1 Introduction

Consider a system of ODEs, potentially coupled and not of the same order. We have shown previously that the system can be expressed as a system of first-order ODEs, where we define intermediate variables for each derivative (e.g. $\xi_0 \equiv x, \xi_1 \equiv x', \xi_2 \equiv x''$) and couple them through additional first-order equations (e.g. $\xi_1 = \xi_0'$).

Therefore, we've reduced the problem to solving a finite number N of coupled first-order ODEs:

$$\frac{d\boldsymbol{\xi}}{dt} = \mathbf{f}\Big(\boldsymbol{\xi}(t)\Big) \quad \text{where} \quad \boldsymbol{\xi}(t) = \Big(\xi_0(t), \xi_1(t), \cdots, \xi_N(t)\Big). \tag{1}$$

In numerical integration, we use a "step" method to compute $\boldsymbol{\xi}(t+h)$ using the values of $\boldsymbol{\xi}(t)$. Given some initial values $\boldsymbol{\xi}(t_0)$ and repeatedly invoking this step, using the previous output as the next input, we can compute $\boldsymbol{\xi}(t_0 + nh)$ for integer n, approximating the function. We consider three different step methods:

1a Explicit Euler method

The explicit Euler method is the simplest: it uses the derivative at the beginning of our timestep to compute the value at the end of the timestep:

$$\boldsymbol{\xi}(t+h) = \boldsymbol{\xi}(t) + h\mathbf{f}\Big(\boldsymbol{\xi}(t)\Big). \tag{2}$$

1b Explicit Euler method

The midpoint method first takes a half-step with the explicit Euler method:

$$\tilde{\boldsymbol{\xi}}(t) \equiv \boldsymbol{\xi}(t) + \frac{h}{2}\mathbf{f}\Big(\boldsymbol{\xi}(t)\Big), \tag{3}$$

then uses the derivative at the midpoint to advance:

$$\boldsymbol{\xi}(t+h) = \boldsymbol{\xi}(t) + h\mathbf{f}\Big(\tilde{\boldsymbol{\xi}}(t)\Big). \tag{4}$$

1c Fourth-order Runge-Kutta

The RK4 algorithm calculates the step that would be given by the derivative at 4 different intermediate values:

$$\mathbf{k}_1 = h\mathbf{f}\Big(\boldsymbol{\xi}(t)\Big), \tag{5}$$

$$\mathbf{k}_2 = h\mathbf{f}\left(\boldsymbol{\xi}(t) + \frac{\mathbf{k}_1}{2}\right), \tag{6}$$

$$\mathbf{k}_3 = h\mathbf{f}\left(\boldsymbol{\xi}(t) + \frac{\mathbf{k}_2}{2}\right), \tag{7}$$

$$\mathbf{k}_3 = h\mathbf{f}\Big(\boldsymbol{\xi}(t) + \mathbf{k}_3\Big), \tag{8}$$

$$\tag{9}$$

then takes a weighted average:

$$\boldsymbol{\xi}(t+h) = \boldsymbol{\xi}(t) + \frac{\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4}{6}. \tag{10}$$

2 Explicit Euler method vs. midpoint method

We consider an exponential function:

$$\frac{dy}{dt} = y \quad \text{and} \quad y(0) = 1 \quad \implies \quad y_{\text{true}}(t) = e^t. \tag{11}$$

We will integrate from $t_0 = 0$ and $t_f = 30$ using the explicit Euler and midpoint methods, and compare the global error at the last computed time, which is $t_f - h$ rather than t_f because this is more convenient in Python:

$$\varepsilon \equiv \left| y_{\text{est}}(t_f - h) - y_{\text{true}}(t_f - h) \right|. \tag{12}$$

Results are shown in Figure 1.

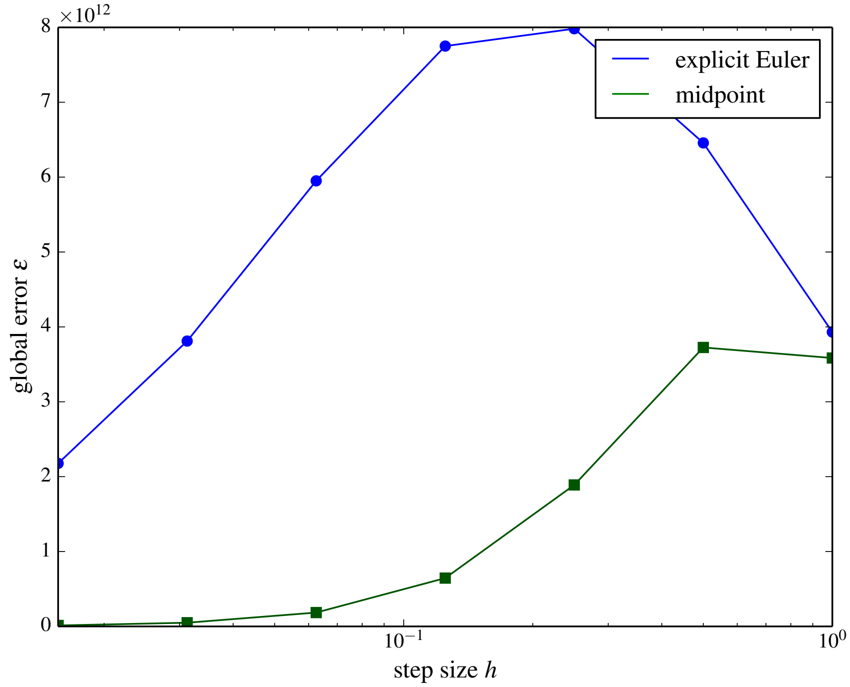

Figure 1: Convergence plot for the explicit Euler and midpoint methods, showing the global error ε (as defined above) as a function of the step size h. The largest step size tried was $h_0 = 2$ on an interval $t_f - t_i = 30$, creating interestingly odd behavior at large h. At small h, the explicit Euler method appears linear and the midpoint method appears quadratic, as expected. Note that the error analysis which led to this expectation assumed h was small when truncating Taylor expansions, so it makes sense that we only see the expected behavior at small h.

3 Runge-Kutta method on a circular orbit

Consider an object of mass m under the gravitational influence of a mass $M \gg m$. Using natural units $m = M = G = 1$ and Cartesian coordinates, the ODEs that describe the object's motion are given by:

$$\dot{x} = v_x, \qquad \dot{y} = v_y, \qquad \dot{v}_x = -\frac{x}{(x^2 + y^2)^{3/2}}, \qquad \dot{v}_y = -\frac{y}{(x^2 + y^2)^{3/2}}. \tag{13}$$

We know we will have a circular orbit when the centripetal acceleration and gravitational force cancel:

$$\frac{v^2}{R} = \frac{1}{R^2} \implies v = \sqrt{\frac{1}{R}}. \tag{14}$$

Thus we set our initial conditions:

$$x(0) = 0, \qquad y(0) = R, \qquad v_x(0) = \sqrt{\frac{1}{R}}, \qquad v_y(0) = 0, \tag{15}$$

which ensures $\mathbf{r}(0) \perp \mathbf{v}(0)$. We let $R = 100.0$ and integrate from $t_0 = 0$ to a full period at $t_f = T = 2\pi R^{3/2}$ using the RK4 algorithm. Results for the orbit are shown in figure 2.

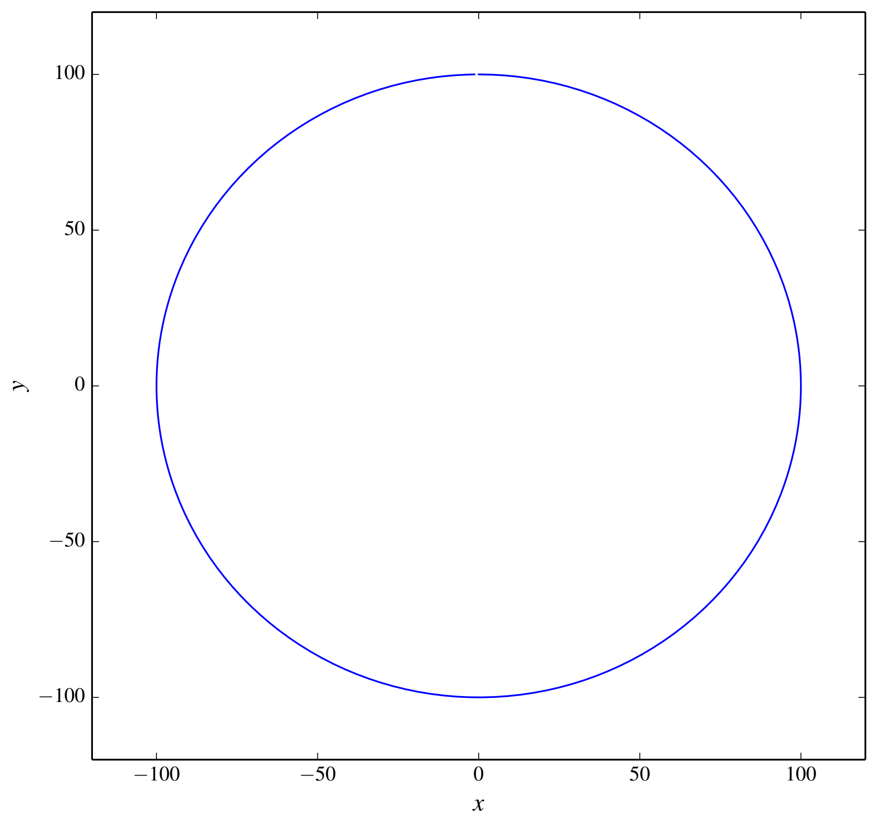

Figure 2: Circular orbit for $R = 100$, $t_f - t_i = T \approx 6283$, using RK4 and a step size $h = 10$.

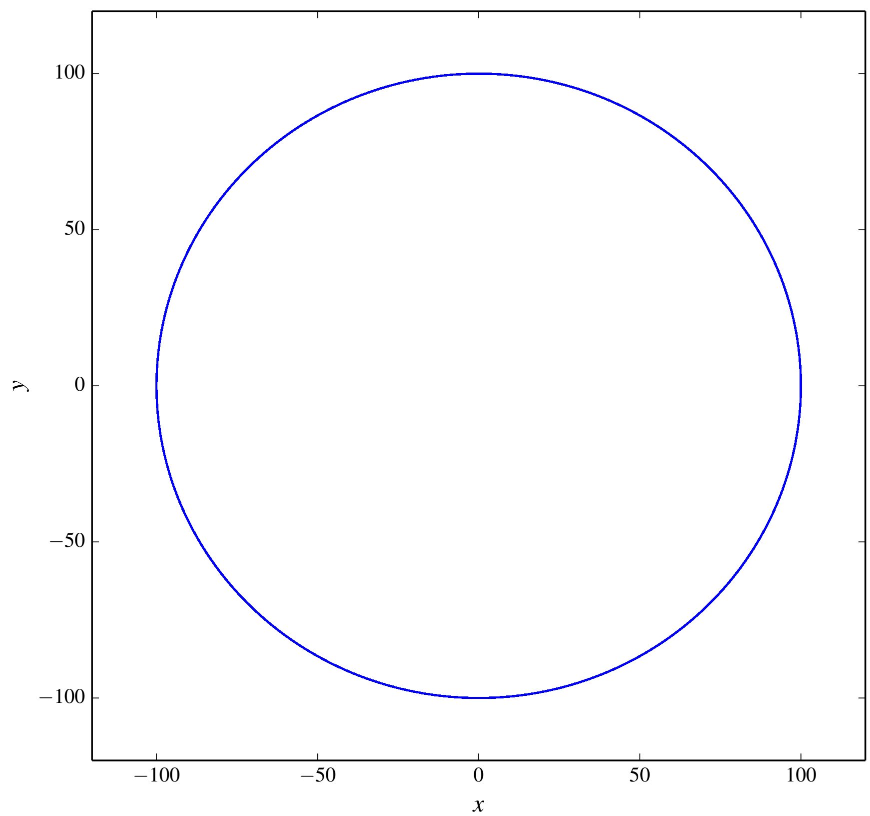

Figure 3: Long-term evolution of the same orbit shown in figure 2 with $h = 10$, plotted for 200 full periods. No drift is discernible. Another calculation for 2000 full periods yielded similar results.