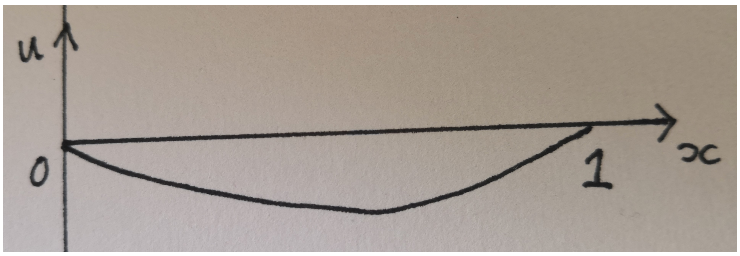
1 Notes on Green's Functions

1.1 Loaded String

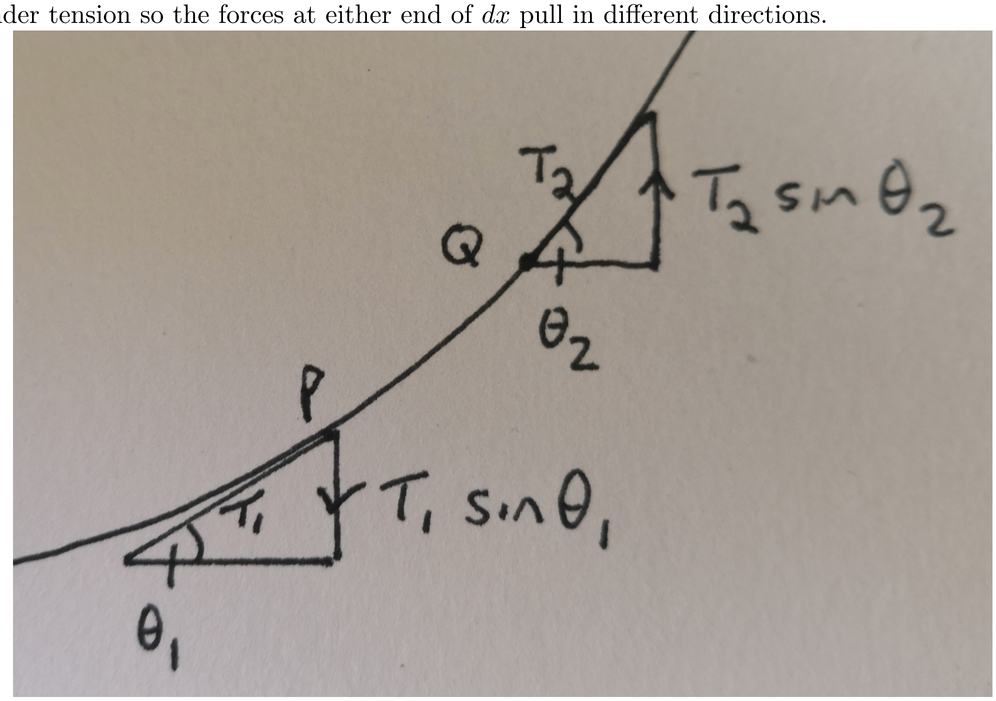


The equation for the displacement $u(x)$ of a string under tension that is subject to a force per unit length $\phi(x)$ is

$$u''(x) = \phi(x) \tag{1}$$

There are a number of available proofs of this but essentially the argument is a follows. Let us consider a string that is displaced downwards by a force.

Consider a small piece of the string dx between points P and Q. The tension in the string at P is T_1 and the tension in the string at Q is T_2. The string is under tension so the forces at either end of dx pull in different directions.



Since the string element dx does not move horizontally, the horizontal component of the tension at point P and point Q must be equal (let us call this constant C) i.e.

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = C \tag{2}$$

Additionally, because the string is stationary, the net vertical component upwards of the tension on element dx must be equal to the net force downwards on dx i.e.

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = \phi(x)\, dx \tag{3}$$

Dividing 3 by the constant C but using the appropriate term in 2

$$\frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} - \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \frac{\phi(x)\, dx}{C}$$

$$\tan\theta_2 - \tan\theta_1 = \frac{\phi(x)\, dx}{C}$$

But the tangent is just the gradient.

$$\frac{du}{dx}|_Q - \frac{du}{dx}|_P = \frac{\phi(x)\, dx}{C}$$

$$u'_Q - u'_P = \frac{\phi(x)\, dx}{C}$$

$$\frac{\left(u'_Q - u'_P\right)}{dx} = \frac{\phi(x)}{C}$$

Including the constant C into the function ϕ,

$$\frac{\left(u'_Q - u'_P\right)}{dx} = \phi(x) \tag{4}$$

we get equation 1.

Note the embedded constant C on the RHS. Intuitively this must exist as the displacement of the string must be related to the tension in the string.

For the following sections I'll use the form

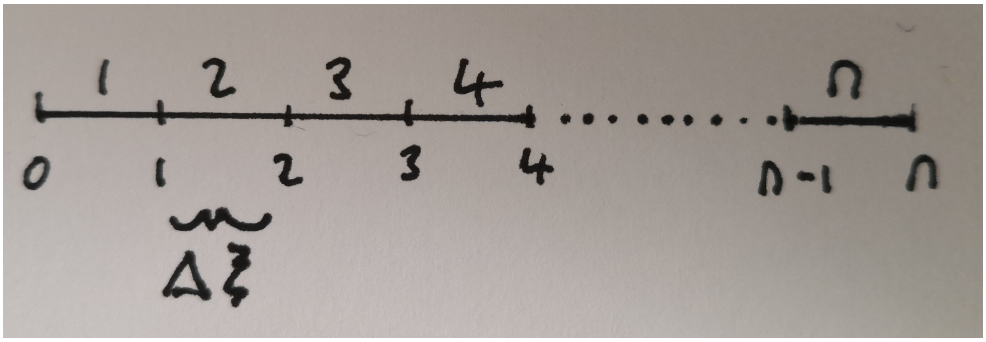
$$u'_Q - u'_P = \phi(x)\, dx \tag{5}$$

1.2 Unit Force

Let the length of the string be 1 and be fixed at both ends.

$$u(0) = u(1) = 0$$

We'll consider n segments of the string. Each segment then has length $= \frac{1}{n}$. Let us call this length $\Delta\xi$. A segment is identified using the index i where $i = 1..n$ and is the interval $[i-1, i]$

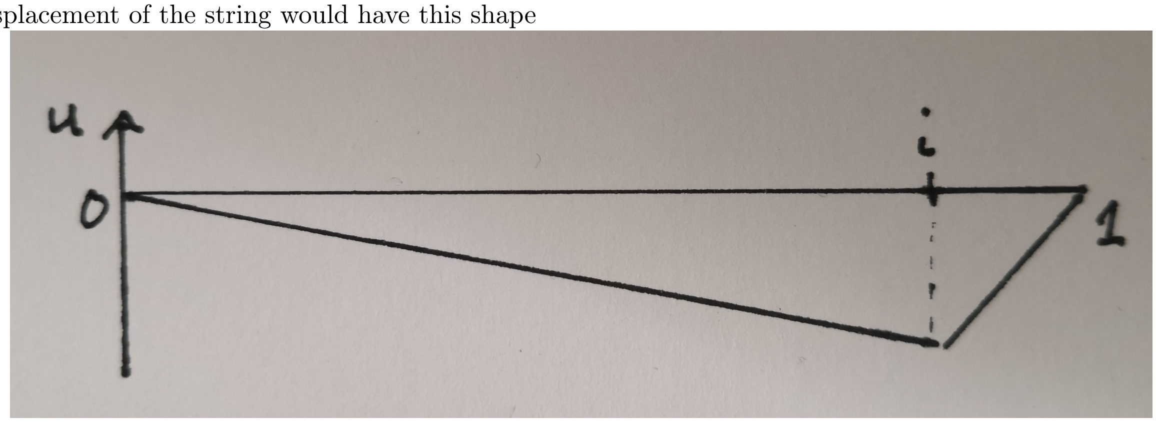


From 5 we have for the i^{th} segment

$$\Delta u_i' = u_i' - u_{i-1}' = \phi\,[i-1,i]\,\Delta\xi$$

From this and the derivation above we can see the change in gradient at segment i only depends on the force per unit length ϕ over the interval $[i-1,i]$.

If we were to make ϕ equal to 0 *outside* the interval $[i-1,i]$ then the RHS becomes 0 and $\Delta u' = 0$ outside the interval $[i-1,i]$. This would imply the displacement of the string would have this shape



From 5 we have

$$u_i' - u_{i-1}' = \phi\,[i-1,i]\,\Delta\xi = F \tag{6}$$

So if the gradient at $i-1$ is m_1 and the gradient at i is m_2 then from 6 we must have

$$m_2 - m_1 = F$$

$$\implies m_1 = m_2 - F$$

We also know that the 2 slopes meet at a point in the interval $[i-1,i]$. Let us call this point ξ. Also the length of the string is 1. u is displaced in the -ve direction so

$$m_1 = \frac{\Delta u}{\xi}$$

$$m_2 = -\frac{\Delta u}{1-\xi}$$

$$\implies \frac{m_1}{m_2} = -\frac{1-\xi}{\xi} = \frac{\xi - 1}{\xi}$$

$$\implies \frac{m_2 - F}{m_2} = \frac{\xi - 1}{\xi}$$

$$\implies -\frac{F}{m_2} = \frac{\xi - 1}{\xi} - 1 = \frac{\xi - 1 - \xi}{\xi} = -\frac{1}{\xi}$$

$$m_2 = F\xi$$

and so

$$m_1 = F\xi - F = F\left(\xi - 1\right)$$

For the first slope

$$u_1\left(x\right) = F\left(\xi - 1\right)x + c_1$$

but $u\left(0\right) = 0$ so $c_1 = 0$ so

$$u_1\left(x\right) = F\left(\xi - 1\right)x$$

For the second slope

$$u_2\left(x\right) = F\xi x + c_2$$

but $u\left(1\right) = 0$ so $c_2 = -F\xi$ so

$$u_2\left(x\right) = F\xi x - F\xi = F\xi\left(x - 1\right)$$

which gives

$$u\left(x\right) = F\begin{cases}\left(\xi - 1\right)x & x \leq \xi \\ \left(x - 1\right)\xi & x \geq \xi\end{cases}$$

For the special case $F = \phi\left(\xi\right)\Delta\xi = 1$ then F is a unit force. Let us use G for the this case, rather than u.

$$G\left(x,\xi\right) = \begin{cases}\left(\xi - 1\right)x & x \leq \xi \\ \left(x - 1\right)\xi & x \geq \xi\end{cases}$$

so

$$u\left(x\right) = FG\left(x,\xi\right) = G\left(x,\xi\right)\phi\left(\xi\right)\Delta\xi$$

1.3 Symmetry of $G(\xi, x)$

$G(x,\xi)$ is the displacement at x due to a unit force at ξ.

$$G(x,\xi) = \begin{cases} (\xi-1)\,x & x \le \xi \\ (x-1)\,\xi & x \ge \xi \end{cases}$$

$G(\xi,x)$ is the displacement at ξ due to a unit force at x.

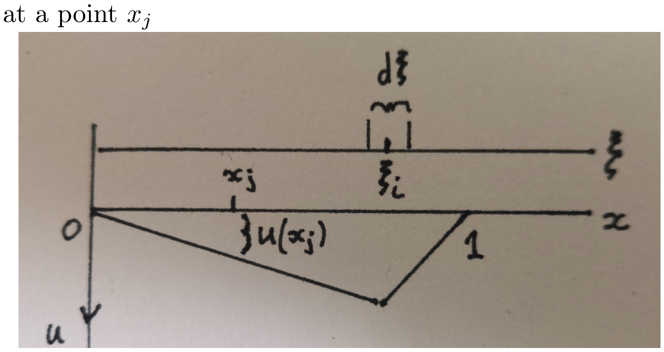
$$G(\xi,x) = \begin{cases} (x-1)\,\xi & \xi \le x \\ (\xi-1)\,x & \xi \ge x \end{cases}$$

We can see from the above that $G(\xi,x) = G(x,\xi)$ i.e. the displacement at ξ due to unit force at x is the same as the displacement at x due to a unit force at ξ. Basically because if $\xi < x$ then $x > \xi$.

1.4 Superposition

$$G(x,\xi) = \begin{cases} (\xi-1)\,x & x \le \xi \\ (x-1)\,\xi & x \ge \xi \end{cases}$$

We can envisage this as below. A unit force f at ξ_i over an interval $d\xi$ so that $f(\xi_i)\,d\xi = 1$ and outside the interval $d\xi$, $f = 0$. This causes a deflection u at a point x_j



Consider applying a force to the i^{th} segment. Let the force on this segment be $\phi(\xi_i)\,\Delta\xi$. The deflection of the string at the point x_j is given by

$$u(x_j) = G(x_j,\xi_i)\,\phi(\xi_i)\,\Delta\xi$$

and because $G(x_j,\xi_i) = G(\xi_i,x_j)$

$$u(x_j) = G(\xi_i,x_j)\,\phi(\xi_i)\,\Delta\xi$$

The total deflection at point x_j due to forces being applied over the complete range of $[0,1]$ is

$$u(x_j) = \sum_{i=1}^{n} G(\xi_i,x_j)\,\phi(\xi_i)\,\Delta\xi$$

In the limit

$$u(x_j) = \int_0^1 G(\xi, x_j)\,\phi(\xi)\,d\xi$$

And more generally

$$u(x) = \int_0^1 G(\xi, x)\,\phi(\xi)\,d\xi$$