1 Theory

1a Self-consistency of the Fourier series

Observation. The following is a valid representation of the Dirac delta function:

$$\sum_{k=-\infty}^{\infty} e^{iky} = 2\pi\delta(y).$$

Lemma. $\delta(ay) = \delta(y)/a$.

Proof. We start with a change of variables $y \mapsto y/a$, which leaves the limits of integraiton unchanged:

$$\int_{-\infty}^{\infty} f(y)\delta(ay)\,dy = \int_{-\infty}^{\infty} f(y/a)\delta(y)\,d(y/a) = \frac{1}{a}\int_{-\infty}^{\infty} f(y/a)\delta(y)\,dy = \frac{f(0)}{a} = \int_{-\infty}^{\infty} f(y)\left[\frac{\delta(y)}{a}\right]dy.$$

□

Claim. The Fourier series, as defined in equations (3) and (2) in the notes, is self-consistent.

Proof. Using the definition of $\tilde{h}_k$ and $f_k \equiv k/L$,

$$\begin{aligned} h(x) &= \sum_{k=-\infty}^{\infty}\left[\frac{1}{L}\int_0^L h(x')e^{2\pi ikx'/L}\,dx'\right]e^{-2\pi ikx/L} \\ &= \int_0^L h(x')\left[\frac{1}{L}\sum_{k=-\infty}^{\infty} e^{2\pi ik(x'-x)/L}\right]dx' \\ &= \int_0^L h(x')\delta(x'-x)\,dx' = h(x). \end{aligned}$$

□

1b Linear combination of exponentials

Lemma. $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

Proof. Using Euler's formula:

$$\begin{aligned} e^{i(\theta+\phi)} &= e^{i\theta}e^{i\phi} \\ \cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi). \end{aligned}$$

Equating imaginary parts yields the desired result. □

Claim. $A\sin(2\pi x/L + \phi)$ is a linear combination of $e^{-2\pi ix/L}$ and $e^{2\pi ix/L}$ over the scalar field $\mathbb{C}$.

Proof. We use the trigonometric identity above and the definitions of sin and cos in terms of complex exponentials (from Euler's formula):

$$\begin{aligned} A\sin(2\pi x/L+\phi) &= (A\cos\phi)\sin(2\pi x/L) + (A\sin\phi)\cos(2\pi x/L) \\ &= (A\cos\phi)\left(\frac{e^{2\pi ix/L} - e^{-2\pi ix/L}}{2i}\right) + (A\sin\phi)\left(\frac{e^{2\pi ix/L} + e^{-2\pi ix/L}}{2}\right) \\ &= \left[\frac{A(\sin\phi - i\cos\phi)}{2}\right]e^{2\pi ix/L} + \left[\frac{A(\sin\phi + i\cos\phi)}{2}\right]e^{-2\pi ix/L}. \end{aligned}$$

□

1c Redundancy in Fourier coefficients of real functions

Claim. For $h(x) \in \mathbb{R}$, the Fourier coefficients $\tilde{h}_k$ satisfy $\tilde{h}_{-k} = \tilde{h}_k^*$.

Proof. Conjugation is linear: for $A, B \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$, $(A\alpha + B\beta)^* = A\alpha^* + B\beta^*$. Therefore, to show that two integral expressions are conjugates, it suffices to show that their integrands are conjugates:

$$\begin{aligned}
\tilde{h}_{-k} &= \frac{1}{L}\int_0^L h(x)e^{-2\pi ikx/L}\,dx \\
&= \frac{1}{L}\int_0^L h(x)\left(e^{2\pi ikx/L}\right)^*\,dx \\
&= \frac{1}{L}\int_0^L \left[h(x)e^{2\pi ikx/L}\right]^*\,dx \\
&= \left[\frac{1}{L}\int_0^L h(x)e^{2\pi ikx/L}\,dx\right]^* = \tilde{h}_k^*.
\end{aligned}$$

□

1d Convolution theorem

Claim. The Fourier coefficients of the product $H(x) = h^{(1)}(x)h^{(2)}(x)$ are given by the convolution:

$$\tilde{H}_k = \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}.$$

Proof. We express $h^{(1)}(x)$ and $h^{(2)}(x)$ as Fourier series and find the Cauchy product $H(x)$:

$$\begin{aligned}
H(x) &= \left[\sum_{k=-\infty}^{\infty}\tilde{h}^{(1)}_k e^{-2\pi ikx/L}\right]\left[\sum_{k'=-\infty}^{\infty}\tilde{h}^{(2)}_{k'} e^{-2\pi ik'x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\tilde{h}^{(2)}_{k'}e^{-2\pi ik'x/L}\right]\left[\tilde{h}^{(1)}_{k-k'}e^{-2\pi i(k-k')x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}e^{-2\pi ikx/L} \\
&= \sum_{k=-\infty}^{\infty}\tilde{H}_k e^{-2\pi ikx/L},
\end{aligned}$$

Equating the cofficients of the sum over k, we have the desired result. □

1e Testing the `numpy` FFT

We compare analytical and numerical methods of obtaining the Fourier series of the following two functions:

$$g(t) = A\cos(ft + \varphi) + C \quad \text{and} \quad h(t) = A\exp\left[-B(t - L/2)^2\right].$$