Dual Method

Dual Problem Formulation

If the original problem is in the standard form, then the dual problem can be formulated using the following rules:

- The number of constraints in the original problem is equal to the number of dual variables, the number of constraints in the dual problem is equal to the number of variables in the original problem.
- The original problem profit coefficients appear on the right hand side of the dual problem constraints
- If the original problem is a maximization problem, then the dual problem is a minimization problem. Similarly, if the original problem is a minimization problem then the dual problem is a maximization problem.
- The original problem has less than or equal to (≤) type of constraints while the dual problem has greater than or equal to (≥) type constraints.
- The coefficients of the constraints of the original problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

Consider the following pair of LP problems

$$\begin{aligned} \text{Maximize} \quad & z = c^T x \\ \text{subject to} \quad & Ax \le b \\ & x \ge 0 \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \text{Minimize} \quad & z' = b^T y \\ \text{subject to} \quad & A^T y \ge c \\ & y \ge 0 \qquad (2) \end{aligned}$$

where A is m x n , b is m x 1, c is n x 1 , "x" is n x 1 and "y" is m x 1

These problems are called **dual problems.** The problem given by (1) is called **primal problem**, the problem given by (2) is called the **dual problem.**

Example

If the primal problem is

Max

$$z = [3\ ,4] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{s.t} \begin{bmatrix} 2 & 3 \\ 3 & -1 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

$$x_1 \ge 0, x_2 \ge 0$$

Then the dual problem

$$\text{Min } z' = [3 \quad 4 \quad 2] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

s.t $\begin{bmatrix} 2 & 3 & 5 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Determine the dual problem

1) Min'

$z' = 3y_1 + 5y_2$

s.t
$2y_1 + 3y_2 \geq 7$
$8y_1 - 9y_2 \geq 12$
$10y_1 + 15y_2 \geq 18$
$y_1 \geq 0, y_2 \geq 0$

$\Rightarrow$

Dual problem

Max $z = 7x_1 + 12x_2 + 18x_3$

s.t
$2x_1 + 8x_2 + 10x_3 \leq 3$
$3x_1 - 9x_2 + 15x_3 \leq 5$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

Rewrite the dual problem

Max $z - 7x_1 - 12x_2 - 18x_3 = 0$

s.t
$2x_1 + 8x_2 + 10x_3 + u = 3$
$3x_1 - 9x_2 + 15x_3 + v = 5$
$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, u \geq 0, v \geq 0$

Then proceed to Simplex Algorithm.

Tableau 1

B. V	x_1	x_2	x_3	u	v	RHS
u	2	8	10	1	0	3
v	3	-9	15	0	1	5
z	-7	-12	-18	0	0	0

$1/k = 1/10$

u-row

$2\left(\frac{1}{10}\right) = \frac{1}{5}$

$8\left(\frac{1}{10}\right) = \frac{4}{5}$

$10\left(\frac{1}{10}\right) = 1$

$1\left(\frac{1}{10}\right) = \frac{1}{10}$

$0\left(\frac{1}{10}\right) = 0$

$3\left(\frac{1}{10}\right) = \frac{3}{10}$

v-row

$3-\left(\frac{1}{5}\right)(15) = 0$

$-9-\left(\frac{4}{5}\right)(15) = -21$

$15-(1)(15) = 0$

$0-\left(\frac{1}{10}\right)(15) = \frac{-3}{2}$

$1-(0)(15) = 1$

$5-\left(\frac{3}{10}\right)(15) = \frac{1}{2}$

z-row

$-7-\left(\frac{1}{5}\right)(-18) = \frac{-17}{5}$

$-12-\left(\frac{4}{5}\right)(-18) = \frac{-12}{5}$

$-18-(1)(-18) = 0$

$0-\left(\frac{1}{10}\right)(-18) = \frac{9}{5}$

$0-(0)(-18) = 0$

$0-\left(\frac{3}{10}\right)(-18) = \frac{27}{5}$

Tableau 2

B. V	x_1	x_2	x_3	u	v	RHS
x_3	1/5	4/5	1	1/10	0	3/10
v	0	-21	0	-3/2	1	1/2
z	-17/5	-12/5	0	9/5	0	27/5

$1/\text{k} = \frac{1}{1/5} = 5$

x_3-row

$\frac{1}{5}(5) = 1$

$\frac{4}{5}(5) = 4$

$1(5) = 5$

$\frac{1}{10}(5) = \frac{1}{2}$

$0(5) = 0$

$\frac{3}{10}(5) = \frac{3}{2}$

v-row

$0 - (1)(0) = 0$

$-21 - (4)(0) = -21$

$0 - (5)(0) = 0$

$\frac{-3}{2} - \left(\frac{1}{2}\right)(0) = 1$

$1 - (0)(0) = 1$

$\frac{1}{2} - \left(\frac{3}{2}\right)(0) = \frac{1}{2}$

z-row

$\frac{-27}{5} - (1)\left(\frac{-17}{5}\right) = 0$

$-\frac{12}{5} - (4)\left(\frac{-17}{5}\right) = 16$

$0 - (5)\left(\frac{-17}{5}\right) = 17$

$\frac{9}{5} - \left(\frac{1}{2}\right)\left(\frac{-17}{5}\right) = \frac{7}{2}$

$0 - (0)\left(\frac{-17}{5}\right) = 0$

$\frac{27}{5} - \left(\frac{3}{2}\right)\left(\frac{-17}{5}\right) = \frac{21}{2}$

Tableau 3

B. V	x_1	x_2	x_3	u	v	RHS
x_1	1	4	5	1/2	0	3/2
v	0	-21	0	-3/2	1	1/2
z	0	16	17	7/2	0	21/2

The optimal solution

$x_1 = 3/2$

$x_2 = 0$

$x_3 = 0$

$u = 0$

$v = ½$ and the value of z is 21/2

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<u>WORKSHEET</u>

General Instruction: Fill out completely the student's information and write all your solutions/answers on the space provided below each item.

Student's Information:

Last Name, First Name M.I.: ____

Student Number: ____________Course – Year: ______________ Date of Submission: ___________ Class ID Number: ___________ Professor/Instructor's Name: ________________________________

A. Solve the linear programming problems by Dual Method