Set 5, Part 2: Symbolic Algebra with *Mathematica*

sin(*x*) and cos(*x*) in series form

We want to compare the series expansions of sin(*x*) and cos(*x*) to the actual functions. We define series versions of sin(*x*) and cos(*x*) about *x* = 0.

In[1]:=
```
SerSin[x_, n_] := Normal[Series[Sin[y], {y, 0, n}]] /. y → x
```

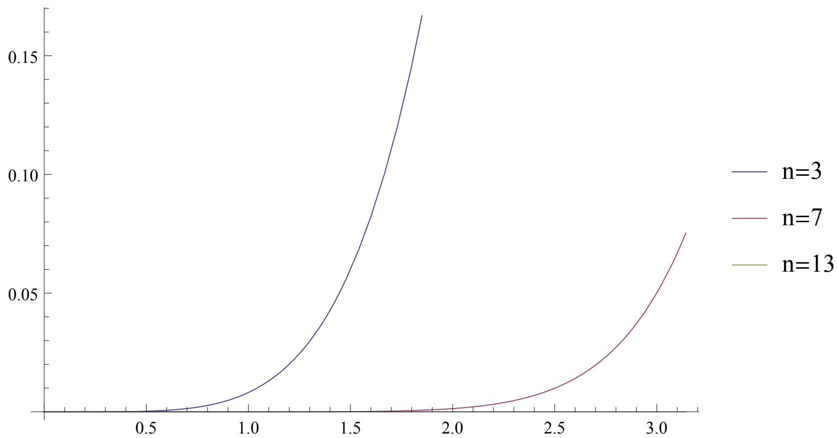
In[2]:=
```
SerCos[x_, n_] := Normal[Series[Cos[y], {y, 0, n}]] /. y → x
```

We compare the series and normal versions of sin(*x*), plotting the error as a function of *x* for various *n.*

In[3]:=
```
SinError[x_, n_] := Abs[N[SerSin[x, n] - Sin[x]]]
```

In[4]:=
```
Plot[{SinError[x, 3], SinError[x, 7], SinError[x, 13]},
 {x, 0, π}, PlotLegends → {"n=3", "n=7", "n=13"}]
```

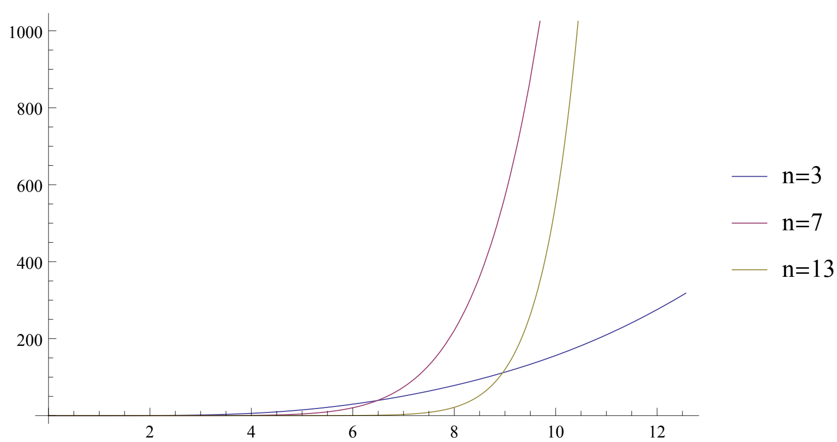
Out[4]=



As we go substantially far away from the point of expansion (*x* = 0), the higher-order approximations are substantially less accurate.

In[5]:=
```
Plot[{SinError[x, 3], SinError[x, 7], SinError[x, 13]},
 {x, 0, 4 π}, PlotLegends → {"n=3", "n=7", "n=13"}]
```

Out[5]=



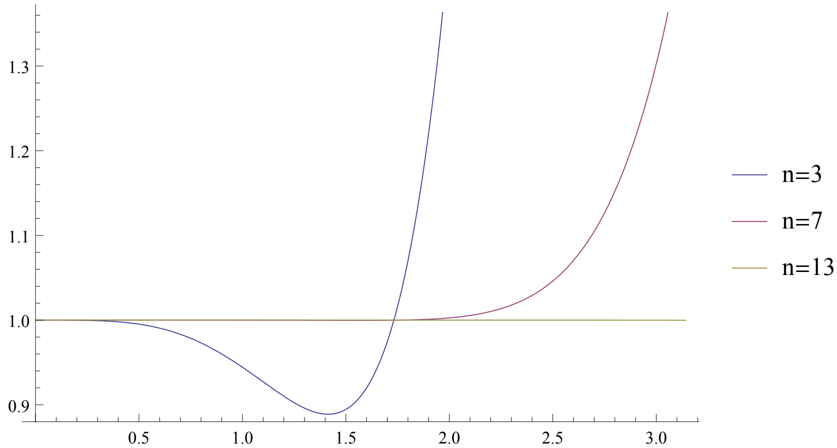
Squaring

We check whether $\sin^2(x) + \cos^2(x) = 1$ holds for the series versions.

In[36]:=
```
Squared1[x_, n_] := (SerSin[x, n])^2 + (SerCos[x, n])^2
```

In[37]:=
```
Plot[{Squared1[x, 3], Squared1[x, 7], Squared1[x, 13]},
 {x, 0, π}, PlotLegends → {"n=3", "n=7", "n=13"}]
```

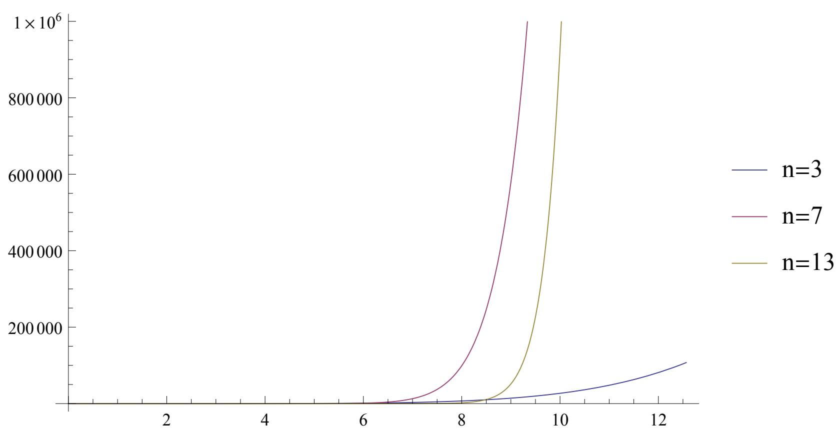
Out[37]=



As expected, when *x* is close to 0 (the point of series expansion), a higher *n* yields a more accurate result.

In[34]:=
```
Plot[{Squared1[x, 3], Squared1[x, 7], Squared1[x, 13]},
 {x, 0, 4 π}, PlotLegends → {"n=3", "n=7", "n=13"}]
```

Out[34]=



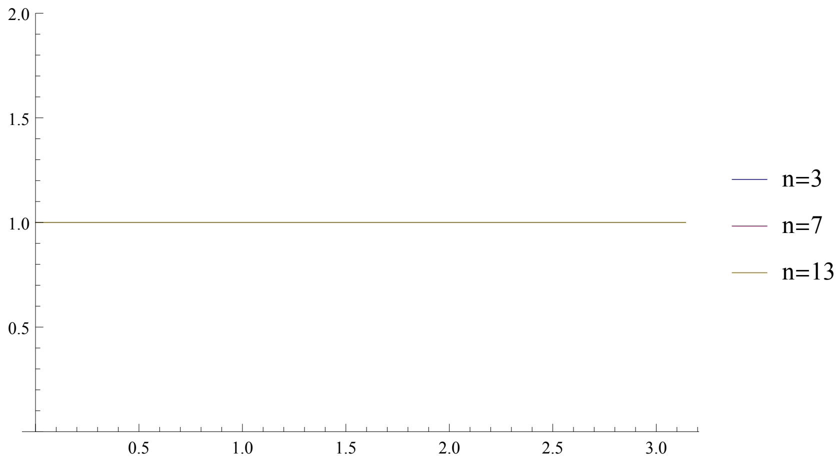
As before, moving far away from the point of expansion causes the higher-order approximations to become far less accurate, yielding values on the order of 10^5 and 10^6 instead of 1. We make another attempt using the series expansion for the trigonometric functions already squared:

In[44]:=
```
SerSinSq[x_, n_] := Normal[Series[(Sin[y])^2, {y, 0, n}]] /. y → x
```

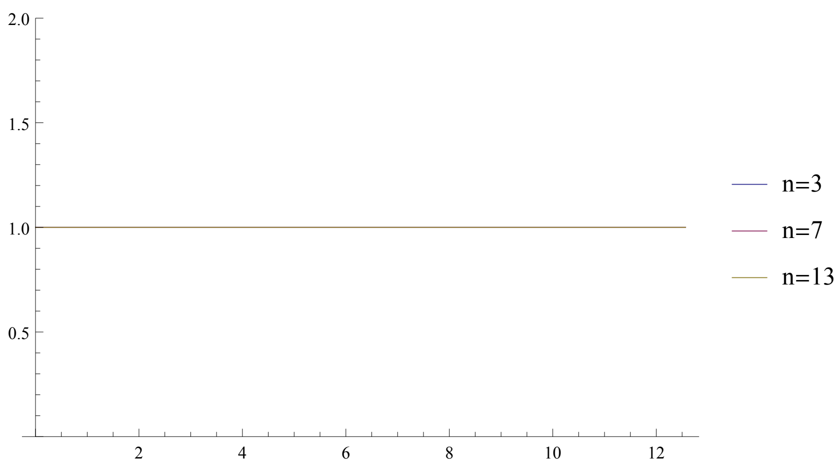
In[45]:=
```
SerCosSq[x_, n_] := Normal[Series[(Cos[y])^2, {y, 0, n}]] /. y → x
```

In[46]:=
```
Squared2[x_, n_] := SerSinSq[x, n] + SerCosSq[x, n]
```

In[39]:=
```
Plot[{Squared2[x, 3], Squared2[x, 7], Squared2[x, 13]},
 {x, 0, π}, PlotLegends → {"n=3", "n=7", "n=13"}, PlotRange → {0, 2}]
```

Out[39]=


In[22]:=
```
Plot[{Squared2[x, 3], Squared2[x, 7], Squared2[x, 13]},
 {x, 0, 4 π}, PlotLegends → {"n=3", "n=7", "n=13"}, PlotRange → {0, 2}]
```

Out[22]=


It seems that using the series expansion of the trigonometric function already squared yields the expected result of 1, while squaring the series expansion creates deviations. This is expected since the series expansion of $\sin^2(x)$ is not the square of the series expansion of $\sin(x)$, as squaring the expansion of $\sin(x)$ would introduces higher-order terms that create large errors when x is far from 0.

Euler Angles

We define 3-dimensional rotation matrices about the x, y, and z axes:

In[61]:= $$\texttt{Rx}[\theta\_] := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \texttt{Cos}[\theta] & \texttt{Sin}[\theta] \\ 0 & -\texttt{Sin}[\theta] & \texttt{Cos}[\theta] \end{pmatrix}$$

In[62]:= $$\texttt{Ry}[\xi\_] := \begin{pmatrix} \texttt{Cos}[\xi] & 0 & \texttt{Sin}[\xi] \\ 0 & 1 & 0 \\ -\texttt{Sin}[\xi] & 0 & \texttt{Cos}[\xi] \end{pmatrix}$$

In[63]:=
```
Rz[ϕ_] := ( Cos[ϕ]   Sin[ϕ]  0
           -Sin[ϕ]  Cos[ϕ]  0
             0        0     1 )
```

A three-dimensional rotation can be expressed as a rotation around the *z* axis, followed by a rotation around the *x* axis, followed by a rotation around the *z* axis:

In[64]:=
```
Rot3[ψ_, θ_, ϕ_] := Rz[ψ].Rx[θ].Rz[ϕ]
```

In[66]:=
```
Simplify[Rot3[ψ, θ, ϕ]]
```

Out[66]=
$$\begin{pmatrix} \cos(\phi)\cos(\psi)-\cos(\theta)\sin(\phi)\sin(\psi) & \cos(\psi)\sin(\phi)+\cos(\theta)\cos(\phi)\sin(\psi) & \sin(\theta)\sin(\psi) \\ -\cos(\theta)\cos(\psi)\sin(\phi)-\cos(\phi)\sin(\psi) & \cos(\theta)\cos(\phi)\cos(\psi)-\sin(\phi)\sin(\psi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\cos(\phi)\sin(\theta) & \cos(\theta) \end{pmatrix}$$

An inverse rotation would simply be undoing these rotations (in reverse order):

In[67]:=
```
Rot3Inverse[ψ_, θ_, ϕ_] := Rz[-ϕ].Rx[-θ].Rz[-ψ]
```

In[68]:=
```
Simplify[Rot3Inverse[ψ, θ, ϕ]]
```

Out[68]=
$$\begin{pmatrix} \cos(\phi)\cos(\psi)-\cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\theta)\cos(\psi)\sin(\phi)-\cos(\phi)\sin(\psi) & \sin(\theta)\sin(\phi) \\ \cos(\psi)\sin(\phi)+\cos(\theta)\cos(\phi)\sin(\psi) & \cos(\theta)\cos(\phi)\cos(\psi)-\sin(\phi)\sin(\psi) & -\cos(\phi)\sin(\theta) \\ \sin(\theta)\sin(\psi) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{pmatrix}$$

As expected, this is equal to the inverse of the original 3D rotation matrix:

In[71]:=
```
Simplify[Inverse[Rot3[ψ, θ, ϕ]]]
```

Out[71]=
$$\begin{pmatrix} \cos(\phi)\cos(\psi)-\cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\theta)\cos(\psi)\sin(\phi)-\cos(\phi)\sin(\psi) & \sin(\theta)\sin(\phi) \\ \cos(\psi)\sin(\phi)+\cos(\theta)\cos(\phi)\sin(\psi) & \cos(\theta)\cos(\phi)\cos(\psi)-\sin(\phi)\sin(\psi) & -\cos(\phi)\sin(\theta) \\ \sin(\theta)\sin(\psi) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{pmatrix}$$

In[70]:=
```
Simplify[Inverse[Rot3[ψ, θ, ϕ]] - Rot3Inverse[ψ, θ, ϕ]]
```

Out[70]=
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$