Primer - Potential outside a Oblate Spheroid

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Notes on a satellite orbiting a non-rotating uniform spheroid.

Contents

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1 Gravitational Potential V in terms of r and ϕ

Outside the spheroid, the gravitational potential satisfies Laplace's equation $\nabla^2 V = 0$. The solution to this is found by separation of variables - looking for a solution, in spherical coordinates, of the form $V = R(r)\Phi(\phi)\Lambda(\lambda)$.

Note the use of geodetic notation for coordinates $[r, \phi, \lambda]$ as opposed to the usual Physics convention of $[r, \theta, \phi]$. The solution can be written as below [Kaula]

$$V = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{r^{l+1}} P_{lm} (\sin\phi) \left[C_{lm} \cos m\lambda + S_{lm} \sin m\lambda\right] \tag{1}$$

where P_{lm} are Legendre associated functions and C_{lm} and S_{lm} are constants. Note the $\sin\phi$ term rather than the usual $\cos\theta$ we normally see in Physics books when discussing Legendre polynomials in spherical coordinates. This is because ϕ is the latitude rather than the usual co-latitude θ we use in spherical coordinates.

$$\sin\phi = \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

Terms involving $m = 0$ i.e. P_{l0} are called zonal terms. Terms involving $m \geq 1$ are called tesseral terms [GeoPotential Model]. I'm interested in the uniform spheroid case, only considering terms $l \leq 2$, where the potential has no λ dependency i.e. $m = 0$. For $m = 0$, the Legendre associated functions $P_{lm}(\sin\phi)$ become the Legendre Polynomials $P_l(\sin\phi)$. The above reduces to (where Legendre Polynomial $P_0(\sin\phi) = 1$)

$$V = C_{00}\frac{1}{r} + C_{20}\frac{1}{r^3} P_2(\sin\phi) \tag{2}$$

Note there is no $l = 1$ term as expected, i.e. P_1, due to symmetry of the spheroid around the equator.

At this point we just have a general solution of $\nabla^2 V = 0$.

1.1 Laplacian Check

Laplacian in polar coordinates [Kaula]

$$r^2\nabla^2 = \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2}$$

There is no λ dependency so we can ignore the last term, V is

$$V = C_{00}\frac{1}{r} + C_{20}\frac{1}{r^3} P_2(\sin\phi)$$

$$= C_{00}\frac{1}{r} + C_{20}\frac{1}{r^3}\left(\frac{1}{2}\left[3\sin^2\phi - 1\right]\right)$$

Applying the Laplacian to this gives

$$r^2\nabla^2 V = \frac{\partial}{\partial r}\left[r^2 C_{00}\left[-\frac{1}{r^2}\right] + r^2 C_{20}\left(\frac{1}{2}\left[3\sin^2\phi - 1\right]\right)\left(-\frac{3}{r^4}\right)\right] + \left[\left(\frac{3}{2}C_{20}\frac{1}{r^3}\right)\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\sin^2\phi\right)\right]$$

$$= \frac{\partial}{\partial r}\left(-C_{00} - C_{20}\left(\frac{1}{2}\left[3\sin^2\phi - 1\right]\right)\left(\frac{3}{r^2}\right)\right) + \left[C_{20}\left(\frac{3}{r^3}\right)\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos^2\phi\sin\phi\right)\right]$$

$$= C_{20}\left(\left[3\sin^2\phi - 1\right]\right)\left(\frac{3}{r^3}\right) + C_{20}\left(\frac{3}{r^3}\right)\frac{1}{\cos\phi}\left(\cos^3\phi - 2\sin^2\phi\cos\phi\right)$$

$$= C_{20}\left(\frac{3}{r^3}\right)\left(\left[3\sin^2\phi - 1\right]\right) + C_{20}\left(\frac{3}{r^3}\right)\left(\cos^2\phi - 2\sin^2\phi\right)$$

$$= C_{20}\left(\frac{3}{r^3}\right)\left(\left[3\sin^2\phi - 1\right]\right) + C_{20}\left(\frac{3}{r^3}\right)\left(\cos^2\phi + \sin^2\phi - 3\sin^2\phi\right)$$

$$= C_{20}\left(\frac{3}{r^3}\right)\left(\left[3\sin^2\phi - 1\right]\right) + C_{20}\left(\frac{3}{r^3}\right)\left(1 - 3\sin^2\phi\right) = 0$$

1.2 Gravitational Potential for a Spheroid

The above is a general solution of $\nabla^2 V = 0$. Let us see what the gravitational potential is for a spheroid - but this time starting from the basic formula for the gravitational potential for a distributed body, which is (from [Potential Uniform Spheroid]. This derivation uses the typical Physics convention for the potential - note the -ve sign)

$$\Phi(\mathbf{r}) = \Phi(r, \theta, \phi) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}'$$

and given that

$$d^3\mathbf{r}' = r'^2 \sin\theta' dr' d\theta' d\phi'$$

and I'm assuming *constant* density ρ and from axial symmetry there is no dependence on ϕ

$$\Phi(r, \theta) = -G\rho \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{r'^2 \sin\theta'}{|\mathbf{r}' - \mathbf{r}|} dr' d\theta' d\phi'$$

Note that$|\mathbf{r}' - \mathbf{r}|$ varies over ϕ' for a given θ'.

$$\Phi(r, \theta) = -G\rho \int_0^\infty \int_0^\pi \left[\int_0^{2\pi} \frac{1}{|\mathbf{r}' - \mathbf{r}|} d\phi'\right] r'^2 \sin\theta' dr' d\theta'$$

1.2.1 Integration over azimuthal angel

Looking at the term $\left[\int_0^{2\pi} \frac{1}{|\mathbf{r}' - \mathbf{r}|} d\phi'\right]$

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = [(\mathbf{r}' - \mathbf{r}) \bullet (\mathbf{r}' - \mathbf{r})]^{-\frac{1}{2}} = \left[r^2 - 2\mathbf{r}' \bullet \mathbf{r} + r'^2\right]^{-\frac{1}{2}}$$

From

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$r' \bullet r = rr' (\sin\theta \cos\phi \sin\theta' \cos\phi' + \sin\theta \sin\phi \sin\theta' \sin\phi' + \cos\theta \cos\theta')$$

but this is a calculation for $\Phi(r, \theta)$ which is independent of ϕ which means $r' \bullet r$ must be independent of ϕ so we can evaluate this at $\phi = 0$

$$r' \bullet r = rr' (\sin\theta \sin\theta' \cos\phi' + \cos\theta \cos\theta') = rr'F$$

where

$$F = \sin\theta \sin\theta' \cos\phi' + \cos\theta \cos\theta'$$

giving

$$\frac{1}{|\mathbf{r}' - \mathbf{r}|} = \left[r^2 - 2rr'F + r'^2\right]^{-\frac{1}{2}} = \frac{1}{r}\left[1 - 2\left(\frac{r'}{r}\right)F + \left(\frac{r'}{r}\right)^2\right]^{-\frac{1}{2}}$$

Binomially expand using $(1+x)^{-\frac{1}{2}} = 1 - \frac{x}{2} + \frac{3}{8}x^2$ and $x = -2\left(\frac{r'}{r}\right)F + \left(\frac{r'}{r}\right)^2$ and keeping terms up to $\left(\frac{r'}{r}\right)^2$

$$\left[1 - 2\left(\frac{r'}{r}\right)F + \left(\frac{r'}{r}\right)^2\right]^{-\frac{1}{2}} \approx 1 + \left(\frac{r'}{r}\right)F - \frac{1}{2}\left(\frac{r'}{r}\right)^2 + \frac{3}{8}4\left(\frac{r'}{r}\right)^2 F^2 = 1 + \left(\frac{r'}{r}\right)F + \frac{1}{2}\left(\frac{r'}{r}\right)^2 \left[3F^2 - 1\right]$$

so

$$\int_0^{2\pi} \frac{1}{|\mathbf{r}' - \mathbf{r}|} d\phi' = \frac{1}{r}\int_0^{2\pi} \left[1 + \left(\frac{r'}{r}\right)F + \frac{1}{2}\left(\frac{r'}{r}\right)^2 \left[3F^2 - 1\right]\right] d\phi'$$

First term is

$$\frac{1}{r}\int_0^{2\pi}[1]\,d\phi' = 2\pi\frac{1}{r}$$

Second term - using $\int_0^{2\pi}\cos\phi' d\phi' = 0$ - is

$$\frac{1}{r}\int_0^{2\pi}\left[\left(\frac{r'}{r}\right)F\right]d\phi' = \frac{1}{r}\left(\frac{r'}{r}\right)\int_0^{2\pi}\left[\left(\sin\theta\sin\theta'\cos\phi'+\cos\theta\cos\theta'\right)\right]d\phi' = \frac{1}{r}\left(\frac{r'}{r}\right)\left(\cos\theta\cos\theta'\right)\phi'\big|_0^{2\pi}$$

$$= 2\pi\frac{1}{r}\left(\frac{r'}{r}\right)\cos\theta\cos\theta'$$

Third term - using $\int_0^{2\pi}\cos^2\phi' d\phi' = \pi$ - is

$$\frac{1}{r}\int_0^{2\pi}\left[\frac{1}{2}\left(\frac{r'}{r}\right)^2\left[3F^2-1\right]\right]d\phi' = \frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\int_0^{2\pi}\left[3F^2-1\right]d\phi'$$

$$= \frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\int_0^{2\pi}\left[3\sin^2\theta\sin^2\theta'\cos^2\phi'+3\cos^2\theta\cos^2\theta'+6\sin\theta\sin\theta'\cos\phi'\cos\theta\cos\theta'-1\right]d\phi'$$

$$= \frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\left[3\pi\sin^2\theta\sin^2\theta'+2\pi 3\cos^2\theta\cos^2\theta'-2\pi\right]$$

$$= 2\pi\frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\left[\frac{3}{2}\sin^2\theta\sin^2\theta'+3\cos^2\theta\cos^2\theta'-1\right]$$

$$= 2\pi\frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\left[\frac{3}{2}\left(1-\cos^2\theta\right)\left(1-\cos^2\theta'\right)+3\cos^2\theta\cos^2\theta'-1\right]$$

$$= 2\pi\frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\left[\frac{3}{2}\left(1+\cos^2\theta\cos^2\theta'-\cos^2\theta-\cos^2\theta'\right)+3\cos^2\theta\cos^2\theta'-1\right]$$

$$= 2\pi\frac{1}{r}\frac{1}{2}\left(\frac{r'}{r}\right)^2\left[\frac{1}{2}+\frac{9}{2}\cos^2\theta\cos^2\theta'-\frac{3}{2}\cos^2\theta-\frac{3}{2}\cos^2\theta'\right]$$

$$2\pi\frac{1}{r}\left(\frac{r'}{r}\right)^2\frac{1}{2}\left(3\cos^2\theta-1\right)\frac{1}{2}\left(3\cos^2\theta'-1\right)$$

Putting the 3 terms together

$$\int_0^{2\pi}\frac{1}{|\mathbf{r}'-\mathbf{r}|}d\phi' = 2\pi\frac{1}{r}\left[1+\left(\frac{r'}{r}\right)\cos\theta\cos\theta'+\left(\frac{r'}{r}\right)^2\frac{1}{2}\left(3\cos^2\theta-1\right)\frac{1}{2}\left(3\cos^2\theta'-1\right)\right]$$

but these are just the Legendre Polynomials

$$\int_0^{2\pi}\frac{1}{|\mathbf{r}'-\mathbf{r}|}d\phi' = 2\pi\frac{1}{r}\left[P_0\left(\cos\theta\right)P_0\left(\cos\theta'\right)+\left(\frac{r'}{r}\right)P_1\left(\cos\theta\right)P_1\left(\cos\theta'\right)+\left(\frac{r'}{r}\right)^2P_2\left(\cos\theta\right)P_2\left(\cos\theta'\right)\right]$$

$$\int_0^{2\pi}\frac{1}{|\mathbf{r}'-\mathbf{r}|}d\phi' = 2\pi\frac{1}{r}\sum_n\left[\left(\frac{r'}{r}\right)^n P_n\left(\cos\theta\right)P_n\left(\cos\theta'\right)\right]$$

1.2.2 Potential term

Going back to

$$\Phi\left(r,\theta\right) = -G\rho\int_0^{\infty}\int_0^{\pi}\left[\int_0^{2\pi}\frac{1}{|\mathbf{r}'-\mathbf{r}|}d\phi'\right]r'^2\sin\theta' dr' d\theta'$$

and using the above

$$\Phi\left(r,\theta\right) = -G\rho\int_0^{\infty}\int_0^{\pi}\left[2\pi\frac{1}{r}\sum_n\left[\left(\frac{r'}{r}\right)^n P_n\left(\cos\theta\right)P_n\left(\cos\theta'\right)\right]\right]r'^2\sin\theta' dr' d\theta'$$

$$\Phi\left(r,\theta\right) = \int_0^{\infty}\int_0^{\pi}\left[-2\pi G\rho\sum_n\frac{1}{r^{n+1}}\left[r'^n P_n\left(\cos\theta\right)P_n\left(\cos\theta'\right)\right]\right]r'^2\sin\theta' dr' d\theta'$$

$$\Phi\left(r,\theta\right) = \sum_n\left\{-\frac{2\pi G\rho}{r^{n+1}}\int_0^{\infty}\int_0^{\pi}r'^{n+2}P_n\left(\cos\theta'\right)\sin\theta' d\theta' dr'\right\}P_n\left(\cos\theta\right)$$

1.2.3 Integrating over radius

From G the surface of the spheroid is described by

$$r'(\theta') \simeq R_\mu \left[1 - \frac{2}{3}\epsilon P_2(\cos\theta')\right]$$

$$[r'(\theta')]^{n+3} \simeq R_\mu^{n+3}\left[1 - \frac{2}{3}\epsilon P_2(\cos\theta')\right]^{n+3} \approx R_\mu^{n+3}\left[P_0(\cos\theta') - \frac{2(n+3)}{3}\epsilon P_2(\cos\theta')\right]$$

$$\Phi(r,\theta) = \sum_n \left\{-\frac{2\pi G\rho}{r^{n+1}}\int_0^\pi \left.\frac{r'^{n+3}}{n+3}\right|_0^{r'(\theta')} P_n\left(\cos\theta'\right)\sin\theta' d\theta'\right\} P_n(\cos\theta)$$

$$\Phi(r,\theta) = \sum_n \left\{-\frac{2\pi G\rho R_\mu^3}{r^{n+1}}\int_0^\pi \frac{R_\mu^n\left[P_0(\cos\theta') - \frac{2(n+3)}{3}\epsilon P_2(\cos\theta')\right]}{n+3} P_n\left(\cos\theta'\right)\sin\theta' d\theta'\right\} P_n(\cos\theta)$$

The orthoganality of Legendre Polynomials is given by

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = \frac{\delta_{nm}}{n+\frac{1}{2}}$$

Let $x = \cos\theta$ so $dx = -\sin\theta d\theta$

$$\int_\pi^0 P_n(\cos\theta) P_m(\cos\theta)(-\sin\theta)\, d\theta = \int_0^\pi P_n(\cos\theta) P_m(\cos\theta)\sin\theta d\theta = \frac{\delta_{nm}}{n+\frac{1}{2}}$$

So from the orthogonality above, the only terms that survive the integration are $n = 0$ and $n = 2$

$$\Phi(r,\theta) = -\frac{4\pi G\rho R_\mu^3}{3r} + \left\{-\frac{2\pi G\rho R_\mu^3}{r^3}\frac{R_\mu^2\left[-\frac{10}{3}\epsilon\right]}{5}\frac{2}{5}\right\} P_2(\cos\theta)$$

$$\Phi(r,\theta) = -\frac{4\pi G\rho R_\mu^3}{3r} + \left\{\frac{4\pi G\rho R_\mu^3}{3r^3} R_\mu^2 \frac{2}{5}\epsilon\right\} P_2(\cos\theta)$$

For a uniform spheroid $M = \frac{4\pi}{3}\rho R_\mu^3$

$$\Phi(r,\theta) = -\frac{GM}{r} + \frac{2}{5}\epsilon\frac{GMR_\mu^2}{r^3} P_2(\cos\theta)$$

The derivation in [Potential Uniform Spheroid] uses the typical Physics convention for the potential as (note the -ve sign)

$$V(\mathbf{r}) = -\frac{GM}{r}$$

and the acceleration is (again note the -ve sign)

$$\mathbf{a} = -\nabla V(\mathbf{r})$$

However [Kaula] uses

$$V(\mathbf{r}) = \frac{GM}{r}$$

and

$$\mathbf{a} = \nabla V(\mathbf{r})$$

and states "V is shown as a positive quantity which is consistent with the sign convention of astronomy and geodesy. In Physics V is conventially taken to be negative"

I'll do the same so

$$V(r,\phi) = \frac{GM}{r} - \frac{2}{5}\epsilon\frac{GMR_\mu^2}{r^3} P_2(\sin\phi) + O(\epsilon^2)$$

and the V_{20} term here is

$$V_{20} = -\frac{2}{5}\epsilon GMR_\mu^2 \frac{1}{r^3} P_2(\sin\phi) \tag{3}$$

1.3 Value of C_{20} constant

Comparing 2 with 3 we can see

$$C_{20} = -\frac{2}{5}\epsilon GMR_\mu^2$$

Typically we make C_{20} nondimensional and introduce a factor GMR_μ^2 i.e. make

$$C_{20} = -\frac{2}{5}\epsilon$$

and rewrite 3 as

$$V_{20} = C_{20}GMR_\mu^2\frac{1}{r^3}P_2(\sin\phi) \tag{4}$$

However for below, unless stated otherwise, C_{20} is the non factored value

1.4 V_{20} Zonal Term

The Legendre associated function $P_{lm}(\sin\phi)$ in 1 is the solution for $\Phi(\phi)$ in $V = R(r)\Phi(\phi)\Lambda(\lambda)$. It can be written as a series expansion [Kaula] where k is the integer part of $\frac{(l-m)}{2}$.

$$P_{lm}(\sin\phi) = \cos^m\phi\sum_{t=0}^{k}T_{lmt}\sin^{l-m-2t}\phi$$

where

$$T_{lmt} = \frac{(-1)^t(2l-2t)!}{2^l t!(l-t)!(l-m-2t)!}$$

For P_{20} this expansion gives

$$P_{20}(\sin\phi) = \sum_{t=0}^{1}\left[T_{20t}\sin^{2-2t}\phi\right] \tag{5}$$

In our case, $l = 2$ and $m = 0$ this reduces to

$$T_{20t} = \frac{(-1)^t(4-2t)!}{4t!(2-t)!(2-2t)!}$$

t ranges from 0 to 1 so we have

$$T_{200} = \frac{3}{2} \tag{6}$$

and

$$T_{201} = -\frac{1}{2} \tag{7}$$

so from 5

$$P_{20} = \frac{3}{2}\sin^2\phi - \frac{1}{2}$$

or (as expected)

$$P_{20} = \frac{1}{2}\left[3\sin^2\phi - 1\right]$$

The $l = 2$ term in 2 can now be re-written as

$$V_{20} = \frac{C_{20}}{r^3}\sum_{t=0}^{1}T_{20t}\sin^{2-2t}\phi \tag{8}$$

Ultimately I am looking at the potential V_{20} at the points along an ellipse that a satellite is travelling along. The ellipse might be at an angle to the x-y plane of the spheroid so using r and ϕ isn't the most convenient. I want to express V_{20} in terms of orbital elements $\{a, e, i, M, \omega, \Omega\}$ i.e. semi-major axis of the ellipse a, eccentricity e, inclination i, mean anomaly M, argument of the pericentre ω and longitude of node Ω. For an explanation of the orbital elements refer to [Orbital Elements].

I want to replace ϕ and r in 8.

Starting with the $\sin\phi$ term. We can use 3-D trigonometry relationships which states that for a triangle on a sphere (see Figure 1)

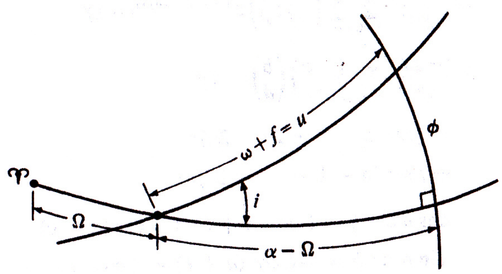


Figure 1: Orbit-equator-meridian triangle (from Fig 4 Kaula)

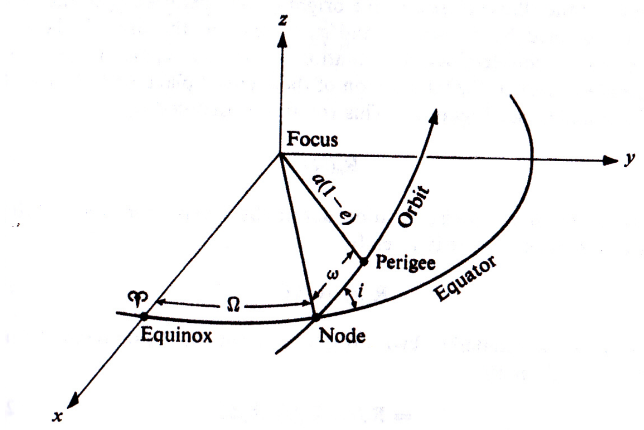


FIGURE 3. ***Orbital orientation.***

Figure 2: Orbital Orientation (from Fig 3 Kaula)

$$\frac{\sin(side)}{\sin(angle)} = const$$

gives

$$\frac{\sin(\omega + f)}{\sin\left(\frac{\pi}{2}\right)} = \frac{\sin\phi}{\sin i}$$

$$\sin\phi = \sin i \sin(\omega + f)$$

where f is the true anomaly.

If the satellite is moving in an ellipse around a central body, then there will be points where it is closest and furthest away. The pericentre is the position in the elliptic path where the satellite is closest to the central body. The apocentre is where the satellite is furthest away from the central body. ω is the argument of the pericentre (see Figure 2). If $\omega = 0$ then the pericentre is at the point where the elliptical path crosses the equatorial plane of the central body.

Eq 8 becomes

$$V_{20} = \frac{C_{20}}{r^3} \sum_{t=0}^{1} T_{20t} \sin^{2-2t} i \sin^{2-2t}(\omega + f) \tag{9}$$

2 Inclination Functions $F_{lmp}(i)$

We are now going to introduce some of the inclination functions that are seen in [Kaula]. These are functions of i i.e. the inclination of the satellite ellipse w.r.t. the spheroid x-y (equatorial) plane. (Imagine the satellite is orbiting in an ellipse in the same plane as the spheroid x-y plane and then tilt the satellite ellipse by an angle i. See Figure 2).

2.1 Prerequisites

Binomial Expansion (positive n)

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{10}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Some special cases

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1$$

$$\binom{n}{n} = \frac{n!}{n!(n-n)!} = 1$$

Sine Power Expansion

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

multiply top and bottom by i and raise to the power of n

$$\sin^n\theta = \left[\frac{-i}{2}\left(e^{i\theta} - e^{-i\theta}\right)\right]^n$$

$$= \frac{(-i)^n}{2^n}\left(e^{i\theta} - e^{-i\theta}\right)^n$$

using 10

$$= \frac{(-i)^n}{2^n}\sum_{k=0}^{n}\binom{n}{k} e^{i\theta(n-k)} e^{-i\theta k}(-1)^k$$

$$= \frac{(-i)^n}{2^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k e^{i\theta(n-2k)}$$

$$\sin^n\theta = \frac{(-i)^n}{2^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\left[\cos(n-2k)\theta + i\sin(n-2k)\theta\right]$$

2.2 V_{20} Inclination Functions

Lets expand the term $\sin^{2-2t}(\omega+f)$ in 9

$$\sin^{2-2t}(\omega+f) = \frac{(-i)^{2-2t}}{2^{2-2t}}\sum_{k=0}^{2-2t}\binom{2-2t}{k}(-1)^k\left[\cos(2-2t-2k)(\omega+f) + i\sin(2-2t-2k)(\omega+f)\right] \tag{11}$$

From 9, t can be either 0 or 1 for V_{20}. For $t=0$, k can have the values 0,1,2. For $t=1$, k can have the value 0. So writing the permutations gives the table below. (The table also shows a variable p which will be used later)

t	k	$2-2t-2k$	$\sin(2-2t-2k)(\omega+f)$	$p=t+k$
0	0	2	$\sin 2(\omega+f)$	0
0	1	0	0	1
0	2	-2	$-\sin 2(\omega+f)$	2
1	0	0	0	1

From the table above, we can see that when $t=0$ then the $k=0$ and $k=2$ cancel out so we can ignore the sin terms. Also

$(-i)^{2-2t}$ for the two cases we have, $t=0$ => -1 and $t=1$ => 1, so lets rewrite $(-i)^{2-2t}$ as $(-1)^{t+1}$. 11 becomes

$$\sin^{2-2t}(\omega+f) = \frac{(-1)^{t+1}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k \left[\cos(2-2t-2k)(\omega+f)\right] \tag{12}$$

Let's introduce a variable $p=t+k$. 12 becomes

$$\sin^{2-2t}(\omega+f) = \frac{(-1)^{t+1}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k \left[\cos(2-2p)(\omega+f)\right] \tag{13}$$

Equation 9 now becomes

$$V_{20} = \frac{C_{20}}{r^3} \sum_{t=0}^{1} T_{20t} \sin^{2-2t} i \frac{(-1)^{t+1}}{2^{2-2t}} \sum_{k=0}^{2-2t} \binom{2-2t}{k} (-1)^k \left[\cos(2-2p)(\omega+f)\right] \tag{14}$$

From the table above, p can have the values 0, 1, 2.

2.3 Inclination function for $p=0$. F_{200}

Just looking at the $p=0$ term, there is only one t and k that gives $p=0$ i.e. $t=0$ and $k=0$. So the $p=0$ term in 14 becomes

$$\frac{C_{20}}{r^3} T_{200} \sin^2 i \frac{-1}{4} \binom{2}{0} (1) \left[\cos(2-2p)(\omega+f)\right]$$

From 6

$$\frac{C_{20}}{r^3} \frac{3}{2} \frac{-1}{4} \sin^2 i \left[\cos(2-2p)(\omega+f)\right]$$

$$\frac{C_{20}}{r^3} \frac{-3}{8} \sin^2 i \left[\cos(2-2p)(\omega+f)\right]$$

So let F_{lmp} be the inclination function. In this case $l=2$, $m=0$ and $p=0$ so

$$F_{200} = \frac{-3}{8} \sin^2 i$$

and the $p=0$ term is

$$\frac{C_{20}}{r^3} F_{200}(i) \left[\cos(2-2p)(\omega+f)\right]$$

2.4 Inclination function for $p=1$. F_{201}

Looking at the $p=1$ term, there are two t and k permutations that give $p=1$ i.e. $t=0$,$k=1$ and $t=1$,$k=0$. So the $p=1$ term in 14 becomes

$$\frac{C_{20}}{r^3} \left[\cos(2-2p)(\omega+f)\right] \left[T_{200} \sin^2 i \frac{-1}{4} \binom{2}{1} (-1) + T_{201}\right]$$

$$\frac{C_{20}}{r^3} \left[\cos(2-2p)(\omega+f)\right] \left[\frac{3}{2}\frac{2}{4} \sin^2 i - \frac{1}{2}\right]$$

$$\frac{C_{20}}{r^3} \left[\cos(2-2p)(\omega+f)\right] \left[\frac{3}{4} \sin^2 i - \frac{1}{2}\right]$$

$$F_{201} = \frac{3}{4}\sin^2 i - \frac{1}{2}$$

and the $p = 1$ term becomes

$$\frac{C_{20}}{r^3} F_{201} \left[\cos\left(2 - 2p\right)\left(\omega + f\right)\right]$$

2.5 General Inclination Functions F_{lmp}

Generally, we can write V_{20} as

$$V_{20} = \frac{C_{20}}{r^3} \sum_{p=0}^{2} F_{20p}(i) \left[\cos\left(2 - 2p\right)\left(\omega + f\right)\right] \tag{15}$$

i.e. we are now writing V_{20} as a sum over p which contains a function of the inclination i. So for an elliptic path, this is easier to work with. However the terms with f and r are still tricky to work with - so we will look to substitute those. We will attempt to use e and M instead i.e. the ellipticity and mean anomaly.

3 Eccentricity Functions G_{lpq}

We are going to introduce a number of functions of e called the eccentricity functions i.e. $G_{lpq}(e)$ where l, p are as before and q is a new index. These functions will be different depending on the values of l, p, q but will still be functions of e alone. The aim is to produce the same $G_{lpq}(e)$ as listed in [Kaula]. Initially I'll also set the argument of the pericentre $\omega = 0$.

When expressing V_{20} in orbital elements [Kaula] gives the below - where he uses a_e the mean equatorial radius of the earth, rather than R_μ. Note that C_{20} here is the nondimensional value discussed in 1.3.

$$V_{20} = \frac{C_{20} G M a_e^2}{a^3} \sum_{p=0}^{2} F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}(e) \cos\left[(2 - 2p + q) M\right] \tag{16}$$

Note in [Kaula], and the above, q ranges from $-\infty$ not 0.

3.1 Prerequisites

e is the eccentricity (unless otherwise stated exp() is used for exponential).

$$e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

$$E - e \sin E = M \tag{17}$$

where E is the eccentric anomaly, M is the mean anomaly.

$$r = a(1 - e\cos E) \tag{18}$$

$$\frac{dM}{dE} = 1 - e\cos E$$

From 18

$$\frac{dM}{dE} = \frac{r}{a}$$

or

$$\boxed{dE = \frac{a}{r} dM} \tag{19}$$

$$\frac{r}{a} = 1 - e\cos E$$

$$\frac{r}{a} = 1 - e\left[\frac{\exp(iE) + \exp(-iE)}{2}\right]$$

Let $e = \frac{2\beta}{(1+\beta^2)}$

$$\frac{r}{a} = 1 - \frac{2\beta}{(1+\beta^2)}\left[\frac{\exp(iE) + \exp(-iE)}{2}\right]$$

$$= 1 - \frac{\beta}{(1+\beta^2)}\left[\exp(iE) + \exp(-iE)\right]$$

$$\boxed{\frac{r}{a} = \frac{1}{(1+\beta^2)}\left[(1 - \beta\exp(iE))(1 - \beta\exp(-iE))\right]} \tag{20}$$

3.2 Eccentricity Functions for $p = 1$ G_{21q}

We have from 15

$$V_{20} = \frac{C_{20}}{r^3} \sum_{p=0}^{2} F_{20p}(i) \left[\cos\left(2 - 2p\right)\left(\omega + f\right)\right]$$

Let's look at the $p = 1$ term initially. We'll use the notation V_{20p} to represent the potential due to the p^{th} term.

$$V_{20p} = \frac{C_{20}}{r^3} F_{20p}(i) \left[\cos\left(2 - 2p\right)\left(\omega + f\right)\right]$$

The cos term $= 1$ for $p = 1$ so

$$V_{201} = \frac{C_{20}}{r^3} F_{201}(i) \tag{21}$$

We would like to remove the reference to r. Lets introduce a the semi-major axis of the ellipse into 21

$$V_{201} = \frac{C_{20}}{a^3} \frac{a^3}{r^3} F_{201}(i)$$

$\frac{a^3}{r^3}$ is a periodic function between 0 and 2π assuming that the satellite is moving in a closed ellipse. Let us expand $\frac{a^3}{r^3}$in an exponential Fourier series w.r.t. the mean anomaly M.

$$\frac{a^3}{r^3} = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

the Fourier coefficients c_k are got by

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} \exp(-ikM) dM \tag{22}$$

so

$$V_{201} = \frac{C_{20}}{a^3} F_{201}(i) \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

and the k term is

$$V_{201k} = \frac{C_{20}}{a^3} F_{201}(i) c_k \exp ikM \tag{23}$$

3.2.1 Eccentricity Function for G_{210} $p = 1$, $q = 0$

For $k = 0$, 22 becomes

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^3}{r^3} dM \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2}{r^2} \frac{a}{r} dM \end{aligned}$$

From 19

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2}{r^2} dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{r}{a}\right)^{-2} dE \end{aligned}$$

From 20

$$\begin{aligned} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(1 + \beta^2\right)^2}{\left[\left(1 - \beta \exp\left(iE\right)\right)\left(1 - \beta \exp\left(-iE\right)\right)\right]^2} dE \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{\left(1 + \beta^2\right)^2}{\left[1 + \beta^2 - \beta(\exp\left(iE\right) + \exp\left(-iE\right))\right]^2} dE \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1+\beta^2\right)^2}{\left[1+\beta^2-2\beta\cos E\right]^2}dE$$

dividing by $\left(1+\beta^2\right)^2$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left[1-\frac{2\beta}{1+\beta^2}\cos E\right]^2}dE$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\left[1-e\cos E\right]^2}dE$$

Using 75

$$c_0 = \frac{1}{2\pi}\frac{2\pi}{\left(1-e^2\right)^{\frac{3}{2}}}$$

$$c_0 = \frac{1}{\left(1-e^2\right)^{\frac{3}{2}}}$$

Compare 16 with 23 implies

$$G_{210} = \left(1-e^2\right)^{-\frac{3}{2}}$$

This is the same as we see in [Kaula].

3.2.2 Eccentricity Function for G_{211} $p = 1$, $q = 1$

Let us look at $k = 1$, so we have

$$c_1 = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^3}{r^3}\exp(-iM)dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2}{r^2}\exp(-iM)\frac{a}{r}dM$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1+\beta^2\right)^2}{\left[\left(1-\beta\exp\left(iE\right)\right)\left(1-\beta\exp\left(-iE\right)\right)\right]^2}\exp -iMdE$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1+\beta^2\right)^2}{\left[\left(1-\beta\exp\left(iE\right)\right)\left(1-\beta\exp\left(-iE\right)\right)\right]^2}\exp\left(-i\left(E-e\sin E\right)\right)dE$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(1+\beta^2\right)^2}{\left[\left(1-\beta\exp\left(iE\right)\right)\left(1-\beta\exp\left(-iE\right)\right)\right]^2}\exp\left(i\left(e\sin E-E\right)\right)dE$$

$$= \frac{\left(1+\beta^2\right)^2}{2\pi}\int_0^{2\pi} \left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\exp\left(i\left(e\sin E-E\right)\right)dE$$

Binomial expansion works for negative powers as well but, instead of ending up with a polynomial with a finite x^n term, we end up with an infinite series.

$$(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$$

The binomial coefficient is given by

$$\binom{k}{n} = \frac{k(k-1)(k-2)...(k-n+1)}{n!}$$

Special case.

$$\binom{k}{0} = 1$$

The cases we will be interested in are

$$\binom{-2}{1} = \frac{(-2)}{1!} = -2$$

$$\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3$$

$$\binom{-2}{3} = \frac{(-2)(-3)(-4)}{3!} = -4$$

binomially expand

$$c_1 = \frac{\left(1+\beta^2\right)^2}{2\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \binom{-2}{b} (-1)^b \beta^b \exp(iEb) \sum_{d=0}^{\infty} \binom{-2}{d} (-1)^d \beta^d \exp(-iEd) \exp(i(e\sin E - E))\, dE$$

rearrange

$$c_1 = \frac{\left(1+\beta^2\right)^2}{2\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} \exp(iEb) \exp(-iEd) \exp(i(e\sin E - E))\, dE$$

combine exp terms

$$c_1 = \frac{\left(1+\beta^2\right)^2}{2\pi} \int_0^{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} \exp(i(e\sin E - (1-b+d)E))\, dE$$

take summations outside integral

$$c_1 = \frac{\left(1+\beta^2\right)^2}{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} \int_0^{2\pi} \exp(i(e\sin E - (1-b+d)E))\, dE \tag{24}$$

The integral is of the form

$$\frac{1}{2\pi} \int_0^{2\pi} \exp(i(e\sin E - nE))dE \tag{25}$$

where n is an integer. The solutions to these integrals have a dependency on n and e are called Bessel Functions $J_n(e)$.

$$c_1 = \left(1+\beta^2\right)^2 \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} J_{1-b+d}(e)$$

Note that in the above for c_1 , it shouldn't matter if we interchange b and d (we'll check this later). so we could also write

$$c_1 = \left(1+\beta^2\right)^2 \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-2}{b} \binom{-2}{d} (-1)^{b+d} \beta^{b+d} J_{1+b-d}(e)$$

The Bessel function can be written as a series solution

$$J_n(e) = \left(\frac{e}{2}\right)^n \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{e}{2}\right)^{2m}}{m!\,(n+m)!} \tag{26}$$

The first few Bessel functions are below. The eccentricity functions are power series in e. I'll only expand as far as e^3

$$J_0(e) = 1 - \frac{e^2}{4} + O(e^4)$$

$$J_1(e) = \frac{e}{2} - \frac{e^3}{16} + O(e^5)$$

$$J_2(e) = \frac{e^2}{8} + O(e^4)$$

$$J_3\left(e\right) = \frac{e^3}{48} + O(e^5)$$

In 24 β is defined as

$$\beta = \frac{\left(1 - \left(1 - e^2\right)^{\frac{1}{2}}\right)}{e}$$

For small e, binomially expand and keep terms up to e^3 as that is as much as I'm going to calculate.

$$\beta \backsimeq \frac{1 - \left(1 - \frac{e^2}{2} - \frac{e^4}{8}\right)}{e} = \frac{e}{2} + \frac{e^3}{8}$$

Similarly

$$\left(1 + \beta^2\right)^2 \backsimeq \left(1 + \frac{e^2}{4}\right)^2 \backsimeq 1 + \frac{e^2}{2}$$

In 24 there is a term β^{b+d} so $b + d \leqq 3$ otherwise we will have O(e^4) terms. Below are the possible combinations.

Row No	b	d	$b+d$	$\binom{-2}{b}$	$\binom{-2}{d}$	$(-1)^{b+d}$	$(\beta)^{b+d}$	$1+b-d$	$J_{1+b-d}\left(e\right)$	$(\beta)^{b+d} * J_{1+b-d}\left(e\right)$ where $O\left(e^3\right)$ and less	$\binom{-2}{b} * \binom{-2}{d} * (-1)^{b+d}$	Multiply by $\binom{-2}{b} * \binom{-2}{d} * (-1)^{b+d}$
a	0	0	0	1	1	1	1	1	$\frac{e}{2} - \frac{e^3}{16}$	$\frac{e}{2} - \frac{e^3}{16}$	1	$\frac{e}{2} - \frac{e^3}{16}$
b	0	1	1	1	-2	-1	$\frac{e}{2} + \frac{e^3}{8}$	0	$1 - \frac{e^2}{4}$	$\frac{e}{2} + \frac{e^3}{8} - \frac{e^3}{8} = \frac{e}{2}$	2	e
c	0	2	2	1	3	1	$\frac{e^2}{4}$	-1	$-\frac{e}{2} + \frac{e^3}{16}$	$-\frac{e^3}{8}$	3	$-\frac{6e^3}{16}$
d	0	3	3	1	-4	-1	$\frac{e^3}{8}$	-2	$\frac{e^2}{8}$	None		
e	1	0	1	-2	1	-1	$\frac{e}{2} + \frac{e^3}{8}$	2	$\frac{e^2}{8}$	$\frac{e^3}{16}$	2	$\frac{2e^3}{16}$
f	1	1	2	-2	-2	1	$\frac{e^2}{4}$	1	$\frac{e}{2} - \frac{e^3}{16}$	$\frac{e^3}{8}$	4	$\frac{8e^3}{16}$
g	1	2	3	-2	3	-1	$\frac{e^3}{8}$	0	$1 - \frac{e^2}{4}$	$\frac{e^3}{8}$	6	$\frac{12e^3}{16}$
h	2	0	2	3	1	1	$\frac{e^2}{4}$	3	$\frac{e^3}{48}$	None		
i	2	1	3	3	-2	-1	$\frac{e^3}{8}$	2	$\frac{e^2}{8}$	None		
j	3	0	3	-4	1	-1	$\frac{e^3}{8}$	4	$O(e^4)$	None		

From inspection we have a $\frac{3e}{2}$ term.
For the e^3 term we have

$$\frac{e^3}{16}\left[-1 - 6 + 2 + 8 + 12\right] = \frac{15e^3}{16}$$

However, because of the $\frac{e^2}{2}$ term in $\left(1 + \frac{e^2}{2}\right)$ there are also $\frac{e^3}{4}$ and $\frac{e^3}{2}$ terms for rows a and b respectively when we multiply through by $\frac{e^2}{2}$ i.e. total of $\frac{12e^3}{16}$.
In total $\frac{27e^3}{16}$. Therefore

$$c_1 = \frac{3e}{2} + \frac{27e^3}{16} + ...$$

This is what [Kaula] lists for G_{211}. Note that in [Kaula] in the case of $p = 1$ his equation reduces to a sum of cosine terms $\cos qM$ and you'd expect the Fourier cosine coefficient a_k to be related to the Fourier complex coefficient c_k by $a_k = c_k + c_{-k}$. Also you'd expect the summation to be over $[0, \infty]$ not $[-\infty, \infty]$. However in [Kaula], the summation of the cosine series is over the range $[-\infty, \infty]$ so he is using c_k and c_{-k}. It'd imply that $G_{211} = G_{21-1}$, which is the case.
The coefficients of a Fourier series are related to the coefficients of the exponential expansion as follows

$$a_0 = c_0$$

$$a_k = c_k + c_{-k}$$

$$b_k = i\left(c_k - c_{-k}\right)$$

3.3 Eccentricity Functions for $p <> 1$ G_{2pq}

For $p <> 1$, we have to deal with the $\cos(2 - 2p)(\omega + f)$ term as well (which disappeared in the $p = 1$ case).

3.3.1 Prerequisites

see Appendix E

$$\boxed{e^{if} = \frac{e^{iE}\left(1-\beta e^{-iE}\right)}{1-\beta e^{iE}}} \tag{27}$$

3.3.2 Expansion of $\cos\left(\left(2-2p\right)\left(\omega+f\right)\right)$

Let's take the case $\omega = 0$ to simplify the maths initially.

$$2\cos\left(\left(2-2p\right)f\right) = \exp\left(i\left(2-2p\right)f\right) + \exp\left(-i\left(2-2p\right)f\right)$$

From 27

$$\exp\left(i\left(2-2p\right)f\right) = \frac{\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}}{\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}}$$

$$2\cos\left(\left(2-2p\right)f\right) = \frac{\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}}{\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}} + \frac{\exp\left(-i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{-(2-2p)}}{\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}}$$

$$\begin{aligned}&= \exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\\&+\exp\left(-i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{-(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}\end{aligned}$$

It follows from the above that the value for $p = 0$ is the same as $p = 2$ i.e. $\cos\left(i2f\right) = \cos\left(-i2f\right)$ as we'd expect. The first and second terms would just interchange.

The potential term is given by

$$V_{20p} = \frac{C_{20}}{a^3}\frac{a^3}{r^3}F_{20p}(i)\left[\cos\left(2-2p\right)f\right]$$

Let's expand the r, f term in a Fourier series - but use the exponential version of the Fourier expansion - note that sum is from $-\infty$ not 0 i.e.

$$function = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$\frac{a^3}{r^3}\left[\cos\left(2-2p\right)f\right] = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$c_k = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}\left[\cos\left(2-2p\right)f\right]\exp -ikMdM$$

(Aside - if, instead of expanding $\frac{a^3}{r^3}\left[\cos\left(2-2p\right)f\right]$, we expand $\frac{a^3}{r^3}\left[\exp\left(2-2p\right)f\right]$, we would get the Hansen coefficients. If we let $m = 2-2p$, we would get the Hansen coefficients $X_k^{-3,m}$.

$$\begin{aligned}c_k = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}&\left[\frac{\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}}{2}\right]\\+&\left[\frac{\exp\left(-i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{-(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}}{2}\right]\exp -ikMdM\end{aligned}$$

(Note the [] term is an even function of E i.e. $f(E) = f(-E)$)

This is the integral of a sum, so let's look at the first term first

$$c_{k_1} = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}\left[\frac{\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}}{2}\right]\exp -ikMdM$$

From previously, we can substitute $\frac{a^3}{r^3}$with $\frac{a^2}{r^2}\frac{a}{r}dM$ or $\frac{a^2}{r^2}dE$ and take the factor $\frac{1}{2}$ out

$$\frac{1}{4\pi}\int_0^{2\pi}\frac{a^2}{r^2}\left[\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\right]\exp -ikMdE$$

substitute $\frac{a^2}{r^2}$

$$=\frac{1}{4\pi}\int_0^{2\pi}\frac{\left(1+\beta^2\right)^2}{\left[\left(1-\beta\exp\left(iE\right)\right)\left(1-\beta\exp\left(-iE\right)\right)\right]^2}$$
$$*\left[\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\right]\exp -ikMdE$$

(Note that the $\frac{a^2}{r^2}$term is also an even function of E.)
Rearrange β terms

$$=\frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}$$
$$*\left[\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\right]\exp -ikMdE$$

substitute M

$$=\frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}$$
$$*\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\exp(-ik\left(E-e\sin E\right))dE$$

take minus sign inside bracket

$$=\frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}$$
$$*\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\exp(ik\left(e\sin E-E\right))dE$$

take k inside bracket

$$=\frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}$$
$$*\exp\left(i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}\exp(i\left[ke\sin E-kE\right]dE$$

rearrange

$$=\frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(iE\right)\right)^{-(2-2p)}$$
$$*\left(1-\beta\exp\left(-iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{(2-2p)}\exp\left(i\left(2-2p\right)E\right)\exp(i\left[ke\sin E-kE\right]dE$$

so the first term is

$$\boxed{c_{k_1}=\frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-(4-2p)}\left(1-\beta\exp\left(-iE\right)\right)^{-2p}\exp\left(i\left(2-2p\right)E\right)\exp(i\left[ke\sin E-kE\right])dE}$$

The 2nd term in the integral is

$$c_{k_2}=\frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}\left[\frac{\exp\left(-i\left(2-2p\right)E\right)\left(1-\beta\exp\left(-iE\right)\right)^{-(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}}{2}\right]\exp(-ikM)dM$$

omitting a few steps for brevity leads to

$$= \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-(2-2p)}\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}$$
$$*\exp\left(-i\left(2-2p\right)E\right)\exp(i\left(ke\sin E-kE\right))dE$$

$$= \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(iE\right)\right)^{(2-2p)}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-(2-2p)}$$
$$*\exp\left(-i\left(2-2p\right)E\right)\exp(i\left(ke\sin E-kE\right))dE$$

$$\boxed{c_{k_2} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2p}\left(1-\beta\exp\left(-iE\right)\right)^{-(4-2p)}\exp\left(-i\left(2-2p\right)E\right)\exp(i\left[ke\sin E-kE\right])dE}$$

Sanity Check 1:
Let $p=1$ and look at the $k=1$ term

$$c_{1_1} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\exp(i\left(e\sin E-E\right))dE$$
$$c_{1_2} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\exp(i\left(e\sin E-E\right))dE$$

so they are both the same. c_1 below is the same as in the previous section.

$$c_1 = c_{1_1}+c_{1_2} = \frac{\left(1+\beta^2\right)^2}{2\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\exp(i\left(e\sin E-E\right))dE$$

Sanity Check 2: For an exponential Fourier expansion of a real function, the coefficient for $k=-1$ should be the complex conjugate of $k=1$.

$$c_{-1_1} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\exp(i\left(-e\sin E+E\right))dE$$
$$c_{-1_2} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\exp(i\left(-e\sin E+E\right))dE$$

which gives c_1 equal to the complex conjugate as expected

$$c_{-1} = c_{-1_1}+c_{-1_2} = \frac{\left(1+\beta^2\right)^2}{2\pi}\int_0^{2\pi}\left(1-\beta\exp\left(-iE\right)\right)^{-2}\left(1-\beta\exp\left(iE\right)\right)^{-2}\exp(-i\left(e\sin E-E\right))dE$$

So continuing with the first term, binomially expand terms

$$c_{k_1} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-(4-2p)}\left(1-\beta\exp\left(-iE\right)\right)^{-2p}$$
$$*\exp\left(i\left(2-2p\right)E\right)\exp(i\left(ke\sin E-kE\right))dE$$

$$c_{k_1} = \frac{\left(1+\beta^2\right)^2}{4\pi}\int_0^{2\pi}\sum_{b=0}^{\infty}\binom{-(4-2p)}{b}\left(-1\right)^b\beta^b\exp\left(iEb\right)\sum_{d=0}^{\infty}\binom{-2p}{d}\left(-1\right)^d\beta^d\exp\left(-iEd\right)$$
$$*\exp(i\left(ke\sin E-\left(k-2+2p\right)E\right))dE$$

$$c_{k_1} = \frac{\left(1+\beta^2\right)^2}{4\pi}\sum_{b=0}^{\infty}\sum_{d=0}^{\infty}\binom{-(4-2p)}{b}\binom{-2p}{d}\left(-1\right)^{b+d}\beta^{b+d}\int_0^{2\pi}\exp(i\left(ke\sin E-\left(k-2+2p-b+d\right)E\right))dE$$

The 2nd term similarly is

$$c_{k_2} = \frac{\left(1+\beta^2\right)^2}{4\pi} \int_0^{2\pi} \left(1-\beta\exp\left(iE\right)\right)^{-2p} \left(1-\beta\exp\left(-iE\right)\right)^{-(4-2p)} \exp\left(-i\left(2-2p\right)E\right)\exp(i\left[ke\sin E - kE\right])dE$$

$$c_{k_2} = \frac{\left(1+\beta^2\right)^2}{4\pi} \int_0^{2\pi} \left(1-\beta\exp\left(iE\right)\right)^{-2p} \left(1-\beta\exp\left(-iE\right)\right)^{-(4-2p)} \exp(i\left[ke\sin E - \left(k+2-2p\right)E\right])dE$$

swapping indices b and d around so we have the same range

$$\begin{aligned} c_{k_2} = & \frac{\left(1+\beta^2\right)^2}{4\pi} \int_0^{2\pi} \sum_{d=0}^{\infty} \binom{-2p}{d} \left(-1\right)^d \beta^d \exp\left(iEd\right) \sum_{b=0}^{\infty} \binom{-\left(4-2p\right)}{b} \left(-1\right)^b \beta^b \exp\left(-iEb\right) \\ & * \exp(i\left[ke\sin E - \left(k+2-2p\right)E\right])dE \end{aligned}$$

$$c_{k_2} = \frac{\left(1+\beta^2\right)^2}{4\pi} \sum_{b=0}^{\infty}\sum_{d=0}^{\infty} \binom{-\left(4-2p\right)}{b} \binom{-2p}{d} \left(-1\right)^{b+d} \beta^{b+d} \int_0^{2\pi} \exp(i\left[ke\sin E - \left(k+2-2p+b-d\right)E\right])dE$$

So we have

$$\begin{aligned} c_{k_1} &= \frac{\left(1+\beta^2\right)^2}{2} \sum_{b=0}^{\infty}\sum_{d=0}^{\infty} \binom{-(4-2p)}{b} \binom{-2p}{d} \left(-1\right)^{b+d} \beta^{b+d} J_{k-2+2p-b+d}\left(ke\right) \\ c_{k_2} &= \frac{\left(1+\beta^2\right)^2}{2} \sum_{b=0}^{\infty}\sum_{d=0}^{\infty} \binom{-\left(4-2p\right)}{b} \binom{-2p}{d} \left(-1\right)^{b+d} \beta^{b+d} J_{k+2-2p+b-d}\left(ke\right) \end{aligned}$$

3.3.3 Relate c_k terms to Hansen Coefficients $X_k^{n,m}$

In the above we expanded as follows

$$\frac{a^3}{r^3}\left[\cos\left(2-2p\right)f\right] = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

and then calculated c_k terms for each p. Let us put $m = 2-2p$

$$\frac{a^3}{r^3}\left[\cos mf\right] = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

This can also be written as

$$\frac{1}{2}\frac{a^3}{r^3}\left[\exp\left(imf\right) + \exp\left(-imf\right)\right] = \sum_{k=-\infty}^{\infty} c_k \exp ikM \tag{28}$$

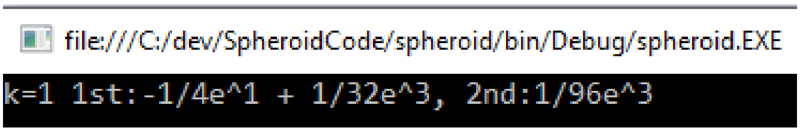
In the previous section we calculate c_k as $c_{k_1} + c_{k_2}$ corresponding to the $\exp\left(imf\right)$ and $\exp\left(-imf\right)$ term respectively. To relate this to Hansen coefficients, we use the following definition of the Hansen coefficient [Hansen Coefficients]

$$\left(\frac{r}{a}\right)^n \exp\left(imf\right) = \sum_k X_k^{n,m}\left(e\right)\exp\left(ikM\right) \tag{29}$$

If $n = -3$ this gives

$$\frac{a^3}{r^3}\exp\left(imf\right) = \sum_k X_k^{-3,m}\left(e\right)\exp\left(ikM\right) \tag{30}$$

Comparing 28 with 30 we can see immediately that $2c_{k_1} = X_k^{-3,m}$ and $2c_{k_2} = X_k^{-3,-m}$
For $p = 0$ or $(m = 2)$ and $k = 1$, we find



$$c_1 = c_{1_1} + c_{1_2} = \left(-\frac{1}{4}e + \frac{1}{32}e^3\right) + \left(\frac{1}{96}e^3\right) = -\frac{1}{4}e + \frac{1}{24}e^3$$

$$X_1^{-3,2} = 2 * c_{1_1} = -\frac{1}{2}e + \frac{1}{16}e^3$$

and

$$X_1^{-3,-2} = 2 * c_{1_2} = \frac{1}{48}e^3$$

3.3.4 Relate Hansen Coefficients $X_k^{n,m}$ to Kaula's eccentricity functions G_{lpq}

Let us consider the $p = 0$ and $k = 1$ term

Using Appendix F we calculate c_1 in the expansion of $(p = 0)$ up to $O\left(e^3\right)$

$$\frac{a^3}{r^3}\left[\cos 2f\right]$$

as

$$c_1 = -\frac{1}{4}e + \frac{1}{24}e^3$$

and

$$c_{-1} = -\frac{1}{4}e + \frac{1}{24}e^3$$

This is as expected for complex Fourier coefficients, because $\frac{a^3}{r^3}\left[\cos 2f\right]$ is real so $c_k = \overline{c}_{-k}$ i.e. the complex conjugate, and because $\frac{a^3}{r^3}\left[\cos 2f\right]$ is also even, c_k is real, so $c_k = c_{-k}$.

So the cosine coefficient $a_1 = c_1 + c_{-1}$ is

$$a_1 = -\frac{1}{2}e + \frac{1}{12}e^3$$

This is what we'd expect for the cosine coefficient in the Fourier expansion. Revisiting Kaula's formula 16

$$V_{20} = \frac{C_{20}GMa_e^2}{a^3}\sum_{p=0}^{2} F_{20p}\left(i\right)\sum_{q=-\infty}^{\infty} G_{2pq}\left(e\right)\cos\left[\left(2-2p+q\right)M\right]$$

we can see that G_{2pq} are cosine coefficients. However his sum is over $[-\infty, \infty]$ whereas we'd expect $[0, \infty]$ for a Fourier cosine expansion. The terms that contribute to the $k = 1$ cosine term are $q = -1$ and $q = -3$ i.e. G_{20-1} and G_{20-3}. So we must have for the cosine $k = 1$ term

$$a_1 = G_{20-1} + G_{20-3} = -\frac{1}{2}e + \frac{1}{12}e^3$$

Aside - in Reference [Hansen Coefficients] the following definition is given

$$G_{lpq} = X_{l-2p+q}^{-l-1,l-2p}$$

Which implies

$$G_{20-1} = X_1^{-3,2} = -\frac{1}{2}e + \frac{1}{16}e^3 \text{ which is what we had above and also what Kaula lists in his book}$$

This also implies (using $X_k^{n,m} = X_{-k}^{n,-m}$)

$$G_{20-3} = X_{-1}^{-3,2} = X_1^{-3,-2} = \frac{1}{48}e^3 \text{ Kaula doesn't list this term in his book}$$

$$G_{20-1} + G_{20-3} = X_1^{-3,2} + X_{-1}^{-3,2} = -\frac{1}{2}e + \frac{4}{48}e^3 = -\frac{1}{2}e + \frac{1}{12}e^3$$

3.4 General G_{2pq} terms

So we have expanded the original in terms of $c_k \exp(ikM)$ and we can consequently also write this as a cosine expansion $(c_k + c_{-k})\, coskM$. However, as we saw from the above, this doesn't necessarily mean that c_k is G_{lpq} and c_{-k} is G_{lp-q}. Going back to the Kaula notation and let $p = 0$

$$V_{20} = \frac{C_{20}}{a^3} \sum_{p=0}^{2} F_{20p}(i) \sum_{q=-\infty}^{\infty} G_{2pq}\,(e) \cos\left((2 - 2p + q)\, M\right)$$

$$V_{200} = \frac{C_{20}}{a^3} F_{200}\,(i) \sum_{q=-\infty}^{\infty} G_{20q}\,(e) \cos\left((2 - 2 * 0 + q)\, M\right)$$

$$V_{200} = \frac{C_{20}}{a^3} F_{200}\,(i) \sum_{q=-\infty}^{\infty} G_{20q}\,(e) \cos\left((2 + q)\, M\right)$$

If I look at the $k = 0$ term, c_0, this would imply $q = -2$. So this would imply that $c_0 = G_{20-2}$. If we calculate c_0 we get a $J_n\,(ke) = J_n\,(0)$ term because $k = 0$ and $n <> 0$

$$c_0 = \frac{\left(1+\beta^2\right)^2}{2\pi} \sum_{b=0}^{\infty} \sum_{d=0}^{\infty} \binom{-4}{b} \binom{0}{d} (-1)^{b+d} \beta^{b+d} \left[J_{-2-b+d}(0) + J_{2+b-d}(0)\right]$$

For a Bessel function, $J_n(0) = 0$ so

$$G_{20-2} = c_0 = 0$$

$G_{20-2} = 0$ agrees with [Kaula]
Similarly if we let $p = 2$

$$V_{202} = \frac{C_{20}}{a^3} F_{202}\,(i) \sum_{q=-\infty}^{\infty} G_{20q}\,(e) \cos\left((2 - 2 * 2 + q)\, M\right)$$

$$V_{202} = \frac{C_{20}}{a^3} F_{202}\,(i) \sum_{q=-\infty}^{\infty} G_{20q}\,(e) \cos\left((-2 + q)\, M\right)$$

If I look at the $k = 0$ term, c_0, this would imply $q = 2$. So this would imply that $c_0 = G_{222}$. If we calculate c_0 we similarly have a $J_n\,(0)$ term so $c_0 = 0$.

$G_{222} = 0$ also agrees with [Kaula]

3.5 Conclusion

Ultimately, we have written the potential term V_{20} at points on the ellipse in terms of the orbital elements $\{a, e, i, M, \omega, \Omega\}$ - albeit simplified because we have omitted ω, Ω. We have also shown the link between Kaula's formula and our derivation. To recap, here is Kaula's formula.

$$V_{20} = \frac{C_{20}}{a^3} \sum_{p=0}^{2} F_{20p}\,(i) \sum_{q=-\infty}^{\infty} G_{2pq}\,(e) \cos\left[(2 - 2p + q)\, M\right]$$

Using the nondimensional version of C_{20} we have

$$V_{20} = \frac{GMR_\mu^2 C_{20}}{a^3} \sum_{p=0}^{2} F_{20p}\,(i) \sum_{q=-\infty}^{\infty} G_{2pq}\,(e) \cos\left[(2 - 2p + q)\, M\right] \tag{31}$$

3.5.1 Sanity Check

Consider $i = 0$ and given G_{lpq} are of $O\left(e^{|q|}\right)$

The e^0 term is G_{210} i.e. $\left(1 - e^2\right)^{-\frac{3}{2}}$. $p = 1$ and $q = 0$. In the cosine term in 31 we see that $\cos\,(2 - 2p + q)\, M = \cos\,(2 - 2 * 1 + 0)\, M = 1$ so the potential related to the G_{210} term has no dependency on the mean anomaly M so is the same for all points on the ellipse.

Given that if $e \neq 0$ we'd expect some variation in the potential V_{20} along the ellipse path. There are 3 G_{2pq} terms that give a dependency on e and M.

1. $G_{20-1} = G_{221} = -\frac{e}{2}$ and gives a $\cos M$ term. However we also need to consider F_{20p} terms as well and $F_{200} = F_{202} = 0$ for $i = 0$ so we can ignore G_{20-1} and G_{221}.
2. $G_{201} = G_{22-1} = \frac{7e}{2}$ and gives a $\cos 3M$ term. However we also need to consider F_{20p} terms and $F_{200} = F_{202} = 0$ for $i = 0$ so we can ignore G_{201} and G_{22-1} as well.
3. $G_{211} = G_{21-1} = \frac{3e}{2}$ and gives a $\cos M$ term. However we also need to consider F_{20p} terms and $F_{201} = -\frac{1}{2}$ for $i = 0$ so we include G_{211} and G_{21-1}.

Therefore in summary we have

$$V_{20} = \frac{GMR_\mu^2 C_{20}}{a^3} \left[\left(-\frac{1}{2}\right) 1 + \left(-\frac{1}{2}\right) \left(\frac{3e}{2}\right) [\cos M + \cos(-M)] \right]$$

$$V_{20} = -\frac{GMR_\mu^2 C_{20}}{2a^3} (1 + 3e \cos M)$$

3.5.2 Sanity Check II

We can also get a similar answer by substituting the solution for the radius of an elliptical orbit $r(\lambda)$

$$\frac{1}{r} = \eta (1 + e \cos \lambda)$$

directly into

$$V_{20} = GMR_\mu^2 C_{20} \frac{1}{r^3} P_2 (\sin \phi) = GMR_\mu^2 C_{20} \frac{1}{r^3} \frac{1}{2} \left(3 \sin^2 \phi - 1\right)$$

where $\phi = 0$ i.e.

$$V_{20} = GMR_\mu^2 C_{20} \left(-\frac{1}{2}\right) \eta^3 (1 + e \cos \lambda)^3 \approx GMR_\mu^2 C_{20} \left(-\frac{1}{2}\right) \left(\frac{1}{a(1-e^2)}\right)^3 (1 + 3e \cos \lambda)$$

$$V_{20} = -\frac{GMR_\mu^2 C_{20}}{2a^3} \left(1 - e^2\right)^{-3} (1 + 3e \cos \lambda)$$

$$V_{20} \approx -\frac{GMR_\mu^2 C_{20}}{2a^3} (1 + 3e \cos \lambda) \tag{32}$$

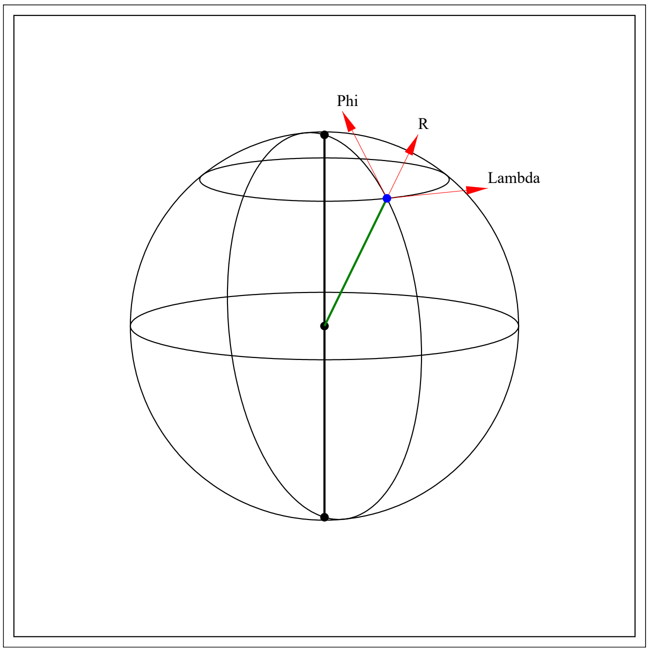


Figure 3: Longitudinal Force Component

4 Forces due to V_{20}

$$V_{20} = C_{20} \frac{1}{r^3} P_2 \left(\sin \phi \right)$$

$$V_{20} = C_{20} \frac{1}{2r^3} \left(3 \sin^2 \phi - 1 \right)$$

$$F_\phi = \frac{1}{r} \frac{\partial V_{20}}{\partial \phi} = C_{20} \frac{1}{r^4} 3 \sin \phi \cos \phi$$

This force is longitudinal. As C_{20} is negative, it is towards the equator, away from the poles (see Figure 3). When describing the motion of a satellite, we can see two frames of reference in Figure 4.

1. $\mathbf{x}$ coordinates - based on the equatorial plane of the spheroid
   There is an inertial reference frame $\mathbf{a}, \mathbf{b}, \mathbf{n}$ with $\mathbf{a}$ and $\mathbf{b}$ in the equatorial plane of the spheroid. Additionally let $\mathbf{a}$ point to the ascending node of the satellite orbit (the blue dot i.e. the point at which the satellite orbit crosses the equatorial plane of the spheroid)

2. $\mathbf{q}$ coordinates - based on the orbiting satellite Keplerian ellipse.
   We can see that there is a reference frame $\mathbf{r}, \mathbf{t}, \mathbf{z}$ with $\mathbf{r}$ and $\mathbf{t}$ in the plane of the satellite orbital Keplerian ellipse.
   See Figure 5
   Note that we need to be a bit more precise when we include the V_{20} potential. Without the V_{20} term then we have a purely inverse square force and the path of the satellite is an ellipse that closes in on itself i.e. it doesn't precess. In this case the meaning of the orbital elements like the semi-major axis a are clear as the ellipse is fixed in space. The satellite follows a path along this ellipse. When we include V_{20} the meaning of a is not so clear. Essentially what we have to do is adjust the ellipse shape at each point in time to reflect how the satellite is actually moving in the inertial $\mathbf{x}$ frame i.e. the satellite is moving on an time dependant ellipse $\xi_t \left(a, e, i, \omega, \Omega \right)$ at time t. This instantaneous ellipse is the osculating orbit - still an ellipse. It describes the orbit the satellite would follow if the perturbing force were not there [Osculating Orbit]. Note the orbital elements, e.g. a, in the osculating orbit may be different to what they would be in the perturbed orbit.

We can describe the motion of the satellite in the inertial frame by position $\{x, y, z\}$ and velocity $\{\dot{x}, \dot{y}, \dot{z}\}$. Alternatively we could describe the motion of the satellite in terms of what are called the Keplerian orbital elements $\{a, e, i, M, \omega, \Omega\}$. For an explanation of the orbital elements refer to [Orbital Elements].

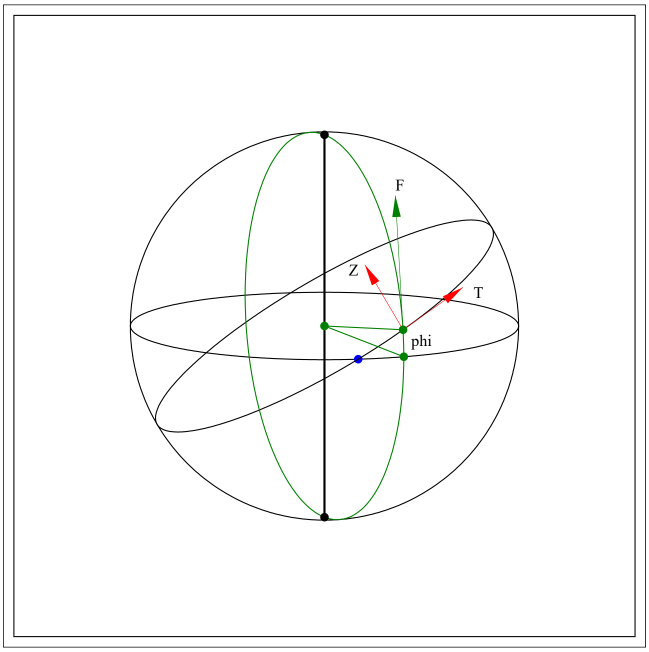


Figure 4: Resolved Longitudinal Force Components along Satellite Ellipse

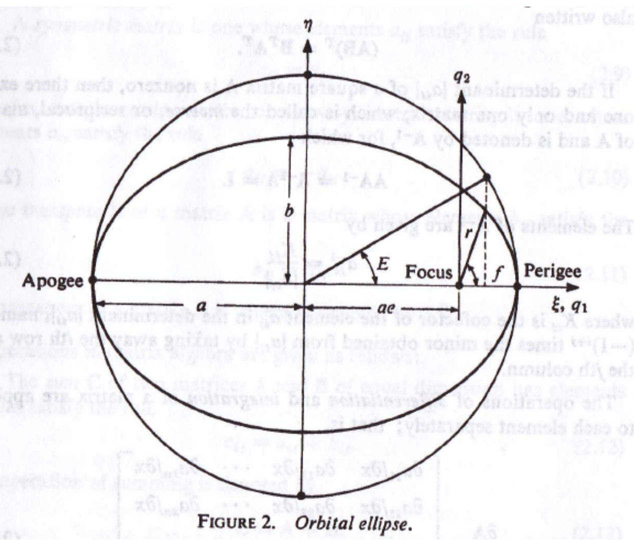


FIGURE 2. *Orbital ellipse.*

Figure 5: Keplerian Orbital Ellipse (from [Kaula])

4.1 Lagrange Bracket

First we will define and calculate a Lagrange bracket and then show why they are useful.

The rotation matrix that relates $\mathbf{x}$ and $\mathbf{q}$ coordinates i.e. $\mathbf{x} = R\mathbf{q}$ is

$$R = \begin{bmatrix} (\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega) & (-\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega) & \sin\Omega\sin i \\ (\sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega) & (-\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega) & -\cos\Omega\sin i \\ \sin i\sin\omega & \sin i\cos\omega & \cos i \end{bmatrix}$$

The $\mathbf{q}$ coordinates are given by

$$q = \begin{bmatrix} a\left(\cos E - e\right) \\ a\sqrt{1-e^2}\sin E \\ 0 \end{bmatrix} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{bmatrix} \frac{na}{1-e\cos E}$$

The Lagrange bracket is defined as below where s_l, s_k can be any of the Keplerian orbital elements. We sum over the i terms i.e. i has values $[1,2,3]$.

$$[s_l, s_k] = \frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial s_k} - \frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial x_i}{\partial s_k}$$

4.1.1 Partial Derivative Meaning e.g. $\frac{\partial x}{\partial e}, \frac{\partial y}{\partial e}, \frac{\partial z}{\partial e}$

The partial derivatives relate changes to coordinates in the inertial frame with changes in the orbital elements e.g. $\frac{\partial x}{\partial e}$ determines how much the x coordinate changes if the eccentricity e changes and we keep all the other orbital elements unchanged.

So, as an example, for the x coordinate, let $\Omega = \omega = i = 0$ so we have the elliptic orbit in the x, y plane of the inertial frame with the pericentre lined up in the direction of the x axis. An increase in the eccentricity ∂e means that the ellipse becomes flatter (a is unchanged) and in effect the focus of the ellipse which was at ae is now at $a\left(e + \partial e\right)$. From the point of view of the origin (i.e. the focus) the pericentre - which was at $a\left(1-e\right)$ is now at $a\left(1 - \left(e + \partial e\right)\right)$ so the change in the x coordinate is $-a\partial e$.

$$\frac{\partial x}{\partial e} = \frac{\partial\left(a\left(\cos E - e\right)\right)}{\partial e} = -a$$

$$\partial x = -a\partial e$$

For the y coordinate, we have

$$\frac{\partial y}{\partial e} = \frac{\partial\left(a\sqrt{1-e^2}\sin E\right)}{\partial e} = a\sin E\frac{1}{2}\left(1-e^2\right)^{-\frac{1}{2}}\left(-2e\right) = -a\sin E\frac{e}{\sqrt{1-e^2}}$$

$$\left.\frac{\partial y}{\partial e}\right|_{E=0} = 0$$

$$\left.\frac{\partial y}{\partial e}\right|_{E=\frac{\pi}{2}} = -a\frac{e}{\sqrt{1-e^2}}$$

so in the second case, y decreases when $E = 90°$ because if e increases then the ellipse becomes flatter.

4.1.2 $[a, \omega]$ Lagrange Bracket

So we can see from $\mathbf{x} = R\mathbf{q}$ that x and it's derivatives can be written in terms of the orbital elements $\{a, e, i, M, \omega, \Omega\}$.

As an example, we'll calculate the $[a, \omega]$ Lagrange Bracket.

$$[a, \omega] = \frac{\partial x_i}{\partial a}\frac{\partial \dot{x}_i}{\partial \omega} - \frac{\partial \dot{x}_i}{\partial a}\frac{\partial x_i}{\partial \omega}$$

$$[\omega, a] = \frac{\partial x_i}{\partial \omega}\frac{\partial \dot{x}_i}{\partial a} - \frac{\partial \dot{x}_i}{\partial \omega}\frac{\partial x_i}{\partial a}$$

so we can see that $[s_l, s_k] = -[s_k, s_l]$. We can also see that $[s_k, s_k] = 0$. In the Lagrange Bracket, we sum over the i terms i.e.

$$[a, \omega] = \left[\frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial \omega} - \frac{\partial \dot{x}}{\partial a}\frac{\partial x}{\partial \omega}\right] + \left[\frac{\partial y}{\partial a}\frac{\partial \dot{y}}{\partial \omega} - \frac{\partial \dot{y}}{\partial a}\frac{\partial y}{\partial \omega}\right] + \left[\frac{\partial z}{\partial a}\frac{\partial \dot{z}}{\partial \omega} - \frac{\partial \dot{z}}{\partial a}\frac{\partial z}{\partial \omega}\right]$$

Looking at the x_1 or x term, the initial position x and velocity $\dot{x}$ is given by the below. However the *change* in position and velocity can be due to *any* of the orbital elements $\{a, e, i, M, \omega, \Omega\}$. changing

$$x = (\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega)\, a\,(\cos E - e) - (\cos\Omega\sin\omega + \sin\Omega\cos i\cos\omega)\, a\sqrt{1-e^2}\sin E$$

$$\dot{x} = \left[(\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega)\,(-\sin E) - (\cos\Omega\sin\omega + \sin\Omega\cos i\cos\omega)\sqrt{1-e^2}\cos E\right]\frac{na}{1 - e\cos E}$$

Lagrange brackets are time invariant [Kaula] so we calculate the Lagrange bracket at the pericentre where $E = 0$

$$q = \begin{bmatrix} a\,(1-e) \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{q} = \begin{bmatrix} 0 \\ \sqrt{1-e^2} \\ 0 \end{bmatrix}\frac{na}{1-e} = \frac{na\sqrt{1-e^2}}{1-e}$$

so just keeping non-zero terms

$$x = (\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega)\, a\,(1-e)$$

$$\dot{x} = (-\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega)\frac{na\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial x}{\partial a} = (\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega)\,(1-e)$$

In the next step be careful with the derivative of na wrt a because $n = \sqrt{\frac{\mu}{a^3}}$ so $na = \left(\frac{\mu}{a}\right)^{\frac{1}{2}}$ and $\frac{\partial(na)}{\partial a} = \frac{\mu^{\frac{1}{2}}\partial(a)^{-\frac{1}{2}}}{\partial a} = -\frac{1}{2}\mu^{\frac{1}{2}}a^{-\frac{3}{2}} = -\frac{1}{2}n$

$$\frac{\partial \dot{x}}{\partial a} = (-\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega)\left(-\frac{n}{2}\right)\frac{\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial x}{\partial \omega} = (-\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega)\, a\,(1-e)$$

$$\frac{\partial \dot{x}}{\partial \omega} = (-\cos\Omega\cos\omega + \sin\Omega\cos i\sin\omega)\frac{na\sqrt{1-e^2}}{1-e}$$

Similarly for y

$$y = (\sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega)\, a\,(1-e)$$

$$\dot{y} = \left(-\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega\right)\frac{na\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial y}{\partial a} = \left(\sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega\right)(1-e)$$

$$\frac{\partial \dot{y}}{\partial a} = \left(-\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega\right)\left(-\frac{n}{2}\right)\frac{\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial y}{\partial \omega} = \left(-\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega\right)a(1-e)$$

$$\frac{\partial \dot{y}}{\partial \omega} = \left(-\sin\Omega\cos\omega - \cos\Omega\cos i\sin\omega\right)\frac{na\sqrt{1-e^2}}{1-e}$$

and for z

$$z = \sin i\sin\omega a(1-e)$$

$$\dot{z} = \sin i\cos\omega\frac{na\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial z}{\partial a} = \sin i\sin\omega(1-e)$$

$$\frac{\partial \dot{z}}{\partial a} = \sin i\cos\omega\left(-\frac{n}{2}\right)\frac{\sqrt{1-e^2}}{1-e}$$

$$\frac{\partial z}{\partial \omega} = \sin i\cos\omega a(1-e)$$

$$\frac{\partial \dot{z}}{\partial \omega} = -\sin i\sin\omega\frac{na\sqrt{1-e^2}}{1-e}$$

Putting this all together

$$[a,\omega] = \left[\frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial \omega} - \frac{\partial \dot{x}}{\partial a}\frac{\partial x}{\partial \omega}\right] + \left[\frac{\partial y}{\partial a}\frac{\partial \dot{y}}{\partial \omega} - \frac{\partial \dot{y}}{\partial a}\frac{\partial y}{\partial \omega}\right] + \left[\frac{\partial z}{\partial a}\frac{\partial \dot{z}}{\partial \omega} - \frac{\partial \dot{z}}{\partial a}\frac{\partial z}{\partial \omega}\right]$$

Look at $na\sqrt{1-e^2}$ terms first $\frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial \omega} + \frac{\partial y}{\partial a}\frac{\partial \dot{y}}{\partial \omega} + \frac{\partial z}{\partial a}\frac{\partial \dot{z}}{\partial \omega}$ i.e. the terms that don't have $\frac{\partial \dot{x}_i}{\partial a}$.

$$\begin{aligned}[a,\omega]_1 = na\sqrt{1-e^2} \\ &\left(\cos\Omega\cos\omega - \sin\Omega\cos i\sin\omega\right)\left(-\cos\Omega\cos\omega + \sin\Omega\cos i\sin\omega\right) \\ &+\left(\sin\Omega\cos\omega + \cos\Omega\cos i\sin\omega\right)\left(-\sin\Omega\cos\omega - \cos\Omega\cos i\sin\omega\right) \\ &-\left(\sin^2 i\sin^2\omega\right)\end{aligned}$$

$$\begin{aligned}&= na\sqrt{1-e^2} \\ &-\left(\cos\Omega\cos\omega\right)^2 - \left(\sin\Omega\cos i\sin\omega\right)^2 + \cos\Omega\cos\omega\sin\Omega\cos i\sin\omega + \sin\Omega\cos i\sin\omega\cos\Omega\cos\omega \\ &-\left(\sin\Omega\cos\omega\right)^2 - \left(\cos\Omega\cos i\sin\omega\right)^2 - \sin\Omega\cos\omega\cos\Omega\cos i\sin\omega - \cos\Omega\cos i\sin\omega\sin\Omega\cos\omega \\ &-\left(\sin^2 i\sin^2\omega\right)\end{aligned}$$

$$= na\sqrt{1-e^2}.\left(-\cos^2\omega - \cos^2 i\sin^2\omega - \sin^2 i\sin^2\omega\right)$$

$$[a,\omega]_1 = -na\sqrt{1-e^2}$$

Now look at $\left(-\frac{na}{2}\right)\sqrt{1-e^2}$ terms $-\frac{\partial \dot{x}}{\partial a}\frac{\partial x}{\partial \omega} - \frac{\partial \dot{y}}{\partial a}\frac{\partial y}{\partial \omega} - \frac{\partial \dot{z}}{\partial a}\frac{\partial z}{\partial \omega}$

$$
\begin{aligned}
[a,\omega]_2 = & \left(-\frac{na}{2}\right)\sqrt{1-e^2} \\
& - \left(-\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega\right)\left(-\cos\Omega\sin\omega - \sin\Omega\cos i\cos\omega\right) \\
& - \left(-\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega\right)\left(-\sin\Omega\sin\omega + \cos\Omega\cos i\cos\omega\right) \\
& - \left(\sin i\cos\omega\sin i\cos\omega\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(-\frac{na}{2}\right)\sqrt{1-e^2} \\
& - \left(\cos\Omega\sin\omega\right)^2 - \left(\sin\Omega\cos i\cos\omega\right)^2 - 2\sin\Omega\cos i\cos\omega\cos\Omega\sin\omega \\
& - \left(\sin\Omega\sin\omega\right)^2 - \left(\cos\Omega\cos i\cos\omega\right)^2 + 2\sin\Omega\sin\omega\cos\Omega\cos i\cos\omega \\
& - \left(\sin^2 i\cos^2\omega\right)
\end{aligned}
$$

$$
\begin{aligned}
= & \left(-\frac{na}{2}\right)\sqrt{1-e^2} \\
& - \sin^2\omega - \cos^2 i\cos^2\omega \\
& - \sin^2 i\cos^2\omega
\end{aligned}
$$

$$
[a,\omega]_2 = \frac{na}{2}\sqrt{1-e^2}
$$

So putting it together

$$
[a,\omega] = [a,\omega]_1 + [a,\omega]_2 = -\frac{na}{2}\sqrt{1-e^2}
$$

4.1.3 All Lagrange Brackets

The full list of non zero Lagrange Brackets are

$$
\begin{aligned}
[\Omega,i] = -[i,\Omega] = & \; -na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i \\
[\Omega,a] = -[a,\Omega] = & \; \left(1-e^2\right)^{\frac{1}{2}}\frac{na}{2}\cos i \\
[\Omega,e] = -[e,\Omega] = & \; -\frac{na^2 e\cos i}{\left(1-e^2\right)^{\frac{1}{2}}} \\
[\omega,a] = -[a,\omega] = & \; \frac{na}{2}\left(1-e^2\right)^{\frac{1}{2}} \\
[\omega,e] = -[e,\omega] = & \; -\frac{na^2 e}{\left(1-e^2\right)^{\frac{1}{2}}} \\
[a,M] = -[M,a] = & \; -\frac{na}{2}
\end{aligned}
$$

Sample Case - Circular motion

$$
x = a\cos M
$$

$$
y = a\sin M
$$

$$
\dot{x} = -a\sin M\frac{dM}{dt} = -na\sin M
$$

$$
\dot{y} = a\cos M\frac{dM}{dt} = na\cos M
$$

$$
[a,M] = \left[\frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial M} - \frac{\partial \dot{x}}{\partial a}\frac{\partial x}{\partial M}\right] + \left[\frac{\partial y}{\partial a}\frac{\partial \dot{y}}{\partial M} - \frac{\partial \dot{y}}{\partial a}\frac{\partial y}{\partial M}\right]
$$

$$= \left[\cos M\left(-na\cos M\right) - \left(\frac{n}{2}\sin M\right)\left(-a\sin M\right)\right] + \left[\sin M\left(-na\sin M\right) - \left(-\frac{n}{2}\cos M\right)\left(a\cos M\right)\right]$$

$$= \left[\left(-na\cos^2 M\right) + \left(\frac{na}{2}\sin^2 M\right)\right] + \left[\left(-na\sin^2 M\right) + \left(\frac{na}{2}\cos^2 M\right)\right]$$

$$= [-na] + \left[\frac{na}{2}\right]$$

$$[a, M] = -\frac{na}{2}$$

Note n isn't constant since $n = \sqrt{\frac{GM}{a^3}}$ so when we are calculating e.g. $\frac{\partial \dot{x}}{\partial a}$ we have to take this into account i.e. if we change a and calculate the change in the length of the arc at $M = 0$ for angle dM

$$ds_{before} = adM = andt$$

$$ds_{after} = (a + da)\sqrt{GM}\frac{1}{\sqrt{(a+da)^3}}dt$$

$$= (a + da)\sqrt{GM}a^{-3}\frac{1}{\sqrt{\left(1 + \frac{da}{a}\right)^3}}dt$$

$$\approx (a + da)\, n\left(1 - \frac{3}{2}\frac{da}{a}\right)dt$$

$$\approx andt + ndadt - n\frac{3}{2}dadt$$

$$\approx andt - \frac{n}{2}dadt$$

$$\dot{x}_{before} = \frac{ds_{before}}{dt} = an$$

$$\dot{x}_{after} = \frac{ds_{after}}{dt} = an - \frac{n}{2}da$$

$$\frac{\partial \dot{x}}{\partial a} = -\frac{n}{2}$$

4.2 Rationale for Lagrange Brackets

In the inertial frame we have

$$\frac{dx_i}{dt} = \dot{x}_i$$

$$\frac{d\dot{x}_i}{dt} = \frac{\partial V}{\partial x_i}$$

We want to express this in terms of Keplerian orbital elements so we can write

$$\sum_{k=1}^{6}\frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt} = \dot{x}_i$$

and

$$\sum_{k=1}^{6}\frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt} = \frac{\partial V}{\partial x_i}$$

On the RHS we still have inertial parameters i.e. $\dot{x_i}$ and $\frac{\partial}{\partial x_i}$. The trick is to multiply the first equation by $-\frac{\partial \dot{x}_i}{\partial s_l}$ and the second equation by $\frac{\partial x_i}{\partial s_l}$ giving the below

$$\sum_{k=1}^{6} -\frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial x_i}{\partial s_k}\frac{ds_k}{dt} = -\frac{\partial \dot{x}_i}{\partial s_l}\dot{x}_i = -\frac{1}{2}\frac{\partial\left(\dot{x}_i\dot{x}_i\right)}{\partial s_l}$$

$$\sum_{k=1}^{6} \frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial s_k}\frac{ds_k}{dt} = \frac{\partial x_i}{\partial s_l}\frac{\partial V}{\partial x_i}$$

In the above, let i range over range $[1, 2, 3]$ i.e. x, y, z and add the two equations together

$$\begin{aligned}\sum_{k=1}^{6}\left[\left[\frac{\partial x}{\partial s_l}\frac{\partial \dot{x}}{\partial s_k} - \frac{\partial \dot{x}}{\partial s_l}\frac{\partial x}{\partial s_k}\right] + \left[\frac{\partial y}{\partial s_l}\frac{\partial \dot{y}}{\partial s_k} - \frac{\partial \dot{y}}{\partial s_l}\frac{\partial y}{\partial s_k}\right] + \left[\frac{\partial z}{\partial s_l}\frac{\partial \dot{z}}{\partial s_k} - \frac{\partial \dot{z}}{\partial s_l}\frac{\partial z}{\partial s_k}\right]\right]\frac{ds_k}{dt}\\ = -\frac{1}{2}\frac{\partial\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)}{\partial s_l} + \frac{\partial x}{\partial s_l}\frac{\partial V}{\partial x} + \frac{\partial y}{\partial s_l}\frac{\partial V}{\partial y} + \frac{\partial z}{\partial s_l}\frac{\partial V}{\partial z}\end{aligned}$$

Let $T = \frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$

$$\sum_{k=1}^{6}\left[\left[\frac{\partial x}{\partial s_l}\frac{\partial \dot{x}}{\partial s_k} - \frac{\partial \dot{x}}{\partial s_l}\frac{\partial x}{\partial s_k}\right] + \left[\frac{\partial y}{\partial s_l}\frac{\partial \dot{y}}{\partial s_k} - \frac{\partial \dot{y}}{\partial s_l}\frac{\partial y}{\partial s_k}\right] + \left[\frac{\partial z}{\partial s_l}\frac{\partial \dot{z}}{\partial s_k} - \frac{\partial \dot{z}}{\partial s_l}\frac{\partial z}{\partial s_k}\right]\right]\frac{ds_k}{dt} = \frac{\partial\left[V-T\right]}{\partial s_l}$$

Use convention that summation happens over the repeated subscript i.

$$\sum_{k=1}^{6}\left[\frac{\partial x_i}{\partial s_l}\frac{\partial \dot{x}_i}{\partial s_k} - \frac{\partial \dot{x}_i}{\partial s_l}\frac{\partial x_i}{\partial s_k}\right]\frac{ds_k}{dt} = \frac{\partial\left[V-T\right]}{\partial s_l}$$

and introducing $[s_l, s_k]$ for the Lagrange Bracket (using the convention the we sum over the repeated k subscript), we finally have

$$[s_l, s_k]\frac{ds_k}{dt} = \frac{\partial\left[V-T\right]}{\partial s_l}$$

If we let $H = V - T$

$$[s_l, s_k]\frac{ds_k}{dt} = \frac{\partial H}{\partial s_l}$$

The key thing in this equation is $\frac{ds_k}{dt}$. We have the possibility now to calculate the change in orbital elements i.e $\frac{d\omega}{dt}$, $\frac{dM}{dt}$, $\frac{d\Omega}{dt}$, etc if we know (1) the values of the Lagrange Brackets and (2) how $(V - T)$ depends on the orbital elements. We have 6 equations and we can solve for $\frac{ds_l}{dt}$ e.g.

$$[a,a]\frac{da}{dt} + [a,e]\frac{de}{dt} + [a,i]\frac{di}{dt} + [a,\omega]\frac{d\omega}{dt} + [a,\Omega]\frac{d\Omega}{dt} + [a,M]\frac{dM}{dt} = \frac{\partial H}{\partial a}$$

$$[i,a]\frac{da}{dt} + [i,e]\frac{de}{dt} + [i,i]\frac{di}{dt} + [i,\omega]\frac{d\omega}{dt} + [i,\Omega]\frac{d\Omega}{dt} + [i,M]\frac{dM}{dt} = \frac{\partial H}{\partial i}$$

etc

4.3 Lagrangian Equations - Solutions of $\frac{ds_l}{dt}$

I shall just quote them here

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial H}{\partial M}$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2 e}\frac{\partial H}{\partial M} - \frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}\frac{\partial H}{\partial \omega}$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\frac{\partial H}{\partial i} + \frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}\frac{\partial H}{\partial e}$$

$$\frac{di}{dt} = \frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\frac{\partial H}{\partial \omega} - \frac{1}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\frac{\partial H}{\partial \Omega}$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\frac{\partial H}{\partial i}$$

$$\frac{dM}{dt} = -\frac{1-e^2}{na^2 e}\frac{\partial H}{\partial e} - \frac{2}{na}\frac{\partial H}{\partial a}$$

4.4 Disturbing Function R

Let us write the potential V as

$$V = \frac{GM}{r} + R$$

i.e. the central term $\frac{GM}{r}$ plus R (the disturbing function) which is the other terms, including V_{20} - which is the only one we are considering, then

$$H = \frac{GM}{r} + R - T$$

However $-\frac{GM}{r} + T$ is the total energy of a unit mass orbiting in the (osculating) ellipse 69. This is equal to $-\frac{GM}{2a}$ so

$$H = \frac{GM}{2a} + R$$

so in the solutions above to $\frac{ds_l}{dt}$ we can replace $\frac{\partial H}{\partial s_l}$ by $\frac{\partial R}{\partial s_l}$ except for $\frac{\partial H}{\partial a}$ where

$$\frac{2}{na}\frac{\partial H}{\partial a} = \frac{2}{na}\frac{GM}{2}\frac{\partial}{\partial a}\left(\frac{1}{a}\right) + \frac{2}{na}\frac{\partial R}{\partial a} = -\frac{2}{na}\frac{GM}{2a^2} + \frac{2}{na}\frac{\partial R}{\partial a} = -\frac{1}{n}\frac{GM}{a^3} + \frac{2}{na}\frac{\partial R}{\partial a}$$

However using Keplers law $n^2a^3 = GM$ or $n^2 = \frac{GM}{a^3}$ gives

$$\frac{2}{na}\frac{\partial H}{\partial a} = -n + \frac{2}{na}\frac{\partial R}{\partial a}$$

so

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}$$

4.5 Apsidal Precession $\frac{d\omega}{dt}$ using orbital elements

Intuitively, if $i = 0$ then the satellite orbits around the equator of the spheroid and due to symmetry we might expect no change in the orientation of the ellipse. However lets do the calculation

4.5.1 Inclination $i = 0$

The full V_{20} term is

$$V_{20} = \frac{GMR_\mu^2 C_{20}}{a^3}\sum_{p=0}^{2} F_{20p}\left(i\right)\sum_{q=-\infty}^{\infty} G_{2pq}\left(e\right)\cos\left[\left(2-2p+q\right)M\right]$$

The only term in the above that doesn't have a $\cos M$ dependency is G_{210}. Terms $G_{20-2} = G_{222} = 0$. The remaining terms have a cosine dependency so over a full orbit will average to zero. Lets consider only the $p = 1, q = 0$ term.

$$R = \frac{GMR_\mu^2 C_{20}}{a^3}F_{201}\left(i\right)G_{210}\left(e\right)$$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\frac{\partial R}{\partial i} + \frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}\frac{\partial R}{\partial e}$$

Calculating $\frac{\partial F_{201}}{\partial i}$ term when $i = 0$

$$\frac{\partial F_{201}}{\partial i} = \frac{\partial\left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)}{\partial i} = \frac{3}{4}2\sin i\cos i$$

$$-\frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}G_{210}\frac{\partial F_{201}}{\partial i} = -\frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\left(1-e^2\right)^{-\frac{3}{2}}\frac{3}{2}\sin i\cos i$$

$$= -\frac{\cos^2 i}{na^2\left(1-e^2\right)^2}\frac{3}{2}$$

$$\frac{GMR_\mu^2 C_{20}}{a^3}\left(-\frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\right)G_{210}\frac{\partial F_{201}}{\partial i} = \frac{GMR_\mu^2 C_{20}}{a^3}\left(-\frac{\cos^2 i}{na^2\left(1-e^2\right)^2}\frac{3}{2}\right)$$

$$= n^2 R_\mu^2 C_{20}\left(-\frac{\cos^2 i}{na^2\left(1-e^2\right)^2}\frac{3}{2}\right)$$

evaluating at $i = 0$

$$= -\frac{3nR_\mu^2 C_{20}}{2a^2\left(1-e^2\right)^2}$$

Calculating $\frac{\partial G_{210}}{\partial e}$ term when $i = 0$

$$\frac{\partial G_{210}}{\partial e} = \frac{\partial\left(1-e^2\right)^{-\frac{3}{2}}}{\partial e} = -\frac{3}{2}\left(1-e^2\right)^{-\frac{5}{2}}(-2e)$$

$$F_{201}\frac{\partial G_{210}}{\partial e} = \left(\frac{3}{4}\sin^2 i - \frac{1}{2}\right)3\left(1-e^2\right)^{-\frac{5}{2}}e$$

$$F_{201}\frac{\partial G_{210}}{\partial e} = \left(-\frac{1}{2}\right)3\left(1-e^2\right)^{-\frac{5}{2}}e$$

$$\frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}F_{201}\frac{\partial G_{210}}{\partial e} = \frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}\left(-\frac{1}{2}\right)3\left(1-e^2\right)^{-\frac{5}{2}}e$$

$$= -\frac{3}{2\left(1-e^2\right)^2 na^2}$$

$$\frac{GMR_\mu^2 C_{20}}{a^3}\frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}F_{201}\frac{\partial G_{210}}{\partial e} = -\frac{GMR_\mu^2 C_{20}}{a^3}\frac{3}{2\left(1-e^2\right)^2 na^2}$$

Using $n^2 = \frac{GM}{a^3}$

$$= -\frac{3nR_\mu^2 C_{20}}{2a^2\left(1-e^2\right)^2}$$

Summing both terms together

$$\frac{d\omega}{dt} = -\frac{3nR_\mu^2 C_{20}}{\left(1-e^2\right)^2 a^2}$$

So we see there is apsidal precession around a spheroid object (but not around a spherical object because $C_{20} = 0$).

The precession over one period is

$$\frac{d\omega}{dt}.T_0 = \frac{d\omega}{dt}.\frac{2\pi}{n} = -\frac{3nR_\mu^2 C_{20}}{\left(1-e^2\right)^2 a^2}.\frac{2\pi}{n} = -\frac{6\pi R_\mu^2 C_{20}}{\left(1-e^2\right)^2 a^2}$$

$$\triangle\omega_p = -\frac{6\pi R_\mu^2 C_{20}}{\left(1-e^2\right)^2 a^2} \tag{33}$$

4.6 Apsidal Precession using Inertial Frame

Because we are looking at $i = 0$ lets see what this looks like in the inertial frame. To recap, we are putting $\phi = 0$ and C_{20} is negative. We'll use the dimensionless C_{20} in what follows. From 4

$$V_{20} = GMR_\mu^2 C_{20} \frac{1}{r^3} P_2 (\sin \phi)$$

$$V_{20} = GMR_\mu^2 C_{20} \frac{1}{r^3} \frac{1}{2} \left(3 \sin^2 \phi - 1\right)$$

$$F_r = \frac{\partial V_{20}}{\partial r} = GMR_\mu^2 C_{20} \frac{-3}{r^4} \frac{1}{2} \left(3 \sin^2 \phi - 1\right) = \frac{3}{2} GMR_\mu^2 C_{20} \frac{1}{r^4}$$

$$F_\phi = \frac{1}{r} \frac{\partial V_{20}}{\partial \phi} = GMR_\mu^2 C_{20} \frac{1}{r^4} 3 \sin \phi \cos \phi = 0$$

So we have an extra inward radial acceleration at the equator due to the negative F_r that we wouldn't have had if the satellite were orbiting a spherical object.

4.6.1 Prerequisites

From $\mathbf{r} = r\mathbf{e}_r$ in polar coordinates and differentiating (see [Newtonian Mechanics]), we have

$$a_r = \frac{d^2 r}{dt^2} - r \left(\frac{d\lambda}{dt}\right)^2$$

$$a_\lambda = r \frac{d^2 \lambda}{dt^2} + 2 \frac{dr}{dt} \frac{d\lambda}{dt}$$

For a central force $a_\lambda = 0$

$$r \frac{d^2 \lambda}{dt^2} + 2 \frac{dr}{dt} \frac{d\lambda}{dt} = 0$$

Multiply through by r

$$r^2 \frac{d^2 \lambda}{dt^2} + 2r \frac{dr}{dt} \frac{d\lambda}{dt} = \frac{d}{dt} \left(r^2 \frac{d\lambda}{dt}\right) = 0$$

$$r^2 \frac{d\lambda}{dt} = const = C = \frac{L}{m} = 2 \frac{dA}{dt}$$

where L is the angular momentum of the mass m

$$\frac{d\lambda}{dt} = \frac{C}{r^2}$$

Let $r = \frac{1}{u}$

$$\frac{d\lambda}{dt} = Cu^2$$

$$\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{dt} = -\frac{1}{u^2} \frac{du}{d\lambda} \frac{d\lambda}{dt} = -\frac{1}{u^2} \frac{du}{d\lambda} Cu^2 = -C \frac{du}{d\lambda}$$

$$\frac{d^2 r}{dt^2} = -C \frac{d^2 u}{d\lambda^2} \frac{d\lambda}{dt} = -C \frac{d^2 u}{d\lambda^2} Cu^2 = -C^2 u^2 \frac{d^2 u}{d\lambda^2}$$

Putting this all together

$$a_r = \frac{d^2 r}{dt^2} - r \left(\frac{d\lambda}{dt}\right)^2 = -C^2 u^2 \frac{d^2 u}{d\lambda^2} - \frac{1}{u} C^2 u^4 = -C^2 u^2 \left(\frac{d^2 u}{d\lambda^2} + u\right)$$

$$F_r = ma_r = -mC^2 u^2 \left(\frac{d^2 u}{d\lambda^2} + u\right)$$

4.6.2 Period T_u

$$F_r = ma_r = -mC^2u^2\left(\frac{d^2u}{d\lambda^2} + u\right)$$

where $C = \frac{L}{m}$

$$F_r = \frac{-mL^2u^2}{m^2}\left(\frac{d^2u}{d\lambda^2} + u\right)$$

$$\frac{d^2u}{d\lambda^2} + u = -\frac{m}{L^2u^2}F(u) \tag{34}$$

Aside - from the above we can see that for an inverse square force $-\frac{GMm}{r^2} = -GMmu^2$, we have $\frac{d^2u}{d\lambda^2} + u = \frac{GMm^2}{L^2}$. The RHS is a constant with units m^{-1} as expected.

$$\frac{d^2u}{d\lambda^2} + u = -\frac{m}{L^2u^2}\left[-GMmu^2 + \frac{3}{2}GMmR_\mu^2C_{20}\frac{1}{r^4}\right]$$

$$\frac{d^2u}{d\lambda^2} + u = -\frac{m}{L^2u^2}\left[-GMmu^2 + \frac{3}{2}GMmR_\mu^2C_{20}u^4\right]$$

$$\frac{d^2u}{d\lambda^2} + u = \frac{GMm^2}{L^2} - \frac{3GMm^2}{2L^2}R_\mu^2C_{20}u^2$$

Let $\eta_p = \frac{GMm^2}{L_p^2}$ where L_p is the angular momentum of the satellite

$$\frac{d^2u}{d\lambda^2} + u = \eta_p - \frac{3}{2}\eta_pRC_{20}u^2$$

Let

$$g(u) = \frac{3}{2}\eta_pR_\mu^2C_{20}u^2$$

so

$$\frac{d^2u}{d\lambda^2} + u = \eta_p - g(u)$$

Let the perturbed solution u be in the vicinity of η_p. Let us expand $g(u)$ around $u = \eta_p$.

$$g_0 = g(\eta_p) = \frac{3}{2}\eta_pR_\mu^2C_{20}\eta_p^2$$

$$g_1 = \left.\frac{dg}{du}\right|_{u=\eta_p} = 3\eta_pR_\mu^2C_{20}\eta_p$$

$$\triangle u = u - \eta_p$$

$$g(u) = g_0 + g_1\triangle u = \frac{3}{2}R_\mu^2C_{20}\eta_p^3 + 3R_\mu^2C_{20}\eta_p^2(u - \eta_p)$$

So

$$\frac{d^2u}{d\lambda^2} + u = \eta_p - \frac{3}{2}R_\mu^2C_{20}\eta_p^3 - 3R_\mu^2C_{20}\eta_p^2(u - \eta_p)$$

$$\frac{d^2u}{d\lambda^2} + \left(1 + 3R_\mu^2C_{20}\eta_p^2\right)u = \eta_p - \frac{3}{2}R_\mu^2C_{20}\eta_p^3 + 3R_\mu^2C_{20}\eta_p^3$$

$$\frac{d^2u}{d\lambda^2} + \left(1 + 3R_\mu^2C_{20}\eta_p^2\right)u = \eta_p + \frac{3}{2}R_\mu^2C_{20}\eta_p^3$$

$$\frac{d^2u}{d\lambda^2} + \left(1 + 3R_\mu^2C_{20}\eta_p^2\right)u = \eta_p\left(1 + \frac{3}{2}R_\mu^2C_{20}\eta_p^2\right) \tag{35}$$

This is an equation of the form $\frac{d^2u}{d\lambda^2} + k^2u = const$ so u is a periodic function $\cos k\lambda$

$$k = \sqrt{1 + 3R_\mu^2C_{20}\eta_p^2}$$

so $u(\lambda)$ reaches it's maximum at

$$\lambda = \frac{2\pi}{\sqrt{1+3R_\mu^2 C_{20}\eta_p^2}} \approx 2\pi\left(1-\frac{3}{2}R_\mu^2 C_{20}\eta_p^2\right)$$

$$\lambda = 2\pi\left(1-\frac{3R_\mu^2 C_{20}}{2a^2\left(1-e_0^2\right)^2}\right)$$

a increased angle of

$$\triangle\lambda_p = -\frac{3\pi R_\mu^2 C_{20}}{a^2\left(1-e_0^2\right)^2} \tag{36}$$

4.7 Apparent Discrepancy in Orbital and Inertial Calculations

Comparing 33 and 36 we seem to have a discrepancy as they differ by a factor 2.

$$\triangle\omega_p = -\frac{6\pi R_\mu^2 C_{20}}{\left(1-e^2\right)^2 a^2}$$

$$\triangle\lambda_p = -\frac{3\pi R_\mu^2 C_{20}}{a^2\left(1-e_0^2\right)^2}$$

To see why this is, we need to revisit the calculation using orbital elements. Instead of using $\Omega=\omega=i=0$, let us instead use $\Omega=\omega=0$ but have $i\approx 0$ i.e. very small but not 0. In this case, we have a well defined ascending node Ω. However Ω also precesses in the following way

$$\frac{d\Omega}{dt} = \frac{3nR_e^2 C_{20}}{2a^2\left(1-e^2\right)^2}\cos i$$

For small i, in one period

$$\triangle\Omega_p = \frac{d\Omega}{dt}.\frac{2\pi}{n} = \frac{3\pi R_e^2 C_{20}}{a^2\left(1-e^2\right)^2}$$

$$\triangle\Omega_p + \triangle\omega_p = -\frac{3\pi R_e^2 C_{20}}{a^2\left(1-e^2\right)^2}$$

i.e.

$$\triangle\lambda_p = \triangle\Omega_p + \triangle\omega_p$$

See also Figure 6

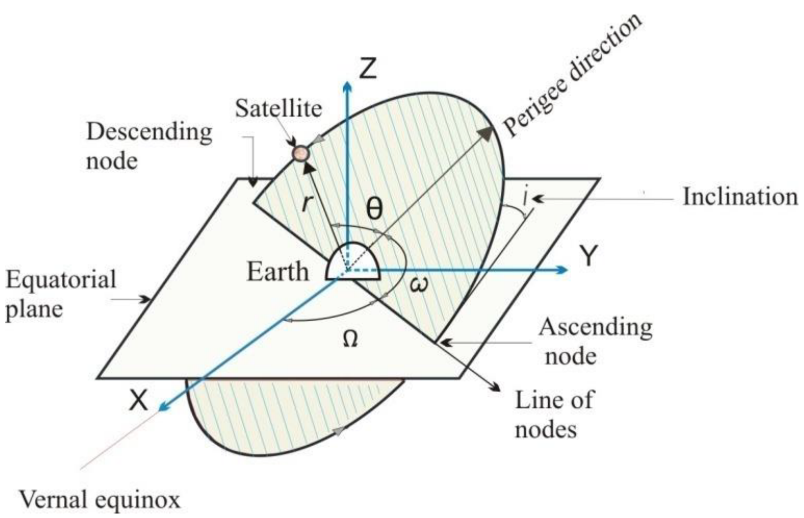


Figure 6: Space orbital parameters (ref [Apsidal Precession Low Earth])

4.8 Lagrangian Equations and Small Eccentricity

If we consider the G_{211} and G_{21-1} terms, we have for the disturbing function component

$$R = \frac{GMR_\mu^2 C_{20}}{a^3}\left(\frac{3\sin^2 i}{4} - \frac{1}{2}\right)\left(\frac{3}{2}e\right)\left(\cos(M) + \cos(-M)\right)$$

$$R = \frac{GMR_\mu^2 C_{20}}{a^3}3e\left(\frac{3\sin^2 i}{4} - \frac{1}{2}\right)\cos M \tag{37}$$

If we look at $\frac{d\omega}{dt}$

$$\frac{d\omega}{dt} = -\frac{\cos i}{na^2\left(1-e^2\right)^{\frac{1}{2}}\sin i}\frac{\partial R}{\partial i} + \frac{\left(1-e^2\right)^{\frac{1}{2}}}{na^2 e}\frac{\partial R}{\partial e}$$

$\frac{\partial R}{\partial e}$ on the $\frac{3}{2}e$ gives $\frac{3}{2}$ so we have removed the e from the numerator and we can see that there will now be a division by e (and e is small for the case we are considering)

$$\frac{d\omega}{dt} \approx ... + \frac{\left(1-\frac{1}{2}e^2\right)}{na^2 e}\frac{\partial R}{\partial e} = \frac{\partial R}{\partial e}\left[\frac{1}{na^2 e} - \frac{e}{2na^2}\right]$$

So we would have a huge increase in the argument of the pericentre ω. However we also have $\frac{dM}{dt}$ which has the same $\frac{\partial R}{\partial e}$ term

$$\frac{dM}{dt} = n - \frac{1-e^2}{na^2 e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a}$$

$$\frac{dM}{dt} \approx ...\frac{\left(-1+e^2\right)}{na^2 e}\frac{\partial R}{\partial e} - ... = \frac{\partial R}{\partial e}\left[-\frac{1}{na^2 e} + \frac{e}{na^2}\right]$$

As $e \to 0$ the $\frac{d\omega}{dt}$ and the $\frac{dM}{dt}$ terms cancel. I think this is what Kaula refers to in section 3.6 Resonance of his book.

4.9 Semi-Major Axis Time Dependency

The total energy of a satellite in an ellipse (osculating or not) is (using subscript $_o$ to indicate osculating)

$$E_o = T - \frac{GM}{r} = -\frac{GM}{2a}$$

However there is also a potential due to the V_{20} term. The total energy for the satellite is therefore

$$E = E_o + V_{20}$$

Using Physics convention for the V_{20} potential (C_{20} is -ve)

$$V_{20} \approx \frac{GMR_\mu^2 C_{20}}{2a^3}\left(1 + 3e\cos\lambda\right)$$

$$E = -\frac{GM}{2a} + \frac{GMR_\mu^2 C_{20}}{2a^3}\left(1 + 3e\cos\lambda\right) \tag{38}$$

However the difference in V_{20} between the apocentre and the pericentre is

$$\triangle V_{20} = V_{20\text{apo}} - V_{20\text{peri}} = -\frac{3GMR_\mu^2 C_{20}}{a^3}e > 0$$

So we have an increase in V_{20} as we go from pericentre to apocentre (it's less negative). Therefore, the semi-major axis a cannot be constant otherwise energy E in 38 would not be conserved.

Lagrange equation for the change in semi-major axis a, due to the disturbing potential, is

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M}$$

In previous calculations 4.5 to calculate the apsidal precession $\frac{d\omega}{dt}$, we considered the disturbing potential due to the G_{210} term, which has no dependency on the mean anomaly M. Therefore

$$\frac{da}{dt} = 0$$

Let us consider the G_{211} and G_{21-1} terms. From 37 and letting $i = 0$

$$R = \frac{GMR_\mu^2 C_{20}}{a^3} 3e \left(-\frac{1}{2}\right) \cos M$$

$$\frac{da}{dt} = \frac{2}{na} \frac{\partial R}{\partial M}$$

$$\frac{da}{dt} = \frac{2}{na} \left(\frac{GMR_\mu^2 C_{20}}{a^3} 3e \left(\frac{1}{2}\right) \sin M\right)$$

$$\frac{da}{dt} = \frac{GMR_\mu^2 C_{20}}{na^4} 3e \sin M$$

$$\int a^4 da = \int \frac{GMR_\mu^2 C_{20}}{n} 3e \sin (nt)\, dt$$

So calculating δa between pericentre and apocenter.

$$\frac{(a + \delta a)^5 - a^5}{5} = -\frac{GMR_\mu^2 C_{20}}{n^2} 3e \cos (M) \Big|_0^\pi$$

Keep $O(\delta a)$

$$a^4 \delta a = \frac{GMR_\mu^2 C_{20}}{n^2} 6e$$

$$\delta a = \frac{GMR_\mu^2 C_{20}}{n^2 a^4} 6e$$

From $n^2 = \frac{GM}{a^3}$

$$\delta a = \frac{R_\mu^2 C_{20}}{a} 6e$$

so a has decreased. Lets work out the change in energy in the osculating ellipse. (From the below it will be more -ve since a is smaller).

$$E_o = -\frac{GM}{2a}$$

$$\frac{dE_o}{da} = \frac{GM}{2a^2}$$

$$\triangle E_o = \frac{GM}{2a^2} \delta a$$

$$= \frac{GMR_\mu^2 C_{20}}{2a^3} 6e$$

$$\triangle E_o = \frac{3GMR_\mu^2 C_{20}}{a^3} e$$

4.10 Mercury Precession due to Sun Oblateness

For Mercury we have the following (see [Mercury: Planet and Orbit]).

Sun Radius (km) $R_{Ⓢ}$	695,700
Mercury e	0.2056
Mercury semi-major axis (km) a	57.91E6
Orbital Period (days) T	87.969
Mean angular velocity n	$\frac{2\pi}{T}$
Inclination i	7°
Sun J_2 (or $-C_{20}$)	2.0E-7

$$\omega_{prec} = \frac{3nR_{Ⓢ}^2 J_2}{2a^2\left(1-e^2\right)^2} = \frac{3nR_{Ⓢ}^2\left(10^{-7}\right)}{2a^2\left(1-e^2\right)^2}\left(\frac{J_2}{10^{-7}}\right)$$

$$\omega_{prec} = 1.68575E-12\left(\frac{J_2}{10^{-7}}\right) \text{ radians per day}$$

$$\omega_{prec} = 6.15299E-08\left(\frac{J_2}{10^{-7}}\right) \text{ radians per century}$$

$$\omega_{prec} = 3.5254E-06\left(\frac{J_2}{10^{-7}}\right) \text{ degrees per century}$$

$$\omega_{prec} = 3.5254E-06\left(\frac{J_2}{10^{-7}}\right) * 60 * 60 \approx 0.012\left(\frac{J_2}{10^{-7}}\right) \text{ arcsec per century}$$

If we choose $J_2 = 2E-7$

$$\omega_{prec} \approx 0.024 \text{ arcsec per century}$$

Note General Relativity contributes $\approx$ 43 arcsec per century. See next section.

4.11 Mercury Precession due to Relativity

The formula for Mercury precession due to Relativity is (see [Mercury: Planet and Orbit])

$$\omega_{prec} = \frac{3GM_{sun}n}{a\left(1-e^2\right)c^2}$$

$GM_{sun}\left(\frac{km^3}{s^2}\right)$	132,712E6
Mercury e	0.2056
Mercury semi-major axis (km) a	57.91E6
Orbital Period (days) T_d	87.969
Orbital Period (sec) T_s	7,600,522
Mean angular velocity (per sec) n	8.27E-07
Speed of light c $\frac{km}{s}$	299,792

$$\omega_{prec} = 6.6\left(10^{-14}\right) \text{radians per second}$$

$$\omega_{prec} = 2.08\left(10^{-4}\right) \text{ radians per century}$$

$$\omega_{prec} = 0.01193 \text{ degrees per century}$$

$$\omega_{prec} = 42.95 \text{ arcsec per century}$$

4.12 Newton's Theorum of Revolving Orbits

We could also account for Mercury's precession by making a small amendment δ to the inverse square law [Newton Apsidal]. Let

$$F(r) = -\frac{GMm}{r^{(2+\delta)}}$$

Let $u = \frac{1}{r}$

$$F(u) = -GMmu^{(2+\delta)} = -GMmu^2u^\delta$$

Using 34

$$\frac{d^2u}{d\lambda^2} + u = -\frac{m}{L^2u^2}F(u) = \left(-\frac{m}{L^2u^2}\right)\left(-GMmu^2u^\delta\right) = \frac{GMm^2u^\delta}{L^2}$$

So we have the following (where if $\delta = 0$ we end up with the standard inverse square law equation).

$$\frac{d^2u}{d\lambda^2} + u = \frac{GMm^2u^\delta}{L^2}$$

$$\frac{d^2u}{d\lambda^2} + u = \eta u^\delta$$

$u^\delta = \exp\left(\ln u^\delta\right)$. Since $\ln u^\delta = \delta \ln u$ then $u^\delta = \exp(\delta \ln u)$. Expanding this to the first order in δ gives

$$u^\delta = 1 + \delta \ln u$$

As previously, let us assume that the solution u is in the region of η (so this derivation is only working for small e). Let

$$g(u) = u^\delta = 1 + \delta \ln u$$

Expand around η

$$g(u) = g_0(\eta) + g_1(\eta) \triangle u$$

$$g_0 = g(\eta) = 1 + \delta \ln \eta$$

$$g_1 = \left.\frac{dg}{du}\right|_{u=\eta} = \frac{\delta}{\eta}$$

$$\triangle u = u - \eta$$

$$g(u) = g_0 + g_1 \triangle u = (1 + \delta \ln \eta) + \frac{\delta}{\eta}(u - \eta)$$

$$\frac{d^2u}{d\lambda^2} + u = \eta g(u) = \eta(1 + \delta \ln \eta) + \delta u - \delta \eta$$

$$\frac{d^2u}{d\lambda^2} + u = \eta(1 - \delta + \delta \ln \eta) + \delta u$$

$$\frac{d^2u}{d\lambda^2} + (1 - \delta)u = \eta(1 - \delta + \delta \ln \eta) \quad (39)$$

This is of the form $\frac{d^2u}{d\lambda^2} + k^2u = const$ so is a periodic function $\cos k\lambda$. $k = \sqrt{1-\delta}$. The maximum is at $\lambda = \frac{2\pi}{\sqrt{1-\delta}} = 2\pi(1-\delta)^{-\frac{1}{2}} \approx 2\pi\left(1 + \frac{\delta}{2}\right) = 2\pi + \delta\pi$

So the increase is

$$\delta\pi \text{ per orbit}$$

For Mercury, letting $\delta = 0.00000016$

$$0.00000016\pi \text{ radians per } 87.969 \text{ days}$$

$$6.6387023\pi\left(10^{-5}\right) \text{ radians per century}$$

$$0.003803696\pi \text{ degrees per century}$$

$$\approx 43 \text{ arcsec per century}$$

Aside 1: Newton's derivation (Proposition 45 in Book I of the Principia) relates the angle between apocentre and pericentre to be $\frac{\pi}{\sqrt{n}}$ where the centripetal force is of magnitude μr^{n-3}. Relating this to the above we have $n-3=-(2+\delta)$ or $n=(1-\delta)$ so the angle is $\frac{\pi}{\sqrt{1-\delta}}$. Therefore the angle from apocentre to apocentre is twice this i.e. $\frac{2\pi}{\sqrt{1-\delta}}$ (as above).

4.12.1 Relationship to c and v

For Mercury, it's orbital velocity is $\sim na$ so from 4.11

$$v = 57.91E6 * 8.27E-07$$

$$v \approx 47.89\text{km/s}$$

Divide by the speed of light is

$$\frac{v}{c} \approx \frac{47.89}{299,792} = 0.0001597441$$

We have a number of interesting relationships given that $\delta = 0.00000016$ from the above section e.g.

$$2\pi\left(\frac{v}{c}\right)^2 \approx 0.00000016$$

and

$$\frac{6}{(1-e^2)}\left(\frac{v}{c}\right)^2 \approx 0.00000016$$

Using the second one

$$v = na$$

$$v^2 = n^2a^2 = \frac{n^2a^3}{a}$$

From Keplers law

$$v^2 = n^2a^2 = \frac{GM}{a}$$

so

$$\frac{v^2}{c^2} = \frac{GM}{ac^2}$$

and

$$\delta = \frac{6}{(1-e^2)}\left(\frac{v}{c}\right)^2 = \frac{6GM}{a(1-e^2)c^2}$$

so we get the General Relativity correction

$$\pi\delta = \frac{6\pi GM}{a(1-e^2)c^2}$$

4.12.2 Semi-Latus Rectum Change

Refer to K.

The above 39 means that the ellipse, besides precessing, has the following particular solution

$$u_p = \eta \frac{(1-\delta+\delta\ln\eta)}{(1-\delta)}$$

So it also has a increased semi-latus rectum $l_{new} = \frac{1}{\eta_{new}}$ i.e.

$$\eta_{new} = \eta\left(\frac{(1-\delta+\delta\ln\eta)}{(1-\delta)}\right) = \left(1+\frac{\delta\ln\eta}{1-\delta}\right)\eta$$

I haven't found any literature on this consequence of changes to the inverse-square force and how significant/observable it is.

In the case of Mercury

$$\eta_{new} = \left(1+\frac{(0.00000016)\ln\left(1.803\left(10^{-8}\right)\right)}{(1-0.00000016)}\right)\eta = 0.9999971467\eta$$

And the semi-latus rectum is

$$l_{new} = \frac{1}{\eta_{new}} = \frac{1}{0.9999971467\eta}$$

semi-latus rectum goes from 55,462,066 to 55,462,224 km. So an increase of ~150km.

5 Mercury Spin-Orbit Coupling.

Mercury has a 3:2 spin orbit coupling i.e. it rotates on it's axis 3 times in the time it takes to orbit twice around the sun. The following references explain it well [Spin Orbit coupling], [Mercury: Planet and Orbit], [Tidal Torques].

The 3:2 spin orbit coupling involves the effects of 1) tidal torques and 2) triaxial torques due to permanent asymmetry in Mercury's equatorial plane i.e. $a > b > c$. Note - the tidal torque does not require a permanent asymmetry in the axis i.e. we could have tidal torque if Mercury was spherical. The tidal torque on it's own would act to slow down Mercury's rotation until it reaches a 1:1 coupling i.e. Mercury would take the same time to rotate on it's axis as it does to orbit the sun. Note the moon is in a 1:1 spin orbit coupling which is why it always presents the same face to the earth. However Mercury hasn't reached the 1:1 state - it spins faster than it orbits and is captured in a 3:2 state.

Lets consider the triaxial torque due to the *permanent* asymmetry of Mercury where the moments of inertia around the 3 principal axis are $C > B > A$. [Spin Orbit coupling] gives a good account of the torques in play and the derivation of subsequent equations(see Fig 7).

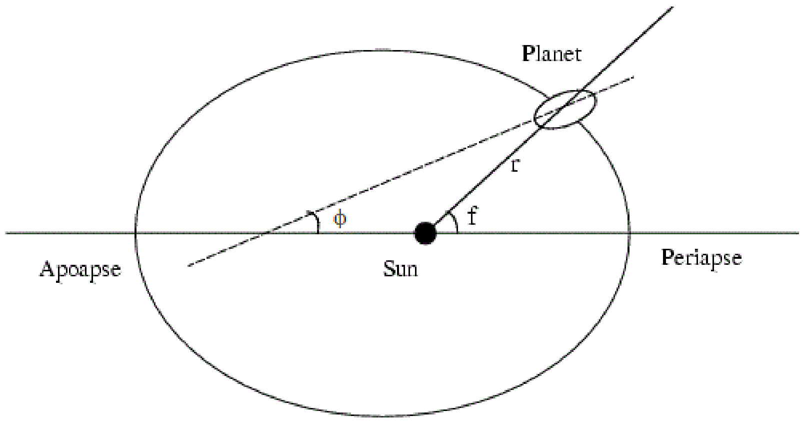


Figure 7: Geometry of spin-orbit coupling (ref [Mercury Main librations]).

Using MacCullagh's formula for the gravitational potential due to Mercury leads to the following formula that describes ϕ. Note $\dot{\phi}$ describes Mercury's spin.

$$\ddot{\phi} + \frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3 \sin\left[2\left(\phi - f\right)\right] = 0 \tag{40}$$

Now 40 is not straightforward to solve. Lets consider the situation where Mercury has been spun down by tidal torques until it is approaching the 3:2 resonance. Let us define

$$\gamma = \phi - \frac{3}{2}M$$

where M is the mean anomaly. So

$$\phi = \gamma + \frac{3}{2}M$$

$$\dot{\phi} = \dot{\gamma} + \frac{3}{2}n$$

$$\ddot{\phi} = \ddot{\gamma}$$

So 40 becomes

$$\ddot{\gamma} + \frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3 \sin\left[2\left(\gamma + \frac{3}{2}M - f\right)\right] = 0$$

$$\ddot{\gamma} + \frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3 \sin\left[2\gamma + 3M - 2f\right] = 0$$

The following term is similar to what we solved in 3.3.

$$\left(\frac{a}{r}\right)^3 \sin\left[2\gamma + 3M - 2f\right]$$

5.1 Prerequisites

Revisiting the complex Fourier expansion of a function we have

$$f = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

If f is real then $c_k = \bar{c}_{-k}$ i.e. the complex conjugate.
If f is even, c_k is real, so $c_k = c_{-k}$.
If f is odd, c_k is imaginary.
The relation between cosine and sin coefficients to the complex coefficients are

$$a_0 = c_0$$

$$a_k = c_k + c_{-k}$$

$$b_k = i\left(c_k - c_{-k}\right)$$

5.2 Expansion of $\frac{a^3}{r^3}\sin\left(2\gamma + 3M - 2f\right)$

$$\sin\left(2\gamma + 3M - 2f\right) = \frac{\exp\left(i\left(2\gamma + 3M - 2f\right)\right) - \exp\left(-i\left(2\gamma + 3M - 2f\right)\right)}{2i}$$

From 27

$$\exp\left(i\left(2\gamma + 3M - 2f\right)\right) = \exp\left(i\left(2\gamma + 3M\right)\right)\exp\left(-i2f\right) = \exp\left(i\left(2\gamma + 3M\right)\right)\frac{\exp\left(-i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{-2}}{\left(1 - \beta\exp\left(iE\right)\right)^{-2}}$$

$$\begin{aligned} 2i\sin\left(2\gamma + 3M - 2f\right) = {} & \exp\left(i\left(2\gamma + 3M\right)\right)\frac{\exp\left(-i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{-2}}{\left(1 - \beta\exp\left(iE\right)\right)^{-2}} \\ & - \exp\left(-i\left(2\gamma + 3M\right)\right)\frac{\exp\left(i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{2}}{\left(1 - \beta\exp\left(iE\right)\right)^{2}} \end{aligned}$$

$$\begin{aligned} & = \exp\left(i\left(2\gamma + 3M\right)\right)\exp\left(-i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{-2}\left(1 - \beta\exp\left(iE\right)\right)^{2} \\ & - \exp\left(-i\left(2\gamma + 3M\right)\right)\exp\left(i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{2}\left(1 - \beta\exp\left(iE\right)\right)^{-2} \end{aligned}$$

Let's expand the r, f term in a Fourier series as before

$$function = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$\frac{a^3}{r^3}\sin\left(2\gamma + 3M - 2f\right) = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

$$c_k = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}\left[\sin\left(2\gamma + 3M - 2f\right)\right]\exp -ikM dM$$

Let's look at the first term first

$$c_{k_1} = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}\left[\frac{\exp\left(i\left(2\gamma + 3M\right)\right)\exp\left(-i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{-2}\left(1 - \beta\exp\left(iE\right)\right)^{2}}{2i}\right]\exp -ikM dM$$

From previously, we can substitute $\frac{a^3}{r^3}$ with $\frac{a^2}{r^2}\frac{a}{r}dM$ or $\frac{a^2}{r^2}dE$ and take constant terms out of the integral ($\gamma = \phi - \frac{3}{2}M$ so γ is approximately constant near the 3:2 resonance.)

$$c_{k_1} = \frac{1}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\frac{a^2}{r^2}\left[\exp\left(i3M\right)\exp\left(-i2E\right)\left(1 - \beta\exp\left(-iE\right)\right)^{-2}\left(1 - \beta\exp\left(iE\right)\right)^{2}\right]\exp -ikM dE$$

substitute $\frac{a^2}{r^2}$

$$= \frac{1}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\frac{\left(1+\beta^2\right)^2}{\left[\left(1-\beta\exp\left(iE\right)\right)\left(1-\beta\exp\left(-iE\right)\right)\right]^2}$$
$$*\left[\exp\left(i3M\right)\exp\left(-i2E\right)\left(1-\beta\exp\left(-iE\right)\right)^{-2}\left(1-\beta\exp\left(iE\right)\right)^2\right]\exp -ikMdE$$

Rearrange terms

$$= \frac{\left(1+\beta^2\right)^2}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}$$
$$*\left[\exp\left(-i2E\right)\left(1-\beta\exp\left(-iE\right)\right)^{-2}\left(1-\beta\exp\left(iE\right)\right)^2\right]\exp -i\left(k-3\right)MdE$$

substitute M and combine iE terms

$$= \frac{\left(1+\beta^2\right)^2}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\left(1-\beta\exp\left(-iE\right)\right)^{-4}$$
$$*\exp\left(-i2E\right)\exp\left(i\left[\left(k-3\right)e\sin E-\left(k-3\right)E\right]\right)dE$$

so the first term is

$$\boxed{c_{k_1} = \frac{\left(1+\beta^2\right)^2}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\left(1-\beta\exp\left(-iE\right)\right)^{-4}\exp\left(i\left[\left(k-3\right)e\sin E-\left(k-1\right)E\right]\right)dE}$$

$$c_{k_1} = \frac{\left(1+\beta^2\right)^2}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\sum_{b=0}^{\infty}\binom{-4}{b}\left(-1\right)^b\beta^b\exp\left(-iEb\right)\exp\left(i\left[\left(k-3\right)e\sin E-\left(k-1\right)E\right]\right)dE$$

$$c_{k_1} = \frac{\left(1+\beta^2\right)^2}{4\pi i}\exp\left(i2\gamma\right)\int_0^{2\pi}\sum_{b=0}^{\infty}\binom{-4}{b}\left(-1\right)^b\beta^b\exp(i\left[\left(k-3\right)e\sin E-\left(k-1+b\right)E\right])dE$$

$$c_{k_1} = \frac{\exp\left(i2\gamma\right)}{2i}\left[\left(1+\beta^2\right)^2\sum_{b=0}^{\infty}\binom{-4}{b}\left(-1\right)^b\beta^bJ_{k-1+b}\left(\left(k-3\right)e\right)\right]$$

For the second term, and omitting some steps

$$c_{k_2} = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^3}{r^3}\left[\frac{\exp\left(-i\left(2\gamma+3M\right)\right)\exp\left(i2E\right)\left(1-\beta\exp\left(-iE\right)\right)^2\left(1-\beta\exp\left(iE\right)\right)^{-2}}{2i}\right]\exp -ikMdM$$

$$= \frac{1}{4\pi i}\exp\left(-i2\gamma\right)\int_0^{2\pi}\left(1+\beta^2\right)^2\left(1-\beta\exp\left(iE\right)\right)^{-2}\left(1-\beta\exp\left(-iE\right)\right)^{-2}$$
$$*\left[\exp\left(-i3M\right)\exp\left(i2E\right)\left(1-\beta\exp\left(-iE\right)\right)^2\left(1-\beta\exp\left(iE\right)\right)^{-2}\right]\exp -ikMdE$$

$$= \frac{\left(1+\beta^2\right)^2}{4\pi i}\exp\left(-i2\gamma\right)\int_0^{2\pi}\left(1-\beta\exp\left(iE\right)\right)^{-4}\exp\left(i\left(\left(k+3\right)e\sin E-\left(k+1\right)E\right)\right)dE$$

$$c_{k_2} = \frac{\exp\left(-i2\gamma\right)}{2i}\left[\left(1+\beta^2\right)^2\sum_{b=0}^{\infty}\binom{-4}{b}\left(-1\right)^b\beta^bJ_{k+1-b}\left(\left(k+3\right)e\right)\right]$$

$$c_k = c_{k_1} - c_{k_2}$$

Using a program, we can calculate the bit in the bracket for both c_{k_1} and c_{k_2} which will be a function of k and e. This is basically the same way we did manually in section 3.2.2).

Here are the results.

```
file:///C:/dev/SpheroidCode/spheroid/bin/Debug/spheroid.EXE
k=0 1st:7/2e^1 + -123/16e^3, 2nd:7/2e^1 + -123/16e^3
k=1 1st:1/1e^0 + -5/2e^2, 2nd:17/2e^2
k=-1 1st:17/2e^2, 2nd:1/1e^0 + -5/2e^2
k=2 1st:-1/2e^1 + 1/16e^3, 2nd:845/48e^3
k=-2 1st:845/48e^3, 2nd:-1/2e^1 + 1/16e^3
```

Figure 8: Calculation of e dependency

Calculate the $k=0$ term - c_0 - keeping only as high as $O(e)$.

$$c_0 \exp i0M = c_{0_1}.1 - c_{0_2}.1 = \frac{7e}{2}\frac{\exp(i2\gamma)}{2i} - \frac{7e}{2}\frac{\exp(-i2\gamma)}{2i}$$

$$c_0 = \frac{7e}{2}\left[\frac{\exp(i2\gamma) - \exp(-i2\gamma)}{2i}\right]$$

$$\boxed{c_0 = \frac{7e}{2}\sin(2\gamma)}$$

Calculate the $k=1$ and $k=-1$ terms, c_1 and c_{-1} keeping only as high as $O(e)$.

$$c_1 \exp iM = c_{1_1}\exp iM - c_{1_2}\exp iM = 1.\frac{\exp(i2\gamma)\exp iM}{2i} - O\left(e^2\right) = \frac{\exp(i(2\gamma+M))}{2i}$$

$$c_{-1}\exp -iM = c_{-1_1}\exp -iM - c_{-1_2}\exp -iM = O\left(e^2\right) - 1.\frac{\exp(-i2\gamma)\exp -iM}{2i} = -\frac{\exp(-i(2\gamma+M))}{2i}$$

$$\boxed{c_1\exp iM + c_{-1}\exp -iM = \sin(2\gamma+M)}$$

Calculate the $k=2$ and $k=-2$ terms, c_2 and c_{-2} keeping only as high as $O(e)$.

$$c_2\exp i2M = c_{2_1}\exp i2M - c_{2_2}\exp i2M = -\frac{e}{2}\frac{\exp(i2\gamma)\exp i2M}{2i} + O\left(e^3\right) = -\frac{e}{2}\frac{\exp(i(2\gamma+2M))}{2i}$$

$$c_{-2}\exp -i2M = c_{-2_1}\exp -i2M - c_{-2_2}\exp -i2M = O\left(e^3\right) + \frac{e}{2}\frac{\exp(-i2\gamma)\exp -i2M}{2i} = +\frac{e}{2}\frac{\exp(-i(2\gamma+2M))}{2i}$$

$$\boxed{c_2\exp i2M + c_{-2}\exp -i2M = -\frac{e}{2}\sin(2\gamma+2M)}$$

Now we can go back to the original expansion

$$\frac{a^3}{r^3}\sin(2\gamma+3M-2f) = \sum_{k=-\infty}^{\infty} c_k \exp ikM$$

and using the values we have calculated for $c_k \exp ikM$ for $k=0,1,2$ we have

$$\frac{a^3}{r^3}\sin(2\gamma+3M-2f) = \frac{7e}{2}\sin(2\gamma) + \sin(2\gamma+M) - \frac{e}{2}\sin(2\gamma+2M) + ...$$

$$\ddot{\gamma} + \frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left(\frac{a}{r}\right)^3\sin\left[2\left(\gamma+\frac{3}{2}M-f\right)\right] = 0$$

$$\ddot{\gamma} + \frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left[\frac{7e}{2}\sin(2\gamma) + \sin(2\gamma+M) - \frac{e}{2}\sin(2\gamma+2M)\right] = 0$$

$$\ddot{\gamma} = -\frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left[\left(\frac{7e}{2} + \cos M - \frac{e}{2}\cos 2M\right)\sin(2\gamma) + \left(\sin M - \frac{e}{2}\sin 2M\right)\cos(2\gamma)\right]$$

$$\ddot{\gamma} = -\frac{3}{2}n^2\left(\frac{B-A}{C}\right)\left[\left(\frac{7e}{2} + \cos nt - \frac{e}{2}\cos 2nt\right)\sin\left(2\gamma\right) + \left(\sin nt - \frac{e}{2}\sin 2nt\right)\cos\left(2\gamma\right)\right] \tag{41}$$

Over an full orbit, the terms with a dependency on t on the RHS in 41 average to zero (check to follow) so we have

$$\ddot{\gamma} + \frac{3}{2}n^2\left(\frac{B-A}{C}\right)\frac{7e}{2}\sin\left(2\gamma\right) = 0$$

$$\ddot{\gamma} + \frac{21}{4}en^2\left(\frac{B-A}{C}\right)\sin\left(2\gamma\right) = 0 \tag{42}$$

Equation 42 is a pendulum equation. For the 3:2 reference, γ as defined is small so $\sin\left(2\gamma\right) \approx 2\gamma$

$$\ddot{\gamma} + \frac{21}{4}en^2\left(\frac{B-A}{C}\right)2\gamma = 0 \tag{43}$$

This is of the form $\ddot{\gamma} + k^2\gamma = 0$ so the orientation of Mercury's longest axis points towards the Sun at the perihelion and librates with a frequency

$$k = \sqrt{\frac{21}{2}en^2\left(\frac{B-A}{C}\right)} = n\sqrt{\frac{21}{2}e\left(\frac{B-A}{C}\right)}$$

To calculate, use $e = 0.2056317$. For $\frac{B-A}{C}$ we have from [Mercury Main Librations Messenger] 0.0002206 (whereas from an earlier paper [Mercury Main librations] we have 0.00012)

$$k \approx n\sqrt{10.5 * 0.2056317 * 0.0002206} = n\sqrt{2.159 * 0.0002206} = 0.0218244n$$

So the period of the librations compared to the orbital period T_M is

$$\frac{\frac{2\pi}{0.0218244n}}{\frac{2\pi}{n}} = \frac{1}{0.0218244} \approx 45.8202615T_M$$

Using $T_M = 88$, the libration period is

$$T_L\left(days\right) = 4032$$

$$T_L\left(years\right) = 11.04$$

If we were to take terms to $O\left(e^3\right)$ in the calculation of $c_{k=0}$ we would have $\frac{7e}{2} - \frac{123e^3}{16} \approx 0.65286$ giving

$$k \approx n\sqrt{3 * 0.65286 * 0.0002206} = 0.020786n$$

$$T_L\left(days\right) = \frac{1}{0.020786}T_M = 48.1087T_M = 4233$$

$$T_L\left(years\right) = 11.59$$

A Asaph Hall Apsidal Precession derivation

Precession due to a force law differing slightly from Inverse-square (this was the hypothesis made by Asaph Hall in 1894). Derivation from [Newton Apsidal]

$$\frac{d^2u}{d\lambda^2} + u = \eta u^{\delta} \tag{44}$$

Let the solution of the above be

$$u = c_h \left(1 + e\cos k_h\lambda\right) + \delta u' \tag{45}$$

where c_h is a constant and k_h represents a rotating ellipse. From [Newton Apsidal]

. Where e is larger, we cannot simply identify a rotating ellipse with the orbit of interest. Instead, we shall employ a rotating ellipse *plus* additional terms; this will allow for deformation of the shape as well as motion of the apse. The additional terms will be given by the perturbation $\delta u'$. We shall restrict this perturbation so as not to contain terms proportional to $\cos \lambda\theta$; thus it will not be implicated in the determination of the constants of the rotating ellipse.

Substituting this into u^{δ} and keeping $O(\delta)$ only

$$u^{\delta} = c_h^{\delta}\left(1 + e\cos k_h\lambda\right)^{\delta}$$

$$\ln u^{\delta} = \ln c_h^{\delta} + \delta\ln\left(1 + e\cos k_h\lambda\right)$$

Taking exponential

$$u^{\delta} = c_h^{\delta}\exp\left(\delta\ln\left(1 + e\cos k_h\lambda\right)\right)$$

Expanding exponential to $O(\delta)$

$$u^{\delta} = c_h^{\delta}\left(1 + \delta\ln\left(1 + e\cos k_h\lambda\right)\right)$$

Expanding the ln

$$u^{\delta} = c_h^{\delta}\left(1 + \delta e\left(\cos k_h\lambda - \frac{e}{2}\cos^2 k_h\lambda + \frac{e^2}{3}\cos^3 k_h\lambda\right)\right)$$

Using the identities

$$\cos 2\theta = 2\cos^2\theta - 1 \implies \cos^2 k_h\lambda = \frac{1}{2} + \frac{1}{2}\cos 2k_h\lambda$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \implies \cos^3 k_h\lambda = \frac{3}{4}\cos k_h\lambda + \frac{1}{4}\cos 3k_h\lambda$$

We get

$$u^{\delta} = c_h^{\delta}\left(1 + \delta e\left(\cos k_h\lambda - \frac{e}{2}\left(\frac{1}{2} + \frac{1}{2}\cos 2k_h\lambda\right) + \frac{e^2}{3}\left(\frac{3}{4}\cos k_h\lambda + \frac{1}{4}\cos 3k_h\lambda\right)\right)\right)$$

$$u^{\delta} = c_h^{\delta}\left(1 + \delta e\left(-\frac{e}{4} + \cos k_h\lambda - \frac{e}{4}\cos 2k_h\lambda + \frac{e^2}{4}\cos k_h\lambda + \frac{e^2}{12}\cos 3k_h\lambda\right)\right)$$

$$u^{\delta} = c_h^{\delta}\left(1 + \delta e\left(-\frac{e}{4} + \left(1 + \frac{e^2}{4}\right)\cos k_h\lambda - \frac{e}{4}\cos 2k_h\lambda + \frac{e^2}{12}\cos 3k_h\lambda\right)\right) \tag{46}$$

Now insert 45 into the left side of 44

$$\delta\left(\frac{d^2u'}{d\lambda^2} + u'\right) - c_h e k_h^2\cos k\lambda + c_h + c_h e\cos k_h\lambda$$

$$\delta\left(\frac{d^2u'}{d\lambda^2} + u'\right) + c_h + c_h e\left(1 - k^2\right)\cos k_h\lambda \tag{47}$$

From 44 and using 47 and 46

$$\delta\left(\frac{d^2u'}{d\lambda^2} + u'\right) + c_h + c_h e\left(1 - k_h^2\right)\cos k_h\lambda = \eta c_h^{\delta}\left(1 + \delta e\left(-\frac{e}{4} + \left(1 + \frac{e^2}{4}\right)\cos k_h\lambda - \frac{e}{4}\cos 2k_h\lambda + \frac{e^2}{12}\cos 3k_h\lambda\right)\right)$$

Equating constant terms

$$c_h = \eta c_h^{\delta}\left(1 - \frac{\delta e^2}{4}\right)$$

$$\eta c_h^{\delta-1} = \left(1 - \frac{\delta e^2}{4}\right)^{-1} \approx \left(1 + \frac{\delta e^2}{4}\right)$$

The coefficients of $\cos k_h \lambda$ must also be equal so

$$c_h e\left(1 - k_h^2\right) = \eta c_h^{\delta} \delta e\left(1 + \frac{e^2}{4}\right)$$

$$1 - k_h^2 = \delta \eta c_h^{\delta-1}\left(1 + \frac{e^2}{4}\right)$$

$$1 - k_h^2 = \delta\left(1 + \frac{\delta e^2}{4}\right)\left(1 + \frac{e^2}{4}\right) \approx \delta\left(1 + \frac{e^2}{4}\right)$$

$$k_h = \left(1 - \delta\left(1 + \frac{e^2}{4}\right)\right)^{\frac{1}{2}}$$

$$k_h \approx 1 - \frac{\delta}{2}\left(1 + \frac{e^2}{4}\right)$$

The precession per orbit is

$$\triangle\lambda = \frac{2\pi}{k_h} = 2\pi\left(1 - \frac{\delta}{2}\left(1 + \frac{e^2}{4}\right)\right)^{-1} - 2\pi$$

$$\triangle\lambda \approx 2\pi\left(1 + \frac{\delta}{2}\left(1 + \frac{e^2}{4}\right)\right) - 2\pi$$

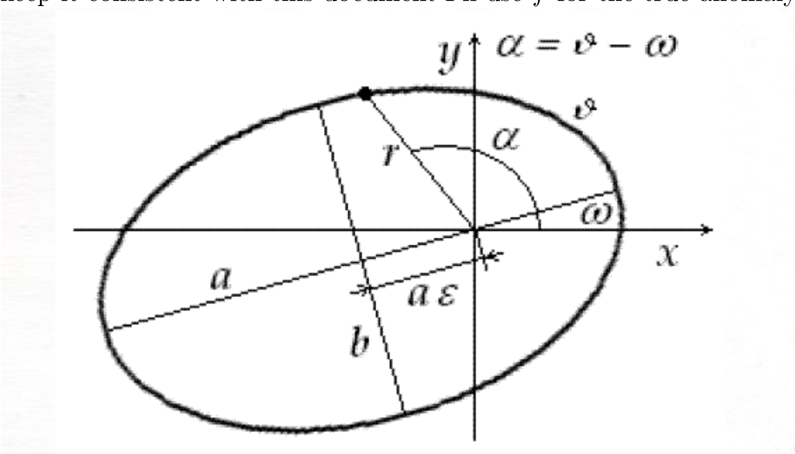
$$\triangle\lambda = \frac{2\pi}{k_h} = \delta\pi\left(1 + \frac{e^2}{4}\right)$$

so the precession increases as the eccentricity increases. The difference is small though in the case of Mercury.

B Gerber's Theory

From [Gerber Theory]. (Still working through this)

The co-ordinate system is as below so we are initially dealing with ϑ where $\vartheta = \omega + \alpha$ (from [Gerber Coords]). To keep it consistent with this document I'll use f for the true anomaly rather than α. So $\vartheta = \omega + f$



Note the below refers to *radial* velocity v_r. I'm also using $\mu = GM$ in the below. Gerber's gravitational Lagrangian potential is

$$S = \frac{\mu}{r\left(1 - \frac{v_r}{c}\right)^2}$$

binomially expand

$$S=\frac{\mu}{r}\left[1+2\frac{v_r}{c}+3\left(\frac{v_r}{c}\right)^2\right]$$

using Lagrangian formula for calculating acceleration

$$a=\frac{\partial S}{\partial r}-\frac{d}{dt}\frac{\partial S}{\partial v_r}$$

$$a=-\frac{\mu}{r^2}\left[1+2\frac{v_r}{c}+3\left(\frac{v_r}{c}\right)^2\right]-\mu\frac{d}{dt}\frac{1}{r}\left[\frac{2}{c}+\frac{6v_r}{c^2}\right]$$

using $\frac{d}{dt}=\frac{dr}{dt}\frac{d}{dr}+\frac{dv_r}{dt}\frac{d}{dv_r}=v_r\frac{d}{dr}+\dot{v_r}\frac{d}{dv_r}$ and

$$a=-\frac{\mu}{r^2}\left[1+2\frac{v_r}{c}+3\left(\frac{v_r}{c}\right)^2\right]-\mu\left(-\frac{v_r}{r^2}\right)\left[\frac{2}{c}+\frac{6v_r}{c^2}\right]-\frac{\mu}{r}\dot{v_r}\left[\frac{6}{c^2}\right]$$

$$a=-\frac{\mu}{r^2}\left[1+2\frac{v_r}{c}+3\frac{v_r^2}{c^2}\right]+\frac{\mu}{r^2}\left[\frac{2v_r}{c}+6\frac{v_r^2}{c^2}-6\frac{r\dot{v_r}}{c^2}\right]$$

$$a=-\frac{\mu}{r^2}\left[1-3\frac{v_r^2}{c^2}+6\frac{r\dot{v_r}}{c^2}\right]$$

Aside: In deriving the above, the $2\frac{v_r}{c}$ term in the binomial expansion (or more generally any $O\left(v_r\right)$ term) doesn't contribute to the acceleration a as the terms cancel as in the above. So we could write Gerber Lagragian potential as

$$S=\frac{\mu}{r}\left[1+3\left(\frac{v_r}{c}\right)^2\right]$$

or Gerber potential energy

$$V=\frac{\mu}{r}\left[1-3\left(\frac{v_r}{c}\right)^2\right]$$

This is similar in form the the Weber potential in electrodynamics 64.
Let

$$\Lambda\left(r,v_r,\dot{v_r}\right)=3\frac{v_r^2}{c^2}-6\frac{r\dot{v_r}}{c^2} \tag{48}$$

$$a=\frac{d^2r}{dt^2}=-\frac{\mu}{r^2}\left[1-\Lambda\right] \tag{49}$$

From 49

$$\frac{d^2x}{dt^2}=-\frac{x\mu}{r^3}\left[1-\Lambda\right]$$

$$\frac{d^2y}{dt^2}=-\frac{y\mu}{r^3}\left[1-\Lambda\right]$$

it follows (and this really is just conservation of angular momentum $\frac{dL}{dt}=0$ see below)

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}=0$$

using $r^2\frac{d\vartheta}{dt}=L\implies\frac{1}{r^2}=\frac{1}{L}\frac{d\vartheta}{dt}$ and $\frac{x}{r}=\cos\vartheta$ and $\frac{y}{r}=\sin\vartheta$, we have

$$\frac{d^2x}{dt^2}=-\frac{\mu}{L}\left[1-\Lambda\right]\cos\vartheta\frac{d\vartheta}{dt}$$

$$\frac{d^2y}{dt^2}=-\frac{\mu}{L}\left[1-\Lambda\right]\sin\vartheta\frac{d\vartheta}{dt}$$

integrating, where M and N are constants, gives

$$\frac{dx}{dt}=-\frac{\mu}{L}\sin\vartheta+\left(M+\frac{\mu}{L}\int\Lambda\cos\vartheta d\vartheta\right)$$

$$\frac{dy}{dt}=\frac{\mu}{L}\cos\vartheta+\left(N+\frac{\mu}{L}\int\Lambda\sin\vartheta d\vartheta\right)$$

But from the case of a circular orbit $e=0$, $a_r = \frac{\mu}{r^2} = \frac{v^2}{r}$. So $v^2 = \frac{\mu}{r}$. And also for a circular orbit $L = rv \implies r = \frac{L}{v}$.

Together this gives $v^2 = \frac{\mu v}{L}$ therefore $v = \frac{\mu}{L}$ so M and N must both be 0.

$$y\frac{dx}{dt} = -\frac{\mu}{L}\left(r\sin\vartheta\right)\sin\vartheta + \left(r\sin\vartheta\right)\left(\mu\int\Lambda\cos\vartheta d\vartheta\right)$$

$$x\frac{dy}{dt} = \frac{\mu}{L}\left(r\cos\vartheta\right)\cos\vartheta + r\cos\vartheta\left(\frac{\mu}{L}\int\Lambda\sin\vartheta d\vartheta\right)$$

from the angular momentum definition, $\hat{L} = \hat{r}\times\hat{p}$, we have $L = x\frac{dy}{dt} - y\frac{dx}{dt}$

$$L = \frac{\mu r}{L} + r\cos\vartheta\left(\frac{\mu}{L}\int\Lambda\sin\vartheta d\vartheta\right) - \left(r\sin\vartheta\right)\left(\frac{\mu}{L}\int\Lambda\cos\vartheta d\vartheta\right)$$

$$L = r\left[\frac{\mu}{L} + \left(\frac{\mu}{L}\int\Lambda\sin\vartheta d\vartheta\right)\cos\vartheta - \left(\frac{\mu}{L}\int\Lambda\cos\vartheta d\vartheta\right)\sin\vartheta\right]$$

multiplying through by $\frac{L}{\mu}$.

$$\frac{L^2}{\mu} = r\left[1 + \left(\int\Lambda\sin\vartheta d\vartheta\right)\cos\vartheta - \left(\int\Lambda\cos\vartheta d\vartheta\right)\sin\vartheta\right]$$

$$r = \frac{\frac{L^2}{\mu}}{1 + \left(\int\Lambda\sin\vartheta d\vartheta\right)\cos\vartheta - \left(\int\Lambda\cos\vartheta d\vartheta\right)\sin\vartheta} \tag{50}$$

Assuming r is an ellipse, then at the pericentre $\vartheta = \omega$ and $r = a\left(1-e\right)$, at the apocentre $\vartheta = \omega + \pi$ and $r = a\left(1+e\right)$ and when $\vartheta = \omega + \frac{\pi}{2}$, $r = a\left(1-e^2\right) = \frac{b^2}{a}$.

Abbreviating with $\left(\int_{\sin\vartheta}\right)$ and $\left(\int_{\cos\vartheta}\right)$ and using trigonometric identities we have

$$a\left(1-e\right) = \frac{\frac{L^2}{\mu}}{1 + \left(\int_{\sin\vartheta}\right)\cos\omega - \left(\int_{\cos\vartheta}\right)\sin\omega}$$

$$a\left(1+e\right) = \frac{\frac{L^2}{\mu}}{1 - \left(\int_{\sin\vartheta}\right)\cos\omega + \left(\int_{\cos\vartheta}\right)\sin\omega}$$

$$\frac{b^2}{a} = \frac{\frac{L^2}{\mu}}{1 - \left(\int_{\sin\vartheta}\right)\sin\omega - \left(\int_{\cos\vartheta}\right)\cos\omega}$$

giving

$$1 + \left(\int_{\sin\vartheta}\right)\cos\omega - \left(\int_{\cos\vartheta}\right)\sin\omega = \frac{L^2}{\mu a\left(1-e\right)} \tag{51}$$

$$1 - \left(\int_{\sin\vartheta}\right)\cos\omega + \left(\int_{\cos\vartheta}\right)\sin\omega = \frac{L^2}{\mu a\left(1+e\right)} \tag{52}$$

using $\frac{1}{a(1-e^2)} = \frac{a}{b^2} = \frac{\mu m^2}{L^2}$

$$1 - \left(\int_{\sin\vartheta}\right)\sin\omega - \left(\int_{\cos\vartheta}\right)\cos\omega = \frac{a}{b^2}\frac{L^2}{\mu} = 1$$

$$\left(\int_{\sin\vartheta}\right)\sin\omega = -\left(\int_{\cos\vartheta}\right)\cos\omega \tag{53}$$

To find L add 51 and 52

$$\frac{L^2}{\mu a\left(1-e\right)} + \frac{L^2}{\mu a\left(1+e\right)} = 2$$

$$\frac{L^2\left(1+e\right)}{\mu a\left(1-e^2\right)} + \frac{L^2\left(1-e\right)}{\mu a\left(1-e^2\right)} = 2$$

$$\frac{2L^2}{\mu a\left(1-e^2\right)} = 2$$

$$L^2 = \mu \frac{b^2}{a}$$

$$\boxed{L = b\sqrt{\frac{\mu}{a}}} \tag{54}$$

To find (...), multiply 51 by $\sin\omega$

$$\sin\omega + \left(\int_{\sin\vartheta}\right)\cos\omega\sin\omega - \left(\int_{\cos\vartheta}\right)\sin^2\omega = \frac{L^2}{\mu a\left(1-e\right)}\sin\omega$$

using 53

$$\sin\omega - \left(\int_{\cos\vartheta}\right)\cos^2\omega - \left(\int_{\cos\vartheta}\right)\sin^2\omega = \frac{L^2}{\mu a\left(1-e\right)}\sin\omega$$

$$\sin\omega - \left(\int_{\cos\vartheta}\right) = \frac{L^2}{\mu a\left(1-e\right)}\sin\omega = \frac{\left(\mu\frac{b^2}{a}\right)}{\mu a\left(1-e\right)}\sin\omega = \frac{b^2}{a^2\left(1-e\right)}\sin\omega$$

$$\left(\int_{\cos\vartheta}\right) = \left(1 - \frac{b^2}{a^2\left(1-e\right)}\right)\sin\omega = \left(1-\left(1+e\right)\right)\sin\omega$$

$$\boxed{\left(\int\Lambda\cos\vartheta d\vartheta\right) = -e\sin\omega}$$

using 53

$$\boxed{\left(\int\Lambda\sin\vartheta d\vartheta\right) = e\cos\omega}$$

Taking the derivative wrt ϑ in the above 2 equations and assuming both e and ω have a dependency on ϑ we have

$$\Lambda\cos\vartheta = \frac{dt}{d\vartheta}\frac{de}{dt}\frac{d}{de}\left(-e\sin\omega\right) + \frac{dt}{d\vartheta}\frac{d\omega}{dt}\frac{d}{d\omega}\left(-e\sin\omega\right)$$

$$\Lambda = -\frac{\sin\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{de}{dt} - \frac{e\cos\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{d\omega}{dt}$$

and

$$\Lambda\sin\vartheta = \frac{dt}{d\vartheta}\frac{de}{dt}\frac{d}{de}\left(e\cos\omega\right) + \frac{dt}{d\vartheta}\frac{d\omega}{dt}\frac{d}{d\omega}\left(e\cos\omega\right)$$

$$\Lambda = \frac{\cos\omega}{\sin\vartheta}\frac{dt}{d\vartheta}\frac{de}{dt} - \frac{e\sin\omega}{\sin\vartheta}\frac{dt}{d\vartheta}\frac{d\omega}{dt}$$

Setting the 2 sides equal

$$-\frac{\sin\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{de}{dt} - \frac{e\cos\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{d\omega}{dt} = \frac{\cos\omega}{\sin\vartheta}\frac{dt}{d\vartheta}\frac{de}{dt} - \frac{e\sin\omega}{\sin\vartheta}\frac{dt}{d\vartheta}\frac{d\omega}{dt}$$

$$-\frac{\sin\omega}{\cos\vartheta}\frac{de}{dt} - \frac{e\cos\omega}{\cos\vartheta}\frac{d\omega}{dt} = \frac{\cos\omega}{\sin\vartheta}\frac{de}{dt} - \frac{e\sin\omega}{\sin\vartheta}\frac{d\omega}{dt}$$

$$\left(-\frac{\sin\omega}{\cos\vartheta} - \frac{\cos\omega}{\sin\vartheta}\right)\frac{de}{dt} = -e\frac{d\omega}{dt}\left(\frac{\sin\omega}{\sin\vartheta} - \frac{\cos\omega}{\cos\vartheta}\right)$$

$$\frac{de}{dt} = e\frac{d\omega}{dt}\frac{\left(\frac{\sin\omega}{\sin\vartheta} - \frac{\cos\omega}{\cos\vartheta}\right)}{\left(\frac{\sin\omega}{\cos\vartheta} + \frac{\cos\omega}{\sin\vartheta}\right)}$$

$$= e\frac{d\omega}{dt}\frac{\left(\sin\omega\cos\vartheta - \cos\omega\sin\vartheta\right)}{\left(\sin\omega\sin\vartheta + \cos\omega\cos\vartheta\right)}$$

$$= e\frac{d\omega}{dt}\frac{\sin\left(\omega-\vartheta\right)}{\cos\left(\omega-\vartheta\right)}$$

$$\frac{de}{dt} = e\frac{d\omega}{dt}\tan(\omega - \vartheta)$$

$$\boxed{\frac{de}{dt} = -e\tan f\frac{d\omega}{dt}}$$

(The $\tan f$ term looks problematic but in the calculation below we end up with $\frac{d\omega}{dt}$ having cos terms which gets rid of the infinities - but I need to review this whole section)

Plugging this back into the formula for Λ

$$\Lambda = -\frac{\sin\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{de}{dt} - \frac{e\cos\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{d\omega}{dt}$$

$\frac{de}{dt} = -e\tan f\frac{d\omega}{dt} = -e\frac{d\omega}{dt}\tan(\vartheta - \omega)$

$$\Lambda = \frac{\sin\omega}{\cos\vartheta}\frac{dt}{d\vartheta}e\frac{1}{\cos f}\sin(\vartheta - \omega)\frac{d\omega}{dt} - e\frac{\cos\omega}{\cos\vartheta}\frac{dt}{d\vartheta}\frac{d\omega}{dt}$$

$$= e\frac{dt}{d\vartheta}\frac{d\omega}{dt}\left(\frac{\sin\omega}{\cos\vartheta}\frac{1}{\cos f}\sin(\vartheta - \omega) - \frac{\cos\omega}{\cos\vartheta}\right)$$

$$= e\frac{dt}{d\vartheta}\frac{d\omega}{dt}\left(\frac{\sin\omega}{\cos\vartheta}\frac{1}{\cos f}\sin(\vartheta - \omega) - \frac{\cos(\vartheta - \omega)\cos\omega}{\cos f\cos\vartheta}\right)$$

$$= e\frac{dt}{d\vartheta}\frac{d\omega}{dt}\frac{1}{\cos f\cos\vartheta}\left(\sin\omega\sin(\vartheta - \omega) - \cos(\vartheta - \omega)\cos\omega\right)$$

$$= e\frac{dt}{d\vartheta}\frac{d\omega}{dt}\frac{1}{\cos f\cos\vartheta}(-\cos\vartheta)$$

$$\boxed{\Lambda = -\frac{e}{\cos f}\frac{dt}{d\vartheta}\frac{d\omega}{dt}} \tag{55}$$

From

$$r = \frac{b^2}{a(1 + e\cos f)}$$

$$\frac{dr}{dt} = \frac{\partial}{\partial e}\left(\frac{b^2}{a(1 + e\cos f)}\right)\frac{de}{dt} + \frac{\partial}{\partial f}\left(\frac{b^2}{a(1 + e\cos f)}\right)\frac{df}{dt}$$

from $\left(\frac{f}{g}\right)' = \frac{f\prime g - fg\prime}{g^2}$ and $g = a(1 + e\cos f) = \frac{b^2}{r}$

$$= \left(\frac{-b^2 a\cos f}{\frac{b^4}{r^2}}\right)\frac{de}{dt} + \left(\frac{b^2 ae\sin f}{\frac{b^4}{r^2}}\right)\frac{df}{dt}$$

$$= \frac{ar^2}{b^2}\left(-\cos f\frac{de}{dt} + e\sin f\frac{df}{dt}\right)$$

from $f = \vartheta - \omega$

$$= \frac{ar^2}{b^2}\left(-\cos f\frac{de}{dt} + e\sin f\frac{d\vartheta}{dt} - e\sin f\frac{d\omega}{dt}\right)$$

$$= \frac{ar^2}{b^2}\left(-\cos f\left(-e\tan f\frac{d\omega}{dt}\right) + e\sin f\frac{d\vartheta}{dt} - e\sin f\frac{d\omega}{dt}\right)$$

$$= \frac{ar^2}{b^2}\left(e\sin f\frac{d\omega}{dt} + e\sin f\frac{d\vartheta}{dt} - e\sin f\frac{d\omega}{dt}\right)$$

$$= e\frac{a}{b^2}\sin f\left(r^2\frac{d\vartheta}{dt}\right) = e\frac{a}{b^2}\sin f(L) = e\frac{a}{b^2}\left(b\frac{\sqrt{\mu}}{\sqrt{a}}\right)\sin f$$

$$\boxed{\frac{dr}{dt} = e\frac{\sqrt{\mu a}}{b}\sin f} \tag{56}$$

$$\frac{d^2r}{dt^2} = \left(\frac{\sqrt{\mu a}}{b}\sin f\right)\frac{de}{dt} + \left(e\frac{\sqrt{\mu a}}{b}\cos f\right)\frac{df}{dt}$$

using $r^2\frac{d\vartheta}{dt} = L$ and $L = \frac{b\sqrt{\mu}}{\sqrt{a}}$ gives $\frac{d\vartheta}{dt} = \frac{b\sqrt{\mu}}{r^2\sqrt{a}}$.
Also $\vartheta = f + \omega$

$$= \left(\frac{\sqrt{\mu a}}{b}\sin f\right)\left(-e\tan f\frac{d\omega}{dt}\right) + \left(e\frac{\sqrt{\mu a}}{b}\cos f\right)\left(\frac{b\sqrt{\mu}}{r^2\sqrt{a}} - \frac{d\omega}{dt}\right)$$

$$= \left(-e\frac{\sqrt{\mu a}}{b}\frac{\sin^2 f}{\cos f} - e\frac{\sqrt{\mu a}}{b}\cos f\right)\frac{d\omega}{dt} + \frac{e\mu}{r^2}\cos f$$

$$\boxed{\frac{d^2r}{dt^2} = -e\frac{\sqrt{\mu a}}{b}\frac{1}{\cos f}\frac{d\omega}{dt} + \frac{e\mu}{r^2}\cos f} \tag{57}$$

Substituting 56 and 57 in 48

$$\Lambda = 3\frac{v_r^2}{c^2} - 6\frac{r\dot{v}_r}{c^2}$$

$$\Lambda = \frac{3e^2\mu a}{c^2b^2}\sin^2 f + \frac{6re\sqrt{\mu a}}{c^2b\cos f}\frac{d\omega}{dt} - \frac{6e\mu}{c^2r}\cos f \tag{58}$$

Equating 55 and 58

$$\frac{e}{\cos f}\frac{dt}{d\vartheta}\frac{d\omega}{dt} = -\frac{3e^2\mu a}{b^2c^2}\sin^2 f - \frac{6er\sqrt{\mu a}}{bc^2\cos f}\frac{d\omega}{dt} + \frac{6e\mu}{rc^2}\cos f$$

using $\frac{df}{dt} = \frac{b\sqrt{\mu}}{r^2\sqrt{a}}$. (Approximating $\frac{d\vartheta}{dt}$ with this - need to check)

$$\frac{er^2\sqrt{a}}{b\sqrt{\mu}\cos f}\frac{d\omega}{dt} = -\frac{3e^2\mu a}{b^2c^2}\sin^2 f - \frac{6er\sqrt{\mu a}}{bc^2\cos f}\frac{d\omega}{dt} + \frac{6e\mu}{rc^2}\cos f$$

multiply through by $\frac{b\sqrt{\mu}\cos f}{er^2\sqrt{a}}$

$$\boxed{\frac{d\omega}{dt} = -\frac{3\sqrt{a}\mu^{\frac{3}{2}}}{r^2c^2b}e\sin^2 f\cos f - \frac{6\mu}{rc^2}\frac{d\omega}{dt} + \frac{6\mu^{\frac{3}{2}}b}{r^3c^2\sqrt{a}}\cos^2 f} \tag{59}$$

multiply by $\frac{c^2}{3\mu}$

$$\frac{c^2}{3\mu}\frac{d\omega}{dt} = -\frac{\sqrt{a}\mu^{\frac{1}{2}}}{r^2b}e\sin^2 f\cos f - \frac{2}{r}\frac{d\omega}{dt} + \frac{2\mu^{\frac{1}{2}}b}{r^3\sqrt{a}}\cos^2 f$$

From $\frac{b^2}{a} = a\left(1-e^2\right)$ and $\frac{1}{r} = \frac{(1+ecosf)}{a(1-e^2)}$ we have $\frac{1}{r} = \frac{a(1+ecosf)}{b^2}$

$$\left(\frac{c^2}{3\mu} + \frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{dt} = -\left(\frac{a}{b^2}\right)^{\frac{1}{2}}\left(\frac{a}{b^2}\right)^2\left(1+e\cos f\right)^2\mu^{\frac{1}{2}}e\sin^2 f\cos f + 2\mu^{\frac{1}{2}}\left(\frac{a}{b^2}\right)^{-\frac{1}{2}}\left(\frac{a}{b^2}\right)^3\left(1+e\cos f\right)^3\cos^2 f$$

$$\left(\frac{c^2}{3\mu} + \frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{dt} = -\left(\frac{a}{b^2}\right)^{\frac{5}{2}}\left(1+e\cos f\right)^2\mu^{\frac{1}{2}}\left[e\sin^2 f\cos f - 2\left(1+e\cos f\right)\cos^2 f\right]$$

$$\left(\frac{c^2}{3\mu} + \frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{dt} = -\left(\frac{a}{b^2}\right)^{\frac{5}{2}}\left(1+e\cos f\right)^2\mu^{\frac{1}{2}}\left[e\left(1-\cos^2 f\right)\cos f - 2\cos^2 f - 2e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu} + \frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{dt} = -\left(\frac{a}{b^2}\right)^{\frac{5}{2}}\left(1+e\cos f\right)^2\mu^{\frac{1}{2}}\left[e\cos f - 2\cos^2 f - 3e\cos^3 f\right]$$

Using $n = \sqrt{\frac{\mu}{a^3}} \implies na^{\frac{3}{2}} = \mu^{\frac{1}{2}}$ and $\frac{df}{dM} = \left(1-e^2\right)^{\frac{1}{2}}\left(\frac{a}{r}\right)^2 =$

$$\frac{d\omega}{dt} = \frac{d\omega}{df}\frac{df}{dM}\frac{dM}{dt} = \frac{d\omega}{df}\left(1-e^2\right)^{\frac{1}{2}}a^2\left(\frac{a}{b^2}\left(1+e\cos f\right)\right)^2 n$$

$$\left(\frac{c^2}{3\mu}+\frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{df}\left(1-e^2\right)^{\frac{1}{2}}\left(\frac{a}{b}\right)^4\left(1+e\cos f\right)^2 n=-\left(\frac{a}{b^2}\right)^{\frac{5}{2}}\left(1+e\cos f\right)^2\mu^{\frac{1}{2}}\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu}+\frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{df}\frac{b}{a}\frac{1}{b^4}\left(a\right)^{\frac{8}{2}}n=-\frac{1}{b^5}\left(a\right)^{\frac{5}{2}}\mu^{\frac{1}{2}}\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu}+\frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{df}\frac{b^2}{a}a^{\frac{3}{2}}n=-\mu^{\frac{1}{2}}\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu}+\frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{df}\frac{b^2}{a}\mu^{\frac{1}{2}}=-\mu^{\frac{1}{2}}\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu}+\frac{2a\left(1+e\cos f\right)}{b^2}\right)\frac{d\omega}{df}\frac{b^2}{a}=-\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu}\frac{b^2}{a}+2\left(1+e\cos f\right)\right)\frac{d\omega}{df}=-\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\left(\frac{c^2}{3\mu}a\left(1-e^2\right)+2\left(1+e\cos f\right)\right)\frac{d\omega}{df}=-\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

If $\Upsilon=\frac{3\mu}{a(1-e^2)}$ then

$$\left(\frac{c^2}{\Upsilon}+2\left(1+e\cos f\right)\right)\frac{d\omega}{df}=-\left[e\cos f-2\cos^2 f-3e\cos^3 f\right]$$

$$\boxed{\frac{d\omega}{df}=\frac{-e\cos f+2\cos^2 f+3e\cos^3 f}{\frac{c^2}{\Upsilon}+\left(2+2e\cos f\right)}}\tag{60}$$

This isn't quite what Gerber has in [Gerber Theory] but it's close enough. If we integrate over $[0,2\pi]$ and just keep the dominant $\frac{c^2}{\Upsilon}$ term in the denominator we get

$$\Delta\omega=\frac{\Upsilon}{c^2}\left(\int_0^{2\pi}\left(-e\cos f+2\cos^2 f+3e\cos^3 f\right)\mathrm{d}f\right)$$

$$=\frac{\Upsilon}{c^2}2\pi$$

$$\boxed{\Psi\approx\frac{6\pi\mu}{c^2 a\left(1-e^2\right)}}$$

B.1 Mercury Radial Velocity

$$\frac{dr}{dt}=\frac{dr}{du}\frac{du}{d\lambda}\frac{d\lambda}{dt}$$

$$=-r^2.\left(-\eta e\sin\lambda\right).\frac{L}{mr^2}=\frac{\eta eL}{m}\sin\lambda$$

The maximum radial velocity of Mercury is at $\lambda=\pi$

$$\frac{dr}{dt}=\frac{\eta eL}{m}$$

$$\left(\frac{dr}{dt}\right)^2=\eta^2\frac{e^2}{m^2}L^2=\frac{1}{\left[a\left(1-e^2\right)\right]^2}\frac{e^2}{m^2}\left[\mu m^2 a\left(1-e^2\right)\right]$$

$$\left(\frac{dr}{dt}\right)^2=\frac{\mu e^2}{a\left(1-e^2\right)}$$

using 4.11

$$\frac{dr}{dt}=\sqrt{\frac{\mu e^2}{a\left(1-e^2\right)}}=10.05\text{ km/sec}$$

As a comparison, Mercury's orbital speed is about 50 km/sec.

B.2 Background

Still working through my thoughts on this. Gerber's potential formula is an interesting historical footnote. I haven't fully understood the reasoning behind it.

B.2.1 Lagrangian Mechanics Review

To arrive at the equations of motion, the aim is to minimise the action $L = T - V$ over a time interval. For more detail see [Kibble]. The below can be extended from cartesian coordinates (x, y, z) to generalised coordinates q_i (e.g. r, θ, ϕ) but I'll stay with just the x component of cartesian coordinates in the below.

$$I = \int_{t_0}^{t_1} L dt$$

The position, velocity and therefore the kinetic energy is a function of time $T(t) = \frac{1}{2}mv^2$. The x component of this is $T_x = \frac{1}{2}mv_x^2$. So considering just $T_x(t)$, the integral below is basically summing the area under the kinetic energy curve T_x over a time interval $[t_0, t_1]$.

$$I = \int_{t_0}^{t_1} T_x(t)\, dt$$

If we make a small change $\delta x(t)$ to $x(t)$ in the interval $[t_0, t_1]$ then we are going to change the kinetic energy T_x. Using $\dot{x}$ for v_x we have a new kinetic energy curve $T_x + \delta T_x = T_x + m\dot{x}\delta\dot{x}$.

$$\delta I = \int_{t_0}^{t_1} m\dot{x}\delta\dot{x}dt$$

We can integrate by parts $\int udv = uv - \int vdu$

$$\delta I = \int_{t_0}^{t_1} m\dot{x}\delta\dot{x}dt = m\left[\dot{x}\delta x|_{t_0}^{t_1} - \int_{t_0}^{t_1} \ddot{x}\delta x\right] dt$$

If we make a condition that $\delta x(t_0) = \delta x(t_1) = 0$ then the first integrated term vanishes and

$$\delta I = -\int_{t_0}^{t_1} m\ddot{x}\delta x dt$$

So the integrand is the work done δW in the x direction i.e. $\delta W = F_x \delta x$

$$\delta I = -\int_{t_0}^{t_1} F_x \delta x dt \tag{61}$$

On the other hand, assuming $T(x, \dot{x})$, the calculus of variations gives the change in $I = \int Tdt$ for a small change δx

$$\delta I = \int_{t_0}^{t_1} \left[\frac{\partial T}{\partial x} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right)\right] \delta x(t)\, dt \tag{62}$$

Equating 61 and 62 we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = F_x$$

If F_x can be written as $F_x = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) - \frac{\partial V}{\partial x}$ then we can say

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{x}}\right) - \frac{\partial V}{\partial x}$$

or

$$\frac{\partial(T - V)}{\partial x} - \frac{d}{dt}\left(\frac{\partial(T - V)}{\partial \dot{x}}\right) = 0$$

$$\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=0$$

(In the case V is conservative and only depends on x this would become $\frac{\partial T}{\partial x}-\frac{d}{dt}\frac{\partial T}{\partial \dot{x}}=\frac{\partial V}{\partial x}$).

This can be extended from a co-ordinate system $\{x,y,z\}$ to generalised co-ordinates (i.e. generalised momentum - the bit in brackets $\frac{d}{dt}(...)$ - and generalised forces) i.e.

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_i}}\right)=0 \tag{63}$$

i.e. instead of using Cartesian co-ordinates $q_i=\{x,y,z\}$ we can use curvilinear co-ordinates e.g. $\{r,\theta,\phi\}$ and if we express the Lagrangian in terms of these co-ordinates then 63 still holds. For co-ordinate system $\{r,\theta,\phi\}$, $q_1=r, q_2=\theta, q_3=\phi$. For a central force only r needs be considered i.e. $q_1=r, \dot{q_1}=v_r$

In Gerbers theory, the potential has a velocity dependency on v_r. So he assumes that the force can be expressed as $F=\frac{d}{dt}\left(\frac{\partial V}{\partial v_r}\right)-\frac{\partial V}{\partial r}$.

B.2.2 Velocity Dependant Forces

Other examples of velocity dependant forces are

1. The non-conservative friction force $F\propto v$.
2. The Lorentz force $\mathbf{F}=q\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)$ is an example of a force which is velocity dependant and can be expressed in Lagrangian form i.e. $F_i=\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q_i}}\right)-\frac{\partial V}{\partial q_i}$. The potential V is $V=q\phi\left(r,t\right)-q\mathbf{v}\cdot\mathbf{A}\left(r,t\right)$. ϕ is the scalar electrostatic potential. $\mathbf{A}$ is the magnetic vector potential such that $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\phi-\frac{\partial \mathbf{A}}{\partial t}$. This force is perpendicular to the velocity so does no work.

B.3 Weber Electrodynamics

There are also velocity and acceleration dependant forces attempts in electromagnetism. Weber's formulation of force between charges can be derived from a potential [Weber's. Electrodynamics]. (See also [Ampere's Force] and [Kirk - Hall Effect]). For generality let's introduce a constant k which for the Weber case we will later put $k=1$.

$$V=\frac{q_iq_j}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(1-k\frac{\dot{r_{ij}}^2}{2c^2}\right) \tag{64}$$

$$V_{k=1}=\frac{q_iq_j}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(1-\frac{\dot{r_{ij}}^2}{2c^2}\right) \tag{65}$$

This gives, using $\frac{d}{dr}\left(\dot{r_{ij}}^2\right)=2\dot{r_{ij}}\frac{d\dot{r_{ij}}}{dr}=2\dot{r_{ij}}\frac{d\dot{r_{ij}}}{dt}\frac{dt}{dr}=2\ddot{r_{ij}}$

$$F=-\frac{dV}{dr}=-\frac{q_iq_j}{4\pi\epsilon_0}\left[-\frac{1}{r_{ij}^2}\left(1-k\frac{\dot{r_{ij}}^2}{2c^2}\right)+\frac{1}{r_{ij}}k\left(-\frac{2\ddot{r_{ij}}}{2c^2}\right)\right]$$

$$=-\frac{q_iq_j}{4\pi\epsilon_0}\left[-\frac{1}{r_{ij}^2}\left(1-k\frac{\dot{r_{ij}}^2}{2c^2}\right)-\frac{1}{r_{ij}^2}k\left(\frac{r_{ij}\ddot{r_{ij}}}{c^2}\right)\right]$$

$$=\frac{q_iq_j}{4\pi\epsilon_0}\frac{1}{r_{ij}^2}\left[1-k\frac{\dot{r_{ij}}^2}{2c^2}+k\frac{r_{ij}\ddot{r_{ij}}}{c^2}\right]$$

In the case $k=1$

$$F=\frac{q_iq_j}{4\pi\epsilon_0}\frac{1}{r_{ij}^2}\left[1-\frac{\dot{r_{ij}}^2}{2c^2}+\frac{r_{ij}\ddot{r_{ij}}}{c^2}\right] \tag{66}$$

Note that this potential conserves energy because if we have

$$T=\frac{1}{2}m_i\dot{r_i}^2+\frac{1}{2}m_j\dot{r_j}^2$$

$$\frac{dT}{dt}=\dot{r_i}m_i\ddot{r_i}+\dot{r_j}m_j\ddot{r_j}$$

The force $m\ddot{r}$ is the same on both charges and in opposite directions

$$\frac{dT}{dt} = F_{ij}\left(\dot{r_j} - \dot{r_i}\right) = F_{ij}\dot{r_{ij}}$$

$$\begin{aligned}\frac{dV}{dt} &= \frac{q_i q_j}{4\pi\epsilon_0}\left[-\frac{1}{r_{ij}^2}\dot{r_{ij}}\left(1 - k\frac{\dot{r_{ij}}^2}{2c^2}\right) + \frac{1}{r_{ij}}k\left(\frac{\dot{r_{ij}}\ddot{r_{ij}}}{c^2}\right)\right] \\ &= -\frac{q_i q_j}{4\pi\epsilon_0}\frac{1}{r_{ij}^2}\dot{r_{ij}}\left[1 - k\frac{\dot{r_{ij}}^2}{2c^2} + k\frac{r_{ij}\ddot{r_{ij}}}{c^2}\right]\end{aligned}$$

from 66 ($k = 1$ but more generally for any k)

$$\frac{dV}{dt} = -\dot{r_{ij}}F_{ij}$$

so

$$\frac{dT}{dt} - \frac{dV}{dt} = 0$$

Comparing Weber and Gerber potentials we have

$$V = \frac{q_i q_j}{4\pi\epsilon_0}\frac{1}{r_{ij}}\left(1 - \frac{\dot{r_{ij}}^2}{2c^2}\right)$$

and

$$V = \frac{\mu}{r}\left[1 - 3\left(\frac{v_r}{c}\right)^2\right]$$

Comparing Weber and Gerber forces we have

$$F = \frac{q_i q_j}{4\pi\epsilon_0}\frac{1}{r_{ij}^2}\left[1 - \frac{\dot{r_{ij}}^2}{2c^2} + \frac{r_{ij}\ddot{r_{ij}}}{c^2}\right]$$

and

$$F = -\frac{\mu m}{r^2}\left[1 - 6\frac{v_r^2}{2c^2} + 6\frac{r\dot{v_r}}{c^2}\right]$$

C Energy in an Elliptic Orbit

From [Newtonian Mechanics]. Note for this section, we are switching to Physics notation for potential energy.

The energy in the orbit is constant so let us evaluate it at the apocentre $r_{max} = a\left(1 + e\right)$.

The potential energy is straightforward.

$$U\left(r_{max}\right) = -\frac{GMm}{a\left(1 + e\right)}$$

At the apocentre, the kinetic energy is purely tangential

$$K\left(r_{max}\right) = \frac{1}{2}mv_\theta^2 = \frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 \tag{67}$$

From Kepler's 2nd law, the satellite sweeps out equal areas in equal time. So we have for an ellipse

$$\frac{dA}{dt} = \frac{\pi ab}{T}$$

$$\left(\frac{dA}{dt}\right)^2 = \frac{\pi^2 a^2 b^2}{T^2}$$

Using Keplers 3rd law $T^2 = \frac{4\pi^2 a^3}{GM}$ and the definition of eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$

$$\left(\frac{dA}{dt}\right)^2 = \frac{\pi^2 a^2 b^2}{4\pi^2 a^3}GM = \frac{1}{4}GM\frac{b^2}{a}$$

$$\left(\frac{dA}{dt}\right)^2 = \frac{1}{4}GMa\left(1-e^2\right) \tag{68}$$

However from geometry we have

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$$

$$\left(\frac{dA}{dt}\right)^2 = \frac{1}{4}r^4\left(\frac{d\theta}{dt}\right)^2$$

therefore

$$r^2\left(\frac{d\theta}{dt}\right)^2 = \frac{4}{r^2}\left(\frac{dA}{dt}\right)^2$$

Plugging this into 67

$$K\left(r_{max}\right) = \frac{1}{2}mr^2\left(\frac{d\theta}{dt}\right)^2 = \frac{1}{2}m\left(\frac{4}{r^2}\left(\frac{dA}{dt}\right)^2\right) = \frac{2m}{r^2}\left(\frac{dA}{dt}\right)^2$$

and using 68

$$K\left(r_{max}\right) = \frac{2m}{r^2}\left(\frac{1}{4}GMa\left(1-e^2\right)\right) = \frac{m}{2r^2}\left(GMa\left(1-e^2\right)\right)$$

which at the apocentre gives

$$K\left(r_{max}\right) = \frac{GMma\left(1-e^2\right)}{2a^2\left(1+e\right)^2} = \frac{GMm\left(1-e\right)}{2a\left(1+e\right)}$$

The total energy is therefore

$$E = K\left(r_{max}\right) + U\left(r_{max}\right) = \frac{GMm\left(1-e\right)}{2a\left(1+e\right)} - \frac{GMm}{a\left(1+e\right)}$$

$$= \frac{GMm\left(1-e\right) - 2GMm}{2a\left(1+e\right)} = \frac{GMm\left(-1-e\right)}{2a\left(1+e\right)} = -\frac{GMm}{2a}$$

$$E = -\frac{GMm}{2a} \tag{69}$$

Note I could have got to this much quicker by using $v_\theta = \frac{L}{mr}$ (e.g. 72).

C.1 Orbital Velocity

From 69 we can calculate the orbital velocity

$$E = T + V$$

$$-\frac{GMm}{2a} = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$\frac{1}{2}mv^2 = GMm\left(\frac{1}{r} - \frac{1}{2a}\right)$$

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

D Centrifugal Force/Radial Acceleration

In polar coordinates we have

$$\boldsymbol{r} = r\boldsymbol{e}_r$$

Using $\frac{d\boldsymbol{e}_r}{dt} = \frac{d\lambda}{dt}\boldsymbol{e}_\lambda$ and $\frac{d\boldsymbol{e}_\lambda}{dt} = -\frac{d\lambda}{dt}\boldsymbol{e}_r$

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dr}{dt}\boldsymbol{e}_r + r\frac{d\boldsymbol{e}_r}{dt}$$

$$\boldsymbol{v} = \frac{dr}{dt}\boldsymbol{e}_r + r\frac{d\lambda}{dt}\boldsymbol{e}_\lambda \tag{70}$$

To get acceleration

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2r}{dt^2}\boldsymbol{e}_r + \frac{dr}{dt}\frac{d}{dt}\boldsymbol{e}_r + \frac{dr}{dt}\frac{d\lambda}{dt}\boldsymbol{e}_\lambda + r\frac{d^2\lambda}{dt^2}\boldsymbol{e}_\lambda + r\frac{d\lambda}{dt}\frac{d}{dt}\boldsymbol{e}_\lambda$$

$$\boldsymbol{a} = \frac{d^2r}{dt^2}\boldsymbol{e}_r + \frac{dr}{dt}\frac{d\lambda}{dt}\boldsymbol{e}_\lambda + \frac{dr}{dt}\frac{d\lambda}{dt}\boldsymbol{e}_\lambda + r\frac{d^2\lambda}{dt^2}\boldsymbol{e}_\lambda - r\frac{d\lambda}{dt}\frac{d\lambda}{dt}\boldsymbol{e}_r$$

$$\boldsymbol{a} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\lambda}{dt}\right)^2\right]\boldsymbol{e}_r + \left[r\frac{d^2\lambda}{dt^2} + 2\frac{dr}{dt}\frac{d\lambda}{dt}\right]\boldsymbol{e}_\lambda \tag{71}$$

Note e.g. the extra $\frac{dr}{dt}\frac{d\lambda}{dt}$ term in 71 than we might have expected from just $\frac{d}{dt}\left(r\frac{d\lambda}{dt}\boldsymbol{e}_\lambda\right)$ in 70.

All the above is general so far. If we now assume a *central* force is causing the acceleration, then there is no $\boldsymbol{e}_\lambda$ acceleration. This means

$$r\frac{d^2\lambda}{dt^2} + 2\frac{dr}{dt}\frac{d\lambda}{dt} = 0$$

and leads to the following (true for *any* central force)

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\lambda}{dt} = \frac{L}{2m} = constant \tag{72}$$

The radial acceleration is given by

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\lambda}{dt}\right)^2$$

Let us now assume that the acceleration is due to the inverse square law.

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\lambda}{dt}\right)^2 = -\frac{GM}{r^2}$$

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + r\left(\frac{d\lambda}{dt}\right)^2$$

D.1 The case where λ doesn't change

$\frac{d\lambda}{dt} = 0$, there is no angular motion and the satellite accelerates straight towards the centre as expected i.e.

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}$$

D.2 The case in the orbit where $r\left(\frac{d\lambda}{dt}\right)^2 = \frac{GM}{r^2}$

In this case we have a situation almost like $r\left(\frac{d\lambda}{dt}\right)^2$ is an equal and opposite force to $-\frac{GM}{r^2}$ and there is no acceleration i.e.

$$\frac{d^2r}{dt^2} = 0$$

Note that for a central force we have, from 72, $\frac{d\lambda}{dt} = \frac{L}{mr^2}$

$$r\frac{L^2}{m^2r^4} = \frac{GM}{r^2}$$

$$\frac{1}{r} = \frac{GMm^2}{L^2} = \eta = \frac{1}{a\left(1-e^2\right)}$$

i.e. at $r = a\left(1-e^2\right)$ (the semi-latus rectum - the radius at $\lambda = \pm 90^{\circ}$), $r\left(\frac{d\lambda}{dt}\right)^2 = \frac{GM}{r^2}$.

$\frac{dr}{dt}$ at the semi-latus rectum is constant (note not zero) so $\frac{d^2r}{dt^2} = \frac{d}{dt}\frac{dr}{dt} = \frac{dv}{dt} = 0$. At this r, the centrifugal force matches the gravitational attraction no matter what the e is.

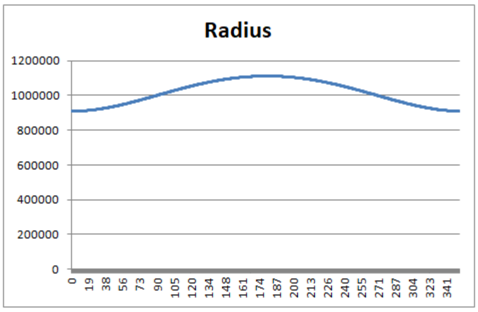
$\frac{d^2r}{dt^2} = 0$ at $\lambda = \pm 90^{\circ}$ also follows by just taking the time derivatives of $\frac{1}{r} = u = \eta\left(1 + e\cos\lambda\right)$ and using $\frac{d\lambda}{dt} = \frac{L}{mr^2}$.

In the case $e = 0$, we have $r = a$ i.e. circular motion.

The following graphs demonstrate how r changes.

Aside - In Excel, I've used $e = 0.1$ and $\eta = 0.000001$ (i.e $\frac{1}{r} = u = 0.000001\left(1 + 0.1\cos\lambda\right)$) and used $\lambda \approx M + \left(2e - \frac{1}{4}e^3\right)\sin M + \frac{5}{4}e^2\sin 2M + \frac{13}{12}e^3\sin 3M$. Then at intervals of M of 0.0048, calculate $\frac{1}{r}$. Then just take the step change in r to approximate v, and likewise take the step change in v to approximate a.

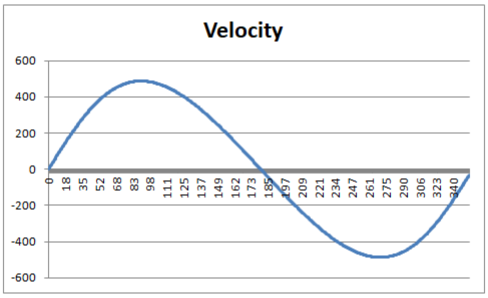
1) Radius - as expected it has it's minimum at $\lambda = 0$ and it's maximum at $\lambda = 180^{\circ}$



2) Velocity - as expected there is no radial velocity at $\lambda = 0$ and $\lambda = 180^{\circ}$ (pericentre and apocentre).

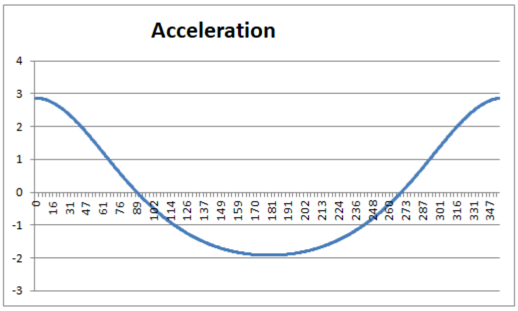
The maximum radial velocity is at $\lambda = 90^{\circ}$ i.e. the satellite is moving away from the central body at it's fastest rate.

The minimum radial velocity is at $\lambda = 270^{\circ}$ i.e. the satellite is moving towards the central body at it's fastest rate.



3) Acceleration - the maximum radial acceleration is at the pericentre $\lambda = 0^{\circ}$ and the minimum acceleration is at the apocentre $\lambda = 180^{\circ}$.

There is zero radial acceleration at $\lambda = 90^{\circ}$ and $\lambda = 270^{\circ}$. At this point the gravitational force and the centrifugal force cancel.



E True to Eccentric Anomaly

For this section, I'll use e instead of exp as there is no confusing e with eccentricity.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}}}{i\left(e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}\right)}$$

times through by $e^{\frac{i\theta}{2}}$

$$\tan\frac{\theta}{2} = \frac{e^{i\theta} - 1}{i\left(e^{i\theta} + 1\right)}$$

The relationship between true and eccentric anomaly is given by [Elliptic Orbits]

$$\tan\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\left(\frac{E}{2}\right)$$

Let $p = \sqrt{\frac{1+e}{1-e}}$

$$\tan\left(\frac{f}{2}\right) = p\tan\left(\frac{E}{2}\right)$$

$$\frac{e^{if} - 1}{i\left(e^{if} + 1\right)} = p\frac{\left(e^{iE} - 1\right)}{i\left(e^{iE} + 1\right)}$$

$$e^{if} - 1 = p\frac{\left(e^{iE} - 1\right)}{\left(e^{iE} + 1\right)}\left(e^{if} + 1\right)$$

$$e^{if} - 1 = p\frac{\left(e^{iE}e^{if} + e^{iE} - e^{if} - 1\right)}{\left(e^{iE} + 1\right)}$$

$$e^{if} - 1 = \frac{pe^{iE}e^{if} + pe^{iE} - pe^{if} - p}{\left(e^{iE} + 1\right)}$$

$$e^{if} - 1 = \frac{e^{if}\left(pe^{iE} - p\right) + pe^{iE} - p}{\left(e^{iE} + 1\right)}$$

$$\left(e^{if} - 1\right)\left(e^{iE} + 1\right) = e^{if}\left(pe^{iE} - p\right) + pe^{iE} - p$$

$$e^{if}e^{iE} + e^{if} - e^{iE} - 1 = e^{if}\left(pe^{iE} - p\right) + pe^{iE} - p$$

$$e^{if}\left(e^{iE} + 1\right) - e^{iE} - 1 = e^{if}\left(pe^{iE} - p\right) + pe^{iE} - p$$

Bring all e^{if} terms to the left

$$e^{if}\left(e^{iE} + 1 + p - pe^{iE}\right) = pe^{iE} - p + e^{iE} + 1$$

$$e^{if}\left((1+p) + (1-p)\,e^{iE}\right) = (1-p) + (1+p)\,e^{iE}$$

$$\boxed{e^{if} = \frac{(1-p) + (1+p)\,e^{iE}}{(1+p) + (1-p)\,e^{iE}}}$$

If we let $\beta = \frac{p-1}{p+1}$ and divide the above by $(1+p)$ we get

$$e^{if} = \frac{-\beta + e^{iE}}{1 - \beta e^{iE}}$$

$$\boxed{e^{if} = \frac{e^{iE}\left(1 - \beta e^{-iE}\right)}{1 - \beta e^{iE}}}$$

F Program to Calculate Coefficients of cosine term

Series Calculator

G Radius of a Spheroid (ellipse)

Because we are considering a spheroid which is got by rotating an ellipse around the z axis, there is no azimuthal λ dependency for r i.e. $r(\phi)$.

The polar form for the radius of the ellipse is given by

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}$$

$$r = \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 \left(1 - \sin^2 \phi\right)}}$$

$$r = \frac{ab}{\sqrt{(a^2 - b^2) \sin^2 \phi + b^2}}$$

$$r = \frac{ab}{b\sqrt{\frac{(a^2-b^2)}{b^2} \sin^2 \phi + 1}}$$

$$r = \frac{a}{\sqrt{\frac{(a^2-b^2)}{b^2} \sin^2 \phi + 1}}$$

$$r = \frac{a}{\sqrt{1 + \frac{(a^2-b^2)}{b^2} \sin^2 \phi}}$$

$$r = a\left[1 + \frac{\left(a^2 - b^2\right)}{b^2} \sin^2 \phi\right]^{-\frac{1}{2}}$$

$$r = a\left[1 + \frac{(a+b)(a-b)}{b^2} \sin^2 \phi\right]^{-\frac{1}{2}}$$

using $\epsilon = \frac{a-b}{R_\mu}$

$$r = a\left[1 + \frac{(a+b)\,\epsilon R_\mu}{b^2} \sin^2 \phi\right]^{-\frac{1}{2}}$$

using $a + b = (a - b) + 2b = \epsilon R_\mu + 2b$

$$r = a\left[1 + \frac{(\epsilon R_\mu + 2b)\,\epsilon R_\mu}{b^2} \sin^2 \phi\right]^{-\frac{1}{2}}$$

$$r = a\left[1 + \left(\frac{2b\epsilon R_\mu}{b^2} \sin^2 \phi + \frac{\epsilon^2 R_\mu^2}{b^2} \sin^2 \phi\right)\right]^{-\frac{1}{2}}$$

Binomial expansion of $(1+x)^{-\frac{1}{2}}$

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2$$

So binomially expand to x term

$$r \simeq a\left[1 - \frac{2b\epsilon R_\mu}{2b^2} \sin^2 \phi - \frac{\epsilon^2 R_\mu^2}{2b^2} \sin^2 \phi\right]$$

Aside. $\sin^2 \phi$ can be expressed in terms of Legendre Polynomials i.e. $P_2 = \frac{3}{2}\sin^2 \phi - \frac{1}{2} = \frac{3}{2}\sin^2 \phi - \frac{1}{2}P_0 \Rightarrow \frac{2}{3}P_2 = \sin^2 \phi - \frac{1}{3}P_0 \Rightarrow \sin^2 \phi = \frac{2}{3}P_2 + \frac{1}{3}P_0$

We will be expressing r as a Legendre Polynomial expansion, however we are just keeping $O(\epsilon)$ terms in our calculation of r

$$r \simeq a\left[1 - \frac{2\epsilon R_\mu}{2b}\sin^2\phi\right]$$

$$r \simeq a - \frac{a\epsilon R_\mu}{b}\sin^2\phi$$

$$r \simeq (2a+b) - \frac{a\epsilon R_\mu}{b}\sin^2\phi - (a+b)$$

using $R_\mu = \frac{2a+b}{3}$

$$r \simeq 3R_\mu - \frac{a\epsilon R_\mu}{b}\sin^2\phi - (a+b)$$

From the definition of flattening f Appendix L, $a/b \simeq 1 + f \simeq 1 + \epsilon$

$$r \simeq 3R_\mu - \epsilon R_\mu \sin^2\phi - (a+b)$$

$$r \simeq 3R_\mu - \frac{2}{3}\epsilon R_\mu \frac{1}{2} 3\sin^2\phi - (a+b)$$

Need $(a+b)$ in terms of $2R_\mu$

$$(a+b) = 2R_\mu + ?$$

$$? = (a+b) - 2R_\mu = (a+b) - 2\frac{(2a+b)}{3} = \frac{3(a+b) - 2(2a+b)}{3} = \frac{-(a-b)}{3} = \frac{-\epsilon R_\mu}{3}$$

so

$$(a+b) = 2R_\mu - \frac{\epsilon R_\mu}{3} \tag{73}$$

$$r \simeq 3R_\mu - \frac{2}{3}\epsilon R_\mu \frac{1}{2} 3\sin^2\phi - 2R_\mu + \frac{\epsilon R_\mu}{3}$$

$$r \simeq R_\mu - \frac{2}{3}\epsilon R_\mu \frac{1}{2} 3\sin^2\phi + \frac{1}{2}\frac{2}{3}\epsilon R_\mu$$

$$r \simeq R_\mu - \frac{2}{3}\epsilon R_\mu \left[\frac{1}{2}\left(3\sin^2\phi - 1\right)\right]$$

$$r \simeq R_\mu\left[1 - \frac{2}{3}\epsilon P_2(\sin\phi)\right]$$

G.1 J_2 term

We have an oblate (flattened) spheroid obtained by rotating an ellipse with semi-major axis a_e and semi-minor axis b_e around around it's minor axis b.

B=The moment of Inertia around b is $\frac{2}{5}Ma_e^2$

A=The moment of Inertia around a is $\frac{1}{5}M\left(a_e^2 + b_e^2\right)$

Note also that J_2 in [GeoPotential Model] is $\frac{(B-A)}{Ma^2}$ (see [GRS80]) where A and B are the principal moments of inertia of the ellipsoid. Note also that J_2 is nondimensional.

$$B - A = \frac{2}{5}Ma_e^2 - \frac{1}{5}M\left(a_e^2 + b_e^2\right) = \frac{1}{5}M\left(a_e^2 - b_e^2\right)$$

The following uses 80 and 73

$$J_2 = \frac{1}{5a_e^2}\left(a_e^2 - b_e^2\right) = \frac{1}{5a_e^2}(a_e - b_e)(a_e + b_e) = \frac{1}{5a_e^2}\epsilon R_\mu (a_e + b_e) = \frac{1}{5a_e^2}\epsilon R_\mu\left(2R_\mu - \frac{\epsilon R_\mu}{3}\right) = \frac{2}{5a_e^2}\epsilon R_\mu^2 + O(\epsilon^2) \simeq \frac{2}{5}\epsilon$$

H Contour Integrals

[Boas] gives a good overview of Contour Integrals and the Residue Theorem.

H.1 Contour Integral I

From [Boas] and [Contour Integral]. Note e is the eccentricity, which is < 1.

$$I = \int_0^{2\pi} \frac{1}{1 - e\cos\theta} d\theta$$

Using $z = \exp(i\theta)$ and $dz = izd\theta$

$$= \oint \frac{1}{iz} \left(\frac{1}{1 - e\left(\frac{z+\frac{1}{z}}{2}\right)} \right) dz$$

$$= \frac{1}{i} \oint \left(\frac{1}{z - \frac{e}{2}z^2 - \frac{e}{2}} \right) dz$$

$$I = -\frac{2}{ie} \oint \left(\frac{1}{z^2 - \frac{2}{e}z + 1} \right) dz$$

$$f(z) = \frac{1}{z^2 - \frac{2}{e}z + 1}$$

The roots to the denominator are

$$z = \frac{\frac{2}{e} \pm \sqrt{\frac{4}{e^2} - 4}}{2}$$

$$z_+ = \frac{1}{e} + \sqrt{\frac{1}{e^2} - 1}$$

$$z_- = \frac{1}{e} - \sqrt{\frac{1}{e^2} - 1}$$

These are 2 simple poles. We want the one that is < 1. Since $e < 1$ the root that lies within the contour is z_-. The residue is

$$R(f, z_0) = \frac{1}{z_- - z_+} = -\frac{1}{2\sqrt{\frac{1}{e^2} - 1}}$$

The Residue theorum is

$$\oint f(z)\, dz = 2\pi i R(f, z_0)$$

so we have

$$I = -\frac{2}{ie} \oint f(z)\, dz = -\frac{2}{ie} 2\pi i R(f, z_0) = \frac{2}{ie} 2\pi i \frac{1}{2\sqrt{\frac{1}{e^2} - 1}}$$

$$I = \frac{2\pi}{e} \frac{1}{\sqrt{\frac{1}{e^2} - 1}}$$

$$I = \frac{2\pi}{\sqrt{1 - e^2}}$$

$$\boxed{\int_0^{2\pi} \frac{1}{1 - e\cos\theta} \mathrm{d}\theta = \frac{2\pi}{\sqrt{1 - e^2}}} \tag{74}$$

H.2 Contour Integral II

I haven't spent much time on Complex Analysis so there are doubtless much more elegant derivations but the below gets there eventually.

$$I = \int_0^{2\pi} \frac{1}{(1 - e\cos\theta)^2} d\theta$$

$$I = \oint \frac{1}{iz} \left(\frac{1}{1 - e\left(\frac{z+\frac{1}{z}}{2}\right)} \right)^2 dz$$

$$I = \oint \frac{1}{iz} \left(\frac{z}{z - \frac{e}{2}z^2 - \frac{e}{2}} \right)^2 dz$$

$$I = \oint \frac{1}{iz} \left(\frac{4}{e^2} \right) \left(\frac{z}{z^2 - \frac{2}{e}z + 1} \right)^2 dz$$

From H.1

$$f(z) = \left(\frac{4}{ie^2 z} \right) \frac{z^2}{(z - z_-)^2 (z - z_+)^2}$$

$$f(z) = \left(\frac{4}{ie^2} \right) \frac{z}{(z - z_-)^2 (z - z_+)^2}$$

From [Boas] section on multiple poles, "Multiply $f(z)$ by $(z - z_0)^m$ where m is an integer greater or equal to the order n of the pole. Differentiate $m - 1$ times, divide by $(m-1)!$ and evaluate resulting expression at $z = z_0$".

So choosing $m = 2$, multiply through by $(z - z_-)^2$

$$R(z_-) = \left(\frac{4}{ie^2} \right) \frac{d}{dz} \frac{z}{(z - z_+)^2} \bigg|_{z_-}$$

$d\left(\frac{f}{g}\right) = \frac{f'g - g'f}{g^2}$ and $\frac{1}{z_- - z_+} = -\frac{1}{2\sqrt{\frac{1}{e^2} - 1}}$

$$= \left(\frac{4}{ie^2} \right) \left[\frac{\left((z - z_+)^2\right) - (2(z - z_+))z}{(z - z_+)^4} \right]$$

$$= \left(\frac{4}{ie^2} \right) \left[\left(-\frac{1}{2\sqrt{\frac{1}{e^2} - 1}} \right)^2 - 2\left(-\frac{1}{2\sqrt{\frac{1}{e^2} - 1}} \right)^3 \left(\frac{1}{e} - \sqrt{\frac{1}{e^2} - 1} \right) \right]$$

$$= \left(\frac{4}{ie^2} \right) \left[\frac{1}{4} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^2 + \frac{1}{4} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^3 \left(\frac{1}{e} - \sqrt{\frac{1}{e^2} - 1} \right) \right]$$

$$= \left(\frac{4}{ie^2} \right) \left[\frac{1}{4} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^2 + \frac{1}{4e} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^3 - \frac{1}{4} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^2 \right]$$

$$= \left(\frac{4}{ie^2} \right) \left[\frac{1}{4e} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^3 \right]$$

$$= \frac{1}{i} \left[\frac{1}{e^3} \left(\frac{1}{\sqrt{\frac{1}{e^2} - 1}} \right)^3 \right]$$

$$= \frac{1}{i}\left[\left(\frac{1}{e\sqrt{\frac{1}{e^2}-1}}\right)^3\right]$$

$$R\left(z_-\right) = \frac{1}{i}\left[\left(\frac{1}{\sqrt{1-e^2}}\right)^3\right]$$

$$2\pi i R\left(z_-\right) = \frac{2\pi}{\left(1-e^2\right)^{\frac{3}{2}}}$$

$$\boxed{\int_0^{2\pi} \frac{1}{\left(1-e\cos\theta\right)^2} \mathrm{d}\theta = \frac{2\pi}{\left(1-e^2\right)^{\frac{3}{2}}}} \tag{75}$$

I Bessel Functions

I.1 Bessel's Equation

In 25 we came across the integral representation of the Bessel Function i.e.

$$J_n\left(e\right) = \frac{1}{2\pi}\int_0^{2\pi} \exp(i\left(e\sin E - nE\right))dE$$

This is the solution to the Bessel equation below - using y for J and x for e

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0$$

To show that J is the solution (cf [Bessel Math Stack])

$$J_n\left(e\right) = \frac{1}{2\pi}\int_0^{2\pi} \exp(i\left(e\sin E - nE\right))dE = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(e\sin E - nE\right) + i\sin\left(e\sin E - nE\right) dE$$

Looking at the cos term first

$$J_1 = \frac{1}{2\pi}\int_0^{2\pi} \cos\left(e\sin E - nE\right)$$

$$J_1' = -\frac{1}{2\pi}\int_0^{2\pi} \sin E \sin\left(e\sin E - nE\right)$$

$$J_1'' = \frac{1}{2\pi}\int_0^{2\pi} -\sin^2 E\cos\left(e\sin E - nE\right) = \frac{1}{2\pi}\int_0^{2\pi} \left(-1+\cos^2 E\right)\cos\left(e\sin E - nE\right)$$

$$= -J_1 + \frac{1}{2\pi}\int_0^{2\pi} \cos^2 E\cos\left(e\sin E - nE\right)$$

$$\implies J_1'' + J_1 = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 E\cos\left(e\sin E - nE\right)$$

So

$$e^2 J_1'' + eJ_1' + \left(e^2 - n^2\right) J_1$$

$$= \left[e^2\left(J_1'' + J_1\right) - n^2 J_1\right] + eJ_1'$$

$$= \left[\frac{1}{2\pi}\int_0^{2\pi} \left(e\cos E + n\right)\left(e\cos E - n\right)\cos\left(e\sin E - nE\right)\right] + eJ_1'$$

Integrating by parts $\int udv = uv - \int vdu$ with $u = \left(e\cos E + n\right)$ and $dv = \left(e\cos E - n\right)\cos\left(e\sin E - nE\right)$

$$v = \int dv = \sin\left(e\sin E - nE\right)$$

$$du = -e\sin E$$

$$= \frac{1}{2\pi}\left[\left[(e\cos E+n)\sin(e\sin E-nE)\right]_0^{2\pi}+\int_0^{2\pi} e\sin E\sin(e\sin E-nE)\right]+eJ_1'$$

$$= e\left[\frac{1}{2\pi}\int_0^{2\pi}\sin E\sin(e\sin E-nE)\right]+eJ_1'$$

$$= -eJ_1^{'}+eJ_1'=0$$

For the second sine term

$$J_1=\frac{1}{2\pi}\int_0^{2\pi}\sin(e\sin E-nE)$$

The integrand is of the form

$$\sin(e\sin x-nx)$$

f=$e\sin x-nx$ is an odd function. And $g=\sin(x)$ is also an odd function. So $f(g(x))=\sin(e\sin x-nx)$ is also an odd function.

Using $\sin(a+b)=\cos b\sin a+\cos a\sin b$.
Using $\sin(\pi-x)=\sin x$
Using $\sin(\pi+x)=-\sin x$
So for $\pi-x$.

$$\sin\left[e\sin(\pi-x)+n(\pi-x)\right]$$

$$=\sin\left[(e\sin(x)-nx)+n\pi\right]$$

$$=\cos n\pi\sin(e\sin(x)-nx)+\cos(e\sin(x)-nx)\sin n\pi$$

$$=\cos n\pi\sin(e\sin(x)-nx)$$

and for $\pi+x$

$$\sin\left[e\sin(\pi+x)+n(\pi+x)\right]$$

$$=\sin\left[-e\sin(x)+n\pi+nx\right]$$

$$=\sin\left[-(e\sin(x)-nx)+n\pi\right]$$

$$=-\cos n\pi\sin(e\sin(x)-nx)+\cos(e\sin(x)-nx)\sin n\pi$$

$$=-\cos n\pi\sin(e\sin(x)-nx)$$

so by the symmetry about π, the integral from 0 to 2π is 0.

I.2 Generating Function

As in 26 we can write the Bessel function as a series solution (e here is the eccentricity of the orbit).

$$J_n(e)=\left(\frac{e}{2}\right)^n\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{e}{2}\right)^{2m}}{m!\,(n+m)!}$$

Let's see how this is related to what is called a generating function below (in the below e is the exp function). cf [Lambers]

$$g(x,z)=e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)}=e^{\left(\frac{xz}{2}\right)}e^{\left(-\frac{x}{2z}\right)}$$

$$=\sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{xz}{2}\right)^l\sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{x}{2z}\right)^m$$

$$= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{x}{2}\right)^{l} z^{l} \sum_{m=0}^{\infty} \frac{1}{m!} \left(-1\right)^{m} \left(\frac{x}{2}\right)^{m} z^{-m}$$

$$= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(-1\right)^{m}}{l!m!} \left(\frac{x}{2}\right)^{l+m} z^{l-m}$$

Let us consider the case where $l \geq m$, and let us introduce $n = l - m$ i.e. $n \geq 0$. The first case we have is $n = 0$ i.e

l	0	1	2	3
m	0	1	2	3

Next $n = 1$

l	1	2	3	4
m	0	1	2	3

Next $n = 2$

l	2	3	4	5
m	0	1	2	3

So from the above we can see that the summation can be expressed as a sum over n and m. We can write the summation as

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\left(-1\right)^{m}}{m!\left(n+m\right)!} \left(\frac{x}{2}\right)^{n+2m} z^{n}$$

$$= \sum_{n=0}^{\infty} \left[\left(\frac{x}{2}\right)^{n} \sum_{m=0}^{\infty} \frac{\left(-1\right)^{m} \left(\frac{x}{2}\right)^{2m}}{m!\left(n+m\right)!} \right] z^{n}$$

The bit in the brackets is $J_n\left(x\right)$.

More generally we have

$$g\left(x,z\right) = e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)} = \sum_{n=-\infty}^{\infty} J_n\left(x\right) z^{n}$$

I.3 Derivation of Integral Form

To see how we can derive the integral form of the Bessel function from the generating function (use m for the summation index)

$$e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)} = \sum_{m=-\infty}^{\infty} J_m\left(x\right) z^{m}$$

divide both side by z^{n+1}

$$\frac{e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)}}{z^{n+1}} = \sum_{m=-\infty}^{\infty} J_m\left(x\right) z^{m-n-1}$$

The RHS is of the form of a Laurent Series i.e.

$$\begin{aligned} f\left(z\right) &= a_0 + a_1\left(z-z_0\right) + a_2\left(z-z_0\right)^2 + a_3\left(z-z_0\right)^3 + ... \\ &+ \frac{b_1}{\left(z-z_0\right)} + \frac{b_2}{\left(z-z_0\right)^2} + \frac{b_3}{\left(z-z_0\right)^3} + ... \end{aligned}$$

and do a contour integral choosing C to be the unit circle centred at $z_0 = 0$

$$\oint_C \frac{e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)}}{z^{n+1}} dz = \oint_C \sum_{m=-\infty}^{\infty} J_m\left(x\right) z^{m-n-1} dz$$

We can use the residue theorum for the second term as there is a pole if $n = m$ and the residue is just the coefficient of $\left(z-z_0\right)^{-1}$

$$\oint_C \frac{e^{\left(\frac{x}{2}\right)\left(z-\frac{1}{z}\right)}}{z^{n+1}} dz = 2\pi i J_n\left(x\right)$$

In the first integral, substitute $z = e^{i\theta}$ and $dz = ie^{i\theta}d\theta$

$$\int_0^{2\pi} \frac{e^{x\left(\frac{e^{i\theta}-e^{-i\theta}}{2}\right)}}{e^{in\theta}e^{i\theta}} ie^{i\theta}d\theta = 2\pi i J_n\left(x\right)$$

$$i\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}d\theta = 2\pi i J_n\left(x\right)$$

$$J_n\left(x\right) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(x\sin\theta - n\theta)}d\theta$$

which is the integral form of the Bessel function.

I.3.1 Contour Integration

The reason why $\oint_C \frac{1}{z}dz = 2\pi i$ is that we are integrating in 2 dimensions along a path in the complex plane. In the case the path is a unit circle, the real component is $x = \cos\theta$ and the imaginary component is $y = \sin\theta$ so

$$z = x + iy = \cos\theta + i\sin\theta$$

and

$$dz = \left(-\sin\theta + i\cos\theta\right)d\theta = -\left(\sin\theta - i\cos\theta\right)d\theta$$

$$\frac{1}{z}dz = -\frac{\left(\sin\theta - i\cos\theta\right)}{\left(\cos\theta + i\sin\theta\right)}d\theta = -\left(\frac{i}{i}\right)\frac{\left(\sin\theta - i\cos\theta\right)}{\left(\cos\theta + i\sin\theta\right)}d\theta$$
$$= i\frac{\left(\sin\theta - i\cos\theta\right)}{\left(-i\cos\theta + \sin\theta\right)}d\theta = id\theta$$

It's more obvious by making the substitution $z = e^{i\theta}$ and therefore $dz = ie^{i\theta}d\theta$ then the complete integral is

$$\oint_C \frac{1}{z}dz = \int_0^{2\pi} e^{-i\theta}ie^{i\theta}d\theta = i\int_0^{2\pi} d\theta = 2\pi i$$

J Mean Radius

This text uses R_μ - the "mean radius" of an ellipsoid (note - not ellipse). It's not obvious what the definition of "mean radius" is. From [Mean Radius Wikipedia]

"In geophysics, the International Union of Geodesy and Geophysics (IUGG) defines the mean radius to be $R_\mu = \frac{2a+b}{3}$"

This is the arithmetic mean of the radius i.e. $\left(a + a + b\right)/3$.

K Semi-latus rectum

Since the solution for the the orbital motion due to the inverse square law is

$$\frac{1}{r} = u = \eta(1 + e\cos\lambda) \tag{76}$$

where $\eta = \frac{GMm^2}{L^2}$, then for $\lambda = 0$ (the periapse), $u = \eta\left(1 + e\right)$. But we know at the periapse $r = \frac{1}{u} = a\left(1 - e\right)$ i.e. $u = \frac{1}{a(1-e)}$. Therefore

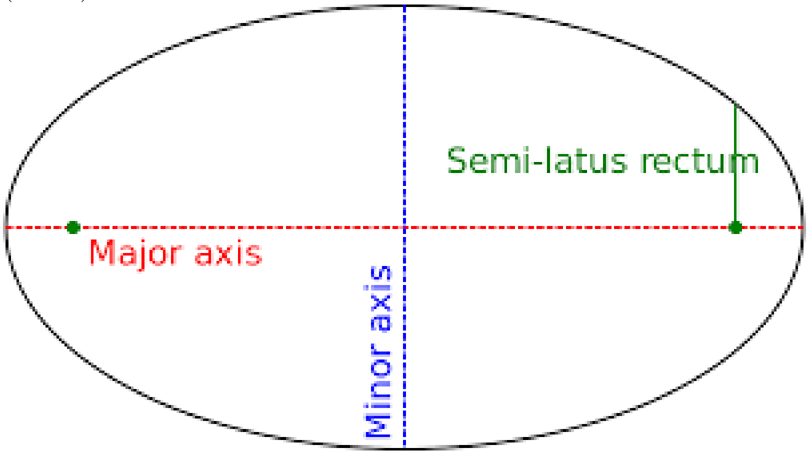
$$\eta\left(1 + e\right) = \frac{1}{a\left(1 - e\right)}$$

$$\eta = \frac{1}{a\left(1 + e\right)\left(1 - e\right)} = \frac{1}{a\left(1 - e^2\right)}$$

i.e.

$$\eta = \frac{GMm^2}{L^2} = \frac{1}{a\left(1-e^2\right)}$$

Note the semi-latus rectum of the ellipse is given by $l = \frac{1}{\eta} = a\left(1-e^2\right)$. We can see that from 76 that this is $r(\lambda = \pi)$.



For Mercury we have

$$\eta = \frac{1}{57,910,000\left(1-0.2056^2\right)}$$

$$\eta = \frac{1}{55,462,066} = 1.803\left(10^{-8}\right)\ km^{-1}$$

L Notation

The notation used varies across different authors/texts. Below is listed some definitions.

a_e semi-major axis of central body/spheroid.
b_e semi-minor axis of central body/spheroid.
a semi-major axis of satellite's orbital ellipse.
b semi-minor axis of satellite's orbital ellipse.
Eccentricity: This is a property of the orbiting satellite.

$$e = \sqrt{\frac{a^2-b^2}{a^2}} \tag{77}$$

Mean Radius of an ellipsoid (not ellipse): see Appendix J

$$R_\mu = \frac{2a_e + b_e}{3} \tag{78}$$

Flattening: This is a property of the central body i.e. spheroid.

$$f = \frac{(a_e - b_e)}{a_e} \tag{79}$$

Ellipticity: This is a property of the central body i.e. spheroid.

$$\epsilon = \frac{(a_e - b_e)}{R_\mu} \tag{80}$$

Note the previous two (flattening and ellipticity) are very similar and are equivalent to $O\left(\epsilon^2\right)$ or $O\left(f^2\right)$ i.e.

$$\epsilon = \frac{3\left(a_e - b_e\right)}{2a_e + b_e}$$

using $b_e = a_e\left(1-f\right)$

$$\epsilon = \frac{3\left(a_e - \left(a_e - a_e f\right)\right)}{2a_e + \left(a_e - a_e f\right)} = \frac{3a_e f}{3a_e - a_e f}$$

Rearrange

$$\frac{f}{\epsilon}=\frac{3a_e-a_ef}{3a_e}=1-\frac{f}{3}$$

$$\epsilon=f\left(1-\frac{f}{3}\right)^{-1}\simeq f+O\left(f^2\right)$$

The following r_0 is seen in [Kaula]. $r_0=a_e\left(1-\frac{f}{3}+O\left(f^2\right)\right)$ which gives the same as R_μ
The following is used in [Kaula]. a_e mean equatorial radius of the earth.
Gravitational constant k or G
The following is used in [Kaula]. $\mu=kM$

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