Cubic Spline Interpolation

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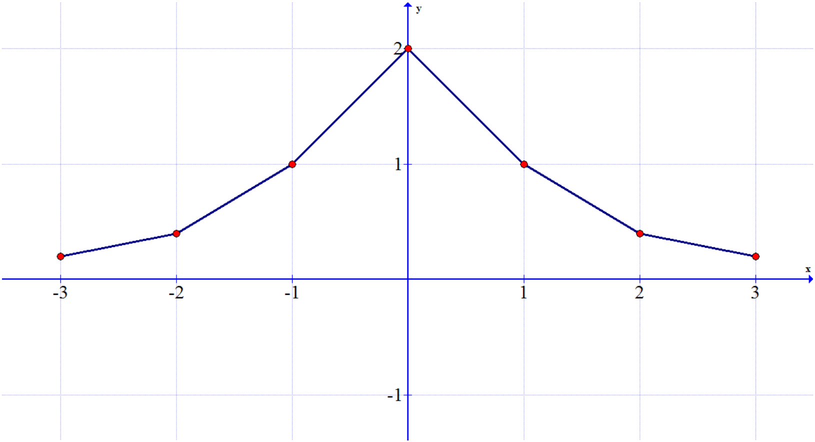
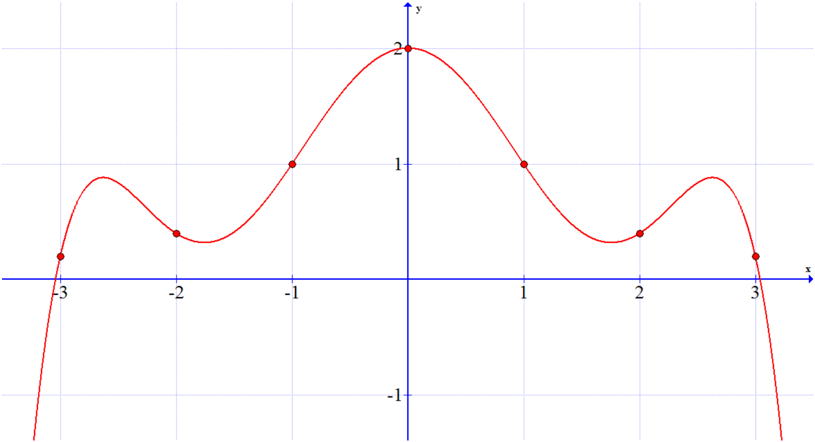
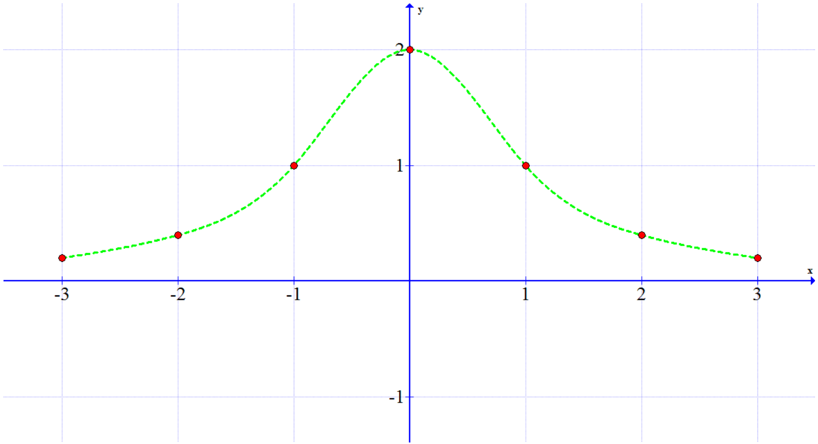
Linear Interpolation



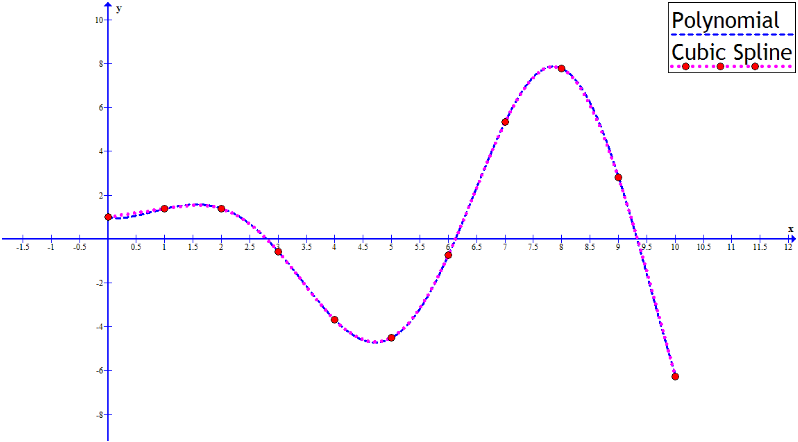
Polynomial Interpolation



Cubic Spline Interpolation



Polynomial Interpolation is not always bad!



Given n+1 data points (x_i, y_i) for $i = 0$ to n, develop n cubic equations between (x_i, y_i) and (x_{i+1}, y_{i+1}) for $i = 0$ to $n-1$, such that the 1st and 2nd derivatives at common points are continuous.

$$f(x) = \begin{cases} f_{0,1}(x) & \text{for} \quad x_0 \le x \le x_1 \\ f_{1,2}(x) & \text{for} \quad x_1 \le x \le x_2 \\ \cdots & \cdots \quad \cdots \\ f_{n-1,n}(x) & \text{for} \quad x_{n-1} \le x \le x_n \end{cases}$$

Given by:

$$\begin{aligned} f_{i,i+1}(x) = &\frac{M_i}{6}\left[h_i(x - x_{i+1}) - \frac{(x - x_{i+1})^3}{h_i}\right] \\ &- \frac{M_{i+1}}{6}\left[h_i(x - x_i) - \frac{(x - x_i)^3}{h_i}\right] \\ &+ \frac{y_{i+1}(x - x_i) - y_i(x - x_{i+1})}{h_i} \end{aligned}$$

Where, $M_i = y_i''$ $\qquad M_{i+1} = y_{i+1}''$ $\qquad h_i = x_{i+1} - x_i$

To find the second derivatives (for $i = 1$ to $n-1$):

$$M_{i-1}(x_i - x_{i-1}) + 2M_i(x_{i+1} - x_{i-1}) + M_{i+1}(x_{i+1} - x_i) = 6\left[\frac{y_{i+1} - y_i}{x_{i+1} - x_i} - \frac{y_i - y_{i-1}}{x_i - x_{i-1}}\right]$$

or,

$$M_{i-1}(h_{i-1}) + 2M_i(h_{i-1} + h_i) + M_{i+1}(h_i) = 6\left[\frac{\Delta y_i}{h_i} - \frac{\Delta y_{i-1}}{h_{i-1}}\right]$$

If $n = 4$:

For $i = 1$: $$M_0(h_0) + 2M_1(h_0 + h_1) + M_2(h_1) = 6\left[\frac{\Delta y_1}{h_1} - \frac{\Delta y_0}{h_0}\right]$$

For $i = 2$: $$M_1(h_1) + 2M_2(h_1 + h_2) + M_3(h_2) = 6\left[\frac{\Delta y_2}{h_2} - \frac{\Delta y_1}{h_1}\right]$$

For $i = 3$: $$M_2(h_2) + 2M_3(h_2 + h_3) + M_4(h_3) = 6\left[\frac{\Delta y_3}{h_3} - \frac{\Delta y_2}{h_2}\right]$$

$$
\begin{bmatrix} h_0 & 2(h_0+h_1) & h_1 & 0 & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & 0 \\ 0 & 0 & h_2 & 2(h_2+h_3) & h_3 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix}
$$

$$
= 6 \begin{bmatrix} \Delta y_1/h_1 - \Delta y_0/h_0 \\ \Delta y_2/h_2 - \Delta y_1/h_1 \\ \Delta y_3/h_3 - \Delta y_2/h_2 \end{bmatrix} = 6 \begin{bmatrix} [x_1, x_2] - [x_0, x_1] \\ [x_2, x_3] - [x_1, x_2] \\ [x_3, x_4] - [x_2, x_3] \end{bmatrix}
$$

If x is equally spaced:

$$
M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2} \Delta^2 y_{i-1}
$$

If $n = 4$,

$$
M_0 + 4M_1 + M_2 = \frac{6}{h^2} \Delta^2 y_0
$$

$$
M_1 + 4M_2 + M_3 = \frac{6}{h^2} \Delta^2 y_1
$$

$$
M_2 + 4M_3 + M_4 = \frac{6}{h^2} \Delta^2 y_2
$$

or,

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} \Delta^2 y_0 \\ \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

In **Natural Cubic Spline**, $M_0 = M_n = 0$

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} \Delta^2 y_0 \\ \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

Example

Using Cubic Spline Interpolation Technique, compute y(3), y(5), and y(7) from the following data:

x	2	4	6	8	10
y	7	6	9	11	8

Solution:

Here, n = 4

and x is in equally-spaced interval i.e., h = 2

Let, M_0, M_1, ..., M_n be the 2nd derivatives at $x = x_0$, x_1, ..., x_n

Thus, we have,

$$\begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} \Delta^2 y_0 \\ \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

In *Natural Cubic Spline*, $M_0 = M_n = 0$

So, the system of equations reduces to:

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} \Delta^2 y_0 \\ \Delta^2 y_1 \\ \Delta^2 y_2 \end{bmatrix}$$

or,

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \frac{6}{4} \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix}$$

or,

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 6.0 \\ -1.5 \\ -7.5 \end{bmatrix}$$

x	y	Δy	$\Delta^2 y$
2	7		
		-1	
4	6		4
		3	
6	9		-1
		2	
8	11		-5
		-3	
10	8		

On solving, we get,

$M_1 = 1.5804 \qquad M_2 = -0.3214 \qquad M_3 = -1.7946$

Thus We now have,

i	0	1	2	3	4
x	2	4	6	8	10
y	7	6	9	11	8
M	0	1.5804	-0.3214	-1.7946	0

To compute y(5), i.e., to compute y at x=5:

Since $x = 5$ lies between x_1 and x_2, we compute y(5) using:

$$f_{1,2}(x) = \frac{M_1}{6}\left[h(x-x_2) - \frac{(x-x_2)^3}{h}\right] - \frac{M_2}{6}\left[h(x-x_1) - \frac{(x-x_1)^3}{h}\right] + \frac{y_2(x-x_1) - y_1(x-x_2)}{h}$$

$$\therefore f(5) = \frac{1.5804}{6}\left[2(5-6) - \frac{(5-6)^3}{2}\right] - \frac{-0.3214}{6}\left[2(5-4) - \frac{(5-4)^3}{2}\right] + \frac{9(5-4) - 6(5-6)}{2}$$

$$= 7.1839$$