Set 3: Numerical methods for ODEs

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1 Implementation of explicit Euler Method

We have:

$$F = m\frac{d^2x}{dt^2} = -kx$$
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Under our simplification that $\frac{k}{m} = 1$:

$$\frac{d^2x}{dt^2} = -x$$

We use the explicit Euler method to approximate a solution to this ODE:

$$x(t+h) \approx x(t) + h\frac{dx}{dt}$$
$$x_{i+1} = x_i + hv_i.$$
$$v(t+h) \approx v(t) + h\frac{dv}{dt}$$
$$v_{i+1} = v_i - hx_i.$$

```
def explicit_Euler(x_0, v_0, h, s, plot=False, plotname='explicit.eps'):
    '''Given a starting position x_0 and velocity v_0, plots the position x
    and velocity v of a spring over time using the explicit Euler method
    with a step size h from t = 0 to t = s.'''

    x = np.array([x_0])
    v = np.array([v_0])
    t = np.arange(0, s, h)

    for i in xrange(len(t) - 1):
        x = np.append(x, x[i] + h*v[i])
        v = np.append(v, v[i] - h*x[i])

    if plot:
        plotter.figure(figsize=(10, 4))
        plotter.plot(t, x, color='blue', label='x')
        plotter.plot(t, v, color='red', label='v')
        plotter.xlabel('t')
        plotter.legend()
        plotter.savefig(plotname, format='eps', bbox_inches='tight', pad_inches=0.1)

    return (t, x, v)
```

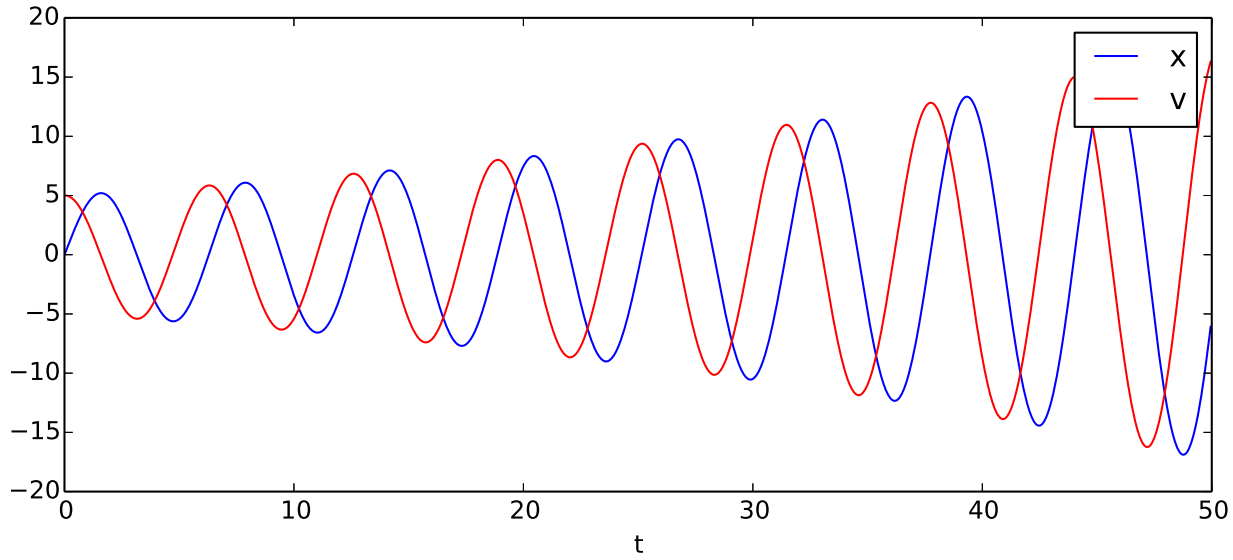


Figure 1: Numerical solutions for $x(t)$ and $v(t)$ as calculated by the explicit Euler method.

With initial conditions $x_0 = 0$, $v_0 = 5$, and a step size $h = 0.05$ with a stop time $s = 50$, the above subroutine produces the numerical solutions for $x(t)$ and $v(t)$ shown in figure 1.

2 Comparison with analytic solution

We show that a generic sinusoidal function is a solution to the given ODE.

$$\begin{aligned} x &= A\sin t + B\cos t \\ v = \frac{dx}{dt} &= A\cos t - B\sin t \\ \frac{d^2x}{dt^2} &= -A\sin t - B\cos t = -x \end{aligned}$$

Using our initial values $x = x_0$ and $v = v_0$ at $t = 0$:

$$\begin{aligned} x_0 &= A\sin 0 + B\cos 0 \\ &= B. \\ v_0 &= A\cos 0 - B\sin 0 \\ &= A. \end{aligned}$$

Thus, the complete analytical solution is:

$$\begin{aligned} x &= v_0\sin t + x_0\cos t. \\ v &= v_0\cos t - x_0\sin t. \end{aligned}$$

Figure 2 shows the global trunctation error, the difference between the explicit Euler method used in section 1 and the analytic method derived here. The error is computed in the `global_error` function:

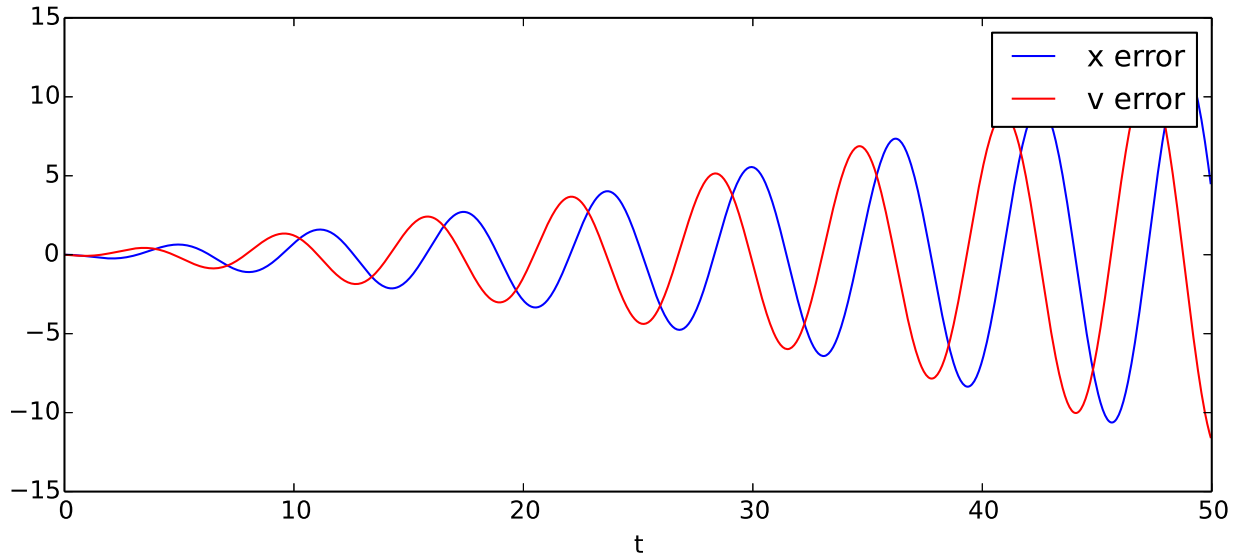


Figure 2: Truncation error: difference between the analytic solution and the solution calculated from the explicit Euler method.

```
def global_error(x_0, v_0, h, s, method_a, method_b, plot=False, plotname=None):
    '''Generates arrays of the global error method_b - method_a for x and v.
    Returns maximum error in x and v.'''

    (t_a, x_a, v_a) = method_a(x_0, v_0, h, s)
    (t_b, x_b, v_b) = method_b(x_0, v_0, h, s)

    x_diff = x_b - x_a
    v_diff = v_b - v_a

    assert np.array_equal(t_a, t_b)
    t = t_a

    if plotname is None:
        plotname = '%s_error.eps' % method_a

    if plot:
        plotter.figure(figsize=(10, 4))
        plotter.plot(t, x_diff, color='blue', label='x error')
        plotter.plot(t, v_diff, color='red', label='v error')
        plotter.xlabel('t')
        plotter.legend()
        plotter.savefig(plotname, format='eps', bbox_inches='tight', pad_inches=0.1)

    max_x_error = np.max(np.absolute(x_diff))
    max_v_error = np.max(np.absolute(v_diff))

    return (max_x_error, max_v_error)
```

3 Relationship between truncation error and step size h

We examine the effect of changing the step size h on the truncation error with the following subroutine `error_vs_h`, which plots the maximum truncation error for various h ranging from a given h_0 to $\frac{h_0}{16}$.

```
def error_vs_h(x_0, v_0, h_0, s, method_a, method_b, plot=False, plotname=None):
    '''Compares global error between method_a and method_b for x and v
    for h ranging from h_0 to h_0/16.'''

    coeff = np.logspace(0, 4, base=2)
    h = h_0 / coeff

    # allow global_error function to take an array of h and s
    global_error_vec = np.vectorize(global_error, excluded=['x_0', 'v_0', \
    's', 'method_a', 'method_b', 'plot'])

    errors = global_error_vec(x_0, v_0, h, s, method_a, method_b)
    x_errors = errors[0]
    v_errors = errors[1]

    if plotname is None:
        plotname = '%s_error_v_h.eps' % method_a

    if plot:
        plotter.figure(figsize=(10, 4))
        plotter.plot(h, x_errors, color='blue', label='x error')
        plotter.plot(h, v_errors, color='red', label='v error')
        plotter.xlabel('h')
        plotter.legend()
        plotter.savefig(plotname, format='eps', bbox_inches='tight', pad_inches=0.1)
```

Figure 3 shows an exponential trend for various, somewhat large h ($h_0 = 0.1$). If we restrict the domain to small h, we expect to see a small portion of this exponential curve, which will appear linear. As expected, figure 4 shows a roughly linear trend when h is small ($h_0 = 0.01$).

4 Energy conservation in explicit Euler method

Given $E = x^2 + v^2$, we can plot energy as a function of time with our numerical solutions for x and v. Figure 5 shows that the numerical solution does not exhibit energy conservation, instead showing energy increasing over time.

Visually, the trend appears to be quadratic or exponential. By generating a log plot in figure 6, we confirm that growth trend of $E(t)$ is indeed exponential.

5 Implicit Euler method

The implicit Euler method uses values at $t + h$ to updates values at $t + h$, as opposed to the explicit Euler method which used values at t to updates values at $t + h$.

$$x_{i+1} = x_i + hv_{i+1}.$$
$$v_{i+1} = v_i - hx_{i+1}.$$

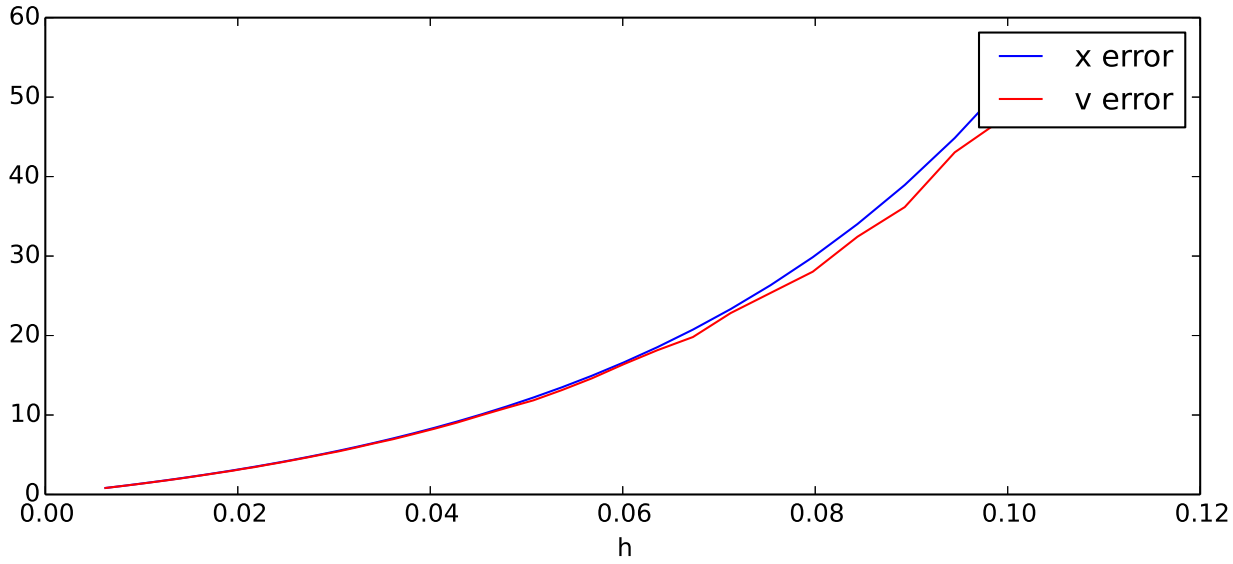


Figure 3: Truncation error vs. h for large values of h.

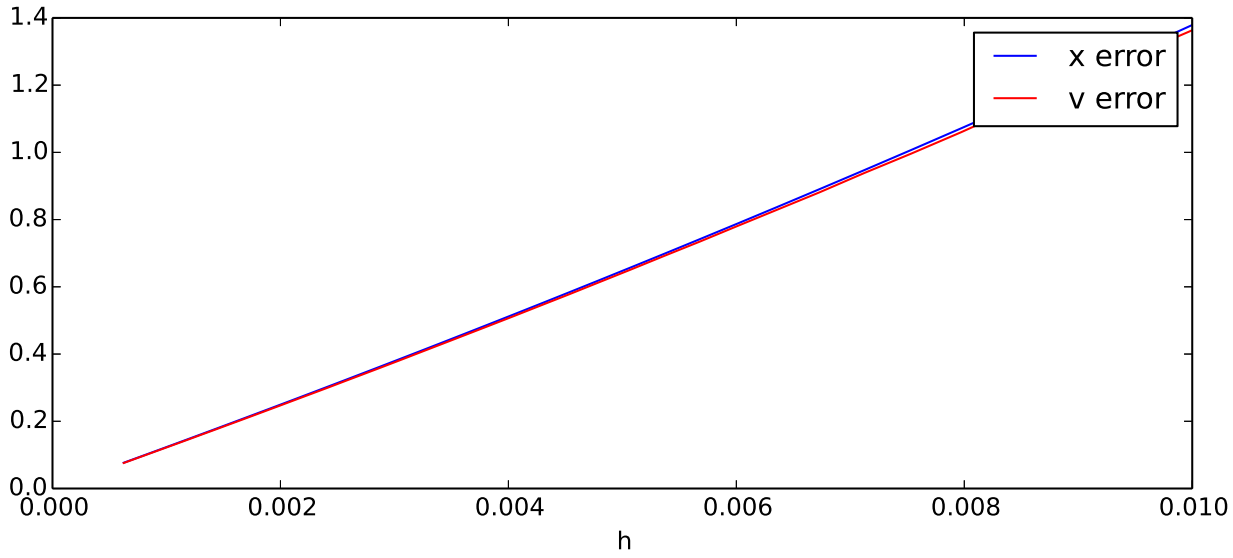


Figure 4: Truncation error vs. h for small values of h.

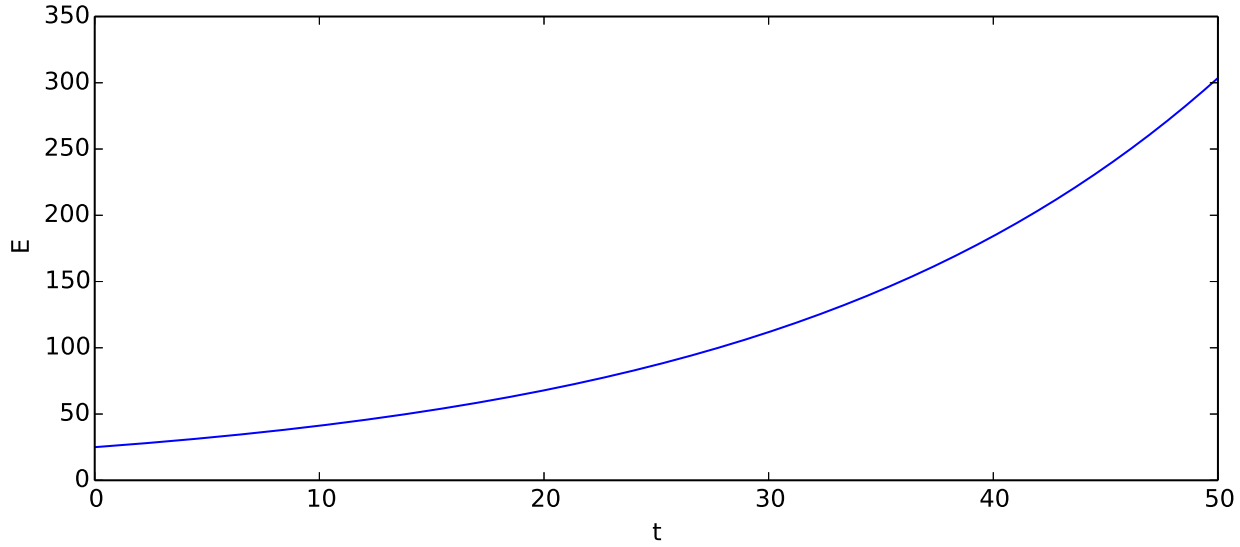


Figure 5: $E(t)$ as determined from explicit-Euler-method numerical solutions to $x(t)$ and $v(t)$.

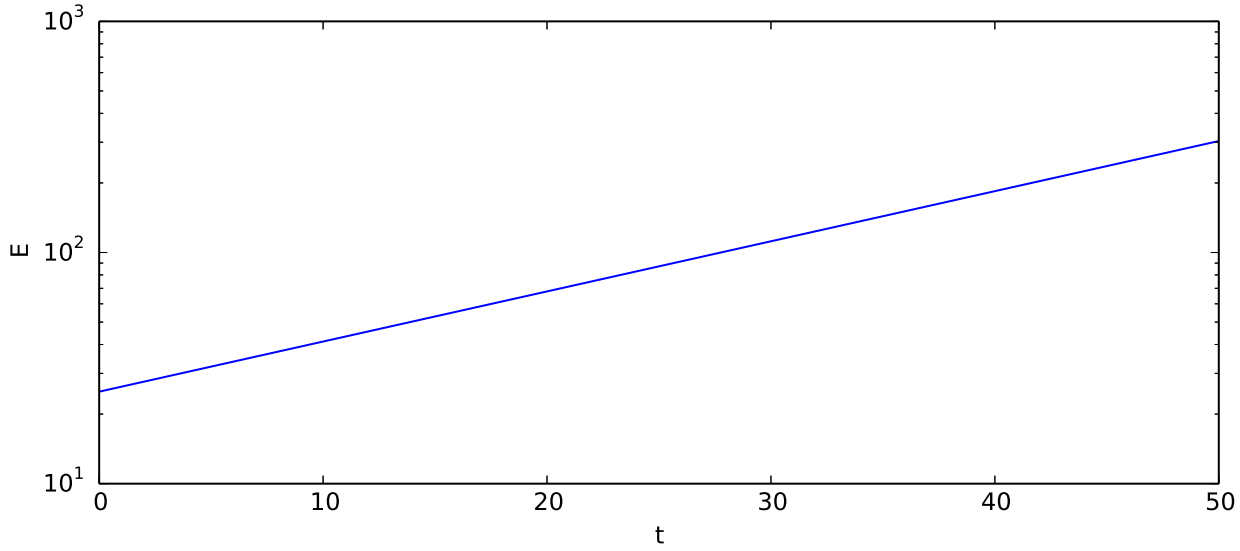


Figure 6: $E(t)$ as determined from explicit-Euler-method numerical solutions, with E plotted on a log scale. The linear graph shows that $E(t)$ increases exponentially with t.

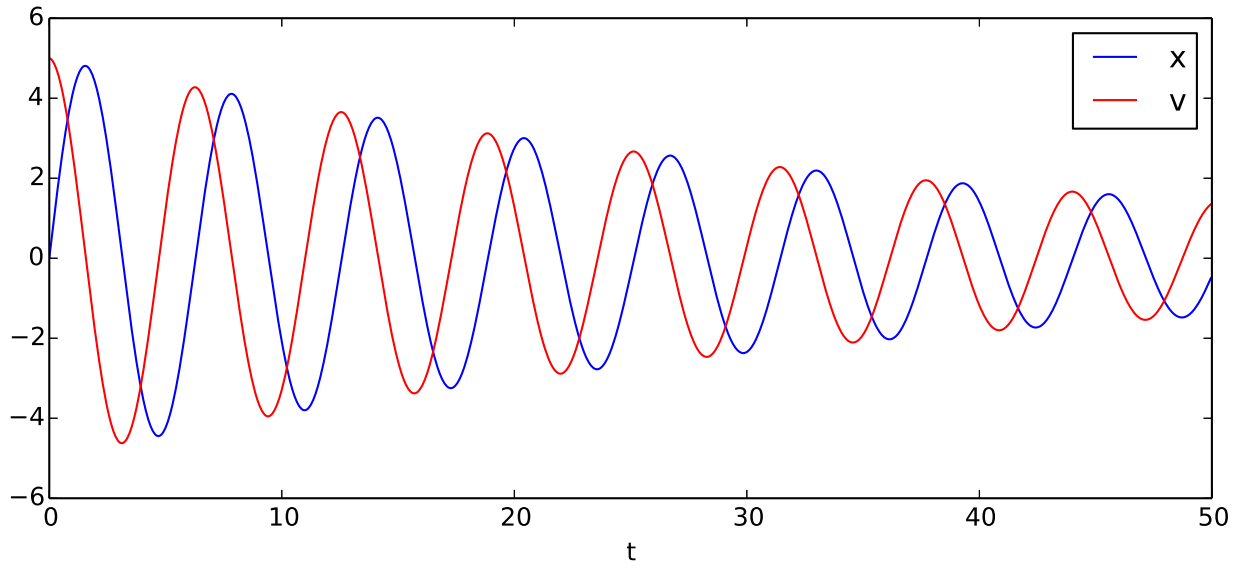


Figure 7: Numerical solutions for $x(t)$ and $v(t)$ as calculated by the implicit Euler method.

We obtain $x(t+h)$ and $v(t+h)$ in terms of $x(t)$ and $v(t)$ with some algebraic manipulation. As demonstrated, the two above equations are equivalent to the linear system:

$$\begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix} \begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ v_i \end{pmatrix}$$

Using *Mathematica*, we find that:

$$x_{i+1} = \frac{x_i + hv_i}{h^2+1}.$$
$$v_{i+1} = \frac{v_i - hx_i}{h^2+1}.$$

Using this method and the same conditions as in figure 1, we can generate new numerical solutions for x and v in figure 7. We also show the global error between the implicit and analytic solutions in figure 8.

With the new numerical solutions, we also generate a new solution for $E(t)$, plotted in figure 9.

6 Phase-space geometry and the non-symplectic Euler methods

Since $E = x^2 + v^2$, and energy should be conserved in an ideal spring, plotting the spring's trajectory in the xv-plane should yield a circle of radius $\sqrt{E}$. Of course, we have already seen that the explicit and implicit Euler methods do not conserve energy, so their phase-space trajectories will be spirals rather than closed circles, as shows in figure 10. These plots are generated with the same conditions as before: $x_0 = 0, v_0 = 5, h = 0.05, s = 50$.

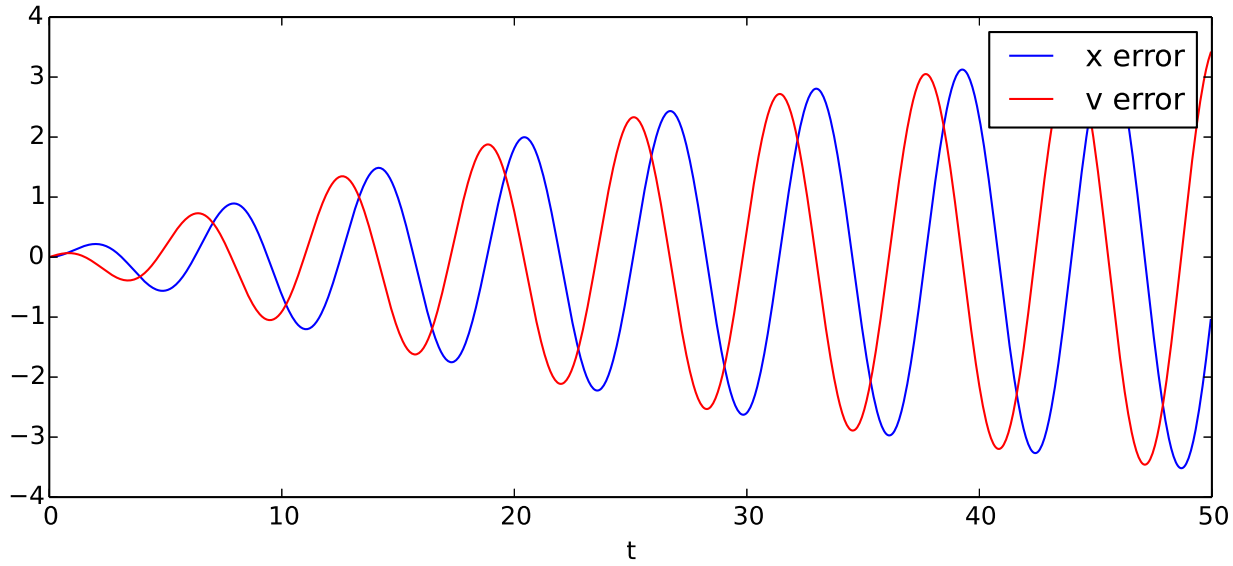


Figure 8: Truncation error for the implicit Euler method.

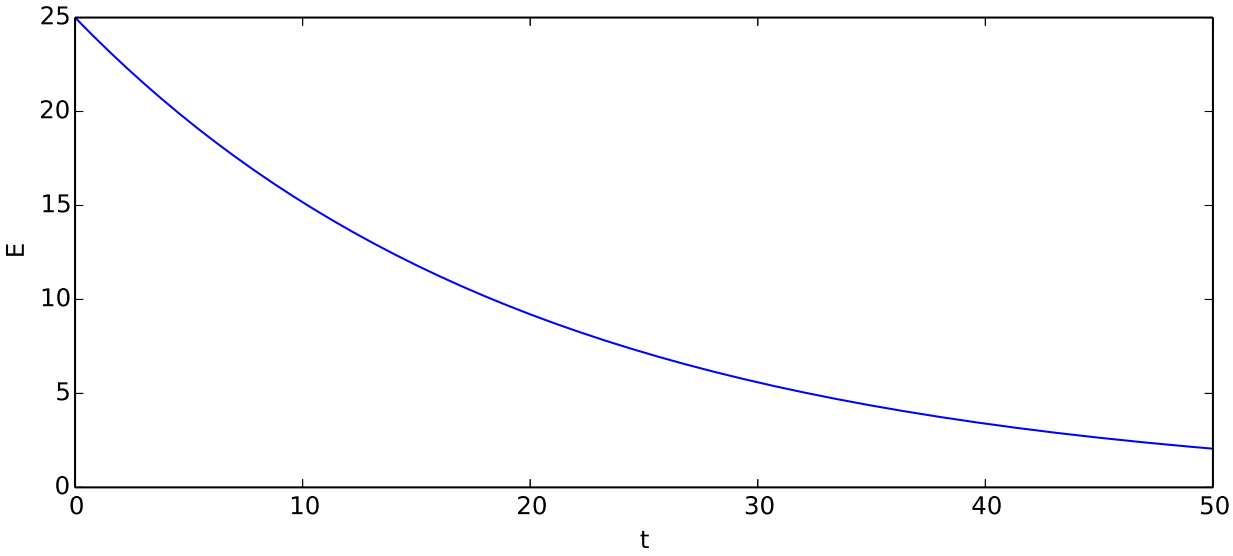
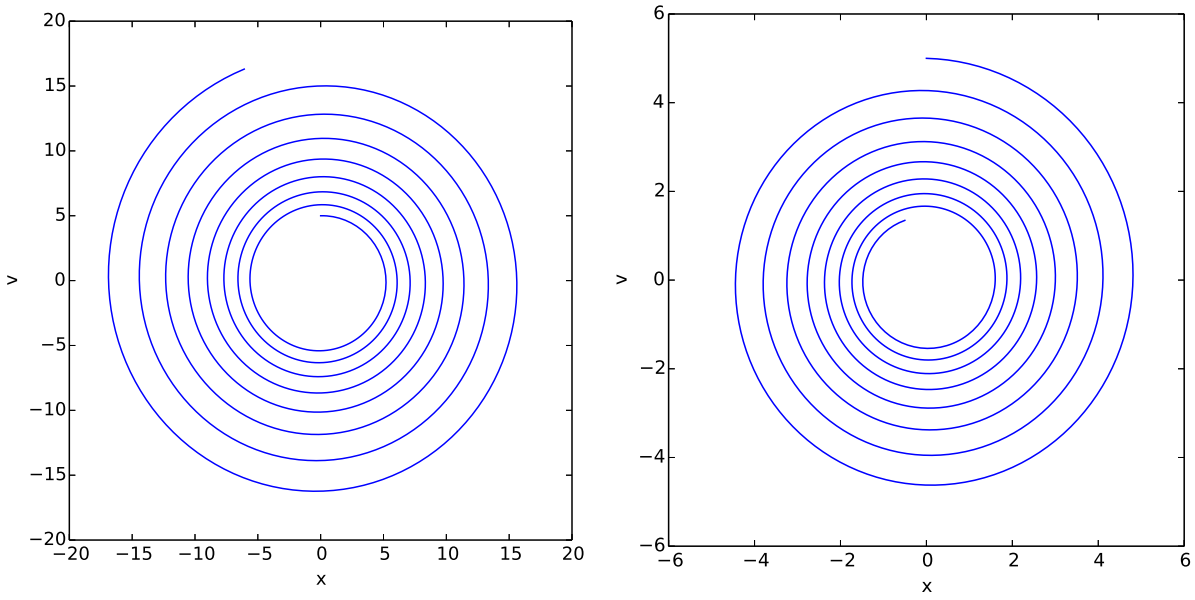


Figure 9: $E(t)$ as determined from implicit-Euler-method numerical solutions to $x(t)$ and $v(t)$.



(a) explicit Euler method

(b) implicit Euler method

Figure 10: Phase-space trajectories for explicit and implicit Euler methods.

7 Implementation of symplectic Euler method

To design a numerical method that conserves area in phase space (and thus conserves energy), we combine the explicit and implicit Euler methods:

$$\begin{aligned} x_{i+1} &= x_i + hv_i. \\ v_{i+1} &= v_i - hx_{i+1} \\ &= v_i - h(x_i + hv_i) \\ &= v_i - hx_i - h^2 v_i. \end{aligned}$$

Figure 11 shows the numerical solutions for $x(t)$ and $v(t)$ obtained through this method. We immediately notice that, as expected from the analytic solution to a spring's motion, the amplitude does not change over time in our time frame ($t = 0$ to $t = 50$).

Figure 12 shows the phase-space trajectory, which appears closed in our time frame. There is a visible offset from the perfect circle traced out by the analytic solution: sometimes the symplectic curve is inside the analytic curve, and other times it is outside. Since the radius of the curve is $\sqrt{E}$, this means we should expect the symplectic Euler method to yield small oscillations in E. This is exactly what happens: see section 8.

8 Energy evolution of symplectic Euler method

Though the phase-space trajectory (figure 12) suggests that energy is conserved with the symplectic Euler method, the calculated energy undergoes small oscillations around the expected constant value of 25.0 (see figure 13).

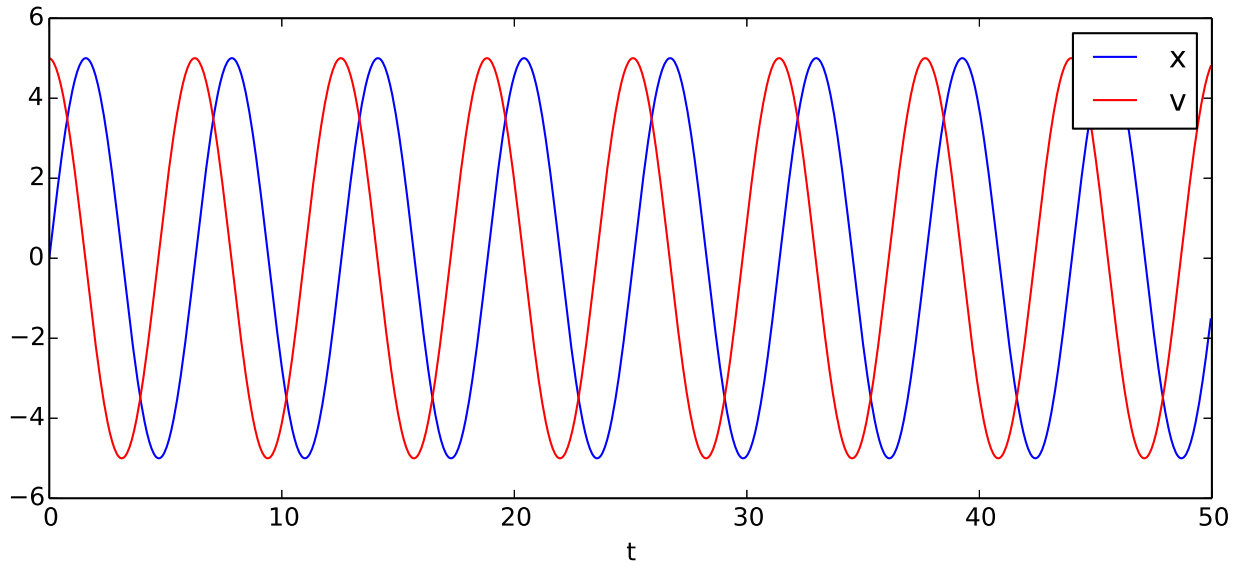


Figure 11: Numerical solutions for $x(t)$ and $v(t)$ as calculated by the symplectic Euler method.

9 Long-term error in symplectic Euler method

In addition to the small oscillations in energy, the symplectic Euler method exhibits another imperfection: the calculated solutions for $x(t)$ and $v(t)$ lag behind the analytic solutions. Figure 14 shows this phenomenon. Note that the figure is plotted from $t = 4950$ to $t = 5000$; the error is not appreciable at small times.

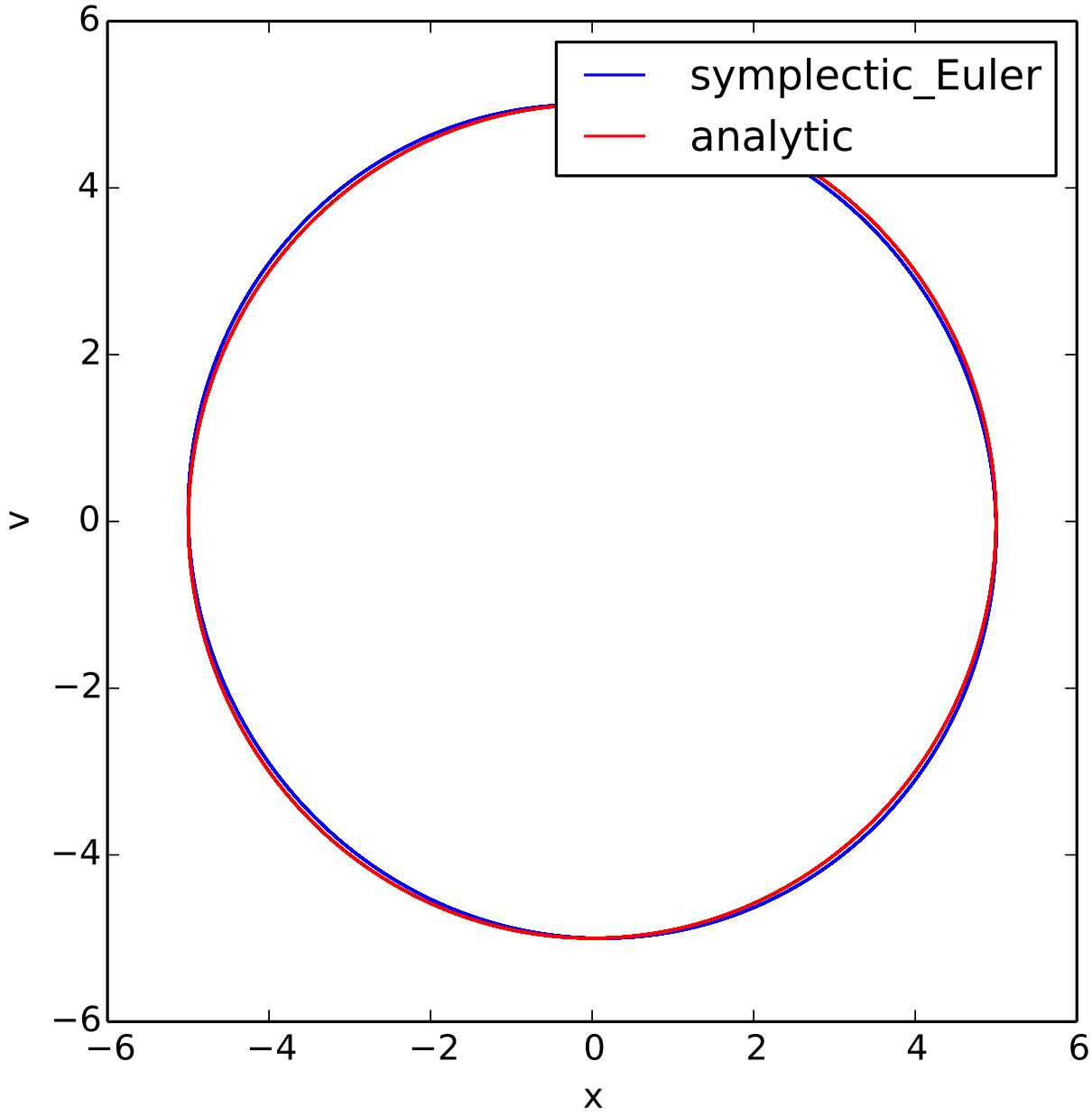


Figure 12: Phase-space trajectory for symplectic Euler and analytic methods.

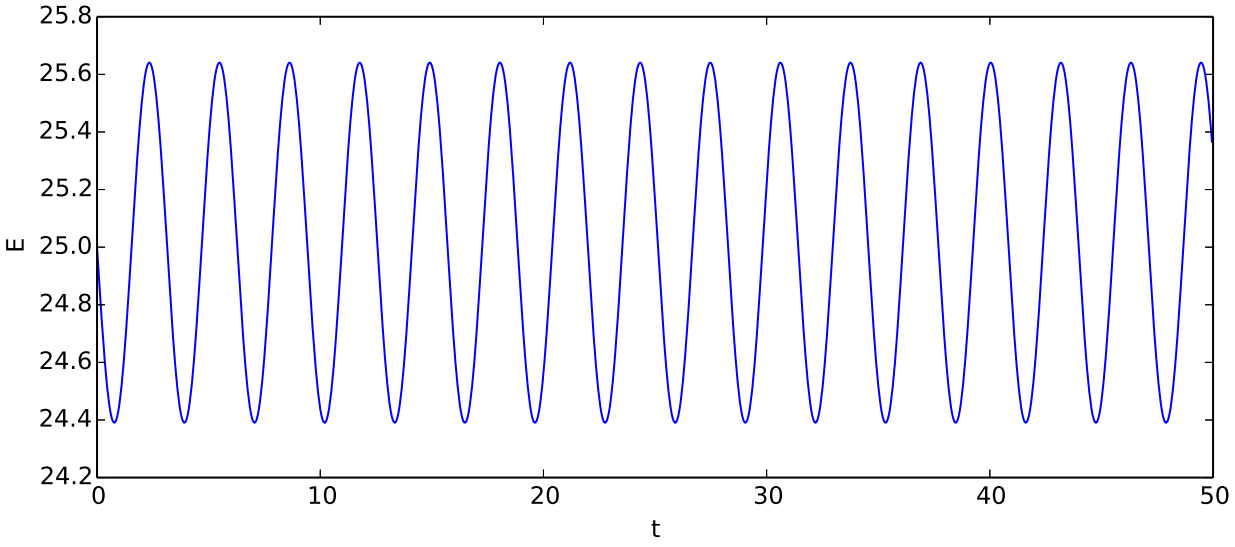


Figure 13: $E(t)$ for symplectic Euler method.

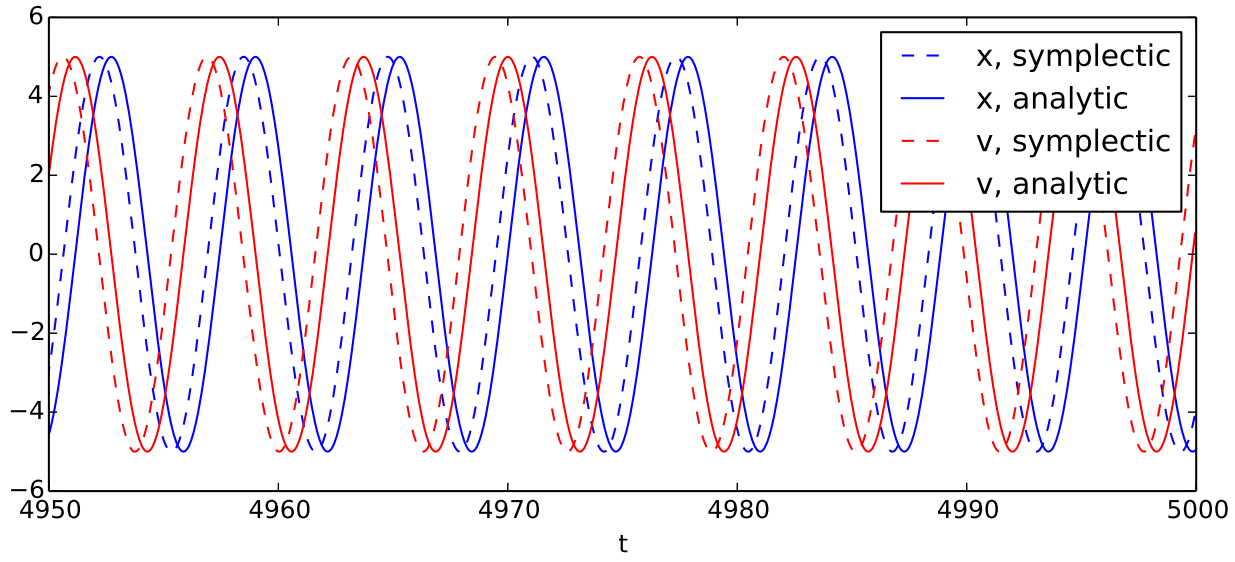


Figure 14: Comparison of numerical and analytic solutions for $x(t)$ and $v(t)$, showing that at large t, the symplectic solution exhibits a noticeable lag.