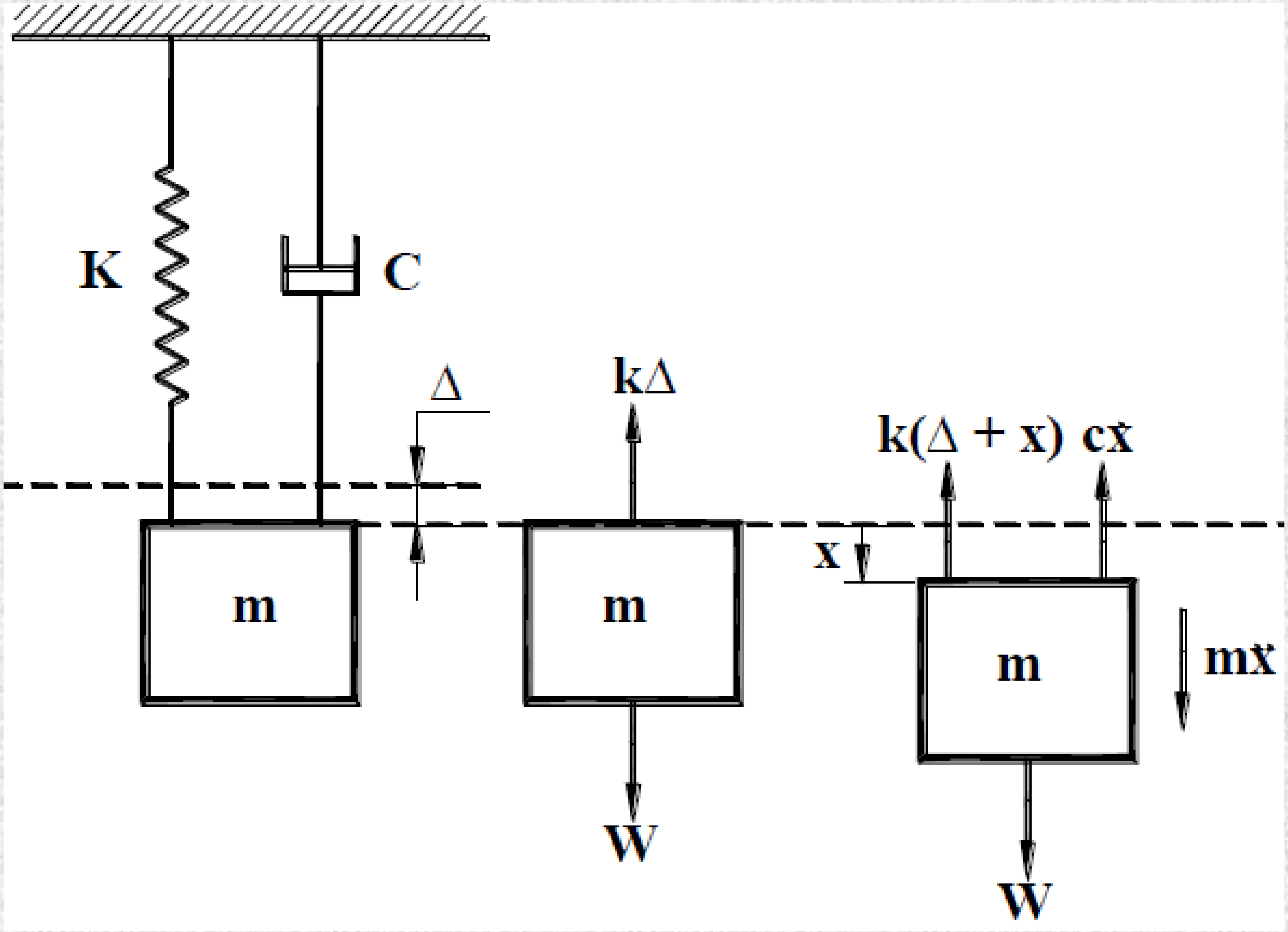
THEORY OF MACHINES AND MECHANISMS II

Mechanical IV/I

Chapter 6

Vibration of Single Degree of Freedom Systems (Damped)

6.4 Damped Free Vibrations of SDOF Systems



The equation of motion can be written as

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \text{.......... (1)}$$

This equation is a linear differential equation of second order and its solution can be written as

$$x = e^{st} \quad \text{.......... (2)}$$

where *s* is a constant to be determined.

Differentiating Equation (2) twice, we get

$$\dot{x} = s\,e^{st} \quad \text{.......... (3)}$$

$$\ddot{x} = s^2\,e^{st} \quad \text{.......... (4)}$$

Substituting Equations (2), (3) and (4) in Equation (1), we get

$$(ms^2 + cs + k)\,e^{st} = 0 \quad \text{.......... (5)}$$

Equation (5) is satisfied for each value of *t* when

$$ms^2 + cs + k \quad = 0 \qquad \ldots\ldots\ldots\ldots (6)$$

The above equation is called the characteristic equation of the system. This equation is quadratic in *s* and has two roots,

$$s_{1,2} \quad = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad \ldots\ldots\ldots\ldots (7)$$

Hence, the general solution is given by the equation

$$x \quad = Ae^{s_1 t} + Be^{s_2 t} \qquad \ldots\ldots\ldots\ldots (8)$$

where *A* and *B* are constants to be determined from the initial conditions.

In order to proceed further it is desirable to define a new term called the critical damping coefficient, denoted by c_c. It is that value of the damping coefficient c that makes the expression within the radical sign of Equation (7) vanishes and thereby gives two equal roots of s.

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad \text{.......... (9)}$$

$$\frac{c_c}{2m} = \sqrt{\frac{k}{m}} = \omega_n \qquad \text{.......... (10)}$$

Another dimensionless factor ξ, called the damping factor or damping ratio, may be defined as the ratio of the damping coefficient to the critical damping coefficient.

$$\xi = \frac{c}{c_c} \qquad \text{.......... (11)}$$

It may be made clear at this stage that the critical damping coefficient is a constant depending upon the actual mass and stiffness of the system, and is independent of the actual amount of damping.

Damping ratio ξ can be introduced into characteristic equation as

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \xi \omega_n \quad \text{..........} \ (12)$$

Therefore, Equation (7) becomes

$$s_{1,2} = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n \quad \text{..........} \ (13)$$

The roots are three different types depending upon the value of ξ. Hence there are three types of responses and consequently three types of systems. The three types of systems are: over-damped, critically damped and under-damped system.

Over-damped Systems

When $c > c_c$ or $\xi > 1$, the characteristic equation will have two real and unequal roots, and the system is called over-damped system.

Critically Damped Systems

When $c = cc$ or $\xi = 1$, the roots will be real and equal, then system is called critically damped system.

Under-damped Systems

When $c < c_c$ or $\xi < 1$, the equation will have complex conjugate pairs of roots, then the system is called under-damped system.

6.4.1 Solution for Over-damped System

In this system the damping is comparatively large and the two values of *s* as given by Equation (13) are

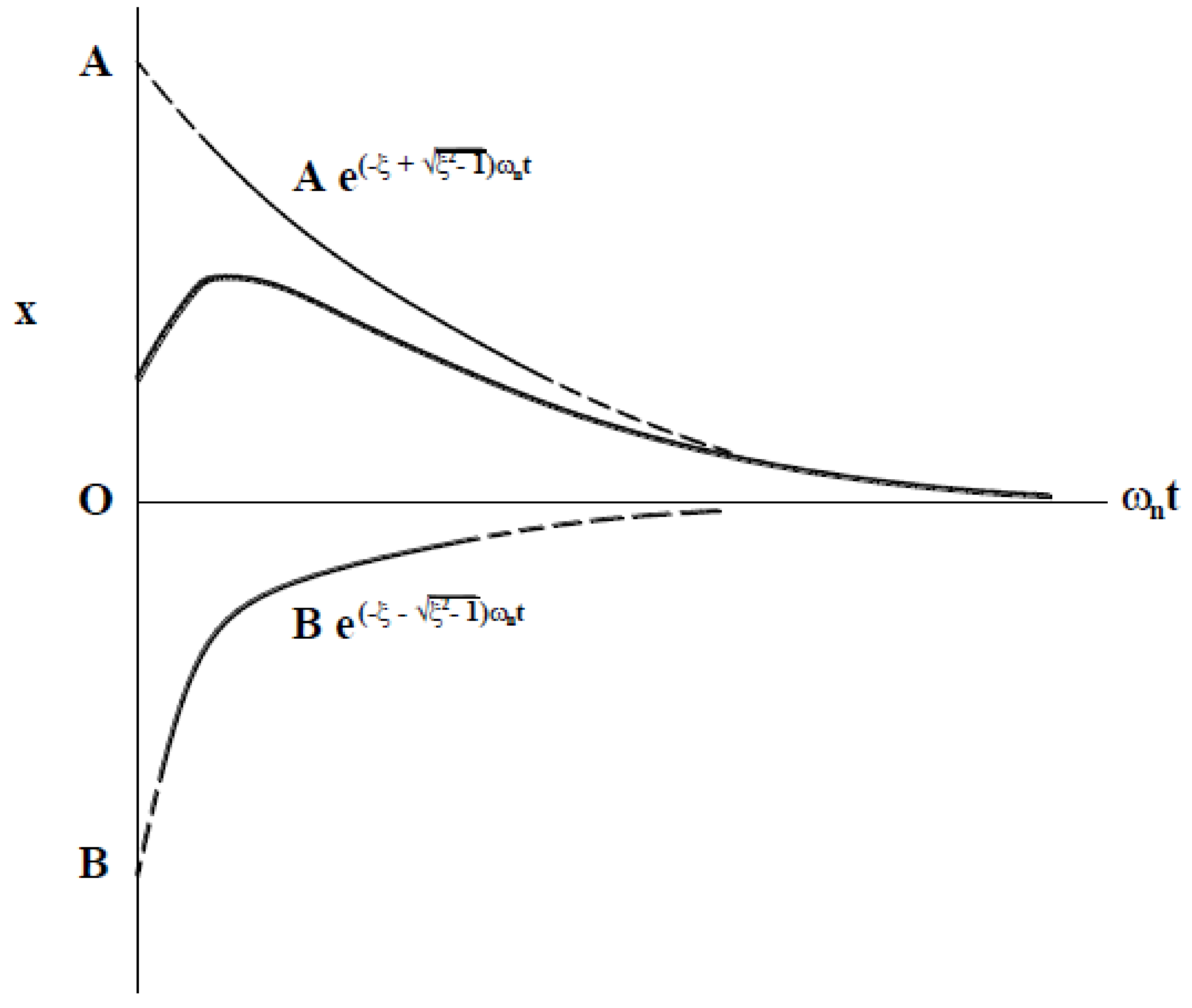
$$s_1 = \left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n$$

$$s_2 = \left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n \quad \ldots\ldots\ldots\ (14)$$

Here, the two roots are real and unequal, one increasing and the other decreasing. The general solution then becomes

$$x = A\, e^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} + B\, e^{\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t} \quad \ldots\ldots\ldots\ (15)$$

The motion is an exponentially decreasing function of time as shown in **Figure**, and is referred as aperiodic.


A
x
O
B
$A\,e^{(-\xi + \sqrt{\xi^2 - 1})\omega_n t}$
$B\,e^{(-\xi - \sqrt{\xi^2 - 1})\omega_n t}$
$\omega_n t$

Solution with initial conditions

Assume that at $t = 0$, $x = x(0)$ and $\dot{x} = \dot{x}(0)$

Substituting $t = 0$ and $x = x(0)$ in Equation (15),

$$A + B = x(0) \quad \ldots\ldots\ldots\ldots (16)$$

Differentiating Equation (15) and substituting, $t = 0$ and $\dot{x} = \dot{x}(0)$

$$\dot{x} = A\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n\, e^{\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t} + B\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n\, e^{\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t}$$

$$A\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n + B\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n = \dot{x}(0) \quad \ldots\ldots\ldots\ldots (17)$$

Solving simultaneous Equations (16) and (17), we get

$$A = \frac{\dot{x}(0) + (\xi + \sqrt{\xi^2 - 1})\omega_n x(0)}{2\omega_n\sqrt{\xi^2 - 1}} \qquad B = \frac{-\dot{x}(0) - (\xi - \sqrt{\xi^2 - 1})\omega_n x(0)}{2\omega_n\sqrt{\xi^2 - 1}}$$

Substituting values of A and B in Equation (15), we have

$$x = \frac{1}{2\omega_n\sqrt{\xi^2-1}}\left[\left\{\dot{x}(0)+(\xi+\sqrt{\xi^2-1})\omega_n x(0)\right\}e^{(-\xi+\sqrt{\xi^2-1})\omega_n t} - \left\{\dot{x}(0)+(\xi-\sqrt{\xi^2-1})\omega_n x(0)\right\}e^{(-\xi-\sqrt{\xi^2-1})\omega_n t}\right] \quad \ldots\ldots (18)$$

6.4.2 Solution for Critically damped System

For the system having critical damping ($\xi = 1$), the two values of s as given by Equation (13) are equal to each other,

$$s_1 = s_2 = -\omega_n \quad \ldots\ldots\ldots\ldots (19)$$

The solution of Equation (1) in case of equal roots is given by

$$x = (A + Bt)e^{-\omega_n t} \quad \ldots\ldots\ldots\ldots (20)$$

Solution with initial conditions

Solution with initial conditions

Assume that at $t = 0$, $x = x(0)$ and $\dot{x} = \dot{x}(0)$

Substituting $t = 0$ and $x = x(0)$ in Equation (20),

$$A = x(0) \quad \ldots\ldots\ldots\ldots (21)$$

Differentiating Equation (20) and substituting, $t = 0$ and $\dot{x} = \dot{x}(0)$

$$\dot{x} = (-A\omega_n + B - B\omega_n t)\, e^{-\omega n t}$$

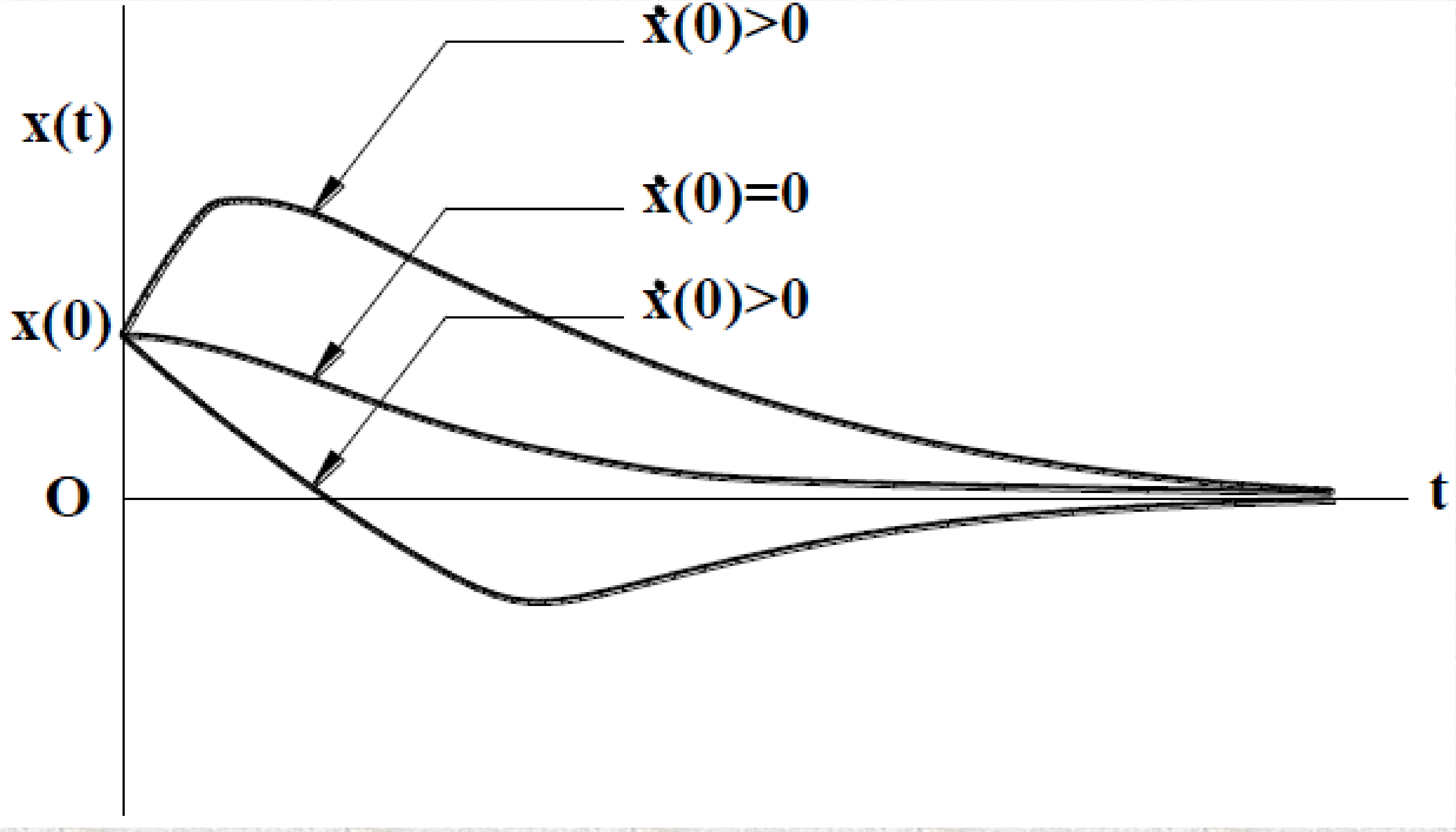
$$-A\omega_n + B = \dot{x}(0)$$

$$B = \dot{x}(0) + A\omega_n = \dot{x}(0) + x(0)\omega_n \quad \ldots\ldots\ldots\ldots (22)$$

Substituting values of *A* and *B* in Equation (20), we have

$$x = \left[x(0) + (\dot{x}(0) + x(0)\omega_n)t\right]e^{-\omega_n t} \quad \ldots\ldots\ldots\ldots (23)$$

Equation (23) that the value of *x* decreases as *t* increases and finally becomes zero as *t* tends to infinity as shown in **Figure**. This is also an aperiodic motion.



6.4.3 Solution for Under-damped System

For under-damped system, the roots of characteristics equation are pair of complex conjugate,

$$s_1 = -\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$s_2 = -\xi\omega_n - j\omega_n\sqrt{1-\xi^2} \qquad \text{……….} \; (24)$$

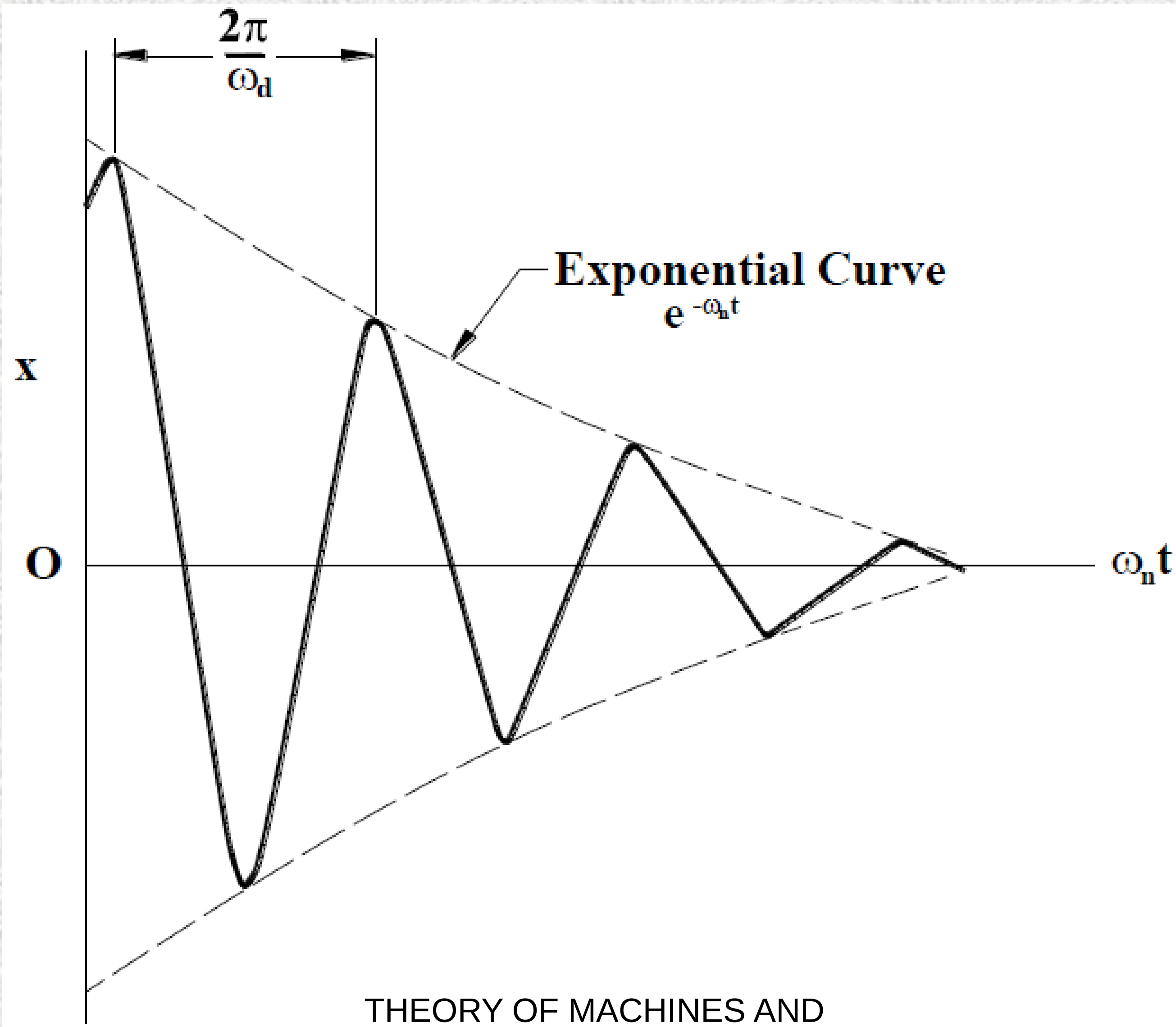
General Solution is then given by Equation (8) as

$$x = e^{-\xi\omega_n t}\left(A\sin\omega_n t\sqrt{1-\xi^2} + B\cos\omega_n t\sqrt{1-\xi^2}\right) \qquad \text{……….} \; (25)$$

The motion governed by the above equation is of oscillatory type with the damped frequency of oscillation given by ω_d,

$$\omega_d = \sqrt{1-\xi^2}\,\omega_n \qquad \text{……….} \; (26)$$

Amplitude decreases in an exponential manner with increase in time as shown in **Figure**.



Solution with initial conditions

Assume that at t = 0, x = x(0) and $\dot{x} = \dot{x}(0)$

Substituting t = 0 and x = x(0) in Equation (25),

$$B = x(0) \quad \text{..........} \; (27)$$

Differentiating Equation (25) and substituting, $t = 0$ and $\dot{x} = \dot{x}(0)$

$$\dot{x} = \left[(A\sqrt{1-\xi^2} - B\xi\omega_n)\cos\sqrt{1-\xi^2}\,t - (B\sqrt{1-\xi^2} + A\xi\omega_n)\sin\sqrt{1-\xi^2}\,t \right] e^{-\xi\omega nt}$$

$$\dot{x}(0) = A\sqrt{1-\xi^2} - B\xi\omega_n$$

$$A = \frac{\dot{x}(0) + B\xi\omega_n}{\sqrt{1-\xi^2}\,\omega_n} = \frac{\dot{x}(0) + x(0)\xi\omega_n}{\sqrt{1-\xi^2}\,\omega_n} \quad \text{..........} \; (28)$$

Substituting values of *A* and *B* in Equation (25), we have

$$x = \left[\frac{\dot{x}(0) + x(0)\xi\omega_n}{\omega_n\sqrt{1-\xi^2}} \sin\sqrt{1-\xi^2}\,\omega_n t + x(0)\cos\sqrt{1-\xi^2}\,\omega_n t \right] e^{-\xi\omega_n t}$$

………. (29)

Logarithmic Decrement

A convenient way to determine the amount of damping present in a system is to measure the rate of decay of free oscillations. The larger the damping the greater will be the rate of decay. The rate of decay is commonly expressed in terms of logarithmic decrement which is defined as the natural logarithm of the ratio of any two successive amplitudes.

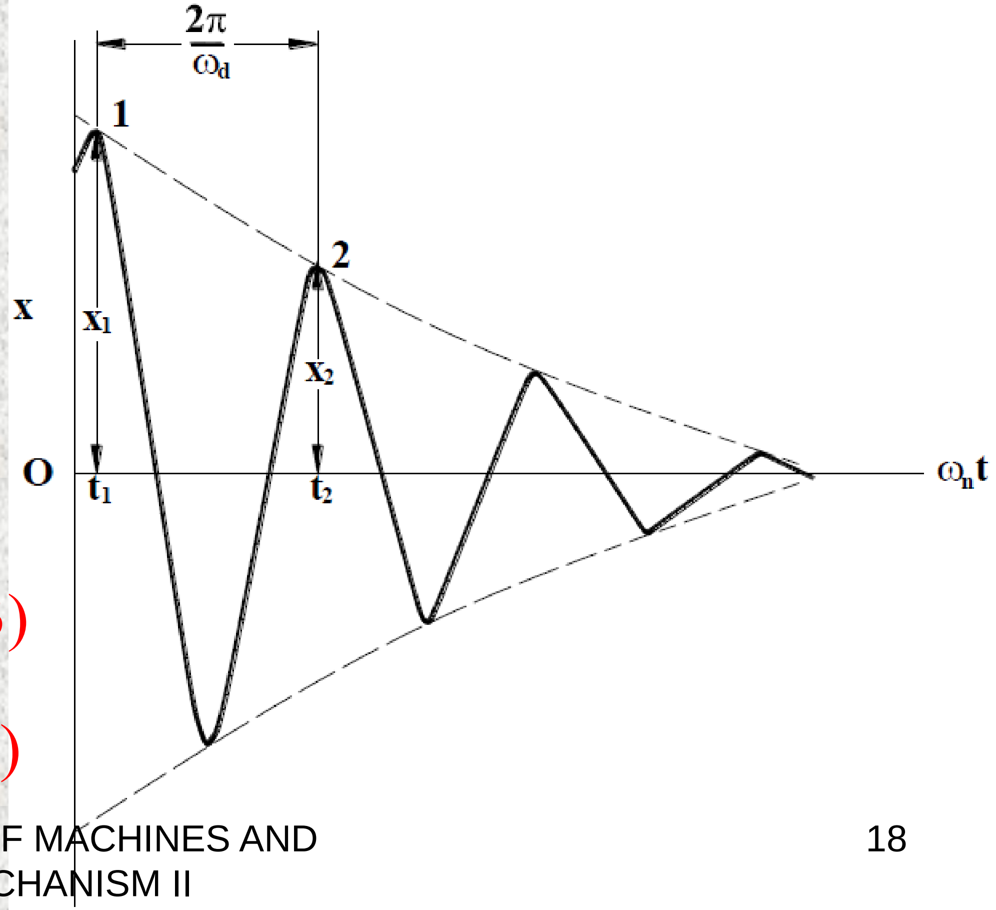
$$\delta = \ln\frac{x_1}{x_2} \qquad \text{……….} \ (30)$$

Consider a damped vibration Equation (29) expressed in sine terms only,

$$x = A_1 \sin\left(\sqrt{1-\xi^2}\,\omega_n t + \phi\right) e^{-\xi\omega_n t} \quad \text{..........} \; (31)$$

Consider two point, 1 and 2, corresponding to the time t_1 and t_2 as shown in **Figure**, where

$$t_2 - t_1 = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\xi^2}\,\omega_n} \quad \text{..........} \; (32)$$



Now the displacements at time t_1 and t_2 are given as,

$$x_1 = A_1 e^{-\xi\omega_n t_1} \quad \text{..........} \; (33)$$

$$x_2 = A_1 e^{-\xi\omega_n t_2} \quad \text{..........} \; (34)$$

Now dividing Equation (34) by Equation (33),

$$\frac{x_1}{x_2} = e^{-\xi\omega_n(t_1-t_2)} = e^{\xi\omega_n(t_2-t_1)} \quad \ldots\ldots\ldots\ldots (35)$$

Now substituting $t_2 - t_1$ from Equation (32) into Equation (35)

$$\frac{x_1}{x_2} = e^{\frac{2\pi\xi}{\sqrt{1-\xi^2}}} \qquad \ln\frac{x_1}{x_2} = \frac{2\pi\xi}{\sqrt{1-\xi^2}}$$

$$\delta = \ln\frac{x_1}{x_2} = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \quad \ldots\ldots\ldots\ldots (36)$$

The logarithmic decrement can also be given by the equation,

$$\delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}} \quad \ldots\ldots\ldots\ldots (37)$$

where x_{n+1} represents the amplitude after n cycles have elapsed.

The amplitude ratio for any two consecutive amplitudes is

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots\ldots\ldots = \frac{x_n}{x_{n+1}} = e^{\delta}$$

$$\text{or,} \quad \frac{x_1}{x_2} \cdot \frac{x_2}{x_3} \cdot \frac{x_3}{x_4} \ldots\ldots\ldots \frac{x_n}{x_{n+1}} = (e^{\delta})^{n}$$

$$\text{or,} \quad \frac{x_1}{x_{n+1}} = (e^{\delta})^{n}$$

$$\text{or,} \quad \ln\frac{x_1}{x_{n+1}} = n\delta$$

$$\therefore \quad \delta = \frac{1}{n}\ln\frac{x_1}{x_{n+1}}$$

Example 6.6

A **100 kg** block is attached to a spring of stiffness **1.5 MN/m** in parallel with a viscous damper of damping coefficient **4900 N.s/m**. The block is given an initial velocity of **5 m/s**. What is its maximum displacement?

30.24 mm

Example 6.7

A **200 kg** block is attached to a spring of stiffness **50000 N/m** in parallel with a viscous damper. The period of free vibration of this system is observed as **0.417 s**. What is the value of damping coefficient?

1.916×10^3 N.s/m

Example 6.8

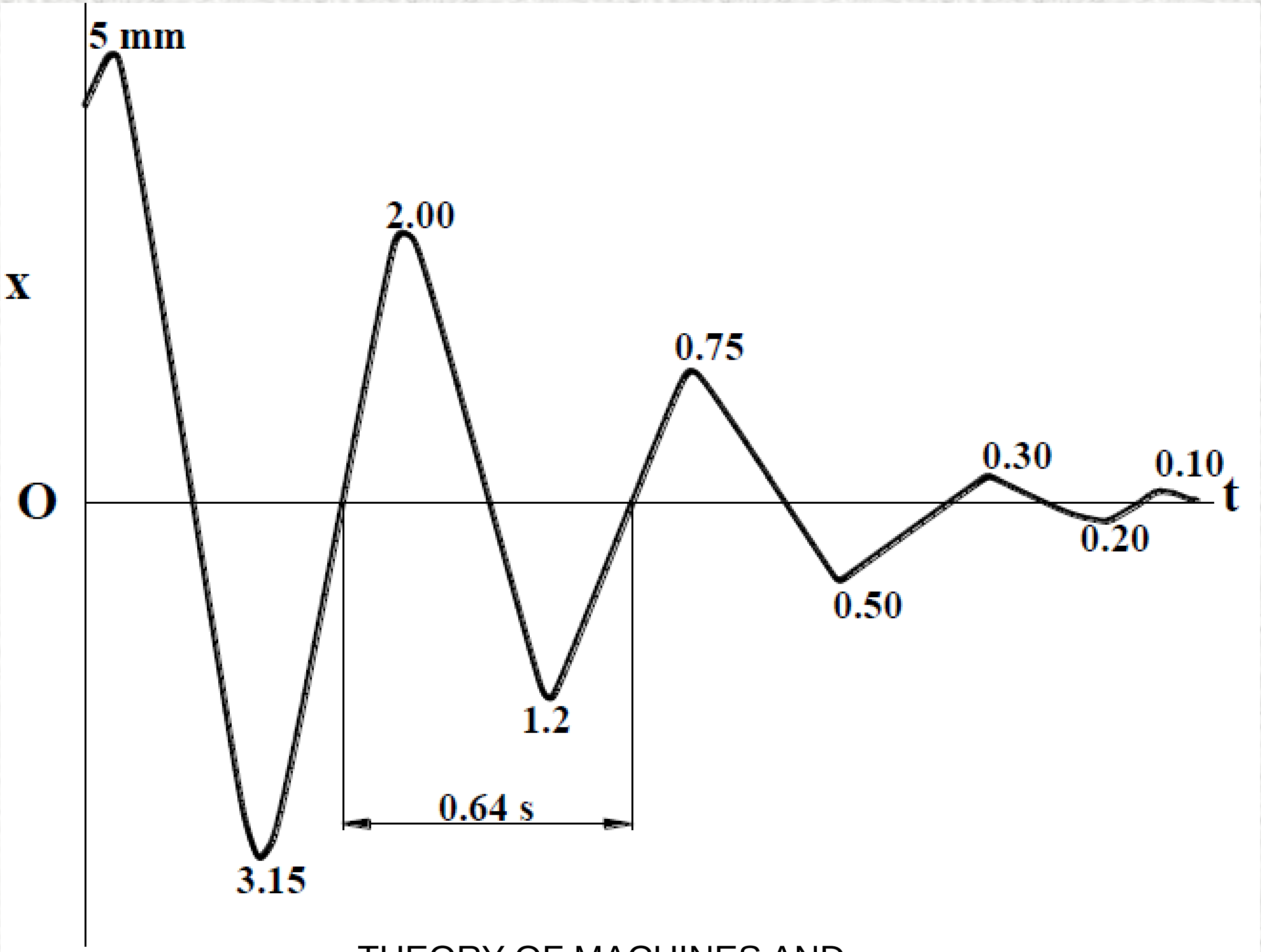
A vibrating system consists of a **5 kg** mass, a spring with constant **k = 3.5 N/mm**, and a dashpot with damping constant **c = 100 Ns/m**. Determine

- the damping factor,
- the damped natural frequency,
- the logarithmic decrement,
- the ratio of two successive amplitudes.

0.378, 24.5 rad/s, 2.56, 13.0

Example 6.9

Free vibration records of **1000 kg** machine mounted on an isolator is shown in **Figure E6.9**. Identify the type of isolator and its spring constant and damping coefficient.



98718 N/m