1 Theory

1a Self-consistency of the Fourier series

Observation. The following is a valid representation of the Dirac delta function:

$$\sum_{k=-\infty}^{\infty} e^{iky} = 2\pi\delta(y).$$

Lemma. $\delta(ay) = \delta(y)/a$.

Proof. We start with a change of variables $y \mapsto y/a$, which leaves the limits of integraiton unchanged:

$$\int_{-\infty}^{\infty} f(y)\delta(ay)\,dy = \int_{-\infty}^{\infty} f(y/a)\delta(y)\,d(y/a) = \frac{1}{a}\int_{-\infty}^{\infty} f(y/a)\delta(y)\,dy = \frac{f(0)}{a} = \int_{-\infty}^{\infty} f(y)\left[\frac{\delta(y)}{a}\right]dy.$$

□

Claim. The Fourier series, as defined in equations (3) and (2) in the notes, is self-consistent.

Proof. Using the definition of $\tilde{h}_k$ and $f_k \equiv k/L$,

$$\begin{aligned}
h(x) &= \sum_{k=-\infty}^{\infty}\left[\frac{1}{L}\int_0^L h(x')e^{2\pi ikx'/L}\,dx'\right]e^{-2\pi ikx/L} \\
&= \int_0^L h(x')\left[\frac{1}{L}\sum_{k=-\infty}^{\infty} e^{2\pi ik(x'-x)/L}\right]dx' \\
&= \int_0^L h(x')\delta(x'-x)\,dx' = h(x).
\end{aligned}$$

□

1b Linear combination of exponentials

Lemma. $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

Proof. Using Euler's formula:

$$\begin{aligned}
e^{i(\theta+\phi)} &= e^{i\theta}e^{i\phi} \\
\cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi).
\end{aligned}$$

Equating imaginary parts yields the desired result. □

Claim. $A\sin(2\pi x/L + \phi)$ is a linear combination of $e^{-2\pi ix/L}$ and $e^{2\pi ix/L}$ over the scalar field $\mathbb{C}$.

Proof. We use the trigonometric identity above and the definitions of sin and cos in terms of complex exponentials (from Euler's formula):

$$\begin{aligned}
A\sin(2\pi x/L + \phi) &= (A\cos\phi)\sin(2\pi x/L) + (A\sin\phi)\cos(2\pi x/L) \\
&= (A\cos\phi)\left(\frac{e^{2\pi ix/L} - e^{-2\pi ix/L}}{2i}\right) + (A\sin\phi)\left(\frac{e^{2\pi ix/L} + e^{-2\pi ix/L}}{2}\right) \\
&= \left[\frac{A(\sin\phi - i\cos\phi)}{2}\right]e^{2\pi ix/L} + \left[\frac{A(\sin\phi + i\cos\phi)}{2}\right]e^{-2\pi ix/L}.
\end{aligned}$$

□

1c Redundancy in Fourier coefficients of real functions

Claim. For $h(x) \in \mathbb{R}$, the Fourier coefficients $\tilde{h}_k$ satisfy $\tilde{h}_{-k} = \tilde{h}_k^*$.

Proof. Conjugation is linear: for $A, B \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$, $(A\alpha + B\beta)^* = A\alpha^* + B\beta^*$. Therefore, to show that two integral expressions are conjugates, it suffices to show that their integrands are conjugates:

$$\begin{aligned}
\tilde{h}_{-k} &= \frac{1}{L}\int_0^L h(x)e^{-2\pi ikx/L}\,dx \\
&= \frac{1}{L}\int_0^L h(x)\left(e^{2\pi ikx/L}\right)^*\,dx \\
&= \frac{1}{L}\int_0^L \left[h(x)e^{2\pi ikx/L}\right]^*\,dx \\
&= \left[\frac{1}{L}\int_0^L h(x)e^{2\pi ikx/L}\,dx\right]^* = \tilde{h}_k^*.
\end{aligned}$$

□

1d Convolution theorem

Claim. The Fourier coefficients of the product $H(x) = h^{(1)}(x)h^{(2)}(x)$ are given by the convolution:

$$\tilde{H}_k = \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}.$$

Proof. We express $h^{(1)}(x)$ and $h^{(2)}(x)$ as Fourier series and find the Cauchy product $H(x)$:

$$\begin{aligned}
H(x) &= \left[\sum_{k=-\infty}^{\infty}\tilde{h}^{(1)}_k e^{-2\pi ikx/L}\right]\left[\sum_{k'=-\infty}^{\infty}\tilde{h}^{(2)}_{k'} e^{-2\pi ik'x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\left[\tilde{h}^{(2)}_{k'}e^{-2\pi ik'x/L}\right]\left[\tilde{h}^{(1)}_{k-k'}e^{-2\pi i(k-k')x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty}\tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}e^{-2\pi ikx/L} \\
&= \sum_{k=-\infty}^{\infty}\tilde{H}_k e^{-2\pi ikx/L},
\end{aligned}$$

Equating the cofficients of the sum over k, we have the desired result. □

1e Testing the `numpy` FFT

We compare analytical and numerical methods of obtaining the Fourier series of the following two functions:

$$g(t) = A\cos(ft + \varphi) + C \quad \text{and} \quad h(t) = A\exp\left[-B(t - L/2)^2\right].$$

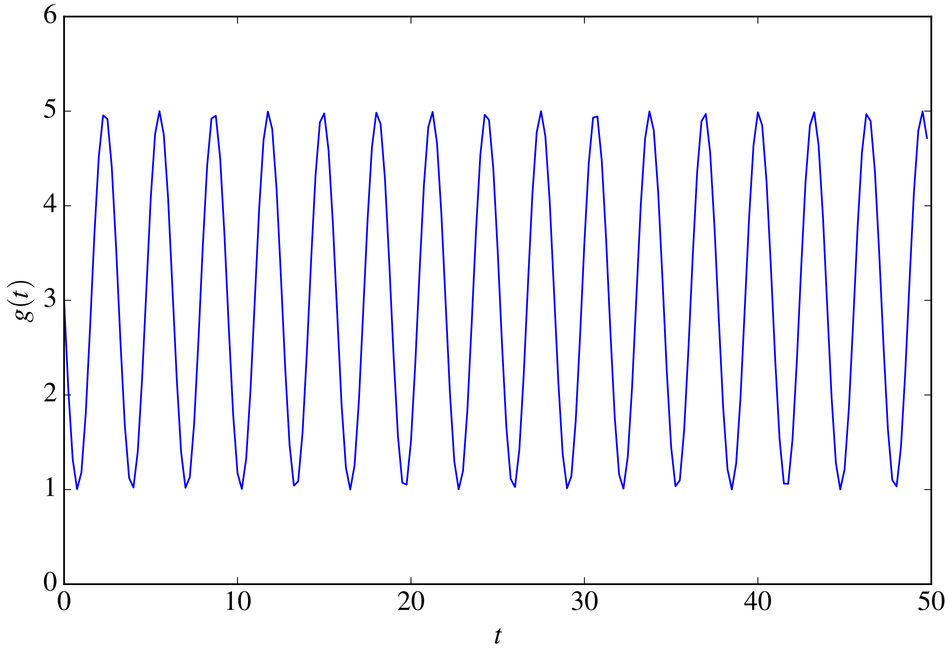

Figure 1: $g(t)$ vs. t with paramaters $A = 2$, $f = 2$, $\varphi = \pi/2$, and $C = 0$.

1e (i) Cosine

Since $\cos t$ has period 2π, $g(t)$ has period $2\pi/f \equiv L$. The Fourier coefficients are:

$$\begin{aligned}
\tilde{g}_k &= \frac{1}{L}\int_0^L [A\cos(ft+\varphi)+C]e^{2\pi i k t/L}\,dt \\
&= \frac{f}{2\pi}\int_0^{2\pi/f} [A\cos(ft+\varphi)+C]e^{ikft}\,dt \\
&= \frac{f}{2\pi}\left[A\int_0^{2\pi/f}\cos(ft+\varphi)e^{ikft}\,dt + C\int_0^{2\pi/f} e^{ikft}\,dt\right] = \begin{cases} C & k=0 \\ \frac{A}{2}e^{-i\varphi} & k=\pm 1 \\ 0 & k\neq 0,\pm 1. \end{cases}
\end{aligned}$$

The first integrand is a product of two waves – a cosine with frequency f and a complex exponential with frequency kf – over full periods of both waves. They will destructively interfere everywhere except when $k = \pm 1$, creating matching frequencies. The second integral evaluates to $f/2\pi$ at $k = 0$, since the integrand is unity, and to 0 everywhere else. We expect isolated peaks at $k = \pm 1$ and a larger peak at $k = 0$. The FFT algorithm in `numpy` makes it easy to convert the k-axis to an angular frequency axis, so we should see the isolated peaks at $\omega = 0, \pm f/2\pi$.

A key difference between our analytical solution and the numerical implementation is that here, we chose L to be the period of the cosine. In `numpy`, L will be the length of our dataset. As long as L is large enough to capture a full oscillation, we should see the same peaks in the transform, but their height will scale with L as the integration interval will be larger.

Figure 1 shows $g(t)$ and figure 2 shows its transform. As expected, there are two isolated peaks in the Fourier transform $\tilde{g}(\omega)$ at $\omega = \pm f/2\pi \approx \pm 0.32$ and a tall peak at $\omega = 0$. The width of the peaks at $\omega = \pm f/2\pi$ is due to the sampling – the input data from $g(t)$ does not perfectly trace a cosine. Inverse-transforming the data in figure 2 returns figure 1 as expected.

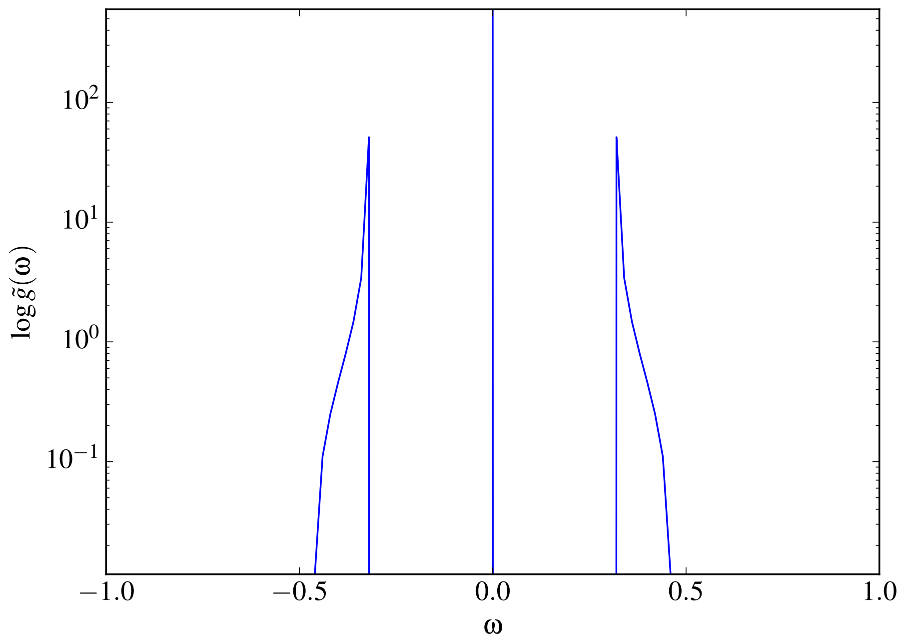

Figure 2: The Fourier transform $\tilde{g}(\omega)$ of the cosine vs. ω.

1e (ii) Gaussian

We are given that $h(t)$ is L-periodic for small enough B – that is, for small enough standard deviation, we can repeat Gaussians centered at multiples of the mean without the distributions bleeding into each other significantly. The Fourier coefficients of the Gaussian are given by:

$$\begin{aligned}\tilde{h}_k &= \frac{1}{L}\int_0^L Ae^{-B(t-L/2)^2} e^{2\pi ikt/L}\,dt \\ &= \frac{A\sqrt{\pi}}{2L\sqrt{B}}\left(\operatorname{erf}\gamma_+ + \operatorname{erf}\gamma_-\right) e^{i\pi k}\exp\left(-\frac{\pi^2 k^2}{BL^2}\right) \quad \text{where} \quad \gamma_\pm \equiv \frac{BL^2 \pm 2\pi ik}{2L\sqrt{B}},\end{aligned}$$

so we expect oscillations of period 2 (due to the $e^{i\pi k}$ factor) in a Gaussian envelope centered at $k = 0$. Figure 3 shows the Gaussian, and figure 4 shows its Fourier transform. The shapes are as expected, and applying the transform and inverse transform in succession returns figure 3.

2 Uniformly-sampled Arecibo data

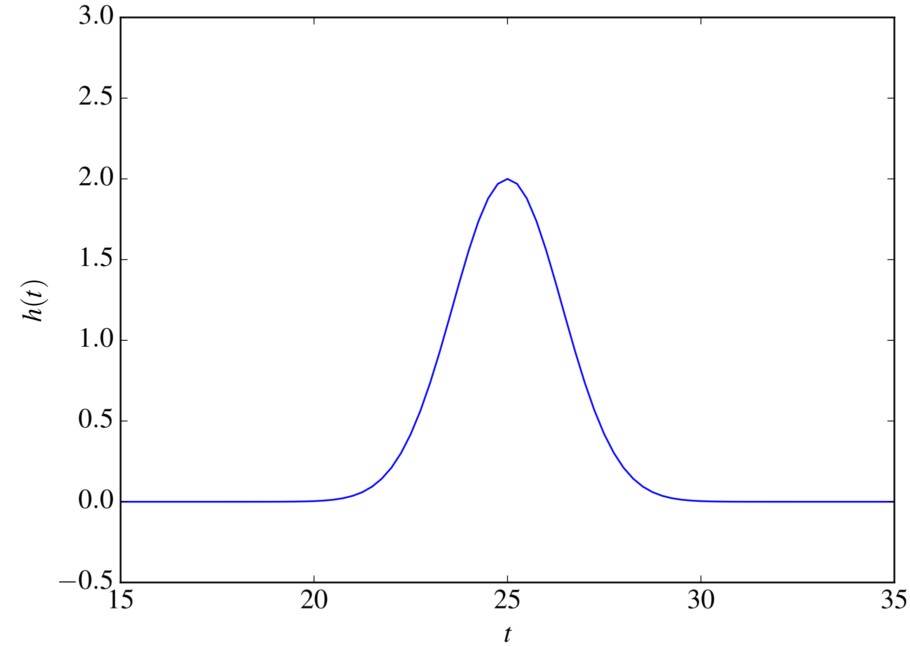

Figure 3: $h(t)$ vs. t with parameters $L = 50$ and $B = 0.25$.

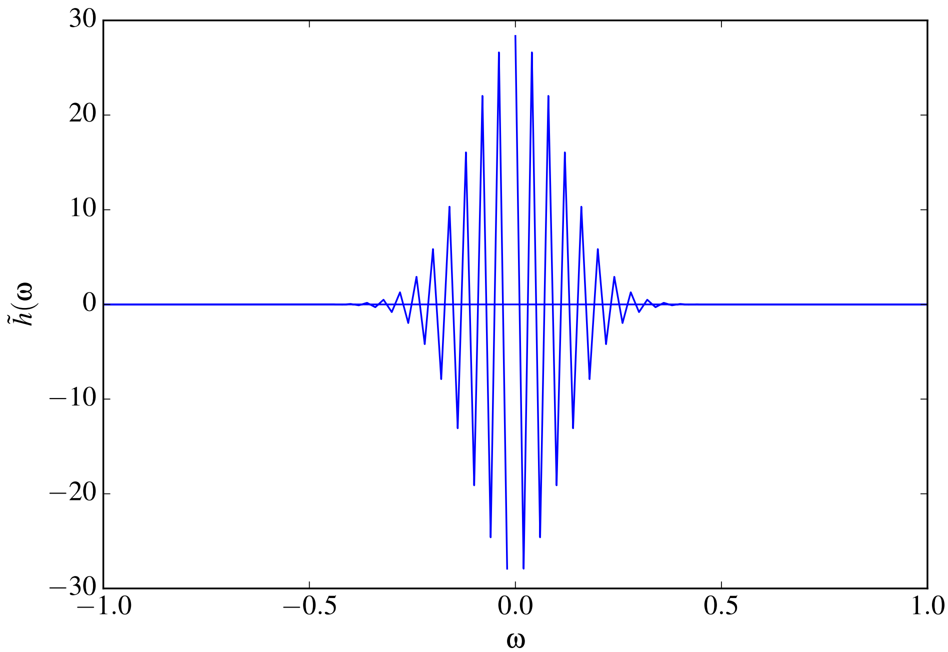

Figure 4: The Fourier transform $\tilde{h}(\omega)$ of the Gaussian vs. ω.