Study Notes on Weber Electrodynamics

October 29, 2024

These are my notes on Weber's Electrodynamics by Andre Koch Torres Assis.

1 Circuital law with displacement current

This section relates to the 2 Maxwell's equations

Divergence

$$\boxed{\begin{gathered}\nabla \cdot \vec{B} = 0 \\ \oiint_S \vec{B} \cdot \vec{da} = 0\end{gathered}}$$

Curl (this is the circuital law with displacement current equation **2.49** in Assis)

$$\boxed{\begin{gathered}\nabla \times \vec{B} = \mu_0 \vec{j} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\ \oint_{C_i} \vec{B}\left(\vec{r}_i, t\right) \cdot \vec{dl}_i = \mu_0 \iint_{S_i} \vec{j}\left(\vec{r}_i, t\right) \cdot \vec{da} + \frac{1}{c^2}\frac{d}{dt}\iint_{S_i} E\left(\vec{r}_i\right) \cdot \vec{da}\end{gathered}}$$

1.1 Ampere force between 2 current elements - Summary.

If I have this right, Ampere's force law is between 2 *currents* I or current elements (and not moving *charges* which is what Weber used). While it leads to the circuital law with displacement current it does not lead to Faraday's law.

Weber's force law does however lead to both.

Note that we see the cross product appear in this section.

The calculation is based on the force between 2 current elements.

1.1.1 Weber's Force

This is introduced in Chapter 3 of Assis.

Weber's force between 2 charges dq_i, dq_j is given by

$$\boxed{d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{ij}\cdot\vec{v}_{ij} - \frac{3}{2}\left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2 + \vec{r}_{ij}\cdot\vec{a}_{ij}\right)\right] \qquad (1)}$$

$\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ is the position of i with respect to j. It points from j to i.

$\vec{a}_{ij} = \vec{a}_i - \vec{a}_j$ is the acceleration of i with respect to j. The dot product with $\hat{r}_{ij}$ gives us the component along the line connecting i and j.

$\vec{F}_{ji}$ is the force on i due to j.

Alternative Representation

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right]$$

This uses the relative radial velocity and relative radial acceleration.

$$\vec{r}_{ij} = (x_i - x_j)\,\hat{x} + (y_i - y_j)\,\hat{y} + (z_i - z_j)\,\hat{z}$$

but it is the magnitude (and change of magnitude and rate of change of magnitude) of $\vec{r}_{ij}$ that we are interested in - not the direction

$$r_{ij} = |\vec{r}_{ij}| = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{\frac{1}{2}}$$

Velocity Term

$$\dot{r}_{ij} = \frac{d}{dt}r_{ij} = \frac{d}{dt}\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{\frac{1}{2}}$$

$$\dot{r}_{ij} = \frac{d}{dt}r_{ij} = \frac{d}{dt}\left[x_{ij}^2 + y_{ij}^2 + z_{ij}^2\right]^{\frac{1}{2}}$$

The $\hat{x}$ component of this is

$$= \frac{1}{2}\left[x_{ij}^2 + y_{ij}^2 + z_{ij}^2\right]^{-\frac{1}{2}} 2x_{ij}\dot{x}_{ij}$$

$$= \frac{x_{ij}\dot{x}_{ij}}{r_{ij}}$$

which when we include the y and z terms leads to

$$\dot{r}_{ij} = \frac{x_{ij}\dot{x}_{ij} + y_{ij}\dot{y}_{ij} + z_{ij}\dot{z}_{ij}}{r_{ij}}$$

$$\dot{r}_{ij} = \hat{r}_{ij}\cdot\vec{v}_{ij} \qquad (2)$$

As expected if $\vec{v}_{ij}$ is perpendicular to $\hat{r}_{ij}$ then $\dot{r}_{ij} = 0$. Also the sign of $\dot{r}_{ij}$ indicates whether i moves towards j or away from j (i.e. +ve means away)

Acceleration Term

$$\dot{r}_{ij} = \frac{x_{ij}\dot{x}_{ij} + y_{ij}\dot{y}_{ij} + z_{ij}\dot{z}_{ij}}{r_{ij}}$$

$$\ddot{r}_{ij} = \frac{d}{dt}\dot{r}_{ij} = \frac{d}{dt}\left(\frac{x_{ij}\dot{x}_{ij} + y_{ij}\dot{y}_{ij} + z_{ij}\dot{z}_{ij}}{r_{ij}}\right)$$

Considering the x term

$$= \frac{\dot{x}_{ij}\dot{x}_{ij}}{r_{ij}} + \frac{x_{ij}\ddot{x}_{ij}}{r_{ij}} + x_{ij}\dot{x}_{ij}\frac{d}{dt}\left(\frac{1}{r_{ij}}\right)$$

and using

$$\frac{d}{dt}\left(\frac{1}{r_{ij}}\right) = \frac{d}{dt}\left[x_{ij}^2 + y_{ij}^2 + z_{ij}^2\right]^{-\frac{1}{2}} = -\frac{1}{2}\left[x_{ij}^2 + y_{ij}^2 + z_{ij}^2\right]^{-\frac{3}{2}} 2x_{ij}\dot{x}_{ij} = -\frac{-x_{ij}\dot{x}_{ij}}{r_{ij}^3}$$

$$= \frac{\dot{x}_{ij}\dot{x}_{ij}}{r_{ij}} + \frac{x_{ij}\ddot{x}_{ij}}{r_{ij}} - x_{ij}\dot{x}_{ij}\frac{x_{ij}\dot{x}_{ij}}{r_{ij}^3} = \frac{\dot{x}_{ij}\dot{x}_{ij}}{r_{ij}} + \frac{x_{ij}\ddot{x}_{ij}}{r_{ij}} - \frac{\left(x_{ij}\dot{x}_{ij}\right)^2}{r_{ij}^3}$$

which when we include the y and z terms leads to

$$= \frac{\vec{v}_{ij}\cdot\vec{v}_{ij}}{r_{ij}} + \frac{\vec{r}_{ij}\cdot\vec{a}_{ij}}{r_{ij}} - \frac{\left(\vec{r}_{ij}\cdot\vec{v}_{ij}\right)^2}{r_{ij}^3}$$

$$= \frac{\vec{v}_{ij}\cdot\vec{v}_{ij}}{r_{ij}} + \frac{\vec{r}_{ij}\cdot\vec{a}_{ij}}{r_{ij}} - \frac{\left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2}{r_{ij}}$$

$$\ddot{r}_{ij} = \frac{1}{r_{ij}}\left(\vec{v}_{ij}\cdot\vec{v}_{ij} + \vec{r}_{ij}\cdot\vec{a}_{ij} - \left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2\right) \tag{3}$$

Intuitively this makes sense e.g. if i moves in a circle with constant velocity, $(\hat{r}_{ij}\cdot\vec{v}_{ij}) = 0$. $(\vec{v}_{ij}\cdot\vec{v}_{ij}) = v^2$, $\vec{a}_{ij}$ is a radial acceleration towards j and has magnitude $\frac{v^2}{r}$ so $\vec{r}_{ij}\cdot\vec{a}_{ij} = -v^2$. Altogether $\ddot{r}_{ij} = 0$. There is no relative radial acceleration.

Vector Form Going back to

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right]$$

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(r_{ij}\ddot{r}_{ij} - \frac{\dot{r}_{ij}^2}{2}\right)\right]$$

$$r_{ij}\ddot{r}_{ij} - \frac{\dot{r}_{ij}^2}{2} = r_{ij}\left[\frac{1}{r_{ij}}\left(\vec{v}_{ij}\cdot\vec{v}_{ij} + \vec{r}_{ij}\cdot\vec{a}_{ij} - \left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2\right)\right] - \frac{\left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2}{2}$$

$$= \vec{v}_{ij} \cdot \vec{v}_{ij} + \vec{r}_{ij} \cdot \vec{a}_{ij} - (\hat{r}_{ij} \cdot \vec{v}_{ij})^2 - \frac{(\hat{r}_{ij} \cdot \vec{v}_{ij})^2}{2}$$

$$= \vec{v}_{ij} \cdot \vec{v}_{ij} + \vec{r}_{ij} \cdot \vec{a}_{ij} - \frac{3}{2} (\hat{r}_{ij} \cdot \vec{v}_{ij})^2$$

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[1 + \frac{1}{c^2} \left(\vec{v}_{ij} \cdot \vec{v}_{ij} - \frac{3}{2} (\hat{r}_{ij} \cdot \vec{v}_{ij})^2 + \vec{r}_{ij} \cdot \vec{a}_{ij} \right) \right]$$

1.1.2 Derivation of Ampere's Force from Weber's Force

This is **Section 4.2** in Assis. The total force on current elements formed of charges dq_{i+}, dq_{i-} and dq_{j+}, dq_{j-} is then just the usual terms $[i+, j+]\,, [i+, j-]\,[i-, j+]\,[i-, j-]$ i.e.

$$d^2\vec{F} = d^2\vec{F}_{i+,j+} + d^2\vec{F}_{i+,j-} + d^2\vec{F}_{i-,j+} + d^2\vec{F}_{i-,j-}$$

This leads to Ampere's force between 2 current elements (although historically it was the other way around i.e. Ampere's force came first). In [Weber's Electrodynamics] this is **Eq 4.24**.

$$\boxed{d^2\vec{F}^A_{ji} = -\frac{\mu_0}{4\pi} I_i I_j \frac{\hat{r}_{ij}}{r_{ij}^2} \left[2 \left(d\vec{l}_i \cdot d\vec{l}_j \right) - 3 \left(\hat{r}_{ij} \cdot d\vec{l}_i \right) \left(\hat{r}_{ij} \cdot d\vec{l}_j \right) \right] \qquad (4)}$$

as opposed to Grassmann

$$d^2\vec{F}^G_{ji} = -\frac{\mu_0}{4\pi} \frac{I_i I_j}{r_{ij}^2} \left[\left(d\vec{l}_i \cdot d\vec{l}_j \right) \hat{r}_{ij} - \left(\hat{r}_{ij} \cdot d\vec{l}_i \right) d\vec{l}_j \right] \qquad (5)$$

Ampere's version has the nice feature that it obeys Newton's 3rd law.

1.1.3 Acceleration Terms

The term $\vec{r}_{ij} \cdot \vec{a}_{ij}$ in Eq 1 does not contribute to the force between the 2 current elements. The $\vec{r}_{ij} \cdot \vec{a}_{ij}$ terms cancel like so

$$\frac{dq_i dq_j}{4\pi\epsilon_0} \frac{1}{c^2} \frac{\hat{r}_{ij}}{r_{ij}^2} [\vec{r}_{ij} \cdot \vec{a}_{ij}]$$

$$= \frac{dq_i dq_j}{4\pi\epsilon_0} \frac{1}{c^2} \frac{\hat{r}_{ij}}{r_{ij}^2} [\vec{r}_{ij} \cdot [(\vec{a}_{i+} - \vec{a}_{j+}) - (\vec{a}_{i+} - \vec{a}_{j-}) - (\vec{a}_{i-} - \vec{a}_{j+}) + (\vec{a}_{i-} - \vec{a}_{j-})]]$$

$$= \frac{dq_i dq_j}{4\pi\epsilon_0} \frac{1}{c^2} \frac{\hat{r}_{ij}}{r_{ij}^2} [\vec{r}_{ij} \cdot [\vec{a}_{i+} - \vec{a}_{j+} - \vec{a}_{i+} + \vec{a}_{j-} - \vec{a}_{i-} + \vec{a}_{j+} + \vec{a}_{i-} - \vec{a}_{j-}]] = 0$$

1.1.4 Biot-Savart Law (closed circuit)

Both Eq 4 and Eq 5 lead to a total force on current element $I_i\vec{dl}_i$ due to circuit C_j which is the same for Ampere and Grassmann and is given by the below. Note from the cross product it is obviously perpendicular to element $I_i\vec{dl}_i$.

$$d\vec{F}^A_{C_j \text{ on } I_i\vec{dl}_i} = d\vec{F}^G_{C_j \text{ on } I_i\vec{dl}_i} = I_i\vec{dl}_i \times \left[\frac{\mu_0}{4\pi}\oint_{C_j}\left(I_j\vec{dl}_j \times \frac{\vec{r}_{ij}}{r^3_{ij}}\right)\right] \tag{6}$$

The term in [] is the Biot-Savart equation and is the magnetic field at $\vec{r}_i$ due to circuit C_j. In Weber's electrodynamics however there is no magnetic field, this term just appears out of Weber's force law.

$$\vec{B}\left(\vec{r}_i\right) \equiv \frac{\mu_0}{4\pi}\oint_{C_j}\left(I_j\vec{dl}_j \times \frac{\vec{r}_{ij}}{r^3_{ij}}\right) \tag{7}$$

Note that from the above, for an infinitely long wire, $\vec{B}\left(\vec{r}_i\right)$ must be perpendicular to the direction of the wire. And because it is also perpendicular to $\vec{r}_{ij}$ the magnetic field must be concentric rings around the wire.

I'm going to work backwards to explain where Eq 6 comes from (but Assis goes the other way - he starts out with Eq 4).

First calculate the integrand in Eq 6. Note that following Assis I will have element $\vec{dl}_i$ at the origin so $\vec{r}_{ij} = -\hat{x}x_j - \hat{y}y_j - \hat{z}z_j$ and we have a minus sign here $-\frac{\mu_0}{4\pi}$

$$\frac{\mu_0}{4\pi}\left(I_j\vec{dl}_j \times \frac{\vec{r_{ij}}}{r^3_{ij}}\right) = -\frac{\mu_0}{4\pi}\frac{I_j}{r^3_{ij}}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ dx_j & dy_j & dz_j\\ x_j & y_j & z_j\end{vmatrix}$$

$$= -\frac{\mu_0}{4\pi}\left(\frac{I_j}{r^3_{ij}}\right)\left[\hat{x}\left(dy_jz_j - dz_jy_j\right) - \hat{y}\left(dx_jz_j - dz_jx_j\right) + \hat{z}\left(dx_jy_j - dy_jx_j\right)\right]$$

$$= -\frac{\mu_0}{4\pi}\left(\frac{I_j}{r^3_{ij}}\right)\left[\hat{x}\left(dy_jz_j - dz_jy_j\right) + \hat{y}\left(dz_jx_j - dx_jz_j\right) + \hat{z}\left(dx_jy_j - dy_jx_j\right)\right]$$

Choose the coordinate axis so $\vec{dl}_i$ is aligned along the z axis.

$$I_i\vec{dl}_i \times \frac{\mu_0}{4\pi}\left(I_j\vec{dl}_j \times \frac{\vec{r_{ij}}}{r^3_{ij}}\right) = -\frac{\mu_0}{4\pi}\frac{I_iI_j}{r^3_{ij}}\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & dz_i\\ \left(dy_jz_j - dz_jy_j\right) & \left(dz_jx_j - dx_jz_j\right) & \left(dx_jy_j - dy_jx_j\right)\end{vmatrix}$$

$$= -\frac{\mu_0}{4\pi}\frac{I_iI_j}{r^3_{ij}}\left[-\left(dz_i\right)\left(dz_jx_j - dx_jz_j\right)\hat{x} + \left(dz_i\right)\left(dy_jz_j - dz_jy_j\right)\hat{y}\right]$$

$$= -\frac{\mu_0}{4\pi} dl_i \frac{I_i I_j}{r_{ij}^3} \left[\hat{x}\left(z_j dx_j - x_j dz_j\right) + \hat{y}\left(z_j dy_j - y_j dz_j\right)\right]$$

$$= \frac{\mu_0}{4\pi} dl_i \frac{I_i I_j}{r_{ij}^3} \left[\hat{x}\left(x_j dz_j - z_j dx_j\right) + \hat{y}\left(y_j dz_j - z_j dy_j\right)\right]$$

This equation equates to Assis **Eq 4.50**. So he goes the other way (Assis **pg 94** onwards) i.e. he starts out using Eq 4 to arrive at this last equation. He shows how exact differential terms disappear when we integrate, leaving us with Eq 6.

1.1.5 Derivation Note

Note that the derivation in **Assis Section 4.2** (Derivation of Ampere's Force from Weber's Force) the terms $\hat{r}_{ij} \cdot \vec{v}_{ij}$ and $\vec{v}_{ij} \cdot \vec{v}_{ij}$ in Eq 1 contribute to the final result.

I had erroneously thought that using the representation $d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0} \frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right]$ it was the velocity term $\frac{\dot{r}_{ij}^2}{2c^2}$ contributing to Ampere's law and the acceleration term $\frac{r_{ij}\ddot{r}_{ij}}{c^2}$ contributing to Faraday's law. (I can't remember why I thought this.)

However if we did just use the term $\frac{\dot{r}_{ij}^2}{2c^2} = \frac{(\hat{r}_{ij}\cdot\vec{v}_{ij})^2}{2c^2}$ (from Eq 2) then

$$d^2\vec{F} = d^2\vec{F}_{i+,j+} + d^2\vec{F}_{i+,j-} + d^2\vec{F}_{i-,j+} + d^2\vec{F}_{i-,j-}$$

gives

$$-\frac{1}{2c^2}\frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left\{\left[\hat{r}_{ij}\cdot\left(\vec{v}_{i+} - \vec{v}_{j+}\right)\right]^2 - \left[\hat{r}_{ij}\cdot\left(\vec{v}_{i+} - \vec{v}_{j-}\right)\right]^2 - \left[\hat{r}_{ij}\cdot\left(\vec{v}_{i-} - \vec{v}_{j+}\right)\right]^2 + \left[\hat{r}_{ij}\cdot\left(\vec{v}_{i-} - \vec{v}_{j-}\right)\right]^2\right\}$$

$$\begin{aligned} = -\frac{1}{2c^2}\frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2} & \left\{\left[\left(\hat{r}_{ij}\cdot\vec{v}_{i+}\right)^2 + \left(\hat{r}_{ij}\cdot\vec{v}_{j+}\right)^2 - 2\left(\hat{r}_{ij}\cdot\vec{v}_{i+}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{j+}\right)\right]\right\} \\ & - \left[\left(\hat{r}_{ij}\cdot\vec{v}_{i+}\right)^2 + \left(\hat{r}_{ij}\cdot\vec{v}_{j-}\right)^2 - 2\left(\hat{r}_{ij}\cdot\vec{v}_{i+}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{j-}\right)\right] \\ & - \left[\left(\hat{r}_{ij}\cdot\vec{v}_{i-}\right)^2 + \left(\hat{r}_{ij}\cdot\vec{v}_{j+}\right)^2 - 2\left(\hat{r}_{ij}\cdot\vec{v}_{i-}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{j+}\right)\right] \\ & + \left[\left(\hat{r}_{ij}\cdot\vec{v}_{i-}\right)^2 + \left(\hat{r}_{ij}\cdot\vec{v}_{j-}\right)^2 - 2\left(\hat{r}_{ij}\cdot\vec{v}_{i-}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{j-}\right)\right] \end{aligned}$$

with $I_i\vec{dl}_i = dq_i\left(\vec{v}_{i+} - \vec{v}_{i-}\right)$ and omitting steps we would eventually get

$$= \frac{\mu_0}{4\pi} I_i I_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)$$

and using the values from Assis **Eq 4.42** $\left(\hat{r}_{ij}\cdot\vec{dl}_i = dz_i\cos\theta_i\right)$, **Eq 4.43** $\left(\hat{r}_{ij}\cdot\vec{dl}_j = dl_j\cos\theta_j\right)$, **Eq 4.45** $\left(\cos\theta_i = -\frac{z_j}{r_j}\right)$ and **Eq 4.46** $\left(\cos\theta_j = -\frac{dr_j}{dl_j}\right)$ and $dz_i = dl_i$

$$= -\frac{\mu_0}{4\pi} I_i I_j dl_i dl_j \frac{x_j \hat{x}_y + y_j \hat{y} + z_j \hat{z}}{r_{ij}^3} \left(\frac{z_j}{r_j} \frac{dr_j}{dl_j} \right)$$

The x component is

$$= \frac{\mu_0}{4\pi} I_i I_j dl_i dl_j \left(-\frac{x_j z_j}{r_{ij}^4} \frac{dr_j}{dl_j} \right)$$

In Assis the equation below is needed to get from the x component $\frac{\mu_0}{4\pi} I_i I_j dl_i dl_j \left(2\frac{x_j}{r_{ij}^3} \frac{dz_j}{dl_j} - 3\frac{x_j z_j}{r_{ij}^4} \frac{dr_j}{dl_j} \right)$ to Assis **Eq 4.50** and the x component we got above will not suffice.

$$\frac{d}{dl_j} \left(\frac{x_j z_j}{r_{ij}^3} \right) = -3\frac{x_j z_j}{r^4} \frac{dr_j}{dl_j} + \frac{z_j}{r_{ij}^3} \frac{dx_j}{dl_j} + \frac{x_j}{r_{ij}^3} \frac{dz_j}{dl_j}$$

1.2 Maxwell Equation (Ampere Law)

We know that Ampere's force law gives

$$d\vec{F}^A_{C_j \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} \oint_{C_j} \left(I_j d\vec{l}_j \times \frac{\vec{r_{ij}}}{r_{ij}^3} \right) \right)$$

$$d^2\vec{F}^A_{I_j d\vec{l}_j \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} \left(I_j d\vec{l}_j \times \frac{\vec{r_{ij}}}{r_{ij}^3} \right) \right)$$

we used $I_j d\vec{l}_j = dq_j \left(\vec{v}_{j+} - \vec{v}_{j-} \right) = dq_j \vec{v}_{j+,j-}$ and similar for i. Substituting $I_j d\vec{l}_j$ back out gives

$$d^2\vec{F}^A_{dq_j \vec{v}_{j+,j-} \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} \left(dq_j \vec{v}_{j+,j-} \times \frac{\vec{r_{ij}}}{r_{ij}^3} \right) \right)$$

Considering currents distributed over a volume, where ρ is the charge density , $dq_j = \rho \left(\vec{r}_j, t \right) dV_j$

$$d^2\vec{F}^A_{dq_j \vec{v}_{j+,j-} \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} \left(\rho \left(\vec{r}_j, t \right) dV_j \vec{v}_{j+,j-} \times \frac{\vec{r_{ij}}}{r_{ij}^3} \right) \right)$$

from $\vec{j} = \rho \vec{v}$

$$d^2\vec{F}^A_{dq_j \vec{v}_{j+,j-} \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} \left(\vec{j} \left(\vec{r}_j, t \right) dV_j \times \frac{\vec{r_{ij}}}{r_{ij}^3} \right) \right)$$

$$d\vec{F}^A_{V \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_0}{4\pi} \iiint_{V_j} \left(\vec{j} \left(\vec{r}_j, t \right) \times \frac{\vec{r_{ij}}}{r_{ij}^3} \right) \right) dV$$

Let us define $\vec{B}(\vec{r}_i, t)$ as

$$\vec{B}(\vec{r}_i, t) \equiv \frac{\mu_0}{4\pi} \iiint_{V_j} \left(\vec{j}(\vec{r}_j, t) \times \frac{\hat{r_{ij}}}{r_{ij}^2} \right) dV$$

Traditionally this is the magnetic field but as stated earlier, in Weber's electrodynamics there is no magnetic field. Let is focus on $\vec{B}(\vec{r}_i, t)$. Note this is the value of a vector at $\vec{r}_i$ due to a circuit C_j.

From $-\nabla_i \left(\frac{1}{r_{ij}} \right) = \frac{\hat{r_{ij}}}{r_{ij}^2}$

$$\vec{B}(\vec{r}_i, t) = -\frac{\mu_0}{4\pi} \iiint_{V_j} \left(\vec{j}(\vec{r}_j, t) \times \nabla_i \left(\frac{1}{r_{ij}} \right) \right) dV$$

from $\nabla \times \left(\phi \vec{G} \right) = \phi \left(\nabla \times \vec{G} \right) - \vec{G} \times (\nabla \phi)$

$$\vec{B}(\vec{r}_i, t) = -\frac{\mu_0}{4\pi} \iiint_{V_j} \left(\frac{1}{r_{ij}} \left(\nabla_i \times \vec{j}(\vec{r}_j, t) \right) - \nabla_i \times \frac{\vec{j}(\vec{r}_j, t)}{r_{ij}} \right) dV$$

the first term is zero because ∇_i operates on terms with label i

$$\vec{B}(\vec{r}_i, t) = \frac{\mu_0}{4\pi} \iiint_{V_j} \left(\nabla_i \times \frac{\vec{j}(\vec{r}_j, t)}{r_{ij}} \right) dV$$

we can take ∇_i outside the integral

$$\vec{B}(\vec{r}_i, t) = \frac{\mu_0}{4\pi} \nabla_i \times \iiint_{V_j} \left(\frac{\vec{j}(\vec{r}_j, t)}{r_{ij}} \right) dV$$

Note applying the $\nabla_i \cdot$ operator and using $\nabla \cdot \nabla \times \vec{G} = 0$ we get the Maxwell equation $\nabla \cdot \vec{B} = 0$ i.e. no magnetic monopoles.

$$\nabla_i \cdot \vec{B}(\vec{r}_i, t) = \nabla_i \cdot \left(\frac{\mu_0}{4\pi} \nabla_i \times \iiint_{V_j} \left(\frac{\vec{j}(\vec{r}_j, t)}{r_{ij}} \right) dV \right) = 0$$

Continuing with

$$\vec{B}(\vec{r}_i, t) = \frac{\mu_0}{4\pi} \nabla_i \times \iiint_{V_j} \left(\frac{\vec{j}(\vec{r}_j, t)}{r_{ij}} \right) dV$$

For the remainder of this section we will be looking at the curl of this vector at i i.e. $\nabla_i \times \vec{B}(\vec{r}_i, t)$

$$\nabla_i \times \vec{B}(\vec{r}_i, t) = \nabla_i \times \left(\frac{\mu_0}{4\pi} \nabla_i \times \iiint_{V_j} \left(\frac{\vec{j}(\vec{r}_j, t)}{r_{ij}} \right) dV \right)$$

using $\nabla \times \left(\nabla \times \vec{G} \right) = \nabla \left(\nabla \cdot \vec{G} \right) - \nabla^2 \vec{G}$

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \nabla_i \cdot \left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right) dV\right) - \frac{\mu_0}{4\pi}\left(\iiint_{V_j} \nabla_i^2 \left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right) dV\right)$$

using $\nabla \cdot \left(f\vec{G}\right) = f\left(\nabla \cdot \vec{G}\right) + (\nabla f)\cdot \vec{G}$ on the first term

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \left[\frac{1}{r_{ij}}\nabla_i \cdot \vec{j}(\vec{r}_j,t) + \nabla_i\left(\frac{1}{r_{ij}}\right)\cdot \vec{j}(\vec{r}_j,t)\right] dV\right) - \frac{\mu_0}{4\pi}\left(\iiint_{V_j} \nabla_i^2 \left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right) dV\right)$$

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \vec{j}(\vec{r}_j,t)\cdot \nabla_i\left(\frac{1}{r_{ij}}\right) dV\right) - \frac{\mu_0}{4\pi}\left(\iiint_{V_j} \nabla_i^2 \left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right) dV\right)$$

For the second term, ∇^2 is the Laplacian of a vector - not $\nabla\cdot(\nabla\phi)$. However, ∇_i doesn't operate on j

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \vec{j}(\vec{r}_j,t)\cdot \nabla_i\left(\frac{1}{r_{ij}}\right) dV\right) - \frac{\mu_0}{4\pi}\left(\iiint_{V_j} \vec{j}(\vec{r}_j,t)\,\nabla_i^2 \left(\frac{1}{r_{ij}}\right) dV\right)$$

using $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{r})$

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \vec{j}(\vec{r}_j,t)\cdot \nabla_i\left(\frac{1}{r_{ij}}\right) dV\right) + \mu_0\vec{j}(\vec{r}_i,t)$$

using $\nabla_i\left(\frac{1}{r_{ij}}\right) = -\nabla_j\left(\frac{1}{r_{ij}}\right)$

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = -\frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \vec{j}(\vec{r}_j,t)\cdot \nabla_j\left(\frac{1}{r_{ij}}\right) dV\right) + \mu_0\vec{j}(\vec{r}_i,t)$$

using $\nabla \cdot \left(f\vec{G}\right) = f\left(\nabla \cdot \vec{G}\right) + (\nabla f)\cdot \vec{G}$

$$\nabla_i \times \vec{B}(\vec{r}_i,t) = -\frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \left[\nabla_j \cdot \left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right) - \left(\frac{1}{r_{ij}}\right)\nabla_j \cdot \vec{j}(\vec{r}_j,t)\right] dV\right) + \mu_0\vec{j}(\vec{r}_i,t)$$

$$= -\frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j} \left[\nabla_j \cdot \left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right)\right] dV - \iiint_{V_j} \left[-\left(\frac{1}{r_{ij}}\right)\nabla_j \cdot \vec{j}(\vec{r}_j,t)\right] dV\right) + \mu_0\vec{j}(\vec{r}_i,t)$$

$$= -\frac{\mu_0}{4\pi}\nabla_i\left(\iint_{S_j}\left(\frac{\vec{j}\left(\vec{r}_j,t\right)}{r_{ij}}\right)\cdot d\vec{a} - \iiint_{V_j}\left[-\left(\frac{1}{r_{ij}}\right)\nabla_j\cdot\vec{j}\left(\vec{r}_j,t\right)\right]dV\right)+\mu_0\vec{j}\left(\vec{r}_i,t\right)$$

If we take the volume V_j as large enough to wholly encompass the circuit C_j then there is no current on S_j so $\iint_{S_j}\left(\frac{\vec{j}(\vec{r}_j,t)}{r_{ij}}\right)\cdot d\vec{a} = 0$.

$$\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) = \mu_0\vec{j}\left(\vec{r}_i,t\right) + \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j}\left[\frac{1}{r_{ij}}\nabla_j\cdot\vec{j}\left(\vec{r}_j,t\right)\right]dV\right)$$

We can use the conservation of charge next i.e. $\nabla\cdot\vec{j}+\frac{\partial\rho}{\partial t} = 0$ (from $-\iiint_{V_j}\frac{\partial\rho}{\partial t}dV = \iint_{S_j}\vec{j}\cdot d\vec{a} = \iiint_{V_j}\nabla\cdot\vec{j}dV$)

$$\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) = \mu_0\vec{j}\left(\vec{r}_i,t\right) - \frac{\mu_0}{4\pi}\nabla_i\left(\iiint_{V_j}\left[\frac{1}{r_{ij}}\frac{\partial\rho\left(\vec{r}_j,t\right)}{\partial t}\right]dV\right)$$

assume circuits fixed in space

$$\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) = \mu_0\vec{j}\left(\vec{r}_i,t\right) - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\nabla_i\left(\iiint_{V_j}\frac{\rho\left(\vec{r}_j,t\right)}{r_{ij}}dV\right)$$

from $\phi\left(\vec{r}_i\right) = \frac{1}{4\pi\epsilon_0}\iiint_{V_j}\frac{\rho(\vec{r}_j)}{r_{ij}}dV$

$$\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) = \mu_0\vec{j}\left(\vec{r}_i,t\right) - \mu_0\epsilon_0\frac{\partial}{\partial t}\nabla_i\left(\phi\left(\vec{r}_i\right)\right)$$

from $E = -\nabla\phi$

$$\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) = \mu_0\vec{j}\left(\vec{r}_i,t\right) + \frac{1}{c^2}\frac{\partial E\left(\vec{r}_i\right)}{\partial t}$$

Note this is Maxwell's equation and we only had to use Ampere's force law and the law of conservation of charge to get it.

For steady currents, if we look at a point in space where there is no current then $\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) = 0$. Note however this does not mean $\vec{B}\left(\vec{r}_i,t\right) = 0$.

Also just because the $\vec{B}$ field curves does not mean that $\nabla_i \times \vec{B}\left(\vec{r}_i,t\right) \neq 0$ cf [Nykamp DQ and Harman C].

2 Faraday's Law

2.1 Coefficient of Mutual Inductance

Before going into Faraday's law, we will introduce M the Coefficient of Mutual Inductance.

In this section, we show how the Weber potential energy U is related to what is called the Coefficient of Mutual Inductance M between 2 circuits.

We end up with path integrals over the 2 circuits.

The Weber potential energy of 2 charges i, j at a distance r_{ij} and a relative velocity v_{ij} is

$$d^2U = \frac{dq_i dq_j}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(1 - \frac{v_{ij}^2}{2c^2}\right) \tag{8}$$

We shall use Eq 8 to derive M.

Consider a current element in circuit C_i consisting of positive and negative moving charges dq_{i+} and dq_{i-} at position $\vec{r}_i$ and similar for another current element in circuit C_j dq_{j+} and dq_{j-}.

There are 4 interactions between the charges in each current element (not including the charges in the same current element).

$[dq_{i+}, dq_{j+}]$, $[dq_{i+}, dq_{j-}]$, $[dq_{i-}, dq_{j+}]$, $[dq_{i-}, dq_{j-}]$

Consider dq_{i+} and dq_{j+} that move with velocities $\vec{v}_{i+}$ and $\vec{v}_{j+}$ respectively. $\vec{v}_{i+j+} = \vec{v}_{i+} - \vec{v}_{j+}$. However I want the relative velocity between dq_i and dq_j which is $\hat{r_{ij}} \cdot \vec{v}_{ij}$

Given that $dq_{i+} = -dq_{i-}$ and that $v_{i+j+}^2 = \left(\hat{r_{ij}} \cdot (\vec{v}_{i+} - \vec{v}_{j+})\right)^2 = \left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)^2 + \left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)^2 - 2\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)$

$$\begin{aligned}
d^2U = &\frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(1 - \frac{\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)^2 + \left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)^2 - 2\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)}{2c^2}\right) \\
&-\frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(1 - \frac{\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)^2 + \left(\hat{r_{ij}} \cdot \vec{v}_{j-}\right)^2 - 2\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j-}\right)}{2c^2}\right) \\
&-\frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(1 - \frac{\left(\hat{r_{ij}} \cdot \vec{v}_{i-}\right)^2 + \left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)^2 - 2\left(\hat{r_{ij}} \cdot \vec{v}_{i-}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)}{2c^2}\right) \\
&+\frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(1 - \frac{\left(\hat{r_{ij}} \cdot \vec{v}_{i-}\right)^2 + \left(\hat{r_{ij}} \cdot \vec{v}_{j-}\right)^2 - 2\left(\hat{r_{ij}} \cdot \vec{v}_{i-}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j-}\right)}{2c^2}\right)
\end{aligned}$$

cancelling terms

$$\begin{aligned}
d^2U = &\frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(\frac{2\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)}{2c^2}\right) + \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(\frac{-2\left(\hat{r_{ij}} \cdot \vec{v}_{i+}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j-}\right)}{2c^2}\right) \\
&+\frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(\frac{-2\left(\hat{r_{ij}} \cdot \vec{v}_{i-}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j+}\right)}{2c^2}\right) + \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0} \frac{1}{r_{ij}} \left(\frac{2\left(\hat{r_{ij}} \cdot \vec{v}_{i-}\right)\left(\hat{r_{ij}} \cdot \vec{v}_{j-}\right)}{2c^2}\right)
\end{aligned}$$

$$
\begin{aligned}
d^2U = {} & \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0 c^2}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot\vec{v}_{i+}\right)\left(\hat{r_{ij}}\cdot\vec{v}_{j+}\right) - \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot\vec{v}_{i+}\right)\left(\hat{r_{ij}}\cdot\vec{v}_{j-}\right) \\
& - \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot\vec{v}_{i-}\right)\left(\hat{r_{ij}}\cdot\vec{v}_{j+}\right) + \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot\vec{v}_{i-}\right)\left(\hat{r_{ij}}\cdot\vec{v}_{j-}\right)
\end{aligned}
$$

$$
\begin{aligned}
d^2U = {} & \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0 c^2}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot\vec{v}_{i+}\right)\left(\hat{r_{ij}}\cdot(\vec{v}_{j+}-\vec{v}_{j-})\right) \\
& - \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot\vec{v}_{i-}\right)\left(\hat{r_{ij}}\cdot(\vec{v}_{j+}-\vec{v}_{j-})\right)
\end{aligned}
$$

$$
d^2U = \frac{dq_{i+}dq_{j+}}{4\pi\epsilon_0 c^2}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot(\vec{v}_{i+}-\vec{v}_{i-})\right)\left(\hat{r_{ij}}\cdot(\vec{v}_{j+}-\vec{v}_{j-})\right)
$$

At this point we go from charges to currents. We also introduce the path integral around both circuits.

$$
d^2U = \frac{I_1 I_2}{4\pi\epsilon_0 c^2}\frac{1}{r_{ij}}\left(\hat{r_{ij}}\cdot d\vec{l}_i\right)\left(\hat{r_{ij}}\cdot d\vec{l}_j\right)
$$

$$
U = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_i}\oint_{C_j}\frac{\left(\hat{r_{ij}}\cdot d\vec{l}_i\right)\left(\hat{r_{ij}}\cdot d\vec{l}_j\right)}{r_{ij}}
$$

from Appendix A

$$
U = \frac{\mu_0}{4\pi} I_1 I_2 \left[\oint_{C_i}\oint_{C_j}\left(\frac{d\vec{l}_i\cdot d\vec{l}_j}{r_{ij}}\right)\right] \tag{9}
$$

$$
U = I_1 I_2 M
$$

where M is the coefficient of mutual inductance between the closed circuits C_i and C_j.

$$
\boxed{M = \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\left(\frac{d\vec{l}_i\cdot d\vec{l}_j}{r_{ij}}\right)} \tag{10}
$$

Assis introduces M in **Eq 4.61** and **Section 4.6** as a way of calculating the force between 2 constant current carrying closed circuits (as an alternative to Ampere's force).

Note that M depends only on the geometry of the circuits. I could calculate M for various r_{ij} and orientations . In a simple case, if I assume circuits are fixed in orientation and the currents are constant but vary r_{ij} , then U will be different and the gradient of U will lead to a force.

2.2 Faraday's Law Introduction

Ampere's force law does not lead to Faraday's law. Whereas Weber's force law does.

This section relates to the Maxwell equation.

$$\boxed{\begin{gathered}\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{a}\end{gathered}}$$

We can induce a current in circuit C_i. Magnetic flux Φ_B is defined, at circuit C_i, as

$$\Phi_{B_i} = \iint_{S_i} \vec{B}\left(\vec{r_i}\right) \cdot d\vec{a}$$

$$emf = -\frac{d}{dt} \Phi_{B_i}$$

$$I_i = \frac{emf}{R_i}$$

Franz Neumann introduced the magnetic potential at circuit C_i due to circuit C_j

$$\vec{A}\left(\vec{r_i}\right) \equiv \frac{\mu_0}{4\pi} \oint_{C_j} \left(I_j \frac{d\vec{l}_j}{r_{ij}} \right) \tag{11}$$

and using equations in Appendix C

$$\nabla_i \times \vec{A}\left(\vec{r_i}\right) = \frac{\mu_0}{4\pi} \oint_{C_j} \left(\frac{\hat{r}_{ij}}{r_{ij}^2} \times I_j d\vec{l}_j \right)$$

but we have seen this before in Eq 6 where we introduce implicitly $\vec{B}\left(\vec{r_i}\right)$. We can now write (remember that $\vec{A}\left(\vec{r_i}\right)$ is due to C_j)

$$emf = -\frac{d}{dt} \iint_{S_i} \vec{B}\left(\vec{r_i}\right) \cdot d\vec{a} = -\frac{d}{dt} \iint_{S_i} \left(\nabla_i \times \vec{A}\left(\vec{r_i}\right) \right) \cdot d\vec{a} = -\frac{d}{dt} \oint_{C_i} \vec{A}\left(\vec{r_i}\right) \cdot d\vec{l}_i$$

$$emf = \oint_{C_i} \left(-\frac{\partial \vec{A}\left(\vec{r_i}\right)}{\partial t} \right) \cdot d\vec{l}_i \tag{12}$$

2.3 Mutual Inductance and Faraday's Law

From

$$M = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \left(\frac{d\vec{l}_i \cdot d\vec{l}_j}{r_{ij}} \right)$$

$$I_j M = \frac{\mu_0}{4\pi} \oint_{C_i} \left[\oint_{C_j} \left(\frac{I_j d\vec{l}_j}{r_{ij}} \right) \right] \cdot d\vec{l}_i$$

from Eq 11

$$I_j M = \oint_{C_i} \vec{A}\left(\vec{r}_i\right) \cdot d\vec{l}_i$$

so

$$-\frac{d}{dt}\left(I_j M\right) = -\frac{d}{dt} \oint_{C_i} \vec{A}\left(\vec{r}_i\right) \cdot d\vec{l}_i$$

which is the same as Eq 12 so

$$emf = -\frac{d}{dt}\left(I_j M\right) \tag{13}$$

2.4 Faraday Law Derivation - Part I

$$d^2\vec{F}_{j>i} = \frac{dq_i dq_j}{4\pi\epsilon_0} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[1 + \frac{1}{c^2} \left(\vec{v}_{ij} \cdot \vec{v}_{ij} - \frac{3}{2} \left(\hat{r}_{ij} \cdot \vec{v}_{ij}\right)^2 + \vec{r}_{ij} \cdot \vec{a}_{ij} \right) \right]$$

2.4.1 Induced Current

Just consider the force on $i-$ due to circuit C_j i.e. $[j+, i-]\,[j-, i-]$ i.e.

$$d^2\vec{F} = d^2\vec{F}_{j+,i-} + d^2\vec{F}_{j-,i-}$$

First Term The first term of Weber's force equation using $\mu_0\epsilon_0 = \frac{1}{c^2}$ is

$$d^2\vec{F}^{1\text{st}}_{j>i} = \frac{\mu_0}{4\pi} dq_i dq_j \frac{\hat{r}_{ij}}{r_{ij}^2} \left(\vec{v}_{ij} \cdot \vec{v}_{ij}\right)^2$$

$$\vec{v}_{ij} = \left(\vec{v}_{id} + \vec{V}_i\right) - \left(\vec{v}_{jd} + \vec{V}_j\right) = \left(\vec{v}_{id} - \vec{v}_{jd}\right) + \vec{V}_{ij}$$

$$\vec{v}_{ij} \cdot \vec{v}_{ij} = \left[v_{id}^2 + v_{jd}^2 - 2\left(\vec{v}_{id} \cdot \vec{v}_{jd}\right) + V_{ij}^2 + 2\left(\vec{v}_{id} - \vec{v}_{jd}\right) \cdot \vec{V}_{ij} \right]$$

So for both $j-$ and $j+$

$$= -\left[v_{id-}^2 + v_{jd+}^2 - 2\left(\vec{v}_{id-} \cdot \vec{v}_{jd+}\right) + V_{ij}^2 + 2\left(\vec{v}_{id-} \cdot \vec{V}_{ij}\right) - 2\left(\vec{v}_{jd+} \cdot \vec{V}_{ij}\right) \right]$$
$$+ \left[v_{id-}^2 + v_{jd-}^2 - 2\left(\vec{v}_{id-} \cdot \vec{v}_{jd-}\right) + V_{ij}^2 + 2\left(\vec{v}_{id-} \cdot \vec{V}_{ij}\right) - 2\left(\vec{v}_{jd-} \cdot \vec{V}_{ij}\right) \right]$$

assuming +ve charges are fixed

$$= \left[v_{jd-}^2 - 2\left(\vec{v}_{id-} \cdot \vec{v}_{jd-}\right) - 2\left(\vec{v}_{jd-} \cdot \vec{V}_{ij}\right) \right]$$

$$= \left[v_{jd-}^2 - 2\left(\vec{v}_{id-} \cdot \vec{v}_{jd-}\right)\right] + \left[-2\left(\vec{v}_{jd-} \cdot \vec{V}_{ij}\right)\right]$$

This is consistent with the derivation leading to Assis **Eq 4.19** because we are only looking at $[j+, i-]$ and $[j-, i-]$ terms in that derivation.

Note it looks like Assis Eq 3.24 and Eq 4.16 are inconsistent. It should be a $+$ sign not - in Eq 4.16.

Second Term

$$d^2\vec{F}_{j>i}^{2nd} = -\frac{3}{2}\frac{\mu_0}{4\pi}dq_i dq_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left(\hat{r}_{ij} \cdot \vec{v}_{ij}\right)^2$$

$$\left(\hat{r}_{ij} \cdot \vec{v}_{ij}\right)^2 = \left(\left[\hat{r}_{ij} \cdot \left(\vec{v}_{id} - \vec{v}_{jd}\right) + \hat{r}_{ij} \cdot \vec{V}_{ij}\right]\right)^2$$

with the same assumptions

$$= -\left[\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)^2 + \left(\hat{r}_{ij} \cdot \vec{v}_{jd+}\right)^2 - 2\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd+}\right) + 2\hat{r}_{ij} \cdot \left(\vec{v}_{id-} - \vec{v}_{jd+}\right)\left(\hat{r}_{ij} \cdot \vec{V}_{ij}\right) + \left(\hat{r}_{ij} \cdot \vec{V}_{ij}\right)^2\right]$$
$$+ \left[\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)^2 + \left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)^2 - 2\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right) + 2\hat{r}_{ij} \cdot \left(\vec{v}_{id-} - \vec{v}_{jd-}\right)\left(\hat{r}_{ij} \cdot \vec{V}_{ij}\right) + \left(\hat{r}_{ij} \cdot \vec{V}_{ij}\right)^2\right]$$

adding terms, assuming fixed $+$ve charges and applying the $-\frac{3}{2}$ factor

$$= \left[-\frac{3}{2}\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)^2 + 3\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)\right] + \left[3\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)\left(\hat{r}_{ij} \cdot \vec{V}_{ij}\right)\right]$$

Total of first and second terms

$$\left[-2\left(\vec{v}_{id-} \cdot \vec{v}_{jd-}\right) + v_{jd-}^2 - \frac{3}{2}\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)^2 + 3\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)\right] + \left[3\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)\left(\hat{r}_{ij} \cdot \vec{V}_{ij}\right) - 2\left(\vec{v}_{jd-} \cdot \vec{V}_{ij}\right)\right]$$

Third Term Assume V_{ij} is a constant.

$$\vec{r}_{ij} \cdot \vec{a}_{ij} = \vec{r}_{ij} \cdot \left(\vec{a}_{id} - \vec{a}_{jd}\right)$$

$$= -\left[\vec{r}_{ij} \cdot \left(\vec{a}_{id-} - \vec{a}_{jd+}\right)\right]$$
$$+ \left[\vec{r}_{ij} \cdot \left(\vec{a}_{id-} - \vec{a}_{jd-}\right)\right]$$

$$= -\vec{r}_{ij} \cdot \vec{a}_{jd-}$$

so $\vec{a}_{jd-}$ is the acceleration of electrons in C_j relative to the wire itself. The acceleration is due to the change in the drift velocity of electrons in C_j.

Total of first, second and third terms gives Assis **Eq 5.52**. It states there that the second square bracket below when integrated is exactly zero but doesn't give a general proof. He mentions 3 circuit orientations where it is proved.

Note also that the minus sign is applied to the [] brackets of both terms.

The sign in Assis Eq 5.27 is different to the below. I think that stems from Eq 5.19 where he uses emf_{12} and $\hat{r}_{12}$ whereas I have stayed with emf_{21} and $\hat{r}_{12}$.

$$d^2\vec{F}_{j>i-} = -\frac{\mu_0}{4\pi}dq_i dq_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2\left(\vec{V}_{ij}\cdot\vec{v}_{jd-}\right) - 3\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd-}\right) + \vec{r}_{ij}\cdot\vec{a}_{jd-}\right] \tag{14}$$

$$-\frac{\mu_0}{4\pi}dq_i dq_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2\left(\vec{v}_{id-}\cdot\vec{v}_{jd-}\right) - v_{jd-}^2 + \frac{3}{2}\left(\hat{r}_{ij}\cdot\vec{v}_{jd-}\right)^2 - 3\left(\hat{r}_{ij}\cdot\vec{v}_{id-}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd-}\right)\right]$$

2.5 Faraday Law Derivation - Part II - First Term

I'll mainly be focusing on the integrands in this section. Using j suffix instead of $j-$, the first term of the above is

$$d^2\vec{F}_{j>i-} = -\frac{\mu_0}{4\pi}dq_i dq_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2\left(\vec{V}_{ij}\cdot\vec{v}_{jd}\right) - 3\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd}\right) + \left(\vec{r}_{ij}\cdot\vec{a}_{jd}\right)\right]$$

Assume the circuits have translational velocity only i.e. $\frac{\partial(\vec{dl})}{\partial t} = 0$.

Let us assume that we have a continuous charge density ρ_j in circuit C_j. Let the wire cross section be dA and consider a length dl_j. In this length the amount of charge is $dq_j = \rho_j dl_j dA$.

$$d^2\vec{F}_{j>i-} = -\frac{\mu_0}{4\pi}dq_i \left(\rho_j dl_j dA\right) \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2\left(\vec{V}_{ij}\cdot\vec{v}_{jd}\right) - 3\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd}\right) + \left(\vec{r}_{ij}\cdot\vec{a}_{jd}\right)\right]$$

The current density $\vec{J} = \rho\vec{v}$ is the charge that passes unit area in unit time. $I_j = \vec{J}\cdot\vec{dA} = \rho\vec{v}\cdot\vec{dA}$ is the charge that passes in unit time.

$$dq_j\left(\vec{r}_{ij}\cdot\vec{a}_{jd}\right) = \vec{r}_{ij}\cdot\left(\rho_j dl_j dA\right)\frac{d}{dt}\left(\vec{v}_{jd}\right) = \vec{r}_{ij}\cdot\frac{\left(\rho_j dl_j dA\right)\vec{v}_{t2} - \left(\rho_j dl_j dA\right)\vec{v}_{t1}}{dt}$$

$\vec{v}_j$ is the same direction as $\vec{dl}_j$. Let $\hat{l}_j$ be the unit vector in the direction of $\vec{v}_j$.

$\left(\rho_j dl_j dA\right)\vec{v}_{jd} = \left(\rho_j \vec{v}_{jd} dA\right)dl_j = \left(\rho_j v_{jd} dA\right)dl_j\hat{l}_j = I_j\vec{dl}_j$

The following is the force exerted by $\vec{dl}_j$ on dq_i.

$$d^2\vec{F}_{j>dq_i} = -\frac{\mu_0}{4\pi}dq_i \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2I_j\left(\vec{V}_{ij}\cdot\vec{dl}_j\right) - 3I_j\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right) + \frac{dI_j}{dt}\left(\vec{r}_{ij}\cdot\vec{dl}_j\right)\right]$$

The total force exerted on dq_i by C_j is

$$d\vec{F}_{C_j>dq_i} = -\frac{\mu_0}{4\pi} dq_i \oint_{C_j} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[2I_j \left(\vec{V}_{ij} \cdot \vec{dl}_j \right) - 3I_j \left(\hat{r}_{ij} \cdot \vec{V}_{ij} \right) \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) + \frac{dI_j}{dt} \left(\vec{r}_{ij} \cdot \vec{dl}_j \right) \right]$$

$$d\vec{E}_{C_j>dq_i} = -\frac{\mu_0}{4\pi} \oint_{C_j} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[2I_j \left(\vec{V}_{ij} \cdot \vec{dl}_j \right) - 3I_j \left(\hat{r}_{ij} \cdot \vec{V}_{ij} \right) \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) + \frac{dI_j}{dt} \left(\vec{r}_{ij} \cdot \vec{dl}_j \right) \right]$$

$$emf = \oint_{C_i} d\vec{E}_{C_j>dq_i} \cdot \vec{dl}_i$$

$$emf = -\frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} \left[2I_j \left(\vec{V}_{ij} \cdot \vec{dl}_j \right) - 3I_j \left(\hat{r}_{ij} \cdot \vec{V}_{ij} \right) \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) + \frac{dI_j}{dt} \left(\vec{r}_{ij} \cdot \vec{dl}_j \right) \right]$$

$$= -\frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \left[2I_j \frac{\left(\vec{V}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} - \frac{3I_j \left(\hat{r}_{ij} \cdot \vec{V}_{ij} \right) \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} + \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}} \frac{dI_j}{dt} \right]$$

From **Assis Eq 4.55** and paragraph after **Assis Eq 5.48** $\oint_{C_i} \frac{\left(\vec{V}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} = \left(\vec{V}_{ij} \cdot \vec{dl}_j \right) \left[\oint_{C_i} \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} \right] = 0$ so we are left with

$$emf = -\frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \left[-\frac{3I_j \left(\hat{r}_{ij} \cdot \vec{V}_{ij} \right) \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} + \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}} \frac{dI_j}{dt} \right] \tag{15}$$

(dimensions $Cs^{-1}ms^{-1}$)

By the chain rule we can see below that $\frac{d}{dt} \left[\frac{I_j \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}} \right]$ contains the acceleration term. Also from the previous section $\oint_{C_i} \oint_{C_j} \frac{\left(\hat{r_{ij}} \cdot \vec{dl}_i \right) \left(\hat{r_{ij}} \cdot \vec{dl}_j \right)}{r_{ij}}$ is the mutual inductance M_{ij}.

$$\frac{d}{dt} \left[\frac{I_j \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}} \right] = -\frac{I_j \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}^2} \frac{dr_{ij}}{dt}$$

$$+ \frac{I_j \left(\hat{r}_{ij} \cdot \vec{dl}_j \right)}{r_{ij}} \frac{d}{dt} \left(\hat{r}_{ij} \cdot \vec{dl}_i \right) + \frac{I_j \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}} \frac{d}{dt} \left(\hat{r}_{ij} \cdot \vec{dl}_j \right) + \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_j \right) \left(\hat{r}_{ij} \cdot \vec{dl}_i \right)}{r_{ij}} \frac{dI_j}{dt}$$

so we can write the last term in Eq 15

$$\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{dI_j}{dt}=\frac{d}{dt}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\right]$$
$$-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)$$
$$+\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}$$

2.5.1 V Term

r_{ij} can change due to relative motion between the 2 circuits. $\frac{\partial\vec{r}_{ij}}{\partial t}=\vec{V}_{ij}$, $\frac{\partial r_{ij}}{\partial t}=\hat{r}_{ij}\cdot\vec{V}_i$. Consider the 3 terms below (i, ii, iii) and using $d\frac{f}{g}=\frac{gf'-fg'}{g^2}$

$$\left[-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\right]_i-\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\right]_{ii}+\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}\right]_{iii}$$

$$\frac{\partial}{\partial t}\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)=\frac{\partial}{\partial t}\left(\frac{\vec{r}_{ij}\cdot\vec{dl}_i}{r_{ij}}\right)=\left[\frac{r_{ij}\vec{V}_{ij}-\vec{r}_{ij}\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}\right]\cdot\vec{dl}_i$$

$$(i)=-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{\partial}{\partial t}=-I_j\left[\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\vec{V}_{ij}\cdot\vec{dl}_i\right)-\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}\right]$$

$$(ii)=-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{\partial}{\partial t}=-I_j\left[\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\vec{V}_{ij}\cdot\vec{dl}_j\right)-\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}\right]$$

$$(iii)=I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{\partial r_{ij}}{\partial t}=I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}$$

Altogether

$$3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}-I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\vec{V}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}-I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\vec{V}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}^2}$$

Likewise when we do the double integral $\oint_{C_i}\oint_{C_j}$ we are left with the term

$$=3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}$$

So going back to the integrand in

$$emf = -\frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\left[-\frac{3I_j\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2} + \frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{dI_j}{dt}\right]$$

$$[] = -\frac{3I_j\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}$$
$$+\frac{d}{dt}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\right]$$
$$+\left\{-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\hat{dl}_i\right) - \frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_j\right) + \frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}\right\}$$

$$[] = -\frac{3I_j\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}$$
$$+\frac{d}{dt}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\right]$$
$$+3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}$$

$$emf = -\frac{\mu_0}{4\pi}\frac{d}{dt}\oint_{C_i}\oint_{C_j}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\right]$$

(dimensions excluding $\frac{\mu_0}{4\pi}$ is $Cs^{-1}ms^{-1}$)

Sanity Check I If we keep the 2 circuits fixed in space (i.e. $V_{ij} = 0$ and r_{ij} doesn't change) and just change the current we see that be below corresponds to Eq 15

$$\frac{d}{dt}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\right] = \frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{dI_j}{dt}$$

Sanity Check II If we keep the current constant and change the velocity i.e. $V_{ij} \neq 0$ we should get

$$emf = -\frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\left[-\frac{3I_j\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\right]$$

$$\frac{d}{dt}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\right]=-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}$$
$$+\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)+\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)$$

which from above is (which also corresponds to Eq 15)

$$=-3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{V}_{ij}\right)}{r_{ij}^2}$$

2.5.2 v_i Term

Not sure about this bit but r_{ij} can also change due to the drift velocity in circuit C_i. $\frac{\partial\vec{r}_{ij}}{\partial t}=\vec{v}_{id}$ and $\frac{\partial r_{ij}}{\partial l_i}=\hat{r}_{ij}\cdot\vec{v}_{id}$

$$\left[-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\right]_i\text{-}\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{d}{dt}\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\right]_{ii}+\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}\right]_{iii}$$

$$-\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)}{r_{ij}}\frac{\partial}{\partial t}\left(\frac{\vec{r}_{ij}}{r_{ij}}\cdot\vec{dl}_i\right)=-I_j\left[\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\vec{v}_{id}\cdot\vec{dl}_i\right)-\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{id}\right)}{r_{ij}^2}\right]$$

$$-I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}}\frac{\partial}{\partial t}\left(\frac{\vec{r}_{ij}}{r_{ij}}\cdot\vec{dl}_j\right)=-I_j\left[\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\vec{v}_{id}\cdot\vec{dl}_j\right)-\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{v}_{id}\right)}{r_{ij}^2}\right]$$

$$+\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}=+I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{id}\right)}{r_{ij}^2}$$

Altogether

$$-\left[\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\vec{v}_{id}\cdot\vec{dl}_i\right)+I_j\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\vec{v}_{id}\cdot\vec{dl}_j\right)-3I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{id}\right)}{r_{ij}^2}\right]$$

The $\oint_{C_i}$ and $\oint_{C_j}$ will remove two of the terms if we assume $\vec{v}_{id}$ is a constant so we have this extra term by looking at the change of r_{ij} due to $\vec{v}_i$

$$= 3I_j \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{v}_{id}\right)}{r_{ij}^2}$$

But if we go back to the second term in 14

$$-\frac{\mu_0}{4\pi} dq_i dq_j \frac{\hat{r}_{ij}}{r_{ij}^2}\left[2\left(\vec{v}_{id-} \cdot \vec{v}_{jd-}\right) - v_{jd-}^2 + \frac{3}{2}\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)^2 - 3\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)\right]$$

In terms of emf the integrand is

$$= \frac{2I_j\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\vec{v}_{id-} \cdot \vec{dl}_j\right) - I_j\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\vec{v}_{jd} \cdot \vec{dl}_j\right) + \frac{3}{2}I_j\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)}{r_{ij}^2}$$

$$-3I_j \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)}{r_{ij}^2}$$

If we assume $\vec{v}_i$and $\vec{v}_j$ is constant then $\oint_{C_i}$ and $\oint_{C_j}$ will remove the first 2 terms leaving

$$\frac{\frac{3}{2}I_j\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd-}\right)}{r_{ij}^2} - 3I_j \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{v}_{id-}\right)}{r_{ij}^2}$$

we can see that the last term here cancels the above.

2.5.3 v_j Term

Likewise r_{ij} can also change due to the drift velocity in circuit C_j. $\frac{\partial \vec{r}_{ij}}{\partial t} = -\vec{v}_{jd}$
$\frac{\partial r_{ij}}{\partial l_i} = -\hat{r}_{ij} \cdot \vec{v}_{jd}$

$$-\frac{I_j\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)}{r_{ij}} \frac{d}{dt}\left(\hat{r}_{ij} \cdot \vec{dl}_i\right) - \frac{I_j\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)}{r_{ij}} \frac{d}{dt}\left(\hat{r}_{ij} \cdot \vec{dl}_j\right) + \frac{I_j\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)}{r_{ij}^2} \frac{dr_{ij}}{dt}$$

$$-\frac{I_j\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)}{r_{ij}} \frac{\partial}{\partial t}\left(\frac{\vec{r}_{ij}}{r_{ij}} \cdot \vec{dl}_i\right) = I_j\left[\frac{\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\vec{v}_{jd} \cdot \vec{dl}_i\right) + \left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd}\right)}{r_{ij}^2}\right]$$

$$-I_j \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)}{r_{ij}} \frac{\partial}{\partial t}\left(\frac{\vec{r}_{ij}}{r_{ij}} \cdot \vec{dl}_j\right) = I_j\left[\frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\vec{v}_{jd} \cdot \vec{dl}_j\right) - \left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\left(\hat{r}_{ij} \cdot \vec{dl}_j\right)\left(\hat{r}_{ij} \cdot \vec{v}_{jd}\right)}{r_{ij}^2}\right]$$

$$+\frac{I_j\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)}{r_{ij}^2}\frac{dr_{ij}}{dt}=-I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd}\right)}{r_{ij}^2}$$

which gives us

$$I_j\left[\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\vec{v}_{jd}\cdot\vec{dl}_i\right)+\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\vec{v}_{jd}\cdot\vec{dl}_j\right)-3\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd}\right)}{r_{ij}^2}\right]$$

The $\oint_{C_i}$ and $\oint_{C_j}$ will remove two of the terms if we assume $\vec{v}_{jd}$ is a constant

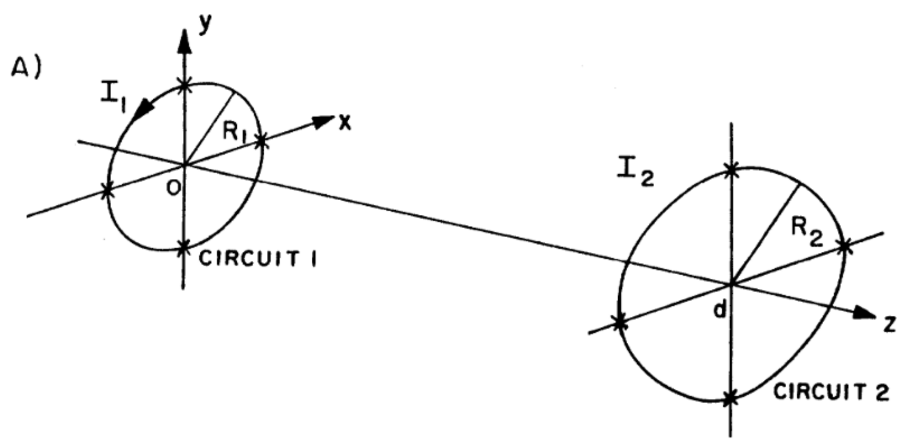
$$-3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd}\right)}{r_{ij}^2}$$

(dimensions $Cs^{-1}ms^{-1}$)

3 Example Circuits

3.1 Parallel Circuits in X-Y plane - Cancellation

2 circular circuits/loops separated by a distance L (from Assis **Fig 5.5 A**)



Choosing θ as the angle from the x axis and R as the radius of the circuit. In the above diagram circuit 1 is C_i and circuit 2 is C_j.

$\vec{r}=\hat{i}R\cos\theta+\hat{j}R\sin\theta$

For $\vec{dl}_i$ and $\vec{dl}_j$ choose the direction as indicated in circuit 1 in the diagram above

$\vec{dl}=-\hat{i}dl\sin\theta+\hat{j}dl\cos\theta$.

$\vec{dl}_i=(-dl_i\sin\theta_i,dl_i\cos\theta_i,0)$

$\vec{dl}_j=(-dl_j\sin\theta_j,dl_j\cos\theta_j,0)$

$\vec{r}_{ij}=(R_i\cos\theta_i-R_j\cos\theta_j,R_i\sin\theta_i-R_j\sin\theta_j,-L)$

$$\left(\vec{r}_{ij}\cdot\vec{dl}_i\right)=\left(R_i\cos\theta_i-R_j\cos\theta_j\right)\left(-dl_i\sin\theta_i\right)+\left(R_i\sin\theta_i-R_j\sin\theta_j\right)\left(dl_i\cos\theta_i\right)$$

$$=\left(-R_idl_i\sin\theta_i\cos\theta_i+R_jdl_i\sin\theta_i\cos\theta_j\right)+\left(R_idl_i\cos\theta_i\sin\theta_i-R_jdl_i\cos\theta_i\sin\theta_j\right)$$

$$=R_jdl_i\sin\theta_i\cos\theta_j-R_jdl_i\cos\theta_i\sin\theta_j$$

$$=R_jdl_i\left[\sin\theta_i\cos\theta_j-\cos\theta_i\sin\theta_j\right]=R_jdl_i\sin\left(\theta_i-\theta_j\right)$$

using $dl_i=R_id\theta_i$

$$\left(\vec{r}_{ij}\cdot\vec{dl}_i\right)=R_iR_jd\theta_i\sin\left(\theta_i-\theta_j\right)$$

$$\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)=\frac{R_iR_jd\theta_i\sin\left(\theta_i-\theta_j\right)}{r_{ij}}$$

$$\left(\vec{r}_{ij}\cdot\vec{dl}_j\right)=\left(R_i\cos\theta_i-R_j\cos\theta_j\right)\left(-dl_j\sin\theta_j\right)+\left(R_i\sin\theta_i-R_j\sin\theta_j\right)\left(dl_j\cos\theta_j\right)$$

$$=R_idl_j\sin\theta_i\cos\theta_j-R_idl_j\cos\theta_i\sin\theta_j$$

$$\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)=\frac{R_iR_jd\theta_j\sin\left(\theta_i-\theta_j\right)}{r_{ij}}$$

$$\left(\hat{r}_{ij}\cdot\vec{v}_j\right)=\frac{R_iv_j\sin\left(\theta_i-\theta_j\right)}{r_{ij}}$$

$$-3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)\left(\hat{r}_{ij}\cdot\vec{v}_{jd}\right)}{r_{ij}^2}=-3I_jv_j\frac{R_i^3R_j^2\sin^3\left(\theta_i-\theta_j\right)d\theta_id\theta_j}{r_{ij}^5}$$

From $\vec{r}_{ij}=\left(R_i\cos\theta_i-R_j\cos\theta_j,R_i\sin\theta_i-R_j\sin\theta_j,-L\right)$,

$$\left|\vec{r}_{ij}\right|^2=R_i^2\cos^2\theta_i+R_j^2\cos^2\theta_j-2R_iR_j\cos\theta_i\cos\theta_j+R_i^2\sin^2\theta_i+R_j^2\sin^2\theta_j-2R_iR_j\sin\theta_i\sin\theta_j+L^2$$

$$\left|\vec{r}_{ij}\right|^2=R_i^2+R_j^2-2R_iR_j\cos\theta_i\cos\theta_j-2R_iR_j\sin\theta_i\sin\theta_j+L^2$$

$$\left|\vec{r}_{ij}\right|^2=R_i^2+R_j^2-2R_iR_j\cos\left(\theta_i-\theta_j\right)+L^2$$

$$-3I_j v_j \frac{R_i^3 R_j^2 \sin^3(\theta_i - \theta_j)\, d\theta_i d\theta_j}{r_{ij}^5} = -3I_j v_j \frac{R_i^3 R_j^2 \sin^3(\theta_i - \theta_j)\, d\theta_i d\theta_j}{\left(R_i^2 + R_j^2 - 2R_iR_j\cos(\theta_i - \theta_j) + L^2\right)^{\frac{5}{2}}}$$

(dimensions $Cs^{-1}ms^{-1}$)

$$\oint_0^{2\pi} \left[\oint_0^{2\pi} \frac{R_i^3 R_j^2 \sin^3(\theta_i - \theta_j)\, d\theta_i}{\left(R_i^2 + R_j^2 - 2R_iR_j\cos(\theta_i - \theta_j) + L^2\right)^{\frac{5}{2}}} \right] d\theta_j$$

Let $\phi = (\theta_i - \theta_j)$ and $d\phi = d\theta_i$
The limits are then from $-\theta_j$ to $2\pi - \theta_j$

$$\oint_0^{2\pi} \left[\oint_{-\theta_j}^{2\pi-\theta_j} \frac{R_i^3 R_j^2 \sin^3(\phi)\, d\phi}{\left(R_i^2 + R_j^2 - 2R_iR_j\cos(\phi) + L^2\right)^{\frac{5}{2}}} \right] d\theta_j$$

$\sin^3(\phi)$ is an odd function so the integral is $= 0$ over any range $[x, x + 2\pi]$. The denominator is an even function.

Integral over *any* 2π range where c is a constant

$$\int_{-c}^{2\pi - c} \sin v dv$$

Let $u = v + c$ and $du = dv$. Limits are now $[0, 2\pi]$

$$\int_0^{2\pi} \sin(u - c)\, du = \cos c \int_0^{2\pi} \sin u du - \sin c \int_0^{2\pi} \cos u du = 0$$

and for the equivalence of range $[0, 2\pi]$ and $[-\pi, \pi]$

$$\int_0^{2\pi} \sin x dx$$

Let $y = x - \pi$

$$\int_0^{2\pi} \sin x dx = \int_{-\pi}^{\pi} \sin(y - \pi)\, dy = -\int_{-\pi}^{\pi} \sin(y)\, dy = 0$$

3.2 Parallel Circuits - Mutual Inductance

$$M = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \frac{\vec{dl}_i \cdot \vec{dl}_j}{r_{ij}}$$

From
$\vec{dl}_i = (-dl_i \sin\theta_i, dl_i \cos\theta_i, 0)$
$\vec{dl}_j = (-dl_j \sin\theta_j, dl_j \cos\theta_j, 0)$

$$M = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \frac{R_i R_j \cos(\theta_i - \theta_j)\, d\theta_i d\theta_j}{r_{ij}}$$

Let $\phi = (\theta_i - \theta_j)$ and $d\phi = d\theta_i$
The limits are then from $-\theta_j$ to $2\pi - \theta_j$

$$M = \frac{\mu_0}{4\pi} \oint_{C_j} \left[\oint_{C_i} \frac{R_i R_j \cos(\phi)\, d\phi}{r_{ij}} \right] d\theta_j$$

From $|\vec{r}_{ij}|^2 = R_i^2 + R_j^2 - 2R_iR_j \cos(\theta_i - \theta_j) + L^2 = R_i^2 + R_j^2 - 2R_iR_j \cos(\phi) + L^2$

$$M = \frac{\mu_0}{4\pi} \oint_{C_j} \left[\oint_{C_i} \frac{R_i R_j \cos(\phi)\, d\phi}{\left(R_i^2 + R_j^2 - 2R_iR_j \cos(\phi) + L^2\right)^{-\frac{1}{2}}} \right] d\theta_j$$

$\left(R_i^2 + R_j^2 - 2R_iR_j \cos(\phi) + L^2\right)^{-\frac{1}{2}} = \left(L^2 \left[\frac{R_i^2+R_j^2-2R_iR_j\cos(\phi)}{L^2} + 1\right]\right)^{-\frac{1}{2}} \approx \frac{1}{L}\left(1 - \frac{R_i^2+R_j^2-2R_iR_j\cos(\phi)}{2L^2}\right)$ assuming $L >> R$.

We need to only keep the cos term $\frac{R_iR_j\cos(\phi)}{L^3}$ because the integral of $\cos(\phi)$ in the numerator of M above is 0 for any range $[x, x+2\pi]$

$$\oint_{C_i} \frac{R_i R_j \cos(\phi)\, d\phi}{\left(R_i^2+R_j^2-2R_iR_j\cos(\phi)+L^2\right)^{\frac{1}{2}}} \approx \oint R_iR_j \cos(\phi) \left(\frac{R_iR_j\cos(\phi)}{L^3}\right) d\phi = \oint_{C_i} \frac{R_i^2R_j^2}{L^3} \cos^2\phi d\phi = \frac{\pi R_i^2 R_j^2}{L^3}$$

$$M = \frac{\mu_0}{4\pi} \oint_{C_j} \left(\frac{\pi R_i^2 R_j^2}{L^3}\right) d\theta_j = \frac{\mu_0}{4\pi} \left(\frac{\pi R_i^2 R_j^2}{L^3}\right) 2\pi = \frac{\mu_0 \pi R_i^2 R_j^2}{2L^3}$$

Note that mutual inductance is a $+ve$ number in this case

3.3 Parallel Circuits - EMF

From

$$emf = -\frac{d}{dt}(I_j M)$$

$$M = \frac{\mu_0 \pi R_i^2 R_j^2}{2L^3}$$

Let the 2 circuits move away from each other with velocity V

$$\frac{dM}{dt} = \frac{dM}{dL}\frac{dL}{dt} = -3\frac{\mu_0 \pi R_i^2 R_j^2}{2L^4} V$$

$$emf = -\frac{d}{dt}(I_j M) = 3 I_j \frac{\mu_0 \pi a^4}{2L^4} V$$

This is a $+ve$ number. Given that $emf = \vec{E} \cdot \vec{dl}_i$ then $\vec{E}$ (and therefore the induced current I_i) must be in the same direction as $\vec{dl}_i$. And from our diagram in the same direction as $\vec{dl}_j$ (and v_j) and therefore I_j. Since I_i and I_j are in the same direction the the loops will attract.

3.4 Parallel Circuits - Mutual Inductance and EMF II

Same as above but suppose this time I choose to traverse C_i in the opposite direction i.e. $\vec{dl}_i$ is in the opposite direction
$\vec{dl}_i = (dl_i \sin\theta_i, -dl_i \cos\theta_i, 0)$
$\vec{dl}_j = (-dl_j \sin\theta_j, dl_j \cos\theta_j, 0)$
$\vec{r}_{ij} = (R_i \cos\theta_i - R_j \cos\theta_j, R_i \sin\theta_i - R_j \sin\theta_j, -L)$

$$M = \frac{\mu_0}{4\pi} \oint_{C_i} \oint_{C_j} \frac{-dl_i dl_j \left(\sin\theta_i \sin\theta_j + \cos\theta_i \cos\theta_j\right)}{\left(R_i^2 + R_j^2 - 2R_i R_j \cos(\theta_i - \theta_j) + L^2\right)^{\frac{1}{2}}}$$

so in this case it's the same except for the $-ve$ sign

$$M = -\frac{\mu_0 \pi R_i^2 R_j^2}{2L^3}$$

$$\frac{dM}{dt} = 3\frac{\mu_0 \pi R_i^2 R_j^2}{2L^4} V$$

$$emf = -\frac{d}{dt}\left(I_j M\right) = -3I_j \frac{\mu_0 \pi R_i^2 R_j^2}{2L^4} V$$

so in this case the induced current I_i is in the opposite direction to $\vec{dl}_i$ (but it's physically the same direction as the previous example - we have just chosen to define $\vec{dl}$ differently)

3.5 Orthogonal Circuits - Mutual Inductance

2 circular circuits/loops separated by a distance L (from Assis **Fig 5.5 C**). In the above diagram circuit 1 is C_i and circuit 2 is C_j.

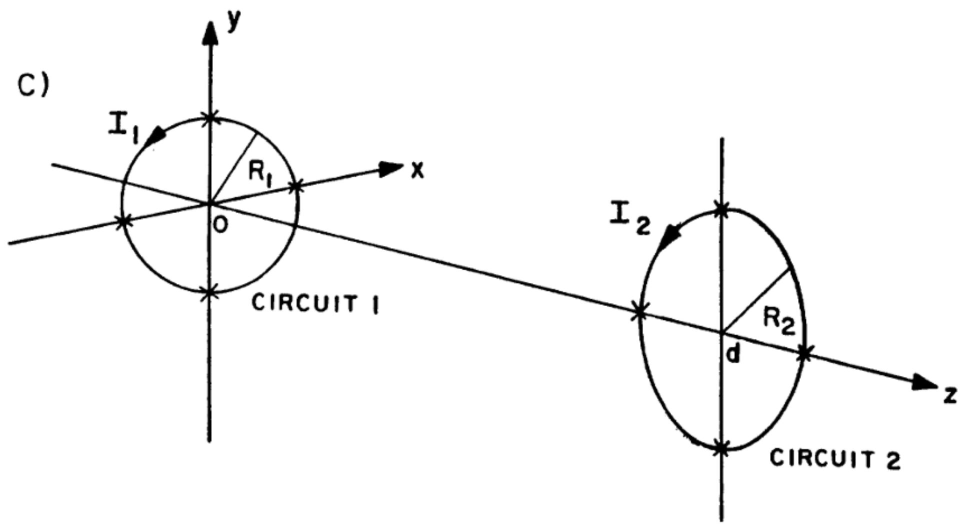


Figure 5.5

C_i is in $[x, y]$ plane, C_j is in $[y, z]$ plane

Choosing θ_i as the angle from the x axis and R_i as the radius of C_i. Same as last example.

$\vec{dl}_i = (-dl_i \sin\theta_i, dl_i \cos\theta_i, 0)$

Choosing θ_j as the angle from the z axis and R_j as the radius of C_j

$\vec{dl}_j = (0, dl_j \cos\theta_j, -dl_j \sin\theta_j)$

For $\vec{r}$ we have

$\vec{r}_i = \hat{i} R_i \cos\theta_i + \hat{j} R_i \sin\theta_i + \hat{k} 0$

$\vec{r}_j = \hat{i} 0 + \hat{j} R_j \sin\theta_j + \hat{k}\left(L + R_j \cos\theta_j\right)$

$\vec{r}_{ij} = \left(R_i \cos\theta_i, R_i \sin\theta_i - R_j \sin\theta_j, -\left(L + R_j \cos\theta_j\right)\right)$

$$\frac{\vec{dl}_i \cdot \vec{dl}_j}{|\vec{r}_{ij}|} = R_i R_j \frac{\cos\theta_i \cos\theta_j}{\left(R_i^2 + R_j^2 + 2R_j L \cos\theta_j + L^2 - 2R_i R_j \sin\theta_i \sin\theta_j\right)^{\frac{1}{2}}} d\theta_i d\theta_j$$

$$= \frac{R_i R_j}{\sqrt{\left(R_i^2 + R_j^2 + 2R_j L \cos\theta_j + L^2\right)}} \frac{\cos\theta_i \cos\theta_j}{\left(\left[1 - \frac{2R_i R_j \sin\theta_i \sin\theta_j}{R_i^2 + R_j^2 + 2R_j L \cos\theta_j + L^2}\right]^{\frac{1}{2}}\right)} d\theta_i d\theta_j$$

let $K = \frac{2R_i R_j \sin\theta_j}{R_i^2 + R_j^2 + 2R_j L \cos\theta_j + L^2}$.

$$= \frac{R_i R_j}{\sqrt{\left(\left(R_i^2 + R_j^2 + 2R_j L \cos\theta_j + L^2\right)\right)}} \frac{\cos\theta_i \cos\theta_j}{\left([1 - K \sin\theta_i]^{\frac{1}{2}}\right)} d\theta_i d\theta_j$$

$K < 1$ if $L > 0$ or $R_i \neq R_j$. So $(1 - K\sin\theta_i)^{\frac{1}{2}}$ can be expanded in a binomial series $\sum_n f(K)^n \sin^n \theta_i$

$$= \int_0^{2\pi} \frac{R_i R_j \cos\theta_j}{\sqrt{\left(\left(R_i^2 + R_j^2 + 2R_j L\cos\theta_j + L^2\right)\right)}} \left[\int_0^{2\pi} \sum_n f(K)^n \cos\theta_i \sin^n\theta_i d\theta_i\right] d\theta_j$$

The integral of $\cos\theta_i \sin^n\theta_i$ is $\frac{\sin^{n+1}(\theta_i)}{n+1}$. The integral over $[0, 2\pi]$ is 0.

3.6 Orthogonal Circuits - Cancellation

As in 3.5
$\vec{dl}_i = (-dl_i \sin\theta_i, dl_i \cos\theta_i, 0)$
$\vec{dl}_j = (0, dl_j \cos\theta_j, -dl_j \sin\theta_j)$
and for $\vec{r}$ we have
$\vec{r}_i = \hat{i} R_i \cos\theta_i + \hat{j} R_i \sin\theta_i + \hat{k} 0$
$\vec{r}_j = \hat{i} 0 + \hat{j} R_j \sin\theta_j + \hat{k}\left(L + R_j \cos\theta_j\right)$
$\vec{r}_{ij} = \left(R_i \cos\theta_i, R_i \sin\theta_i - R_j \sin\theta_j, -\left(L + R_j \cos\theta_j\right)\right)$

$$\left(\vec{r}_{ij} \cdot \vec{dl}_i\right) = \left(R_i \cos\theta_i\right)\left(-dl_i \sin\theta_i\right) + \left(R_i \sin\theta_i - R_j \sin\theta_j\right)\left(dl_i \cos\theta_i\right)$$

$$= \left(R_i dl_i \sin\theta_i \cos\theta_i - R_j dl_i \sin\theta_j \cos\theta_i - R_i dl_i \cos\theta_i \sin\theta_i\right) = -R_j dl_i \sin\theta_j \cos\theta_i$$

$$\left(\vec{r}_{ij} \cdot \vec{dl}_i\right) = -R_i R_j \sin\theta_j \cos\theta_i d\theta_i$$

$$\left(\vec{r}_{ij} \cdot \vec{dl}_j\right) = \left(R_i \sin\theta_i - R_j \sin\theta_j\right)\left(dl_j \cos\theta_j\right) + \left(L + R_j \cos\theta_j\right)\left(dl_j \sin\theta_j\right)$$

$$= \left(R_i dl_j \cos\theta_j \sin\theta_i - R_j dl_j \cos\theta_j \sin\theta_j\right) + \left(L dl_j \sin\theta_j + R_j dl_j \sin\theta_j \cos\theta_j\right)$$

$$= R_i dl_j \cos\theta_j \sin\theta_i + L dl_j \sin\theta_j$$

$$\left(\vec{r}_{ij} \cdot \vec{dl}_j\right) = R_i R_j \cos\theta_j \sin\theta_i d\theta_j + L R_j \sin\theta_j d\theta_j$$

$$\left(\vec{r}_{ij} \cdot \vec{v}_j\right) = \left(R_i \sin\theta_i - R_j \sin\theta_j\right)\left(v_j \cos\theta_j\right) + \left(L + R_j \cos\theta_j\right)\left(v_j \sin\theta_j\right)$$

$$= \left(R_i v_j \cos\theta_j \sin\theta_i - R_j v_j \cos\theta_j \sin\theta_j\right) + \left(L v_j \sin\theta_j + R_j v_j \sin\theta_j \cos\theta_j\right)$$

$$(\vec{r}_{ij}\cdot\vec{v}_j) = v_j R_i \cos\theta_j \sin\theta_i + L v_j \sin\theta_j$$

$$-3I_j \frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)(\hat{r}_{ij}\cdot\vec{v}_{jd})}{r_{ij}^2} =$$

$$\frac{(-R_iR_j\sin\theta_j\cos\theta_i d\theta_i)(R_iR_j\cos\theta_j\sin\theta_i d\theta_j + LR_j\sin\theta_j d\theta_j)(v_jR_i\cos\theta_j\sin\theta_i + v_jL\sin\theta_j)}{\left(R_i^2 + R_j^2 + 2R_jL\cos\theta_j + L^2 - 2R_iR_j\sin\theta_i\sin\theta_j\right)^{\frac{5}{2}}}$$

As per 3.5, the numerator only has the single $\cos\theta_i$ and $\sin^n\theta_i$ terms. Likewise the denominator can be expanded in a binomial series $\sum_n f(K)^n \sin^n\theta_i$. Altogether we have $\left[\int_0^{2\pi}\sum_n f(K)^n \cos\theta_i \sin^n\theta_i d\theta_i\right]$ which is 0.

3.7 Circuit y-Axis Rotation

If we start off with parallel circuits in the $[x, y]$ plane
$\vec{r_i} = \hat{i}R_i\cos\theta_i + \hat{j}R_i\sin\theta_i$
$\vec{r_j} = \hat{i}R_j\cos\theta_j + \hat{j}R_j\sin\theta_j + L$
$\vec{dl} = -\hat{i}dl\sin\theta + \hat{j}dl\cos\theta.$
$\vec{dl}_i = (-dl_i\sin\theta_i, dl_i\cos\theta_i, 0)$
$\vec{dl}_j = (-dl_j\sin\theta_j, dl_j\cos\theta_j, 0)$
$\vec{r}_{ij} = (R_i\cos\theta_i - R_j\cos\theta_j, R_i\sin\theta_i - R_j\sin\theta_j, -L)$

If we rotate C_j around the y axis by an angle ϕ_j and rotate C_i around the y axis by an angle ϕ_i

$$\vec{r_i} = \hat{i}R_i\cos\theta_i\cos\phi_i + \hat{j}R_i\sin\theta_i + \hat{k}R_i\cos\theta_i\sin\phi_i$$

$$\vec{r_j} = \hat{i}R_j\cos\theta_j\cos\phi_j + \hat{j}R_j\sin\theta_j + \hat{k}R_j\cos\theta_j\sin\phi_j + L$$

$$\vec{r}_{ij} = (R_i\cos\theta_i\cos\phi_i - R_j\cos\theta_j\cos\phi_j, R_i\sin\theta_i - R_j\sin\theta_j, R_i\cos\theta_i\sin\phi_i - R_j\cos\theta_j\sin\phi_j - L)$$

$$\begin{aligned}|\vec{r}_{ij}|^2 = {} & \left(R_i^2\cos^2\theta_i\cos^2\phi_i + R_j^2\cos^2\theta_j\cos^2\phi_j - 2R_iR_j\cos\theta_i\cos\theta_j\cos\phi_i\cos\phi_j\right)\\ & +R_i^2\sin^2\theta_i + R_j^2\sin^2\theta_j - 2R_iR_j\sin\theta_i\sin\theta_j \qquad (16)\\ & +R_i^2\cos^2\theta_i\sin^2\phi_i + R_j^2\cos^2\theta_j\sin^2\phi_j - 2R_iR_j\cos\theta_j\sin\phi_j\cos\theta_i\sin\phi_i + L^2\\ & -2LR_i\cos\theta_i\sin\phi_i + 2LR_j\cos\theta_j\sin\phi_j\end{aligned}$$

If we don't rotate C_i then $\phi_i = 0$ and it is in the $[x, y]$ plane

$$|\vec{r}_{ij}|^2 = \left(R_i^2 + R_j^2\cos^2\theta_j\cos^2\phi_j - 2R_iR_j\cos\theta_i\cos\theta_j\cos\phi_j\right)$$
$$+R_j^2\sin^2\theta_j - 2R_iR_j\sin\theta_i\sin\theta_j$$
$$+R_j^2\cos^2\theta_j\sin^2\phi_j + L^2$$
$$+2LR_j\cos\theta_j\sin\phi_j$$

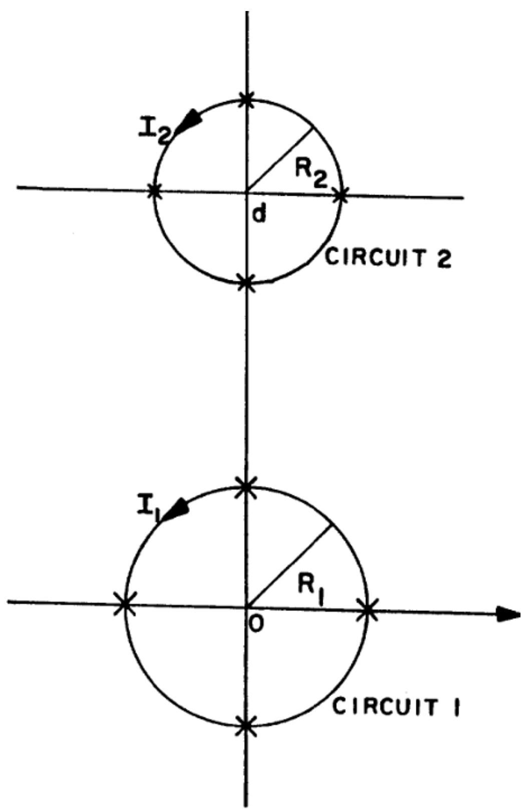
Sanity Check I: When $\phi_j = 0$ we have the parallel circuits example

$$|\vec{r}_{ij}|^2 = R_i^2 + R_j^2 - 2R_iR_j\left(\cos\theta_i\cos\theta_j + \sin\theta_i\sin\theta_j\right) + L^2$$

Sanity Check II: When $\phi_j = \frac{\pi}{2}$ we have the orthogonal circuits example

$$|\vec{r}_{ij}|^2 = R_i^2 + R_j^2 + 2LR_j\cos\theta_j + L^2 - 2R_iR_j\sin\theta_i\sin\theta_j$$

3.8 Same Plane Circuits



If $\phi_i = \phi_j = \frac{\pi}{2}$ we have 2 circuits in the same $[y, z]$ plane (from Assis **Fig 5.5 B**)

From above

$d\vec{l}_i = (0, R_i\cos\theta_i d\theta_i, -R_i\sin\theta_i d\theta_i)$

$\vec{dl_j} = (0, R_j \cos\theta_j d\theta_j, -R_j \sin\theta_j d\theta_j)$
$\vec{r_i} = \hat{i}0 + \hat{j}R_i \sin\theta_i + \hat{k}R_i \cos\theta_i$
$\vec{r_j} = \hat{i}0 + \hat{j}R_j \sin\theta_j + \hat{k}R_j \cos\theta_j + L$
$\vec{r}_{ij} = (0, R_i \sin\theta_i - R_j \sin\theta_j, R_i \cos\theta_i - R_j \cos\theta_j - L)$

$$\begin{aligned}|\vec{r}_{ij}|^2 = &R_i^2 \sin^2\theta_i + R_j^2 \sin^2\theta_j - 2R_iR_j \sin\theta_i \sin\theta_j \\ &+R_i^2 \cos^2\theta_i + R_j^2 \cos^2\theta_j - 2R_iR_j \cos\theta_j \cos\theta_i + L^2 \\ &-2LR_i \cos\theta_i + 2LR_j \cos\theta_j\end{aligned}$$

$$\begin{aligned}|\vec{r}_{ij}|^2 = &R_i^2 + R_j^2 - 2R_iR_j \sin\theta_i \sin\theta_j - 2R_iR_j \cos\theta_j \cos\theta_i + L^2 \\ &-2LR_i \cos\theta_i + 2LR_j \cos\theta_j\end{aligned}$$

Sanity Check I:
If $\theta_i = \theta_j = 0$ and $\phi_i = \phi_j = \frac{\pi}{2}$

$$|\vec{r}_{ij}|^2 = R_i^2 + R_j^2 - 2R_iR_j + L^2 - 2LR_i + 2LR_j$$

$$|\vec{r}_{ij}|^2 = [(R_j + L) - R_i]^2$$

$$|\vec{r}_{ij}| = R_j + L - R_i$$

Sanity Check II:
If $\theta_i = \theta_j = \frac{\pi}{2}$ and $\phi_i = \phi_j = \frac{\pi}{2}$

$$|\vec{r}_{ij}|^2 = R_i^2 + R_j^2 - 2R_iR_j + L^2$$

$$|\vec{r}_{ij}|^2 = (R_j - R_i)^2 + L^2$$

3.9 Same Plane Circuits - Mutual Inductance

From above
$\vec{dl_i} = (0, R_i \cos\theta_i d\theta_i, -R_i \sin\theta_i d\theta_i)$
$\vec{dl_j} = (0, R_j \cos\theta_j d\theta_j, -R_j \sin\theta_j d\theta_j)$

$$\frac{\left(\vec{dl_i} \cdot \vec{dl_j}\right)}{|\vec{r}_{ij}|} = \frac{R_iR_j \cos\theta_i \cos\theta_j d\theta_i d\theta_j + R_iR_j \sin\theta_i \sin\theta_j d\theta_i d\theta_j}{\left(R_i^2 + R_j^2 - 2R_iR_j \sin\theta_i \sin\theta_j - 2R_iR_j \cos\theta_j \cos\theta_i + L^2 - 2LR_i \cos\theta_i + 2LR_j \cos\theta_j\right)^{\frac{1}{2}}}$$

$$= \left[\frac{R_i R_j \cos\theta_i \cos\theta_j + R_i R_j \sin\theta_i \sin\theta_j}{\left(R_i^2 + R_j^2 + L^2\right)^{\frac{1}{2}} \left(1 + \frac{-2R_i R_j \sin\theta_i \sin\theta_j - 2R_i R_j \cos\theta_i \cos\theta_j - 2LR_i \cos\theta_i + 2LR_j \cos\theta_j}{R_i^2 + R_j^2 + L^2}\right)^{\frac{1}{2}}} \right] d\theta_i d\theta_j$$

assume $L >> R$

$$= \left[\frac{R_i R_j \cos\theta_i \cos\theta_j + R_i R_j \sin\theta_i \sin\theta_j}{L \left(1 + \frac{-2R_i R_j \sin\theta_i \sin\theta_j - 2R_i R_j \cos\theta_i \cos\theta_j - 2LR_i \cos\theta_i + 2LR_j \cos\theta_j}{L^2}\right)^{\frac{1}{2}}} \right] d\theta_i d\theta_j$$

From $(1+x)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}x + \frac{3}{8}x^2$
and $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd$
and $x = \left(\frac{-2R_i R_j \sin\theta_i \sin\theta_j - 2R_i R_j \cos\theta_i \cos\theta_j - 2LR_i \cos\theta_i + 2LR_j \cos\theta_j}{L^2}\right)$

Taking into account only those terms in the expansion that aren't going to zero when integrated with the numerator terms.

$$(...)^{-\frac{1}{2}} \approx 1 + \left[\frac{R_i R_j \sin\theta_i \sin\theta_j + R_i R_j \cos\theta_i \cos\theta_j}{L^2}\right] + \left[\frac{3}{8} \frac{(-8)\, L^2 R_i R_j \cos\theta_i \cos\theta_j}{L^4}\right]$$

$$1 + \left[\frac{R_i R_j \sin\theta_i \sin\theta_j + R_i R_j \cos\theta_i \cos\theta_j}{L^2}\right] - \left[\frac{3R_i R_j \cos\theta_i \cos\theta_j}{L^2}\right]$$

$$1 + \left[\frac{R_i R_j \sin\theta_i \sin\theta_j - 2R_i R_j \cos\theta_i \cos\theta_j}{L^2}\right]$$

applying to the numerator and keeping just the terms that don't integrate to 0

$$M = \frac{\mu_0}{4\pi} \int \int \left[\frac{R_i^2 R_j^2 \sin^2\theta_i \sin^2\theta_j - 2R_i^2 R_j^2 \cos^2\theta_i \cos^2\theta_j}{L^3}\right] d\theta_i d\theta_j$$

$$= \frac{\mu_0}{4\pi} \left[\frac{R_i^2 R_j^2 \pi^2 - 2R_i^2 R_j^2 \pi^2}{L^3}\right] = -\frac{\mu_0 \pi R_i^2 R_j^2}{4L^3}$$

$$M = -\frac{\mu_0 \pi R_i^2 R_j^2}{4L^3}$$

3.10 General Circuits Calculation

I think instead of the above I could have just used even/odd function logic for $f(\theta_i,\theta_j)$ over the $[0,2\pi]$ integral ranges.

If we reconsider ϕ_i and ϕ_j we have the following for $|\vec{r}_{ij}|^2$ from Eq 16 and we have $f(\theta_i,\theta_j)=f(-\theta_i,-\theta_j)$ so it is an even function

$$
\begin{aligned}
|\vec{r}_{ij}|^2 = &\left(R_i^2\cos^2\theta_i\cos^2\phi_i + R_j^2\cos^2\theta_j\cos^2\phi_j - 2R_iR_j\cos\theta_i\cos\theta_j\cos\phi_i\cos\phi_j\right)\\
&+R_i^2\sin^2\theta_i + R_j^2\sin^2\theta_j - 2R_iR_j\sin\theta_i\sin\theta_j\\
&+R_i^2\cos^2\theta_i\sin^2\phi_i + R_j^2\cos^2\theta_j\sin^2\phi_j - 2R_iR_j\cos\theta_j\sin\phi_j\cos\theta_i\sin\phi_i + L^2\\
&-2LR_i\cos\theta_i\sin\phi_i + 2LR_j\cos\theta_j\sin\phi_j
\end{aligned}
$$

More generally taking ϕ into account we have for $\vec{dl}$ with $\phi=0$ meaning that a circuit is in the $[x,y]$ plane

$\vec{dl_i} = (-dl_i\sin\theta_i\cos\phi_i, dl_i\cos\theta_i, -dl_i\sin\theta_i\sin\phi_i)$
$\vec{dl_j} = (-dl_j\sin\theta_j\cos\phi_j, dl_j\cos\theta_j, -dl_j\sin\theta_j\sin\phi_j)$
$\vec{r_i} = \hat{\imath}R_i\cos\theta_i\cos\phi_i + \hat{\jmath}R_i\sin\theta_i + \hat{k}R_i\cos\theta_i\sin\phi_i$
$\vec{r_j} = \hat{\imath}R_j\cos\theta_j\cos\phi_j + \hat{\jmath}R_j\sin\theta_j + \hat{k}R_j\cos\theta_j\sin\phi_j + L$
$\vec{r}_{ij} = (R_i\cos\theta_i\cos\phi_i - R_j\cos\theta_j\cos\phi_j, R_i\sin\theta_i - R_j\sin\theta_j, R_i\cos\theta_i\sin\phi_i - (R_j\cos\theta_j\sin\phi_j + L))$
and the following 3 terms are all odd $f(\theta_i,\theta_j) = -f(-\theta_i,-\theta_j)$

$$
\begin{aligned}
\left(\vec{r}_{ij}\cdot\vec{dl}_i\right) = &-(R_i\cos\theta_i\cos\phi_i - R_j\cos\theta_j\cos\phi_j)(R_id\theta_i\sin\theta_i\cos\phi_i) + (R_i\sin\theta_i - R_j\sin\theta_j)(R_id\theta_i\cos\theta_i)\\
&-(R_i\cos\theta_i\sin\phi_i - (R_j\cos\theta_j\sin\phi_j + L))(R_id\theta_i\sin\theta_i\sin\phi_i)
\end{aligned}
$$

$$
\begin{aligned}
\left(\vec{r_{ij}}\cdot\vec{dl}_j\right) = &-(R_i\cos\theta_i\cos\phi_i - R_j\cos\theta_j\cos\phi_j)(R_jd\theta_j\sin\theta_j\cos\phi_j) + (R_i\sin\theta_i - R_j\sin\theta_j)(R_jd\theta_j\cos\theta_j)\\
&-(R_i\cos\theta_i\sin\phi_i - (R_j\cos\theta_j\sin\phi_j + L))(R_jd\theta_j\sin\theta_j\sin\phi_j)
\end{aligned}
$$

$$
\begin{aligned}
(\vec{r_{ij}}\cdot\vec{v}_j) = &-v_j(R_i\cos\theta_i\cos\phi_i - R_j\cos\theta_j\cos\phi_j)(\sin\theta_j\cos\phi_j) + v_j(R_i\sin\theta_i - R_j\sin\theta_j)(\cos\theta_j)\\
&-v_j(R_i\cos\theta_i\sin\phi_i - (R_j\cos\theta_j\sin\phi_j + L))(\sin\theta_j\sin\phi_j)
\end{aligned}
$$

which would give for any ϕ_i and ϕ_j

$$\int_{C_i}\int_{C_j}\left[-3I_j\frac{\left(\hat{r}_{ij}\cdot\vec{dl}_j\right)\left(\hat{r}_{ij}\cdot\vec{dl}_i\right)(\hat{r}_{ij}\cdot\vec{v}_{jd})}{r_{ij}^2}\right]=0$$

Sanity Check I: $\phi_i=\phi_j=0$ we get as before for parallel circuits 3.1

$$\left(\vec{r}_{ij}\cdot\vec{dl}_i\right) = -(R_i\cos\theta_i - R_j\cos\theta_j)(R_id\theta_i\sin\theta_i) + (R_i\sin\theta_i - R_j\sin\theta_j)(R_id\theta_i\cos\theta_i)$$

$$\left(\vec{r_{ij}} \cdot d\vec{l}_j\right) = -\left(R_i \cos\theta_i - R_j \cos\theta_j\right)\left(R_j d\theta_j \sin\theta_j\right) + \left(R_i \sin\theta_i - R_j \sin\theta_j\right)\left(R_j d\theta_j \cos\theta_j\right)$$

$$(\vec{r_{ij}} \cdot \vec{v}_j) = -v_j \left(R_i \cos\theta_i - R_j \cos\theta_j\right)\left(\sin\theta_j\right) + v_j \left(R_i \sin\theta_i - R_j \sin\theta_j\right)\left(\cos\theta_j\right)$$

Sanity Check II: $\phi_i = 0$ and $\phi_j = \frac{\pi}{2}$ we get as before for orthogonal circuits 3.6 e.g. $\left(\vec{r}_{ij} \cdot d\vec{l}_i\right)$

$$\left(\vec{r}_{ij} \cdot d\vec{l}_i\right) = -\left(R_i \cos\theta_i\right)\left(R_i d\theta_i \sin\theta_i\right) + \left(R_i \sin\theta_i - R_j \sin\theta_j\right)\left(R_i d\theta_i \cos\theta_i\right)$$

$$\left(\vec{r}_{ij} \cdot d\vec{l}_i\right) = -R_i R_j \sin\theta_j \cos\theta_i d\theta_i$$

A Assis Equation 4.66. Derivation

$$\oint_{C_i}\oint_{C_j}\frac{\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)\left(\hat{r}_{ij}\cdot d\vec{l}_j\right)}{r_{ij}}$$

$$=\oint_{C_i}\oint_{C_j}\left[\frac{\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)}{r_{ij}}\hat{r}_{ij}\right]\cdot d\vec{l}_j$$

from Stokes theorem

$$=\oint_{C_i}\iint_{S_j}\left\{\nabla_j\times\left[\frac{\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)}{r_{ij}}\hat{r}_{ij}\right]\right\}\cdot d\vec{a}_j$$

from $\nabla\times\left(\phi\vec{G}\right)=\phi\left(\nabla\times\vec{G}\right)-\vec{G}\times(\nabla\phi)$

$$=\oint_{C_i}\iint_{S_j}\left\{\frac{\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)}{r_{ij}}\left(\nabla_j\times\hat{r}_{ij}\right)-\hat{r}_{ij}\times\nabla_j\left[\frac{\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)}{r_{ij}}\right]\right\}\cdot d\vec{a}_j$$

from $\nabla_j\times\hat{r}_{ij}=\nabla_j\times\frac{\vec{r_{ij}}}{|\vec{r_{ij}}|}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0$

$$=-\oint_{C_i}\iint_{S_j}\left\{\hat{r}_{ij}\times\nabla_j\left[\frac{\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)}{r_{ij}}\right]\right\}\cdot d\vec{a}_j$$

from

$\nabla_j\left(\hat{r}_{ij}\cdot d\vec{l}_i\right)=\nabla_j\left(\frac{\vec{r_{ij}}}{r_{ij}}\cdot d\vec{l}_i\right)=\nabla_j\left(\frac{1}{r_{ij}}\left(\vec{r_{ij}}\cdot d\vec{l}_i\right)\right)=\frac{1}{r_{ij}}\nabla_j\left(\vec{r_{ij}}\cdot d\vec{l}_i\right)+\left(\vec{r_{ij}}\cdot d\vec{l}_i\right)\left(\nabla_j\left(\frac{1}{r_{ij}}\right)\right)$

first term

$\frac{1}{r_{ij}}\nabla_j\left(\vec{r_{ij}}\cdot d\vec{l}_i\right)$

$=\frac{1}{r_{ij}}\nabla_j\left(\left(x_i-x_j\right)\hat{x}+\left(y_i-y_j\right)\hat{y}+\left(z_i-z_j\right)\hat{z}\right)\cdot\left(dl_x\hat{x}+dl_y\hat{y}+dl_z\hat{z}\right)$

$=\frac{1}{r_{ij}}\nabla_j\left(dl_x\left(x_i-x_j\right)+dl_y\left(y_i-y_j\right)+dl_z\left(z_i-z_j\right)\right)$

$=\frac{1}{r_{ij}}\left(\hat{x}\frac{\partial}{\partial x_j}+\hat{y}\frac{\partial}{\partial y_j}+\hat{z}\frac{\partial}{\partial z_j}\right)\left(dl_x\left(x_i-x_j\right)+dl_y\left(y_i-y_j\right)+dl_z\left(z_i-z_j\right)\right)$

$=-\frac{\vec{dl}}{r_{ij}}$

second term

$\left(\vec{r_{ij}}\cdot d\vec{l}_i\right)\left(\nabla_j\left(\frac{1}{r_{ij}}\right)\right)$

$=\left(\vec{r_{ij}}\cdot d\vec{l}_i\right)\left(\left(\hat{x}\frac{\partial}{\partial x_j}+\hat{y}\frac{\partial}{\partial y_j}+\hat{z}\frac{\partial}{\partial z_j}\right)\left[\left(x_i-x_j\right)^2+\left(y_i-y_j\right)^2+\left(z_i-z_j\right)^2\right]^{-\frac{1}{2}}\right)$

$$= \vec{r_{ij}} \cdot \vec{dl}_i \left(-\frac{1}{2}\right) \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{-\frac{3}{2}}$$

$$[\hat{x}2\,(x_i - x_j)\,(-1) + \hat{y}2\,(y_i - y_j)\,(-1) + \hat{z}2\,(z_i - z_j)\,(-1)]$$

$$= \left(\vec{r_{ij}} \cdot \vec{dl}_i\right) \left(\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{-\frac{3}{2}} [\hat{x}\,(x_i - x_j) + \hat{y}\,(y_i - y_j) + \hat{z}\,(z_i - z_j)]\right)$$

$$= \left(\vec{r_{ij}} \cdot \vec{dl}_i\right) \frac{\vec{r_{ij}}}{{r_{ij}}^3}$$

$$= \left(\vec{r_{ij}} \cdot \vec{dl}_i\right) \frac{\hat{r}_{ij}}{{r_{ij}}^2}$$

$$= \left(\hat{r}_{ij} \cdot \vec{dl}_i\right) \frac{\hat{r}_{ij}}{r_{ij}}$$

$$\nabla_j \left(\hat{r}_{ij} \cdot \vec{dl}_i\right) = -\frac{\vec{dl}_i}{r_{ij}} + \left(\hat{r}_{ij} \cdot \vec{dl}_i\right) \frac{\hat{r}_{ij}}{r_{ij}}$$

So going back to

$$= -\oint_{C_i} \iint_{S_j} \left\{\hat{r}_{ij} \times \nabla_j \left[\frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right)}{r_{ij}}\right]\right\} \cdot \vec{da}_j$$

$$= -\oint_{C_i} \iint_{S_j} \left\{\left[\hat{r}_{ij} \times \frac{1}{r_{ij}} \nabla_j \left(\hat{r}_{ij} \cdot \vec{dl}_i\right)\right] + \left[\hat{r}_{ij} \times \left(\hat{r}_{ij} \cdot \vec{dl}_i\right) \nabla_j \left(\frac{1}{r_{ij}}\right)\right]\right\} \cdot \vec{da}_j$$

$$= -\oint_{C_i} \iint_{S_j} \left\{\left[\hat{r}_{ij} \times \frac{1}{r_{ij}} \left(-\frac{\vec{dl}_i}{r_{ij}} + \left(\hat{r}_{ij} \cdot \vec{dl}_i\right) \frac{\hat{r}_{ij}}{r_{ij}}\right)\right] + \left[\hat{r}_{ij} \times \left(\hat{r}_{ij} \cdot \vec{dl}_i\right) \left(\frac{\hat{r}_{ij}}{r_{ij}^2}\right)\right]\right\} \cdot \vec{da}_j$$

terms with $\hat{r}_{ij} \times \hat{r}_{ij} = 0$ so we are left with

$$= \oint_{C_i} \iint_{S_j} \left\{\frac{\hat{r}_{ij}}{r_{ij}^2} \times \vec{dl}_i\right\} \cdot \vec{da}_j$$

from $\nabla \times \left(\phi \vec{G}\right) = \phi \left(\nabla \times \vec{G}\right) - \vec{G} \times (\nabla \phi)$

$\nabla_j \times \left(\frac{\vec{dl}_i}{r_{ij}}\right) = \frac{1}{r_{ij}} \left(\nabla_j \times \vec{dl}_i\right) - \vec{dl}_i \times \left(\nabla_j \frac{1}{r_{ij}}\right) = -\vec{dl}_i \times \frac{\hat{r}_{ij}}{r_{ij}^2} = \frac{\hat{r}_{ij}}{r_{ij}^2} \times \vec{dl}_i$

$$\oint_{C_i} \iint_{S_j} \left\{\nabla_j \times \left(\frac{\vec{dl}_i}{r_{ij}}\right)\right\} \cdot \vec{da}_j$$

$$= \oint_{C_i} \oint_{C_j} \left(\frac{\vec{dl}_i}{r_{ij}}\right) \cdot \vec{dl}_j$$

$$\boxed{\oint_{C_i} \oint_{C_j} \frac{\left(\hat{r}_{ij} \cdot \vec{dl}_i\right) \left(\hat{r}_{ij} \cdot \vec{dl}_j\right)}{r_{ij}} = \oint_{C_i} \oint_{C_j} \left(\frac{\vec{dl}_i \cdot \vec{dl}_j}{r_{ij}}\right)}$$

B Useful Equations

$$d^2U = \frac{dq_n dq_m}{4\pi\epsilon_0}\frac{1}{r_{nm}}\left(1 - \frac{v_{nm}^2}{2c^2}\right) \tag{17}$$

$$M_{ij} = \frac{\mu_0}{4\pi}\oint_{C_i}\oint_{C_j}\left(\frac{d\vec{l}_i \cdot d\vec{l}_j}{r_{ij}}\right) \tag{18}$$

There are a number of formulations of Weber's force. Here is the vectorial one. See **Assis Eq 3.5** for another.

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{ij}\cdot\vec{v}_{ij} - \frac{3}{2}\left(\hat{r}_{ij}\cdot\vec{v}_{ij}\right)^2 + \vec{r}_{ij}\cdot\vec{a}_{ij}\right)\right] \tag{19}$$

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\epsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 - \frac{\dot{r}^2}{2c^2} + \frac{r\ddot{r}}{c^2}\right] \tag{20}$$

$$d^2\vec{F}_{ji}^A = -\frac{\mu_0}{4\pi}\frac{I_i I_j dl_i dl_j \hat{r}_{ij}}{r_{ij}^2}\left[2\cos\epsilon + 3\cos\theta\cos\theta^{'}\right] \tag{21}$$

$$\vec{A}_2\left(\vec{r_2}\right) \equiv \frac{\mu_0}{4\pi}\oint_{C_1} I_1\frac{d\vec{l}_1}{r_{12}} \tag{22}$$

$$\vec{B}\left(\vec{r_2}\right) = \nabla_2 \times \vec{A}_2\left(\vec{r_2}\right) = \frac{\mu_0}{4\pi}\oint_{C_1}\frac{\hat{r}_{12}}{r_{12}^2}\times I_1 d\vec{l}_1$$

C Useful Vector Formulae

$$\nabla \times \left(\phi\vec{G}\right) = \phi\left(\nabla \times \vec{G}\right) - \vec{G} \times (\nabla\phi)$$

$$\nabla_i\left(\frac{1}{r_{ij}}\right) = -\nabla_j\left(\frac{1}{r_{ij}}\right) = -\frac{\hat{r}_{ij}}{r_{ij}^2}$$

$$\nabla \cdot \left(f\vec{G}\right) = f\left(\nabla \cdot \vec{G}\right) + (\nabla f)\cdot\vec{G}$$

Laplacian of function $\nabla^2 = \nabla \cdot (\nabla\phi)$

$$\nabla \times \left(\nabla \times \vec{G}\right) = \nabla\left(\nabla \cdot \vec{G}\right) - \nabla^2\vec{G}$$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta\left(\vec{r}\right)$$

$$\phi\left(\vec{r_0}\right) = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho\left(\vec{r_j}\right)}{r_{0j}}dV$$

$$\oint_{C_1}\frac{\hat{r}_{12}}{r_{12}^2}\cdot d\vec{l}_1 = 0$$

D Exact Differential

The integral of an exact differential around a closed loop is equal to 0.

$$\frac{\mu_0}{4\pi} I_i I_j dl_i dl_j \left(2\frac{z_j}{r_j^3}\frac{dz_j}{dl_j} - 3\frac{z_j^2}{r_j^4}\frac{dr_j}{dl_j} \right)$$

$$= \frac{\mu_0}{4\pi} I_i I_j dl_i dl_j \frac{d}{dl_j}\left(\frac{z^2}{r^3}\right) = \frac{\mu_0}{4\pi} I_i I_j dl_i d\left(\frac{z^2}{r^3}\right)$$

$$d\phi = 2\frac{z_j}{r_j^3} dz_j - 3\frac{z_j^2}{r_j^4} dr_j$$

$$d\phi = M(r,z)\, dz_j + N(r,z)\, dr_j$$

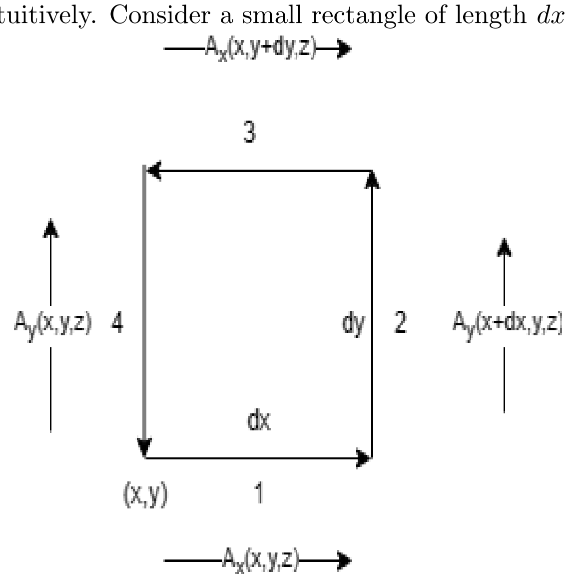
If $d\phi$ is an exact differential then $\frac{\partial M}{\partial r} = \frac{\partial N}{\partial z}$

$$\frac{\partial M}{\partial r} = \frac{\partial}{\partial r}\left(2\frac{z_j}{r_j^3}\right) = -6\frac{z_j}{r_j^4}$$

$$\frac{\partial N}{\partial z} = \frac{\partial}{\partial z}\left(-3\frac{z_j^2}{r_j^4}\right) = -6\frac{z_j}{r_j^4}$$

E Stokes Theorem

Intuitively. Consider a small rectangle of length dx and height dy



$$\sum \vec{A}\cdot\vec{dl} = A_x dx + \left(A_y + \frac{\partial A_y}{\partial x}dx\right)dy - \left(A_x + \frac{\partial A_x}{\partial y}dy\right)dx - A_y dy$$

$$\sum \vec{A}\cdot\vec{dl}=\frac{\partial A_y}{\partial x}dxdy-\frac{\partial A_x}{\partial y}dxdy$$

From the definition of curl $\nabla\times\vec{A}$

$\hat{i}$	$\hat{j}$	$\hat{k}$
$\frac{\partial}{\partial x}$	$\frac{\partial}{\partial y}$	$\frac{\partial}{\partial z}$
A_x	A_y	A_z

$\hat{k}$ component is

$$\hat{k}\left[\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right]$$

from

$$\vec{da}=\vec{k}\,(dxdy)$$

$$\left(\nabla\times\vec{A}\right)\cdot\vec{da}=\left[\frac{\partial A_y}{\partial x}dxdy-\frac{\partial A_x}{\partial y}dxdy\right]$$

leading to

$$\boxed{\oint_C\vec{A}\cdot\vec{dl}=\iint_S\left(\nabla\times\vec{A}\right)\cdot\vec{da}\qquad(23)}$$

F Gauss's Theorem / Divergence Theorem

$$\boxed{\oiint_S\vec{G}\cdot\vec{da}=\iiint_V\left(\nabla\cdot\vec{G}\right)dV\qquad(24)}$$

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