Reviewing Matrix Operations

General Form of a Matrix of Size *m* x *n* with *m* Rows and *n* Columns

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots. a_{1n} \\ a_{21} & a_{22} & a_{23} \dots. a_{2n} \\ \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} \dots a_{mn} \end{bmatrix}$$

Square Matrix

If *m* = *n*, the matrix is said to be *square.* In this case, the matrix could be designated as an m x m matrix.

$$\mathbf{A}_{\mathbf{m,m}}$$

Vectors

A matrix having only one row is called a *row matrix.* A matrix having only one column is called a *column matrix.*

A matrix of either form is called a *vector.* The *row matrix* is called a *row vector* and the *column matrix* is called a *column vector*.

Scalars

A 1 x 1 matrix is called a *scalar* and is the form of most variables considered in simple algebraic forms.

Transpose

The transpose of a matrix **A** is denoted as **A′** and is obtained by interchanging the rows and columns.

Thus, if **A** has a size of m x n, **A′** will have a size of n x m. If the transpose operation is applied twice, the original matrix is restored.

Example 1. Determine the size of the matrix shown below.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix}$$

The matrix has 2 rows and 3 columns. Its size is 2 x 3.

Example 2. Determine the size of the matrix shown below.

$$\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix}$$

The matrix has 3 rows and 2 columns. Its size is 3 x 2.

Example 3. Determine the size of the matrix shown below.

$$\mathbf{C} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 6 & 1 \\ -5 & 2 & 1 \end{bmatrix}$$

The matrix has 3 rows and 3 columns. Its size is 3 x 3. It is a *square* matrix.

Example 4. Express the integer values of time from 0 to 5 s as a row vector.

$$\mathbf{t} = [0\ 1\ 2\ 3\ 4\ 5]$$

The size of the vector is 1 x 6.

Example 5. Express the variables x_1, x_2, and x_3 as a column vector.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Example 6. Determine the transpose of the matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix}$$

$$\mathbf{A'} = \begin{bmatrix} 2 & -1 \\ -3 & 4 \\ 5 & 6 \end{bmatrix}$$

Addition and Subtraction of Matrices

Matrices can be added together or subtracted from each other ***if and only if they are of the same size***. Corresponding elements are added or subtracted.

$$\mathbf{C}_{\mathbf{m,n}} = \mathbf{A}_{\mathbf{m,n}} \pm \mathbf{B}_{\mathbf{m,n}}$$

Multiplication of Two Matrices

Two matrices can be multiplied together ***only if the number of columns of the first matrix is equal to the number of rows of the second matrix***. This means that

$$\mathbf{AB} \neq \mathbf{BA}$$

Multiplication of Two Matrices (Continuation)

The number of rows in the product matrix is equal to the number of rows in the first matrix and the number of columns in the product matrix is equal to the number of columns in the second matrix.

$$\mathbf{A}_{\mathbf{m,n}}\mathbf{B}_{\mathbf{n,k}} = \mathbf{C}_{\mathbf{m,k}}$$

Multiplication of Two Matrices (Continuation)

An element in the product matrix is obtained by summing successive products of elements in the row of the first with elements of the column of the second.

$$c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$$

Division of Matrices ?

There is no such thing as division of matrices. However, *matrix inversion* can be viewed in some sense as a procedure similar to division. This process will be considered later.

Example 7. Determine **C = A + B** for the matrices shown below.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 9 & x \\ 6 & -3 & y \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 7 & 11 & 1+x \\ 2 & 2 & 6+y \end{bmatrix}$$

Example 8. Determine **D = A - B** for the matrices shown below.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 4 & 9 & x \\ 6 & -3 & y \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} -1 & -7 & 1-x \\ -10 & 8 & 6-y \end{bmatrix}$$

Example 9. For the matrices of Examples 1 and 2, determine possible orders of multiplication.

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \mathbf{A}_{2,3}\mathbf{B}_{3,2} = \mathbf{C}_{2,2}$$

$$\mathbf{BA} = \mathbf{B}_{3,2}\mathbf{A}_{2,3} = \mathbf{D}_{3,3}$$

Example 10. For the matrices of Examples 1 and 2, determine **C=AB**.

$$\mathbf{A} = \mathbf{A}_{2,3} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix} \qquad \mathbf{B} = \mathbf{B}_{3,2} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix}$$

Example 10. Continuation.

$$c_{11} = (2)(2) + (-3)(7) + (5)(3) = 4 - 21 + 15 = -2$$

$$c_{12} = (2)(1) + (-3)(-4) + (5)(1) = 2 + 12 + 5 = 19$$

$$c_{21} = (-1)(2) + (4)(7) + (6)(3) = -2 + 28 + 18 = 44$$

$$c_{22} = (-1)(1) + (4)(-4) + (6)(1) = -1 - 16 + 6 = -11$$

$$\mathbf{C} = \begin{bmatrix} -2 & 19 \\ 44 & -11 \end{bmatrix}$$

Example 11. For the matrices of Examples 1 and 2, determine **D=BA**.

$$\mathbf{A} = \mathbf{A}_{2,3} = \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix} \qquad \mathbf{B} = \mathbf{B}_{3,2} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{BA} = \begin{bmatrix} 2 & 1 \\ 7 & -4 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 5 \\ -1 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 16 \\ 18 & -37 & 11 \\ 5 & -5 & 21 \end{bmatrix}$$

Determinants

The determinant of a matrix **A** can be determined only for a *square* matrix. It is a *scalar* value. Various representations are shown as follows:

$$\det(\mathbf{A}) \qquad \left|\mathbf{A}\right| \qquad \Delta$$

Determinant of 2 x 2 Matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}$$

Example 13. Determine the determinant of the matrix shown below.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -4 & 5 \end{bmatrix}$$

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & 2 \\ -4 & 5 \end{vmatrix} = (3)(5) - (2)(-4)$$

$$= 15 + 8 = 23$$

Identity Matrix

$$\mathbf{I} = \begin{bmatrix} 1\ 0\ 0 \ldots . 0 \\ 0\ 1\ 0 \ldots . 0 \\ 0\ 0\ 1 \ldots . 0 \\ : \qquad\qquad : \\ 0\ 0\ 0\ \ldots . 1 \end{bmatrix}$$

Inverse Matrix

The inverse of a matrix **A** is denoted by $\mathbf{A}^{-1}$ and is defined by the equation that follows.

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Inverse of a 2 x 2 Matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{\det(\mathbf{A})}$$

$$= \begin{bmatrix} \dfrac{a_{22}}{\det(\mathbf{A})} & \dfrac{-a_{12}}{\det(\mathbf{A})} \\[2ex] \dfrac{-a_{21}}{\det(\mathbf{A})} & \dfrac{a_{11}}{\det(\mathbf{A})} \end{bmatrix}$$

Example 16. Determine the inverse of the matrix **A** below.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$\det(\mathbf{A}) = (2)(5) - (3)(4) = -2$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}}{-2} = \begin{bmatrix} -2.5 & 1.5 \\ 2 & -1 \end{bmatrix}$$