Overview

In the previous module, we used graphical method to solve linear programming problem while here, we will be using another method which is called as simplex method. Simplex methods are tools used to solve linear programming problems which provides an algorithm which is based on the fundamental theorem of linear programming. This is another method used to obtain the optimal solution of a linear system of constraints, given a linear objective function.

The simplex algorithm has many steps and rules, so it is helpful to understand the logic behind these steps and rules with a simple example before proceeding with the formal algorithm.

Learning Outcomes

After working on this module, you will be able to:

1. define linear programming;
2. demonstrate simplex algorithm
3. solve linear programming problem by simplex method;
4. determine an optimal solution for each linear programming problem by simplex method;

Simplex Method

How to solve Linear Programming Problems(LPP) using Simplex Method?

Step 1: Convert the LPP into the standard form before we can solve this using simplex method.

Standard Form of an LPP

The following are the characteristics of a LPP in standard form.

i) All the constraints should be expressed as equations by adding slack or surplus and/ or artificial variables.

ii) The right hand side of each constraint should be made non negative if it is not, this should be done by multiplying both sides of the resulting constraints by -1.

iii) The objective function should be of the maximization type

Key Terms

Objective Function - The objective function is a function that defines some quantity that should be minimized or maximized. The arguments of the objective function are the same variables that are used in the constraints. In order for linear programming techniques to work, the objective function should be linear.

Constraints - A constraint is an inequality that defines how the values of the variables in a problem are limited. In order for linear programming techniques to work, all constraints should be linear inequalities.

Decision variable - The *variables* in a linear program are a set of quantities that need to be determined in order to solve the problem; i.e., the problem is solved when the best values of the variables have been identified. The variables are sometimes called *decision variables* because the problem is to decide what value each variable should take. Typically, the variables represent the amount of a resource to use or the level of some activity. In this module, if there will be only two variables, we will be using x and y, but if there are more than two variables we will be using x_1, x_2, . . ., x_n.

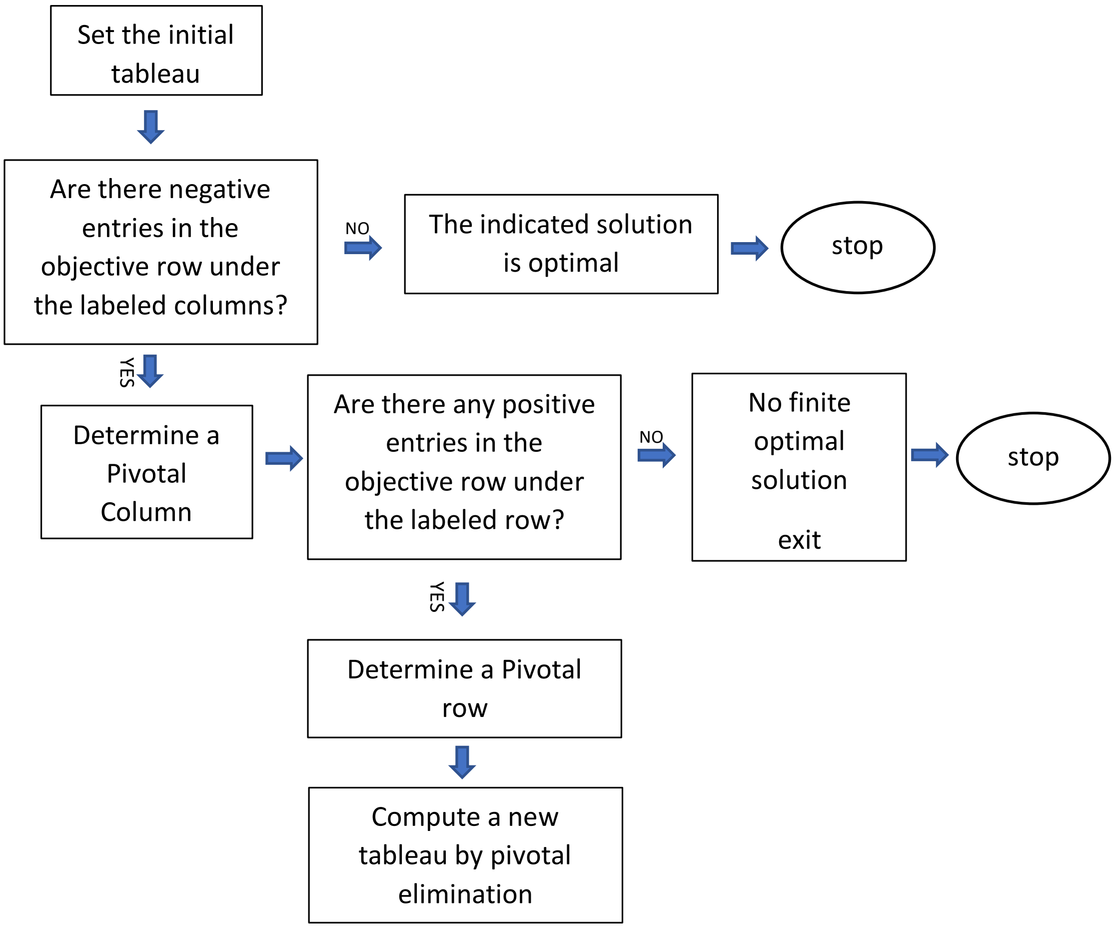
Slack Variable – a slack variable represents unused resource, either in the form of time on a machine, labor hours, money, warehouse space or any number of such resources in various business problems. Since these variables yield no profit. Therefore, such variables are added to the original objective function with zero coefficients. Slack variables are also defined as the non-negative variables which are added in the left hand side of the constraints to convert the inequality "≤" into an equation. This we will be designated as u, v . . . :

Surplus Variable – A surplus variable represents amount by which solution values exceed a resource. These variables are also called negative slack variables. Surplus variables, like slack variable carry a zero coefficient in the objective function. Surplus variables which are removed from the left hand side of the constraints to convert the inequality "≥" into an equation. This we will be designated as S:

Artificial Variable - The artificial variable for each equality and "≥ type" constraint is introduced to obtain an initial basic feasible solution for the auxiliary problem. These variables have no physical meaning and need to be eliminated from the problem. To eliminate the artificial variables from the problem, we define an auxiliary cost function called the *artificial cost function* and minimize it subject to the constraints of the problem and the non-negativity of all of the variables. The artificial cost function is simply a sum of all the artificial variables and will be designated as R:

Step 2: Find the initial solution by using simplex algorithm

Structure of an algorithm



In the simplex method, a start is made with a basic feasible solution, which we shall bet by assuming that the objective function value Z= 0. This will be so when decision variables $x_1, x_2, \dots, x_n$ each equal to zero. These variables are called non-basic variables.

Example: Using simplex method, find the values of x and y that will

maximize $z = 8x + 10y$

subject to the constraints:

$2x + y \leq 50$
$x + 2y \leq 70$
$x, y \geq 0$

Solution

Step 1: To convert the LPP into standard form, we need to rewrite the problem and add the slack or surplus variables to the constraints such that

$z - 8x - 10y = 0$, transpose the expression in the right hand side to the left hand side of the equation and equate it to zero

subject to the constraints

$2x + y + u = 50$, add the slack variable "u" because this constraint is in "≤" type.
$x + 2y + v = 70$, add another slack variable "v" for the second constraint because this is also in "≤" type.
$x, y, u, v \geq 0$, take note that decision, slack and surplus variables must not be negative

Step 2: Set the Initial Tableau

Tableau 1:

B.V	x	y	u	v	RHS
u	2	1	1	0	50
v	1	2	0	1	70
z	-8	-10	0	0	0

Put in the table the coefficients of the variables in each constraint and objective function

Note: B.V – Basic Variables
RHS – Right Hand Side of the equation
x & y – Decision variables
u & v – slack variables
z- objective function

Step 3: Find the most negative in the objective function row and encircle the column along this. In this problem we have, (this is called as **pivotal column**)

Tableau 1:

B.V	x	y	u	v	RHS
u	2	1	1	0	50
v	1	2	0	1	70
z	-8	-10	0	0	0

Step 3: Determine the pivotal row by getting the smallest ratio of the positive entries in the pivotal column with the right hand side entries.

Tableau 1:

Entering variable

B.V	x	y	u	v	RHS
u	2	1	1	0	50
v	1	2	0	1	70
z	-8	-10	0	0	0

Exiting variable

pivot

50/1 = 50

70/2 = 35

Since the smallest ratio is 70/2 = 35, thus the pivotal row is the v-row.

The intersection of the pivotal row and column is called **pivot**. In this case, the pivot is 2.

Step 4: (Pivotal Elimination) Multiply 1/k (where k is the pivot) to the entries of the pivotal row.

v-row	u-row	z-row
$1\left(\frac{1}{2}\right) = \frac{1}{2}$	$2 - \left(\frac{1}{2}\right)(1) = \frac{1}{2}$	$-8 - \left(\frac{1}{2}\right)(-10) = -3$
$2\left(\frac{1}{2}\right) = 1$	$1 - (1)(1) = 1$	$-10 - (1)(-10) = 0$
$0\left(\frac{1}{2}\right) = 0$	$1 - (0)(1) = 0$	$0 - (0)(-10) = 0$
$1\left(\frac{1}{2}\right) = \frac{1}{2}$	$0 - \left(\frac{1}{2}\right)(1) = \frac{1}{2}$	$0 - \left(\frac{1}{2}\right)(-10) = 5$
$70\left(\frac{1}{2}\right) = 35$	$50 - (35)(1) = 15$	$0 - (35)(-10) = 350$

Note that we always need to calculate first the pivotal row before the other rows. Then the entries in the next row would be subtracted by the product of its pivot and the answers from the first row.

Step 5: Make another tableau based on the results in step 4. Then repeat this step until the objective function row won't have negative entries.

Take note of the entering and exiting variables. Entering variable is the variable in pivotal column and exiting variable is the variable in pivotal row. In tableau 1, we have y and v respectively. (You may refer to previous tableau). We just need to replace the exiting variable v by the entering variable y.

Tableau 2:

B.V	x	y	u	v	RHS
u	3/2	0	1	1/2	15
y	1/2	1	0	1/2	35
z	-3	0	0	5	350

Follow the same procedure in tableau 1. Here pivot is 3/2. Thus, 1/k is 2/3.

Example

u-row	y-row	z-row
$\frac{3}{2}\left(\frac{2}{3}\right) = 1$	$\frac{1}{2} - (1)\left(\frac{1}{2}\right) = 0$	3 - (1) (-3)= 0
$0\left(\frac{2}{3}\right) = 0$	$1 - (0)\left(\frac{1}{2}\right) = 1$	0 - (0) (-3)= 0
$1\left(\frac{2}{3}\right) = 2/3$	$0 - \left(\frac{2}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{3}$	$0 - \left(\frac{2}{3}\right)(-3) = 2$
$\frac{-1}{2}\left(\frac{2}{3}\right) = -\frac{1}{3}$	$\frac{1}{2} - \left(-\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{2}{3}$	$5 - \left(\frac{-1}{3}\right)(-3) = 4$
$15\left(\frac{2}{3}\right) = 10$	$35 - (10)\left(\frac{1}{2}\right) = 30$	350 - (10)(-3) = 380

Tableau 3:

B.V	x	y	u	v	RHS
x	1	0	2/3	-1/3	10
y	0	1	-1/3	2/3	30
z	0	0	2	4	380

As you can see, the objective function row has no negative entry, thus this ends our solution. Therefore, the optimal solution is $x = 10$, $y = 30$ and $z = 380$.
We can check this by substituting the values in the original equation, such that

$$\begin{aligned} z &= 8x + 10y \\ 380 &= 8(10) + 10(30) \\ 380 &= 80 + 300 \\ 380 &= 380 \end{aligned}$$

Therefore, our solution is correct.

References

Nocon, et.al, Essential Mathematics for the Modern World
Hoppin, Operations Research and Management Information Science
H.Lyeme,et.al, Introduction to Operations Research : Theory and Applications
https://www.techopedia.com/definition/20403/linear-programming-lp
https://www.sciencedirect.com/topics/social-sciences/quantitative-methodlpp

Suggested Readings

https://www.analyticsvidhya.com/blog/2017/02/lintroductory-guide-on

2 WORKSHEET

General Instruction: Fill out completely the student's information and write all your solutions/answers on a separate sheet of bond paper then attach to this worksheet.

Student's Information:
Last Name, First Name M.I.: ____
Student Number: ____________Course – Year: ______________ Date of Submission: ___________
Class ID Number: ___________ Professor/Instructor's Name: ________________________________

A. Find the optimal solution of each linear programming problem using simplex method.

1. Max $z = 120x + 100y$

   s.t $2x + 2y \leq 8$
   $5x + 3y \leq 15$
   $x, y \geq 0$

2. Max $z = 3x + 7y$

   s.t $3x - 2y \leq 7$
   $2x + 5y \leq 6$
   $2x + 3y \leq 8$
   $x, y \geq 0$

3. Max $z = 3x + 4y$

   subject to the constraints

   $2x - 3y \leq 5$
   $4x + y \leq 3$
   $x, y \geq 0$

   (You can compare your answer in previous module solved by graphical method)