

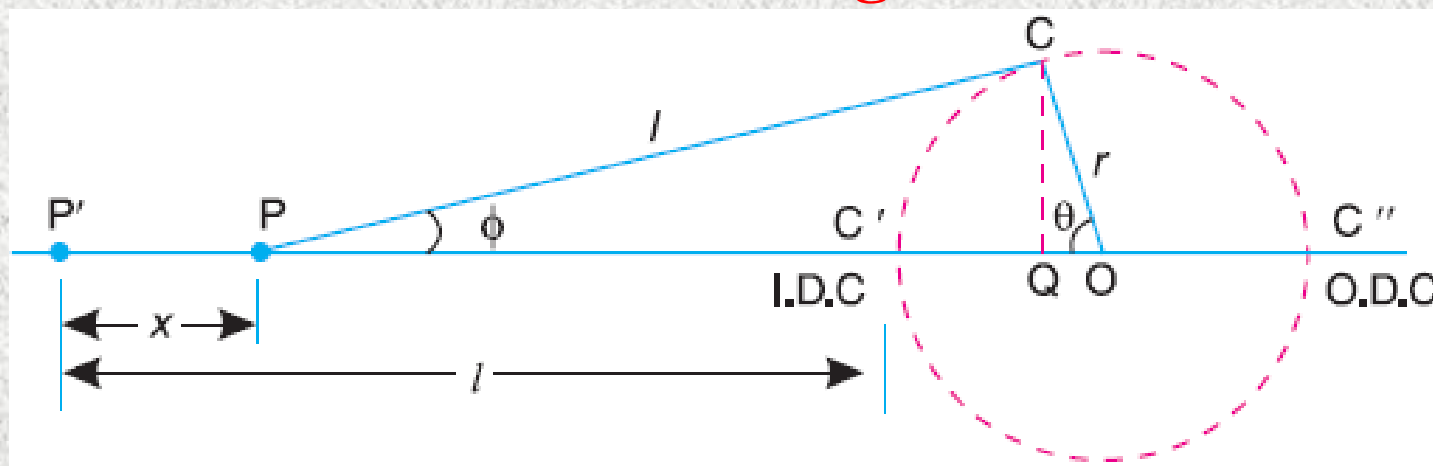
THEORY OF MECHANISM AND MACHINE II

Mechanical IV/I

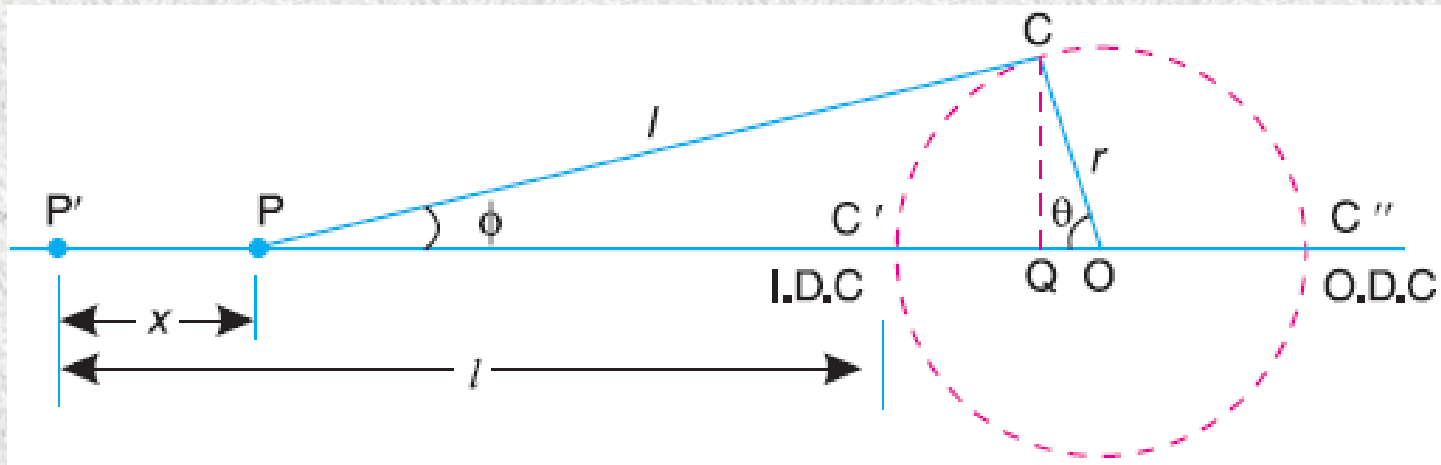
Chapter 1

Engine Force Analysis

1.1 Analytical Method for Velocity and Acceleration of the Piston and the Connecting Rod



Let OC be the crank and PC the connecting rod. Let the crank rotates with angular velocity of ω rad/s and the crank turns through an angle θ from the inner dead centre. Let x be the displacement of a reciprocating body P from I.D.C. after time t seconds, during which the crank has turned through an angle θ .



$l =$ Length of connecting rod between the centers,

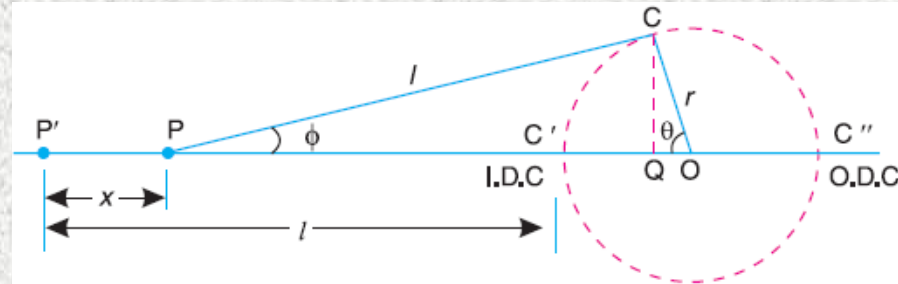
$r =$ Radius of crank or crank pin circle,

$\phi =$ Inclination of connecting rod to the line of stroke PO , and

$n =$ Ratio of length of connecting rod to the radius of crank $= l/r$.

Velocity of the piston

From the geometry of Figure 1.1,



$$\begin{aligned}
 x &= P'P = OP' - OP = (P'C' + C'O) - (PQ + QO) \\
 &= (l + r) - (l \cos \phi + r \cos \theta) \\
 &= r (1 - \cos \theta) + l (1 - \cos \phi) \\
 &= r \left[(1 - \cos \theta) + \frac{l}{r} (1 - \cos \phi) \right] \\
 &= r [(1 - \cos \theta) + n (1 - \cos \phi)] \quad \dots\dots\dots [1.1]
 \end{aligned}$$

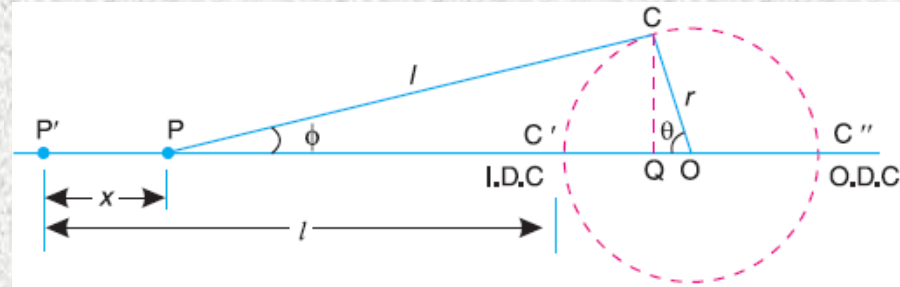
From triangles CPQ and CQO,

$$\begin{aligned}
 CQ &= l \sin \phi = r \sin \theta \\
 \text{or, } \frac{l}{r} &= \frac{\sin \theta}{\sin \phi} = n
 \end{aligned}$$

$$\therefore \sin \phi = \frac{\sin \theta}{n} \quad \dots\dots\dots [1.2]$$

We know that,

$$\cos \phi = (1 - \sin^2 \phi)^{\frac{1}{2}} = \left(1 - \frac{\sin^2 \theta}{n^2}\right)^{\frac{1}{2}}$$



Expanding the above expression by binomial theorem and neglecting higher order terms, we get

$$\cos \phi = 1 - \frac{1}{2} \frac{\sin^2 \theta}{n^2} + \dots \dots \dots [1.3]$$

Substituting $\cos \phi$ into **Equation [1.1]**, we get

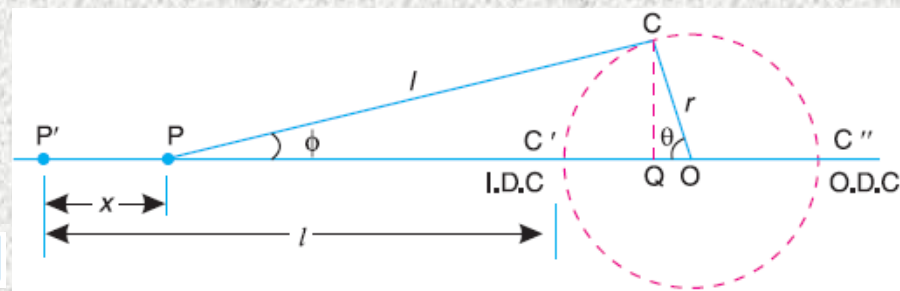
$$x = r \left[(1 - \cos \theta) + n \frac{\sin^2 \theta}{2n^2} \right] = r \left[(1 - \cos \theta) + \frac{\sin^2 \theta}{2n} \right] \dots \dots \dots [1.4]$$

Differentiating **Equation [1.4]** with respect to θ , we get

$$\frac{dx}{d\theta} = r \left[\sin \theta + \frac{1}{2n} \times 2 \sin \theta \cos \theta \right] = r \left[\sin \theta + \frac{\sin 2\theta}{2n} \right] \dots \dots \dots [1.5]$$

Velocity of P with respect to O or velocity of the piston P is then given by

$$V_{PO} = V_P = \frac{dx}{dt} = \frac{dx}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{dx}{d\theta} \dots\dots\dots [1.6]$$



$$V_{PO} = V_P = \omega r \left[\sin\theta + \frac{\sin 2\theta}{2n} \right] \dots\dots\dots [1.7]$$

Acceleration of the piston

Since the acceleration is the rate of change of velocity, therefore acceleration of the piston P,

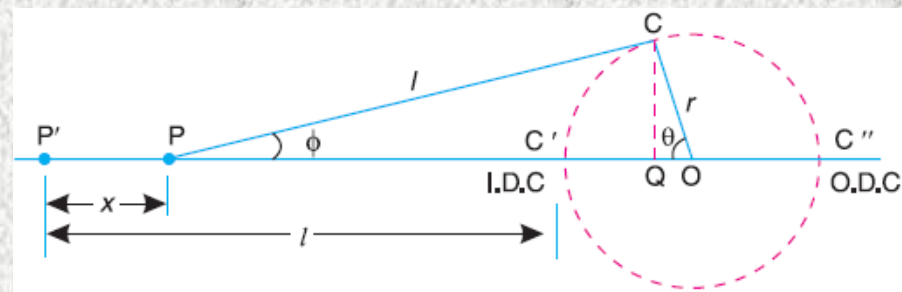
$$a_P = \frac{dV_P}{dt} = \frac{dV_P}{d\theta} \cdot \frac{d\theta}{dt} = \omega \frac{dV_P}{d\theta} \dots\dots\dots [1.8]$$

Differentiating **Equation [1.7]** with respect to θ , we get

$$\frac{dV_P}{d\theta} = \omega r \left[\cos\theta + \frac{\cos 2\theta}{n} \right] \dots\dots\dots [1.9]$$

Substituting $dV_P/d\theta$ from **Equation [1.8]** into **Equation [1.9]**, we get

$$a_P = \omega^2 r \left[\cos\theta + \frac{\cos 2\theta}{n} \right] \dots\dots\dots [1.10]$$



Angular Velocity and Acceleration of the Connecting Rod

From **Equation 1.2**,

$$\sin\phi = \frac{\sin\theta}{n}$$

Differentiating both sides with respect to time t ,

$$\cos\phi \frac{d\phi}{dt} = \frac{\cos\theta}{n} \frac{d\theta}{dt} = \frac{\cos\theta}{n} \omega \dots\dots\dots [1.11]$$

Since the angular velocity of the connecting rod PC is same as the angular velocity of point P with respect to C and is equal to $d\phi/dt$, therefore angular velocity of the connecting rod

$$\omega_{PC} = \frac{d\phi}{dt} = \frac{\omega}{n} \frac{\cos\theta}{\cos\phi} \dots\dots\dots [1.12]$$

We know that,

$$\begin{aligned} \cos\phi &= (1 - \sin^2\phi)^{\frac{1}{2}} = \left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}} \\ \therefore \omega_{PC} &= \frac{\omega}{n} \frac{\cos\theta}{\left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}} = \frac{\omega \cos\theta}{(n^2 - \sin^2\theta)^{\frac{1}{2}}} \dots\dots\dots [1.13] \end{aligned}$$

Angular acceleration of the connecting rod PC is then given by

$$\alpha_{PC} = \frac{d\omega_{PC}}{dt} = \frac{d\omega_{PC}}{d\theta} \frac{d\theta}{dt} = \omega \frac{d\omega_{PC}}{d\theta} \dots\dots\dots [1.14]$$

Now differentiating **Equation [1.13]**, we get

$$\begin{aligned}\frac{d\omega_{PC}}{d\theta} &= \frac{d}{d\theta} \left[\frac{\omega \cos\theta}{(n^2 - \sin^2\theta)^{\frac{1}{2}}} \right] \\ &= \omega \left[\frac{(n^2 - \sin^2\theta)^{\frac{1}{2}} \times (-\sin\theta) - \cos\theta \times \frac{1}{2}(n^2 - \sin^2\theta)^{-\frac{1}{2}} \times -2\sin\theta \cos\theta}{n^2 - \sin^2\theta} \right] \\ &= -\omega \sin\theta \left[\frac{(n^2 - \sin^2\theta)^{\frac{1}{2}} - (n^2 - \sin^2\theta)^{-\frac{1}{2}} \times 2\cos^2\theta}{n^2 - \sin^2\theta} \right]\end{aligned}$$

$$\frac{d\omega_{PC}}{d\theta} = -\omega \sin\theta \left[\frac{(n^2 - \sin^2\theta) - \cos^2\theta}{(n^2 - \sin^2\theta)^{\frac{3}{2}}} \right] = -\frac{\omega \sin\theta (n^2 - 1)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}} \dots\dots\dots [1.15]$$

Substituting $d\omega_{PC}/d\theta$, from **Equation [1.15]** into **Equation [1.14]**, we get

$$\alpha_{PC} = -\frac{\omega^2 \sin\theta (n^2 - 1)}{(n^2 - \sin^2\theta)^{\frac{3}{2}}} \dots\dots\dots [1.16]$$

Example 1.1

The stroke of a steam engine is 600 mm and the length of connecting rod is 1.5 m. The crank rotates at 180 r.p.m. Determine:

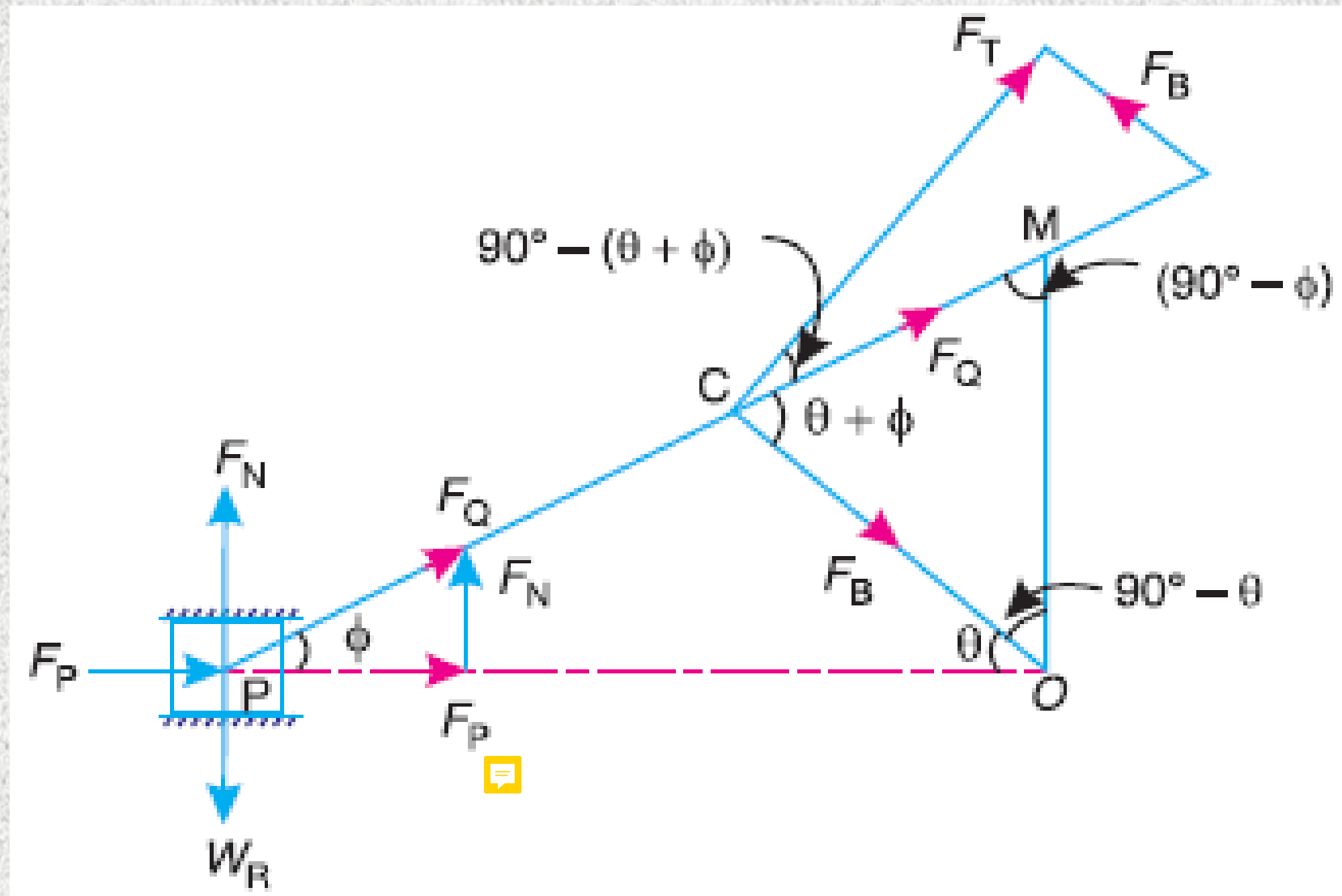
- a) velocity and acceleration of the piston when crank has travelled through an angle of 40° from inner dead centre, and
- b) the position of the crank for zero acceleration of the piston.

4.1918 m/s,

85.356 m/s²,

79.27°

1.2 Forces on the Reciprocating Parts of an Engine

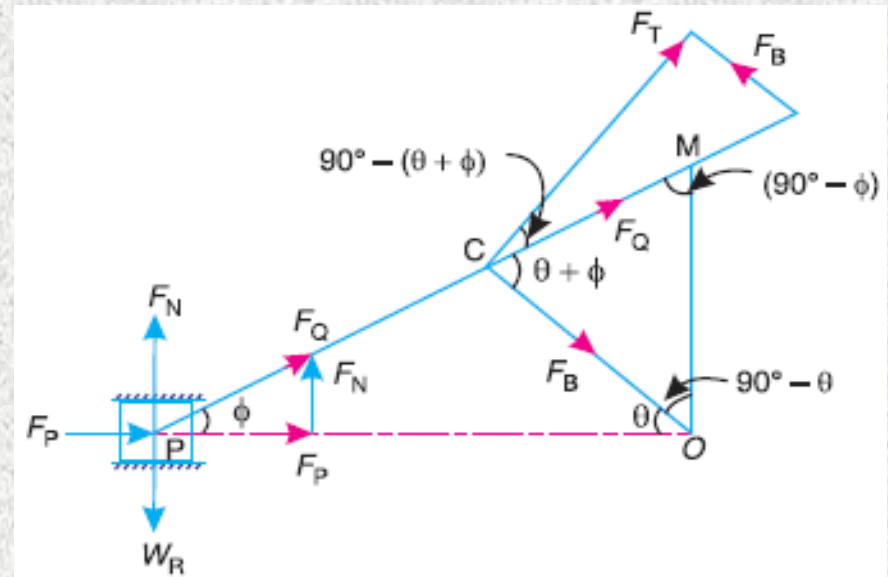


Piston effort

It is the net force acting on the piston or crosshead pin, along the line of stroke. It is denoted by F_P .

We know that acceleration of the reciprocating parts

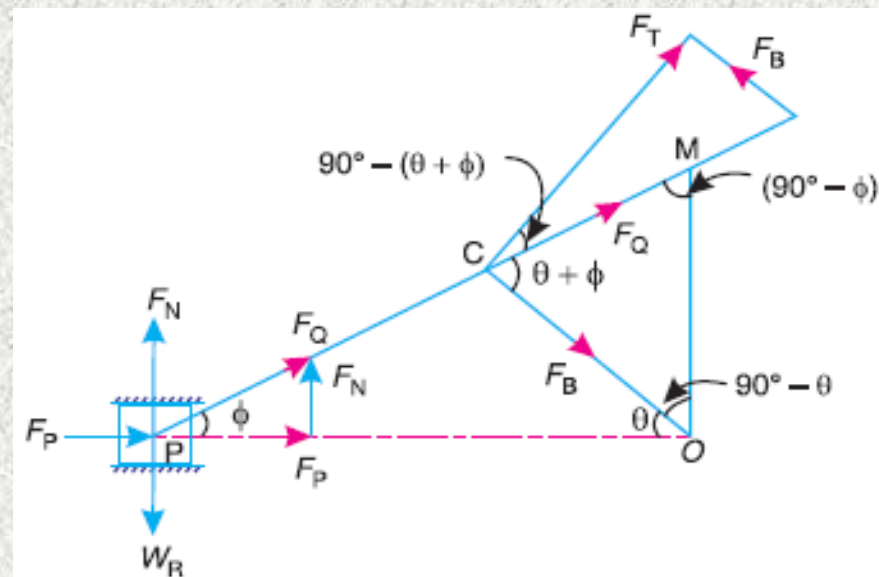
$$a_R = a_P = \omega^2 r \left[\cos\theta + \frac{\cos 2\theta}{n} \right]$$



Accelerating force or inertia force of the reciprocating parts

$$F_I = m_R a_R = m_R \omega^2 r \left[\cos\theta + \frac{\cos 2\theta}{n} \right]$$

It may be noted that in a horizontal engine, the reciprocating parts are accelerated from rest, when the piston moves from inner dead centre to outer dead centre. It is, then, retarded when the piston moves from outer dead centre to inner dead centre.




The inertia force due to the acceleration of the reciprocating parts, opposes the force on the piston due to the difference of pressures in the cylinder on the two sides of the piston. On the other hand, the inertia force due to retardation of the reciprocating parts, helps the force on the piston.

$$\begin{aligned}
 \text{Piston effort, } F_P &= \text{Net load on the piston} \mp \text{Inertia force} \\
 &= F_L \mp F_I \quad (\text{Neglecting frictional resistance}) \\
 &= F_L \mp F_I - R_F \quad (\text{Considering frictional resistance})
 \end{aligned}$$

where R_F = Frictional resistance

The – ve sign is used when the piston is accelerated, and + ve sign is used when the piston is retarded.

In a **double acting reciprocating steam engine**, net load on the piston, 

$$F_L = p_1 A_1 - p_2 A_2 = p_1 A_1 - p_2 (A_1 - a)$$

where p_1, A_1 = Pressure and cross-sectional area on the back end side of the piston,
 p_2, A_2 = Pressure and cross-sectional area on the crank end side of the piston,
 a = Cross-sectional area of the piston rod.

If ' p ' is the **net pressure of steam or gas on the piston** and D is diameter of the piston, then net load on the piston,

$$F_L = \text{Pressure} \times \text{Area} = p \times \frac{\pi}{4} \times D^2$$

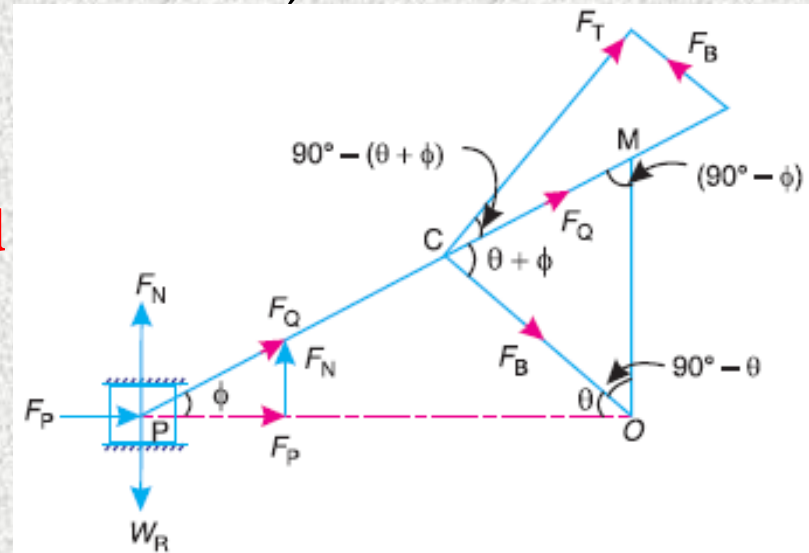
In case of a **vertical engine**, the weight of the reciprocating parts assists the piston effort during the downward stroke (i.e. when the piston moves from top dead centre to bottom dead centre) and opposes during the upward stroke of the piston (i.e. when the piston moves from bottom dead centre to top dead centre).

$$F_P = F_L \mp F_I \pm W_R - R_F$$

Force acting along the connecting rod

It is denoted by F_Q .

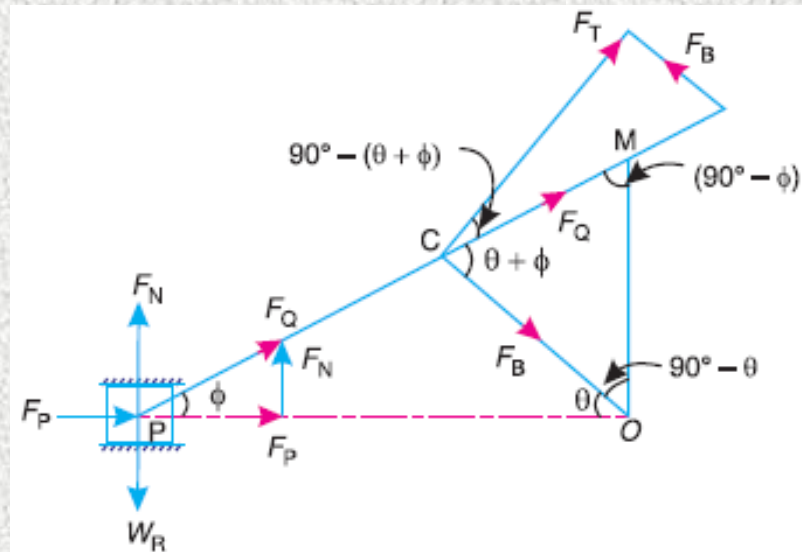
$$F_Q = \frac{F_P}{\cos \phi} = \frac{F_P}{\sqrt{1 - \frac{\sin^2 \theta}{n^2}}}$$



Thrust on the sides of the cylinder walls or normal reaction on the guide bars

It is denoted by F_N .

$$F_N = F_Q \sin \phi = \frac{F_P}{\cos \phi} \times \sin \phi = F_P \tan \phi$$



Crank-pin effort and thrust on crank shaft bearings

The force acting on the connecting rod F_Q may be resolved into two components, one perpendicular to the crank and the other along the crank. The component of F_Q perpendicular to the crank is known as crank-pin effort and it is denoted by F_T . The component of F_Q along the crank produces a thrust on the crank shaft bearings and it is denoted by F_B .

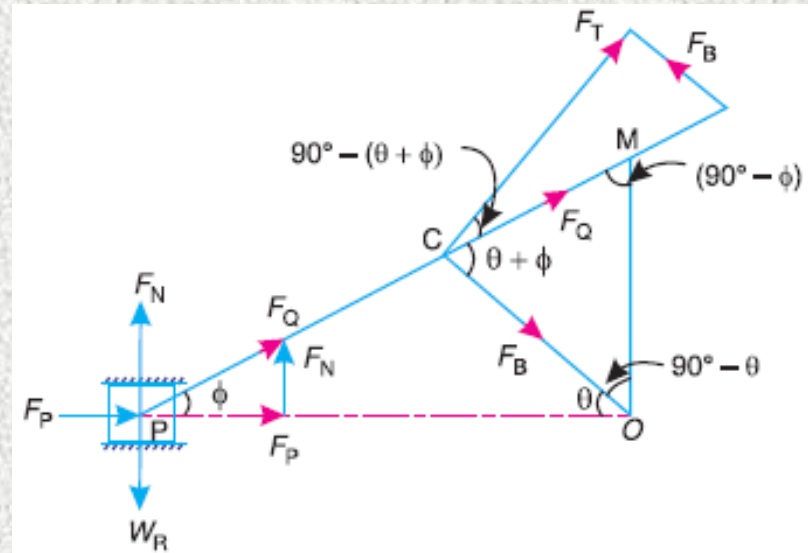
$$F_T = F_Q \sin(\theta + \phi) = \frac{F_P}{\cos \phi} \times \sin(\theta + \phi)$$

$$F_B = F_Q \cos(\theta + \phi) = \frac{F_P}{\cos \phi} \times \cos(\theta + \phi)$$

Crank effort or turning moment or torque on the crank shaft

The product of the crankpin effort (F_T) and the crank pin radius (r) is known as crank effort or turning moment or torque on the crank shaft

$$\begin{aligned} \text{Crank effort, } T &= F_T \times r \\ &= \frac{F_P}{\cos \phi} \times \sin(\theta + \phi) \times r \end{aligned}$$



$$\begin{aligned}
 \text{Crank effort, } T &= F_T \times r \\
 &= \frac{F_P}{\cos\phi} \times \sin(\theta + \phi) \times r \\
 &= \frac{F_P(\sin\theta \cos\phi + \cos\theta \sin\phi)}{\cos\phi} \times r \\
 &= F_P(\sin\theta + \cos\theta \tan\phi) \times r
 \end{aligned}$$

As we know

$$\sin\phi = \frac{\sin\theta}{n}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sin\phi}{\sqrt{1 - \sin^2\phi}} = \frac{\sin\theta}{n} \times \frac{n}{\sqrt{n^2 - \sin^2\theta}} = \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Substituting the value of $\tan\phi$, we get the expression for crank effort as

$$T = F_P \left(\sin\theta + \cos\theta \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}} \right) \times r$$

$$\therefore T = F_P \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$

Example 1.2

A petrol engine 90 mm in diameter and 120 mm stroke has a connecting rod of 240 mm length. The piston has a mass of 1 kg and the speed is 1800 rpm. On the explosion stroke with the crank at 30° from top dead centre, the gas pressure is 0.5 N/mm^2 . Determine:

- the resultant load on the gudgeon pin,
- the thrust on the cylinder walls
- the crank effort at the given position of the crank, and
- the speed, above which other things remaining same, the gudgeon pin load would be reversed in direction.

1076.604 N, 135.639 N, 39.04 N.m, 2208.64 rpm

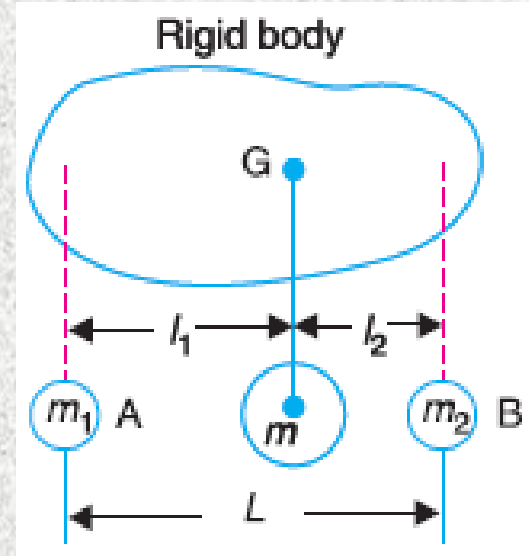
1.3 Equivalent Dynamical System

In order to determine the motion of a rigid body, under the action of external forces, it is usually convenient to replace the rigid body by two masses placed at a fixed distance apart, in such a way that,

- the sum of their masses is equal to the total mass of the body;
- the centre of gravity of the two masses coincides with that of the body ; and
- the sum of mass moment of inertia of the masses about their centre of gravity is equal to the mass moment of inertia of the body.

When these three conditions are satisfied, then it is said to be an **equivalent dynamical system**.

Consider a rigid body, having its centre of gravity at G , as shown in **Figure**.



Let	$m =$	Mass of the body,
	$k_G =$	Radius of gyration about its centre of gravity G ,
	m_1 and $m_2 =$	Two masses which form a dynamical equivalent system,
	$l_1 =$	Distance of mass m_1 from G ,
	$l_2 =$	Distance of mass m_2 from G , and
	$L =$	Total distance between the masses m_1 and m_2 .

Thus, for the two masses to be dynamically equivalent,

$$m_1 + m_2 = m \quad \text{..... [1.28]}$$

$$m_1 l_1 = m_2 l_2 \quad \text{..... [1.29]}$$

$$\text{and} \quad m_1 l_1^2 + m_2 l_2^2 = m k_G^2 \quad \text{..... [1.30]}$$

From **Equations [1.28] and [1.29]**,

$$m_1 = \frac{m l_2}{l_1 + l_2} \quad \text{..... [1.31]}$$

$$\text{and} \quad m_2 = \frac{m l_1}{l_1 + l_2} \quad \text{..... [1.32]}$$

Substituting the value of m_1 and m_2 in **Equation [1.32]**, we have

$$\frac{m l_2}{l_1 + l_2} l_1^2 + \frac{m l_1}{l_1 + l_2} l_2^2 = m k_G^2$$

$$\text{or, } \frac{l_1 l_2 (l_1 + l_2)}{(l_1 + l_2)} = k_G^2$$

$$\therefore l_1 l_2 = k_G^2 \dots\dots\dots [1.33]$$

This equation gives the essential condition of placing the two masses, so that the system becomes dynamical equivalent. The distance of one of the masses (i.e. either l_1 or l_2) is arbitrary chosen and the other distance is obtained from **Equation [1.33]**.

When the radius of gyration k_G is not known, then the position of the second mass may be obtained by considering the body as a compound pendulum. As we know that the length of the simple pendulum which gives the same frequency as the rigid body (i.e. compound pendulum) is

$$L = \frac{k_G^2 + h^2}{h}$$

Replacing h by l_1 ,

$$L = \frac{k_G^2 + l_1^2}{l_1}$$

We also know that $l_1 l_2 = k_G^2$

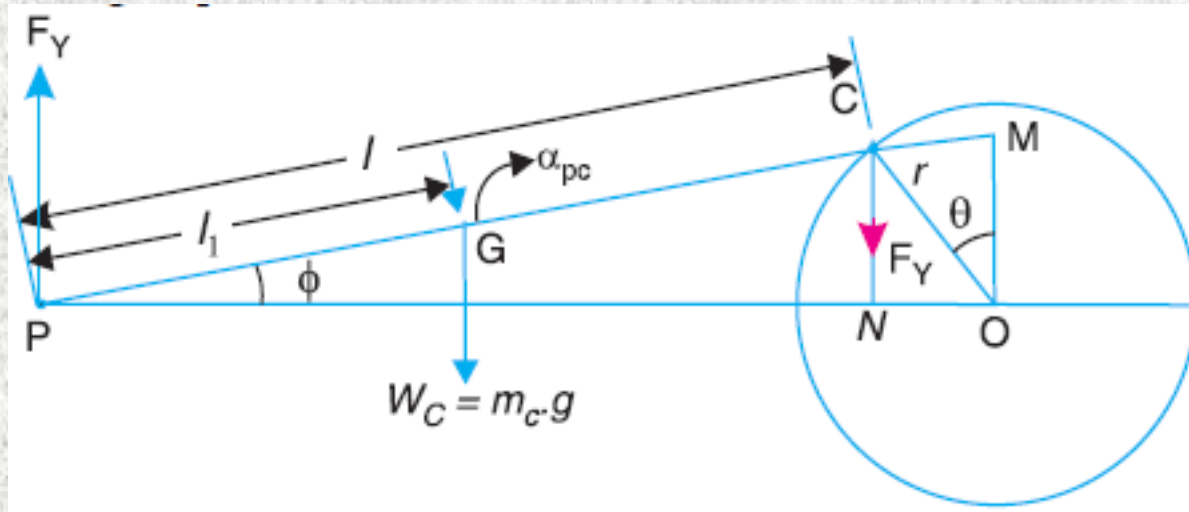
$$\therefore L = \frac{l_1 l_2 + l_1^2}{l_1} = l_1 + l_2$$

This means that the second mass is situated at the centre of oscillation or percussion of the body, which is at a distance of

$$l_2 = k_G^2 / l_1$$

1.4 Analytical Method for Inertia Torque (Considering Weight of the Connecting Rod)

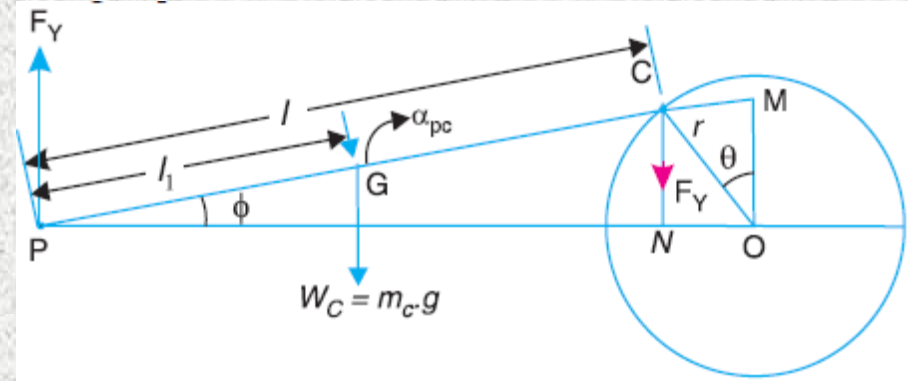
The effect of the inertia of the connecting rod on the crankshaft torque may be obtained by using equivalent dynamic system.



The mass of the connecting rod (m_c) is divided into two masses. One of the mass is placed at the crosshead pin P and the other at the crankpin C as shown in **Figure**, so that the centre of gravity of these two masses coincides with the centre of gravity of the rod G .

Mass of the connecting rod at P

$$= \frac{l - l_1}{l} \times m_c$$



The mass of the reciprocating parts (m_R) is also acting at P . Therefore, total equivalent mass of the reciprocating parts acting at P

$$= m_R + \frac{l - l_1}{l} \times m_c$$

Total inertia force of the equivalent mass acting at P ,

$$F_I = \left(m_R + \frac{l - l_1}{l} \times m_c \right) a_R$$

$$F_I = \left(m_R + \frac{l - l_1}{l} \times m_c \right) \omega^2 r \left[\cos \theta + \frac{\cos 2\theta}{n} \right]$$

Then, the corresponding torque exerted on the crank shaft,

$$T_I = F_I \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$