}{24k} \end{bmatrix}$$

Example 8.2

For the system shown in **Figure E8.2**, determine the natural frequency and corresponding mode shapes.



$$\begin{bmatrix} \dfrac{k}{m} \\[2ex] \dfrac{2k}{m} \\[2ex] \dfrac{3k}{m} \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$