# THEORY OF MACHINES AND MECHANISMS II

# Mechanical IV/I

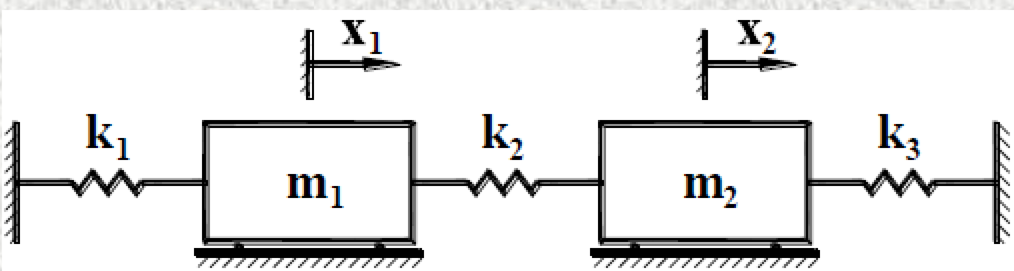
## Chapter 7

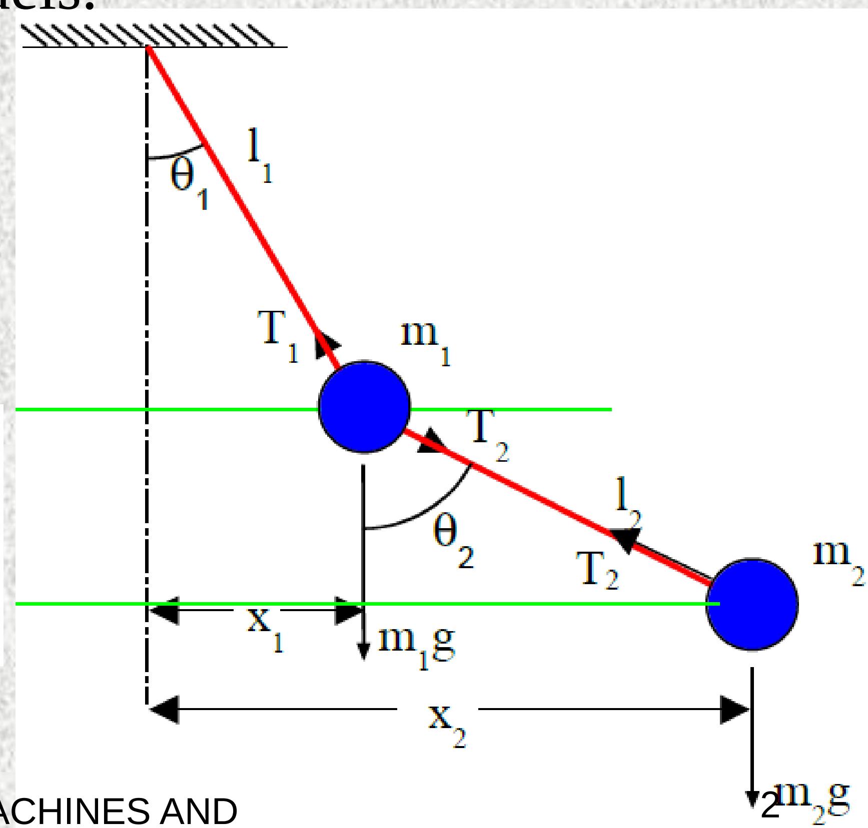
# Vibration of Two Degree of Freedom Systems

General mechanical systems require several degrees of freedom for a meaningful model. The single degree of freedom model, besides dealing with certain practical examples, it helped to formulate the basic theory of vibrations of mechanical systems.

A natural extension of single degree of freedom model systems is to consider two degrees of freedom models.

When a system requires two coordinates to describe its motion, it is said to have two degrees of freedom.



A two degrees of freedom system will have two natural frequencies.

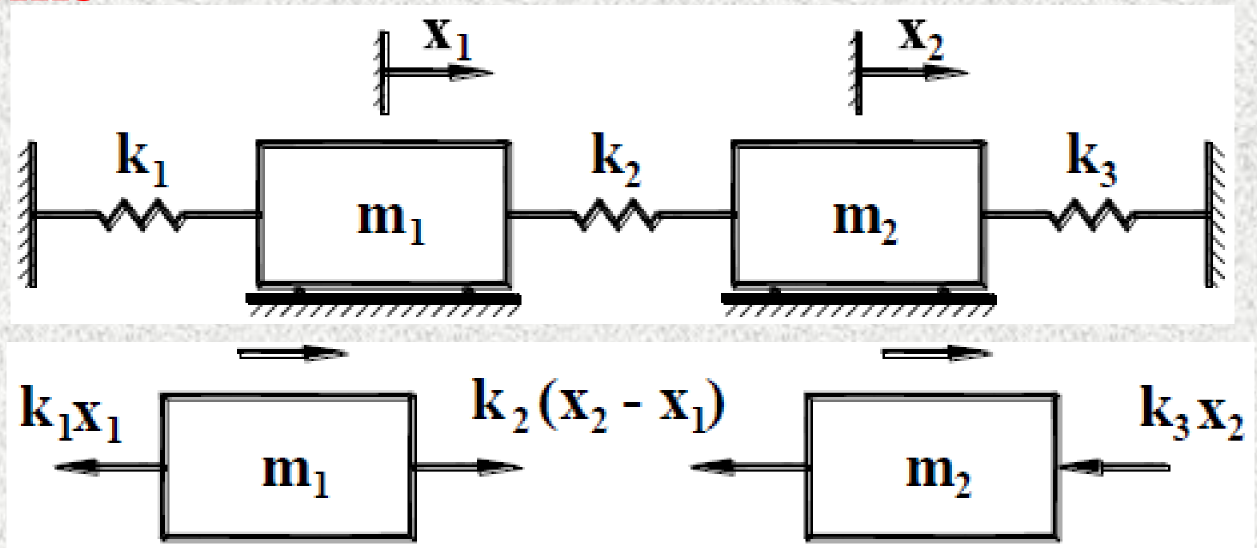
When free vibration takes place at one of these natural frequencies, a definite relationship exists between the amplitudes of the two coordinates, and the configuration is referred to as the normal mode.

The two degrees of freedom system will then have two normal mode vibrations corresponding to the two natural frequencies.

Free vibration initiated under any condition will in general be the superposition of the two normal mode vibrations.

However, forced harmonic vibration will take place at the frequency of the excitation and the amplitude of the two coordinates will tend to a maximum at the two natural frequencies

# 7.1 Vibrations of Undamped Two Degrees of Freedom Systems




The equation of motion for the two masses can be written as

$$m_1\ddot{x}_1 = -k_1x_1 + k_2(x_2 - x_1)$$
$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3x_2 \quad \text{.......... (1)}$$

These equations may be rewritten as,

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 = k_2x_2$$
$$m_2\ddot{x}_2 + (k_2 + k_3)x_2 = k_2x_1 \quad \text{.......... (2)}$$

Let us now assume the solutions for $x_1$ and $x_2$ under steady state conditions as harmonic vibrations; i.e.

$$\left.\begin{aligned} x_1 &= A_1 \sin \omega t \\ x_2 &= A_2 \sin \omega t \end{aligned}\right] \quad \text{..........} \ (3)$$

where $A_1$ and $A_2$ are the amplitudes of the two masses respectively and $\omega$ is the frequency of harmonic motion of both the masses.

Substituting Equations (2) in Equations (1), we have

$$\begin{aligned} \left[-m_1\omega^2 + (k_1 + k_2)\right]A_1 - k_2 A_2 &= 0 \\ -k_2 A_1 + \left[-m_2\omega^2 + (k_2 + k_3)\right]A_2 &= 0 \end{aligned} \quad \text{..........} \ (4)$$

Equations (4) can be rearranged as

$$\frac{A_1}{A_2} = \frac{k_2}{(k_1 + k_2) - m_1\omega^2} \quad \text{....} \ (5a) \qquad \frac{A_1}{A_2} = \frac{(k_2 + k_3) - m_2\omega^2}{k_2} \quad \text{....} \ (5b)$$

Equating Equations (5a) and (5b)

$$\frac{k_2}{(k_1+k_2)-m_1\omega^2}=\frac{(k_2+k_3)-m_2\omega^2}{k_2} \quad \text{.......... (6)}$$

$$[(k_1+k_2)-m_1\omega^2][(k_2+k_3)-m_2\omega^2]-{k_2}^2=0 \quad \text{.......... (7)}$$

**Alternative Method**

Equations (4) can be expressed matrix form as

$$\begin{bmatrix}(k_1+k_2)-m_1\omega^2 & -k_2\\ -k_2 & (k_2+k_3)-m_2\omega^2\end{bmatrix}\begin{Bmatrix}A_1\\ A_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \quad \text{.......... (8)}$$

These are homogeneous linear algebraic equations in $A_1$ and $A_2$. The solution is obtained by equating to zero the determinant of the coefficients $A_1$ and $A_2$, i.e.,

$$\begin{vmatrix}(k_1+k_2)-m_1\omega^2 & -k_2\\ -k_2 & (k_2+k_3)-m_2\omega^2\end{vmatrix}=0 \quad \text{.......... (9)}$$

The expansion of Equation (9) results the Equation (7).

Letting $\omega^2 = \lambda$, the above determinant leads to the characteristic equation

$$\lambda^2 - \left[\frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2}\right]\lambda + \frac{k_1 k_2 + k_2 k_3 + k_3 k_1}{m_1 m_2} = 0 \quad \text{..........} (10)$$

The above equation is quadratic in $\lambda$ (= $\omega^2$) and gives two values of $\lambda$ (= $\omega^2$), and therefore two positive values of $\omega$ corresponding to the two natural frequencies of the system. Equation (10) is called Frequency Equation since the roots of this equation give the natural frequencies of the system.

If we assume $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, frequency equation can be written as

$$\lambda^2 - \frac{4k}{m}\lambda + \frac{3k^2}{m^2} = 0 \quad \text{..........} (11)$$

The two roots of this equation are

$$\lambda_{1,2} = \frac{2k}{m} \pm \frac{1}{2}\sqrt{\left(\frac{4k}{m}\right)^2 - 12\frac{k^2}{m^2}} = \frac{2k}{m} \pm \frac{k}{m} \quad \text{.......... (12)}$$

$$\therefore \quad \lambda_1 = \frac{k}{m}, \text{ and } \quad \lambda_2 = \frac{3k}{m} \quad \text{.......... (13)}$$

The natural frequencies of the system are then found to be

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{m}}, \quad \text{and} \quad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{3k}{m}} \quad \text{....... (14)}$$

Amplitudes ratio can be determined by substituting natural frequencies in Equation (5a) or (5b). For $\lambda_1 = \omega_1^2 = k/m$, we obtain

$$\left(\frac{A_1}{A_2}\right)_1 = \frac{k}{2k - m\omega_1^2} = \frac{k}{2k-k} = 1 \quad \text{.......... (15)}$$

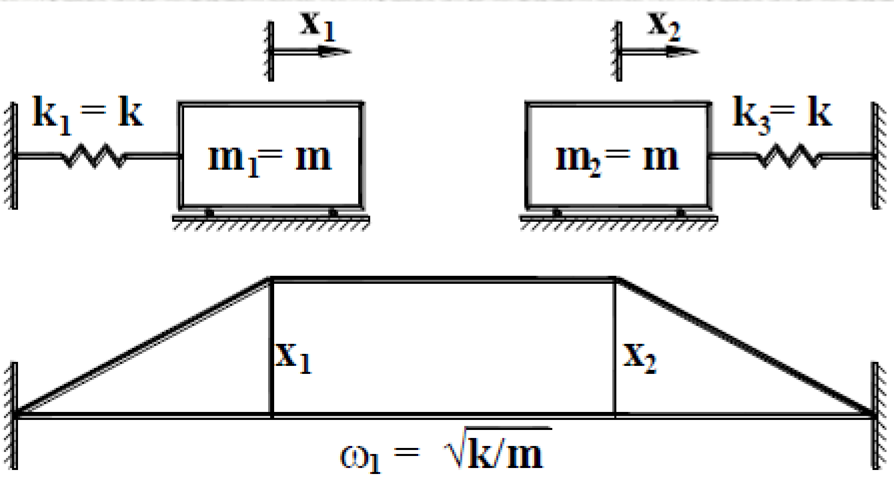
which is the amplitude ratio or mode shape corresponding to the first natural frequency.

The amplitude ratio being 1 means that the amplitudes of vibration of two masses are equal and the two motions are in phase.

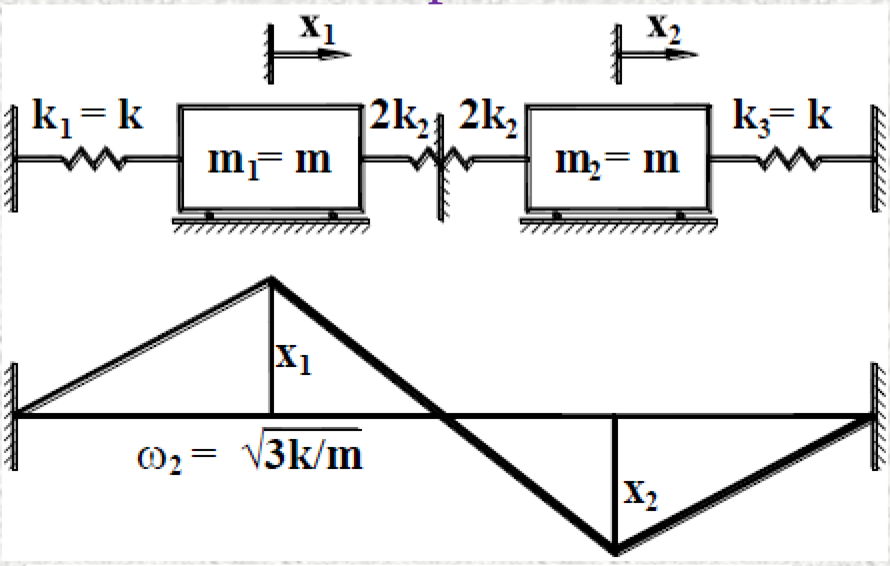
Again for $\lambda_2 = \omega_2^2 = 3k/m$, we obtain

$$\left(\frac{A_1}{A_2}\right)_2 = \frac{k}{2k - m\omega_2^2} = \frac{k}{2k - 3k} = -1 \quad \text{..........} \ (16)$$

for mode shape corresponding to the second normal mode. The amplitude ratio being *–1* means that the amplitudes of vibration of two masses are equal but the two motions are out of phase.




**(a) First mode of vibration**




**(b) Second mode of vibration**

If the two masses are given equal initial displacements in the same direction and released, they will vibrate in the first principle mode of vibration with the first natural frequency. Also if, they are given equal initial displacements in the opposite directions and released, they will vibrate in the second mode of vibration with the second natural frequency.

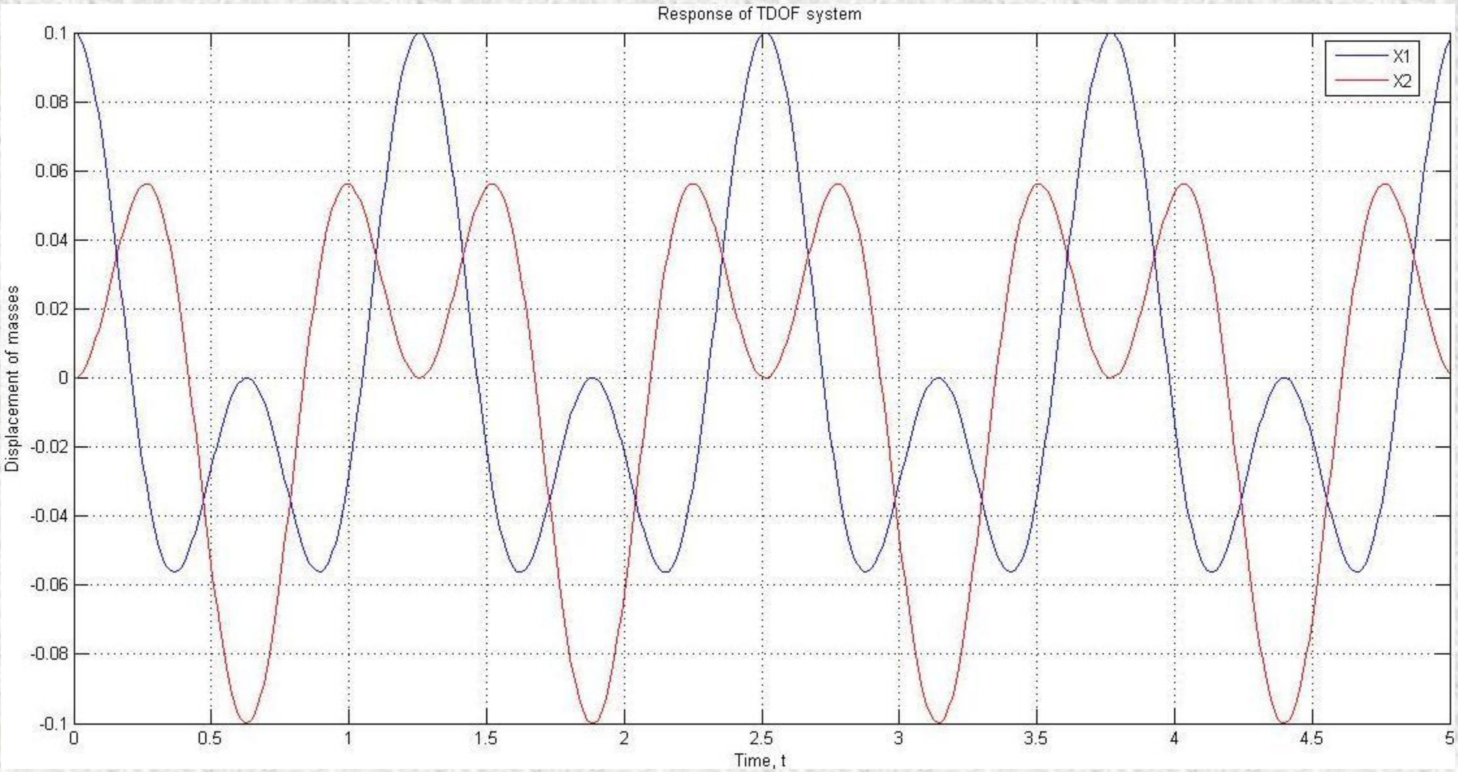
If, however, the two masses are given unequal initial displacements in any direction, their motion will be superposition of two harmonic motions corresponding to the two natural frequencies as given below

$$
\begin{aligned}
x_1 &= A \sin(\omega_1 t + \psi_1) - B \sin(\omega_2 t + \psi_2) \\
x_2 &= A \sin(\omega_1 t + \psi_1) + B \sin(\omega_2 t + \psi_2)
\end{aligned}
\qquad \text{.......... (17)}
$$

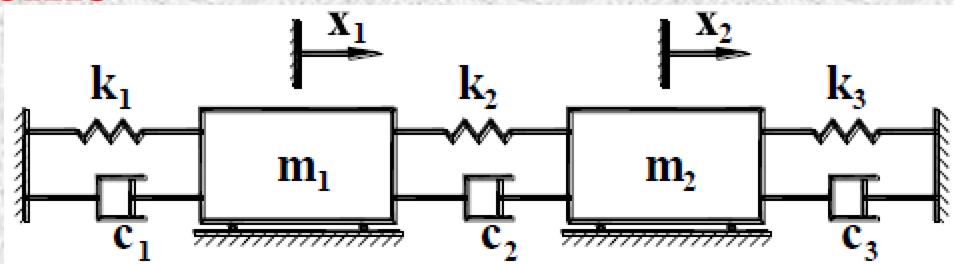
It should be noted here that the first terms on the right correspond to the first normal mode at natural frequency $\omega_1$. Its amplitude ratio is also $A_1/A_2 = 1 = A/A$, which is the first normal mode shape.

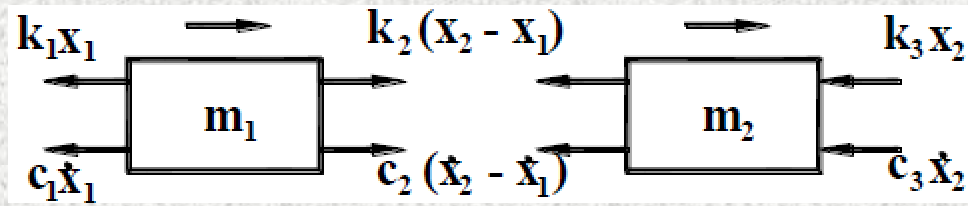
The second terms oscillate at frequency $\omega_2$ with amplitude ratio $A_1/A_2 = -1 = -B/B$, in conformity with the second normal mode vibration.

The phase $\psi_1$ and $\psi_2$ simply allows the freedom of shifting the time origin and does not alter the character of the normal modes. The constants $A$, $B$, $\psi_1$ and $\psi_2$ are sufficient to satisfy the four initial conditions $\{x_1(0), x_2(0), \dot{x}_1(0) \text{ and } \dot{x}_2(0)\}$.



# 7.2 Vibrations of Damped Two Degrees of Freedom Systems




The differential equations of motion for the two masses are given as

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) \\ m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) - k_3x_3 - c_3\dot{x}_2 \end{aligned} \quad \text{.......... (18)}$$

Rearranging the above equations

$$\begin{aligned} &\left[m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1 + k_2)x_1\right] - \left[c_2\dot{x}_2 + k_2x_2\right] = 0 \\ &\left[m_2\ddot{x}_2 + (c_2 + c_3)\dot{x}_2 + (k_2 + k_3)x_2\right] - \left[c_2\dot{x}_1 + k_2x_1\right] = 0 \end{aligned} \quad \text{.......... (19)}$$

These are two coupled linear differential equations of second order and their solution may be of the form

$$\begin{aligned} x_1 &= A_1 e^{st} \\ x_2 &= A_2 e^{st} \end{aligned} \quad \ldots\ldots\ldots\ldots (20)$$

Substituting Equation (20) in Equations (19), we get

$$\begin{aligned} \left[m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2)\right]A_1 - \left[c_2 s + k_2\right]A_2 &= 0 \\ -\left[c_2 s + k_2\right]A_1 + \left[m_2 s^2 + (c_2 + c_3)s + (k_2 + k_3)\right]A_2 &= 0 \end{aligned} \quad \ldots\ldots\ldots\ldots (21)$$

The above equations will have values of $A_1$ and $A_2$ are different form zero only if the determinant formed from their coefficients is zero, i.e., if

$$\begin{vmatrix} m_1 s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_1 s + k_1) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2 + c_3)s + (k_2 + k_3) \end{vmatrix} = 0 \quad \ldots\ldots (22)$$

Expanding and rearranging the above equation,

$$s^4 + \left[\frac{c_1 + c_2}{m_1} + \frac{c_2 + c_3}{m_2}\right]s^3 + \left[\frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2} + \frac{c_1c_2 + c_2c_3 + c_3c_1}{m_1m_2}\right]s^2$$
$$+ \left[\frac{k_1(c_2 + c_3) + k_2(c_3 + c_1) + k_3(c_1 + c_2)}{m_1m_2}\right]s + \left[\frac{k_1k_2 + k_2k_3 + k_3k_1}{m_1m_2}\right] = 0$$

…. (23)

This is called the characteristic equation of the system and the values of $s$ have to be obtained from this equation. There will be four values of $s$ for which Equations (20) will be solutions of Equations (19), and so the general solutions are

$$x_1 = A_{11}e^{s1t} + A_{12}e^{s2t} + A_{13}e^{s3t} + A_{14}e^{s4t}$$
$$x_2 = A_{21}e^{s1t} + A_{22}e^{s2t} + A_{23}e^{s3t} + Ae^{s4t}$$

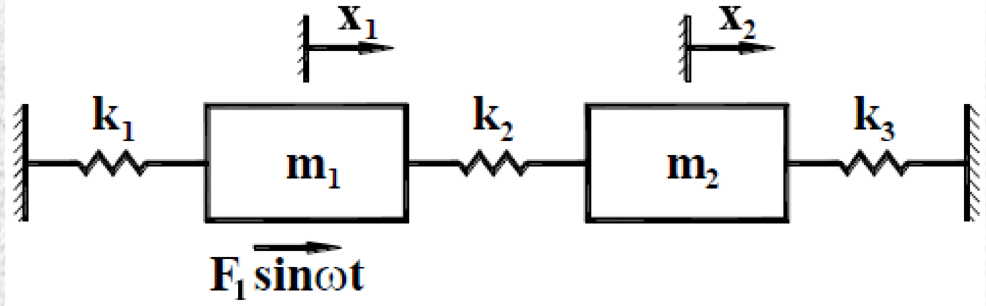
………. (24)

where the coefficients $A_{11}$ in the first of these equations are four arbitrary constants to be determined from the initial conditions and the coefficients $A_{21}$ in the second equation are related to $A_{11}$ from either of the Equations (21) by substituting for $s$ the corresponding value of $s_1$ obtained from Equation (23), i.e.,

$$A_{21} = \frac{\left[m_1 s_1^{\,2} + (c_1 + c_2)s_1 + (k_1 + k_2)\right]}{c_2 s_1 + k_2} A_{11} \quad \text{..........} \quad (24)$$

# 7.3 Forced Harmonic Vibrations of Two Degrees of Freedom Systems

Consider the system shown in **Figure** with the exciting force $F_1 \sin\omega t$ acting on mass $m_1$. The differential equation of motion can be written as




$$\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 &= F_1 \sin\omega t \\ m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 &= 0 \end{aligned} \quad \text{..........} \ (25)$$

For the steady state solution

$$\left.\begin{aligned} x_1 &= X_1 \sin\omega t \\ x_2 &= X_2 \sin\omega t \end{aligned}\right] \quad \text{..........} \ (26)$$

Substituting Equation (26) in Equations (25), we get

$$\begin{aligned} \left[-m_1\omega^2 + (k_1 + k_2)\right]X_1 - k_2 X_2 &= F_1 \\ -k_2 X_1 + \left[-m_2\omega^2 + (k_2 + k_3)\right]X_2 &= 0 \end{aligned} \quad \text{..........} \ (27)$$

Solving for $X_1$ and $X_2$ from the above equations, we get

$$X_1 = \frac{\left[k_2 + k_3 - m_2\omega^2\right]F_1}{\left[m_1 m_2 \omega^4 - \{m_1(k_2 + k_3) + m_2(k_1 + k_2\}\omega^2 + (k_1 k_2 + k_2 k_3 + k_3 k_1)\right]}$$

$$X_2 = \frac{k_2 F_1}{\left[m_1 m_2 \omega^4 - \{m_1(k_2 + k_3) + m_2(k_1 + k_2)\}\omega^2 + (k_1 k_2 + k_2 k_3 + k_3 k_1)\right]}$$

……….. (28)

If we assume $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, Equations (28) can be rewritten as,

$$X_1 = \frac{(2k - m\omega^2)F_1}{m^2\omega^4 - 4mk\omega^2 + 3k^2}$$

$$X_2 = \frac{kF_1}{m^2\omega^4 - 4mk\omega^2 + 3k^2}$$

……….. (29)

As we know

$$\omega_1 = \sqrt{\lambda_1} = \sqrt{\frac{k}{m}}, \qquad \text{and} \qquad \omega_2 = \sqrt{\lambda_2} = \sqrt{\frac{3k}{m}}$$

The denominator of Equations (29) can be expressed as

$$m^2\omega^4 - 4km\omega^2 + 3k^2 = m^2 (\omega_1^2 - \omega^2) (\omega_2^2 - \omega^2)$$

Then Equations (28) become

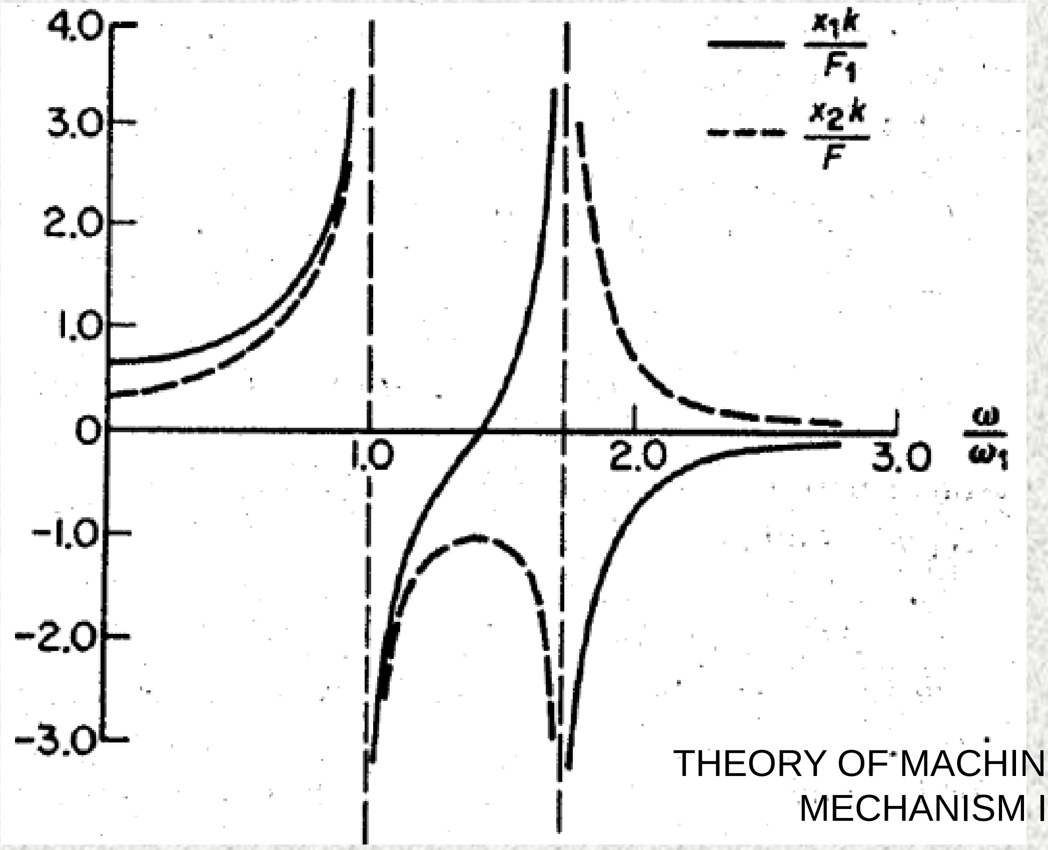
$$X_1 = \frac{(2k - m\omega^2)F_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)}$$

$$X_2 = \frac{kF_1}{m^2(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2)} \qquad \text{......... (30)}$$

Alternative forms of $X_1$ and $X_2$ are then

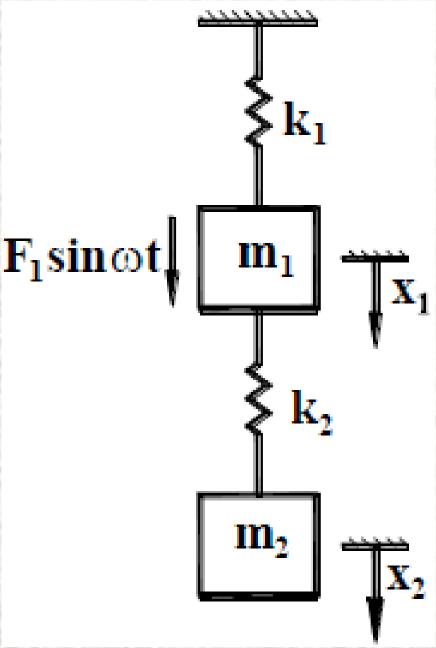
$$X_1 = \frac{F_1}{2m}\left[\frac{1}{\omega_1{}^2-\omega^2}+\frac{1}{\omega_2{}^2-\omega^2}\right] = \frac{F_1}{2m}\left[\frac{1}{1-(\omega/\omega_1)^2}+\frac{1}{3-(\omega/\omega_1)^2}\right]$$

$$X_2 = \frac{F_1}{2m}\left[\frac{1}{1-(\omega/\omega_1)^2}-\frac{1}{3-(\omega/\omega_1)^2}\right] \qquad \text{..........(31)}$$



# 7.4 Vibration Absorber

The expressions for $X_1$ and $X_2$ can be obtained by putting $k_3 = 0$ in Equation (28)




$$X_1 = \frac{\left[k_2 - m_2\omega^2\right]F_1}{\left[m_1 m_2 \omega^4 - \{m_1 k_2 + m_2(k_1 + k_2)\}\omega^2 + k_1 k_2\right]}$$

$$X_2 = \frac{k_2 F_1}{\left[m_1 m_2 \omega^4 - \{m_1 k_2 + m_2(k_1 + k_2)\}\omega^2 + k_1 k_2\right]} \quad \ldots\ldots\ldots (32)$$

The numerator of the expression of $X_1$ of Equation (32) vanishes when $\omega = \sqrt{k_2 / m_2}$, thereby making the mass $m_1$ motionless at this frequency. No such stationary condition exists for mass $m_2$.

The fact that mass which is being excited can have zero amplitude of vibration under certain conditions by coupling it to another spring mass system form the principle of dynamic vibration absorber.

Divide numerators and denominators of Equations (32) by $k_1 k_2$ and introduce the following notations

$$\omega_{11} = \sqrt{\frac{k_1}{m_1}} = \text{natural frequency of the main system alone,}$$

$$\omega_{22} = \sqrt{\frac{k_2}{m_2}} = \text{natural frequency of the absorber system alone,}$$

$$\mu = \frac{m_2}{m_1} = \text{ratio of the absorber mass to the main mass.}$$

Equations (32) can then be rewritten in the dimensionless form as

$$\frac{X_1 k_1}{F_1} = \frac{\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right]}{\frac{\omega^4}{\omega_{11}{}^2 \omega_{22}{}^2} - \left[(1+\mu)\left(\frac{\omega}{\varpi_{11}}\right)^2 + \left(\frac{\omega}{\omega_{22}}\right)^2\right] + 1} \qquad \frac{X_2 k_1}{F_1} = \frac{1}{\frac{\omega^4}{\omega_{11}{}^2 \omega_{22}{}^2} - \left[(1+\mu)\left(\frac{\omega}{\varpi_{11}}\right)^2 + \left(\frac{\omega}{\omega_{22}}\right)^2\right] + 1}$$

………. (33)

First part of the Equations (33) clearly shows that $X_1 = 0$ when $\omega = \omega_{22}$; that is when the excitation frequency is equal to the natural frequency of the absorber, the main system amplitude becomes zero even though it is excited by a harmonic force.

Similarly substituting $\omega = \omega_{22}$ in the second part of Equation (33) , we get

$$\frac{X_2 k_1}{F_1} = -\frac{1}{\mu\left(\frac{\omega_{22}}{\omega_{11}}\right)^2} = -\frac{1}{\frac{m_2}{m_1} \cdot \frac{k_2}{m_2} \cdot \frac{m_1}{k_1}} = -\frac{k_1}{k_2}$$

$$\therefore \quad F_1 = -k_2 X_2 \quad \ldots\ldots\ldots\ldots (34)$$

The above equation shows that the spring force $k_2X_2$ on the main mass due to the amplitude $X_2$ of the absorber mass is equal and opposite to the exciting force on the main mass resulting in no motion of the main system. The main system vibrations have been reduced to zero and these vibrations have been taken up by the absorber system.

The addition of a vibration absorber to main system is not much meaningful unless the main system is operating at resonance or at least near it. Under these conditions we have $\omega = \omega_{11}$. But for the absorber to be effective $\omega = \omega_{22}$. Therefore, for the effectiveness of the absorber at the operating frequency corresponding to the natural frequency of the main system alone, we have

$$\omega_{11} = \omega_{22}, \text{ or } \quad \frac{k_1}{m_1} = \frac{k_2}{m_2} \qquad \text{……….} (35)$$

When this condition is fulfilled, the absorber is known to be a tuned absorber. For a tuned absorber, Equations (33) now become

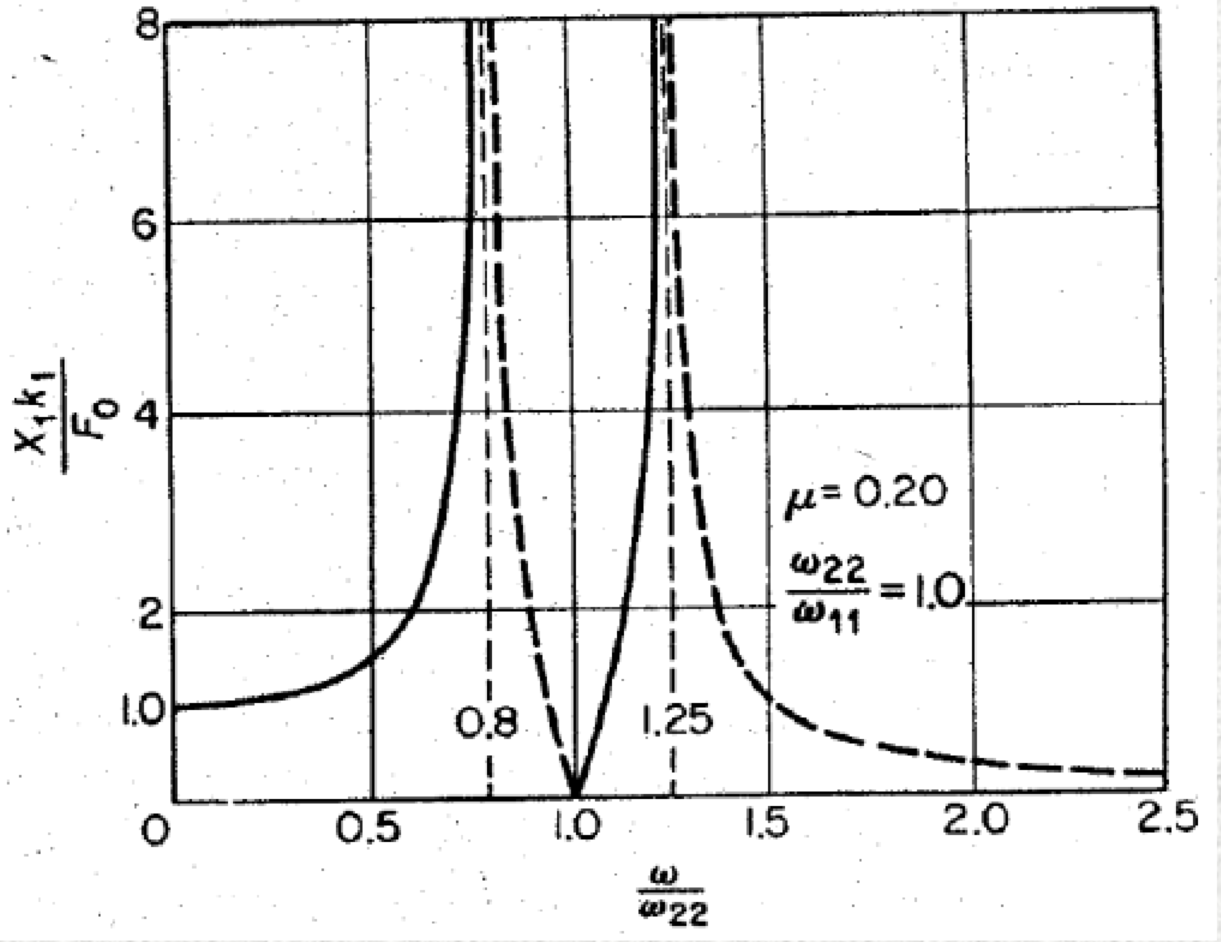
$$\frac{X_1 k_1}{F_1} = \frac{\left[1 - \left(\frac{\omega}{\omega_{22}}\right)^2\right]}{\left(\frac{\omega}{\omega_{22}}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_{22}}\right)^2 + 1} \qquad \frac{X_2 k_1}{F_1} = \frac{1}{\left(\frac{\omega}{\omega_{22}}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_{22}}\right)^2 + 1}$$

$$\text{……….} (36)$$

To have a tuned absorber, we can have many combinations of $k_2$, $m_2$ as long as their ratio is equal to $k_1/m_1$ to satisfy Equation (35). We can have a small spring $k_2$ and a small mass $m_2$ or $k_2$ large and $m_2$ large.

However, Equation (34) shows that for the same exciting force the amplitude of the absorber mass is inversely proportional to its spring constant. In order to have small amplitude of absorber mass $m_2$, we must have a large $k_2$ and therefore large $m_2$ which may not be desirable from practical considerations.

Small mass $m_2$ would be the best from practical considerations but that is associated with small $k_2$ and therefore large amplitude $X_2$ of vibration of the absorber mass. So a compromise is usually made between the amplitude and mass ratio $\mu$. The mass ratio is usually kept between 0.05 to 0.25.



8
6
4
2
1.0
0
$\frac{X_1 k_1}{F_0}$
0.5
1.0
1.5
2.0
2.5
0.8
1.25
$\mu = 0.20$
$\frac{\omega_{22}}{\omega_{11}} = 1.0$
$\frac{\omega}{\omega_{22}}$

The denominators of Equations (36) are identical. At a value of $\omega$ when these denominators are zero, the two masses have infinite amplitudes of vibration. The expression for the denominators is a quadratic in $\omega^2$, and therefore there are two values of $\omega$ for which these expressions vanish.

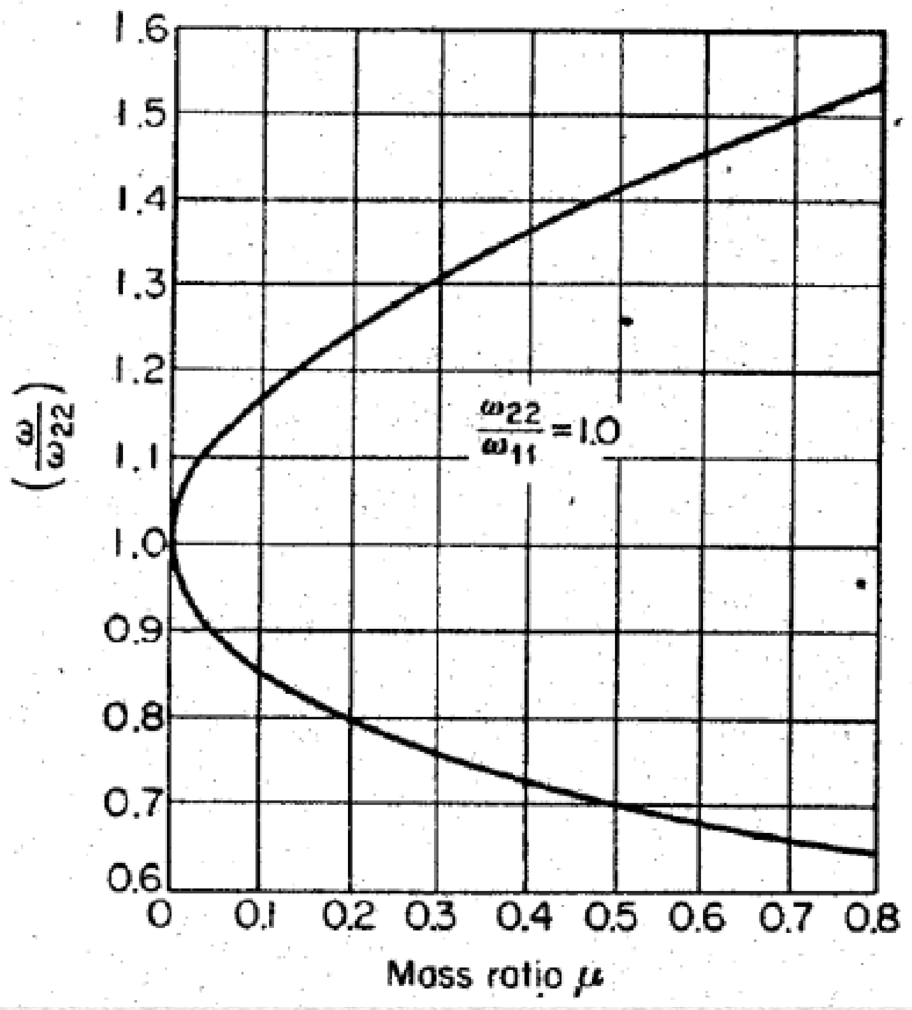
These two frequencies are the resonant frequencies or the natural frequencies of the system. When the excitation frequency equals any of the natural frequencies of the system, all the points in the system have infinite amplitudes of vibration, or the system is in resonance.

To find the two resonant frequencies of the systems when $\omega_{11} = \omega_{22}$, we equate denominator of either of Equations (36) to zero,

$$\left(\frac{\omega}{\omega_{22}}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_{22}}\right)^2 + 1 = 0 \quad \text{..........} (37)$$

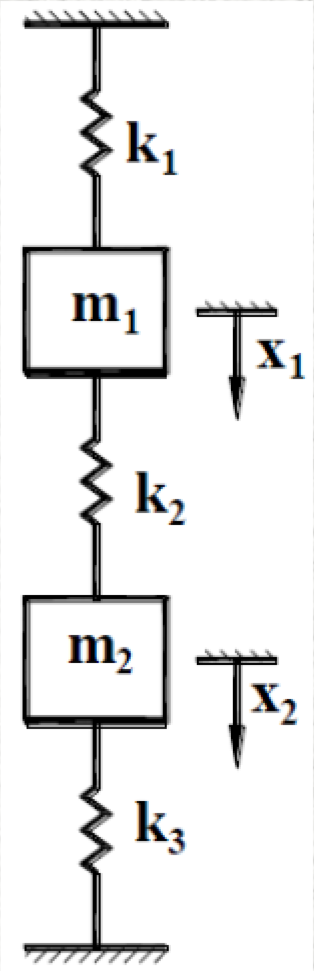
Solving for $(\omega/\omega_{22})$ we have

$$\frac{\omega}{\omega_{22}} = \left(1+\frac{\mu}{2}\right) \pm \sqrt{\mu + \frac{\mu^2}{4}} \quad \text{……….. (38)}$$



## Example 7.1

Determine the two natural frequencies and the modes of vibration of the system shown in **Figure E7.1**. Given $k_1 = k_3 = k$; $k_2 = 2k$ and $m_1 = m_2 = m$.



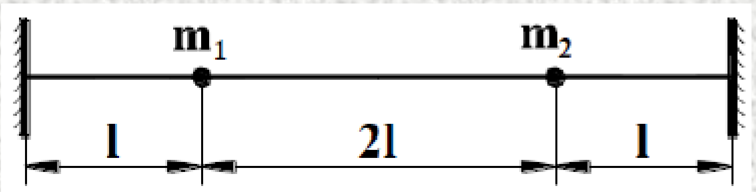

If $x_1(0) = 1,\ \dot{x}_1(0) = 0;\ x_2(0) = 0,\ \dot{x}_2(0) = 0,$ determine the response of the system.

$$\sqrt{\frac{k}{m}},\ \sqrt{\frac{5k}{m}}\ ;\ 1, -1;$$

$$0.5 \cos\sqrt{\frac{k}{m}}\,t + 0.5 \cos\sqrt{\frac{5k}{m}}\,t,\ 0.5 \cos\sqrt{\frac{k}{m}}\,t - 0.5 \cos\sqrt{\frac{5k}{m}}\,t$$

## Example 7.2

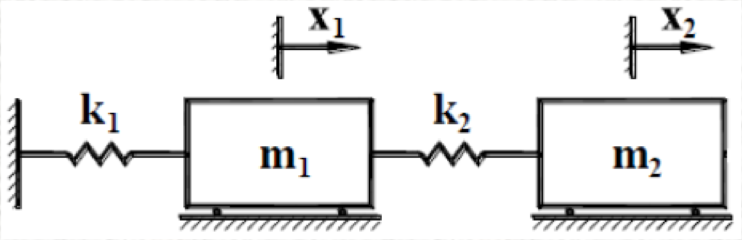
Determine the two natural frequencies and the modes of vibration of the system shown in **Figure E7.2**. The two equal masses are under tension *T*, which is large.




$$\sqrt{\frac{T}{ml}},\ \sqrt{\frac{2T}{ml}}\ ;\ 1,\ -1$$

## Example 7.3

In the system shown in **Figure E7.3**, the mass $m_1$ is excited by a harmonic force having a maximum value of 50 N and a frequency of 2 Hz. Find the forced amplitude of each mass for $m_1$ = 10 kg, $m_2$ = 5 kg, $k_1$ = 8000 N/m, and $k_2$ = 2000 N/m.




**9.7728 mm**, **16.1476 mm**