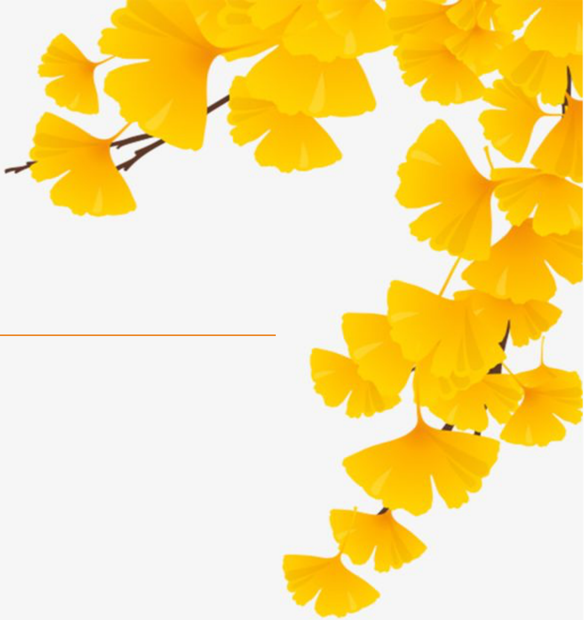
量 子 光 学

Quantum Optics



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前情回顾

- 一个自由度为 n 的系统由 n 对正则变量 $(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n)$ 描述，每对正则共轭变量 (q_i, p_i) 符合正则方程:

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i}$$

所有物理量都可用正则变量表示，比如哈密顿

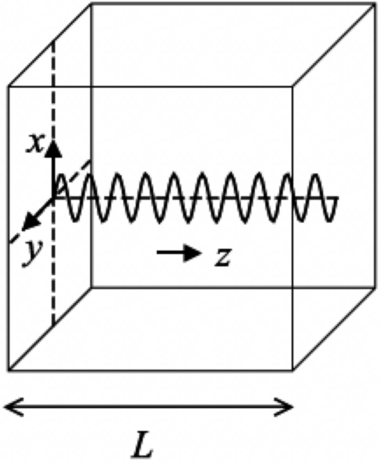
$$H(q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n)$$

- 正则量子化:
  - 写出经典哈密顿;
  - 哈密顿正则化;
  - 正则变量算符化; 哈密顿算符化 (其他物理量也可算符化);
  - 把哈密顿算符代入薛定谔方程求解.

第 4-5 讲：光场量子化

Contents

考虑空腔中的线性极化电磁波，如图



经典解:

- 电场: $E_x(z,t) = \sum_{l=1}^{\infty} E_0 \sin\omega_l t \sin k_l z = \sum_{l=1}^{\infty} a_l q_l(t) \sin(k_l z)$
- 磁场: $H_y(z,t) = \sum_{l=1}^{\infty} H_0 \cos\omega_l t \cos k_l z = \sum_{l=1}^{\infty} a_l \frac{\varepsilon_0}{k_l} q_l'(t) \cos(k_l z)$

电磁场的能量密度:

$$\omega = \frac{1}{2}(\varepsilon_0 E_x^2 + \mu_0 H_y^2)$$

经典哈密顿:

$$H = \frac{1}{2}\int_V (\varepsilon_0 E_x^2 + \mu_0 H_y^2) dV$$

代入电磁场经典解, 利用腔模正交性, 得:

$$
\begin{aligned}
H &= \frac{1}{2}\int_V (\mu_0 \mathbf{H}^2 + \varepsilon_0 \mathbf{E}^2) dV \\
&= \sum_l \frac{1}{2}(p_l^2 + \omega_l^2 q_l^2) \\
&= \sum_l H_l
\end{aligned}
$$

电磁场总哈密顿是单模哈密顿的线性叠加.

单模量子化

单模哈密顿:

$$H_l = \frac{1}{2}(p_l^2 + \omega_l^2 q_l^2)$$

写出密顿运动方程

$$\frac{\mathrm{d}p_l}{\mathrm{d}t} = -\frac{\partial H_l}{\partial q_l} = -\omega_l^2 q_l$$

$$\frac{\mathrm{d}q_l}{\mathrm{d}t} = \frac{\partial H_l}{\partial p_l} = p_l$$

说明 p_l 和 q_l 是一对正则量. 可进行正则量子化!

量子化条件

$$[\hat{q}_l, \hat{p}_l] = i\hbar$$

写在一起:

$$\hat{H}_l = \frac{1}{2}(\hat{p}_l^2 + \omega_l^2 \hat{q}_l^2) \qquad \text{with} \quad [\hat{q}_l, \hat{p}_l] = i\hbar$$

代入薛定谔方程, 既可求得场的波函数!

求解单模光场

单模哈密顿

$$\hat{H}_l = \frac{1}{2}(\hat{p}_l^2 + \omega_l^2 \hat{q}_l^2) \qquad \text{with} \quad [\hat{q}_l, \hat{p}_l] = i\hbar$$

谐振子哈密顿

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 \qquad \text{with} \quad [\hat{x}, \hat{p}] = i\hbar$$

比较：如果令谐振子质量 $m = 1$，则形式完全相同.
单模光场可视为单位质量的谐振子. 并称 $\hat{q}_l, \hat{p}_l$ 为电场和磁场算符

* 机械振子通过动-势能的相互转换形成振荡，光场通过电场能和磁场能的相互转换形成振荡!

- 具体求解与谐振子相同，令:

$$\hat{Q}_l = \sqrt{\frac{\omega}{\hbar}}\hat{q}_l, \qquad \hat{P}_l = \sqrt{\frac{1}{\hbar\omega}}\hat{p}_l$$

有:

$$\hat{H}_l = \frac{\hbar\omega}{2}(\hat{Q}_l^2 + \hat{P}_l^2) \qquad \text{with} \quad [\hat{Q}_l, \hat{P}_l] = i$$

令:

$$\hat{a}_l = \frac{1}{\sqrt{2}}(\hat{Q}_l + i\hat{P}_l), \qquad \hat{a}_l^\dagger = \frac{1}{\sqrt{2}}(\hat{Q}_l - i\hat{P}_l)$$

有

$$\hat{a}_l = \frac{1}{\sqrt{2\hbar\omega_l}}(\omega_l\hat{q}_l + i\hat{p}_l), \qquad \hat{a}_l^\dagger = \frac{1}{\sqrt{2\hbar\omega_l}}(\omega_l\hat{q}_l - i\hat{p}_l)$$

可得产生湮灭形式:

$$\hat{H}_l = \hbar\omega\left(\hat{a}_l^\dagger\hat{a}_l + \frac{1}{2}\right) \qquad \text{with} \quad [\hat{a}_l, \hat{a}_l^\dagger] = 1$$

量子谐振子中取质量 $m=1$, 即得单模光场 (ω_l) 解

能量本征值

$$E_n=(n+\frac{1}{2})\hbar\omega_l$$

能量本征态

$$|n\rangle$$

位置表象的能量本征态

$$\langle x\,|n\rangle=\frac{1}{\sqrt{n!2^n}}(\frac{\alpha}{\pi})^{\frac{1}{4}}e^{-\frac{1}{2}\xi^2}H_n(\xi)$$

式中

$$\xi=\alpha x,\qquad \alpha=\sqrt{\frac{m\omega}{\hbar}}=\sqrt{\frac{\omega}{\hbar}}$$

解毕!

电磁场的其他物理量

根据

$$\hat{a}_l = \frac{1}{\sqrt{2\hbar\omega_l}}(\omega_l\hat{q}_l + i\hat{p}_l), \qquad \hat{a}_l^\dagger = \frac{1}{\sqrt{2\hbar\omega_l}}(\omega_l\hat{q}_l - i\hat{p}_l)$$

反向求得:

$$\hat{q}_l = \sqrt{\frac{\hbar}{2\omega_l}}(\hat{a}_l + \hat{a}_l^\dagger)$$

$$\hat{p}_l = -i\sqrt{\frac{\hbar\omega_l}{2}}(\hat{a}_l - \hat{a}_l^\dagger)$$

若光场的某物理量 F 的经典表示为: $F(q_l, p_l)$, 则其量子力学算符为: $F(\hat{q}_l, \hat{p}_l)$, 其产生湮灭算符形式也可求得.

即: 电磁场的所有经典物理量都可经由产生湮灭算求得!

例-1. 求单模腔场电磁场分量的算符形式:

解: 经典单模腔场解:

$$E_l(z,t) = a_l q_l(t)\sin(k_l z)$$
$$H_l(z,t) = b_l p_l(t)\cos(k_l z)$$

算符:

$$\hat{E}_{x,l}(z,t) = a_l \hat{q}_l(t)\sin(k_l z)$$
$$\hat{H}_{y,l}(z,t) = b_l \hat{p}_l(t)\cos(k_l z)$$

代入

$$\hat{q}_l = \sqrt{\frac{\hbar}{2\omega_l}}(\hat{a}_l + \hat{a}_l^\dagger), \qquad \hat{p}_l = -i\sqrt{\frac{\hbar\omega_l}{2}}(\hat{a}_l - \hat{a}_l^\dagger)$$

大氣大度 求實求真

产生湮灭算符描述的电场算符

$$\hat{\mathbf{E}}_{x,l}(z,t) = E_l^0 \left(\hat{a}_l(t) + \hat{a}_l^\dagger(t)\right) \sin(k_l z)$$

式中

$$E_l^0 = \sqrt{\frac{\hbar\omega_l}{\varepsilon_0 V}}$$

三维形式

$$\hat{\mathbf{E}}_l(\mathbf{r},t) = E_l^0 \left(\hat{a}_l(t) + \hat{a}_l^\dagger(t)\right) \mathbf{E}_l(\mathbf{r})$$

例-2. 试证明电场算符与占据数算符不对易:

$$[\hat{n}, \hat{E}_x] = E_0 \sin kz(\hat{a}^\dagger - \hat{a})$$

解: 考虑单模场:

$$\begin{aligned}
[\hat{n}, \hat{E}_x] &= \left[\hat{a}_l^\dagger \hat{a}_l, E_l^0 \left(\hat{a}_l + \hat{a}_l^\dagger\right) \sin(k_l z)\right] \\
&= E_l^0 \sin(k_l z) \left[\hat{a}_l^\dagger \hat{a}_l, \left(\hat{a}_l + \hat{a}_l^\dagger\right)\right] \\
&= E_l^0 \sin(k_l z) \hat{a}_l^\dagger \left[\hat{a}_l, \left(\hat{a}_l + \hat{a}_l^\dagger\right)\right] + E_l^0 \sin(k_l z) \left[\hat{a}_l^\dagger, \left(\hat{a}_l + \hat{a}_l^\dagger\right)\right] \hat{a}_l \\
&= E_l^0 \sin(k_l z) \hat{a}_l^\dagger - E_l^0 \sin(k_l z) \hat{a}_l \\
&= E_0 \sin kz(\hat{a}^\dagger - \hat{a})
\end{aligned}$$

证毕!

说明: 电场越确定, 则光场的光子数越不确定, 也即场强与相位不能同时确定。

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求實求真
大氣大為

例-3. 试证明 *Fock* 态下电磁场的电场强度平均值为零:

证明: 设电磁场处于 *Fock* 态 $|n\rangle$, 取三维腔模进行计算

$$
\begin{aligned}
\langle n|\mathbf{E}(\mathbf{r},\mathbf{t})|n\rangle &= \langle n|E^0(\hat{a}+\hat{a}^\dagger)\mathbf{E}(\mathbf{r})|n\rangle \\
&= E^0\mathbf{E}(\mathbf{r})\langle n|\hat{a}|n\rangle + E^0\mathbf{E}(\mathbf{r})\langle n|\hat{a}^\dagger|n\rangle \\
&= 0+0 \\
&= 0
\end{aligned}
$$

* 相位随机性导致测量平均值为零! (证明见后)

Fock 态表象一般用于处理小粒子数的情况.

例-4. 考虑一维单模驻波场, 求电场和磁场强度的量子涨落:

解: 一维单模驻波场的电场和磁场算符为

$$E_x(z,t) = E_0\left(a(t) + a^\dagger(t)\right)\sin(kz)$$

设光场处于 Fock 态 $|n\rangle$, 有:

$$\langle n\left|E_x(z,t)\right|n\rangle = \left\langle n\left|H_y(z,t)\right|n\right\rangle = 0$$

$$
\begin{aligned}
\langle n \left| E_x^2 \right| n \rangle &= \left\langle n \left| E_0^2 \sin^2(kz)(a^\dagger(t) - a(t))^2 \right| n \right\rangle \\
&= E_0^2 \sin^2(kz) \left\langle n \left| 2a^\dagger a + 1 \right| n \right\rangle \\
&= 2E_0^2 \sin^2(kz) \left\langle n \left| n + \frac{1}{2} \right| n \right\rangle \\
&= 2(n + \frac{1}{2}) E_0^2 \sin^2(kz)
\end{aligned}
$$

量子涨落:

$$
\begin{aligned}
\Delta E_x &= \sqrt{\langle E_x^2 \rangle - \langle E_x \rangle^2} \\
&= \sqrt{2}\sqrt{n + \frac{1}{2}} E_0 \left| \sin kz \right|
\end{aligned}
$$

即使没有激发 (n=0), 依然存在真空涨落 $E_0 \left| \sin kz \right|$

课堂作业

试证明如下重要结论

$$\langle n|\hat{a}^\dagger\hat{a}^\dagger|n\rangle = 0$$

$$\langle n|\hat{a}\hat{a}|n\rangle = 0$$

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = n$$

$$\langle n|\hat{a}\hat{a}^\dagger|n\rangle = n+1$$

Contents

- 海森堡方程描述算符随时间的演化规律:

$$i\hbar\frac{\mathrm{d}F}{\mathrm{d}t}=[F,H]$$

把 $F=a$ 和光场哈密顿代入上式

$$\begin{aligned}\frac{\mathrm{d}a}{\mathrm{d}t}&=\frac{1}{i\hbar}\left[a,\hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)\right]\\&=i\omega(a^{\dagger}aa-aa^{\dagger}a)\\&=i\omega[a,a^{\dagger}]a\\&=-i\omega a\end{aligned}$$

解得:

$$a(t)=a(0)e^{-i\omega t}=ae^{-i\omega t}$$

同理,

$$a^{\dagger}(t)=a^{\dagger}(0)e^{i\omega t}=a^{\dagger}e^{i\omega t}$$

场算符随时间的演化:

$$q_l(t) = \sqrt{\frac{\hbar}{2\omega_l}}\left(a_l e^{-i\omega_l t} + a_l^\dagger e^{i\omega_l t}\right)$$

$$p_l(t) = -i\sqrt{\frac{\hbar\omega_l}{2}}\left(a_l e^{-i\omega_l t} - a_l^\dagger e^{i\omega_l t}\right)$$

场强随时间的演化

$$E_x(z,t) = \sum_l \mathcal{E}_l^{(s)} \left(a_l e^{-i\omega t} + a_l^\dagger e^{i\omega t} \right) \sin(k_l z)$$

$$B_y(z,t) = -i \sum_l \mathcal{B}_l^{(s)} \left(a_l e^{-i\omega t} - a_l^\dagger e^{i\omega t} \right) \cos(k_l z)$$

式中 $\mathcal{E}_l^{(s)} = \sqrt{\dfrac{\hbar\omega_l}{\varepsilon_0 V}}, \quad \mathcal{B}_l^{(s)} = \dfrac{\mu_0}{k_l}\sqrt{\dfrac{\varepsilon_0 \hbar \omega_l^3}{V}}$

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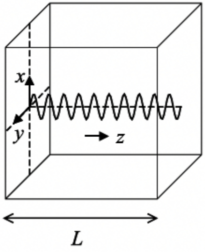
考虑空腔中的一个线性极化的电磁波，如图



场分量如下：

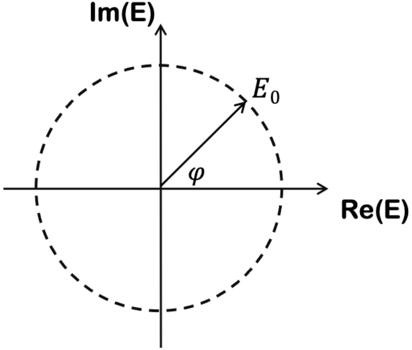
$$\begin{cases} E_x(z,t) = E_0 \sin \omega t \sin kz \\ B_y(z,t) = B_0 \cos \omega t \cos kz, \quad \text{with} \quad B_0 = E_0/c \end{cases}$$

初始条件决定一个初始相位 φ

- 复函表示:

$$\begin{aligned} E_x(z,t) &= E_0 \sin(\omega t + \varphi)\sin kz \\ &= E_0(z,t)e^{i\varphi} \\ &= E_0(z,t)\cos\varphi + iE_0(z,t)\sin\varphi \\ &= E_1(z,t) + iE_2(z,t) \end{aligned}$$

其中 $E_1(z,t), E_2(z,t)$ 是电场的实部和虚部



$$\begin{cases} Re(E) = E_0(z,t)\cos\varphi \\ Im(E) = E_0(z,t)\sin\varphi \end{cases}$$

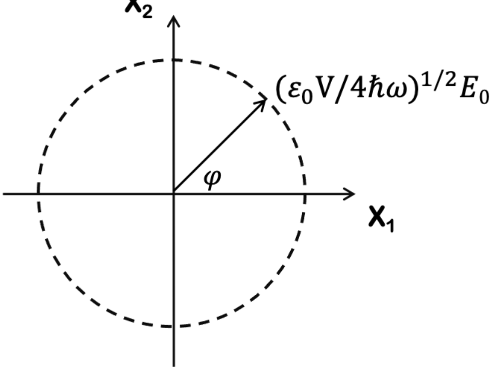
电磁场的大小和相位角都是确定的.

定义电场的正交分量 (field quadratures)

$$\begin{cases} X_1(t) = \sqrt{\dfrac{\omega}{2\hbar}}q(t) = \sqrt{\dfrac{\varepsilon_0 V}{4\hbar\omega}}E_0 \sin\omega t \\ X_2(t) = \sqrt{\dfrac{1}{2\hbar\omega}}p(t) = \sqrt{\dfrac{\varepsilon_0 V}{4\hbar\omega}}E_0 \cos\omega t \end{cases}$$

代入

$$\begin{aligned} E_x(z,t) &= E_0 \sin kz \sin(\omega t + \varphi) \\ &= E_0 \sin kz(\cos\varphi \sin\omega t + \sin\varphi\cos\omega t) \\ &= \sqrt{\frac{4\hbar\omega}{\varepsilon_0 V}} \sin kz \left(\cos\varphi X_1(t) + \sin\varphi X_2(t)\right) \\ &= X_1(z,t) + iX_2(z,t) \end{aligned}$$


X_2
$(\varepsilon_0 V/4\hbar\omega)^{1/2}E_0$
φ
X_1

量子化相图

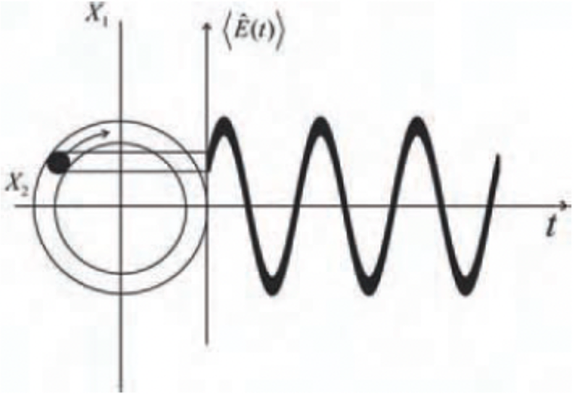
场正交分量可量子化:

$$\begin{cases} \hat{X}_1(t) = \sqrt{\dfrac{\omega_l}{2\hbar}}\hat{q}(t) = \dfrac{1}{2}\left(a + a^{\dagger}\right) \\ \hat{X}_2(t) = \sqrt{\dfrac{1}{2\hbar\omega_l}}\hat{p}(t) = \dfrac{1}{2i}\left(a - a^{\dagger}\right) \end{cases}$$

$$\text{with } [X_1, X_2] = \frac{i}{2}$$

计算不确定度:

$$\Delta X_1 \Delta X_2 = \frac{1}{2\hbar}\Delta q \Delta p \geq \frac{1}{2\hbar}\frac{\hbar}{2} = \frac{1}{4}$$



$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}$$

电磁场的大小和相位角都有一定的不确定性. 黑色区描述不可区分的简并态.

例-5. 试证明真空态是最小不确定度乘积态:

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

解: 不确定度计算公式

$$\Delta x = \sqrt{\overline{x^2} - \overline{x}^2}$$

$$\begin{aligned}
\overline{X_1} &= \left\langle n \left| \frac{1}{2}\left(a + a^\dagger\right) \right| n \right\rangle \\
&= 0 \\
\overline{X_2} &= \left\langle n \left| \frac{1}{2i}\left(a - a^\dagger\right) \right| n \right\rangle \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\overline{X_1^2} &= \left\langle n \left| \frac{1}{4}\left(a + a^\dagger\right)^2 \right| n \right\rangle \\
&= \frac{1}{4}\left\langle n \left| \left(aa + a^\dagger a^\dagger + aa^\dagger + a^\dagger a\right) \right| n \right\rangle \\
&= \frac{1}{4}(2n+1) \\
\overline{X_2^2} &= \left\langle n \left| -\frac{1}{4}\left(a - a^\dagger\right)^2 \right| n \right\rangle \\
&= -\frac{1}{4}\left\langle n \left| \left(aa + a^\dagger a^\dagger - aa^\dagger - a^\dagger a\right) \right| n \right\rangle \\
&= \frac{1}{4}(2n+1)
\end{aligned}
$$

$$
\Delta X_1 = \Delta X_2 = \sqrt{\overline{X^2} - \overline{X}^2} = \frac{1}{2}\sqrt{2n+1} \geq \frac{1}{2}
$$

对于真空态

$$\Delta X_1 = \Delta X_2 = \frac{1}{2}$$

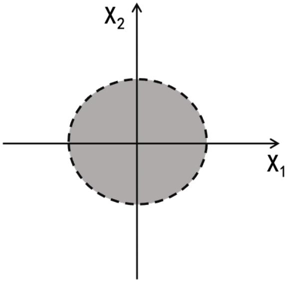
即:

$$\Delta X_1 \Delta X_2 = \frac{1}{4}$$

证毕!

真空态相图

真空态的经典电场能 $E_0 = 0$, 电场矢量大小为零, 处于原点. 是相位完全不确定的态.



$$\Delta X_1 = \Delta X_2 = \frac{1}{2}$$

真空态的量子涨落为量子噪音的极限 (极小值 $\frac{1}{4}$)

Contents

相位算符的定义

经典光场的复振幅

$$a = |a|\, e^{i\varphi}$$

经典光场的正则分量

$$q(t) = ae^{-\omega t} + ae^{\omega t} = 2\,|a| \cos(\omega t + \varphi)$$

(1) 那是否可写出一个相位算符呢?, 比如把湮灭算符写成

$$\hat{a} = \hat{g}e^{i\hat{\varphi}}$$

这种写法是不行的, 因为 $e^{i\hat{\varphi}}$ 与 $e^{-i\hat{\varphi}}$ 并不是幺正的, 会导致 $\hat{\varphi}$ 不是厄密的, 因而没有可测量意义.

(2) 为保证厄密, Susskind–Glogower, 令

$$|\varphi\rangle = \sum_{n=0}^{\infty} e^{i(n\varphi)} |n\rangle$$

定义指数算符

$$\hat{e}\,|\varphi\rangle = e^{i\varphi}\,|\varphi\rangle$$

$$\langle\varphi|\,e^{-i\varphi} = \langle\varphi|\,\hat{e}^{\dagger}$$

可得指数算符与产生湮灭算符的关系

$$\hat{e} = \frac{1}{\sqrt{\hat{n}+1}}\hat{a} = \frac{1}{\sqrt{\hat{a}\hat{a}^{\dagger}}}\hat{a}$$

$$\hat{e}^{\dagger} = \hat{a}^{\dagger}\frac{1}{\sqrt{\hat{n}+1}} = \hat{a}^{\dagger}\frac{1}{\sqrt{\hat{a}\hat{a}^{\dagger}}}$$

定义新的相位算符

$$\hat{C} = \frac{1}{2}(\hat{e} + \hat{e}^{\dagger}) = \frac{1}{2}\left[\frac{1}{\sqrt{\hat{n}+1}}a + a^{\dagger}\frac{1}{\sqrt{\hat{n}+1}}\right]$$

$$\hat{S} = \frac{1}{2i}(\hat{e} - \hat{e}^{\dagger}) = \frac{1}{2i}\left[\frac{1}{\sqrt{\hat{n}+1}}a - a^{\dagger}\frac{1}{\sqrt{\hat{n}+1}}\right]$$

反向求得

$$\hat{a} = \sqrt{\hat{n}+1}(\hat{C} + i\hat{S})$$

$$\hat{a}^{\dagger} = (\hat{C} - i\hat{S})\sqrt{\hat{n}+1}$$

例-6. 试证明指数算符有如下性质:

$$\hat{e}\,|n\rangle = |n-1\rangle\,, \qquad \hat{e}^{\dagger}\,|n\rangle = |n+1\rangle$$

证明:

$$\begin{aligned}
\hat{e}\,|n\rangle &= \frac{1}{\sqrt{\hat{n}+1}}\hat{a}\,|n\rangle \\
&= \frac{1}{\sqrt{\hat{n}+1}}\sqrt{n}\,|n-1\rangle \\
&= \frac{1}{\sqrt{(n-1)+1}}\sqrt{n}\,|n-1\rangle \\
&= |n-1\rangle
\end{aligned}$$

性质 (1):

$$S\left|n\right\rangle = \frac{1}{2i}[\left|n-1\right\rangle + \left|n-1\right\rangle]$$

$$S\left|0\right\rangle = -\frac{1}{2i}\left|1\right\rangle$$

$$S^2\left|n\right\rangle = \frac{1}{4}[\left|n\right\rangle + \left|n\right\rangle - \left|n-2\right\rangle - \left|n+2\right\rangle]$$

性质 (2):

$$[C,S]=\frac{1}{2}i\,|0\rangle\langle 0|$$

$$[C,\hat{n}]=iS$$

$$[S,\hat{n}]=-iC$$

$$C^2+S^2=1-\frac{1}{2}\,|0\rangle\langle 0|$$

(3) 数态的相位均值

$$\langle n|C|n\rangle=\langle n|S|n\rangle=0$$

这正是光场电磁分量均值为零的原因

$$\langle n|C^2|n\rangle=\langle n|S^2|n\rangle=\begin{cases}\frac{1}{2} & n\geq 1\\[2em] \frac{1}{4} & n=0\end{cases}$$

例-7. 设光场处于数态叠加态，求相位分布函数:

解: 光场

$$|\psi\rangle = \sum c_n |n\rangle$$

方法 (1) :

相位表象的波函数

$$\langle\varphi|\psi\rangle = \sum_{n=0}^{\infty} e^{-i(n\varphi)} c_n \langle n|n\rangle$$

相位分布

$$P(\varphi) = \frac{1}{2\pi} |\langle\varphi|\psi\rangle|^2 = \frac{1}{2\pi} \left|\sum_{n=0}^{\infty} e^{-i(n\varphi)} c_n\right|^2$$

对于具体的叠加态，可得分布函数

方法 (2) :

密度算符

$$\rho = |\psi\rangle\langle\psi|$$

相位分布

$$\begin{aligned} P(\varphi) &= \frac{1}{2\pi} \langle \varphi | \rho | \varphi \rangle \\ &= \frac{1}{2\pi} \langle \varphi | \psi \rangle \langle \psi | \varphi \rangle \\ &= \cdots \end{aligned}$$

* 密度算符方法算混态更方便...

$$\rho = \sum_j P_j |\psi_j\rangle\langle\psi_j|$$

Contents

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大氣大為

自由光场的单模量子化

自由空间电磁场为平面波（行波解）：

$$\hat{e}_\sigma exp(\pm i(\omega_k t - \mathbf{k}\cdot\mathbf{r})), \qquad k^2 = \frac{\omega_k^2}{c^2}$$

式中 $\hat{e}_\sigma$ 为偏振方向上的单位矢量，$\sigma = \pm$ 代表两个振动方向，它们相互正交且都与波矢 $\mathbf{k}$ 正交.

经箱归一化，可离散化行波，得

$$\mathbf{k} = \frac{2\pi}{L}(l_1\mathbf{i} + l_2\mathbf{j} + l_3\mathbf{k}), \qquad l_i = 0, \pm1, \pm2, \cdots$$

行波本征模为

$$\mathbf{u}_{k\sigma}(\mathbf{r}) = \hat{e}_\sigma e^{i\mathbf{k}\cdot\mathbf{r}}$$

自由空间电磁场行波展开

$$\mathbf{E}(\mathbf{r},t) = i\sum_{k,\sigma}^{\infty}\left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2}\hat{e}_\sigma[a_{k\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - a^*_{k\sigma}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}]$$

$$\mathbf{H}(\mathbf{r},t) = i\sum_{k,\sigma}^{\infty}\left(\frac{\hbar\omega_k}{2\mu_0 V}\right)^{1/2}(\hat{e}_k \times \hat{e}_\sigma)[a_{k\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - a^*_{k\sigma}(t)e^{-i\mathbf{k}\cdot\mathbf{r}}]$$

箱内总能量:

$$\begin{aligned} H &= \frac{1}{2}\int_V(\mu_0\mathbf{H}^2 + \epsilon_0\mathbf{E}^2)dV \\ &= \frac{1}{2}\sum_{k,\sigma}^{\infty}(p^2_{k\sigma} + \omega_k^2 q^2_{k\sigma}) \\ &= \sum_{k,\sigma}^{\infty} H_{k\sigma} \end{aligned}$$

单色行波的哈密顿

$$H_{k\sigma} = \frac{1}{2}(p_{k\sigma}^2 + \omega_k^2 q_{k\sigma}^2)$$

哈密顿运动方程

$$\frac{\mathrm{d}p_{k\sigma}}{\mathrm{d}t} = -\frac{\partial H_{k\sigma}}{\partial q_{k\sigma}} = -\omega_{k\sigma}^2 q_{k\sigma}$$

$$\frac{\mathrm{d}q_{k\sigma}}{\mathrm{d}t} = \frac{\partial H_{k\sigma}}{\partial p_{k\sigma}} = p_{k\sigma}$$

同理, 正则量子化

$$\hat{H}_{k\sigma} = \frac{1}{2}(\hat{p}_{k\sigma}^2 + \omega_k^2 \hat{q}_{k\sigma}^2) \qquad \text{with} \quad [\hat{p}_{k\sigma}, \hat{q}_{k\sigma}] = i\hbar$$

$$\hat{H}_{k\sigma} = \hbar\omega_{k\sigma}\left(\hat{a}_{k\sigma}^\dagger \hat{a}_{k\sigma} + \frac{1}{2}\right) \qquad \text{with} \quad [\hat{a}_{k\sigma}, \hat{a}_{k\sigma}^\dagger] = 1$$

行波电场和磁场算符:

$$\begin{cases} \hat{\mathbf{E}}(\mathbf{r},t) & = i\sum_{k,\sigma}^{\infty}(\frac{\hbar\omega_k}{2\epsilon_0 V})^{1/2}\hat{e}_\sigma[\hat{a}_{k\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}_{k\sigma}^\dagger(t)e^{-i\mathbf{k}\cdot\mathbf{r}}] \\ & = \hat{\mathbf{E}}^+(\mathbf{r},t) + \hat{\mathbf{E}}^-(\mathbf{r},t) \\ \hat{\mathbf{H}}(\mathbf{r},t) & = i\sum_{k,\sigma}^{\infty}(\frac{\hbar\omega_k}{2\mu_0 V})^{1/2}(\hat{e}_k \times \hat{e}_\sigma)[\hat{a}_{k\sigma}(t)e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}_{k\sigma}^\dagger(t)e^{-i\mathbf{k}\cdot\mathbf{r}}] \end{cases}$$

例-8. 考虑单模行波场，求电场和磁场强度的量子涨落：

解： 单模行波场的电场为

$$\mathbf{E}_l(\mathbf{r},t) = E_0(\hat{a}_l \exp^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_l^\dagger \exp^{i\omega t - i\mathbf{k}\cdot\mathbf{r}})$$

设光场处于 $Fock$ 态 $|n\rangle$，电场的平均值：

$$\begin{aligned}
\langle \mathbf{E}_l(\mathbf{r},t)\rangle &= \langle n\,|\mathbf{E}_l(\mathbf{r},t)|\,n\rangle \\
&= \left\langle n \left| E_0(\hat{a}_l \exp^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_l^\dagger \exp^{i\omega t - i\mathbf{k}\cdot\mathbf{r}}) \right| n \right\rangle \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\left\langle \mathbf{E}_l^2(\mathbf{r},t)\right\rangle &= \left\langle n \left| \mathbf{E}_l^2(\mathbf{r},t) \right| n \right\rangle \\
&= E_0^2 \left\langle n \left| (\hat{a}_l \exp^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_l^\dagger \exp^{i\omega t - i\mathbf{k}\cdot\mathbf{r}})^2 \right| n \right\rangle \\
&= E_0^2 \left\langle n \left| \hat{a}_l \hat{a}_l^\dagger + \hat{a}_l^\dagger \hat{a}_l \right| n \right\rangle \\
&= E_0^2 \left\langle n \left| \sqrt{(n+1)(n+1)} + \sqrt{nn} \right| n \right\rangle \\
&= E_0^2 (2n+1)
\end{aligned}
$$

量子涨落:

$$
\begin{aligned}
\Delta \mathbf{E}_l &= \sqrt{\langle \mathbf{E}_l^2 \rangle - \langle \mathbf{E}_l \rangle^2} \\
&= E_0 \sqrt{2n+1}
\end{aligned}
$$

即使没有激发 (n=0), 依然存在真空涨落 E_0

能量是对所有模求和

$$\hat{H} = \sum_l \hbar\omega_l(\hat{n}_l + \frac{1}{2}) = \sum_l \frac{1}{2}\hbar\omega_l(a_l^\dagger a_l + a_l a_l^\dagger)$$

能量本征态是各模数态的张量积

$$|\{n_l\}\rangle = |n_1\rangle \otimes |n_2\rangle \otimes |n_3\rangle \cdots |n_l\rangle \cdots = |n_1, n_2, n_3, \cdots, n_l, \cdots\rangle$$

能量本征方程

$$\hat{H}\,|\{n_l\}\rangle = E\,|\{n_l\}\rangle$$

根据张量空间运算法则

$$\hat{a}_l\,|\{n_l\}\rangle = \sqrt{n_l}\,|n_1, n_2, n_3, \cdots, n_l - 1, \cdots\rangle$$

$$\langle n_1, n_2, n_3, \cdots, n_l + 1, \cdots|\,\sqrt{n_l + 1} = \langle\{n_l\}|\,\hat{a}_l^\dagger$$

例-9. 计算双模场算符 Q 的均值:

$$Q=\Omega\sum_{i,j,k,l}a_i^\dagger a_j^\dagger a_k a_l,\qquad i,j,k,l=1,2$$

解:双模应对双光子态进行计算

$$\begin{aligned}\overline{Q}&=\langle n_1n_2|Q|n_2n_1\rangle\\&=\Omega[\left\langle n_1n_2|a_1^\dagger a_1^\dagger a_1a_1|n_2n_1\right\rangle+\left\langle n_1n_2|a_1^\dagger a_2^\dagger a_1a_2|n_2n_1\right\rangle\\&+\left\langle n_1n_2|a_1^\dagger a_2^\dagger a_1a_2|n_2n_1\right\rangle+\left\langle n_1n_2|a_2^\dagger a_2^\dagger a_2a_2|n_2n_1\right\rangle]\\&=\Omega[(n_1^2-n_1)+(n_2^2-n_2)+4n_1n_2]\end{aligned}$$

* 计算第一项

$$
\begin{aligned}
\left\langle n_1 n_2 | a_1^\dagger a_1^\dagger a_1 a_1 | n_2 n_1 \right\rangle &= \left\langle n_1 | a_1^\dagger a_1^\dagger a_1 a_1 | n_1 \right\rangle \\
&= \left\langle n_1 | a_1^\dagger (a_1 a_1^\dagger - 1) a_1 | n_1 \right\rangle \\
&= \left\langle n_1 | (a_1^\dagger a_1 a^\dagger a_1 - a^\dagger a_1) | n_1 \right\rangle \\
&= \left\langle n_1 | (a_1^\dagger a_1)^2 | n_1 \right\rangle - \langle n_1 | a^\dagger a_1 | n_1 \rangle \\
&= \langle n_1 | \hat{n}_1^2 | n_1 \rangle - \langle n_1 | \hat{n}_1 | n_1 \rangle \\
&= n_1^2 - n_1
\end{aligned}
$$

* 课堂作业：计算第二项

小结:

(1) 光场哈密顿可分解为一系列场谐振子哈密顿

$$\hat{H}_{k\sigma} = \sum_{k,\sigma} \hbar\omega_{k\sigma} \left(\hat{a}^{\dagger}_{k\sigma}\hat{a}_{k\sigma} + \frac{1}{2} \right) \qquad \text{with} \quad [\hat{a}_{k\sigma}, \hat{a}^{\dagger}_{k\sigma}] = 1$$

(2) 自由光场可表示为一系列量子化的平面波

$$\begin{aligned} \hat{\mathbf{E}}(\mathbf{r},t) &= i \sum_{k,\sigma}^{\infty} \left(\frac{\hbar\omega_k}{2\epsilon_0 V}\right)^{1/2} \hat{e}_{\sigma} \left[\hat{a}_{k\sigma}(t) e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^{\dagger}_{k\sigma}(t) e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \\ &= \hat{\mathbf{E}}^{+}(\mathbf{r},t) + \hat{\mathbf{E}}^{-}(\mathbf{r},t) \end{aligned}$$

(3) 腔场则可表示为一系列量子化的驻波

课外作业

1. 利用腔模正交性证明

$$H = \sum_l H_l$$

2. 计算电磁场真空态的量子涨落
3. 试证明，只有光场处于两相临数态的叠加态时，电场算符的均值才不为零
4. 已知光场处于如下混态，求相位分布

$$\rho = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

Thanks for your attention!

A & Q

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