Supplementary Material for "Robust bilinear factor analysis based on the matrix-variate *t* distribution"

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Abstract

In this supplementary material, we provide detailed proofs of some results in the main paper.

S1. The Proofs

S1.1. Proof of Theorem 1

Recalling that vec $(\mathbf{X}_n) \sim t_{d_c d_r}(\text{vec}(\mathbf{W}), \mathbf{\Sigma}_r \otimes \mathbf{\Sigma}_c, \nu)$, $\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) = \text{tr}\{\mathbf{\Sigma}_c^{-1}(\mathbf{X}_n - \mathbf{W})\mathbf{\Sigma}_r^{-1}(\mathbf{X}_n - \mathbf{W})'\} \sim d_c d_r F\,(d_c d_r, \nu)$ and $\nu/(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})) \sim \text{Beta}\,(\nu/2, d_c d_r/2)$. With reference to these properties and the work by Wang et al. (2017), we can readily demonstrate that (a)-(e). Hereafter, we will provide the proof solely for (f).

(f) Let θ_c^i represent the ith entry of $\boldsymbol{\theta}_c$, θ_r^k denote the k-th entry of $\boldsymbol{\theta}_r$, $\dot{\mathbf{\Sigma}}_c^i = \partial \mathbf{\Sigma}_c / \partial \theta_c^i$ for $i \in \{1, \ldots, d_c(d_c+1)/2\}$ and $\dot{\mathbf{\Sigma}}_r^k = \partial \mathbf{\Sigma}_r / \partial \theta_r^k$ for $k \in \{1, \ldots, d_r(d_r+1)/2\}$ throughout the following proof. Due to

$$\mathbb{E}\left\{\frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \theta_c^i \partial \theta_r^k}\right\} = -\mathbb{E}\left\{\frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i}\frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k}\right\}, \tag{S.1}$$

we first derive

$$\begin{aligned}\frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \theta_c^i \partial \theta_r^k} =& \frac{1}{2}(\nu + d_c d_r)\frac{\boldsymbol{\epsilon}_n'\left(\mathbf{\Sigma}_r^{-1}\dot{\mathbf{\Sigma}}_r^k\mathbf{\Sigma}_r^{-1} \otimes \mathbf{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\mathbf{\Sigma}_r^{-1} \otimes \left(\mathbf{\Sigma}_c^{-1}\dot{\mathbf{\Sigma}}_c^i\mathbf{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \\ & -\frac{1}{2}(\nu + d_c d_r)\frac{\text{tr}\left\{\left(\left(\mathbf{\Sigma}_r^{-1}\dot{\mathbf{\Sigma}}_r^k\mathbf{\Sigma}_r^{-1}\right) \otimes \left(\mathbf{\Sigma}_c^{-1}\dot{\mathbf{\Sigma}}_c^i\mathbf{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\right\}}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})},\end{aligned} \tag{S.2}$$

and

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$$
\begin{aligned}
&-\left\{\frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i}\right\}\left\{\frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k}\right\} = -\frac{1}{4} d_c d_r \mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right) \\
&+\frac{1}{4} d_r\left(\nu+d_c d_r\right)\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right) \\
&+\frac{1}{4} d_c\left(\nu+d_c d_r\right)\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{v+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right) \\
&-\frac{1}{4}\left(\nu+d_c d_r\right)^2\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right\}.
\end{aligned} \tag{S.3}
$$

Taking the expectations for (S.2) and (S.3), we obtain

$$
\begin{aligned}
\mathbb{E}\left\{\frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial \theta_c^i \partial \theta_r^k}\right\} =&\frac{1}{2}\left(\nu+d_c d_r\right)\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right\} \\
&-\frac{1}{2}\left(\nu+d_c d_r\right)\mathbb{E}\left\{\frac{\mathrm{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\right\}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\} \\
=&\frac{1}{2}\left(\nu+d_c d_r\right)\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right\} \\
&-\frac{1}{2}\mathrm{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)\right\},
\end{aligned} \tag{S.4}
$$

$$
\begin{aligned}
&-\mathbb{E}\left[\left\{\frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_c^i}\right\}\left\{\frac{\partial \ln p(\mathbf{X}_n)}{\partial \theta_r^k}\right\}\right] \\
&\quad=-\frac{1}{4}d_c d_r \mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)+\frac{1}{4}d_r\left(\nu+d_c d_r\right) \\
&\qquad\cdot\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\}\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right) \\
&\qquad+\frac{1}{4}d_c\left(\nu+d_c d_r\right)\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\}\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right) \\
&\qquad-\frac{1}{4}\left(\nu+d_c d_r\right)^2\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right\} \\
&\quad=-\frac{1}{4}d_c d_r \mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)+\frac{1}{4}d_c d_r \mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right) \\
&\qquad+\frac{1}{4}d_c d_r \mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right) \\
&\qquad-\frac{1}{4}\left(\nu+d_c d_r\right)^2\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right\}. \qquad \text{(S.5)}
\end{aligned}
$$

Additional details regarding the terms in (S.4) and (S.5) are

$$
\begin{aligned}
&-\frac{1}{2}\left(\nu+d_c d_r\right)\mathbb{E}\left\{\frac{\mathrm{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\right\}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\} \\
&\quad=-\frac{1}{2}\left(\nu+d_c d_r\right)\mathbb{E}\left[\mathrm{tr}\left\{\frac{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\}\right] \\
&\quad=-\frac{1}{2}\left(\nu+d_c d_r\right)\mathrm{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\mathbb{E}\left(\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right\} \\
&\quad=-\frac{1}{2}\left(\nu+d_c d_r\right)\mathrm{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\left(\frac{\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c}{\nu+d_c d_r}\right)\right\} \\
&\quad=-\frac{1}{2}\mathrm{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{4}d_r\left(\nu+d_cd_r\right)\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\}\\
&\quad=\frac{1}{4}d_r\left(\nu+d_cd_r\right)\mathbb{E}\left\{\operatorname{tr}\left(\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right\}\\
&\quad=\frac{1}{4}d_r\left(\nu+d_cd_r\right)\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\mathbb{E}\left(\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right\}\\
&\quad=\frac{1}{4}d_cd_r\operatorname{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{4}d_c\left(\nu+d_cd_r\right)\mathbb{E}\left\{\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right\}\\
&\quad=\frac{1}{4}d_c\left(\nu+d_cd_r\right)\mathbb{E}\left\{\operatorname{tr}\left(\frac{\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right\}\\
&\quad=\frac{1}{4}d_c\left(\nu+d_cd_r\right)\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\mathbb{E}\left(\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right\}\\
&\quad=\frac{1}{4}d_cd_r\operatorname{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)
\end{aligned}
$$

respectively. Consequently, employing the information from (S.1) results in:

$$
\begin{aligned}
&\mathbb{E}\left\{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^{-2}\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\right\}\\
&\quad=\left(\nu+d_cd_r\right)^{-1}\left(\nu+d_cd_r+2\right)^{-1}\left\{d_cd_r\operatorname{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right)\operatorname{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)\right.\\
&\qquad\left.+2\operatorname{tr}\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)\right)\right\}.
\end{aligned}
$$

(g) and (h) can be proven in a similar way. This completes the proof of Theorem 1.

S1.2. Proof of Theorem 2

The score vector:

The score vector $\mathbf{s}\left(\boldsymbol{\theta}\mid\mathbf{x}_n\right)$ is the vector of the first derivatives of $\ln p(\mathbf{X}_n)$ w.r.t. $\boldsymbol{\theta}=(\boldsymbol{\mu},\boldsymbol{\theta}_c,\boldsymbol{\theta}_r,\nu)$.

It comprises the following entries:

$$
\begin{aligned}
\mathbf{s}^{\mu'} &= \sum_{n=1}^{N} \frac{(\nu + d_c d_r)}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))} \boldsymbol{\epsilon}_n' \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right), \\
\mathbf{s}^{\theta_c^i} &= -\frac{1}{2} \sum_{n=1}^{N} \left[d_r \mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i\right) - (\nu + d_c d_r)\, \mathrm{tr}\left\{ \frac{\boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \right], \\
\mathbf{s}^{\theta_r^k} &= -\frac{1}{2} \sum_{n=1}^{N} \left[d_c \mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k\right) - (\nu + d_c d_r)\, \mathrm{tr}\left\{ \frac{\boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right\} \right], \\
\mathbf{s}^{\nu} &= \frac{1}{2} \sum_{n=1}^{N} \left[\psi\left(\frac{\nu + d_c d_r}{2}\right) - \psi\left(\frac{\nu}{2}\right) + \ln(\nu) + 1 - \ln\left(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right) - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \right].
\end{aligned}
$$

The elements of the Hessian matrix are:

$$
\begin{aligned}
\mathbf{H}^{\boldsymbol{\mu}\boldsymbol{\mu}} &= \sum_{n=1}^{N} \left[\frac{2(\nu + d_c d_r)}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right) \boldsymbol{\epsilon}_n \boldsymbol{\epsilon}_n' \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right) - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right) \right], \\
\mathbf{H}^{\boldsymbol{\mu}\theta_c^i} &= \sum_{n=1}^{N} \left[\frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \mathrm{tr}\left\{ \boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \right\} \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right) \boldsymbol{\epsilon}_n \right. \\
&\quad \left. - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left\{ \boldsymbol{\Sigma}_r^{-1} \otimes \left(\boldsymbol{\Sigma}_c^{-1} \dot{\boldsymbol{\Sigma}}_c^i \boldsymbol{\Sigma}_c^{-1}\right) \right\} \boldsymbol{\epsilon}_n \right], \\
\mathbf{H}^{\boldsymbol{\mu}\theta_r^k} &= \sum_{n=1}^{N} \left[\frac{\nu + d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \mathrm{tr}\left\{ \boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1} (\mathbf{X}_n - \mathbf{W})' \boldsymbol{\Sigma}_c^{-1} (\mathbf{X}_n - \mathbf{W}) \right\} \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right) \boldsymbol{\epsilon}_n \right. \\
&\quad \left. - \frac{\nu + d_c d_r}{\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta})} \left\{ \left(\boldsymbol{\Sigma}_r^{-1} \dot{\boldsymbol{\Sigma}}_r^k \boldsymbol{\Sigma}_r^{-1}\right) \otimes \boldsymbol{\Sigma}_c^{-1} \right\} \boldsymbol{\epsilon}_n \right], \\
\mathbf{H}^{\boldsymbol{\mu}\nu} &= \sum_{n=1}^{N} \frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{(\nu + \delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \left(\boldsymbol{\Sigma}_r^{-1} \otimes \boldsymbol{\Sigma}_c^{-1}\right) \boldsymbol{\epsilon}_n,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{H}^{\nu\nu} &= -\frac{1}{2}\sum_{n=1}^{N}\left[\frac{1}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}+\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta})-d_c d_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\right]+\frac{N}{4}\left[\mathrm{TG}(\frac{\nu+d_c d_r}{2})-\frac{1}{4}\mathrm{TG}(\frac{\nu}{2})+\frac{2}{\nu}\right],\\
\mathbf{H}^{\theta_c^i\theta_c^j} &= -\frac{1}{2}\sum_{n=1}^{N}\Big[d_r\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}-\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)\\
&\quad+\frac{\nu+d_c d_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))}\mathrm{tr}\Big\{2\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\\
&\quad-\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\Big\}\\
&\quad-\frac{\nu+d_c d_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\right\}\\
&\quad\cdot\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right\}\Big],\\
\mathbf{H}^{\theta_c^i\theta_r^k} &= -\frac{1}{2}\sum_{n=1}^{N}\Big[\frac{\nu+d_c d_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\\
&\quad-\frac{\nu+d_c d_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\\
&\quad\cdot\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\right\}\Big],\\
\mathbf{H}^{\theta_c^i\nu} &= \frac{1}{2}\sum_{n=1}^{N}\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta})-d_c d_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right\},\\
\mathbf{H}^{\theta_r^k\theta_r^s} &= -\frac{1}{2}\sum_{n=1}^{N}\Big[d_c\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}-\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right)\\
&\quad+\frac{\nu+d_c d_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))}\mathrm{tr}\Big\{2\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\\
&\quad-\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\Big\}\\
&\quad-\frac{\nu+d_c d_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right\}\\
&\quad\cdot\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right\}\Big],\\
\mathbf{H}^{\theta_r^k\nu} &= \frac{1}{2}\sum_{n=1}^{N}\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta})-d_c d_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right\},
\end{aligned}
$$

where $\ddot{\boldsymbol{\Sigma}}_c^{ij}=\partial^2\boldsymbol{\Sigma}_c/\partial\theta_c^i\partial\theta_c^j$ for each $i,j\in\{1,\ldots,d_c(d_c+1)/2\}$, $\ddot{\boldsymbol{\Sigma}}_r^{ks}=\partial^2\boldsymbol{\Sigma}_r/\partial\theta_r^k\partial\theta_r^s$ for each $k,s\in\{1,\ldots,d_r(d_r+1)/2\}$.

By Theorem 1, it can be shown that

$$
\begin{aligned}
\mathbf{I}_N^{\theta_c^i\theta_r^k} =& -\mathbb{E}\left[\mathbf{H}^{\theta_c^i\theta_r^k}\right]\\
=& \frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\text{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\right.\\
&-\frac{\nu+d_cd_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\text{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\\
&\left.\cdot\,\text{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\right\}\right]\\
=& \frac{1}{2}\sum_{n=1}^{N}\left\{\left(\nu+d_cd_r\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n'\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right\}\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]\right.\\
&\left.-\left(\nu+d_cd_r\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right]\right\}\\
=& \frac{1}{2}\sum_{n=1}^{N}\left\{\left(\nu+d_cd_r\right)\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right\}\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]\right.\\
&\left.-\left(\nu+d_cd_r\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right]\right\}\\
=& \frac{1}{2}\sum_{n=1}^{N}\left[\left(\nu+d_cd_r\right)\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\right\}\right.\\
&-\left(\nu+d_cd_r\right)\left(\nu+d_cd_r\right)^{-1}\left(\nu+d_cd_r+2\right)^{-1}\left\{d_cd_r\text{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)\text{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\right.\\
&\left.\left.+2\text{tr}\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\right)\right\}\right]\\
=& \frac{N}{2(\nu+d_cd_r+2)}\left[\left(\nu+d_cd_r\right)\text{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\right\}-d_cd_r\text{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right)\text{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right)\right].
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\theta_c^i\theta_c^j} =& -\mathbb{E}\left[\mathbf{H}^{\theta_c^i\theta_c^j}\right] \\
=& \frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[d_r\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}-\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)+\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))}\mathrm{tr}\left\{2\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right.\right. \\
& \cdot\left.(\mathbf{X}_n-\mathbf{W})\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})'-\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}\boldsymbol{\Sigma}_c^{-1}(\mathbf{X}_n-\mathbf{W})\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})'\right\}-\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2} \\
& \left.\cdot\,\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}(\mathbf{X}_n-\mathbf{W})\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\right\}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}(\mathbf{X}_n-\mathbf{W})\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right\}\right] \\
=& \frac{1}{2}\sum_{n=1}^{N}\left\{d_r\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}-\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)+\mathbb{E}\left[\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\mathrm{tr}\left\{\left(2\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\right)\right.\right.\right.\right. \\
& \left.\left.\left.-\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\right\}\right] \\
& \left.-\mathbb{E}\left[\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\mathrm{tr}\left\{\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\right\}\right]\right\} \\
=& \frac{1}{2}\sum_{n=1}^{N}\left\{d_r\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}-\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)+(\nu+d_cd_r)\left[\mathrm{tr}\left\{\left(2\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\right)\right.\right.\right.\right. \\
& \left.\left.\left.-\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\ddot{\boldsymbol{\Sigma}}_c^{ij}\boldsymbol{\Sigma}_c^{-1}\right)\right)\frac{(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c)}{\nu+d_cd_r}\right\}\right] \\
& \left.-(\nu+d_cd_r)\left[\frac{d_r^2\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right\}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\right\}+2d_r\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right\}}{(\nu+d_cd_r)(\nu+d_cd_r+2)}\right]\right\} \\
=& \frac{Nd_r}{2(\nu+d_cd_r+2)}\left\{(\nu+d_cd_r)\,\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\right)-d_r\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{i}\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^{j}\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\theta_c^i \nu} =& -\mathbb{E}\left[\mathbf{H}^{\theta_c^i \nu}\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta}) - d_c d_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\operatorname{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right\}\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\operatorname{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right\}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left(1-\frac{\nu+d_c d_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\right\}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left(1-\frac{\nu+d_c d_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\left\{\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]\right\}\right. \\
& \left.-\left(\nu+d_c d_r\right)\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right]\right\}\right\} \\
=& -\frac{1}{2}\sum_{n=1}^{N}\left\{\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\frac{\left(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c\right)}{\nu+d_c d_r}\right\}\right. \\
& \left.-\left(\nu+d_c d_r\right)\operatorname{tr}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right)\frac{\left(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c\right)}{\left(\nu+d_c d_r\right)\left(\nu+d_c d_r+2\right)}\right\}\right\} \\
=& \frac{-N d_r}{(\nu+d_c d_r)(\nu+d_c d_r+2)}\operatorname{tr}\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\right).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\theta_r^k\theta_r^s} =& -\mathbb{E}\left[\mathbf{H}^{\theta_r^k\theta_r^s}\right] \\
=& \frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[d_c\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}-\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right)+\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right. \\
&\cdot\mathrm{tr}\left\{2\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)-\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\right\} \\
&-\frac{\nu+d_cd_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right\} \\
&\left.\cdot\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right\}\right] \\
=& \frac{1}{2}\sum_{n=1}^{N}\left\{d_c\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}-\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right)+\mathbb{E}\left[\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\mathrm{tr}\left\{\left(2\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right.\right.\right.\right. \\
&\left.\left.\left.-\left(\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\right\}\right] \\
&\left.-\mathbb{E}\left[\frac{\nu+d_cd_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\epsilon}_n'\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right\}\right]\right\} \\
=& \frac{1}{2}\sum_{n=1}^{N}\left\{d_c\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}-\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right)+\left(\nu+d_cd_r\right)\left[\mathrm{tr}\left\{\left(\left(2\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right.\right.\right.\right. \\
&\left.\left.-\left(\boldsymbol{\Sigma}_r^{-1}\ddot{\boldsymbol{\Sigma}}_r^{ks}\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\frac{\left(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c\right)}{\nu+d_cd_r}\right\}\right] \\
&\left.-\left(\nu+d_cd_r\right)\left[\frac{d_c^2\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right\}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right\}+2d_c\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right\}}{\left(\nu+d_cd_r\right)\left(\nu+d_cd_r+2\right)}\right]\right\} \\
=& \frac{Nd_c}{2(\nu+d_cd_r+2)}\left\{\left(\nu+d_cd_r\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right)-d_c\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{k}\right)\mathrm{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^{s}\right)\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\theta_r^k \nu} =& -\mathbb{E}\left[\mathbf{H}^{\theta_r^k \nu}\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta})-d_c d_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\operatorname{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right\}\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\operatorname{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right\}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left(1-\frac{\nu+d_c d_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{\operatorname{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}\right\}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left(1-\frac{\nu+d_c d_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right)\right] \\
=& -\frac{1}{2}\sum_{n=1}^{N}\left\{\operatorname{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]\right\}\right. \\
&\left. -(\nu+d_c d_r)\operatorname{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n^{'}}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\right]\right\}\right\} \\
=& -\frac{1}{2}\sum_{n=1}^{N}\left\{\operatorname{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\frac{(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c)}{\nu+d_c d_r}\right\}\right. \\
&\left. -(\nu+d_c d_r)\operatorname{tr}\left\{\left(\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right)\frac{(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c)}{(\nu+d_c d_r)(\nu+d_c d_r+2)}\right\}\right\} \\
=& \frac{-N d_c}{(\nu+d_c d_r)(\nu+d_c d_r+2)}\operatorname{tr}\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\right).
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\nu\nu} =& -\mathbb{E}\left[\mathbf{H}^{\nu\nu}\right] \\
=& \frac{1}{2}\sum_{n=1}^{N}\mathbb{E}\left[\frac{1}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}+\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta})-d_c d_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\right]-\frac{N}{4}\left[\mathrm{TG}(\frac{\nu+d_c d_r}{2})-\mathrm{TG}(\frac{\nu}{2})+\frac{2}{\nu}\right] \\
=& \frac{1}{2}\sum_{n=1}^{N}\left[\frac{2}{\nu+d_c d_r}+\frac{\nu+2}{\nu(\nu++d_c d_r+2)}\right]-\frac{N}{4}\left[\mathrm{TG}(\frac{\nu+d_c d_r}{2})-\mathrm{TG}(\frac{\nu}{2})+\frac{2}{\nu}\right] \\
=& N\left\{\frac{1}{4}\mathrm{TG}(\frac{\nu}{2})-\frac{1}{4}\mathrm{TG}(\frac{\nu+d_c d_r}{2})-\frac{d_c d_r(\nu+d_c d_r+4)}{2\nu(\nu+d_c d_r)(\nu+d_c d_r+2)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\boldsymbol{\mu}\boldsymbol{\mu}} =& -\mathbb{E}\left[\mathbf{H}^{\boldsymbol{\mu}\boldsymbol{\mu}}\right] \\
=& -\sum_{n=1}^{N}\mathbb{E}\left[\frac{2\left(\nu+d_cd_r\right)}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)-\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\right] \\
=& -\sum_{n=1}^{N}\left\{2\left(\nu+d_cd_r\right)\mathbb{E}\left[\frac{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\boldsymbol{\epsilon}_n'\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\right]-\left(\nu+d_cd_r\right)\mathbb{E}\left[\frac{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]\right\} \\
=& -\sum_{n=1}^{N}\left\{2\left(\nu+d_cd_r\right)\frac{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\left(\boldsymbol{\Sigma}_r\otimes\boldsymbol{\Sigma}_c\right)\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)}{\left(\nu+d_cd_r\right)\left(\nu+d_cd_r+2\right)}-\left(\nu+d_cd_r\right)\left[\frac{\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)}{\nu+d_cd_r}\right]\right\} \\
=& N\frac{\nu+d_cd_r}{\nu+d_cd_r+2}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{I}_N^{\boldsymbol{\mu}\theta_c^i} =& -\mathbb{E}\left[\mathbf{H}^{\boldsymbol{\mu}\theta_c^i}\right] \\
=& -\sum_{n=1}^{N}\mathbb{E}\left[\frac{\nu+d_cd_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right. \\
& \left.-\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left\{\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right\}\boldsymbol{\epsilon}_n\right] \\
=& -\sum_{n=1}^{N}\mathbb{E}\left[\frac{\nu+d_cd_r}{\left(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})\right)^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right] \\
& -\left(\nu+d_cd_r\right)\left\{\boldsymbol{\Sigma}_r^{-1}\otimes\left(\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\right)\right\}\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right] \\
=& 0,
\end{aligned}
$$

we can deduce that $\mathbb{E}\left[\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_c^{-1}\dot{\boldsymbol{\Sigma}}_c^i\boldsymbol{\Sigma}_c^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)\boldsymbol{\Sigma}_r^{-1}\left(\mathbf{X}_n-\mathbf{W}\right)'\right\}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right] = 0$ by using the property of the expectation involving the second derivative of the log-likelihood, $\mathbb{E}\left\{\frac{\partial^2\ln p(\mathbf{X}_n)}{\partial\boldsymbol{\mu}\partial\theta_c^i}\right\} = -\mathbb{E}\left\{\frac{\partial\ln p(\mathbf{X}_n)}{\partial\boldsymbol{\mu}}\frac{\partial\ln p(\mathbf{X}_n)}{\partial\theta_c^i}\right\}$.

$$
\begin{aligned}
\mathbf{I}_N^{\boldsymbol{\mu}\theta_r^k} =& -\mathbb{E}\left[\mathbf{H}^{\boldsymbol{\mu}\theta_r^k}\right] \\
=& -\sum_{n=1}^{N}\mathbb{E}\left[\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})'\,\boldsymbol{\Sigma}_c^{-1}(\mathbf{X}_n-\mathbf{W})\right\}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right. \\
& \left.-\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right\}\boldsymbol{\epsilon}_n\right] \\
=& -\sum_{n=1}^{N}\mathbb{E}\left[\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})\,\boldsymbol{\Sigma}_c^{-1}(\mathbf{X}_n-\mathbf{W})'\right\}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right] \\
& -(\nu+d_cd_r)\left\{\left(\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}\right)\otimes\boldsymbol{\Sigma}_c^{-1}\right\}\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right] \\
=& 0
\end{aligned}
$$

we can deduce that $\mathbb{E}\left[\frac{\nu+d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\mathrm{tr}\left\{\boldsymbol{\Sigma}_r^{-1}\dot{\boldsymbol{\Sigma}}_r^k\boldsymbol{\Sigma}_r^{-1}(\mathbf{X}_n-\mathbf{W})\,\boldsymbol{\Sigma}_c^{-1}(\mathbf{X}_n-\mathbf{W})'\right\}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right] = 0$ by using the property of the expectation involving the second derivative of the log-likelihood, $\mathbb{E}\left\{\frac{\partial^2 \ln p(\mathbf{X}_n)}{\partial\boldsymbol{\mu}\partial\theta_r^k}\right\} = -\mathbb{E}\left\{\frac{\partial \ln p(\mathbf{X}_n)}{\partial\boldsymbol{\mu}}\frac{\partial \ln p(\mathbf{X}_n)}{\partial\theta_r^k}\right\}$.

$$
\begin{aligned}
\mathbf{I}_N^{\boldsymbol{\mu}\nu} =& -\mathbb{E}\left[\mathbf{H}^{\boldsymbol{\mu}\nu}\right] \\
=& -\sum_{n=1}^{N}\mathbb{E}\left[\frac{\delta_{\mathbf{X}_n}(\boldsymbol{\theta})-d_cd_r}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))^2}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\boldsymbol{\epsilon}_n\right] \\
=& -\sum_{n=1}^{N}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n}{(\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta}))}(1-\frac{\nu+d_cd_r}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})})\right] \\
=& -\sum_{n=1}^{N}\left(\boldsymbol{\Sigma}_r^{-1}\otimes\boldsymbol{\Sigma}_c^{-1}\right)\left\{\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]-(\nu+d_cd_r)\,\mathbb{E}\left[\frac{\boldsymbol{\epsilon}_n}{\nu+\delta_{\mathbf{X}_n}(\boldsymbol{\theta})}\right]\right\} \\
=& 0
\end{aligned}
$$

This completes the proof of Theorem 2.

References

Wang, W.L., Castro, L.M., Lin, T.I., 2017. Automated learning of t factor analysis models with complete and incomplete data. J. Multivar. Anal. 161, 157–171.