1 Restricted three-body problem

We study an asteroid of mass m in the Jupiter-Sun system (masses M_1 and M_2 respectively). Since $m \ll M_1, M_2$, we approximate Jupiter and the Sun as fixed in the corotating frame, which rotates with angular velocity Ω:

$$\Omega = \sqrt{\frac{G(M_1 + M_2)}{R^3}}, \text{ where } R_3 \text{ is the Jupiter-Sun separation,} \tag{1}$$

with respect to an inertial frame. In this frame, Jupiter and the Sun are fixed at:

$$\mathbf{r_1} = \left(\frac{M_2 R}{M_1 + M_2}, 0\right), \qquad \mathbf{r_2} = \left(-\frac{M_1 R}{M_1 + M_2}, 0\right). \tag{2}$$

The acceleration of the asteroid at $\mathbf{r} = (x, y)$ is given by:

$$\ddot{\mathbf{r}} = -GM_1 \frac{\mathbf{r} - \mathbf{r_1}}{|\mathbf{r} - \mathbf{r_1}|^3} - GM_2 \frac{\mathbf{r} - \mathbf{r_2}}{|\mathbf{r} - \mathbf{r_2}|^3} + 2\Omega(\dot{y}, -\dot{x}) + \Omega^2 \mathbf{r}, \tag{3}$$

with gravitational contributions from Jupiter and the Sun, a Coriolis force contribution, and a centrifugal force contribution. Lagrange's solution identifies two *stable* Lagrange points $\mathbf{L_4}$ and $\mathbf{L_5}$ located at:

$$\mathbf{L_{4,5}} = R\left(\frac{M_2 - M_1}{M_1 + M_2} \cos\alpha_{4,5}, \sin\alpha_{4,5}\right) \quad \text{where} \quad \alpha_{4,5} = \pm\frac{\pi}{3}. \tag{4}$$

To observe the stability of these starting points, we will start our asteroid at nearby points $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\tilde{\alpha}_{4,5} \equiv \alpha_{4,5} \pm \Delta\alpha$. For small $\Delta\alpha$, we should see small oscillations around $\mathbf{L_{4,5}}$, but for large $\Delta\alpha$, we should see more variation in the asteroid's orbit. We use an RK4 integration routine to evolve the position $\mathbf{r}$ of the asteroid over several periods $T = 2\pi/\Omega$.

Figures 1 through 4 show asteroid orbits for $\Delta\alpha = 0.01, 0.05, 0.5, 1.5$. No orbit is shown for $\Delta\alpha = 0$ because there is none: the asteroid remains stationary at its starting position. As expected, the asteroid's orbit amplitude increases dramatically with $\Delta\alpha$.

2 General three-body problem with equal masses

In a general multi-body problem, the acceleration of the i-th object is given by:

$$\ddot{\mathbf{r}}_\mathbf{i} = -G \sum_{j \neq i} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \tag{5}$$

We consider a case with three bodies of equal mass. We will use natural units, where $G = M_{1,2,3} = 1$.

2a Lagrange solution

We want to verify the Lagrange solution, where the masses move along the vertices of a rotating equilateral triangle of side length d and velocity v. First, we will confirm this solution analytically.

Let $R \equiv d/\sqrt{3}$, the distance of a point on the equilateral triangle from the center. This is the radius of the centripetal motion of each mass. Without loss of generality, let one of the masses begin at $(x, y) = (0, R)$. By symmetry, the net gravitational force on this mass is in the y-direction (towards the center) and its magnitude is:

$$F = 2\left(\frac{GM^2}{d^2}\right)\cos 30^\circ = \frac{GM^2}{d^2}\sqrt{3} = \frac{GM^2}{dR}. \tag{6}$$

The velocity of centripetal motion is related to the centripetal force by $F = v^2/R$, so:

$$v = \sqrt{RF} = \sqrt{\frac{GM}{d}}. \square \tag{7}$$

We simulate this orbit using our RK4 integrator. Initial conditions are:

$$(x_i, y_i) = (R\sin\theta_i, R\cos\theta_i), \qquad (\dot{x}_i, \dot{y}_i) = (v\cos\theta_i, -v\sin\theta_i), \quad \text{with} \quad \{\theta_1, \theta_2, \theta_3\} = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}. \tag{8}$$

The result of the simulation is saved in `lagrange.mp4` in this directory.

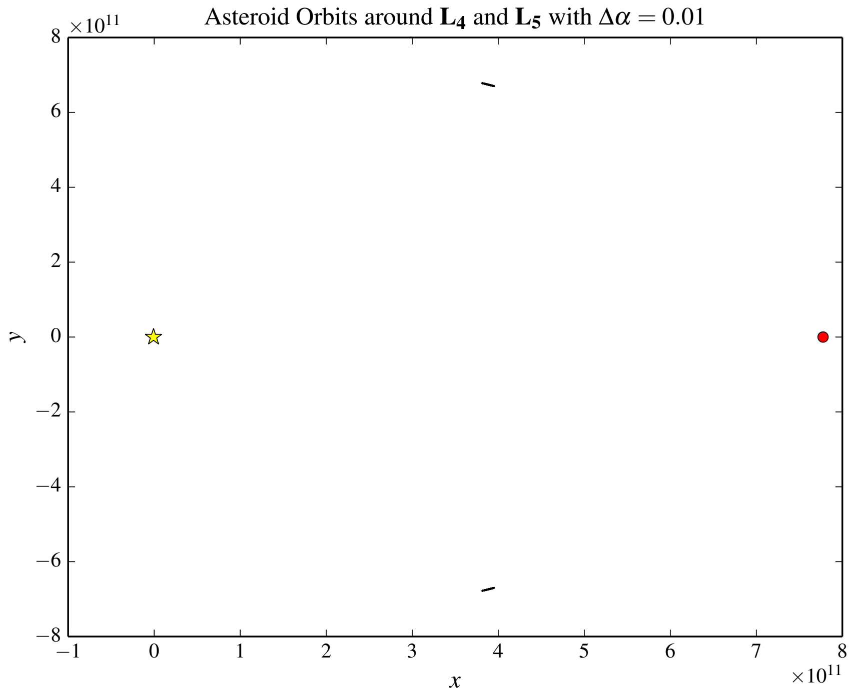

Figure 1: Small-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\Delta\alpha = 0.01$. Yellow star denotes the Sun, and red circle denotes Jupiter.

2b Choreographic orbit

Using the same RK4 routine, we use the given conditions for a choreographic orbit. The simulation result is saved in `choreographic.mp4` in this directory.

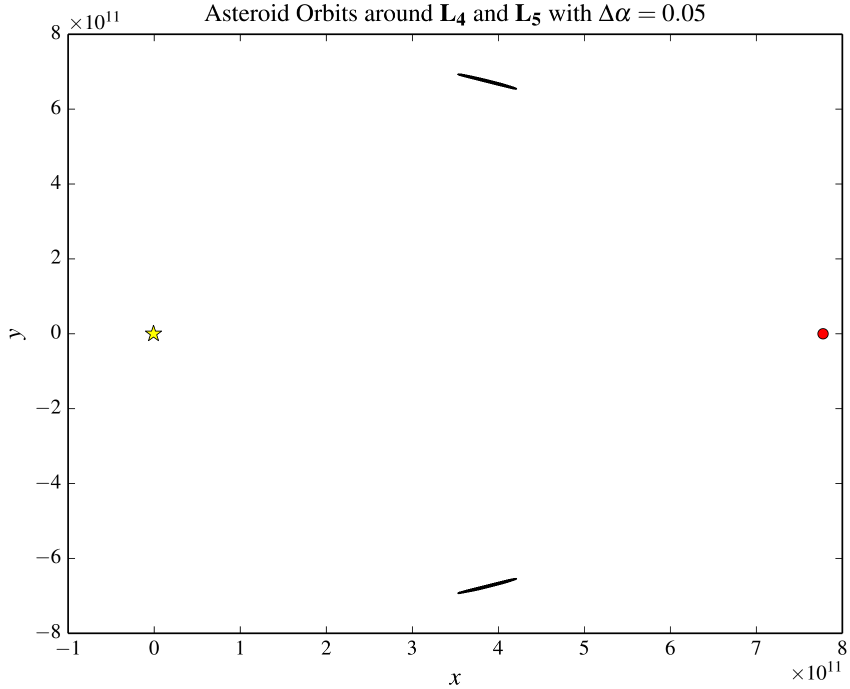

Figure 2: Small-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\Delta\alpha = 0.05$. Yellow star denotes the Sun, and red circle denotes Jupiter.

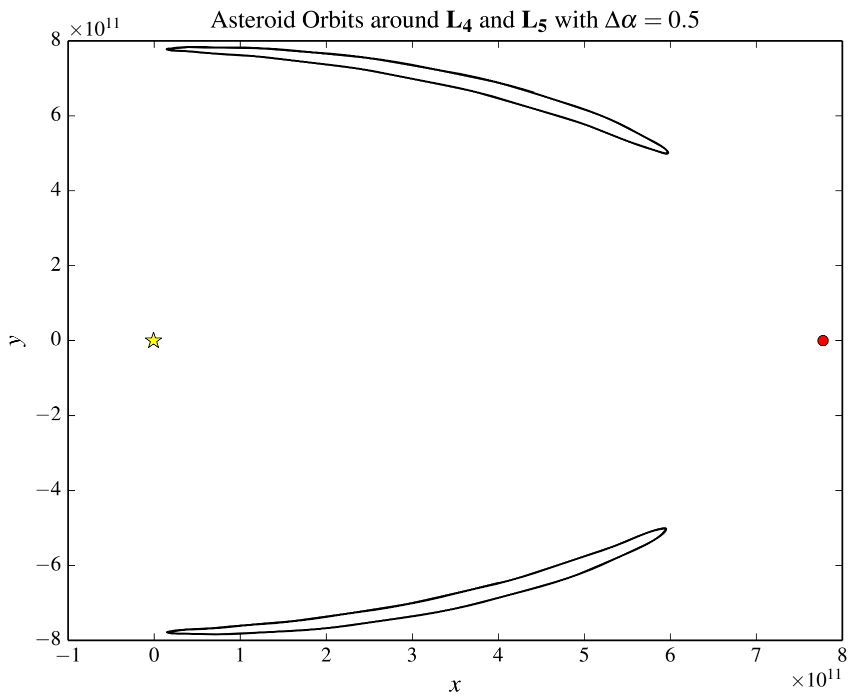

Figure 3: Larger-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\Delta\alpha = 0.5$. Yellow star denotes the Sun, and red circle denotes Jupiter.

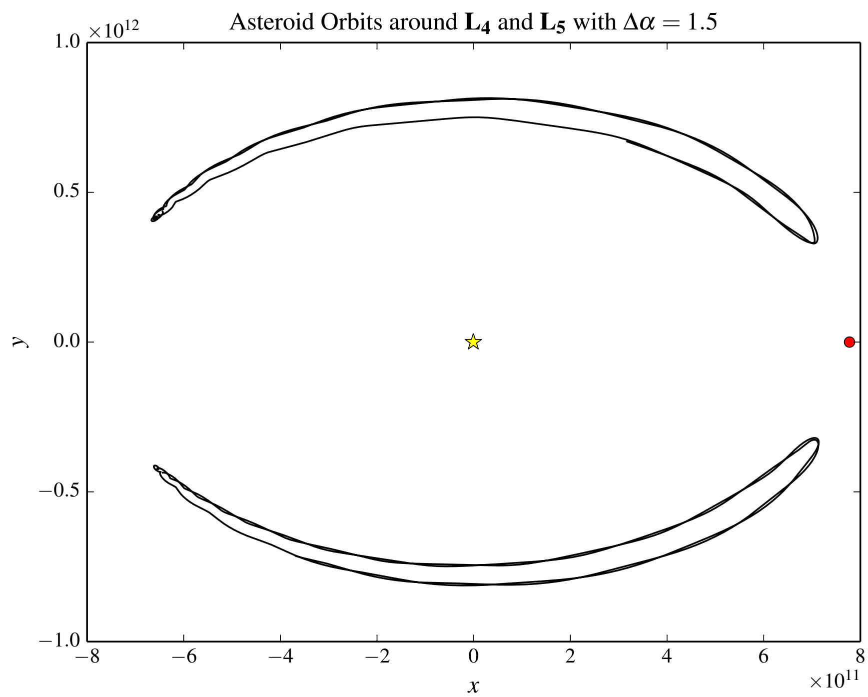

Figure 4: Larger-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\Delta\alpha = 1.5$. Yellow star denotes the Sun, and red circle denotes Jupiter.