1 Convergence of the secant method

Claim. The order of convergence of the secant method is the golden ratio $\phi = (1 + \sqrt{5})/2$.

Proof. We are trying to find a root of a function $f(x)$. Let $\hat{x}$ be the root we seek, such that $f(\hat{x}) = 0$. The Newton-Rhaphson method produces a series of guesses x_i which iterate as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{1}$$

In the secant method, we approximate $f'(x_i)$ using the slope of a secant:

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}. \tag{2}$$

Let $\varepsilon_i \equiv x_i - \hat{x}$. After our guesses x_i are close, we can approximate $f(x_i)$ with a Taylor expansion in the error:

$$f(x_i) = f(\hat{x} + \varepsilon_i) \approx \cancel{f(\hat{x})} + \varepsilon_i f'(\hat{x}) + \varepsilon_i^2 \frac{f''(\hat{x})}{2}. \tag{3}$$

We can express (2) in terms of the errors and use this approximation to obtain a recursion relation for the errors ε_i:

$$\varepsilon_{i+1} = \varepsilon_i - f(x_i)\frac{\varepsilon_i - \varepsilon_{i-1}}{f(x_i) - f(x_{i-1})}. \tag{4}$$

We'll anticipate the geometric nature of the recurrence relation by isolating the ratio $\varepsilon_{i+1}/\varepsilon_i$. For simplicity, let's abstract away the constants $A \equiv f'(\hat{x})$, $B \equiv f''(\hat{x})/2$, and $C \equiv B/A$:

$$\frac{\varepsilon_{i+1}}{\varepsilon_i} = 1 - (A + B\varepsilon_i)\frac{\varepsilon_i - \varepsilon_{i-1}}{A(\varepsilon_i - \varepsilon_{i-1}) + B(\varepsilon_i^2 - \varepsilon_{i-1}^2)}. \tag{5}$$

Factoring $\varepsilon_i - \varepsilon_{i-1}$ out of the fraction yields:

$$\frac{\varepsilon_{i+1}}{\varepsilon_i} = 1 - (A + B\varepsilon_i)\frac{1}{A + B(\varepsilon_i + \varepsilon_{i-1})} = 1 - (1 + C\varepsilon_i)\frac{1}{1 + C(\varepsilon_i + \varepsilon_{i-1})}. \tag{6}$$

Here we note that $\eta \equiv C(\varepsilon_i + \varepsilon_{i-1}) \ll 1$ and use $(1+\eta)^{-1} \approx 1 - \eta$:

$$\frac{\varepsilon_{i+1}}{\varepsilon_i} = 1 - (1 + C\varepsilon_i)(1 - C\varepsilon_i - C\varepsilon_{i-1}) = 1 - \left[1 - C\varepsilon_i - C\varepsilon_{i-1} + C\varepsilon_i + O(\varepsilon_i^2)\right] \approx C\varepsilon_{i-1}. \tag{7}$$

Thus we have our desired recurrence relation:

$$\varepsilon_{i+1} \approx C\varepsilon_i\varepsilon_{i-1}. \tag{8}$$

Now we assume that $\varepsilon_{i+1} = D\varepsilon_i^r$ for all i, which allows us to replace:

$$\varepsilon_{i+1} = D\varepsilon_i^r = D(D\varepsilon_{i-1}^r)^r = D^{r+1}\varepsilon_{i-1}^{r^2}. \tag{9}$$

This yields:

$$D^{r+1}\varepsilon_{i-1}^{r^2} = C(D\varepsilon_{i-1}^r)\varepsilon_{i-1} = CD\varepsilon_{i-1}^{r+1}. \tag{10}$$

Since this equation must be true for all i, corresponding powers and coefficients of ε_{i-1} must match and we get:

$$D^{r+1} = CD, \quad \text{and} \quad r^2 = r + 1. \tag{11}$$

The latter is the defining polynomial for the golden ratio: $r = \phi$ is its positive solution. We exclude the negative solution since $\varepsilon_i \ll 1$, so $r < 0$ would have $\varepsilon_{i+1} > \varepsilon$ and the method would not converge. □