1 Theory

1a Self-consistency of the Fourier series

Observation. The following is a valid representation of the Dirac delta function:

$$\sum_{k=-\infty}^{\infty} e^{iky} = 2\pi\delta(y).$$

Lemma. $\delta(ay) = \delta(y)/a$.

Proof. We start with a change of variables $y \mapsto y/a$, which leaves the limits of integraiton unchanged:

$$\int_{-\infty}^{\infty} f(y)\delta(ay)\,dy = \int_{-\infty}^{\infty} f(y/a)\delta(y)\,d(y/a) = \frac{1}{a}\int_{-\infty}^{\infty} f(y/a)\delta(y)\,dy = \frac{f(0)}{a} = \int_{-\infty}^{\infty} f(y)\left[\frac{\delta(y)}{a}\right]dy.$$

□

Claim. The Fourier series, as defined in equations (3) and (2) in the notes, is self-consistent.

Proof. Using the definition of $\tilde{h}_k$ and $f_k \equiv k/L$,

$$\begin{aligned}
h(x) &= \sum_{k=-\infty}^{\infty}\left[\frac{1}{L}\int_0^L h(x')e^{2\pi ikx'/L}\,dx'\right]e^{-2\pi ikx/L} \\
&= \int_0^L h(x')\left[\frac{1}{L}\sum_{k=-\infty}^{\infty} e^{2\pi ik(x'-x)/L}\right]dx' \\
&= \int_0^L h(x')\delta(x'-x)\,dx' = h(x).
\end{aligned}$$

□

1b Linear combination of exponentials

Lemma. $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

Proof. Using Euler's formula:

$$\begin{aligned}
e^{i(\theta+\phi)} &= e^{i\theta}e^{i\phi} \\
\cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\
&= \cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi).
\end{aligned}$$

Equating imaginary parts yields the desired result. □

Claim. $A\sin(2\pi x/L + \phi)$ is a linear combination of $e^{-2\pi ix/L}$ and $e^{2\pi ix/L}$ over the scalar field $\mathbb{C}$.

Proof. We use the trigonometric identity above and the definitions of sin and cos in terms of complex exponentials (from Euler's formula):

$$\begin{aligned}
A\sin(2\pi x/L + \phi) &= (A\cos\phi)\sin(2\pi x/L) + (A\sin\phi)\cos(2\pi x/L) \\
&= (A\cos\phi)\left(\frac{e^{2\pi ix/L} - e^{-2\pi ix/L}}{2i}\right) + (A\sin\phi)\left(\frac{e^{2\pi ix/L} + e^{-2\pi ix/L}}{2}\right) \\
&= \left[\frac{A(\sin\phi - i\cos\phi)}{2}\right]e^{2\pi ix/L} + \left[\frac{A(\sin\phi + i\cos\phi)}{2}\right]e^{-2\pi ix/L}.
\end{aligned}$$

□

1c Redundancy in Fourier coefficients of real functions

Claim. For $h(x) \in \mathbb{R}$, the Fourier coefficients $\tilde{h}_k$ satisfy $\tilde{h}_{-k} = \tilde{h}_k^*$.

Proof. Conjugation is linear: for $A, B \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{C}$, $(A\alpha + B\beta)^* = A\alpha^* + B\beta^*$. Therefore, to show that two integral expressions are conjugates, it suffices to show that their integrands are conjugates:

$$\begin{aligned}
\tilde{h}_{-k} &= \frac{1}{L}\int_0^L h(x)e^{-2\pi ikx/L}\,dx \\
&= \frac{1}{L}\int_0^L h(x)\left(e^{2\pi ikx/L}\right)^*\,dx \\
&= \frac{1}{L}\int_0^L \left[h(x)e^{2\pi ikx/L}\right]^*\,dx \\
&= \left[\frac{1}{L}\int_0^L h(x)e^{2\pi ikx/L}\,dx\right]^* = \tilde{h}_k^*.
\end{aligned}$$

□

1d Convolution theorem

Claim. The Fourier coefficients of the product $H(x) = h^{(1)}(x)h^{(2)}(x)$ are given by the convolution:

$$\tilde{H}_k = \sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'}.$$

Proof. We express $h^{(1)}(x)$ and $h^{(2)}(x)$ as Fourier series and find the Cauchy product $H(x)$:

$$\begin{aligned}
H(x) &= \left[\sum_{k=-\infty}^{\infty} \tilde{h}^{(1)}_k e^{-2\pi ikx/L}\right]\left[\sum_{k'=-\infty}^{\infty} \tilde{h}^{(2)}_{k'} e^{-2\pi ik'x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} \left[\tilde{h}^{(2)}_{k'} e^{-2\pi ik'x/L}\right]\left[\tilde{h}^{(1)}_{k-k'} e^{-2\pi i(k-k')x/L}\right] \\
&= \sum_{k=-\infty}^{\infty}\sum_{k'=-\infty}^{\infty} \tilde{h}^{(1)}_{k-k'}\tilde{h}^{(2)}_{k'} e^{-2\pi ikx/L} \\
&= \sum_{k=-\infty}^{\infty} \tilde{H}_k e^{-2\pi ikx/L},
\end{aligned}$$

Equating the cofficients of the sum over k, we have the desired result. □

1e Testing the `numpy` FFT

We compare analytical and numerical methods of obtaining the Fourier series of the following two functions:

$$g(t) = A\cos(ft + \varphi) + C \quad \text{and} \quad h(t) = A\exp\left[-B(t - L/2)^2\right].$$

1e (i) Cosine

Since $\cos t$ has period 2π, $g(t)$ has period $2\pi/f \equiv L$. The Fourier coefficients are:

$$\begin{aligned}\tilde{g}_k &= \frac{1}{L}\int_0^L [A\cos(ft+\varphi)+C]e^{2\pi ikt/L}\,dt \\ &= \frac{f}{2\pi}\int_0^{2\pi/f}[A\cos(ft+\varphi)+C]e^{ikft}\,dt \\ &= \frac{f}{2\pi}\left[A\int_0^{2\pi/f}\cos(ft+\varphi)e^{ikft}\,dt + C\int_0^{2\pi/f} e^{ikft}\,dt\right] = \begin{cases} C & k=0 \\ \frac{A}{2}e^{-i\varphi} & k=\pm 1 \\ 0 & k\neq 0,\pm 1. \end{cases}\end{aligned}$$

The first integrand is a product of two waves – a cosine with frequency f and a complex exponential with frequency kf – over full periods of both waves. They will destructively interfere everywhere except when $k = \pm 1$, creating matching frequencies. The second integral evaluates to $f/2\pi$ at $k = 0$, since the integrand is unity, and to 0 everywhere else. We expect isolated peaks at $k = \pm 1$ and a larger peak at $k = 0$. The FFT algorithm in `numpy` makes it easy to convert the k-axis to an angular frequency axis, so we should see the isolated peaks at $\omega = 0, \pm f/2\pi$.

A key difference between our analytical solution and the numerical implementation is that here, we chose L to be the period of the cosine. In `numpy`, L will be the length of our dataset. As long as L is large enough to capture a full oscillation, we should see the same peaks in the transform, but their height will scale with L as the integration interval will be larger.

Figure 1 shows $g(t)$ and figure 2 shows its transform. As expected, there are two isolated peaks in the Fourier transform $\tilde{g}(\omega)$ at $\omega = \pm f/2\pi \approx \pm 0.32$ and a tall peak at $\omega = 0$. The width of the peaks at $\omega = \pm f/2\pi$ is due to the sampling – the input data from $g(t)$ does not perfectly trace a cosine.

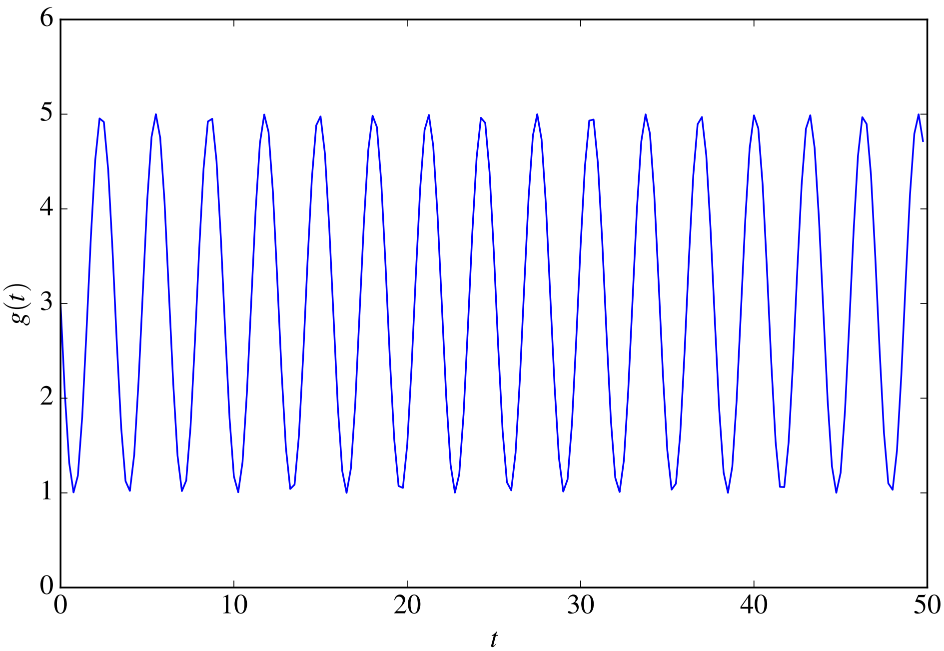

Figure 1: $g(t)$ vs. t with paramaters $A = 2$, $f = 2$, $\varphi = \pi/2$, and $C = 0$.

1e (ii) Gaussian

We are given that $h(t)$ is L-periodic for small enough B – that is, for small enough standard deviation, we can repeat Gaussians centered at multiples of the mean without the distributions bleeding into each other significantly. The

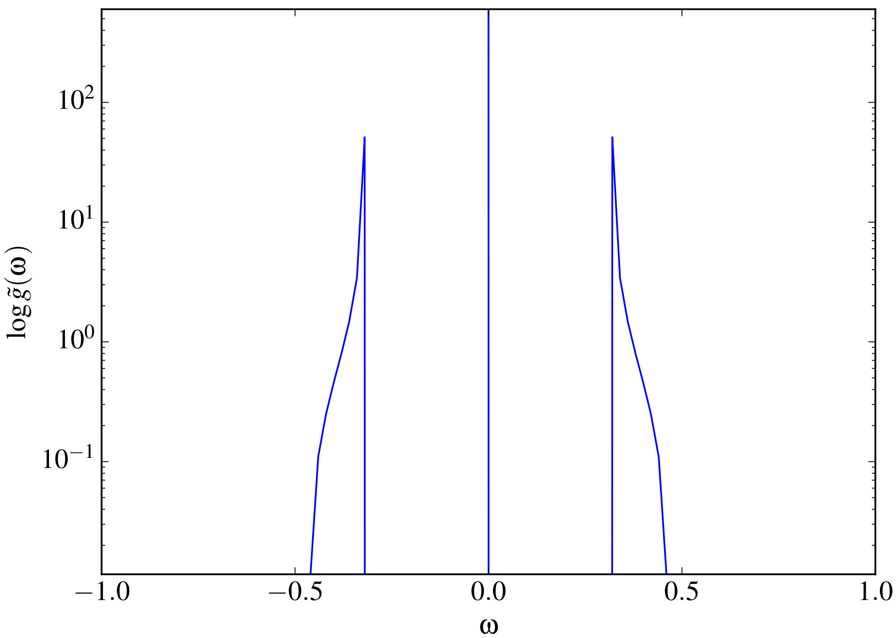

Figure 2: The Fourier transform $\tilde{g}(\omega)$ of the cosine vs. ω.

Fourier coefficients are given by:

$$\begin{aligned}\tilde{h}_k &= \frac{1}{L}\int_0^L Ae^{-B(t-L/2)^2} e^{2\pi i k t/L}\,dt \\ &= \frac{A\sqrt{\pi}}{2L\sqrt{B}}\left(\operatorname{erf}\gamma_+ + \operatorname{erf}\gamma_-\right) e^{i\pi k}\exp\left(-\frac{\pi^2 k^2}{BL^2}\right) \quad \text{where} \quad \gamma_\pm \equiv \frac{BL^2 \pm 2\pi i k}{2L\sqrt{B}}.\end{aligned}$$