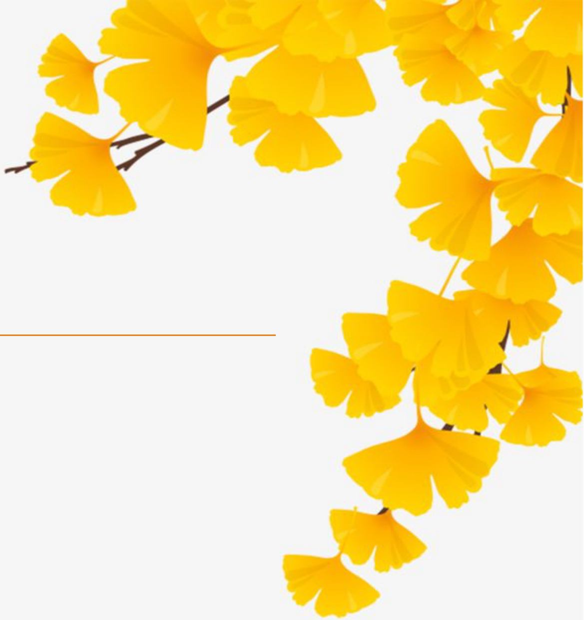
# 工 程 数 学

## Engineering Mathematics



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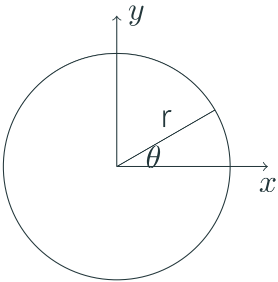
# 第五章 贝塞尔函数（8学时）

# Contents

### 例 1、建立贝塞尔方程

对于半径为 $r_0$ 的侧面绝缘的薄均匀圆盘，边界温度始终保持为 0 度，当盘的初始温度已知时 $(\Psi(x,y))$，求体系的温度分布函数。



解: 这是一个温度场, 是非稳恒场, 服从热传导方程:

$$\begin{cases} u_t = a^2[u_{xx} + u_{yy}] \quad (0 < x^2 + y^2 < r_0^2, t > 0) \\ u(x,y,t)|_{x^2+y^2=r_0^2} = 0 \\ u(x,y,t)|_{t=0} = \Psi(x,y) \end{cases}$$

考虑到圆域边界条件, 得使用极坐标描述

大氣大居 求實求真

试证明极坐标拉普拉斯算子为:

$$[u_{xx} + u_{yy}] = [\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}]$$

证明: 基本变换关系为

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

对于函数 $u(x, y) = u(r\cos\theta, r\sin\theta)$ 计算导数:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial u}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\frac{\partial u}{\partial y}$$

改写成矩阵形式:

$$
\begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix}
$$

$$
\begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix}
$$

$$
\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/r \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial u}{\partial \theta} \end{bmatrix}
$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{bmatrix} = \begin{bmatrix} \vec{e}_r & \vec{e}_\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} \end{bmatrix}$$

$$\nabla = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}$$

$$\begin{aligned} \nabla^2 &= \nabla \cdot \nabla \\ &= (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}) \cdot (\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}) \end{aligned}$$

$$
\begin{aligned}
\nabla^2 &= \vec{e}_r \cdot \frac{\partial}{\partial r}(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}) + \vec{e}_\theta \cdot \frac{1}{r}\frac{\partial}{\partial \theta}(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}) \\
&= \vec{e}_r \cdot (\frac{\partial \vec{e}_r}{\partial r}\frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial r^2} + \frac{\partial \vec{e}_\theta}{\partial r}\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\theta \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial \theta})) \\
&\quad + \vec{e}_\theta \cdot (\frac{1}{r}\frac{\partial \vec{e}_r}{\partial \theta}\frac{\partial}{\partial r} + \vec{e}_r \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial \vec{e}_\theta}{\partial \theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial}{\partial \theta})) \\
&= \vec{e}_r \cdot (0\frac{\partial}{\partial r} + \vec{e}_r \frac{\partial^2}{\partial r^2} + 0\frac{\partial}{\partial \theta} + \vec{e}_\theta \frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial \theta})) \\
&\quad + \vec{e}_\theta \cdot (\vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial r} + \vec{e}_r \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial}{\partial r} + \frac{1}{r}(-\vec{e}_r)\frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\theta \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}) \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}
\end{aligned}
$$

Tips: $\frac{\partial \vec{e}_\theta}{\partial \theta} = -\vec{e}_r; \quad \frac{\partial \vec{e}_\theta}{\partial r} = 0; \quad \frac{\partial \vec{e}_r}{\partial r} = 0; \quad \frac{\partial \vec{e}_r}{\partial \theta} = \vec{e}_\theta$

## 极坐标系下的方程

$$\begin{cases} u_t = a^2[\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}], & (0 < r < r_0, t > 0) \\ u(r_0, \theta) = 0, & 0 < \theta < 2\pi \\ u(r, \theta, t = 0) = \Psi(r, \theta), & 0 < \theta < 2\pi \end{cases}$$

令: $u(r, \theta, t) = R(r)\Theta(\theta)T(t)$, 代回原方程, 得:

$$R\Theta T' = a^2[R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta'' T(t)]$$

整理:

$$\frac{T'}{a^2 T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda$$

转化为两个方程:

$$T'(t) + \lambda a^2 T(t) = 0 \quad \ldots\ldots. \quad (1)$$

方程 1 是衰减模型, 已求解!

$$r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\lambda r^2+\frac{\Theta''(\theta)}{\Theta(\theta)}=0 \qquad \ldots\ldots \quad (2)$$

方程 2 是固有值问题，可继续分离变量:

$$r^2\frac{R''(r)}{R(r)}+r\frac{R'(r)}{R(r)}+\lambda r^2=-\frac{\Theta''(\theta)}{\Theta(\theta)}=\mu$$

角向固有值问题: $\begin{cases}\Theta''(\theta)+\mu\Theta(\theta)=0\\ \Theta(\theta)=\Theta(\theta+2\pi)\\ \Theta'(\theta)=\Theta'(\theta+2\pi)\end{cases}$

径向固有值问题: $\begin{cases}r^2R''(r)+rR'(r)+(\lambda r^2-\mu)R(r)=0\\ R(r_0)=0\end{cases}$

角向固有值问题有解,
固有值:

$$\mu = n^2, \qquad (n = 0, 1, 2, 3\ldots)$$

固有函数:

$$\Theta_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta), \qquad (n = 0, 1, 2, 3\ldots)$$

$$\Theta_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{-in\theta}, \qquad (n = 0, 1, 2, 3\ldots)$$

把 $\mu = n^2$, 代回径向方程，得径向固有值问题:

$$\boxed{\begin{cases} r^2 R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0 \\ R(r_0) = 0 \end{cases}}$$

称为贝塞尔方程

## 例 2、试证明圆域波动方程的径向固有值问题也是贝塞尔方程：

**证明：** 圆域波动方程如下：

$$\begin{cases} u_{tt} = a^2[u_{xx} + u_{yy}] \quad (0 < x^2 + y^2 < r_0^2, t > 0) \\ u(x,y,t)|_{x^2+y^2=r_0^2} = 0 \\ u(x,y,t)|_{t=0} = \Psi(x,y) \\ u_t(x,y,t)|_{t=0} = \varphi(x,y) \end{cases}$$

考察方程，发现当使用极坐标拉普拉斯算子后，整个的分离变量过得与热传导方程高度一致,(*P115)

其固有值问题也是贝塞尔方程！

$$\begin{cases} r^2R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0 \\ R(r_0) = 0 \end{cases}$$

## 结论:

贝塞尔方程是圆域极坐标条件下的一个普适的径向本征方程!

$$\begin{cases} r^2R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0 \\ R(r_0) = 0 \end{cases}$$

在正式求解之前, 先预处理一下

令:

$$x = \sqrt{\lambda} r, \quad y(x) = R(r) = R(\frac{x}{\sqrt{\lambda}})$$

有:

$$\frac{dy}{dx} = \frac{dR}{dr}\frac{dr}{dx} = \frac{1}{\sqrt{\lambda}}\frac{dR}{dr}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\lambda}\frac{dR^2}{dr^2}$$

代回原方程,

得:

$$\boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0}$$

称为 n 整数阶贝塞尔方程.

- 与欧拉方程比较:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+n^2y=0$$

欧拉方程是一个变系数微分方程, 可通过变量代换 (令 $t=\ln x$) 转化为常系数微分方程

$$\frac{d^2y}{dt^2}-n^2y=0$$

若同样令 $t = \ln x$, 贝塞尔方程转化为

$$\frac{d^2y}{dt^2} + (x^2 - n^2)y = 0$$

依然是变系数微分方程, 无法进一步进行求解.

因此, 贝塞尔方程没有通常意义的初等函数表达式解!

## 例 2、求解贝塞尔方程

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$$

**解:** 没有通常意义的初等函数表达式解，即没有通常意义的级数解，尝试，设方程有如下非一般意义的级数解:

$$y=\sum_{k=0}^{\infty}a_kx^{s+k}, \qquad (a_0\neq 0)$$

求导:

$$y' = \sum_{k=0}^{\infty}(s+k)a_k x^{s+k-1}$$

$$y'' = \sum_{k=0}^{\infty}(s+k)(s+k-1)a_k x^{s+k-2}$$

代回原方程 (注意脚标的变化)，得:

$$\sum_{k=0}^{\infty}[(s+k)^2 - n^2]a_k x^{s+k} + \sum_{k=2}^{\infty} a_{k-2}x^{s+k} = 0$$

第一项 (k=0) 系数应为零:

$$(s+k)^2 - n^2 = 0, \quad \to s_1 = -n, \qquad s_2 = n.$$

第二项 (k=1) 系数应为零:

$$[(s+k)^2 - n^2]a_1 = 0, \qquad \to a_1 = 0.$$

后面各项系数都为零:

$$[(s+k)^2 - n^2]a_k + a_{k-2} = 0, \quad (k = 2, 3, 4, ...)$$

存在递推关系:

$$a_k = -\frac{1}{(s+k)^2 - n^2} a_{k-2}$$

由 $a_1 = 0$, 可推出奇数项为零

$$a_{2m+1} = 0$$

现取 $s = n$, (-n 不影响解题过程), 得:

$$a_{2m} = \frac{-1}{(n+2m)^2 - n^2} a_{2m-2} = \frac{-1}{2m(2n+2m)} a_{2m-2}, \qquad (m = 1, 2, 3, ...)$$

归纳，得:

$$a_{2m} = (-1)^m \frac{1}{2^{2m} m!(n+m)(n+m-1)...(n+1)} a_0$$

取：$a_0 = 1/2^n n!$，得:

$$a_{2m} = (-1)^m \frac{1}{2^{2m+n} m!(n+m)!}$$

贝塞尔方程有级数特解:

$$y(x) = \sum_{m=0}^{\infty} a_{2m} x^{n+2m}$$

分析收敛性，发现:

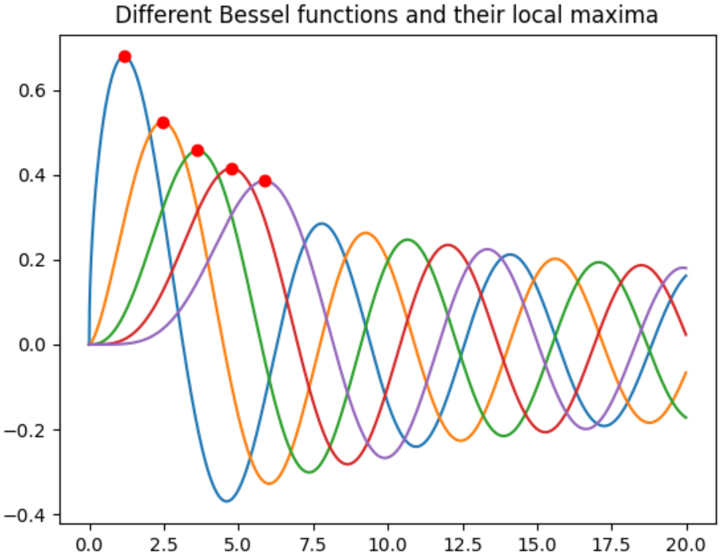
$$\lim_{m\to\infty} |\frac{a_{2m+2}}{a_{2m}}| = \lim_{m\to\infty} \frac{1}{4(m+1)(n+m+1)} = 0$$

说明此级数必为某函数的展开式，称之为贝塞尔函数。

记为:

$$\boxed{J_n(x) = \sum_{m=0}^{\infty} a_{2m} x^{n+2m}}$$

称为 n 整数阶贝塞尔函数!


Different Bessel functions and their local maxima
0.6
0.4
0.2
0.0
−0.2
−0.4
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0

贝塞尔（Bessel，Friedrich Wilhelm，1784～1846）德国天文学家，数学家，天体测量学的奠基人. 提出贝塞尔函数，讨论该函数的一系列性质及其求值方法，为解决物理学、天文学和信息学有关问题提供了重要工具。

1、由圆域波动方程导出贝塞尔方程

2、求衰减模型

$$T'(t) + \lambda a^2 T(t) = 0$$

3、求角向固有值及归一化的固有函数:

$$\begin{cases} \Theta''(\theta) + \mu\Theta(\theta) = 0 \\ \Theta(\theta) = \Theta(\theta + 2\pi) \\ \Theta'(\theta) = \Theta'(\theta + 2\pi) \end{cases}$$

4、求欧拉方程

# Contents

# 贝塞尔函数

零阶贝塞尔函数:

$$J_0(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!m!} (\frac{x}{2})^{2m}$$

n 阶贝塞尔函数:

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(n+m)!} (\frac{x}{2})^{n+2m}$$

第二类贝塞尔函数:

$$Y_\alpha(x) = \frac{J_\alpha(x)\cos(\alpha\pi) - J_{-\alpha}(x)}{\sin(\alpha\pi)}$$

贝塞尔函数在除 $x = 0$ 点外的整个实数轴上收敛。

为进一步讨论贝塞尔函数的性质，先讨论 Gamma 函数.

- 1728 年，哥德巴赫在考虑数列插值的问题, 比如我们可以计算 2!,3!, 是否可以计算 2.5! 呢?
- 他写信请教尼古拉斯·伯努利和他的弟弟丹尼尔·伯努利, 欧拉当时正好与丹尼尔·伯努利在一块，他因此得知了这个问题
- 1729 年欧拉解决了这个问题!

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad (x > 0)$$

Euler 创造的 Γ 函数, 将积分和阶乘联系在起来

**例-1. 试证明:**

$$\Gamma(1) = 1$$



**证明:**

$$\Gamma(1) = \int_0^{\infty} t^{1-1} e^{-t} dt$$

$$= \int_0^{\infty} e^{-t} dt = 1$$

## Γ 函数的性质

**性质 1:** 试证明 Gamma 函数有如下递推式:

$$\Gamma(x+1) = x\Gamma(x), \qquad (x > 0)$$

**证明:**

$$\begin{aligned}
\Gamma(x+1) &= \int_0^{\infty} t^{x+1-1} e^{-t} dt \\
&= -t^x e^{-t} \Big|_0^{\infty} + x \int_0^{\infty} t^{x-1} e^{-t} dt \\
&= x \int_0^{\infty} t^{x-1} e^{-t} dt \\
&= x\Gamma(x)
\end{aligned}$$

**推论:**

$$\Gamma(x+n) = (x+n-1)(x+n-2)\cdots(x+1)x\Gamma(x)$$

**性质 2:** 试证明自变量为正整数的 Gamma 函数与阶乘有如下关系:

$$\Gamma(n+1) = n!$$

**证明 1:** 由递推公式

$$\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \\
\Gamma(n+1) &= n\Gamma(n) \\
&= n(n-1)\cdots 1\Gamma(1) \\
&= n!\int_0^{\infty} e^{-t}dt \\
&= n!
\end{aligned}$$

**证明 2:** 在推论中：取 $x=1$

$$
\begin{aligned}
\Gamma(x+n) &= (x+n-1)(x+n-2)\cdots(x+1)x\Gamma(x) \\
\Gamma(1+n) &= (1+n-1)(1+n-2)\cdots(1+1)1\Gamma(1) \\
\Gamma(n+1) &= n!
\end{aligned}
$$

*Tips:* 阶乘只是 $\Gamma(x)$ 函数的特例!$(x=1,2,3,\cdots)$ 对应 $(0!,1!,2!,\cdots)$

性质 3: 试证明半正整数 Γ 函数有如下性质

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

$$\Gamma(n+\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

证明:(1) 由 Gamma 函数定义

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt, \qquad (x>0)$$

$$\Gamma(\frac{1}{2}) = \int_0^{\infty} t^{\frac{1}{2}-1}e^{-t}dt$$

$$= \int_0^{\infty} \frac{1}{\sqrt{t}}e^{-t}dt$$

令 $x = \sqrt{t}$

$$
\begin{aligned}
\Gamma(\frac{1}{2}) &= 2\int_0^{\infty} e^{-x^2}dx \\
&= 2\sqrt{\iint_0^{\infty} e^{-x^2}e^{-y^2}dxdy} \\
&= 2\sqrt{\int_0^{\frac{\pi}{2}}\int_0^{\infty} e^{-r^2}rdrd\theta} \\
&= 2\sqrt{\frac{\pi}{2}\int_0^{\infty} re^{-r^2}dr} \\
&= 2\sqrt{\frac{\pi}{2}[-\frac{1}{2}e^{-r^2}]_0^{\infty}} \\
&= \sqrt{\pi}
\end{aligned}
$$

* x,y 皆取正数, 是第一象限!

(2) 由递推公式 $\Gamma(1+x)=x\Gamma(x), \qquad (x>0)$

$$
\begin{aligned}
\Gamma(1+\frac{1}{2}) &= \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi} \\
\Gamma(2+\frac{1}{2}) &= \Gamma(1+\frac{3}{2}) = \frac{3}{2}\Gamma(1+\frac{1}{2}) = \frac{3}{2}\frac{1}{2}\sqrt{\pi} \\
\Gamma(3+\frac{1}{2}) &= \frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi} \\
\Gamma(n+\frac{1}{2}) &= \frac{(2n-1)!!}{2^n}\sqrt{\pi} \\
&= \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}
\end{aligned}
$$

证毕!

性质 4: 试证明半正整数的阶乘为

$$(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$$

$$(m-\frac{1}{2})! = \frac{(2m)!}{2^{2m}m!}\sqrt{\pi}$$

证明: 把如下公式从正整数向正实数延拓!

$$\Gamma(n+1) = n! \qquad \rightarrow \qquad \Gamma(x+1) = x!, (x>0)$$

(1) 取: $x = m - \frac{1}{2}, \qquad (m = 1, 2, 3, \cdots)$

$$\begin{aligned}(m-\frac{1}{2})! &= \Gamma(m-\frac{1}{2}+1)\\ &= \Gamma(m+\frac{1}{2})\\ &= \frac{(2m)!}{2^{2m}m!}\sqrt{\pi}\end{aligned}$$

(2) 在上式中取 $m=1$

$$(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$$

* 一般分数的阶乘如何求?(余元公式,Gamma 函数定义式)

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin x\pi}, \qquad 0 < x < 1$$

$$\Gamma(\frac{1}{2}+x)\Gamma(\frac{1}{2}-x) = \frac{\pi}{\cos x\pi}, \qquad 0 < x < \frac{1}{2}$$

* 例如, 求 $(\pi)!$

$$(\pi)! = \Gamma(\pi+1) = \int_0^{\infty} t^{\pi} e^{-t} dt = ?$$

性质 5: 试求半负整数 $\Gamma$ 函数的值

$$\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$$
$$\Gamma(-\frac{3}{2}) = \frac{4}{3}\sqrt{\pi}$$

解: (1) 把递推公式 $\Gamma(1+x) = x\Gamma(x)$ 从正实数向实数延拓!

并令 $x = -\frac{1}{2}$

$$\begin{aligned}\Gamma(-\frac{1}{2}) &= -2\Gamma(1-\frac{1}{2}) \\ &= -2\Gamma(\frac{1}{2}) \\ &= -2\sqrt{\pi}\end{aligned}$$

(2) 在递推公式 $\Gamma(1+x) = x\Gamma(x)$ 中, 令 $x = -\frac{3}{2}$

$$
\begin{aligned}
\Gamma(-\frac{3}{2}) &= -\frac{2}{3}\Gamma(1-\frac{3}{2}) \\
&= -\frac{2}{3}\Gamma(-\frac{1}{2}) \\
&= (-\frac{2}{3})(-2\sqrt{\pi}) \\
&= \frac{4}{3}\sqrt{\pi}
\end{aligned}
$$

* 从正实数 $x>0$ 向实数延拓 $x\in\mathbb{R}$, 导致严重的问题, 比如:

$$n! = (n-1)(n-2)(n-3)\cdots 1$$

$$(-1)! = (-1)(-2)(-3)\cdots - \infty?$$

$$\Gamma(0) = (-1)! \quad ?, \qquad \Gamma(-1) = ?, \cdots$$

性质 6: 试证明 Gamma 函数在非正整数点的极限为无穷大

$$\lim_{x\to -n}\Gamma(x) = \infty, \qquad (n = 0, 1, 2, \cdots)$$

$$\frac{1}{\Gamma(-n)} = 0, \qquad (n = 0, 1, 2, \cdots)$$

**证明:** 由递推公式

$$\Gamma(x) = \frac{1}{x}\Gamma(x+1)$$

$$\lim_{x\to 0}\Gamma(x) = \lim_{x\to 0}\frac{1}{x}\Gamma(x+1) = \infty$$

$$\lim_{x\to -1}\Gamma(x) = \lim_{x\to -1}\frac{1}{x}\Gamma(x+1) = \lim_{x\to 0}\frac{1}{x-1}\Gamma(x) = \infty$$

$$\cdots\cdots$$

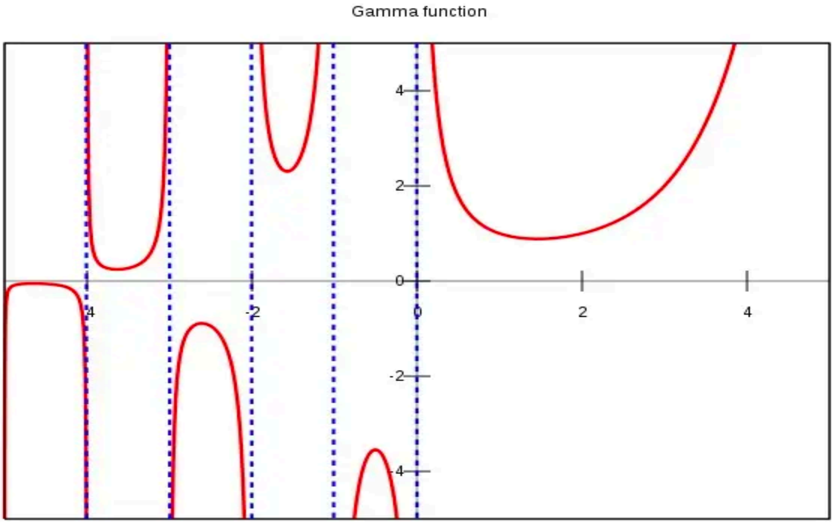
$$\lim_{x\to -n}\Gamma(x) = \lim_{x\to -n}\frac{1}{x}\Gamma(x+1) = \lim_{x\to -(n-1)}\frac{1}{x-1}\Gamma(x) = \infty$$

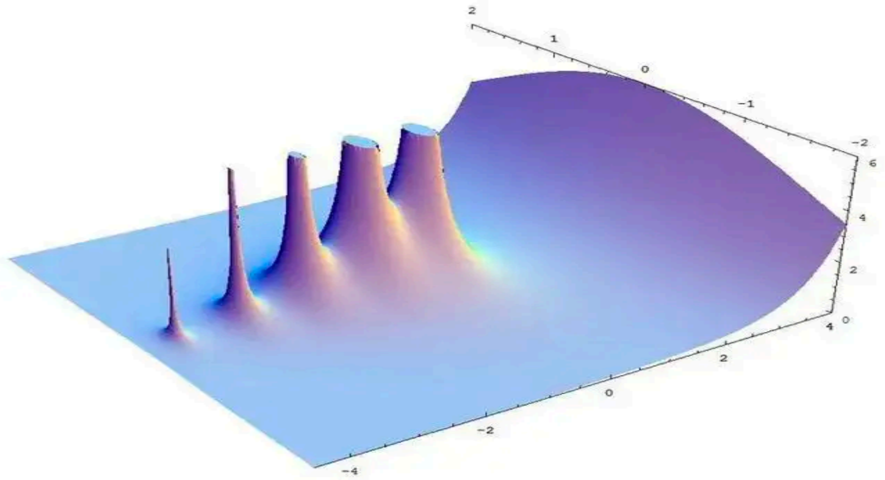
证毕!

*Tips:*

延拓到实数域后, $\Gamma$ 函数存在奇点 $x = 0, -1, -2, \cdots$, 如果进一步延拓到复数域, 这些奇点依然存在. 它们是 $\Gamma$ 函数的一阶极点!

# 实数域 Gamma 函数






网络教学资源，点这里

# Gamma 函数的应用实例

**例-1. 计算如下积分:**

$$(1)\ \int_0^{\infty} x^2 e^{-x^2} dx, \qquad (2)\ \int_0^{\infty} x^2 e^{-\frac{1}{2}x^2} dx$$

**解:** 比较 (1) 式与 Gamma 函数的定义式的结构,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt, \qquad (x > 0)$$

$$\begin{aligned}\int_0^{\infty} x^2 e^{-x^2} dx &= \int_0^{\infty} (x^2)^1 e^{-(x^2)} \frac{1}{2x} d(x^2) \\ &= \frac{1}{2} \int_0^{\infty} (x^2)^{\frac{1}{2}} e^{-(x^2)} d(x^2) \\ &= \frac{1}{2} \int_0^{\infty} t^{\frac{1}{2}} e^{-t} d(t)\end{aligned}$$

$$
\begin{aligned}
\int_0^{\infty} x^2 e^{-x^2} dx &= \frac{1}{2}\int_0^{\infty} t^{\frac{3}{2}-1} e^{-t} d(t) \\
&= \frac{1}{2}\Gamma(\frac{3}{2}) \\
&= \frac{1}{2}\Gamma(1+\frac{1}{2}) \\
&= \frac{1}{2}\frac{1}{2}\Gamma(\frac{1}{2}) \\
&= \frac{\sqrt{\pi}}{4}
\end{aligned}
$$

1. 求 0!, $$\int_0^{\infty} x^2 e^{-\frac{1}{2}x^2} dx$$

2. 试证明 $$\frac{1}{\Gamma(-n)} = 0, \qquad (n = 0, 1, 2, \cdots)$$

3. 试证明半正整数 $\Gamma$ 函数值的一般性公式

$$(m - \frac{1}{2})! = \frac{(2m)!}{2^{2m} m!}\sqrt{\pi} \qquad (m = 1, 2, 3, \cdots)$$

4. 试求出半负整数 $\Gamma$ 函数值的一般性公式

$$\Gamma(m - \frac{1}{2}) = ?, \qquad (m = 0, -1, -2, -3, \cdots)$$

# Contents

现在讨论贝塞尔函数的性质:

**性质 1:** 负数阶贝塞尔函数与正数阶贝塞尔函数有如下关系

$$J_{-n}(x) = (-1)^n J_n(x)$$

**证明:** 用 $\Gamma$ 函数重写出贝塞尔函数:

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!(n+m)!} \left(\frac{x}{2}\right)^{n+2m}$$

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

负数阶塞尔函数可写成

$$J_{(-n)}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(-n+m+1)} \left(\frac{x}{2}\right)^{-n+2m}$$

对于 $m < n$ 的项，分母中的 Gamma 函数为无穷大，因此都为零, 要去除:

$$J_{(-n)}(x) = \sum_{m=n}^{\infty} (-1)^m \frac{1}{m!\Gamma(-n+m+1)} (\frac{x}{2})^{-n+2m}$$

令 $m - n = k$, 有 $m = n + k$,

$$J_{(-n)}(x) = (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{1}{(n+k)!\Gamma(k+1)} (\frac{x}{2})^{n+2k}$$

$$J_{-n}(x) = (-1)^n \sum_{k=0}^{\infty} (-1)^k \frac{1}{(n+k)!k!} (\frac{x}{2})^{n+2k} = (-1)^n J_n(x)$$

证毕!

**性质 2:** 半整数阶贝塞尔函数与三角函数有如下关系

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

**证明:** 基于 Gamma 函数，可以写出半整数阶贝塞尔函数

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(n+m+1)} (\frac{x}{2})^{n+2m}$$

$$J_{1/2}(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(1/2+m+1)} (\frac{x}{2})^{1/2+2m}$$

其中，

$$
\begin{aligned}
\Gamma(1/2+m+1) &= (\frac{2m+1}{2})\Gamma(1/2+m) \\
&= (\frac{2m+1}{2}\frac{2m-1}{2})\Gamma(1/2+m-1) \\
&\cdots\cdots \\
&= \frac{(2m+1)!!}{2^{m+1}}\Gamma(1/2) \\
&= \frac{(2m+1)!!}{2^{m+1}}\sqrt{\pi}
\end{aligned}
$$

代回，有：

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m+1}}{(2m+1)!}$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$

同理，有

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

证毕！

# 课堂作业

1. 求

$$\Gamma(-\frac{1}{2}), (-\frac{1}{2})!$$

2. 证明:

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

## 性质 3: 贝塞尔函数的导数与递推式

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \qquad \cdots(1)$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x), \qquad \cdots(2)$$

$$2nJ_n(x) = xJ_{n-1}(x) + xJ_{n+1}(x), \qquad \cdots(3)$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \qquad \cdots(4)$$

由贝塞尔函数的 $\Gamma$ 函数形式

$$J_n(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(n+m+1)} \left(\frac{x}{2}\right)^{n+2m}$$

证明 (1) 式：上等式两端同乘 $x^n$ 再求导:

$$\frac{d}{dx}\left[x^n J_n(x)\right] = \frac{d}{dx} \sum_{m=0}^{\infty} (-1)^m \frac{1}{m!\Gamma(n+m+1)} \left(\frac{x^{2n+2m}}{2^{n+2m}}\right)$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n+m+1)}(\frac{(2n+2m)x^{2n-1+2m}}{2^{n+2m}})$$

$$= x^n \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(n-1+m+1)}(\frac{x^{n-1+2m}}{2^{n-1+2m}})$$

$$= x^n J_{n-1}(x), \qquad \cdots(1)$$

证明 (2) 式: 同理, 得:

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x), \qquad \cdots(2)$$

**证明 (3) 和 (4) 式**: 把 (1)(2) 二式左端求导

$$\frac{d}{dx}[x^n J_n(x)] = nx^{n-1}J_n(x) + x^n J_n'(x) = x^n J_{n-1}(x)$$
$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -nx^{-n-1}J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$$

两式消去 $J_n'(x)$ 得:

$$2nJ_n(x) = xJ_{n-1}(x) + xJ_{n+1}(x), \qquad \cdots(3)$$

两式消去 $J_n(x)$ 得:

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \qquad \cdots(4)$$

**证毕!**

性质 4: 证明 n 阶贝塞尔函数有如下零点近似公式

$$\mu_m^n \approx m\pi + \frac{n\pi}{2} + \frac{3\pi}{4}$$

解:对 n(整数) 阶贝塞尔方程

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

做变量代换

$$y = \frac{u}{\sqrt{x}}$$

得到 u(x) 的方程:

$$u'' + [1 + \frac{\frac{1}{4} - n^2}{x^2}]u = 0$$

当 $x \to \infty$ 有方程:

$$u'' + u = 0$$

通解为:

$$u = A\cos(x+\theta)$$

确定 A 和 $\theta$ (不证), 得 n 阶贝塞尔函数的渐近公式

$$J_n(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{n\pi}{2} - \frac{\pi}{4}\right)$$

令 $J_n(x) = 0$, 由上式可得 n 阶贝塞尔函数的零点近似公式:

$$\mu_m^n \approx m\pi + \frac{n\pi}{2} + \frac{3\pi}{4}$$

对于圆域热传导方程或波动方程，其径向方程为

$$\begin{cases} r^2R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0 \\ R(r_0) = 0 \end{cases}$$

解为 n 阶贝塞尔函数 $R(r) = J_n(x) = J_n(\sqrt{\lambda} r)$

- 零边界条件 $R(r_0) = 0$ 对应 $J_n(\sqrt{\lambda} r_0) = 0$

因此零边界条件下的本征解，正是贝塞尔函数的零点！

基此确定:

(1) 固有值:

$$\sqrt{\lambda} r_0 = \mu_m^n \qquad \rightarrow \qquad \lambda_m^n = (\frac{\mu_m^n}{r_0})^2$$

(2) 固有函数:

$$R_m^n(r) = J_n(\frac{\mu_m^n}{r_0} r)$$

**性质 5:** 试证明 n 阶贝塞尔函数有如下零点递推式

$$J_n'(\mu_m^n) = -J_{n+1}(\mu_m^n)$$

**证明:** 由微分公式 (2)

$$\begin{aligned}
\frac{d}{dx}\left[x^{-n}J_n(x)\right] &= -\,x^{-n}J_{n+1}(x), \qquad \cdots (2) \\
-nx^{-n-1}J_n(x) + x^{-n}J_n'(x) &= -x^{-n}J_{n+1}(x) \\
-nJ_n(x) + xJ_n'(x) &= -xJ_{n+1}(x) \\
-n0 + xJ_n'(\mu_m^n) &= -xJ_{n+1}(\mu_m^n) \\
J_n'(\mu_m^n) &= -J_{n+1}(\mu_m^n)
\end{aligned}$$

## 性质 6: 贝塞尔函数正交归一性

固有函数体现塞尔函数的正交归一性:

$$\int_0^{r_0} rJ_n(\frac{\mu_m^n}{r_0}r)J_n(\frac{\mu_k^n}{r_0}r)dr = ?$$

**证明:**对径向方程做等价变换

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0$$

$$rR'' + R' + ((\frac{\mu_m^n}{r_0})^2 r - \frac{n^2}{r})R = 0$$

$$(rR')' + ((\frac{\mu_m^n}{r_0})^2 r - \frac{n^2}{r})R = 0$$

令:

$$J_n(\frac{\mu_m^n}{r_0}r) = R_1, \qquad J_n(\frac{\mu_k^n}{r_0}r) = R_2$$

有

$$(rR_1')' + ((\frac{\mu_m^n}{r_0})^2 r - \frac{n^2}{r})R_1 = 0 \cdots (1)$$

$$(rR_2')' + ((\frac{\mu_m^n}{r_0})^2 r - \frac{n^2}{r})R_2 = 0 \cdots (2)$$

$(1) \times R_2, (2) \times R_1$, 所得两次相减, 并做积分, 有

$$\int_0^{r_0} \left[ \left( \frac{\mu_m^{(n)}}{r_0} \right)^2 - \left( \frac{\mu_k^{(n)}}{r_0} \right)^2 \right] rR_1R_2dr = \int_0^{r_0} [R_1 \left(rR_2'\right)' - R_2 \left(rR_1'\right)']dr$$

$$
\begin{aligned}
=&[rR_1R_2']\big|_0^{r_0} - [rR_2R_1']\big|_0^{r_0} + \int_0^{r_0} rR_2'R_1 dr - \int_0^{r_0} rR_1'R_2 dr \\
=&\int_0^{r_0} rR_2'R_1 dr - \int_0^{r_0} rR_1'R_2 dr \\
=&0
\end{aligned}
$$

$$
\rightarrow \qquad \int_0^{r_0} rR_1R_2 dr = 0
$$

正交性，证毕！

证明归一性:

$$r^2R'' + rR' + (\lambda r^2 - n^2)R = 0$$

$$2r^2R'R'' + 2r(R')^2 + (\lambda r^2 - n^2)R'R = 0$$

整理:

$$[r^2(R')^2 + (\lambda r^2 - n^2)R^2]' = 2\lambda r R^2$$

$$
\begin{aligned}
\int_0^{r_0} rR^2 dr =& \frac{1}{2\lambda}\int_0^{r_0}[r^2(R')^2 + (\lambda r^2 - n^2)R^2]' dr \\
=& \frac{1}{2\lambda}|r^2(R')^2 + (\lambda r^2 - n^2)R^2|_0^{r_0} \\
=& \frac{1}{2\lambda} r_0^2 (R'(r_0))^2 \\
=& \frac{1}{2} r_0^2 [J_n'(\mu_m^n)]^2 \\
=& \frac{r_0^2}{2}[J_{n+1}(\mu_m^n)]^2
\end{aligned}
$$

1. 试证明

$$\Gamma(1/2) = \sqrt{\pi}, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

2. 试证明零点递推公式

$$J_n'(\mu_m^n) = -J_{n+1}(\mu_m^n)$$

3. 设函数 $f(r)$ 的贝塞尔展开式为

$$f(r) = \sum_{m=1}^{\infty} c_m J_n(\frac{\mu_m^n}{r_0} r)$$

试证明其展开系数为:

$$c_m = \frac{2}{r_0^2 J_{n+1}^2(\mu_m^n)} \int_0^{r_0} f(r) J_n(\frac{\mu_m^n}{r_0} r) r dr$$

# Contents

求解圆域热传导问题

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \left( \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} \right), 0 < r < R, 0 < \theta < 2\pi \\ u\big|_{r=R} = 0,\ u\big|_{t=0} = \varphi(r,\theta) \end{cases}$$

解: 令

$$u(r,\theta,t) = T(t)V(r,\theta)$$

代入方程，进行第一次分离变量，得衰减方程:

$$T' + \lambda a^2 T = 0, \qquad \cdots(1)$$

及亥姆霍兹方程:

$$\begin{cases} \dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} + \lambda V = 0, 0 < r < R, 0 \leq \theta \leq 2\pi \\ V\big|_{r=R} = 0, 0 \leq \theta \leq 2\pi \end{cases}$$

令

$$V(r,\theta) = F(r)G(\theta)$$

, 代入亥姆霍兹方程，得两个方程

$$G'' + \mu G = 0, \qquad \cdots (2)$$

$$r^2 F'' + rF' + (\lambda r^2 - \mu)F = 0, \qquad \cdots (3)$$

方程 (1) 的解为:

$$T(t) = Ae^{-\lambda a^2 t}$$

方程 (2) 的解为:

$$G(\theta) = C_1 \cos\sqrt{\mu}\theta + C_2 \sin\sqrt{\mu}\theta$$

由周期性边界条件, 有 $G(2\pi) = G(0)$, 必有 $\cos\sqrt{\mu}\theta = 1$, 得固有值:

$$\mu = n^2, \qquad (n = 0, 1, 2, \cdots)$$

固有函数:

$$G_0(\theta) = \frac{1}{2}a_0, G_n(\theta) = a_n \cos n\theta + b_n \sin n\theta, \qquad (n = 0, 1, 2, \cdots)$$

固有函数也可写成 $G_n(\theta) = a_n e^{-in\theta} = \dfrac{1}{\sqrt{2\pi}} e^{-in\theta}$

将固有值代入方程 (3)，得方程

$$r^2F'' + rF' + (\lambda r^2 - n^2)F = 0$$

令 $x = \sqrt{\lambda}r, y(x) = F(x/\sqrt{\lambda})$, 方程转化为标准整数贝赛尔方程:

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

则方程 (3) 的零边界条件解用贝赛尔函数的零点表示:
固有值:

$$\lambda_m^n = (\frac{\mu_m^n}{R})^2$$

固有函数:

$$F_m^n(r) = J_n(\frac{\mu_m^n}{R}r)$$

原方程的基本解为:

$$u(r,\theta,t) = F_m^n(r)G_n(\theta)e^{-\lambda_m a^2 t}$$

叠加解为:

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_m^n F_m^n(r)G_n(\theta)e^{-\lambda_m a^2 t}$$

应用初值条件,

$$\varphi(r,\theta) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_m^n F_m^n(r)G_n(\theta)$$

利用正交归一性确定系数 $A_m^n$

$$\int_0^{2\pi} G_k^*(\theta)\varphi(r,\theta)d\theta = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_m^n F_m^n(r)\int_0^{2\pi} G_n(\theta)G_k(\theta)d\theta$$

$$\int_0^{2\pi} G_n^*(\theta)\varphi(r,\theta)d\theta = \sum_{m=0}^{\infty} A_m^n F_m^n(r)$$

$$\int_0^{R}\int_0^{2\pi} G_n^*(\theta) r J_n(\frac{\mu_k^n}{R} r)\varphi(r,\theta)d\theta dr = \sum_{m=0}^{\infty} A_m^n \int_0^{R} r J_n(\frac{\mu_k^n}{R} r) J_n(\frac{\mu_m^n}{R} r) dr$$

$$\int_0^{R}\int_0^{2\pi} G_n^*(\theta) r J_n(\frac{\mu_m^n}{R} r)\varphi(r,\theta)d\theta dr = A_m^n \frac{R^2}{2}[J_{n+1}(\mu_m^n)]^2$$

$$A_m^n = \frac{2}{R^2[J_{n+1}(\mu_m^n)]^2}\int_0^{R}\int_0^{2\pi} G_n^*(\theta) r J_n(\frac{\mu_m^n}{R} r)\varphi(r,\theta)d\theta dr$$

# 课外作业

1、证明

$$J_{3/2}(x)=\sqrt{\frac{2}{\pi x}}[\frac{1}{x}\sin x-\cos x]$$

2、证明

$$\frac{d}{dx}[x^{-n}J_n(x)]=-x^{-n}J_{n+1}(x)$$

3、用分离变量法求解圆域热传导方程

$$\begin{cases} u_t=a^2(u_{xx}+u_{yy}), & (0<r<1) \\ u\mid_{r=1}=0 \\ u\mid_{t=0}=\cos^2\theta \end{cases}$$

4、求柱面坐标系的拉普拉斯并用分离变量法求解柱面热传导方程

# Contents

- 设有一条质量为 1，长度为 $2l$ 的均匀直线 (段). 则直线的线密度为 $\rho = 1/2l$

若将直线的中点放置于坐标轴的原点，则密度函数为：

$$\rho(x) = \begin{cases} \dfrac{1}{2l}, & (-l \le x \le l) \\ 0, & (x < -l, x > l) \end{cases}$$

积分得质量:

$$\int_{-\infty}^{+\infty} \rho(x) dx = 1$$

显然:

$$\int_a^b \rho(x) dx = 1, \qquad (-l, l) \subset (a, b)$$

## $\delta$ 函数的定义

- 考虑当 $l \to 0$ 时，线段变为质点. 质点的密度函数记为 $\delta(x)$，有

$$\delta(x) = \begin{cases} \infty, & (x = 0) \\ 0, & (x \neq 0) \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

量子力学中，这么定义：

$$\delta(x) = \begin{cases} 1, & (x = 0) \\ 0, & (x \neq 0) \end{cases}$$

$$\int_{-\infty}^{+\infty} \delta(x) dx = 1$$

考虑到量子化，一般写成一个连续实函数的序列:

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}\delta_n(x)dx = 1$$

*Tips:* 在很多问题中，我们需要用数学语言描述“点”的相关性质，这就需要推广古典函数概念，引入广义函数（a generalized function）.Dirac 函数是历史上第一个广义函数，也是使用最广的.

大气大为 求实求真

# $\delta$ 函数的性质

$$\int_a^b \delta(x)dx = 1, \quad 0 \in (a,b)$$

$$\int_{-\infty}^{+\infty} \delta(x)\psi(x)dx = \psi(0)$$

$$\int_a^b \delta(x)\psi(x)dx = \psi(0), \quad 0 \in (a,b)$$

$$\int_{-\infty}^{+\infty} \delta(x-x_0)\psi(x)dx = \psi(x_0)$$

$$\delta(-x) = \delta(x)$$

$$\delta'(-x) = -\delta'(x)$$

$$\delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

$$x\delta'(x) = -\delta(x)$$

$$x\delta_\lambda(x) = \lambda\delta_\lambda(x)$$

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

$$\delta(\sin x) = \sum_{n=0}^{\infty} \delta(x - n\pi)$$

**例-1. 试证明 $\delta$ 函数是偶函数:**

$$\delta(-x) = \delta(x)$$

**证明:**

$$\begin{aligned}
\int_{-\infty}^{+\infty} \delta(-x)\psi(x)dx &= \int_{+\infty}^{-\infty} \delta(x')\psi(-x')d(-x') \\
&= \int_{-\infty}^{+\infty} \delta(x')\psi(-x')d(x') \\
&= \psi(-x')|_{x'=0} \\
&= \psi(0) \\
&= \int_{-\infty}^{+\infty} \delta(x)\psi(x)dx
\end{aligned}$$

得证!

例-2. 试证明 $\delta$ 函数的导数是奇函数:

$$\delta'(-x) = -\delta'(x)$$

证明:

$$\begin{aligned}\int_{-\infty}^{+\infty} \delta'(x)\psi(x)dx &= \delta(x)\psi(x)|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(x)\psi'(x)dx \\ &= -\int_{-\infty}^{+\infty} \delta(x)\psi'(x)dx \\ &= -\psi'(0)\end{aligned}$$

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \delta'(-x)\psi(x)dx &= \int_{+\infty}^{-\infty} \delta'(x')\psi(-x')d(-x') \\
&= \int_{-\infty}^{+\infty} \delta'(x')\psi(-x')d(x') \\
&= \psi'(-x')_{x'=0} \\
&= \psi'(0)
\end{aligned}
$$

评毕!

推论:

$$
\int_{-\infty}^{+\infty} \delta^{(n)}(-x)\psi(x)dx = (-1)^n \psi^{(n)}(0)
$$

**例-3. 试证明:**

$$x\delta(x) = 0$$

**证明:**

$$\begin{aligned}\int_{-\infty}^{+\infty} x\delta(x)\psi(x)dx &= \int_{-\infty}^{+\infty} \delta(x)[x\psi(x)]dx \\ &= [x\psi(x)]\,|_{x=0} \\ &= 0\psi(0) \\ &= 0\end{aligned}$$

证毕!

**例-5. 试证明 $\delta$ 函数的傅里叶变换公式:**

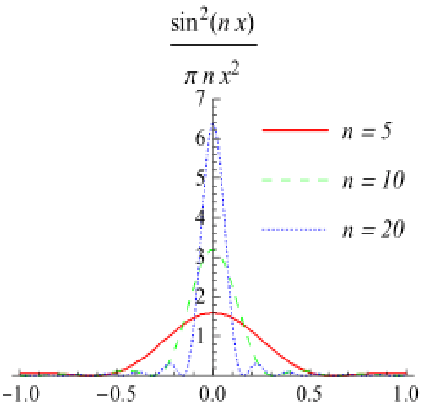
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ikx}dx = \delta(k), \qquad \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}p_x x}dx = \delta(p_x)$$

**证明:**

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{\frac{i}{\hbar}p_x x}dx &= \int_{-\infty}^{+\infty} e^{i\frac{p_x}{\hbar}x}dx \\
&= 2\pi\delta(\frac{p_x}{\hbar}) \\
&= 2\pi\left|\hbar\right|\delta(p_x) \\
&= 2\pi\hbar\delta(p_x)
\end{aligned}$$

## 例-6. 试证明 $\delta$ 函数的极限公式:

$$
\begin{aligned}
\delta(x) &= \lim_{n\to\infty} \frac{\sin^2(nx)}{\pi n x^2} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2} \\
&= \lim_{\epsilon\to 0} \frac{\epsilon}{\pi(x^2+\epsilon^2)} \\
&= \lim_{n\to\infty} \frac{n}{\pi(n^2x^2+1)}
\end{aligned}
$$



## 例-7. 试证明 $\delta$ 函数的微分公式:

$$\delta(x) = \frac{dH(x)}{dx}, \qquad H(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x >= 0) \end{cases}$$

**证明:**

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dH(x)}{dx}\psi(x)dx &= H(x)\psi(x) \mid_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} H(x)\psi'(x)dx \\
&= \psi(+\infty) - \int_{0}^{+\infty} \psi'(x)dx \\
&= \psi(+\infty) - \psi(+\infty) + \psi(0) \\
&= \psi(0) \\
&= \int_{-\infty}^{+\infty} \delta(x)\psi(x)dx
\end{aligned}$$

证毕!

例-8. 试证明 $\delta$ 函数的展开公式:

$$\begin{aligned}\delta(\vec{r}-\vec{r}_0) &= \delta(x-x_0)\delta(y-y_0)\delta(z-z_0)\\ &= \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0)\\ &= \frac{1}{r^2\sin\theta}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)\end{aligned}$$

## 1. 试证明:

$$\int_{-\infty}^{+\infty} 1 \cdot e^{2\pi ikx} dx = \delta(k)$$

## 2. 求动量表象波函数

已知坐标表象的波函数如下，现基于 $\delta$ 函数求动量表象的波函数 $c(p)$

$$\Psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} c(p) e^{\frac{i}{\hbar}px} dp$$

## 3. 证明第一类贝塞尔函数的正交性公式

$$\int_{0}^{+\infty} J_n(kr) J_n(k'r) r dr = \frac{\delta(k-k')}{k}, \qquad (n \geq -1; k, k' > 0)$$

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# Thanks for your attention!

# A & Q

求實求真
大氣大為