Overview

In a wide range of industries, including transportation, supply chain management, and others, quantitative methods are employed in decision-making. There are various techniques that fall under the category of quantitative approaches, but in this module, we will only cover mathematical techniques, particularly linear programming, which refers to the formulation and solution of optimization problems using mathematical models and algorithms. To get the greatest results, this is used to figure out the best strategy to handle operating difficulties, especially in business and industrial engineering. Within all of its constraints, including those related to money, energy, people, machine resources, time, space, and several other factors, this entails either increasing profit or output or reducing cost or consumption. This topic focuses on the mathematical formulation, graphic analysis, and solution of linear programming problems.

Learning Outcomes

After working on this module, you will be able to:

1. define linear programming;
2. formulate linear programming problem;
3. formulate mathematical model of linear programming problems;
4. determine an optimal solution for each linear programming problem by graphical method;

Introduction to Linear Programming

Optimization is a part of our everyday lives. We all have limited resources and time, and we all want to maximize them. Everything uses optimization, from how you spend your time to how you solve supply chain issues for your company.

Linear programming (LP) is one of the most basic methods of optimization. By making a few simplifying assumptions, it can assist you in solving some very complex optimization problems. We will undoubtedly encounter applications and issues that can be solved by Linear Programming. One example is diet problem which we have to decide the food combination we have to buy with cheaper price but satisfies the requirement of our diet. Another non-profit linear programming problem is the space allocation. It has known to be the allocation of resources to areas of space such as rooms. One example is the study of Frimpong and Owusu about the classroom space allocation which refers to the distribution of the available areas of classroom space among a number of courses with different sizes of student population so as to ensure the optimal space utilization.

With the help of linear programming, business processes will be more efficient, room spaces and storage will be utilized well, cost will be minimized without compromising the quality of the product.

This chapter begins with a review of linear inequalities in two variables.

3.1 Linear Inequalities in Two Variables

Definition

A linear inequality in two variables is a statement in the form $Ax + By > C$, $Ax + By < C$, $Ax + By \geq C$, or $Ax + By \leq C$ where A and B are not both 0.

The solution of linear inequality in two variables is an ordered pair (x, y) where x is the first component

Example 1 Inequality in Two Variables

1. Is (2,3) a solution to the inequality $2x + 5y > 3$?
2. Is (4,2) a solution to the inequality $3x - 2y < 4$?
3. Is (5,1) a solution to the inequality $9x + 7y \leq 5$?
4. Is (7,2) a solution to the inequality $4x - 3y \geq 6$?

and y is the second component. The statement is true when the values of x and y satisfy the inequality.

Solution

1. Given is an ordered pair (2,3), then $x = 2$ and $y = 3$. We need to show that $2x + 5y > 3$ when the values of x and y are substituted to this inequality.

   Answer:

   $$\begin{aligned} 2x + 5y &> 3 \\ 2(2) + 5(3) &> 3 \\ 4 + 15 &> 3 \\ 19 &> 3 \end{aligned}$$

   Since the statement is true, thus (2,3) is a solution to the inequality $2x + 5y > 3$.

2. Given is an ordered pair (4,2), then x= 4 and y= 2. We need to show that $3x - 2y < 4$ when the values of x and y are substituted to this inequality.

   Answer:

   $$\begin{aligned} 3x - 2y &< 4 \\ 3(4) - 2(2) &< 4 \\ 12 - 4 &< 4 \\ 8 &< 4 \end{aligned}$$

   Since the statement is false, thus (4,2) is not a solution to the inequality $3x - 2y < 4$.

3. Given is an ordered pair (5,1), then x= 5 and y= 1. We need to show that $9x + 7y \le 5$ when the values of x and y are substituted to this inequality. Note that $9x + 7y \le 5$ means either $9x + 7y < 5$ or $9x + 7y = 5$.

   Answer:

   $$\begin{aligned} 9x + 7y &\le 5 \\ 9(5) + 7(1) &\le 5 \\ 45 + 7 &\le 5 \\ 52 &\le 5 \end{aligned}$$

   Since 32 is not less than and not equal to 5, hence the statement is false. Thus (5,1) is not a solution to the inequality $9x + 7y \le 5$.

4. Given is an ordered pair (7,2), then x= 7 and y= 2. We need to show that $4x - 3y \ge 6$ when the values of x and y are substituted to this inequality. Note that $4x - 3y \ge 6$ means either $4x - 3y > 6$ or $4x - 3y = 6$.

   Answer:

   $$\begin{aligned} 4x - 3y &\ge 6 \\ 4(7) - 3(2) &\ge 6 \\ 28 - 6 &\ge 6 \\ 22 &\ge 6 \end{aligned}$$

Since 22 is greater than 6, hence the statement is true. Thus (7,2) is a solution to the inequality
$4x - 3y \ge 6$.

Graphing Linear Inequality in Two Variables

Step 1: Write the inequality into linear equation. Then graph the linear equation.
Step 2: Find the x and y intercepts of the equation. Let x equal to zero or let y equal to zero
Step 3: Test the inequality by selecting a point that is not on the line
Step 4: Plot the intercept points. Then connect the two points.
Step 5: Shade the region on the other side of the boundary line.

2

Example

Graph the following linear inequalities

1. $y > x + 2$
2. $y > 2x + 3$
3. $y \geq -3x + 6$
4. $x \geq -4$

Solution

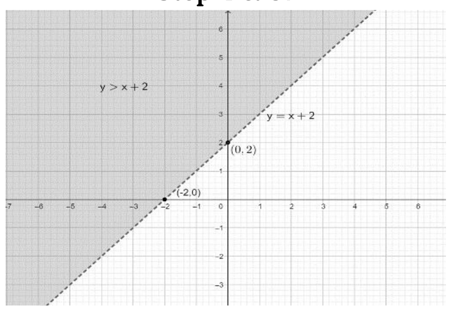
1) **Step 1:** Consider the equation $y = x + 2$. (Graph of this is shown in Step 4 & 5)

Step 2: To get the x and y intercepts,

Let $x=0$
$y = (0) + 2$
$y = 2$, thus (0, 2)

Let $y = 0$
$0 = x + 2$
$x =-2$, thus (-2, 0)

Step 4 & 5:



Plot the points (0,2) and (-2,0), then connect these points as shown in the figure.

Step 3: Let's pick a test point (0,0). Substitute $x=0$ and $y=0$ into $y > x + 2$,

$0 > 0 + 2$
$0 > 2$

The statement is false. Thus, (0,0) does not satisfy the inequality, so the shaded region does not contain (0,0).
The graph of inequality $y > x + 2$ is the shaded region shown in the figure, not including the line $y = x + 2$

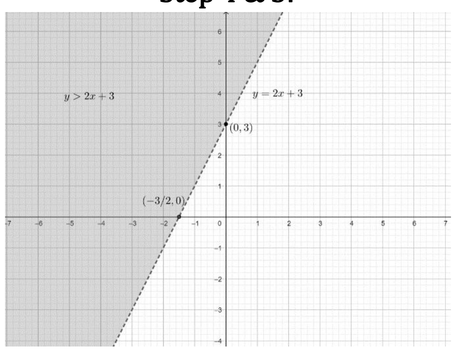
2) **Step 1:** Consider the equation $y = 2x + 3$. (Graph of this is shown in the figure below)

Step 2: To get the x and y intercepts,

Let $x = 0$
$y = 2(0) + 3$
$y = 3$, thus (0,3)

Let $y = 0$
$0 = 2x + 3$
$- 3 = 2x$
$x = -3/2$, thus (-3/2, 0)

Step 4 & 5:



Plot the points (0,3) and (-3/2, 0), then connect these points as shown in the figure.

Step 3: Let's pick a test point (0,0). Substitute x=0 and y=0 into y > 2x + 3,

0>2(0) + 3

0>3

The statement is false. Thus, (0,0) does not satisfy the inequality, so the shaded region does not contain (0,0).

The graph of inequality y > 2x + 3 is the shaded region, not including the line y = 2x + 3.

3) **Step 1:** Consider the equation y = -3x + 6. (Graph of this is shown in the figure below)

Step 2: Then to get the x and y intercepts,

Let x = 0

y = -3(0) + 6

y = 6, thus (0,6)

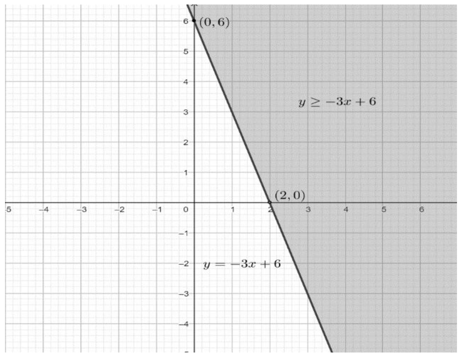
Let y = 0

0 = -3x + 6

3x = 6

x = 2, thus (2,0)

Step 4 & 5:



Plot the points (0,6) and (2,0), then connect these points as shown in the figure.

Step 3: Let's pick a test point (0,0). Substitute x=0 and y=0 into y ≥ -3x + 6,

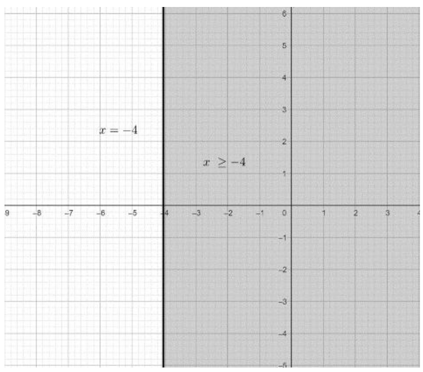
0≥-3(0) + 6

0≥6

The statement is false. Thus, (0,0) does not satisfy the inequality, so the shaded region does not contain (0,0).

The graph of inequality y ≥ -3x + 6 is the shaded region, including the line y = -3x + 6.

4) Consider the equation x =-4. This only means that the value of y varies while x = -4. (Graph of this equation is shown in the figure below)



The graph of inequality x ≥ -4 is the shaded region, including the line x = -4.

System of Linear Inequalities in Two Variables

The system of linear inequalities is composed of two or more linear inequalities. In solving this system, we need to graph linear inequalities together on the same rectangular coordinate system, by following the same procedure presented in the previous lesson. The solution set would be the shaded region common to the given inequalities.

3

Example

Solve the system of inequalities by graphing

1. $4x + y \geq 8$
   $x \leq 3$

2. $3x + 2y \leq 3$
   $4x + 5y \geq 4$

Solution

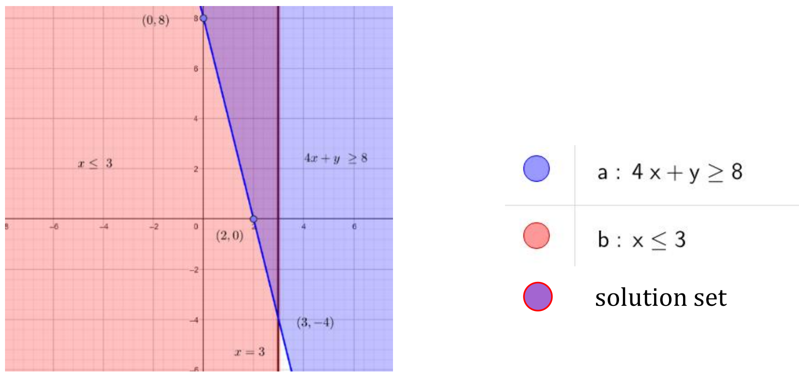
1. Consider the equations $4x + y = 8$ and $x = 3$, then to get the x and y intercepts,

Let x= 0

$$4x + y = 8$$
$$4(0) + y = 8$$
$$y = 8, \text{ thus } (0,8)$$

Let y = 0

$$4x + 0 = 8$$
$$4x = 8$$
$$x = 2, \text{ thus } (2,0)$$



Thus, the solution set of this system are the points on and inside the shaded region common to the inequalities $4x + y \geq 8$ and $x \leq 3$.

Let's check by using a point (5/2, 4) that is inside the region. Since 5/2 is less than 3, thus it satisfies the inequality $x \leq 3$. Then show that this point also satisfies the inequality $4x + y \geq 8$.

Answer:

a) $$4x + y \geq 8$$
$$4(5/2) + 4 \geq 8$$
$$10 + 4 \geq 8$$
$$14 \geq 8$$

Since the statement is true, thus (5/2, 4) is a solution of the system.

2. Consider the equations $3x + 2y = 3$ and $4x + 5y = 4$, then to get the x and y intercepts,

Let x = 0

$$3(0) + 2y = 3$$
$$2y = 3$$
$$y = 3/2, \text{ thus } (0,3/2)$$

Let x = 0

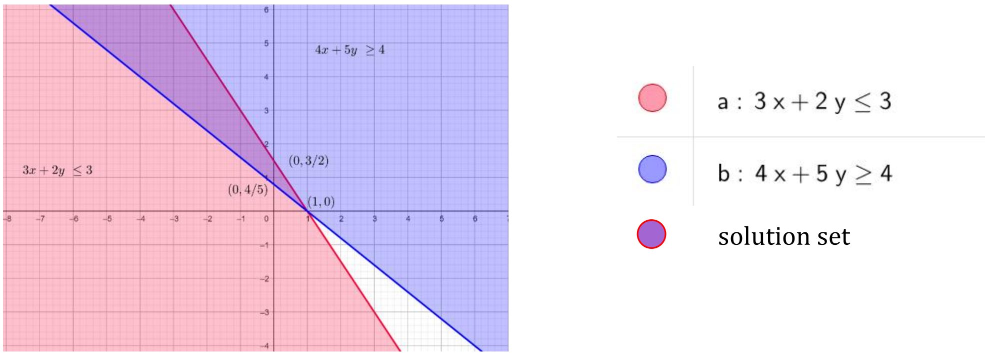
$$4(0) + 5y = 4$$
$$5y = 4$$
$$y = 4/5, \text{ thus } (0,4/5)$$

Let y = 0

$$3x + 2(0) = 3$$
$$3x = 3$$
$$x = 1, \text{ thus } (1,0)$$

Let y = 0

$$4x + 5(0) = 4$$
$$4x = 4$$
$$x = 1, \text{ thus } (1,0)$$



Thus, the solution set of this system are the points on and inside the shaded region common to the inequalities $3x + 2y \leq 3$ and $4x + 5y \geq 4$.

Let's check by using a point (-1, 2) that is inside the region. Show that this point satisfies the inequalities $3x + 2y \leq 3$ and $4x + 5y \geq 4$.

Answer:

a)
$$3x + 2y \leq 3$$
$$3(-1) + 2(2) \leq 3$$
$$-3 + 4 \leq 3$$
$$1 \leq 3$$

b)
$$4x + 5y \geq 4$$
$$4(-1) + 5(2) \geq 4$$
$$-4 + 10 \geq 4$$
$$6 \geq 4$$

Since the statements a) and b) are true, thus (-1,2) is a solution of the system.

3.2 Self-Assessment Activity 1

Graph each system of inequalities and give at least two solution points.

1. $2x + y < 5$
   $x > 1$

2. $x + y > 5$
   $x - y < 3$

3. $3x + y \geq 9$
   $y \leq 6$

The concept of linear inequalities in two variables has an important role in dealing with linear programming problem. In the next lesson, we shall apply the systems of linear inequalities or equations to solve real life problems which all relations are linear and to find an optimal solution using tools and techniques under linear programming.

In this chapter, we will study some of the linear programming problems and their solution by graphical method.

In linear programming problem, the linear function that has to be optimized is called the **objective function** and the limitations given as the system of inequalities are referred as **constraints**.

Rules in mathematical formulation of linear programming problem is presented below.

3.3 Mathematical Formulation of Linear Programming Model

a. Objective Function- refers to the goal of either maximizing profit or minimizing cost. This is in the form of a linear function.

b. Decision variables – are the controllable factors or quantities. When there is at most two quantities given, we can use x and y as decision variables but if it is more than two, we can use $x_1, x_2 \ldots x_n$ and $y_1, y_2 \ldots y_n$ as decision variables.

c. Constraints/Restrictions – refers to the limitations such as time, space, money and so on. The decision variables that will be using in objective function must be the same with the constraints.

General formulation can be stated as follows; Find $x_1, x_2 \ldots x_n$ which optimize the linear function,

$$z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to the constraints

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots a_{1n}x_n \ (\leq = \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots a_{2n}x_n \ (\leq = \geq)\ b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots a_{in}x_n \ (\leq = \geq)\ b_i \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mi}x_j + \ldots a_{mn}x_n \ (\leq = \geq) b_n \end{array}$$

Where all a_{ij}, b_i and C_j are constants and x_i are variables.

Procedure in Formulating Linear Programming Model

Step 1: Write down the decision variables of the problem
Step 2: Formulate the objective function to be optimized (Maximized or Minimized) as a linear function of the decision variable.
Step 3: Formulate the constraints based on the problem
Step 4: Add the statement of non-negativity of the decision variables.

4 Linear Programming Problems

Example

Determine the Mathematical model of the given Linear Programming Problems.

(Product-Mix Problem)

1. A company makes to two types of soaps: bath soap and laundry soap. It cost \$5 and takes 3 hours to produce a bath soap. It costs \$3 and takes 4 hours to produce a laundry soap. The factory has \$200 and 100 hours this week to produce these products. If each bath soap sells for \$6 and each laundry soap sells for \$4, then how many of each product should be produced to maximize the profit?

(Diet-Problem) – This is to find a low-cost diet that would meet the nutritional needs of a person.

2. Lylia is trying to decide on lowest cost diet that provides sufficient amount of proteins, fats and carbohydrates. Suppose that there are three foods available, fish, fruit and bread. Dietary instructions are that she needs at least 40 grams of protein, 36 grams of carbohydrate and 25 grams of fat per week.

   The availability of protein, fats and carbohydrates (in grams) per kg of fish, fruit and bread is as given below.

	Protein	Fat	Carbohydrate	Price/kg
Fish	26	20	13	80
Fruit	18	10	30	50
Bread	20	7	12	60

Formulate a suitable mathematical model for the above diet mix in order to minimize the cost. How many kg of each food she needs to eat per week?

Solution:

1) **Step 1: Decision Variables**

   Let x be the number of bath soaps
   Let y be the number of laundry soaps

   Step 2: Objective Function

   Each bath soap sells for \$6 and each laundry soap sells for \$4. Thus to maximize the profit, (in dollars) we have the following objective function,

$$z = 6x + 4y$$

   Step 3: Constraints

   The cost of bath soaps and laundry soaps is \$3 and \$2 respectively. This can be written as,

$$3x + 2y$$

   Since the company has \$200, thus the first constraint can be expressed as,

$$3x + 2y \leq 200$$

   It takes 3 hours to produce a bath soap and 4 hours to produce a laundry soap. This can be written as,

$$3x + 4y$$

   Since the available time per week is 100 hours, thus the second constraint can be expressed as,

$$3x + 4y \leq 100$$

   Step 4: Non-negativity of decision variables

   Since there is no negative production, the values of the decision variables must be positive. This is in addition to the above constraints, which may be represented as

$$x \geq 0$$
$$y \geq 0$$

Therefore,

Linear Programming Model

Find the value of x and y that will maximize

$$\text{Max} \quad z = 6x + 4y$$

subject to the constraints

$$3x + 2y \leq 200$$
$$3x + 4y \leq 100$$
$$x \geq 0, y \geq 0$$

Solution:

2) **Step 1: Decision Variables**

Let x_1 be the no. of kilos of fish
Let x_2 be the no. of kilos of fruit
Let x_3 be the no. of kilos of rice

Step 2: Objective Function

The cost of fish, fruit and rice is 80, 50 and 60 pesos respectively. Thus to minimize the cost, we have the following objective function,

$$z = 80x_1 + 50x_2 + 60x_3$$

Step 3: Constraints

The number of protein (in grams) present per kilo of fish, fruit and rice can be written as,

$$26x_1 + 18x_2 + 20x_3$$

Since lylia needs at least 40 grams of protein per week, thus

$$26x_1 + 18x_2 + 20x_3 \geq 40$$

The number of fats (in grams) present per kilo of fish, fruit and rice can be written as,

$$20x_1 + 10x_2 + 7x_3$$

Since lylia needs at least 36 grams of fats per week, thus

$$20x_1 + 10x_2 + 7x_3 \geq 36$$

The number of carbohydrates (in grams) present per kilo of fish, fruit and rice can be written as,

$$13x_1 + 30x_2 + 12x_3$$

Since lylia needs at least 25 grams of carbohydrates per week, thus

$$13x_1 + 30x_2 + 12x_3 \geq 25$$

Step 4: Non-negativity of decision variables

Since there is no negative production, the values of the decision variables must be positive. This is in addition to the above constraints, which may be represented as

$$x_1 \geq 0$$
$$x_2 \geq 0$$
$$x_3 \geq 0$$

Therefore,

Linear Programming Model

Find the value of x_1, x_2 and x_3 that will minimize the cost (in peso),

$$\text{Min} \quad z = 80x_1 + 50x_2 + 60x_3$$

subject to the constraints

$$26x_1 + 18x_2 + 20x_3 \geq 40$$
$$20x_1 + 10x_2 + 7x_3 \geq 36$$
$$13x_1 + 30x_2 + 12x_3 \geq 25$$
$$x_1, x_2, x_3 \geq 0$$

Solving Linear Programming Problem

3.4 Corner Principle

Definition

Let $f(x, y) = ax + by + c$ be an objective function subject to constraints on two variables x and y. If the constraints form a system of linear inequalities, the maximum and minimum values of f, if there are any, can be found by:

1. Graphing systems of inequalities and locating the vertices (the points of intersection); and
2. Finding the value of the objective function f at each vertex.

The maximum and minimum values of f are the largest and the smallest values obtained from step 2, respectively.

Note: In this case, we use $f(x, y)$ in place of z which are both represent the objective function either maximizing or minimizing.

4 Find the maximum values of the following function by graphing method

Example

$$\text{Max } f(x, y) = 3x + 4y$$

subject to the constraints

$$2x - 3y \leq 5$$
$$4x + y \leq 3$$
$$x, y \geq 0$$

Solution:

Graph the system of linear inequalities. (In graphing system of linear inequalities, you may refer to our previous topics)

Consider the equations $2x - 3y = 5$ and $4x + y = 3$

Finding the x and y intercepts:

$\boxed{2x - 3y = 5}$

Let $x = 0$

$2(0) - 3y = 5$

$y = -5/3$, thus $(0, -5/3)$

Let $y = 0$

$2x - 3(0) = 5$

$2x = 5$

$x = 5/2$, thus $(5/2, 0)$

$\boxed{4x + y = 3}$

Let $x = 0$

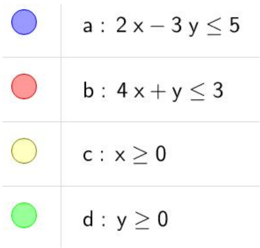
$4(0) + y = 3$

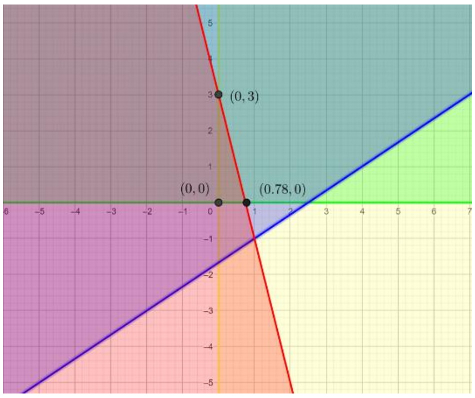
$y = 3$, thus $(0,3)$

Let $y = 0$

$4x + 0 = 3$

$x = 3/4$, thus $(3/4, 0)$





The region shaded by all colors is called the **feasible region**, which all possible solutions to the problem are on and inside this region. But according to the corner principle, the maximum value of the objective function f is at the vertices of this. So we need to try which among the three vertices will arrive at the maximum value of f.

Vertex (x ,y)	f (x, y) = 3x + 4y
a. (0,0)	f(0,0) = 3(0) + 4(0) = **0**
b. (0.78,0)	f(0.78, 0) = 3 (0.78) + 4(0) = **2.34**
c. (0,3)	f(0,3) = 3(0)+4(3)= **12**

As you can see in the table, the maximum value of f is 12 at point (0,3). It only means that x must be equal to zero and y must equal to 3 in order to maximize $f(x, y)= 3x + 4y$.

3.3	Self-Assessment Activity 3
	Find the maximum and minimum values of objective function in example 4 nos. 1 and 2 respectively.

References

Nocon, et.al, Essential Mathematics for the Modern World
Hoppin, Operations Research and Management Information Science
H.Lyeme,et.al, Introduction to Operations Research : Theory and Applications
https://www.techopedia.com/definition/20403/linear-programming-lp
https://www.sciencedirect.com/topics/social-sciences/quantitative-methodlpp

Other Materials

https://youtube.be/Bzzqx1F23a8
https://youtu.be/8IRrgDoV8Eo

Suggested Readings

https://www.analyticsvidhya.com/blog/2017/02/lintroductory-guide-on

1 WORKSHEET

General Instruction: Fill out completely the student's information and write all your solutions/answers on the space provided below each item.

Student's Information:
Last Name, First Name M.I.: ____
Student Number: ____________Course – Year: ______________ Date of Submission: ____________
Class ID Number: ___________ Professor/Instructor's Name: ________________________________

A. Find at least five (5) solution points of each of the following inequalities. Show your complete solution.

1. $2x - 3y < 12$

Solution:
a. b. c. d. e.

Solution Set:

x					
y					

2. $3x + 7y > 30$

Solution:
a. b. c. d. e.

Solution Set:

x					
y					

3. $y \geq \frac{1}{2}x - 7$

Solution:
a. b. c. d. e.

Solution Set:

x					
y					

a. x ≤ 8

Solution:

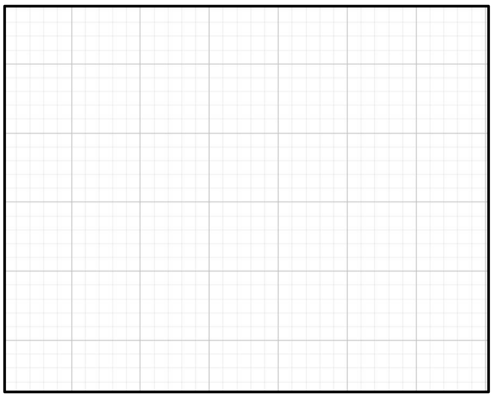
a. b. c. d. e.

Solution Set:

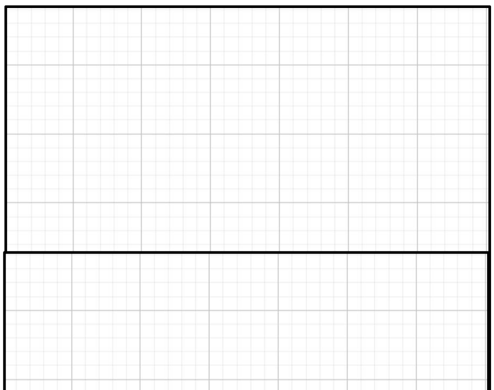
x					
y					

B. Graph each system of inequalities and give at least two solution points.

1. $2x + y < 6$
   $x \geq 1$



2. $x + y \leq 5$
   $x – y > - 3$



3. $x + y > 2$
   $y \leq 6$

C. Create your own (1) product mix problem and (1) diet problem.

1. Product Mix Problem   2. Diet Problem

D. Formulate a linear programming model for each problem given below. Then find the optimal solution using graphical method.

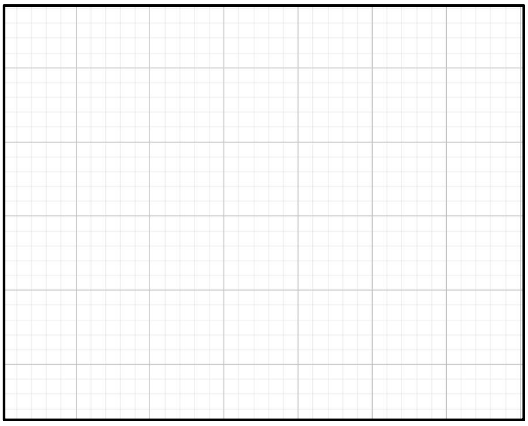
1. A businessman plans to sell two models of an item at costs of \$260 and \$300. The \$260 model yields a profit of \$85 and the \$300 model yields a profit of \$90. The total demand per month for the two models will not exceed 135. Find the number of units of each model that should be stocked each month in order to maximize the profit.

    a. Define Variables

    b. Write the Objective Function

    c. Write inequalities for constraints

d. Graph



e. Linear Programming Model

f. Find the optimal solution or the maximum values for the objective function. Show the complete solution.

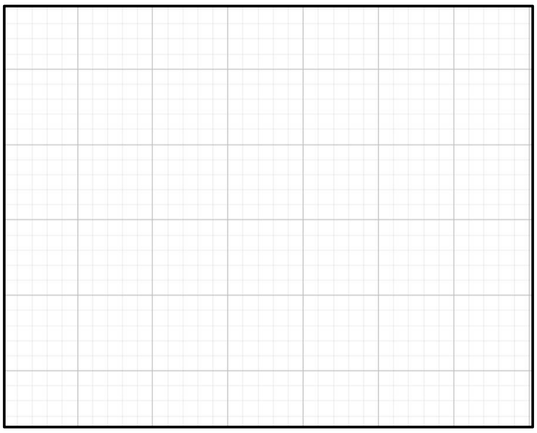
2. Rose bakes two breads, A and B. One batch of A uses 6 pounds of oats and 4 pounds of flour. One batch of B uses 3 pounds of oats and 5 pounds of flour. The company has 150 pounds of oats and 120 pounds of flour available. One batch of A yields a profit of $40 and one batch of B yields a profit of $30. Find how many batches of each bread should she bakes to maximize the profits from bread sales.

   a. Define Variables

   b. Write the Objective Function

c. Write inequalities for constraints

d. Graph



e. Linear Programming Model

f. Find the optimal solution or maximum values for the objective function. Show the complete solution.

3. On a piggery farm, the pigs are given a healthy diet to gain weight. The pigs have to consume a minimum of 20 units of Substance A and another 12 units of Substance B. In

the market there are only two classes of compounds: Type X, with a composition of 2 units of A to 6 units of B, and another type, Y, with a composition of 6 units of A to 2 of B. The price of Type X is $15 and Type Y, $35. What are the quantities of each type of compound that have to be purchased to cover the needs of the diet with a minimal cost?

e. Define Variables

f. Write the Objective Function

g. Write inequalities for constraints

h. Graph

f. Linear Programming Model

f. Find the optimal solution or maximum values for the objective function. Show the complete solution.