Numerical solution of the 1D Euler equations

The objective of this course work assignment is the development of a computer code and the assessment of its results for the solution of the 1D Euler equations using the Flux Vector Splitting method. The method will be applied to the solution of the Riemann shock tube problem.

Riemann shock tube problem

The 1D Euler equations representing the flow of an inviscid perfect gas are written in conservation form as

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = 0 \qquad \equiv \qquad \frac{\partial}{\partial t}\begin{bmatrix} \rho \\ \rho u \\ \rho E \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

with

$$p = (\gamma - 1)\,\rho\,(E - \frac{u^2}{2}\,)\,; \qquad H = E + \frac{p}{\rho}$$

Take $\gamma = 1.4$ for air.

The Riemann or shock tube problem is set as follows. Consider, using the notation of Figure 1, a long duct of constant sectional area containing air at different conditions, a driver part (2) and a driven part (1), which are initially at rest and are separated by a diaphragm. The diaphragm position is $x = 0.$, and, at time $t = 0$, the values of the flow variables for the driver air ($x < 0$) are: p_2, ρ_2 and $u_2 = 0$, and for the driven air ($x > 0$) are: p_1, ρ_1 and $u_1 = 0$. A solution of the the 1D Euler equations, corresponding to the initial conditions of the Riemann problem, is sought in the interval $-2 \leq x \leq 2$, for a pressure ratio $p_2/p_1 = 10$ and a density ratio $\rho_2/\rho_1 = 8$.

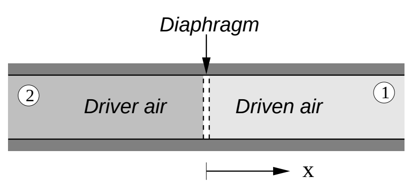

Figure 1: Notation for the Riemann shock tube problem.

Exact solution of the shock tube problem

The MATLAB code for the analytical solution of the shock tube problem is called `Shock_tube_analytic.m` and can be downloaded from Blackboard Learn. The code will interactively ask for the pressure and density ratios, and for the time at which you want to compute the solution. It will produce the file `shock_analytic.dat` that contains the analytical solution.

The Flux Vector Splitting method

The Flux Vector Splitting method splits the flux vector $\mathbf{F}$ into 2 components, $\mathbf{F}^+$ and $\mathbf{F}^-$ such that

$$\mathbf{F}(\mathbf{U}) = \mathbf{F}^+(\mathbf{U}) + \mathbf{F}^-(\mathbf{U}) \tag{2}$$

where

$$\mathbf{F}^+(\mathbf{U}) = \mathbf{A}^+\mathbf{U}, \quad \mathbf{F}^-(\mathbf{U}) = \mathbf{A}^-\mathbf{U} \tag{3}$$

The resulting 1D Euler equations become

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}^+(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{F}^-(\mathbf{U})}{\partial x} = 0 \tag{4}$$

The matrices $\mathbf{A}^+$ and $\mathbf{A}^-$ result from the splitting of the Jacobian $\mathbf{A}$ in such a way that

$$\mathbf{A} = \mathbf{A}^+ + \mathbf{A}^- \tag{5}$$

Matrix $\mathbf{A}$ can be written as $\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}$ where $\mathbf{\Lambda}$ is a diagonal matrix that contains the eigenvalues λ_i $(i = 1, 2, 3)$ of $\mathbf{A}$ and $\mathbf{R}$ is the matrix of right eigenvectors, stacked column by column. Suppose that we split the eigenvalues λ_i as

$$\lambda_i = \lambda_i^+ + \lambda_i^- \tag{6}$$

sush that $\lambda_i^+ \geq 0$ and $\lambda_i^- \leq 0$. Then $\mathbf{\Lambda}$ can be split as

$$\mathbf{\Lambda} = \mathbf{\Lambda}^+ + \mathbf{\Lambda}^- \tag{7}$$

where

$$\mathbf{\Lambda}^+ = \begin{bmatrix} \lambda_1^+ & & 0 \\ & \lambda_2^+ & \\ 0 & & \lambda_3^+ \end{bmatrix}, \quad \mathbf{\Lambda}^- = \begin{bmatrix} \lambda_1^- & & 0 \\ & \lambda_2^- & \\ 0 & & \lambda_3^- \end{bmatrix}$$

Matrices $\mathbf{A}^+$ and $\mathbf{A}^-$ can be computed from

$$\mathbf{A}^+ = \mathbf{R}\mathbf{\Lambda}^+\mathbf{R}^{-1}, \quad \mathbf{A}^- = \mathbf{R}\mathbf{\Lambda}^-\mathbf{R}^{-1} \tag{8}$$

and from equation (3) one can compute $\mathbf{F}^+(\mathbf{U})$ and $\mathbf{F}^-(\mathbf{U})$. Steger and Warming (1981) proposed a splitting of the eigenvalues as:

$$\lambda_i^+ = \frac{1}{2}\left(\lambda_i + |\lambda_i|\right) \quad \lambda_i^- = \frac{1}{2}\left(\lambda_i - |\lambda_i|\right) \tag{9}$$

Assume that the eigenvalues are ordered as $\lambda_1 = u - c$, $\lambda_2 = u$, $\lambda_3 = u + c$. After some algebra, the fluxes $\mathbf{F}^\pm(\mathbf{U})$ are found to be:

$$\mathbf{F}^\pm(\mathbf{U}) = \frac{\rho}{2\gamma} \begin{bmatrix} \lambda_1^\pm + 2(\gamma-1)\lambda_2^\pm + \lambda_3^\pm \\ (u-c)\lambda_1^\pm + 2(\gamma-1)u\lambda_2^\pm + (u+c)\lambda_3^\pm \\ (H-uc)\lambda_1^\pm + (\gamma-1)u^2\lambda_2^\pm + (H+uc)\lambda_3^\pm \end{bmatrix} \tag{10}$$

1. Calculate the analytical solution of the Riemann problem by running the MATLAB code `Shock_tube_analytic.m`. Set time $t = 0.5$. [5%]

2. Discretise system (4) using appropriate first order upwind schemes for the derivatives $\frac{\partial \mathbf{F}^+(\mathbf{U})}{\partial x}$ and $\frac{\partial \mathbf{F}^-(\mathbf{U})}{\partial x}$. [5%]

3. Compute a numerical solution, using three different meshes with $N = 100, 200$ and 300 equally spaced points in the interval $-2 \leq x \leq 2$. For simplicity, take the numerical boundary conditions to be $\mathbf{U}_0 = \mathbf{U}_1$ and $\mathbf{U}_{N+1} = \mathbf{U}_N$ at the end points $x = -2$ and $x = 2$ respectively. [50%]

4. Explain how you have obtained a suitable value of Δt to guarantee the stability of the numerical scheme for the 3 meshes. [10%]

5. Plot the numerical solution versus the analytical solution computed at time $t = 0.5$. Comment on the computed results and the accuracy of the scheme. [30%]

Submit a report that includes your answers to the above questions as well as relevant plots and discussion. The report should not exceed 4-5 pages. Upload also the code that you have written to compute the numerical solution. Make sure that your code is clear and contains appropriate comments.

Reference

Steger,J.L. and Warming,R.F. (1981) Flux Vector Splitting of the inviscid Gas Dynamic Equations with Applications to Finite Difference methods. J. Comput. Physics, vol. 40, pp 263-293.