1 Simulation of N-body systems

1a Simulation framework

We take an object-oriented approach to simulating N bodies under a mutual gravitational attraction, although the approach is more general. A `Particle` class holds a 3-dimensional vector position and velocity, a mass, and a radius. Methods are included to set a particle's position and velocity in both Cartesian and spherical coordinates; the latter is particularly useful for generating large numbers of particles distributed in a sphere.

`Particle` objects are contained within a `Cluster` object which is responsible for the interaction between `Particle` objects. `Cluster` objects also contain a reference to a force function, which takes two `Particle` objects as arguments and returns the force vector on the first `Particle`. In this way, the force mechanism is easily expandable: for example, we could add a charge attribute to the `Particle` class and define an `elec_force(p1, p2)` function which uses `p1.charge` and `p2.charge`. Then, the `Cluster` could easily include both gravitational and electric forces simply by using `Cluster.force = grav_force + elec_force`.

The `Cluster` object contains an `evolve(dt)` method, which brute-force calculates the force on every object and evolves their motion for a timestep `dt`. The `Cluster` can also run a visualization of its state using `vpython`, and can save its state to a file using `cPickle`.

1b Higher-order symplectic Euler method

In symplectic Euler methods with two dynamical variables, the current value of one variable is used to update the other, and then the updated value is used to update the former. Thus, we have two possible sympletic Euler methods in our case:

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}_i\,dt, \qquad \mathbf{v}_{i+1} = \mathbf{v}_i + \mathbf{a}(\mathbf{r}_{i+1})\,dt. \tag{1}$$

$$\mathbf{r}_{i+1} = \mathbf{r}_i + \mathbf{v}_{i+1}\,dt, \qquad \mathbf{v}_{i+1} = \mathbf{v}_i + \mathbf{a}(\mathbf{r}_i)\,dt. \tag{2}$$

The behavior of these numerical integration methods was studied on the three-body choreographic orbit (Ph22.3) using visualization. It was observed that method (1) resulted in a collapsing orbit, while method (2) resulted in an expanding orbit. Inspired by these results, the `Cluster` method employs a higher-order method which alternates the two symplectic steps, resulting in a stable three-body choreographic orbit. A visualization can be seen using `vs_sims.py`.

1c Force softening

Since the gravitational force is inversely proportional to R^2 where $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ is the separation vector, extremely large forces are calculated when the particles come very close to each other. In reality, gravity would not be the dominant effect at these close distances, as collision and absorption would have to be considered in our model. However, in our simplistic model, it is sufficient to prevent the simulation from stalling at these distances.

We add a force softening parameter a:

$$\mathbf{F}' = \frac{Gm_1m_2\mathbf{R}}{(R+a)^3}, \text{ which under } a \to 0 \text{ reduces to } \mathbf{F} = \frac{Gm_1m_2\hat{\mathbf{R}}}{R^2}. \tag{3}$$

This ensures that a finite "fake force" is calculated even when the particles occupy the same position. We choose $a = 0.1$, $G = 1$ for our simulations.

2 Spherical distributions

Long-timescale simulations were run on two initial conditions: $N = 1000$ particles distributed in an unit sphere, (1) with unit speeds in random directions, and (2) with the particles initially at rest. Figures 1 and 2 show radial position distributions of the particles at large timesteps.

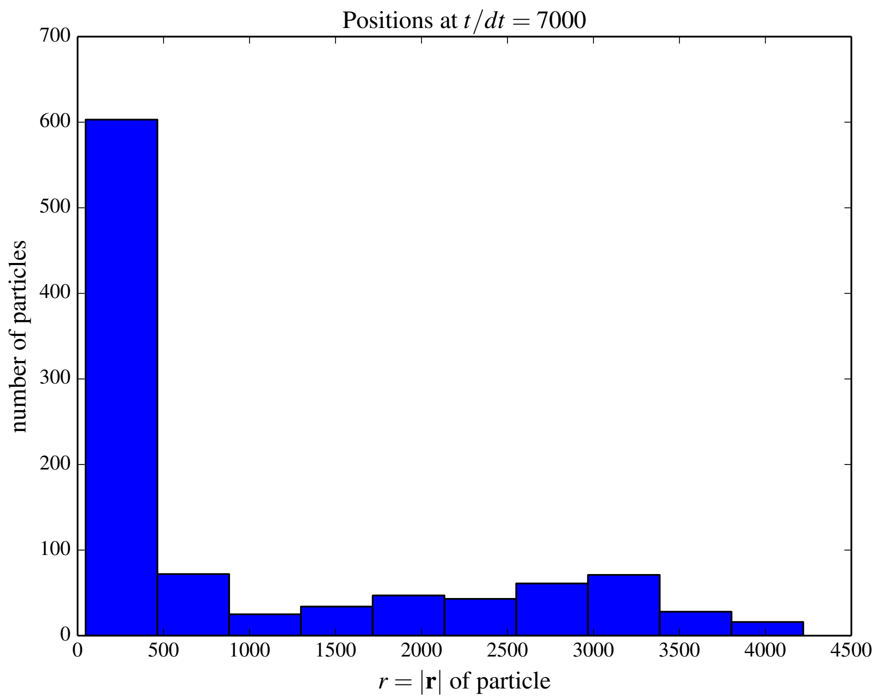

Figure 1: Radial position distribution of particles, with particles initially distributed with unit velocities in random directions in an unit sphere and evolved for 7000 timesteps. The formation of a core in $r < 1000$ and a halo near $r = 3000$ is apparent.

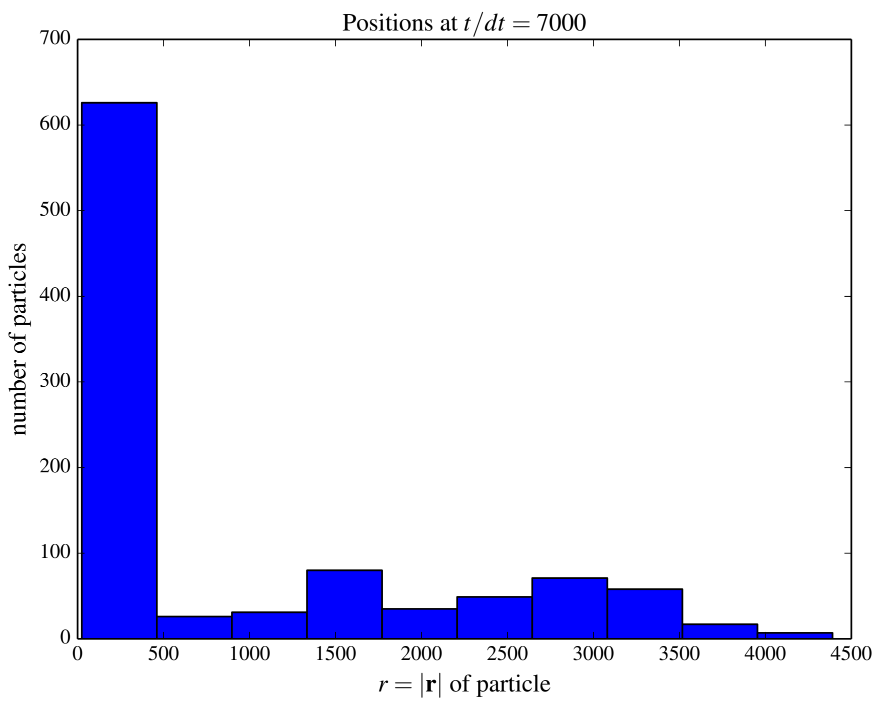

Figure 2: Radial position distribution of particles, with particles at rest initially distributed uniformly in an unit sphere and evolved for 7000 timesteps. The formation of a core in $r < 1000$ is still visible, as well as a smaller halo at $r = 1500$ and a wider halo at $r = 3000$.