Set 6: Numerical Computations with *Mathematica*

First, we integrate the equations of motion without drag.

In[409]:=
```
EqNoDrag = {X''[T] == 0, Y''[T] == -g}
```

Out[409]= $\{X''(T) = 0,\ Y''(T) = -g\}$

In[410]:=
```
Init = {X[0] == 0, Y[0] == 0, X'[0] == V Cos[θ], Y'[0] == V Sin[θ]}
```

Out[410]= $\{X(0) = 0,\ Y(0) = 0,\ X'(0) = V\cos(\theta),\ Y'(0) = V\sin(\theta)\}$

In[411]:=
```
NoDragSystem = Join[EqNoDrag, Init]
```

Out[411]= $\{X''(T) = 0,\ Y''(T) = -g,\ X(0) = 0,\ Y(0) = 0,\ X'(0) = V\cos(\theta),\ Y'(0) = V\sin(\theta)\}$

In[412]:=
```
NoDragSolution = Simplify[DSolve[NoDragSystem, {X[T], Y[T]}, T]]
```

Out[412]= $$\left\{\left\{X(T) \to T\,V\cos(\theta),\ Y(T) \to T\,V\sin(\theta) - \frac{g\,T^2}{2}\right\}\right\}$$

In[413]:=
```
x[t_] := (X[T] /. NoDragSolution[[1]]) /. T → t
```

In[414]:=
```
y[t_] := (Y[T] /. NoDragSolution[[1]]) /. T → t
```

We derive the range of the projectile and show that it is maximized with at 45-degree firing angle.

In[415]:=
```
NoDragTimeAtGround = Solve[y[t] == 0, t]
```

Out[415]= $$\left\{\{t \to 0\},\ \left\{t \to \frac{2\,V\sin(\theta)}{g}\right\}\right\}$$

In[416]:=
```
NoDragXPositionsAtGround = x[t] /. NoDragTimeAtGround
```

Out[416]= $$\left\{0,\ \frac{2\,V^2\sin(\theta)\cos(\theta)}{g}\right\}$$

In[417]:=
```
NoDragXRange = NoDragXPositionsAtGround[[2]] - NoDragXPositionsAtGround[[1]]
```

Out[417]= $$\frac{2\,V^2\sin(\theta)\cos(\theta)}{g}$$

In[418]:=
```
Solve[(∂θ NoDragXRange) == 0, θ]
```

Out[418]= $$\left\{\left\{\theta \to \text{ConditionalExpression}\left[2\pi c_1 - \frac{3\pi}{4},\ c_1 \in \mathbb{Z}\right]\right\},\ \left\{\theta \to \text{ConditionalExpression}\left[2\pi c_1 - \frac{\pi}{4},\ c_1 \in \mathbb{Z}\right]\right\},\ \left\{\theta \to \text{ConditionalExpression}\left[2\pi c_1 + \frac{\pi}{4},\ c_1 \in \mathbb{Z}\right]\right\},\ \left\{\theta \to \text{ConditionalExpression}\left[2\pi c_1 + \frac{3\pi}{4},\ c_1 \in \mathbb{Z}\right]\right\}\right\}$$

These solutions all correspond to 45 degrees. We want to reach PCC, which is 1000 m away. With the optimal firing angle, we find the required velocity.

In[419]:=
```
StandardFiring = {θ → π / 4}
```

Out[419]= $$\left\{\theta \to \frac{\pi}{4}\right\}$$

In[420]:=
```
PhysicalConditions = {g → 9.81}
```

Out[420]= $\{g \to 9.81\}$

In[421]:=
```
VelocityNoDrag =
  (NSolve[NoDragXRange == 1000, V] /. Join[StandardFiring, PhysicalConditions])[[2]]
```

Out[421]= $\{V \to 99.0454\}$

We also show the optimal performance of the 45-degree angle visually.

In[422]:=
```
Nx[t_, θ_] := x[t] /. Join[PhysicalConditions, VelocityNoDrag]
```

In[423]:=
```
Ny[t_, θ_] := y[t] /. Join[PhysicalConditions, VelocityNoDrag]
```

In[424]:=
```
ToRad[Θ_] := (2 π)/360 * Θ
```

In[425]:=
```
FiringAngles = {ToRad[15], ToRad[30], ToRad[45], ToRad[60], ToRad[75]}
```

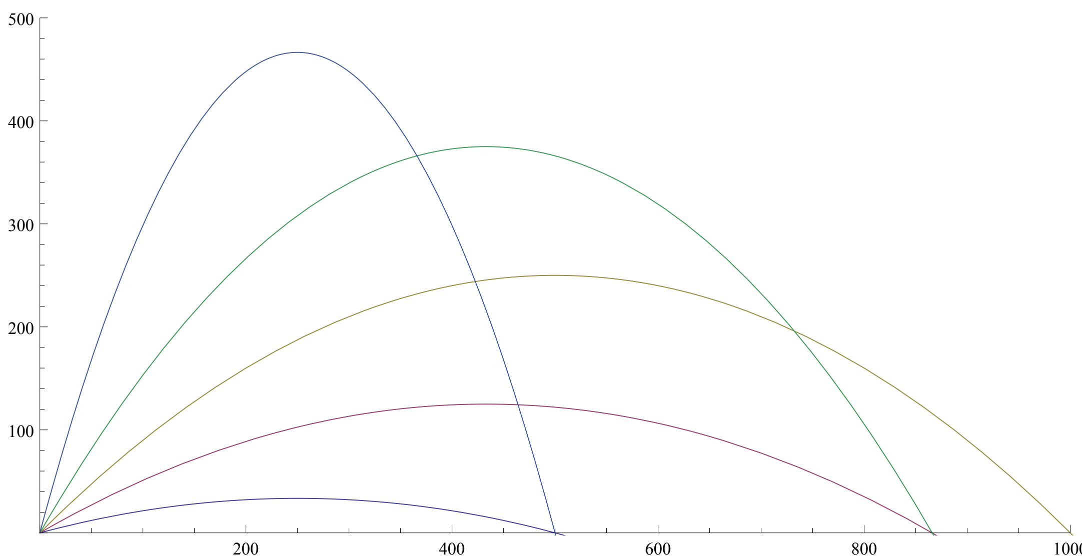
Out[425]= $\left\{\frac{\pi}{12}, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}\right\}$

In[426]:=
```
Traj = Transpose[{Nx[t, θ], Ny[t, θ]} /. θ → FiringAngles]
```

Out[426]= $$\begin{pmatrix} 95.6706\,t & 25.6348\,t - 4.905\,t^2 \\ 85.7759\,t & 49.5227\,t - 4.905\,t^2 \\ 70.0357\,t & 70.0357\,t - 4.905\,t^2 \\ 49.5227\,t & 85.7759\,t - 4.905\,t^2 \\ 25.6348\,t & 95.6706\,t - 4.905\,t^2 \end{pmatrix}$$

In[427]:=
```
ParametricPlot[Traj, {t, 0, 20}, PlotRange → {{0, 1000}, {0, 500}}]
```

Out[427]=



Clearly, the 45-degree firing angle maximizes the projectile's range. We now include the drag force in our equations of motion and generate a more accurate solution.

In[486]:=
```
FDrag = -1/2 ρ r^2 v^2
```
Out[486]= $-\frac{1}{2}\rho r^2 \left(X'(T)^2 + Y'(T)^2\right)$

In[487]:=
```
v = Sqrt[(X'[T])^2 + (Y'[T])^2]
```
Out[487]= $\sqrt{X'(T)^2 + Y'(T)^2}$

In[491]:=
```
EqDrag = {X''[T] == FDrag/m * X'[T]/v, Y''[T] == -g + (FDrag/m * Y'[T]/v)}
```
Out[491]= $\left\{X''(T) = -\frac{\rho r^2 X'(T)\sqrt{X'(T)^2 + Y'(T)^2}}{2m},\ Y''(T) = -g - \frac{\rho r^2 Y'(T)\sqrt{X'(T)^2 + Y'(T)^2}}{2m}\right\}$

In[492]:=
```
Init = {X[0] == 0, Y[0] == 0, X'[0] == V Cos[θ], Y'[0] == V Sin[θ]}
```
Out[492]= $\{X(0) = 0,\ Y(0) = 0,\ X'(0) = V\cos(\theta),\ Y'(0) = V\sin(\theta)\}$

In[493]:=
```
DragSystem = Join[EqDrag, Init]
```
Out[493]= $\left\{X''(T) = -\frac{\rho r^2 X'(T)\sqrt{X'(T)^2 + Y'(T)^2}}{2m},\right.$

$\left.Y''(T) = -g - \frac{\rho r^2 Y'(T)\sqrt{X'(T)^2 + Y'(T)^2}}{2m},\ X(0) = 0,\ Y(0) = 0,\ X'(0) = V\cos(\theta),\ Y'(0) = V\sin(\theta)\right\}$

In[494]:=
```
DragConditions = {g → 9.81, ρ → 1.3, r → 0.05, m → 0.5}
```
Out[494]= $\{g \to 9.81,\ \rho \to 1.3,\ r \to 0.05,\ m \to 0.5\}$

In[495]:=
```
DragSolution[argV_, argθ_] :=
 NDSolve[(DragSystem /. Join[DragConditions, {V → argV, θ → argθ}]),
  {X[T], Y[T]}, {T, 0, 1000}]
```
In[496]:=
```
x[V_, θ_, t_] := (X[T] /. (DragSolution[V, θ])[[1]]) /. T → t
```
In[497]:=
```
y[V_, θ_, t_] := (Y[T] /. (DragSolution[V, θ])[[1]]) /. T → t
```
We compute the time of impact with a starting guess equivalent to the time of impact in the non-drag case.

In[498]:=
```
DragImpactTime[V_, θ_] := FindRoot[y[V, θ, t] == 0, {t, 2 V Sin[θ]/9.81}]
```
In[499]:=
```
DragXRange[V_, θ_] := x[V, θ, (t /. DragImpactTime[V, θ])]
```
In[500]:=
```
DragXRange[(V /. VelocityNoDrag), (θ /. StandardFiring)]
```
Out[500]= 333.33

Thus, with the same initial speed and 45-degree angle that we found optimal in the no-drag case, the projectile travels one-third of the desired distance. We now want to find the optimal firing conditions with drag - i.e. the firing angle that requires the least initial speed.

To do this, we first define a function that, given a firing angle, finds the minimum velocity required to travel a given range by iterating through possible velocities.

```
In[503]:= DragMinVelocityForRange[θ_, D_] := Module[
   {V},
   V = 0;
   While[DragXRange[V, θ] < D,
    V = V + 500];
   V  =  V - 500;
   While[DragXRange[V, θ] < D,
    V = V + 100];
   V = V - 100;
   While[DragXRange[V, θ] < D,
    V = V + 10];
   V = V - 10;
   While[DragXRange[V, θ] < D,
    V = V + 1];
   V = V - 1;
   While[DragXRange[V, θ] < D,
    V = V + 0.1];
   V = V - 1;
   V]
```

```
In[504]:= DragMinVelocityForRange[π / 4, 1000]
```

Out[504]= 1320.9

With a 45-degree firing angle, we would need to fire the projectile at a whopping 1320.9 m/s to reach PCC. We now iterate through various firing angles, find the minimum velocity required for each angle, find the lowest among these minimum velocities, and return the firing angle associated with that velocity - the optimum firing angle.

```
In[514]:= DragOptimalAngle[D_] := Module[
   {θ},
   θ = ToRad[1];
   While[DragMinVelocityForRange[θ + ToRad[10], D] < DragMinVelocityForRange[θ, D],
    θ = θ + ToRad[10]];
   While[DragMinVelocityForRange[θ + ToRad[1], D] < DragMinVelocityForRange[θ, D],
    θ = θ + ToRad[1]];
   θ]
```

```
In[516]:= DragOptimalAngle[1000]
```

Out[516]= $\frac{2\pi}{15}$

In[518]:= `ToDeg[θ_] :=` $\frac{360}{2\pi} * \theta$

```
In[519]:= ToDeg[2 π / 15]
```

Out[519]= 24

Thus, we find that a 24-degree angle is optimal. Figure 2 below shows the trajectories for various firing angles, with the initial velocities adjusted to be the minimum velocity required to travel 1000 m.

In[520]:=
```
DragFiringAngles = {ToRad[15], ToRad[24], ToRad[30], ToRad[40], ToRad[50]}
```

Out[520]= $\left\{\frac{\pi}{12}, \frac{2\pi}{15}, \frac{\pi}{6}, \frac{2\pi}{9}, \frac{5\pi}{18}\right\}$

In[536]:=
```
DragTraj = Table[{x[(DragMinVelocityForRange[θ, 1000]), θ, t],
    y[(DragMinVelocityForRange[θ, 1000]), θ, t]}, {θ, DragFiringAngles}]
```

Out[536]=

InterpolatingFunction[(0. 1000.), <>](t)	InterpolatingFunction[(0. 1000.), <>](t)
InterpolatingFunction[(0. 1000.), <>](t)	InterpolatingFunction[(0. 1000.), <>](t)
InterpolatingFunction[(0. 1000.), <>](t)	InterpolatingFunction[(0. 1000.), <>](t)
InterpolatingFunction[(0. 1000.), <>](t)	InterpolatingFunction[(0. 1000.), <>](t)
InterpolatingFunction[(0. 1000.), <>](t)	InterpolatingFunction[(0. 1000.), <>](t)

In[539]:=
```
ParametricPlot[DragTraj, {t, 0, 100}, PlotRange → {{0, 1000}, {0, 800}}]
```

Out[539]=