1 Convergence of the secant method

We are trying to find a root of a function $f(x)$. Let $\hat{x}$ be the root we seek, such that $f(\hat{x}) = 0$. The Newton-Rhaphson method produces a series of guesses x_i which iterate as:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}. \tag{1}$$

In the secant method, we approximate $f'(x_i)$ using the slope of a secant:

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}. \tag{2}$$

Let $\varepsilon_i \equiv x_i - \hat{x}$. After our guesses x_i are close, we can approximate $f(x_i)$ with a second-order Taylor expansion in the error:

$$f(x_i) = f(\hat{x} + \varepsilon_i) \approx \cancel{f(\hat{x})} + \varepsilon_i f'(\hat{x}) + \varepsilon_i^2 \frac{f''(\hat{x})}{2}. \tag{3}$$

We can express (2) in terms of the errors and use this approximation to obtain a recursion relation for the errors ε_i:

$$\varepsilon_{i+1} = \varepsilon_i - f(x_i)\frac{\varepsilon_i - \varepsilon_{i-1}}{f(x_i) - f(x_{i-1})}. \tag{4}$$

Since we only care about the order of convergence, we'll clean this up with $A \equiv f'(\hat{x})$ and $B \equiv f''(\hat{x})/2$:

$$\begin{aligned}
\varepsilon_{i+1} &= \varepsilon_i - (A\varepsilon_i + B\varepsilon_i^2)\frac{\varepsilon_i - \varepsilon_{i-1}}{A(\varepsilon_i - \varepsilon_{i-1}) + B(\varepsilon_i^2 - \varepsilon_{i-1}^2)} && (5)\\
&= \varepsilon_i - \frac{A\varepsilon_i + B\varepsilon_i^2}{A}\left(\frac{1}{1+\eta_i}\right) \quad \text{where} \quad \eta_i \equiv \frac{B}{A}\cdot\frac{\varepsilon_i^2 - \varepsilon_{i-1}^2}{\varepsilon_i - \varepsilon_{i-1}} \sim \varepsilon_i \text{ is small} && (6)\\
&\approx \varepsilon_i - \left(\varepsilon_i + \frac{B}{A}\varepsilon_i^2\right)\left(1 - \frac{B}{A}\cdot\frac{\varepsilon_i^2 - \varepsilon_{i-1}^2}{\varepsilon_i - \varepsilon_{i-1}}\right) && (7)\\
&= -\frac{B}{A}\varepsilon_i^2 + \frac{B}{A}\left(\varepsilon_i\cdot\frac{\varepsilon_i^2 - \varepsilon_{i-1}^2}{\varepsilon_i - \varepsilon_{i-1}}\right) + \frac{B^2}{A^2}\left(\varepsilon_i^2\cdot\frac{\varepsilon_i^2 - \varepsilon_{i-1}^2}{\varepsilon_i - \varepsilon_{i-1}}\right) && (8)
\end{aligned}$$

Assume that $\varepsilon_{i+1} = C\varepsilon_i^r$ and thus $\varepsilon_{i-1} = (\varepsilon_i/C)^{1/r}$. This gives us: