

# Supplementary Material for “Robust bilinear factor analysis based on the matrix-variate $t$ distribution”

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## Abstract

In this supplementary material, we provide detailed proofs of some results in the main paper.

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## S1. The Proofs

### S1.1. Proof of Theorem 1

Recalling that  $\text{vec}(\mathbf{X}_n) \sim t_{d_c d_r}(\text{vec}(\mathbf{W}), \Sigma_r \otimes \Sigma_c, \nu)$ ,  $\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) = \text{tr}\{\Sigma_c^{-1}(\mathbf{X}_n - \mathbf{W})\Sigma_r^{-1}(\mathbf{X}_n - \mathbf{W})'\} \sim d_c d_r F(d_c d_r, \nu)$  and  $\nu/(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})) \sim \text{Beta}(\nu/2, d_c d_r/2)$ . With reference to these properties and the work by Wang et al. (2017), we can readily demonstrate that (a)-(e). Hereafter, we will provide the proof solely for (f).

(f) Let  $\theta_c^i$  represent the  $i$ th entry of  $\boldsymbol{\theta}_c$ ,  $\theta_r^k$  denote the  $k$ -th entry of  $\boldsymbol{\theta}_r$ ,  $\dot{\Sigma}_c^i = \partial \Sigma_c / \partial \theta_c^i$  for  $i \in \{1, \dots, d_c(d_c + 1)/2\}$  and  $\dot{\Sigma}_r^k = \partial \Sigma_r / \partial \theta_r^k$  for  $k \in \{1, \dots, d_r(d_r + 1)/2\}$  throughout the following proof. Due to

$$\mathbb{E} \left\{ \frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \theta_c^i \partial \theta_r^k} \right\} = -\mathbb{E} \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i} \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k} \right\}, \quad (\text{S.1})$$

we first derive

$$\begin{aligned} \frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \theta_c^i \partial \theta_r^k} &= \frac{1}{2} (\nu + d_c d_r) \frac{\boldsymbol{\epsilon}_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \boldsymbol{\epsilon}_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \\ &\quad - \frac{1}{2} (\nu + d_c d_r) \frac{\text{tr} \left\{ \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})}, \end{aligned} \quad (\text{S.2})$$

and

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$$\begin{aligned}
& - \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i} \right\} \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k} \right\} = -\frac{1}{4} d_c d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \\
& + \frac{1}{4} d_r (\nu + d_c d_r) \frac{\epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \\
& + \frac{1}{4} d_c (\nu + d_c d_r) \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \\
& - \frac{1}{4} (\nu + d_c d_r)^2 \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right\}. \quad (\text{S.3})
\end{aligned}$$

Taking the expectations for (S.2) and (S.3), we obtain

$$\begin{aligned}
\mathbb{E} \left\{ \frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \theta_c^i \partial \theta_r^k} \right\} &= \frac{1}{2} (\nu + d_c d_r) \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right\} \\
& - \frac{1}{2} (\nu + d_c d_r) \mathbb{E} \left\{ \frac{\text{tr} \left\{ \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \\
&= \frac{1}{2} (\nu + d_c d_r) \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right\} \\
& - \frac{1}{2} \text{tr} \left\{ \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \right\}, \quad (\text{S.4})
\end{aligned}$$

$$\begin{aligned}
& -\mathbb{E} \left[ \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i} \right\} \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k} \right\} \right] \\
& = -\frac{1}{4} d_c d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) + \frac{1}{4} d_r (\nu + d_c d_r) \\
& \quad \cdot \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \\
& \quad + \frac{1}{4} d_c (\nu + d_c d_r) \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \\
& \quad - \frac{1}{4} (\nu + d_c d_r)^2 \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right\} \\
& = -\frac{1}{4} d_c d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) + \frac{1}{4} d_c d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \\
& \quad + \frac{1}{4} d_c d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \\
& \quad - \frac{1}{4} (\nu + d_c d_r)^2 \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right\}. \quad (\text{S.5})
\end{aligned}$$

Additional details regarding the terms in (S.4) and (S.5) are

$$\begin{aligned}
& -\frac{1}{2} (\nu + d_c d_r) \mathbb{E} \left\{ \frac{\text{tr} \left\{ \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \\
& = -\frac{1}{2} (\nu + d_c d_r) \mathbb{E} \left[ \text{tr} \left\{ \frac{\left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \right] \\
& = -\frac{1}{2} (\nu + d_c d_r) \text{tr} \left\{ \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \mathbb{E} \left( \frac{\epsilon_n \epsilon_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right\} \\
& = -\frac{1}{2} (\nu + d_c d_r) \text{tr} \left\{ \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \left( \frac{\Sigma_r \otimes \Sigma_c}{\nu + d_c d_r} \right) \right\} \\
& = -\frac{1}{2} \text{tr} \left\{ \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \right\},
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{4}d_r(\nu + d_c d_r) \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \\
&= \frac{1}{4}d_r(\nu + d_c d_r) \mathbb{E} \left\{ \text{tr} \left( \frac{\epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right\} \\
&= \frac{1}{4}d_r(\nu + d_c d_r) \text{tr} \left\{ \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \mathbb{E} \left( \frac{\epsilon_n \epsilon_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right\} \\
&= \frac{1}{4}d_c d_r \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{4}d_c(\nu + d_c d_r) \mathbb{E} \left\{ \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \\
&= \frac{1}{4}d_c(\nu + d_c d_r) \mathbb{E} \left\{ \text{tr} \left( \frac{\epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right\} \\
&= \frac{1}{4}d_c(\nu + d_c d_r) \text{tr} \left\{ \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \mathbb{E} \left( \frac{\epsilon_n \epsilon_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right\} \\
&= \frac{1}{4}d_c d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right)
\end{aligned}$$

respectively. Consequently, employing the information from (S.1) results in:

$$\begin{aligned}
& \mathbb{E} \left\{ (\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^{-2} \epsilon_n' \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \otimes \Sigma_c^{-1} \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \right\} \\
&= (\nu + d_c d_r)^{-1} (\nu + d_c d_r + 2)^{-1} \left\{ d_c d_r \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \right. \\
&\quad \left. + 2 \text{tr} \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \right) \right\}.
\end{aligned}$$

(g) and (h) can be proven in a similar way. This completes the proof of Theorem 1.

### S1.2. Proof of Theorem 2

#### The score vector:

The score vector  $\mathbf{s}(\boldsymbol{\theta} \mid \mathbf{x}_n)$  is the vector of the first derivatives of  $\ln p(\mathbf{X}_n)$  w.r.t.  $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\theta}_c, \boldsymbol{\theta}_r, \nu)$ .

It comprises the following entries:

$$\begin{aligned}
\mathbf{s}^{\mu'} &= \sum_{n=1}^N \frac{(\nu + d_c d_r)}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))} \boldsymbol{\epsilon}'_n (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}), \\
\mathbf{s}^{\theta^i_c} &= -\frac{1}{2} \sum_{n=1}^N \left[ d_r \text{tr} \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right) - (\nu + d_c d_r) \text{tr} \left\{ \frac{\boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \right], \\
\mathbf{s}^{\theta^k_r} &= -\frac{1}{2} \sum_{n=1}^N \left[ d_c \text{tr} \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right) - (\nu + d_c d_r) \text{tr} \left\{ \frac{\boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \right], \\
\mathbf{s}^\nu &= \frac{1}{2} \sum_{n=1}^N \left[ \psi \left( \frac{\nu + d_c d_r}{2} \right) - \psi \left( \frac{\nu}{2} \right) + \ln(\nu) + 1 - \ln(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})) - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right].
\end{aligned}$$

The elements of the Hessian matrix are:

$$\begin{aligned}
\mathbf{H}^{\mu\mu} &= \sum_{n=1}^N \left[ \frac{2(\nu + d_c d_r)}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}'_n (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \right], \\
\mathbf{H}^{\mu\theta^i_c} &= \sum_{n=1}^N \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right. \\
&\quad \left. - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left\{ \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right\} \boldsymbol{\epsilon}_n \right], \\
\mathbf{H}^{\mu\theta^k_r} &= \sum_{n=1}^N \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right. \\
&\quad \left. - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \boldsymbol{\Sigma}_c^{-1} \right\} \boldsymbol{\epsilon}_n \right], \\
\mathbf{H}^{\mu\nu} &= \sum_{n=1}^N \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n,
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}^{\nu\nu} &= -\frac{1}{2} \sum_{n=1}^N \left[ \frac{1}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} + \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] + \frac{N}{4} \left[ \text{TG}\left(\frac{\nu + d_c d_r}{2}\right) - \frac{1}{4} \text{TG}\left(\frac{\nu}{2}\right) + \frac{2}{\nu} \right], \\
\mathbf{H}^{\theta_c^i \theta_c^j} &= -\frac{1}{2} \sum_{n=1}^N \left[ d_r \text{tr} \left( \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} - \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \right. \\
&\quad + \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))} \text{tr} \left\{ 2 \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right. \\
&\quad \left. - \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} \\
&\quad - \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} \dot{\Sigma}_c^j \right\} \\
&\quad \left. \cdot \text{tr} \left\{ \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} \dot{\Sigma}_c^i \right\} \right], \\
\mathbf{H}^{\theta_c^i \theta_r^k} &= -\frac{1}{2} \sum_{n=1}^N \left[ \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \text{tr} \left\{ \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} \right. \\
&\quad - \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} \\
&\quad \left. \cdot \text{tr} \left\{ \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} \right], \\
\mathbf{H}^{\theta_c^i \nu} &= \frac{1}{2} \sum_{n=1}^N \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} \dot{\Sigma}_c^i \right\}, \\
\mathbf{H}^{\theta_r^k \theta_r^s} &= -\frac{1}{2} \sum_{n=1}^N \left[ d_c \text{tr} \left( \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} - \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \right. \\
&\quad + \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))} \text{tr} \left\{ 2 \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right. \\
&\quad \left. - \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} \\
&\quad - \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} \dot{\Sigma}_r^s \right\} \\
&\quad \left. \cdot \text{tr} \left\{ \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} \dot{\Sigma}_r^k \right\} \right], \\
\mathbf{H}^{\theta_r^k \nu} &= \frac{1}{2} \sum_{n=1}^N \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} \dot{\Sigma}_r^k \right\},
\end{aligned}$$

where  $\ddot{\Sigma}_c^{ij} = \partial^2 \Sigma_c / \partial \theta_c^i \partial \theta_c^j$  for each  $i, j \in \{1, \dots, d_c(d_c + 1)/2\}$ ,  $\ddot{\Sigma}_r^{ks} = \partial^2 \Sigma_r / \partial \theta_r^k \partial \theta_r^s$  for each  $k, s \in \{1, \dots, d_r(d_r + 1)/2\}$ .

By Theorem 1, it can be shown that

$$\begin{aligned}
\mathbf{I}_N^{\theta_c^i \theta_r^k} &= -\mathbb{E} \left[ \mathbf{H}^{\theta_c^i \theta_r^k} \right] \\
&= \frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} \right. \\
&\quad - \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} \\
&\quad \cdot \text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} \left. \right] \\
&= \frac{1}{2} \sum_{n=1}^N \left\{ (\nu + d_c d_r) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n' \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right\} \boldsymbol{\epsilon}_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \right. \\
&\quad \left. - (\nu + d_c d_r) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n' \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1} \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \boldsymbol{\epsilon}_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] \right\} \\
&= \frac{1}{2} \sum_{n=1}^N \left\{ (\nu + d_c d_r) \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right\} \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \right. \\
&\quad \left. - (\nu + d_c d_r) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n' \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1} \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \boldsymbol{\epsilon}_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] \right\} \\
&= \frac{1}{2} \sum_{n=1}^N \left[ (\nu + d_c d_r) \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right) \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right) \right\} \right. \\
&\quad - (\nu + d_c d_r) (\nu + d_c d_r)^{-1} (\nu + d_c d_r + 2)^{-1} \left\{ d_c d_r \text{tr} \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right) \text{tr} \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right) \right. \\
&\quad \left. \left. + 2 \text{tr} \left( \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right) \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right) \right) \right\} \right] \\
&= \frac{N}{2(\nu + d_c d_r + 2)} \left[ (\nu + d_c d_r) \text{tr} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right) \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right) \right\} - d_c d_r \text{tr} \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right) \text{tr} \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right) \right].
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_N^{\theta_c^i \theta_c^j} &= -\mathbb{E} \left[ \mathbf{H}^{\theta_c^i \theta_c^j} \right] \\
&= \frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ d_r \text{tr} \left( \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} - \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \right) + \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))} \text{tr} \left\{ 2 \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right. \right. \\
&\quad \cdot (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' - \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \left. \right\} - \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \\
&\quad \cdot \text{tr} \left\{ \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} \dot{\Sigma}_c^j \right\} \text{tr} \left\{ \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} \dot{\Sigma}_c^i \right\} \left. \right] \\
&= \frac{1}{2} \sum_{n=1}^N \left\{ d_r \text{tr} \left( \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} - \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \right) + \mathbb{E} \left[ \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \text{tr} \left\{ \left( 2 \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \right) \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \right\} \right] \right. \\
&\quad \left. - \mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \right) \right) \epsilon_n \epsilon_n' \left( \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \right) \right) \epsilon_n \right\} \right] \right\} \\
&= \frac{1}{2} \sum_{n=1}^N \left\{ d_r \text{tr} \left( \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} - \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \right) + (\nu + d_c d_r) \left[ \text{tr} \left\{ \left( 2 \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \right) \right. \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \Sigma_r^{-1} \otimes \left( \Sigma_c^{-1} \ddot{\Sigma}_c^{ij} \Sigma_c^{-1} \right) \right) \frac{(\Sigma_r \otimes \Sigma_c)}{\nu + d_c d_r} \right\} \right] \\
&\quad \left. - (\nu + d_c d_r) \left[ \frac{d_r^2 \text{tr} \left\{ \Sigma_c^{-1} \dot{\Sigma}_c^i \right\} \text{tr} \left\{ \Sigma_c^{-1} \dot{\Sigma}_c^j \right\} + 2 d_r \text{tr} \left\{ \Sigma_c^{-1} \dot{\Sigma}_c^j \Sigma_c^{-1} \dot{\Sigma}_c^i \right\}}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} \right] \right\} \\
&= \frac{N d_r}{2(\nu + d_c d_r + 2)} \left\{ (\nu + d_c d_r) \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \Sigma_c^{-1} \dot{\Sigma}_c^j \right) - d_r \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^i \right) \text{tr} \left( \Sigma_c^{-1} \dot{\Sigma}_c^j \right) \right\}.
\end{aligned}$$



$$\begin{aligned}
\mathbf{I}_N^{\theta_c^i \nu} &= -\mathbb{E} \left[ \mathbf{H}_c^{\theta_c^i \nu} \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right\} \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left( 1 - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\text{tr} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left( 1 - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \left\{ \text{tr} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \right\} \right. \\
&\quad \left. - (\nu + d_c d_r) \text{tr} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n'}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] \right\} \right\} \\
&= -\frac{1}{2} \sum_{n=1}^N \left\{ \text{tr} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \frac{(\boldsymbol{\Sigma}_r \otimes \boldsymbol{\Sigma}_c)}{\nu + d_c d_r} \right\} \right. \\
&\quad \left. - (\nu + d_c d_r) \text{tr} \left\{ \left( \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right) \frac{(\boldsymbol{\Sigma}_r \otimes \boldsymbol{\Sigma}_c)}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} \right\} \right\} \\
&= \frac{-N d_r}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} \text{tr} \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \right).
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_N^{\theta_r^k \theta_r^s} &= -\mathbb{E} \left[ \mathbf{H}^{\theta_r^k \theta_r^s} \right] \\
&= \frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ d_c \text{tr} \left( \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} - \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \dot{\Sigma}_r^k \right) + \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right. \\
&\quad \cdot \text{tr} \left\{ 2 \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) - \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} \\
&\quad - \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} \dot{\Sigma}_r^s \right\} \\
&\quad \cdot \text{tr} \left\{ \Sigma_r^{-1} (\mathbf{X}_n - \mathbf{W})' \Sigma_c^{-1} (\mathbf{X}_n - \mathbf{W}) \Sigma_r^{-1} \dot{\Sigma}_r^k \right\} \Big] \\
&= \frac{1}{2} \sum_{n=1}^N \left\{ d_c \text{tr} \left( \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} - \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \dot{\Sigma}_r^s \right) + \mathbb{E} \left[ \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \text{tr} \left\{ \left( 2 \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \right) \otimes \Sigma_c^{-1} \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \left( \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} \Sigma_r^{-1} \right) \otimes \Sigma_c^{-1} \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \right\} \right] \right. \\
&\quad \left. - \mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\epsilon}_n' \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \right) \otimes \Sigma_c^{-1} \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \left( \left( \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \right) \otimes \Sigma_c^{-1} \right) \boldsymbol{\epsilon}_n \right\} \right] \right\} \\
&= \frac{1}{2} \sum_{n=1}^N \left\{ d_c \text{tr} \left( \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} - \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \dot{\Sigma}_r^s \right) + (\nu + d_c d_r) \left[ \text{tr} \left\{ \left( \left( 2 \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \dot{\Sigma}_r^s \Sigma_r^{-1} \right) \otimes \Sigma_c^{-1} \right. \right. \right. \right. \\
&\quad \left. \left. \left. - \left( \Sigma_r^{-1} \ddot{\Sigma}_r^{ks} \Sigma_r^{-1} \right) \otimes \Sigma_c^{-1} \right) \frac{(\Sigma_r \otimes \Sigma_c)}{\nu + d_c d_r} \right\} \right] \right. \\
&\quad \left. - (\nu + d_c d_r) \left[ \frac{d_c^2 \text{tr} \left\{ \Sigma_r^{-1} \dot{\Sigma}_r^k \right\} \text{tr} \left\{ \Sigma_r^{-1} \dot{\Sigma}_r^s \right\} + 2 d_c \text{tr} \left\{ \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \dot{\Sigma}_r^s \right\}}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} \right] \right\} \\
&= \frac{N d_c}{2(\nu + d_c d_r + 2)} \left\{ (\nu + d_c d_r) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \Sigma_r^{-1} \dot{\Sigma}_r^s \right) - d_c \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^k \right) \text{tr} \left( \Sigma_r^{-1} \dot{\Sigma}_r^s \right) \right\}.
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_N^{\theta_r^k \nu} &= -\mathbb{E} \left[ \mathbf{H}^{\theta_r^k \nu} \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right\} \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left( 1 - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{\text{tr} \left\{ \left( \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \boldsymbol{\Sigma}_c^{-1} \right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left( 1 - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right] \\
&= -\frac{1}{2} \sum_{n=1}^N \left\{ \text{tr} \left\{ \left( \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \boldsymbol{\Sigma}_c^{-1} \right) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n'}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \right\} \right. \\
&\quad \left. - (\nu + d_c d_r) \text{tr} \left\{ \left( \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \boldsymbol{\Sigma}_c^{-1} \right) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n'}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] \right\} \right\} \\
&= -\frac{1}{2} \sum_{n=1}^N \left\{ \text{tr} \left\{ \left( \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \boldsymbol{\Sigma}_c^{-1} \right) \frac{(\boldsymbol{\Sigma}_r \otimes \boldsymbol{\Sigma}_c)}{\nu + d_c d_r} \right\} \right. \\
&\quad \left. - (\nu + d_c d_r) \text{tr} \left\{ \left( \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} \right) \otimes \boldsymbol{\Sigma}_c^{-1} \right) \frac{(\boldsymbol{\Sigma}_r \otimes \boldsymbol{\Sigma}_c)}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} \right\} \right\} \\
&= \frac{-N d_c}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} \text{tr} \left( \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \right).
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_N^{\nu \nu} &= -\mathbb{E} [\mathbf{H}^{\nu \nu}] \\
&= \frac{1}{2} \sum_{n=1}^N \mathbb{E} \left[ \frac{1}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} + \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] - \frac{N}{4} \left[ \text{TG}\left(\frac{\nu + d_c d_r}{2}\right) - \text{TG}\left(\frac{\nu}{2}\right) + \frac{2}{\nu} \right] \\
&= \frac{1}{2} \sum_{n=1}^N \left[ \frac{2}{\nu + d_c d_r} + \frac{\nu + 2}{\nu(\nu + d_c d_r + 2)} \right] - \frac{N}{4} \left[ \text{TG}\left(\frac{\nu + d_c d_r}{2}\right) - \text{TG}\left(\frac{\nu}{2}\right) + \frac{2}{\nu} \right] \\
&= N \left\{ \frac{1}{4} \text{TG}\left(\frac{\nu}{2}\right) - \frac{1}{4} \text{TG}\left(\frac{\nu + d_c d_r}{2}\right) - \frac{d_c d_r(\nu + d_c d_r + 4)}{2\nu(\nu + d_c d_r)(\nu + d_c d_r + 2)} \right\}
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_N^{\mu\mu} &= -\mathbb{E} [\mathbf{H}^{\mu\mu}] \\
&= -\sum_{n=1}^N \mathbb{E} \left[ \frac{2(\nu + d_c d_r)}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \right] \\
&= -\sum_{n=1}^N \left\{ 2(\nu + d_c d_r) \mathbb{E} \left[ \frac{(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1})}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \right] - (\nu + d_c d_r) \mathbb{E} \left[ \frac{(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1})}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \right\} \\
&= -\sum_{n=1}^N \left\{ 2(\nu + d_c d_r) \frac{(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) (\boldsymbol{\Sigma}_r \otimes \boldsymbol{\Sigma}_c) (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1})}{(\nu + d_c d_r)(\nu + d_c d_r + 2)} - (\nu + d_c d_r) \left[ \frac{(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1})}{\nu + d_c d_r} \right] \right\} \\
&= N \frac{\nu + d_c d_r}{\nu + d_c d_r + 2} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1})
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}_N^{\mu\theta_c^i} &= -\mathbb{E} [\mathbf{H}^{\mu\theta_c^i}] \\
&= -\sum_{n=1}^N \mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right. \\
&\quad \left. - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left\{ \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right\} \boldsymbol{\epsilon}_n \right] \\
&= -\sum_{n=1}^N \mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right] \\
&\quad - (\nu + d_c d_r) \left\{ \boldsymbol{\Sigma}_r^{-1} \otimes \left( \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} \right) \right\} \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \\
&= 0,
\end{aligned}$$

we can deduce that  $\mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right] = 0$  by using the property of the expectation involving the second derivative of the log-likelihood,  $\mathbb{E} \left\{ \frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \boldsymbol{\mu} \partial \theta_c^i} \right\} = -\mathbb{E} \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \boldsymbol{\mu}} \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i} \right\}.$

$$\begin{aligned}
\mathbf{I}_N^{\mu\theta_r^k} &= -\mathbb{E} \left[ \mathbf{H}^{\mu\theta_r^k} \right] \\
&= -\sum_{n=1}^N \mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right. \\
&\quad \left. - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left\{ (\boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1}) \otimes \boldsymbol{\Sigma}_c^{-1} \right\} \boldsymbol{\epsilon}_n \right] \\
&= -\sum_{n=1}^N \mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right] \\
&\quad - (\nu + d_c d_r) \left\{ (\boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1}) \otimes \boldsymbol{\Sigma}_c^{-1} \right\} \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \\
&= 0
\end{aligned}$$

we can deduce that  $\mathbb{E} \left[ \frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \text{tr} \left\{ \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right] = 0$  by using the property of the expectation involving the second derivative of the log-likelihood,  $\mathbb{E} \left\{ \frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \boldsymbol{\mu} \partial \theta_r^k} \right\} = -\mathbb{E} \left\{ \frac{\partial \ln p(\mathbf{X}_n)}{\partial \boldsymbol{\mu}} \frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k} \right\}$ .

$$\begin{aligned}
\mathbf{I}_N^{\mu\nu} &= -\mathbb{E} [\mathbf{H}^{\mu\nu}] \\
&= -\sum_{n=1}^N \mathbb{E} \left[ \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \boldsymbol{\epsilon}_n \right] \\
&= -\sum_{n=1}^N (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))} \left( 1 - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right) \right] \\
&= -\sum_{n=1}^N (\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}) \left\{ \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] - (\nu + d_c d_r) \mathbb{E} \left[ \frac{\boldsymbol{\epsilon}_n}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right] \right\} \\
&= 0
\end{aligned}$$

This completes the proof of Theorem 2.

## References

Wang, W.L., Castro, L.M., Lin, T.I., 2017. Automated learning of  $t$  factor analysis models with complete and incomplete data. *J. Multivar. Anal.* 161, 157–171.