1 Convergence of the secant method

Claim. The order of convergence of the secant method is the golden ratio $\phi = (1 + \sqrt{5})/2$.

Proof. We are trying to find a root of a function $f(x)$. Let $\hat{x}$ be the root we seek, such that $f(\hat{x}) = 0$. The Newton-Rhaphson method produces a series of guesses x_i which iterate as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{1}$$

In the secant method, we approximate $f'(x_i)$ using the slope of a secant:

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}. \tag{2}$$

Let $\varepsilon_i \equiv x_i - \hat{x}$. After our guesses x_i are close, we can approximate $f(x_i)$ with a Taylor expansion in the error:

$$f(x_i) = f(\hat{x} + \varepsilon_i) \approx \cancel{f(\hat{x})} + \varepsilon_i f'(\hat{x}) + \varepsilon_i^2 \frac{f''(\hat{x})}{2}. \tag{3}$$

We can express (2) in terms of the errors and use this approximation to obtain a recursion relation for the errors ε_i:

$$\varepsilon_{i+1} = \varepsilon_i - f(x_i)\frac{\varepsilon_i - \varepsilon_{i-1}}{f(x_i) - f(x_{i-1})}. \tag{4}$$

We'll anticipate the geometric nature of the recurrence relation by isolating the ratio $\varepsilon_{i+1}/\varepsilon_i$. For simplicity, let's abstract away the constants $A \equiv f'(\hat{x})$, $B \equiv f''(\hat{x})/2$, and $C \equiv B/A$:

$$\frac{\varepsilon_{i+1}}{\varepsilon_i} = 1 - (A + B\varepsilon_i)\frac{\varepsilon_i - \varepsilon_{i-1}}{A(\varepsilon_i - \varepsilon_{i-1}) + B(\varepsilon_i^2 - \varepsilon_{i-1}^2)}. \tag{5}$$

Factoring $\varepsilon_i - \varepsilon_{i-1}$ out of the fraction yields:

$$\frac{\varepsilon_{i+1}}{\varepsilon_i} = 1 - (A + B\varepsilon_i)\frac{1}{A + B(\varepsilon_i + \varepsilon_{i-1})} = 1 - (1 + C\varepsilon_i)\frac{1}{1 + C(\varepsilon_i + \varepsilon_{i-1})}. \tag{6}$$

Here we note that $\eta \equiv C(\varepsilon_i + \varepsilon_{i-1}) \ll 1$ and use $(1 + \eta)^{-1} \approx 1 - \eta$:

$$\frac{\varepsilon_{i+1}}{\varepsilon_i} = 1 - (1 + C\varepsilon_i)(1 - C\varepsilon_i - C\varepsilon_{i-1}) = 1 - \left[1 - C\varepsilon_i - C\varepsilon_{i-1} + C\varepsilon_i + O(\varepsilon_i^2)\right] \approx C\varepsilon_{i-1}. \tag{7}$$

Thus we have our desired recurrence relation:

$$\varepsilon_{i+1} \approx C\varepsilon_i\varepsilon_{i-1}. \tag{8}$$

Now we assume that $\varepsilon_{i+1} = D\varepsilon_i^r$ for all i, which allows us to replace:

$$\varepsilon_{i+1} = D\varepsilon_i^r = D(D\varepsilon_{i-1}^r)^r = D^{r+1}\varepsilon_{i-1}^{r^2}. \tag{9}$$

This yields:

$$D^{r+1}\varepsilon_{i-1}^{r^2} = C(D\varepsilon_{i-1}^r)\varepsilon_{i-1} = CD\varepsilon_{i-1}^{r+1}. \tag{10}$$

Since this equation must be true for all i, corresponding powers and coefficients of ε_{i-1} must match and we get:

$$D^{r+1} = CD, \quad \text{and} \quad r^2 = r + 1. \tag{11}$$

The latter is the defining polynomial for the golden ratio: $r = \phi$ is its positive solution. We exclude the negative solution since $\varepsilon_i \ll 1$, so $r < 0$ would have $\varepsilon_{i+1} > \varepsilon$ and the method would not converge. □

Method	Initial Guesses	Result	Difference from Analytical
Analytical		0.00400001066674	0
Bisection	-0.5, 1.0	0.00400000996888	$-6.97866629068 \times 10^{-10}$
Newton-Raphson	1.0	0.00400001066674	$-8.67361737988 \times 10^{-19}$
Secant	-0.5, 1.0	0.00400001066674	0.0

Table 1: Results for $f_c(x) = \sin x - c$ with $c = 4 \times 10^{-3}$ and requested tolerance 10^{-8}.

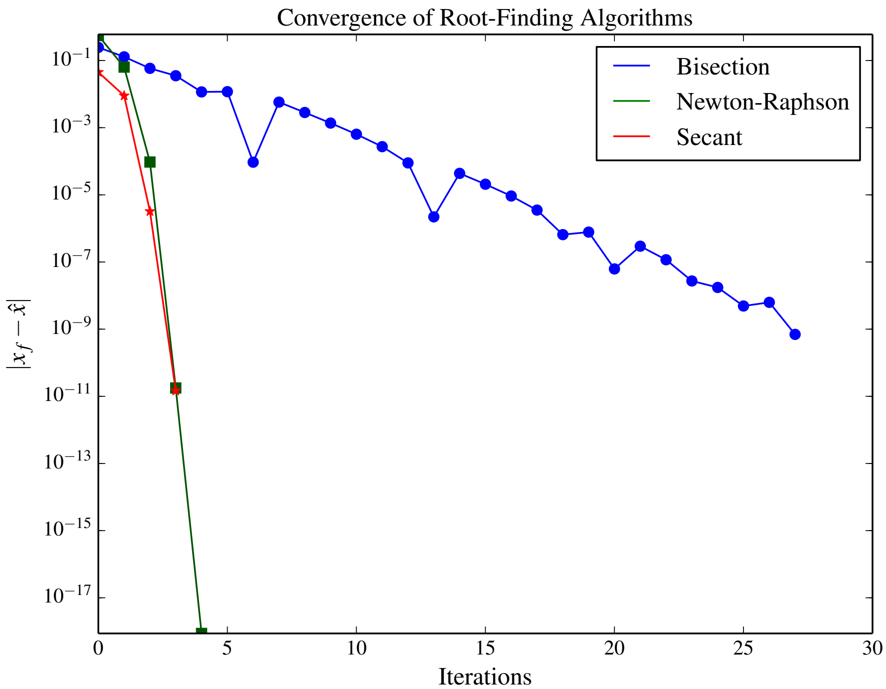

Figure 1: Absolute difference $|x_f - \hat{x}|$, where x_f is the final result and $\hat{x}$ is the analytical root, as a function of number of iterations for all methods.

2 Testing convergence on root-finding methods

We define a test function $f_c(x) = \sin x - c$, with an analytical root of $\hat{x} = \arcsin c$. We will test the convergence rates of our three root-finding algorithms (bisection, Newton-Rhaphson, and secant) on this function with $c = 4 \times 10^{-3}$.

3 Orbit of 1913+16 binary pulsar

Elliptical orbits can be parametrized in terms of the eccentric anomaly, or angle around the ellipse, ξ:

$$x(\xi) = a(\cos \xi - e), \qquad y(\xi) = a\sqrt{1 - e^2} \sin \xi, \qquad t(\xi) = \frac{T}{2\pi}(\xi - e \sin \xi), \tag{12}$$

where a is the projected semimajor axis, T the period, and e the eccentricity. Since ξ varies from 0 to 2π, we need not include a time calculation to calculate the shape of the orbit. However, to obtain $x(t)$ and $y(t)$ numerically, we use the secant method to invert the last equation and numerically find $\xi(t)$, then inserting this parameter into $x(\xi)$ and $y(\xi)$. The resulting orbit is shown in Figure 2. A projection angle of $\phi = 3\pi/2$ resulted in the best visual agreement for the radial velocity curve, shown in Figure 3.

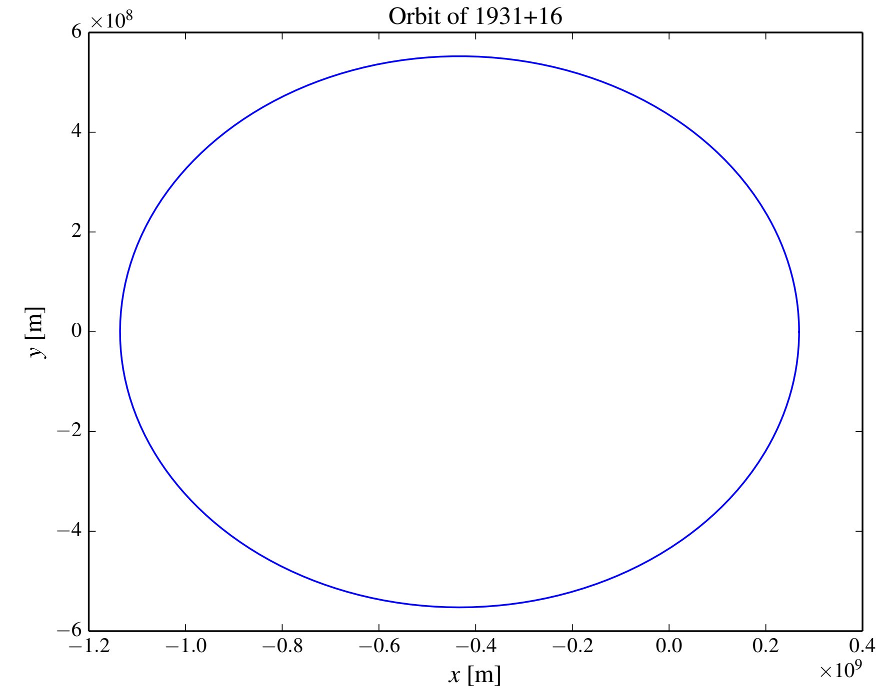

Figure 2: Orbit shape calculated through time.

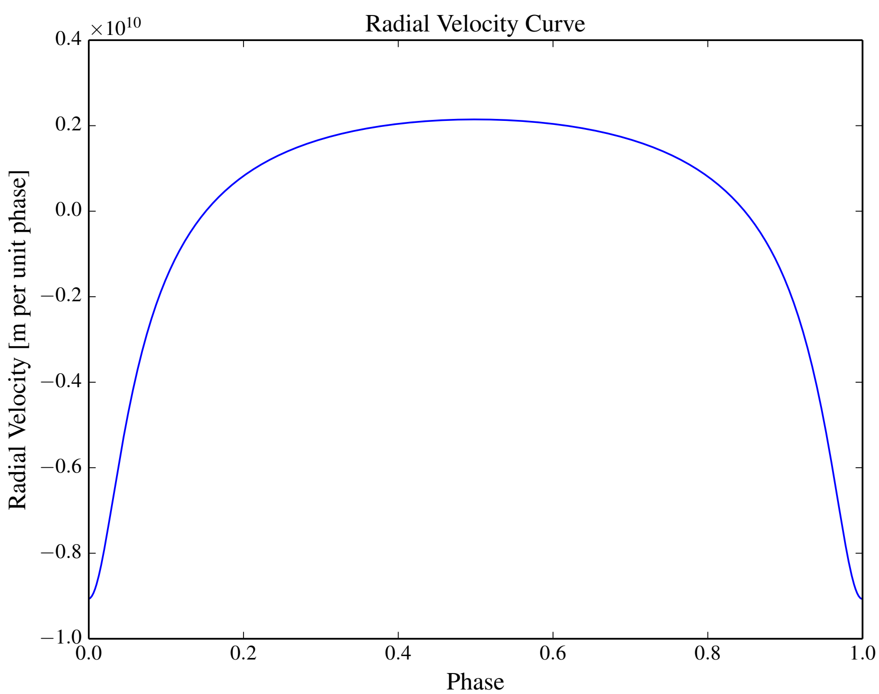

Figure 3: Radial velocity curve, projected at a viewing angle of $3\pi/2$.