THEORY OF MACHINES AND MECHANISMS II

Mechanical IV/I

Chapter 10

Vibration of Continuous Systems

In the previous chapters mass, stiffness and damping of vibrating systems were assumed to be acting only at certain discrete points. It was an ideal approach to analyze the vibrating system.

There are systems such as beams, cables, rods, etc., which have their mass and elasticity distributed continuously through the system. These systems are assumed to be homogeneous and isotropic, obeying Hooke's law within the elastic limit.

To specify the position of every particle in the elastic body, an infinite number of coordinates are necessary, and such therefore posses an infinite number of degrees of freedom.

The vibratory motions of continuous systems are described by space and time and partial differential equations are formulated for analysis of the systems. Partial differential equations consist of many constants which can be determined from boundary conditions and initial conditions as well.

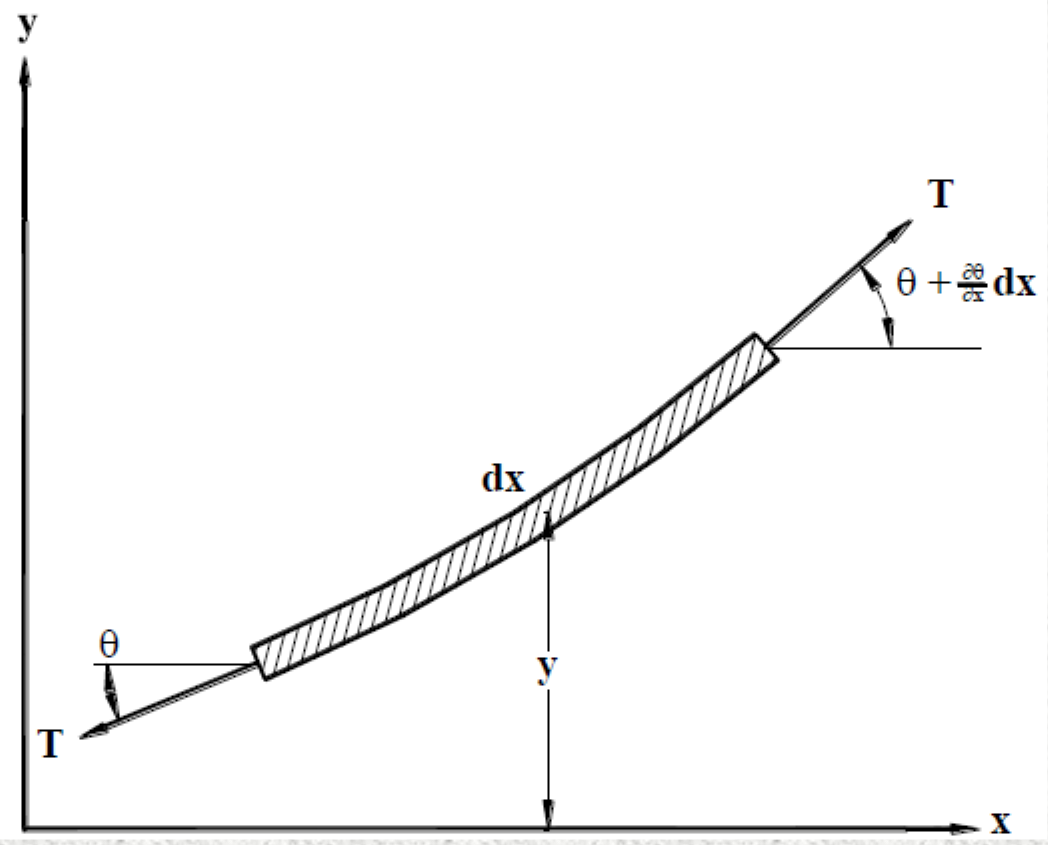
10.1 Lateral Vibration of a String

A flexible string of mass ρ per unit length is stretched under tension T. By assuming the lateral deflection y of the string to be small, the change in tension with deflection is negligible and can be ignored.

A free body diagram of an elementary length dx of the string is shown in **Figure**. Assuming small deflections and slopes, the equation of motion in the y- direction is

$$T\left(\theta + \frac{\partial \theta}{\partial x}dx\right) - T\theta = \rho dx \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$$



Since the slope of the string is $\theta = \partial y/\partial x$, the above equation reduces to

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

This is one dimensional wave equation and where $c = \sqrt{T/\rho}$ is the velocity of wave propagation along the string.

One method of solving partial differential equations is that of separation of variables. In this method the solution is assumed in the form

$$y(x, t) = Y(x)\, G(t)$$

By substituting into the partial differential equation, we obtain

$$\frac{1}{Y}\frac{d^2 Y}{dx^2} = \frac{1}{c^2}\frac{1}{G}\frac{d^2 G}{dt^2}$$

Sine the left side of this equation is independent of t, whereas the right side is independent of x, it follows that each side must be a constant. Letting this constant be $-(\omega/c)^2$ we obtain two ordinary differential equations

$$\frac{d^2Y}{dx^2} + \left(\frac{\omega}{c}\right)^2 Y = 0$$

$$\frac{d^2G}{dt^2} + \omega^2 G = 0$$

with the general solutions

$$Y = A \sin\frac{\omega}{c}x + B \cos\frac{\omega}{c}x$$

$$G = C \sin \omega t + D \cos \omega t$$

Hence the general solution can be written as

$$y(x, t) = [A \sin\frac{\omega}{c}x + B \cos\frac{\omega}{c}x]\,[C \sin \omega t + D \cos \omega t]$$

The arbitrary constants A, B, C, D in the above equation can be determined from the boundary conditions and the initial conditions.

For example, if the string is stretched between two fixed points with distance *l* between them, the boundary conditions are between them, the boundary conditions are *y (0, t)* = *y (l, t)* = 0.

The condition that *y (0, t)* = 0 will require that *B* = 0 so that the solution will appear as

$$y(x,t) = \left[C\sin\omega t + D\cos\omega t\right]\sin\frac{\omega}{c}x$$

The condition *y (l, t)* = 0 then leads to the equation

$$\sin\frac{\omega}{c}l = 0$$

$$\text{or,} \quad \frac{\omega_n l}{c} = \frac{2\pi l}{\lambda} = n\pi, \quad n = 1, 2, 3, \ldots\ldots$$

and $\lambda = c/f$ is the wave length of and f is the frequency of oscillation. Each n represents a normal mode vibration with natural frequency determined from the equation

$$f_n = \frac{n}{2l}c = \frac{n}{2l}\sqrt{\frac{T}{\rho}},$$

The mode shape is sinusoidal with the distribution

$$Y = \sin n\pi \frac{x}{l}$$

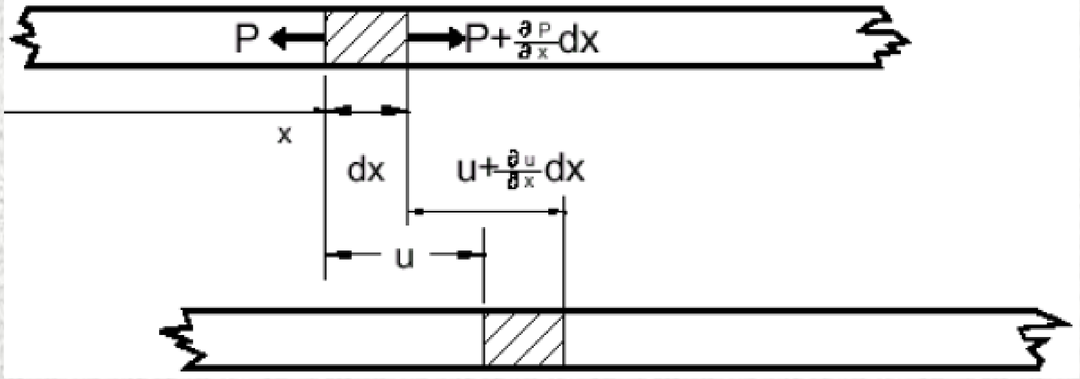
In the more general case of free vibration initiated in any manner, the solution will contain many of the normal modes and the equation for the displacement may be written as

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \sin \omega_n t + D_n \cos \omega_n t\right)\sin n\pi \frac{x}{l}$$

Fitting this to the initial conditions, the C_n and D_n can be evaluated.

10.2 Longitudinal Vibration in Rods

Consider a thin rod which is uniform along its length. Due to axial forces there will be displacements *u* along the rod will be a function of both the position *x* and the time *t*.



Since from Hooke's law the ratio of unit stress to unit strain is equal to the modulus of elasticity *E*, we can write

$$\frac{\partial u}{\partial x} = \frac{P}{AE}$$

Differentiating with respect to *x*

$$AE\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x}$$

We now apply Newton's law of motion for the element and equate the unbalanced force to the product of mass and acceleration of the element

$$\frac{\partial P}{\partial x}dx = \rho A dx \frac{\partial^2 u}{\partial t^2}$$

Eliminating $\partial P/\partial x$ between the above Equations, we obtain the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{E}{\rho}\right)\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The velocity of propagation of the displacement or stress wave in the rod is then equal to

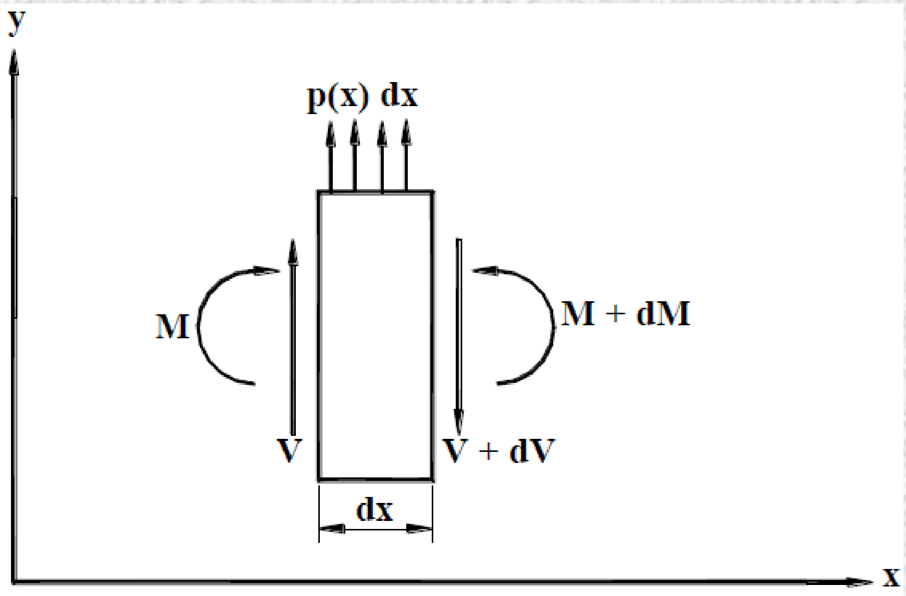
$$c = \sqrt{\frac{E}{\rho}}$$

The general solution becomes

$$u(x, t) = [A \sin\frac{\omega}{c}x + B \cos\frac{\omega}{c}x]\,[C \sin \omega t + D \cos \omega t]$$

10.3 Lateral Vibration in Beams

To determine the differential equation for the lateral vibration of beams, consider the forces and moments acting on an element of the beam shown in **Figure**.



V and *M* are shear force and bending moments, respectively, and *p(x)* represents the load per unit length of the beam.

By summing the forces in *y*-direction

$$dV - p(x)dx \quad = 0$$

By summing the moments about any point on the right face of the element

$$dM - Vdx - \frac{1}{2}p(x)(dx)^2 \qquad = 0$$

In the limiting process these equations result in the following relationships

$$\frac{dV}{dx} = p(x); \qquad \frac{dM}{dx} = V$$

From the above Equations, we obtain the following

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = p(x)$$

The bending moment is related to the curvature by flexure equation,

$$M = EI\frac{d^2y}{dx^2}$$

From the above Equations

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) = p(x)$$

For a beam vibrating about its static equilibrium position under its own weight, the load per unit length is equal to the inertia load due to its mass and acceleration. Since the inertia force is in the same direction as *p(x)* we have, by assuming harmonic motion

$$p(x) \quad = m\omega^2 y$$

where *m* is the mass per unit length of the beam.

Using this relation, the equation for the lateral vibration of the beam reduces to

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2 y}{dx^2}\right) - m\omega^2 y \qquad = 0$$

In this special case where the flexure rigidity *EI* is a constant, the above equation may be written as

$$EI\,\frac{d^4 y}{dx^4} - m\omega^2 y \qquad = 0$$

On substituting $\beta^4 = \dfrac{m\omega^2}{EI}$

we obtain the fourth-order differential equation

$$\frac{d^4 y}{dx^4} - \beta^4 y = 0$$

To find its solution, let us assume $y = e^{ax}$

Substituting *y* into the differential equation, we get

$a = \pm\beta$, and $a = \pm i\beta$

Since $e^{\pm\beta x} = \cosh\beta x \pm \sinh\beta x$ and $e^{\pm i\beta x} = \cos\beta x \pm \sin\beta x$

The general solution can be written as

$$y = A\cosh\beta x + B\sinh\beta x + C\cos\beta x + D\sin\beta x$$

The natural frequencies of vibration are found to be

$$\omega_n = \beta^2\sqrt{\frac{EI}{m}}$$

where β depends on the boundary conditions of the problem.