1 Theory

1a Self-consistency of the Fourier series

Observation. The following is a valid representation of the Dirac delta function:

$$\sum_{k=-\infty}^{\infty} e^{iky} = 2\pi\delta(y).$$

Lemma. $\delta(ay) = \delta(y)/a$.

Proof. We start with a change of variables $y \mapsto y/a$, which leaves the limits of integraiton unchanged:

$$\int_{-\infty}^{\infty} f(y)\delta(ay)\,dy = \int_{-\infty}^{\infty} f(y/a)\delta(y)\,d(y/a) = \frac{1}{a}\int_{-\infty}^{\infty} f(y/a)\delta(y)\,dy = \frac{f(0)}{a} = \int_{-\infty}^{\infty} f(y)\left[\frac{\delta(y)}{a}\right] dy.$$

□

Claim. The Fourier series, as defined in equations (3) and (2) in the notes, is self-consistent.

Proof. Using the definition of $\tilde{h}_k$ and $f_k \equiv k/L$,

$$\begin{aligned} h(x) &= \sum_{k=-\infty}^{\infty} \left[\frac{1}{L}\int_0^L h(x')e^{2\pi i k x'/L}\,dx'\right] e^{-2\pi i k x/L} \\ &= \int_0^L h(x')\left[\frac{1}{L}\sum_{k=-\infty}^{\infty} e^{2\pi i k(x'-x)/L}\right] dx' \\ &= \int_0^L h(x')\delta(x'-x)\,dx' = h(x). \end{aligned}$$

□

1b Linear combination of exponentials

Lemma. $\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

Proof. Using Euler's formula:

$$\begin{aligned} e^{i(\theta+\phi)} &= e^{i\theta}e^{i\phi} \\ \cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi). \end{aligned}$$

Equating imaginary parts yields the desired result. □

Claim. $A\sin(2\pi x/L + \phi)$ is a linear combination of $e^{-2\pi i x/L}$ and $e^{2\pi i x/L}$ over the scalar field $\mathbb{C}$.

Proof. We use the trigonometric identity above and the definitions of sin and cos in terms of complex exponentials (from Euler's formula):

$$\begin{aligned} A\sin(2\pi x/L + \phi) &= (A\cos\phi)\sin(2\pi x/L) + (A\sin\phi)\cos(2\pi x/L) \\ &= (A\cos\phi)\left(\frac{e^{2\pi i x/L} - e^{-2\pi i x/L}}{2i}\right) + (A\sin\phi)\left(\frac{e^{2\pi i x/L} + e^{-2\pi i x/L}}{2}\right) \\ &= \left[\frac{A(\sin\phi - i\cos\phi)}{2}\right] e^{2\pi i x/L} + \left[\frac{A(\sin\phi + i\cos\phi)}{2}\right] e^{-2\pi i x/L}. \end{aligned}$$

□

1c Redundancy in Fourier coefficients of real functions

Claim. For $h(x) \in \mathbb{R}$, the Fourier coefficients $\tilde{h}_k$ satisfy $\tilde{h}_{-k} = \tilde{h}_k^*$.

Proof. Conjugation is linear: for $A, B \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$, $(A\alpha + B\beta)^* = A\alpha^* + B\beta^*$. Therefore, to show that two integral expressions are conjugates, it suffices to show that their integrands are conjugates:

$$\begin{aligned}
\tilde{h}_{-k} &= \frac{1}{L}\int_0^L h(x)e^{-2\pi ikx/L}\,dx \\
&= \frac{1}{L}\int_0^L h(x)\left(e^{2\pi ikx/L}\right)^*\,dx \\
&= \frac{1}{L}\int_0^L \left[h(x)e^{2\pi ikx/L}\right]^*\,dx \\
&= \left[\frac{1}{L}\int_0^L h(x)e^{2\pi ikx/L}\,dx\right]^* = \tilde{h}_k^*.
\end{aligned}$$

□

1d Convolution theorem

Claim. The Fourier coefficients of the product $H(x) = h^{(1)}(x)h^{(2)}(x)$ are given by the convolution:

$$\tilde{H}_k = \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}.$$

Proof. We express $h^{(1)}(x)$ and $h^{(2)}(x)$ as Fourier series and find the Cauchy product $H(x)$:

$$\begin{aligned}
H(x) &= \left[\sum_{k=-\infty}^{\infty}\tilde{h}^{(1)}_k e^{-2\pi ikx/L}\right]\left[\sum_{k'=-\infty}^{\infty}\tilde{h}^{(2)}_{k'} e^{-2\pi ik'x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\tilde{h}^{(2)}_{k'}e^{-2\pi ik'x/L}\right]\left[\tilde{h}^{(1)}_{k-k'}e^{-2\pi i(k-k')x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}e^{-2\pi ikx/L} \\
&= \sum_{k=-\infty}^{\infty}\tilde{H}_k e^{-2\pi ikx/L},
\end{aligned}$$

Equating the cofficients of the sum over k, we have the desired result. □

1e Testing the `numpy` FFT

We compare analytical and numerical methods of obtaining the Fourier series of the following two functions:

$$g(t) = A\cos(ft + \varphi) + C \quad \text{and} \quad h(t) = A\exp\left[-B(t - L/2)^2\right].$$

1e (i) Cosine

Since $\cos t$ has period 2π, $g(t)$ has period $2\pi/f \equiv L$. The Fourier coefficients are:

$$\begin{aligned}
\tilde{g}_k &= \frac{1}{L}\int_0^L [A\cos(ft+\varphi)+C]e^{2\pi ikt/L}\,dt \\
&= \frac{f}{2\pi}\int_0^{2\pi/f} [A\cos(ft+\varphi)+C]e^{ikft}\,dt \\
&= \frac{f}{2\pi}\left[A\int_0^{2\pi/f}\cos(ft+\varphi)e^{ikft}\,dt + C\int_0^{2\pi/f} e^{ikft}\,dt\right] \\
&= \frac{f}{2\pi}\left[A\int_0^{2\pi/f}\cos(ft+\varphi)e^{ikft}\,dt\right] = \begin{cases} \frac{A}{2}e^{-i\varphi} & k=1 \\ 0 & k\neq 1. \end{cases}
\end{aligned}$$

1e (ii) Gaussian

We are given that $h(t)$ is L-periodic for small enough B – that is, for small enough standard deviation, we can repeat Gaussians centered at multiples of the mean without the distributions bleeding into each other significantly. The Fourier coefficients are given by:

$$\begin{aligned}
\tilde{h}_k &= \frac{1}{L}\int_0^L Ae^{-B(t-L/2)^2}e^{2\pi ikt/L}\,dt \\
&= \frac{A\sqrt{\pi}}{2L\sqrt{B}}\left(\operatorname{erf}\gamma_+ + \operatorname{erf}\gamma_-\right)e^{i\pi k}\exp\left(-\frac{\pi^2k^2}{BL^2}\right) \quad \text{where} \quad \gamma_\pm \equiv \frac{BL^2 \pm 2\pi ik}{2L\sqrt{B}}.
\end{aligned}$$