1 Restricted three-body problem

We study an asteroid of mass m in the Jupiter-Sun system (masses M_1 and M_2 respectively). Since $m \ll M_1, M_2$, we approximate Jupiter and the Sun as fixed in the corotating frame, which rotates with angular velocity Ω:

$$\Omega = \sqrt{\frac{G(M_1 + M_2)}{R^3}}, \quad \text{where } R_3 \text{ is the Jupiter-Sun separation,} \tag{1}$$

with respect to an inertial frame. In this frame, Jupiter and the Sun are fixed at:

$$\mathbf{r_1} = \left(\frac{M_2 R}{M_1 + M_2}, 0\right), \qquad \mathbf{r_2} = \left(-\frac{M_1 R}{M_1 + M_2}, 0\right). \tag{2}$$

The acceleration of the asteroid at $\mathbf{r} = (x, y)$ is given by:

$$\ddot{\mathbf{r}} = -GM_1 \frac{\mathbf{r} - \mathbf{r_1}}{|\mathbf{r} - \mathbf{r_1}|^3} - GM_2 \frac{\mathbf{r} - \mathbf{r_2}}{|\mathbf{r} - \mathbf{r_2}|^3} + 2\Omega(\dot{y}, -\dot{x}) + \Omega^2 \mathbf{r}, \tag{3}$$

with gravitational contributions from Jupiter and the Sun, a Coriolis force contribution, and a centrifugal force contribution. Lagrange's solution identifies two *stable* Lagrange points $\mathbf{L_4}$ and $\mathbf{L_5}$ located at:

$$\mathbf{L_{4,5}} = R\left(\frac{M_2 - M_1}{M_1 + M_2}\cos\alpha_{4,5}, \sin\alpha_{4,5}\right) \quad \text{where} \quad \alpha_{4,5} = \pm\frac{\pi}{3}. \tag{4}$$

To observe the stability of these starting points, we will start our asteroid at nearby points $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\tilde{\alpha}_{4,5} \equiv \alpha_{4,5} \pm \Delta\alpha$. For small $\Delta\alpha$, we should see small oscillations around $\mathbf{L_{4,5}}$, but for large $\Delta\alpha$, we should see more variation in the asteroid's orbit. We use an RK4 integration routine to evolve the position $\mathbf{r}$ of the asteroid over several periods $T = 2\pi/\Omega$.

Figures 1 through 4 show asteroid orbits for $\Delta\alpha = 0.01, 0.05, 0.5, 1.5$. No orbit is shown for $\Delta\alpha = 0$ because there is none: the asteroid remains stationary at its starting position. As expected, the asteroid's orbit amplitude increases dramatically with $\Delta\alpha$.

2 General three-body problem with equal masses

In a general multi-body problem, the acceleration of the i-th object is given by:

$$\ddot{\mathbf{r}}_\mathbf{i} = -G \sum_{j \neq i} M_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}. \tag{5}$$

We consider a case with three bodies of equal mass. We will use natural units, where $G = M_{1,2,3} = 1$.

2a Lagrange solution

We want to verify the Lagrange solution, where the masses move along the vertices of a rotating equilateral triangle of side length d and velocity v. First, we will confirm this solution analytically.

Let $R \equiv d/\sqrt{3}$, the distance of a point on the equilateral triangle from the center. This is the radius of the centripetal motion of each mass. Without loss of generality, let one of the masses begin at $(x, y) = (0, R)$. By symmetry, the net gravitational force on this mass is in the y-direction (towards the center) and its magnitude is:

$$F = 2\left(\frac{GM^2}{d^2}\right)\cos 30^\circ = \frac{GM^2}{d^2}\sqrt{3} = \frac{GM^2}{dR}. \tag{6}$$

The velocity of centripetal motion is related to the centripetal force by $F = v^2/R$, so:

$$v = \sqrt{RF} = \sqrt{\frac{GM}{d}}. \square \tag{7}$$

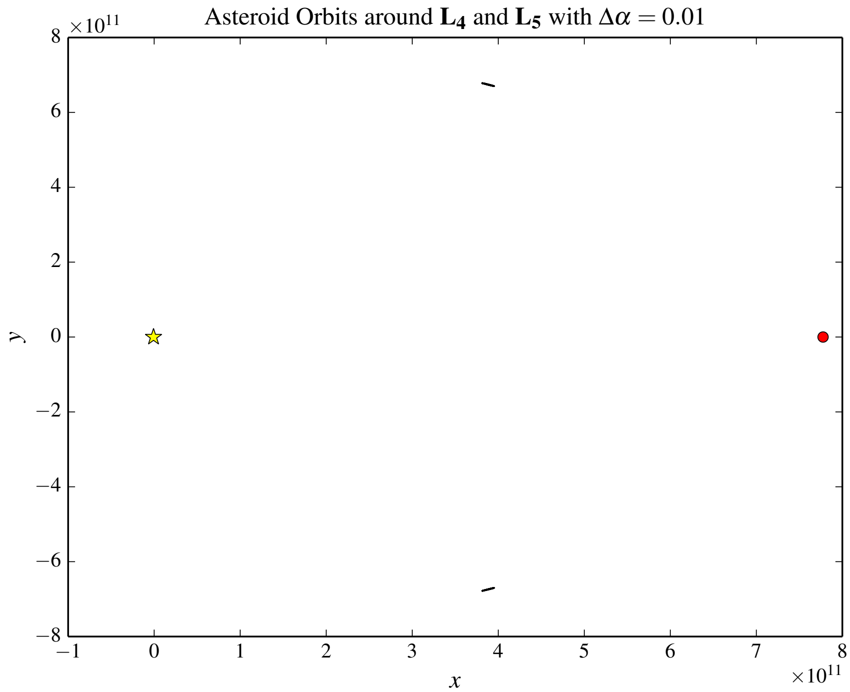

Figure 1: Small-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\Delta\alpha = 0.01$. Yellow star denotes the Sun, and red circle denotes Jupiter.

We simulate this orbit using our RK4 integrator. Initial conditions are:

$$(x_i, y_i) = (R\sin\theta_i, R\cos\theta_i), \qquad (\dot{x}_i, \dot{y}_i) = (v\cos\theta_i, -v\sin\theta_i), \quad \text{with} \quad \{\theta_1, \theta_2, \theta_3\} = \left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}. \tag{8}$$

The result of the simulation is saved in `lagrange.mp4` in this directory.

As with all numerical simulations, our initial conditions are approximations to the exact stable parameters. Figure 5 shows the long-term evolution of the system: the bodies eventually leave their stable orbit and drift outward.

2b Choreographic orbit

Using the same RK4 routine, we use the given conditions for a choreographic orbit. The simulation result is saved in `choreographic.mp4` in this directory.

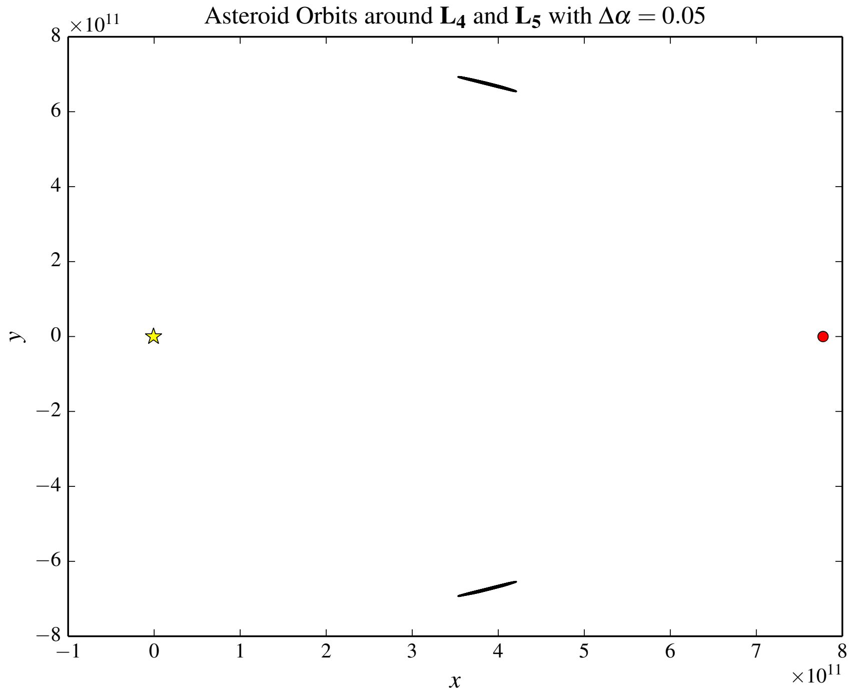

Figure 2: Small-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{4,5}$ with $\Delta\alpha = 0.05$. Yellow star denotes the Sun, and red circle denotes Jupiter.

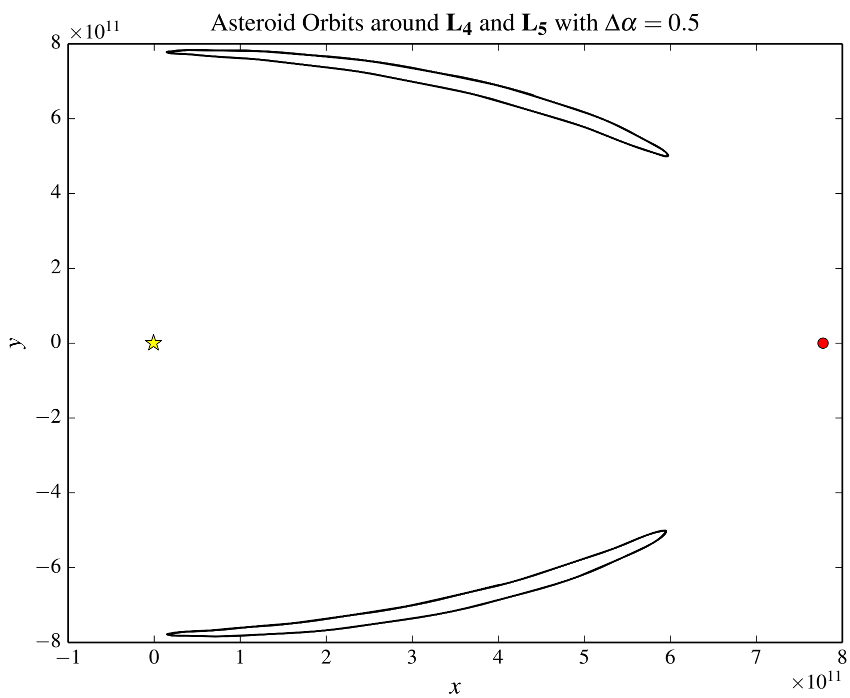

Figure 3: Larger-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{4,5}$ with $\Delta\alpha = 0.5$. Yellow star denotes the Sun, and red circle denotes Jupiter.

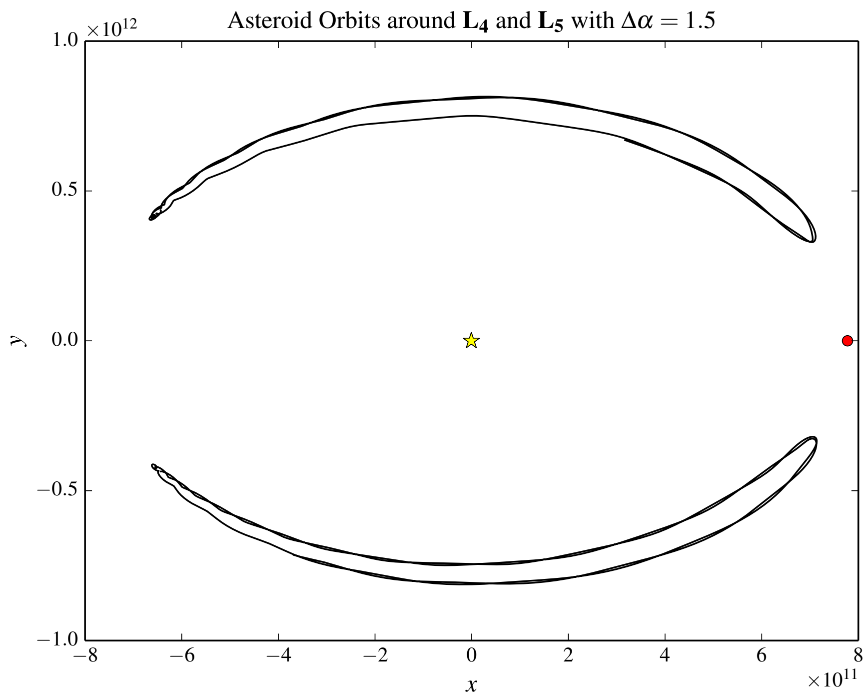

Figure 4: Larger-oscillation orbits of an asteroid in the Jupiter-Sun system, placed at starting positions $\tilde{\mathbf{L}}_{\mathbf{4,5}}$ with $\Delta\alpha = 1.5$. Yellow star denotes the Sun, and red circle denotes Jupiter.

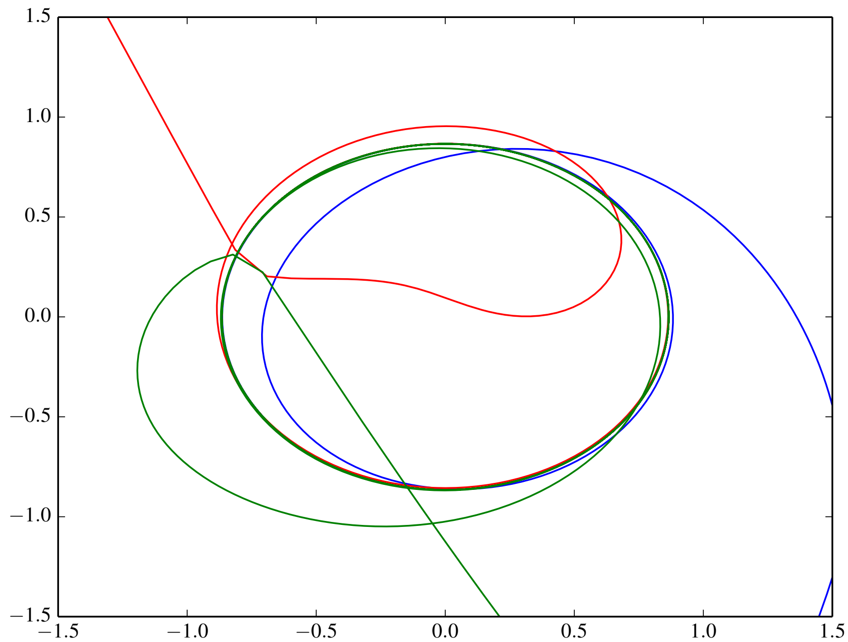

Figure 5: Long-term evolution of the Lagrange solution. The three bodies follow the stable circle for a long time, but eventually drift outwards, as seen in the blue path. The red and green bodies meet during their outward spirals and slingshot off of each other, shooting off to infinity.