# 第10章  弦振动方程的达朗贝尔解

# 弦振动方程的初值问题的达朗贝尔解

**基本思想：像求解常微分方程一样求出其通解，然后根据定解条件定出符合要求的特解。**

一维：达朗贝尔公式

二维：泊松公式

三维：克希霍夫公式

定解问题的求解思路

原则：由已知猜未知

方法：类比法

步骤：由泛定方程求通解，由条件定特解。

# 一、泛定方程的求解

- 常微分方程
  - 方程：$u' = 2a\, x$
    - 通解：$u = a\, x^2 + C$
- 偏微分方程
  - 方程：$u_x = 2y\, x$
    - 通解：$u = y\, x^2 + C(y)$
- 二阶方程：$u_{xy} = 0$
    - 对y偏积分：$u_x = C(x)$
    - 通解：$u = \int C(x)\, dx + D(y) = f(x) + g(y)$

# 二、达朗贝尔公式的推导

定解问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) \end{cases}$$

通解

$$u(x,t) = f_1(x-at) + f_2(x+at)$$

特解

$$u(x,t) = \tfrac{1}{2}[\varphi(x-at) + \varphi(x+at)]$$

$$+ \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)\, d\xi$$

## 1、 通解

波动方程为： $u_{tt} - a^2 u_{xx} = 0$

即：$(\frac{\partial}{\partial t} + a\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - a\frac{\partial}{\partial x})u = 0$

作变换：$\begin{cases} \xi = x + at \\ \eta = x - at \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(\xi + \eta) \\ t = \frac{1}{2a}(\xi - \eta) \end{cases}$

$$\frac{\partial}{\partial \xi}=\frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial \xi}+\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \xi}=\frac{1}{2a}\cdot\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}$$

$$=\frac{1}{2a}\left(\frac{\partial}{\partial t}+a\frac{\partial}{\partial x}\right)$$

$$\frac{\partial}{\partial \eta}=\frac{\partial}{\partial t}\cdot\frac{\partial t}{\partial \eta}+\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \eta}=-\frac{1}{2a}\cdot\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial}{\partial x}$$

$$=-\frac{1}{2a}\left(\frac{\partial}{\partial t}-a\frac{\partial}{\partial x}\right)$$

得到：$u_{\xi\eta}=0$

对 $\eta$ 偏积分得：$u_{\xi} = f_1'(\xi)$

再对 $\xi$ 偏积分得：$u = \int f_1'(\xi) d\xi + f_2(\eta)$

$$= f_1(\xi) + f_2(\eta) = f_1(x+at) + f_2(x-at)$$

## 2、特解

由初始条件得：$\begin{cases} f_1(x) + f_2(x) = \varphi(x) \\ af_1'(x) - af_2'(x) = \psi(x) \end{cases}$

对第二式作定积分得：

$$f_1(x)-f_2(x)=\frac{1}{a}\int_{x_0}^{x}\psi(\xi)d\xi+f_1(x_0)-f_2(x_0)$$

由此解得：

$$\begin{cases} f_1(x)=\frac{1}{2}\varphi(x)+\frac{1}{2a}\int_{x_0}^{x}\psi(\xi)d\xi+\dfrac{1}{2}\left[f_1(x_0)+f_2(x_0)\right] \\ f_2(x)=\frac{1}{2}\varphi(x)-\frac{1}{2a}\int_{x_0}^{x}\psi(\xi)d\xi-\dfrac{1}{2}\left[f_1(x_0)+f_2(x_0)\right] \end{cases}$$

代入通解得：

$$u(x,\ t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

## 三、达朗贝尔公式的物理意义

(1)设 $u_1 = \frac{1}{2}[\varphi(x+at)+\varphi(x-at)]$

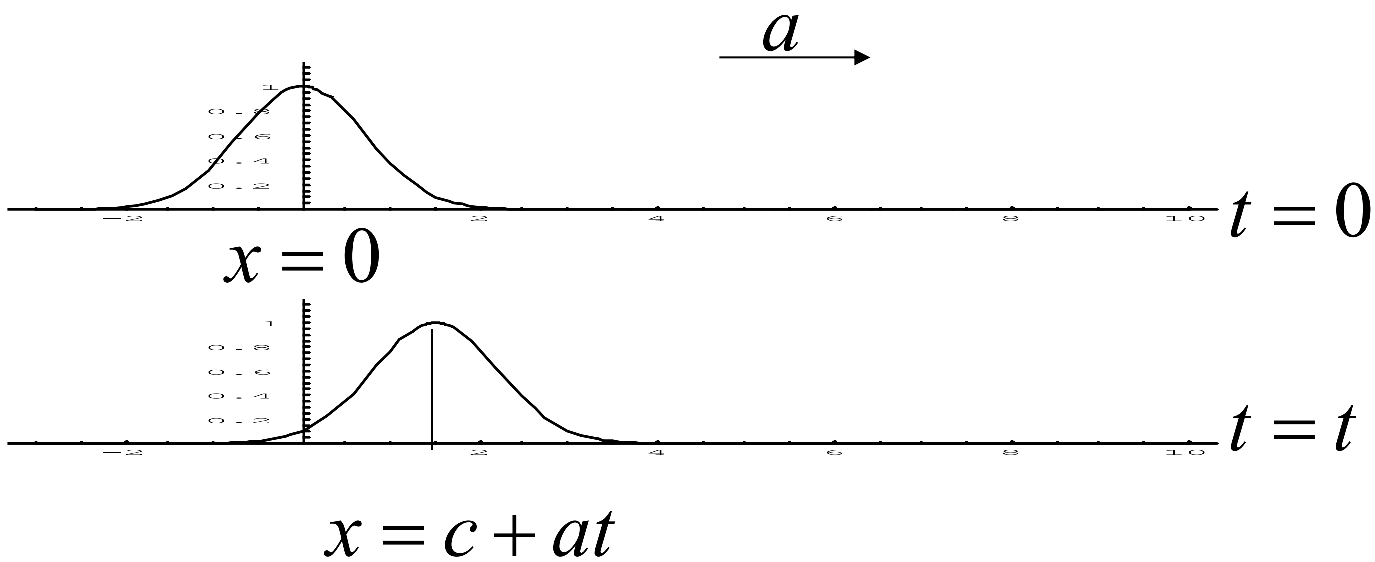
设人在 $t=0$ 时在x=c处看到：

$$\varphi(x-at)=\varphi(c-0)=\varphi(c)$$

若人以速度$a$行走，则t时他在x=c+at处将看到

$$\varphi(x-at)=\varphi(c+at-at)=\varphi(c)$$

如图所示:




$\varphi(x-at)$： 正行波

$\varphi(x+at)$： 反行波

则$u_1$为两列行波的叠加

$$(2)\text{设 } u_2 = \frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

$$\text{若令}\Psi(x) = \frac{1}{2a}\int_{x_0}^{x}\psi(\xi)d\xi$$

则 $u_2 = \Psi(x+at) - \Psi(x-at)$，也表示两列行波的叠加

综合（1）、（2）可知，达朗贝尔解$u = u_1 + u_2$表示正行波和反行波的叠加

**所以达朗贝尔公式法也称为行波法**

另外：从达朗贝尔公式

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

可以看出，波动方程度解，是初始条件的演化。方程本身并不可能产生出超出初始条件的、额外的形式来。

而这种演化又受到边界条件的限制。

这就说明了初始条件和边界条件在确定波动方程解时的重要性。

# 四、达朗贝尔公式的直接应用举例

**例1**
$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty \\ u|_{t=0} = \sin x, \quad u_t|_{t=0} = x^2 \end{cases}$$

解：将初始条件代入达朗贝尔公式

$$u(x,t) = \tfrac{1}{2}[\sin(x+at) + \sin(x-at)] + \tfrac{1}{2a}\int_{x-at}^{x+at} \xi^2 d\xi$$

$$= \sin x \cos at + \frac{t}{3}(3x^2 + a^2 t^2)$$

## 例2

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty \\ u|_{t=0} = e^{-x^2}, \quad u_t|_{t=0} = 2axe^{-x^2} \end{cases}$$

解：将初始条件代入达朗贝尔公式

$$u(x,t) = \tfrac{1}{2}[e^{-(x+at)^2} + e^{-(x-at)^2}] + \frac{1}{2a}\int_{x-at}^{x+at} 2a\xi e^{-s^2} d\xi$$

$$= \tfrac{1}{2}[e^{-(x+at)^2} + e^{-(x-at)^2}] + \tfrac{1}{2}\int_{x-at}^{x+at} e^{-\xi^2} d\xi^2$$

$$= \tfrac{1}{2}[e^{-(x+at)^2} + e^{-(x-at)^2}] + \tfrac{1}{2}(-e^{-\xi^2})\Big|_{x-at}^{x+at}$$

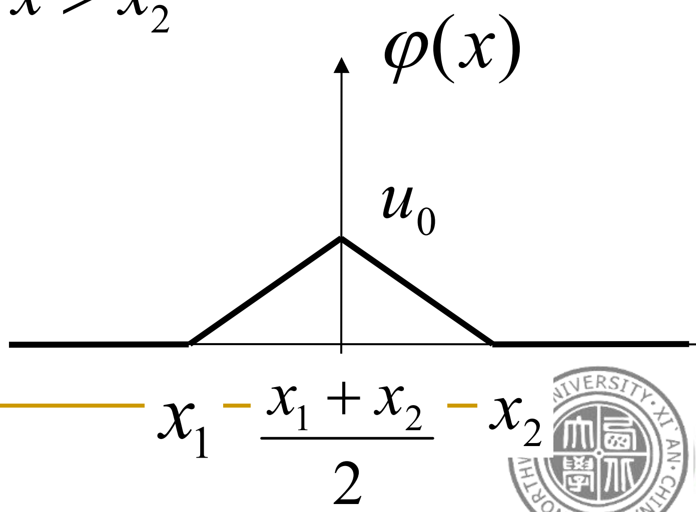
$$= e^{-(x-at)^2}$$

**例3**

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad -\infty < x < \infty \\ u|_{t=0} = \varphi(x), \quad u_t|_{t=0} = \psi(x) \end{cases}$$

$$\text{其中}\ \varphi(x) = \begin{cases} 2u_0 \dfrac{x - x_1}{x_2 - x_1}, & x_1 \le x \le \dfrac{x_1 + x_2}{2} \\ 2u_0 \dfrac{x - x_2}{x_2 - x_1}, & \dfrac{x_1 + x_2}{2} \le x \le x_2 \\ 0 & x < x_1, or, x > x_2 \end{cases}$$

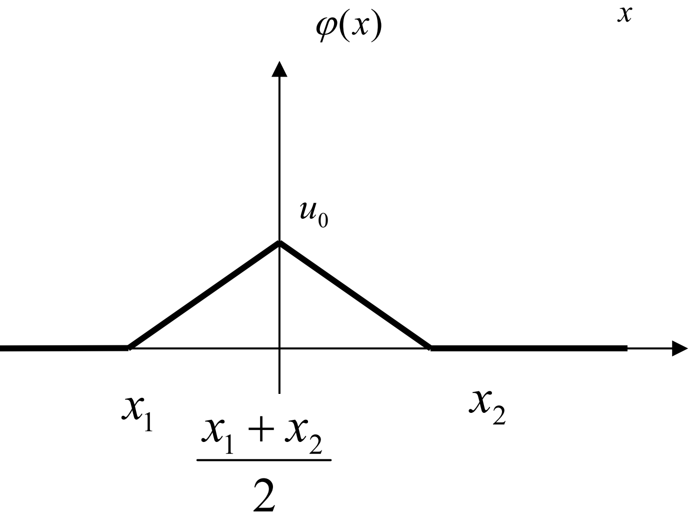
$$\psi(x) = 0$$

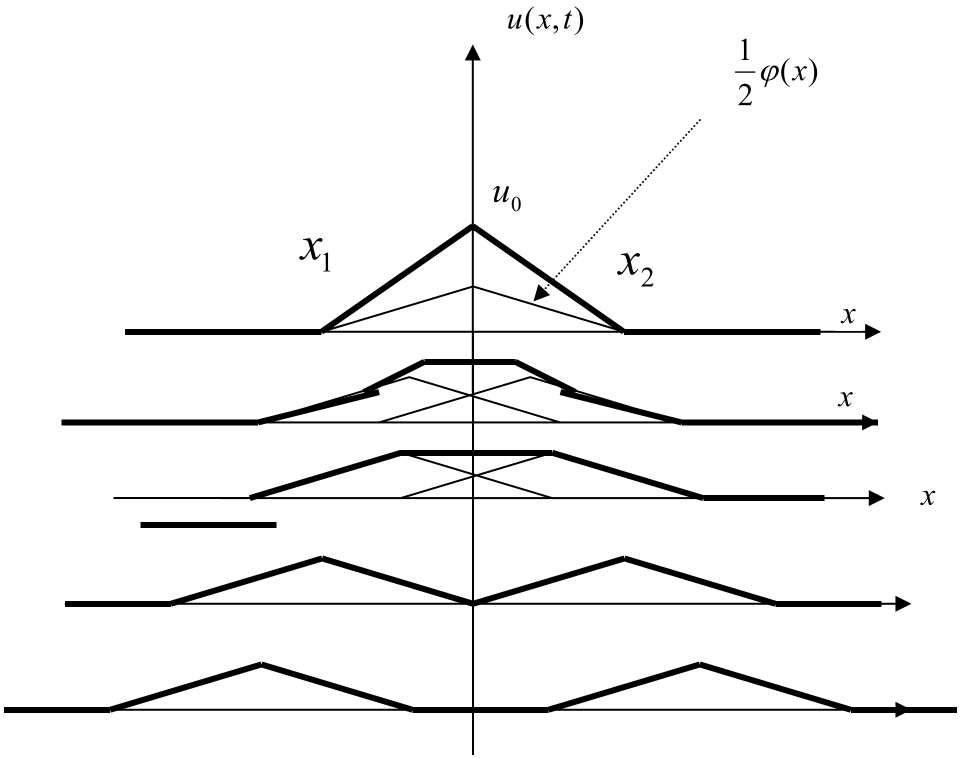



解：由达朗贝尔公式得：

$$u(x,t) = \frac{1}{2}[\varphi(x+at) + \varphi(x-at)]$$

$$u(x,t)=\frac{1}{2}[\varphi(x+at)+\varphi(x-at)]$$




弦的振动完全由初始位移引起，波通过的地区，振动消失,弦静止在平衡位置.

**例4** $\begin{cases} u_{tt} - a^2 u_{xx} = 0, & -\infty < x < \infty \\ u|_{t=0} = \varphi(x), & u_t|_{t=0} = \psi(x) \end{cases}$

其中 $\varphi(x) = 0$ $\quad \psi(x) = \begin{cases} \psi_0 & x_1 \le x \le x_2 \\ 0 & x_1 \ge x, x_2 \le x \end{cases}$

**解：**

$$u(x,t) = \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi = \frac{1}{2a}\int_{-\infty}^{x+at} \psi(\xi)d\xi - \frac{1}{2a}\int_{-\infty}^{x-at} \psi(\xi)d\xi$$

令 $\Psi(x) = \frac{1}{2a}\int_{-\infty}^{x} \psi(\xi)d\xi$

则 $x < x_1$ $\quad \Psi(x) = \frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi = \frac{1}{2a}\int_{-\infty}^{x} 0 d\xi \quad = 0$

$x_1 < x < x_2$ $\quad \Psi(x) = \frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi = \frac{1}{2a}[\int_{-\infty}^{x_1} 0 d\xi + \psi_0 \int_{x_1}^{x} 1 d\xi] = \frac{\psi_0}{2a}(x - x_1)$

$x > x_2$ $\quad \Psi(x) = \frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi = \frac{1}{2a}[\int_{-\infty}^{x_1} 0 d\xi + \psi_0 \int_{x_1}^{x_2} 1 d\xi + \int_{x_2}^{x} 0 d\xi] = \frac{\psi_0}{2a}(x_2 - x_1)$

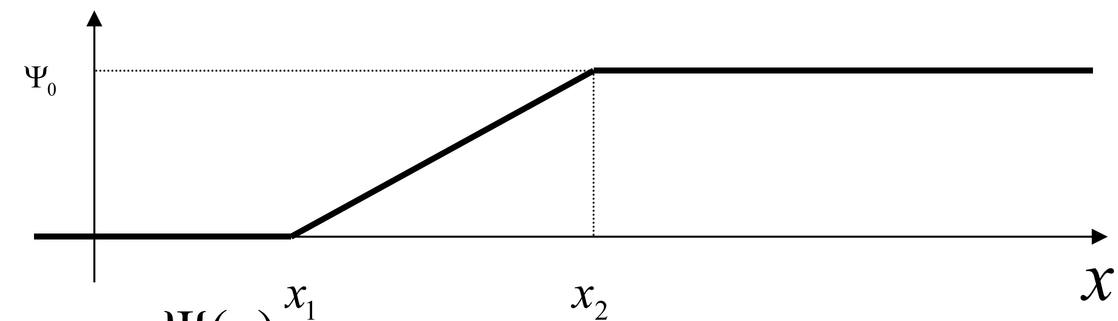
$$\Psi(x)=\frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi$$

$x<x_1$ $$\Psi(x)=\frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi=\frac{1}{2a}\int_{-\infty}^{x}0d\xi \qquad =0$$

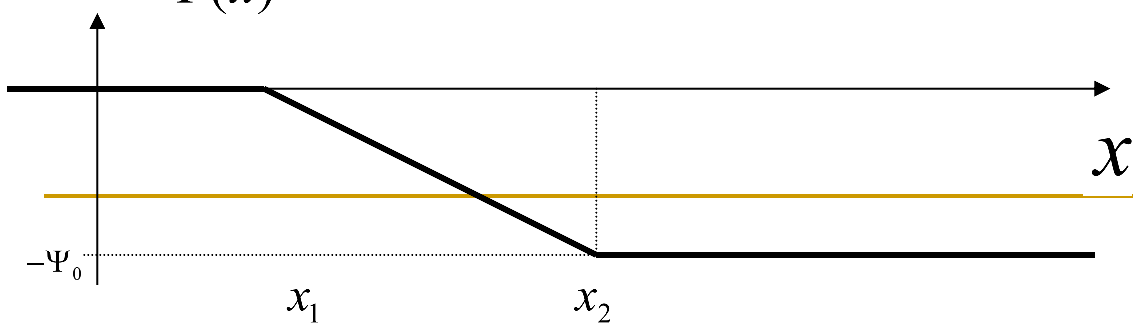
$x_1<x<x_2$ $$\Psi(x)=\frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi=\frac{1}{2a}\left[\int_{-\infty}^{x_1}0d\xi+\psi_0\int_{x_1}^{x}1d\xi\right] \quad =\frac{\psi_0}{2a}(x-x_1)$$

$x>x_2$ $$\Psi(x)=\frac{1}{2a}\int_{-\infty}^{x}\psi(\xi)d\xi=\frac{1}{2a}\left[\int_{-\infty}^{x_1}0d\xi+\psi_0\int_{x_1}^{x_2}1d\xi+\int_{x_2}^{x}0d\xi\right] \quad =\frac{\psi_0}{2a}(x_2-x_1)$$

$\Psi(x)$



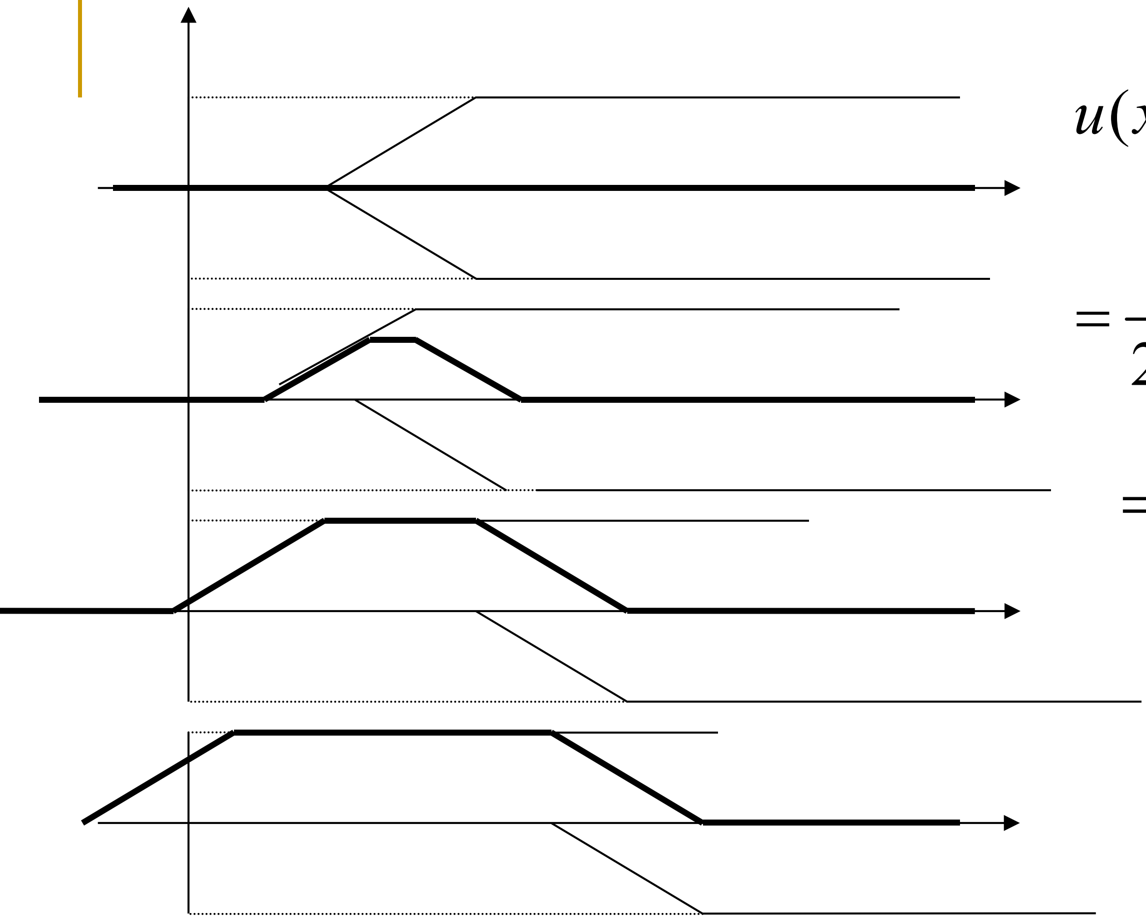

$-\Psi(x)$




$$u(x,t)=\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

$$=\frac{1}{2a}\int_{-\infty}^{x+at}\psi(\xi)d\xi-\frac{1}{2a}\int_{-\infty}^{x-at}\psi(\xi)d\xi$$

$$=\Psi(x+at)-\Psi(x-at)$$



$$u(x,t)=\frac{1}{2a}\int_{x-at}^{x+at}\psi(\xi)d\xi$$

$$=\frac{1}{2a}\int_{-\infty}^{x+at}\psi(\xi)d\xi-\frac{1}{2a}\int_{-\infty}^{x-at}\psi(\xi)d\xi$$

$$=\Psi(x+at)-\Psi(x-at)$$

振动完全由初始速度引起，波通过的地区,振动消失,但弦偏离了原来的平衡位置.

# 五、达朗贝尔公式的间接应用

**例1**

$$\begin{cases} u_{xx} - u_{yy} = 0, \quad -\infty < x < \infty \\ u|_{y=0} = x, \quad u_y|_{y=0} = 0 \end{cases}$$

解：化成类似于波动方程的初值问题

$$\begin{cases} u_{yy} - u_{xx} = 0, \quad -\infty < x < \infty \\ u|_{y=0} = x, \quad u_y|_{y=0} = 0 \end{cases}$$

将定解条件代入达朗贝尔公式，得

$$u(x,t) = \tfrac{1}{2}[(x+at)+(x-at)] = x$$

**例2** 解定解问题

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + 2\dfrac{\partial^2 u}{\partial x \partial y} - 3\dfrac{\partial^2 u}{\partial y^2} = 0, & y > 0, -\infty < x < +\infty \\ u(x,0) = e^{-x^2}, \dfrac{\partial u(x,0)}{\partial y} = 0, & -\infty < x < +\infty \end{cases}$$

**解:**

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)u = 0$$

令 $x = \dfrac{1}{4}(\xi + \eta)$, $y = \dfrac{1}{4}(-\xi + 3\eta)$ 则 $\xi = 3x - y$, $\eta = x + y$

且 $$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi} = \frac{1}{4}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta} = \frac{1}{4}\left(\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

$$\therefore u = f_1(\xi) + f_2(\eta) \quad = f_1(3x - y) + f_2(x + y)$$

$$\therefore u(x,0) = e^{-x^2} \quad = f_1(3x) + f_2(x)$$

$$\therefore u(x,0) = e^{-x^2} = f_1(3x) + f_2(x)$$

$$\frac{\partial u(x,0)}{\partial y} = 0 = -f_1'(3x) + f_2'(x)$$

积分得：$-\frac{1}{3} f_1(3x) + f_2(x) = C$

$$f_1(3x) = \frac{3}{4}\left(e^{-x^2} - C\right) \qquad \therefore f_2(x) = \frac{1}{4}\left(e^{-x^2} + 3C\right)$$

$$u = f_1(3x - y) + f_2(x + y)$$

$$= \frac{3}{4} e^{-\frac{1}{9}(3x-y)^2} - \frac{3}{4} C + \frac{1}{4} e^{-(x+y)^2} + \frac{3}{4} C$$

$$= \frac{3}{4} e^{-\frac{1}{9}(3x-y)^2} + \frac{1}{4} e^{-(x+y)^2}$$

# 六、小结

达朗贝尔公式(行波法):

1、它基于波动的特点，引入坐标变换简化方程

利用偏积分的方法先求出通解，然后利用定解条件，得到定解形式。

2、优点：求解方式易于理解，求解波动方程十分方便；

缺点：只适合求解一维无界的齐次波动方程（初值问题）