Use of Julia/JuMP for Multi-Objective Optimization

Prof. Dr. Xavier Gandibleux

Nantes Université, France

June 2025

Why this adventure with Julia and JuMP for MOO?

2015: Kick-off of the ANR/DFG research project vOpt

Sub-task 2.2: Experimental Analysis and Prototype Development.

↳ The project of a multi-objective MILP solver was born!

Purposes of the solver (1/2)

Expectations:

Easy to

- formulate a problem
- provide the data
- solve a problem
- query the results
- analyze the solutions

- install (no need of being a geek)

↳ Julia programming language: free, open source, multi-platform

JuMP modeling language: expressive, efficient, evolutive

Purposes of the solver (1/2)

Expectations:

Easy to

- formulate a problem
- provide the data
- solve a problem
- query the results
- analyze the solutions

- install (no need of being a geek)

↳ Julia programming language: free, open source, multi-platform

JuMP modeling language: expressive, efficient, evolutive

Purposes of the solver (2/2)

Public and usages:

Natural, intuitive use for mathematicians, informaticians, engineers

- for research:
  support and primitives for the development of new algorithms
- for experimentation:
  methods and algorithms for performing numerical experiments
- for teaching:
  environment for practicing of theories and algorithms

2017: Introduction of vOptSolver

Xavier Gandibleux, Gauthier Soleilhac, Anthony Przybylski, Stefan Ruzika. vOptSolver: an open source software environment for multiobjective mathematical optimization. *IFORS2017: 21st Conference of the International Federation of Operational Research Societies*. July 17-21, 2017. Quebec City (Canada).

https://github.com/vOptSolver

Purposes of the solver (2/2)

Public and usages:

Natural, intuitive use for mathematicians, informaticians, engineers

- for research:
  support and primitives for the development of new algorithms
- for experimentation:
  methods and algorithms for performing numerical experiments
- for teaching:
  environment for practicing of theories and algorithms

2017: Introduction of vOptSolver

Xavier Gandibleux, Gauthier Soleilhac, Anthony Przybylski, Stefan Ruzika. vOptSolver: an open source software environment for multiobjective mathematical optimization. *IFORS2017: 21st Conference of the International Federation of Operational Research Societies*. July 17-21, 2017. Quebec City (Canada).

`https://github.com/vOptSolver`

From `vOptGeneric` to `MultiObjectiveAlgorithms`

`vOptSolver` is composed of two packages:

- `vOptGeneric.jl`
- `vOptSpecific.jl`

2015: Kick-off of vOpt ANR/DFG project

2017: Introduction of vOptSolver (vOptGeneric, vOptSpecific)

2018: Julia 1.0

2019: End of vOpt project

2019: First discussions on discourse/new design of MOO in JuMP

2022: JuMP 1.0

2023: Introduction of `MultiObjectiveAlgorithms.jl`

2023: End of vOptGeneric

- part concerning the MO model $\longrightarrow$ `JuMP.jl`
- part concerning MO algorithms $\longrightarrow$ `MultiObjectiveAlgorithms.jl`

From vOptGeneric to MultiObjectiveAlgorithms

vOptSolver is composed of two packages:

- vOptGeneric.jl
- vOptSpecific.jl

2015: Kick-off of vOpt ANR/DFG project

2017: Introduction of vOptSolver (vOptGeneric, vOptSpecific)

2018: Julia 1.0

2019: End of vOpt project

2019: First discussions on discourse/new design of MOO in JuMP

2022: JuMP 1.0

2023: Introduction of MultiObjectiveAlgorithms.jl

2023: End of vOptGeneric

- part concerning the MO model → JuMP.jl
- part concerning MO algorithms → MultiObjectiveAlgorithms.jl

Agenda

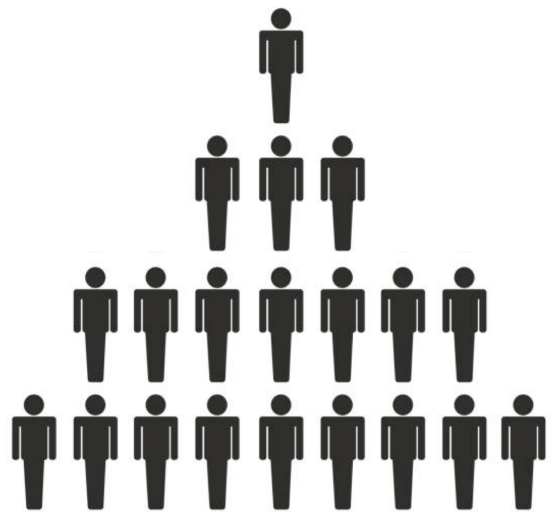
1. Few words about Julia
2. Brief introduction to JuMP
3. Optimization with MultiObjectiveAlgorithms

Julia

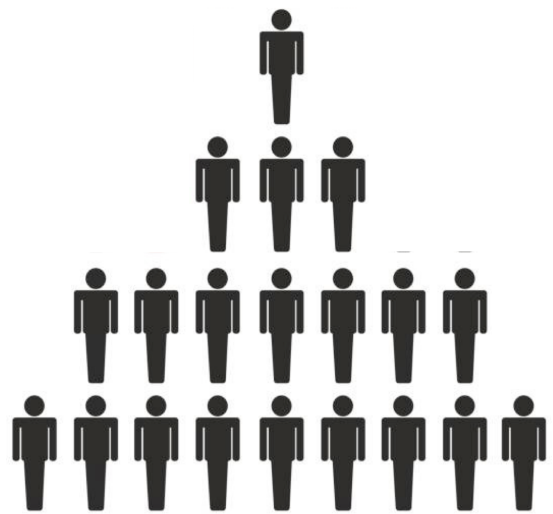
Coding and Running
Scientific Computing Algorithms

`https://julialang.org/`

Julia

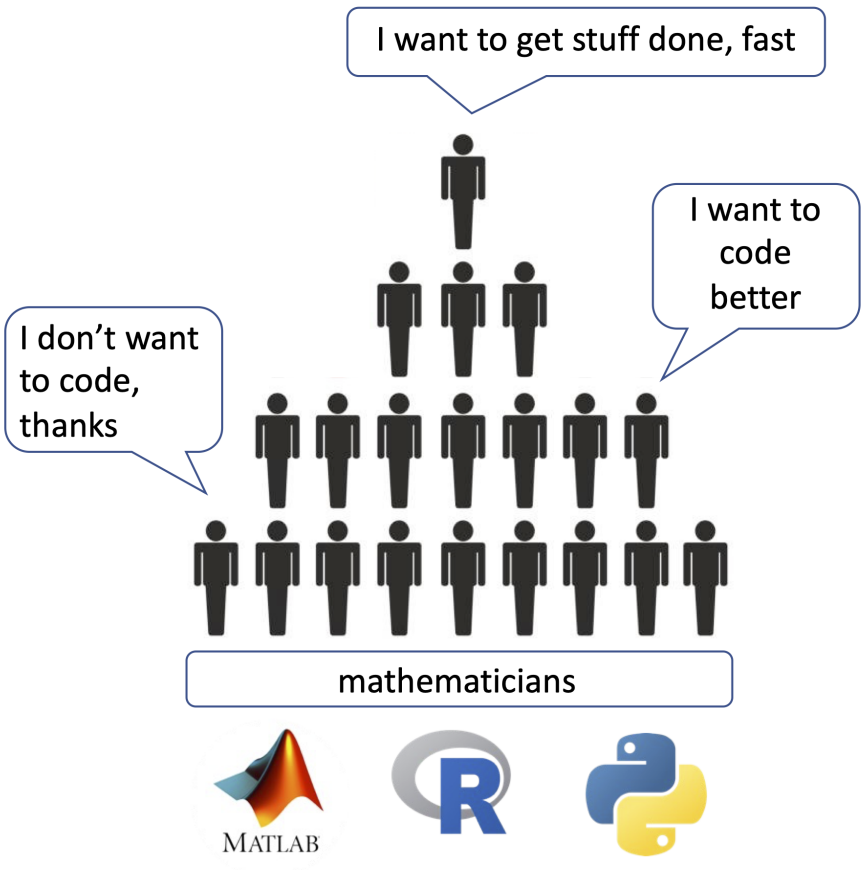


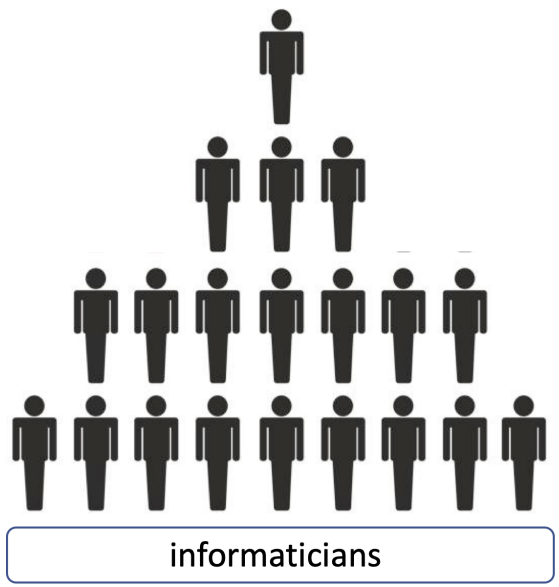
mathematicians



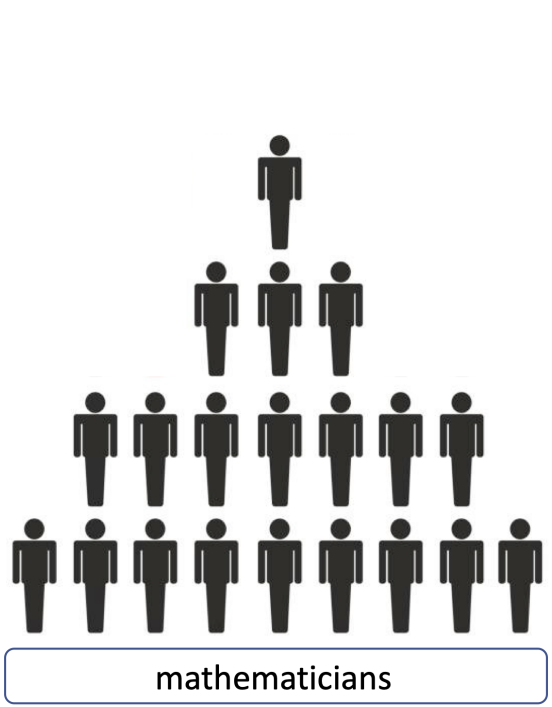
informaticians

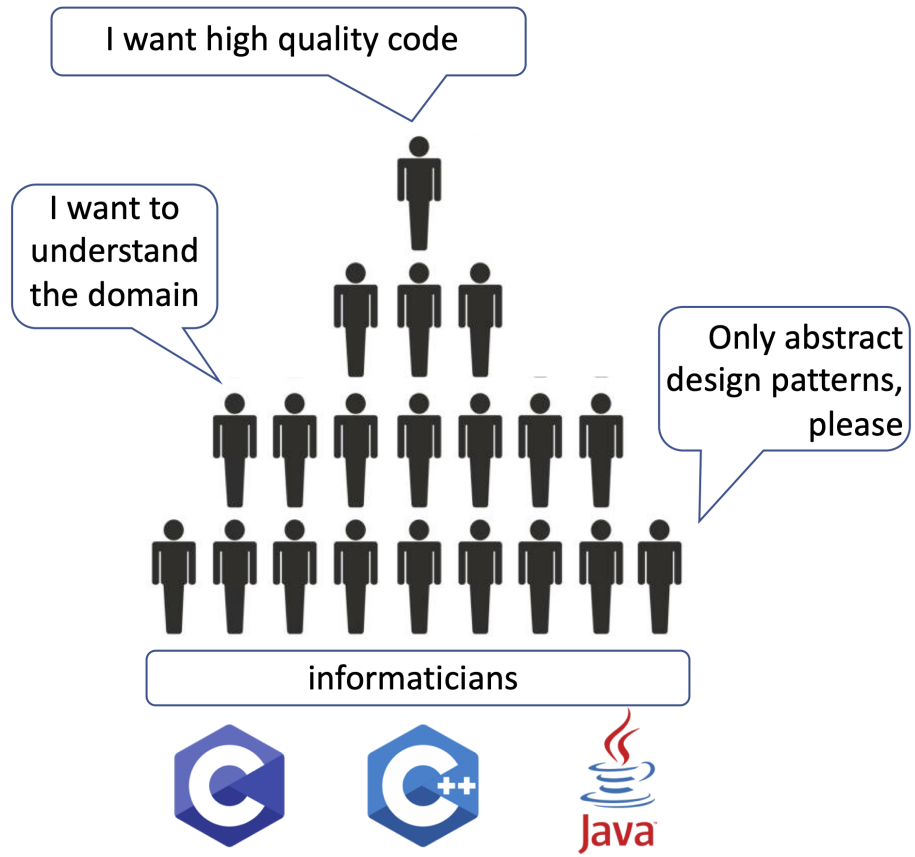
Julia


I want to get stuff done, fast
I want to code better
I don't want to code, thanks
mathematicians
MATLAB


informaticians

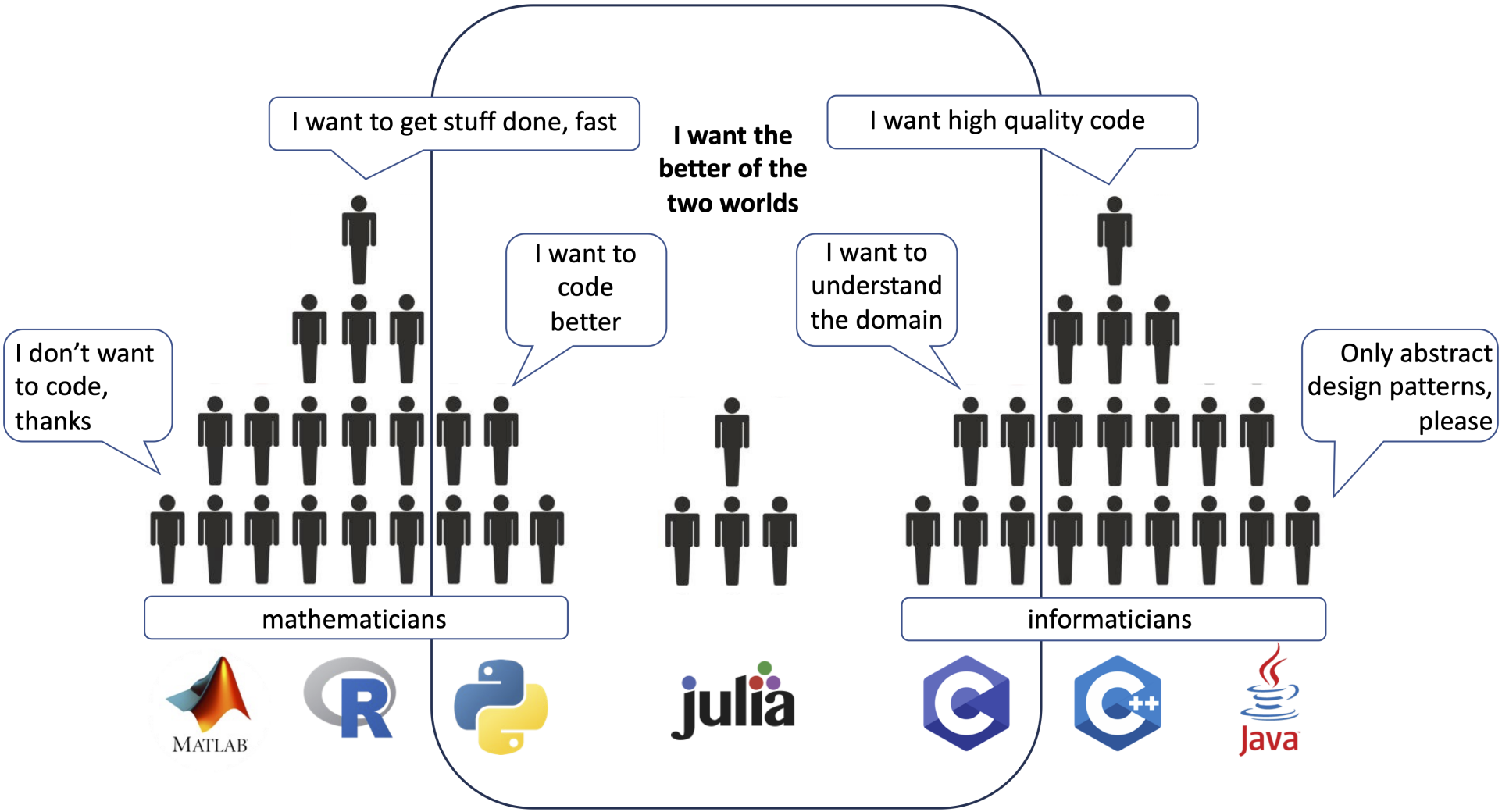
Julia


mathematicians


I want high quality code
I want to understand the domain
Only abstract design patterns, please
informaticians
Java

Julia

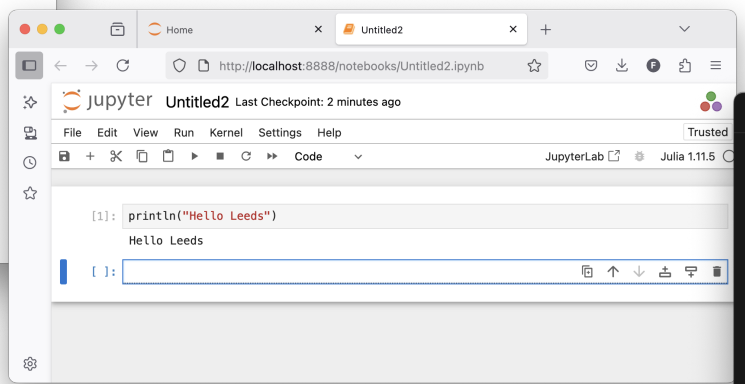
A programming language for optimization



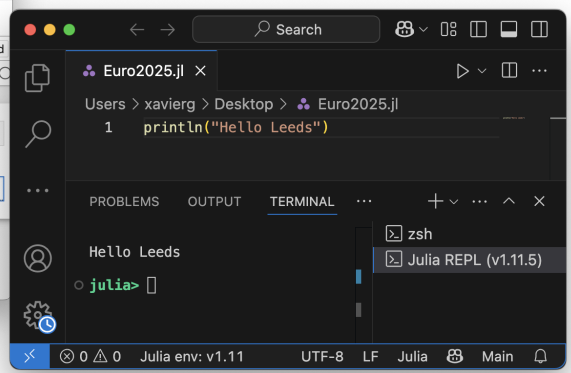
Running a Julia code into...

...the Julia REPL

...a notebook
(e.g. jupyter)

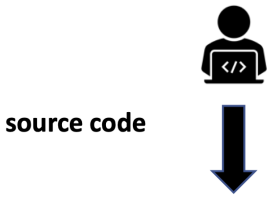


...an IDE
(e.g. VScode)



... or also in command line into a terminal.

Julia

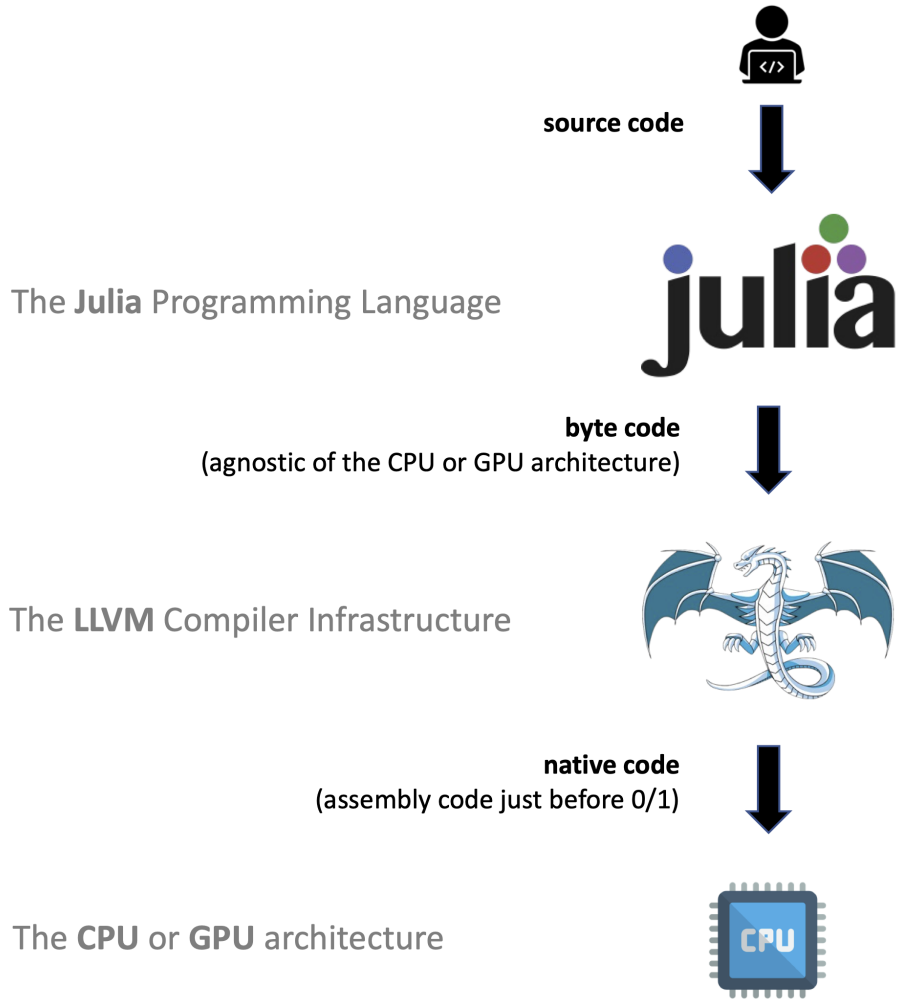



```
julia> 1+2
3
```

The **Julia** Programming Language

Julia

LLVM: middleman between source code and compiled native code




```
julia> 1+2
3
```

```
julia> @code_llvm 1+2
;  @ int.jl:87 within `+`
define i64 @"julia_+_115"(i64 signext %0, i64 signext %1) #0 {
top:
  %2 = add i64 %1, %0
  ret i64 %2
}
```

```
julia> @code_native 1+2
        .section        __TEXT,__text,regular,pure_instructions
        .build_version macos, 13, 0
        .globl  "_julia_+_150"                  ; -- Begin function julia_+_150
        .p2align        2
"_julia_+_150":                         ; @"julia_+_150"
; ┌ @ int.jl:87 within `+`
        .cfi_startproc
; %bb.0:                                ; %top
        add     x0, x1, x0
        ret
        .cfi_endproc
; └
                                        ; -- End function
.subsections_via_symbols
```

```
01101100
01101111
01110110
01100101
```

Example (1/2)

MATLAB

```
tic

    A = zeros(100,100,100);
    for loop = 1:1000
        for i = 1:100
            for j = 1:100
                for k = 1:100
                    A(i,j,k) = loop;
                end
            end
        end
        disp(A(100,100,100));
    end

toc
```

Julia

```
@time begin

    A = zeros(100,100,100)
    for loop = 1:1000
        for i = 1:100
            for j = 1:100
                for k = 1:100
                    A[i,j,k] = loop
                end
            end
        end
        println(A[100,100,100])
    end

end
```

C

```
#include <stdio.h>
#include <stdlib.h>

int main()
{
    int A[100][100][100];
    int i,j,k,loop;
    for (loop=0; loop <1000; loop++){
        for (i=0; i<100; i++){
            for (j=0; j<100; j++){
                for (k=0; k<100; k++){
                    A[i][j][k] = loop;
                }
            }
        }
        printf("%d\n", A[99][99][99]);
    }
    return 0;
}
```

Example (2/2)

- C

  without optimization option

  `1.286582 seconds`

  with optimization option `-O3`

  `0.087682 seconds`

- Julia

  1-to-1 straightforward implementation

  `26.732819 seconds (489.05 M allocations: 7.297 GiB, 0.29% gc time)`

  julia style compliant implementation

  `0.252995 seconds (22.39 k allocations: 8.499 MiB)`

  polished implementation

  `0.114872 seconds (20.75 k allocations: 4.318 MiB)`

↳ https://github.com/xgandibleux/EURO2025/tree/main/2.examples%20jl

Example (2/2)

- C

  without optimization option

  `1.286582 seconds`

  with optimization option `-O3`

  `0.087682 seconds`

- Julia

  1-to-1 straightforward implementation

  `26.732819 seconds (489.05 M allocations: 7.297 GiB, 0.29% gc time)`

  julia style compliant implementation

  `0.252995 seconds (22.39 k allocations: 8.499 MiB)`

  polished implementation

  `0.114872 seconds (20.75 k allocations: 4.318 MiB)`

↳ https://github.com/xgandibleux/EURO2025/tree/main/2.examples%20jl

Example (2/2)

- C

  without optimization option

  `1.286582 seconds`

  with optimization option `-O3`

  `0.087682 seconds`

- Julia

  1-to-1 straightforward implementation

  `26.732819 seconds (489.05 M allocations: 7.297 GiB, 0.29% gc time)`

  julia style compliant implementation

  `0.252995 seconds (22.39 k allocations: 8.499 MiB)`

  polished implementation

  `0.114872 seconds (20.75 k allocations: 4.318 MiB)`

↳ https://github.com/xgandibleux/EURO2025/tree/main/2.examples%20jl

Example (2/2)

- C

  without optimization option

  `1.286582 seconds`

  with optimization option `-O3`

  `0.087682 seconds`

- Julia

  1-to-1 straightforward implementation

  `26.732819 seconds (489.05 M allocations:  7.297 GiB, 0.29% gc time)`

  julia style compliant implementation

  `0.252995 seconds (22.39 k allocations:  8.499 MiB)`

  polished implementation

  `0.114872 seconds (20.75 k allocations:  4.318 MiB)`

↳ `https://github.com/xgandibleux/EURO2025/tree/main/2.examples%20jl`

More (1/2)...

Jeff Bezanson, Alan Edelman, Stefan Karpinski and Viral B. Shah. Julia: A Fresh Approach to Numerical Computing. *SIAM Review*, 59: 65–98. 2017.

Resources:

The Julia Programming Language (`https://julialang.org/`):
download, documentation, resources, community, etc.

Conferences:

JuliaCon Global (`https://juliacon.org/`):
Annual International Conference since 2014 (Chicago, Cambridge, Berkeley, London, Baltimore, online, Eindhoven). **2025: Pittsburgh**

JuliaCon Local (`https://juliacon.org/`):
Annual International Conference since 2023 (Eindhoven). **2025: Paris**

JuliaDays:
the Julia & Optimization days in France since 2019 (Nantes, Paris, Toulouse).

More (2/2)...

Example of material for optimizers:

Course

An introduction to Julia and JuMP for Operations Research
Xavier Gandibleux
ÖGOR Summer-Workshop for PhD-candidates and Post-Docs (Krems, Austria)
`https://github.com/xgandibleux/Krems2023`

Book

Algorithms for Optimization
Mykel J. Kochenderfer, and Tim A. Wheeler
The MIT Press Cambridge, Massachusetts London, England
`https://algorithmsbook.com/optimization/files/optimization.pdf`

Practice

1. Installing Julia on your computer

   ↳ `https://julialang.org/`

2. Discovering the four modes of the REPL

3. Installing the package IJulia
4. Running Julia code into Jupyter notebook

5. Installing the Julia extension into VScode
6. Editing and running Julia code into VScode

JuMP

Coding and Solving
Mathematical Programming Models

`JuMP.jl`

What is JuMP?

An Algebraic Modeling Language (AML) among the existing ones . . .

... for mathematical optimization (linear, mixed-integer, conic, semidefinite, nonlinear) written in the Julia language.

Overview of JuMP

- **User friendliness**: syntax that mimics natural mathematical expressions.
- **Speed**: similar speeds to special-purpose modeling languages such as AMPL.
- **Solver independence**: Currently 56 solvers are supported.

  Among the (MI)LP solvers: GLPK, Cbc, Clp, HiGHS, SCIP, CPLEX, Gurobi, FICO Xpress, COPT...
- **Ease of embedding**: JuMP itself is written purely in Julia. Solvers are the only binary dependencies.

Overview of JuMP

- **User friendliness**: syntax that mimics natural mathematical expressions.
- **Speed**: similar speeds to special-purpose modeling languages such as AMPL.
- **Solver independence**: Currently 56 solvers are supported.

  Among the (MI)LP solvers: GLPK, Cbc, Clp, HiGHS, SCIP, CPLEX, Gurobi, FICO Xpress, COPT...
- **Ease of embedding**: JuMP itself is written purely in Julia. Solvers are the only binary dependencies.

Overview of JuMP

- **User friendliness**: syntax that mimics natural mathematical expressions.
- **Speed**: similar speeds to special-purpose modeling languages such as AMPL.
- **Solver independence**: Currently 56 solvers are supported.

  Among the (MI)LP solvers: GLPK, Cbc, Clp, HiGHS, SCIP, CPLEX, Gurobi, FICO Xpress, COPT...
- **Ease of embedding**: JuMP itself is written purely in Julia. Solvers are the only binary dependencies.

Overview of JuMP

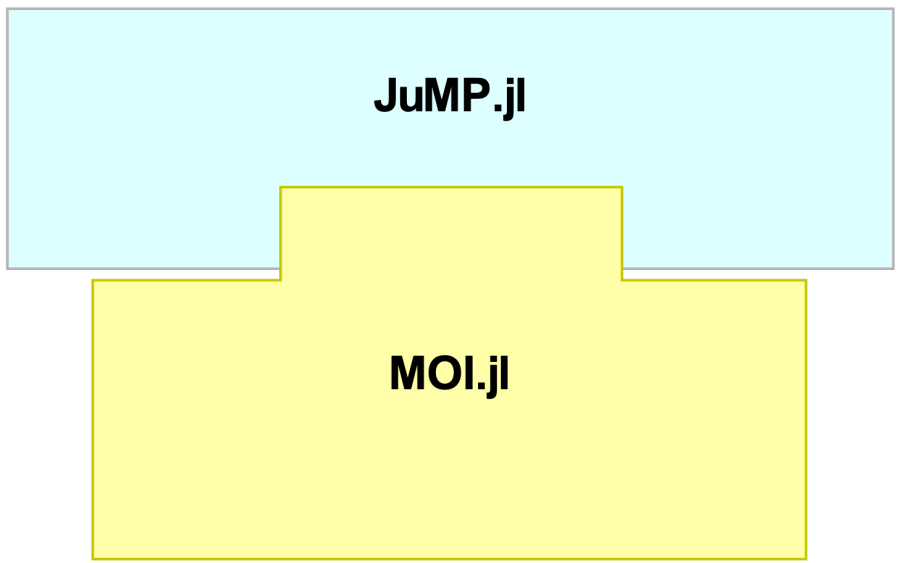
- **User friendliness**: syntax that mimics natural mathematical expressions.
- **Speed**: similar speeds to special-purpose modeling languages such as AMPL.
- **Solver independence**: Currently 56 solvers are supported.

  Among the (MI)LP solvers: GLPK, Cbc, Clp, HiGHS, SCIP, CPLEX, Gurobi, FICO Xpress, COPT...
- **Ease of embedding**: JuMP itself is written purely in Julia. Solvers are the only binary dependencies.

Getting started with JuMP

Getting started

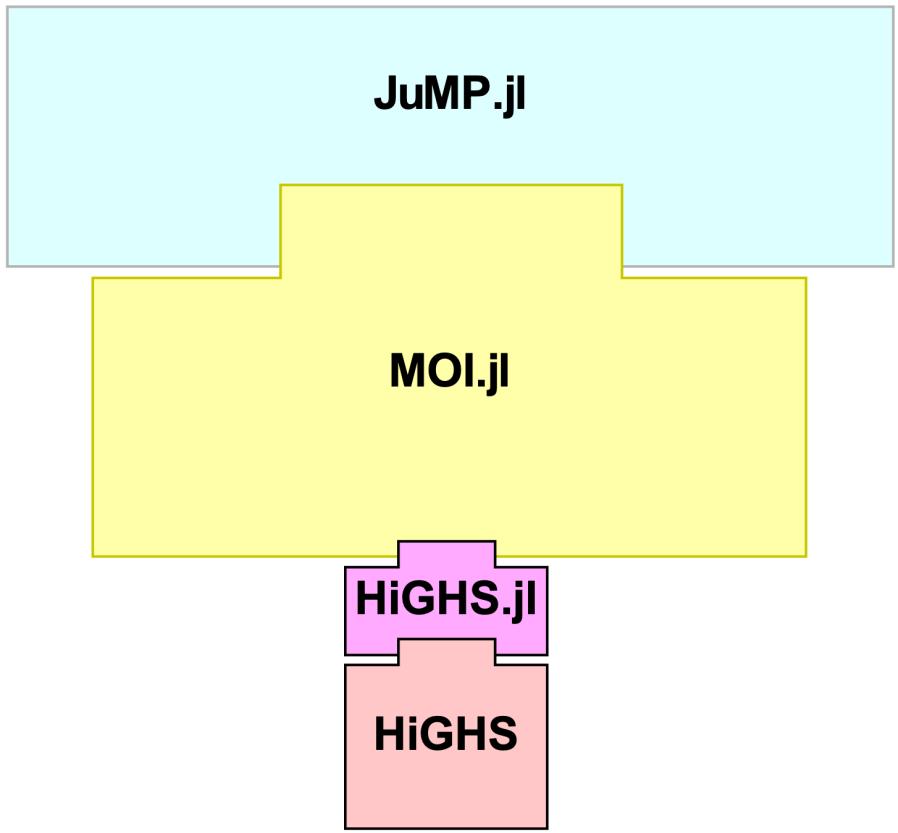
```
julia> using Pkg
julia> Pkg.add("JuMP")
```



JuMP uses *MathOptInterface (MOI)*, an abstraction layer designed to provide a unified interface to mathematical optimization solvers.

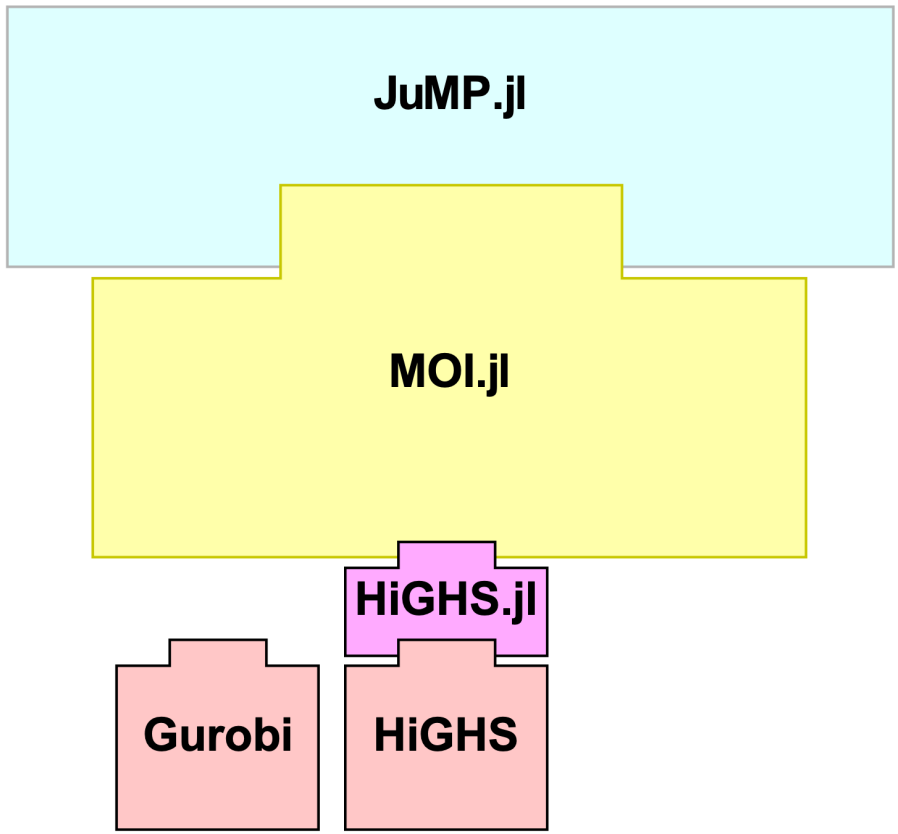
Getting started

```
julia> Pkg.add("HiGHS")
```



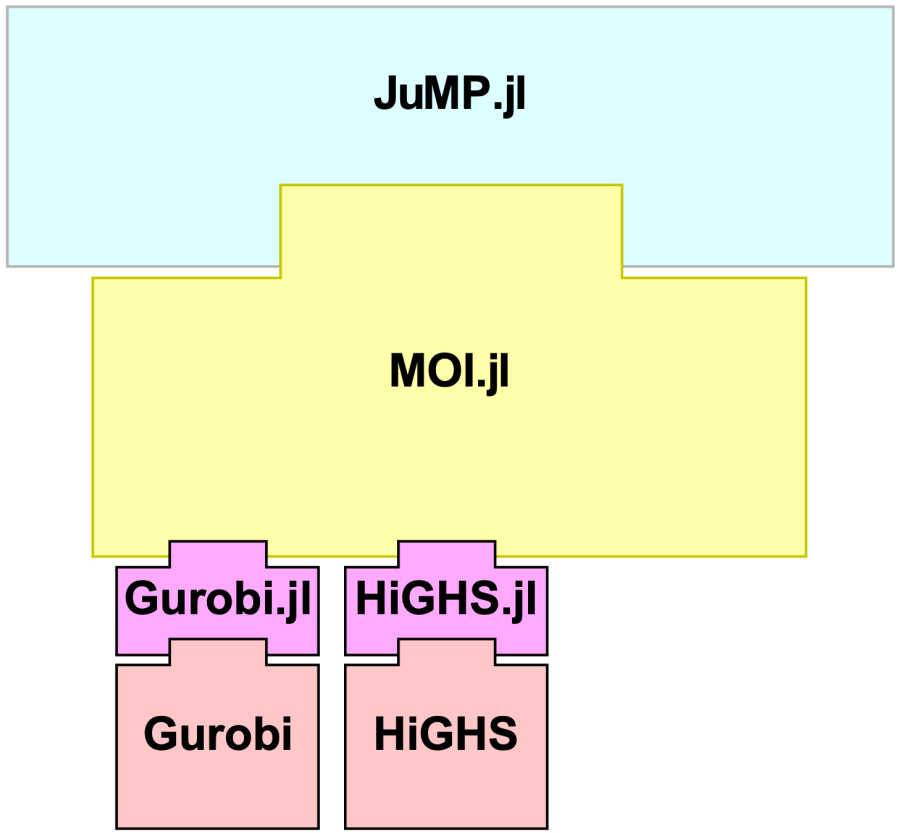
Getting started

With Gurobi or CPLEX or Xpress MP solver properly installed on your computer...



Getting started

```
julia> Pkg.add("Gurobi")
```



Modeling and solving a (MI)LP with JuMP

The optimization problem

$$
\left[\begin{array}{rrrcrcl}
\max z(x) = & x_1 & + & 3x_2 & & & (0) \\
s.t & x_1 & + & x_2 & \leqslant & 14 & (1) \\
& -2x_1 & + & 3x_2 & \leqslant & 12 & (2) \\
& 2x_1 & - & x_2 & \leqslant & 12 & (3) \\
& x_1 & , & x_2 & \geqslant & 0 & (4)
\end{array}\right]
$$

Declaring the packages to use

```
julia> using JuMP, HiGHS
```

Coding the optimization problem (1/4)

Creating a model:

```
modelId = Model( )
```

```
julia> modLP = Model( )
```

Result:

$$\begin{bmatrix} & \end{bmatrix}$$

Coding the optimization problem (2/4)

Defining a variable:

```
@variable(modelId, variable)
```

```
julia> @variable(modLP, x1 >= 0)
julia> @variable(modLP, x2 >= 0)
```

Result:

$$x_1 \quad , \quad x_2 \quad \geqslant \quad 0 \qquad (4)$$

Coding the optimization problem (3/4)

Defining an objective function:

```
@objective(modelId, sense, function)
```

```
julia> @objective(modLP, Max, x1 + 3x2)
```

Result:

$$\begin{bmatrix} \max z(x) = & x_1 + 3x_2 & & (0) \\ & & & \\ & x_1 \quad , \quad x_2 \geqslant 0 & & (4) \end{bmatrix}$$

Coding the optimization problem (4/4)

Defining a constraint:

```
@constraint(modelId, id, constraint)
```

```
julia> @constraint(modLP, cst1, x1 + x2 <= 14)
julia> @constraint(modLP, cst2, -2x1 + 3x2 <= 12)
julia> @constraint(modLP, cst3, 2x1 - x2 <= 12)
```

Result:

$$\left[\begin{array}{rrrrrrr} \max z(x) = & x_1 & + & 3x_2 & & & (0) \\ s.t & x_1 & + & x_2 & \leqslant & 14 & (1) \\ & -2x_1 & + & 3x_2 & \leqslant & 12 & (2) \\ & 2x_1 & - & x_2 & \leqslant & 12 & (3) \\ & x_1 & , & x_2 & \geqslant & 0 & (4) \end{array}\right]$$

Solving the optimization problem

```
set_optimizer(modelId, solver )
```

```
julia> set_optimizer(modLP, HiGHS.Optimizer)
```

```
set_silent(modelId )
```

```
julia> set_silent(modLP)
```

```
optimize!(modelId)
```

```
julia> optimize!(modLP)
```

Solving the optimization problem

```
set_optimizer(modelId, solver )
```

```
julia> set_optimizer(modLP, HiGHS.Optimizer)
```

```
set_silent(modelId )
```

```
julia> set_silent(modLP)
```

```
optimize!(modelId)
```

```
julia> optimize!(modLP)
```

Solving the optimization problem

```
set_optimizer(modelId, solver )
```

```
julia> set_optimizer(modLP, HiGHS.Optimizer)
```

```
set_silent(modelId )
```

```
julia> set_silent(modLP)
```

```
optimize!(modelId)
```

```
julia> optimize!(modLP)
```

Querying the result

```
julia> @show is_solved_and_feasible(modLP)
```

```
julia> @show termination_status(modLP)
```

```
julia> @show objective_value(modLP)
```

```
julia> @show value(x1)
julia> @show value(x2)
```

Querying the result

```
julia> @show is_solved_and_feasible(modLP)
```

```
julia> @show termination_status(modLP)
```

```
julia> @show objective_value(modLP)
```

```
julia> @show value(x1)
julia> @show value(x2)
```

Querying the result

```
julia> @show is_solved_and_feasible(modLP)
```

```
julia> @show termination_status(modLP)
```

```
julia> @show objective_value(modLP)
```

```
julia> @show value(x1)
julia> @show value(x2)
```

Summary (explicit model)

```
julia> using JuMP, HiGHS

julia> modLP = Model( )
julia> @variable(modLP, x1 >= 0)
julia> @variable(modLP, x2 >= 0)
julia> @objective(modLP, Max, x1 + 3x2)
julia> @constraint(modLP, cst1, x1 + x2 <= 14)
julia> @constraint(modLP, cst2, -2x1 + 3x2 <= 12)
julia> @constraint(modLP, cst3, 2x1 - x2 <= 12)

julia> set_optimizer(modLP, HiGHS.Optimizer)
julia> set_silent(modLP)
julia> optimize!(modLP)

julia> @show is_solved_and_feasible(modLP)
julia> @show objective_value(modLP)
julia> @show value(x1), value(x2)
```

↳ https://github.com/xgandibleux/EURO2025/blob/main/2.examples%20jl/1IP.ipynb

More...

Miles Lubin, Oscar Dowson, Joaquim Dias Garcia, Joey Huchette, Benoît Legat, Juan Pablo Vielma. JuMP 1.0: Recent improvements to a modeling language for mathematical optimization. *Mathematical Programming Computation*. 2023.

Miles Lubin. The state of JuMP. *JuMP-dev 2024 - Co-Located with ISMP'2024*, July 19-21, 2024. GERAD research center, Montréal, Canada.
`https://www.youtube.com/watch?v=cMUTdfoAMpo`

Resources:

The JuMP website (`https://jump.dev/`):
documentation, resources, community, etc.

Workshops:

JuMP-dev (`https://jump.dev/meetings/jumpdev2025/`):
Regular meeting since 2017: Cambridge (USA), Bordeaux (France), Santiago (Chile), online, Montréal (Canada). **2025: Auckland (New Zealand)**

Practice

1. Installing the packages JuMP and HiGHS
2. Coding and solving an IP problem
   ↳ `https://github.com/xgandibleux/EURO2025/blob/main/3.practice/Part2.ipynb`

MOA

Moving from
single to multiple objectives MIP optimization

`MultiObjectiveAlgorithms.jl`

Multiobjective Optimization Problem (MOP) notations

$$\begin{array}{ll} \min & F(x) \\ s.t. & x \in X \end{array}$$

where:

$$\begin{array}{rcl} X \subseteq \mathbb{R}^n & \longrightarrow & \text{the set of feasible solutions} \\ Y = F(X) \subseteq \mathbb{R}^p & \longrightarrow & \text{the set of images} \end{array}$$

Computing sets X_E and Y_N for MOP:

$$y = F(x)$$

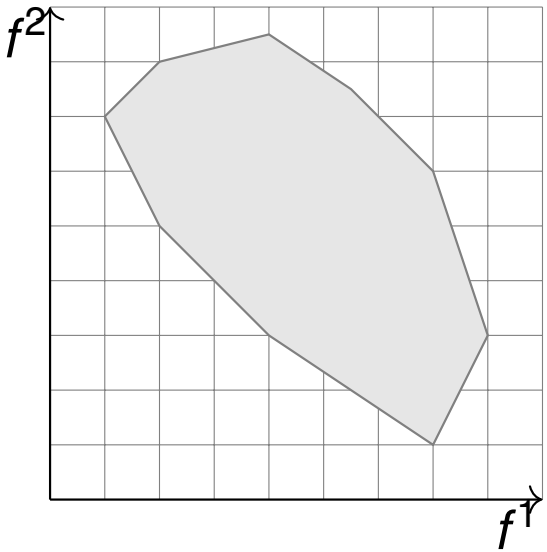
$y^* \in Y$ is nondominated, if $\nexists\, y \in Y$ such that $y_i \leqslant y_i^*,\ \forall i$ and $y \neq y^*$.
Y_N is the set of nondominated points.

$x^* \in X$ is efficient if y^* is nondominated.
X_E is a complete set of efficient solutions.

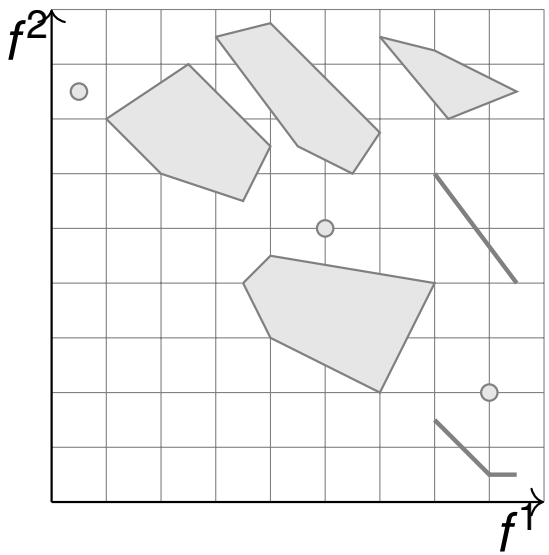
Examples of Y and Y_N when $p = 2$

Computing Y_N for Multiobjective (linear) Optimization Problems (MOP):

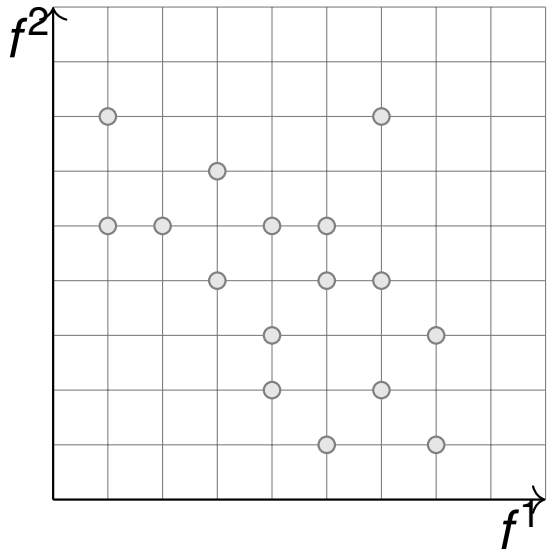
$x \in \mathbb{R}^{n_1}$



$x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$



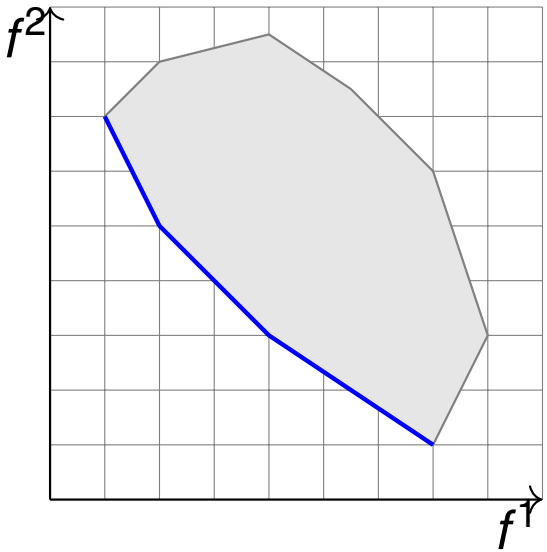
$x \in \mathbb{Z}^{n_2}$



Examples of Y and Y_N when $p = 2$

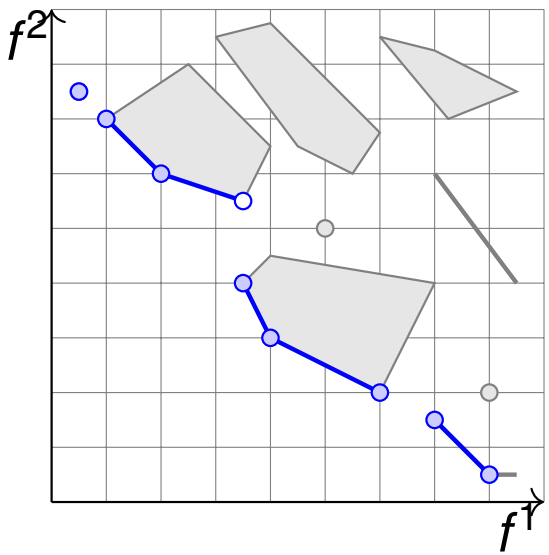
Computing Y_N for Multiobjective (linear) Optimization Problems (MOP):

$x \in \mathbb{R}^{n_1}$



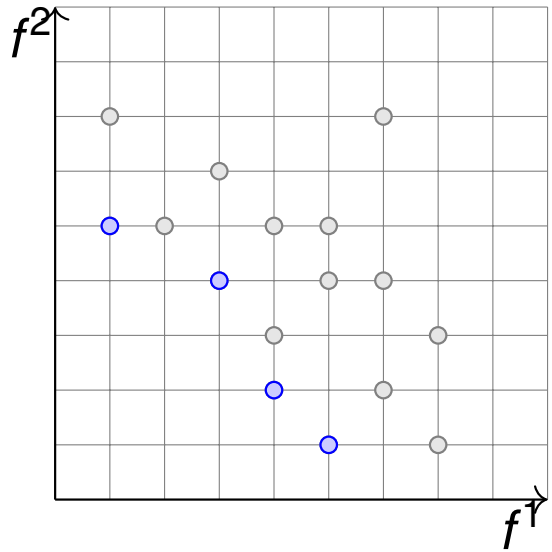
Continuous nondominated set (edges)

$x \in \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$



Nondominated set composed of edges that are either closed, half-open, open or reduced to a point

$x \in \mathbb{Z}^{n_2}$



Discrete set of nondominated points

Existing MP Software for MOP

ADBASE	MOLP	(1975)
BENSOLVE (with GLPK)	MOLP	(2014; last rev 2017)
polySCIP (with SCIP)	MOLP, MOIP	(2016; last rev 2017)
inner (with GLPK)	MOLP	(2016?; last rev 2024)
JuMP + vOptSolver|MOA + solvers	MOIP	(2017, last rev 2025)
PaMILO (with Gurobi/CPLEX)	MOMILP(Y_{SN})	(2020?; last rev'25)
HiGHS	...	(since Ver1.10?, 2025?)

- Gurobi Optimization (since Ver7.0, 2016)
- IBM ILOG CPLEX Optimisation (since Ver12.7, 2016)
- FICO Xpress Optimization (since Ver9.0, 2022)
- Hexaly Optimizer (since Ver12.0, 2023)

Existing MP Software for MOP

- ADBASE — MOLP — (1975)
- BENSOLVE (with GLPK) — MOLP — (2014; last rev 2017)
- polySCIP (with SCIP) — MOLP, MOIP — (2016; last rev 2017)
- inner (with GLPK) — MOLP — (2016?; last rev 2024)
- JuMP + vOptSolver|MOA + solvers — MOIP — (2017, last rev 2025)
- PaMILO (with Gurobi/CPLEX) — MOMILP(Y_{SN}) — (2020?; last rev'25)
- HiGHS — ... — (since Ver1.10?, 2025?)

- Gurobi Optimization — (since Ver7.0, 2016)
- IBM ILOG CPLEX Optimisation — (since Ver12.7, 2016)
- FICO Xpress Optimization — (since Ver9.0, 2022)
- Hexaly Optimizer — (since Ver12.0, 2023)

Existing MP Software for MOP

ADBASE	MOLP	(1975)
BENSOLVE (with GLPK)	MOLP	(2014; last rev 2017)
polySCIP (with SCIP)	MOLP, MOIP	(2016; last rev 2017)
inner (with GLPK)	MOLP	(2016?; last rev 2024)
JuMP + vOptSolver|MOA + solvers	MOIP	(2017, last rev 2025)
PaMILO (with Gurobi/CPLEX)	MOMILP(Y_{SN})	(2020?; last rev'25)
HiGHS	...	(since Ver1.10?, 2025?)

- Gurobi Optimization (since Ver7.0, 2016)
- IBM ILOG CPLEX Optimisation (since Ver12.7, 2016)
- FICO Xpress Optimization (since Ver9.0, 2022)
- Hexaly Optimizer (since Ver12.0, 2023)

Getting Started (con't)

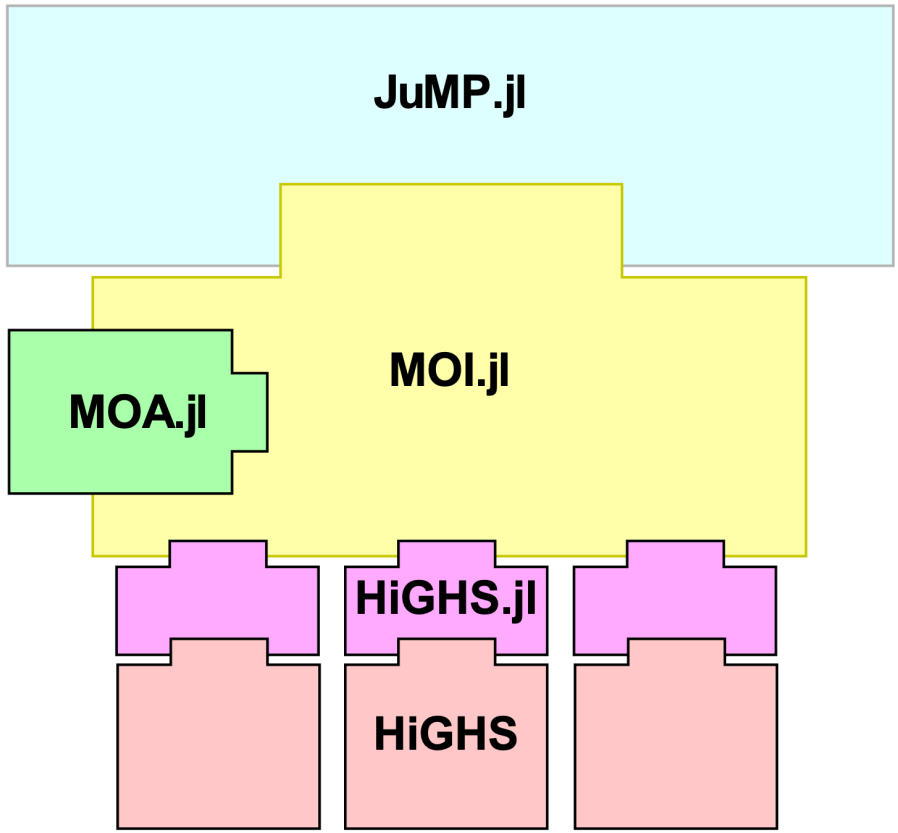
Installing the package:

```
julia> using Pkg
```

```
julia> Pkg.status()
Status `~/.julia/environments/v1.11/Project.toml`
  [60bf3e95] HiGHS v1.18.1
  [4076af6c] JuMP v1.26.0
```

```
julia> Pkg.add("MultiObjectiveAlgorithms")
```

Getting Started (con't)



Algorithms currently available (v1.4.3)

1. MOA.Lexicographic() [default] $p \geqslant 2$
2. MOA.Dichotomy() $p = 2$
3. MOA.EpsilonConstraint() $p = 2$
4. MOA.Hierarchical() $p \geqslant 2$
5. MOA.Chalmet() $p = 2$
6. MOA.KirlikSayin() $p \geqslant 3$
7. MOA.DominguezRios() $p \geqslant 3$
8. MOA.TambyVanderpooten() $p \geqslant 3$

+ optimization attributes coming with a given algorithm

Algorithms currently available (v1.4.3)

1. MOA.Lexicographic() [default] $p \geqslant 2$
2. MOA.Dichotomy() $p = 2$
3. MOA.EpsilonConstraint() $p = 2$
4. MOA.Hierarchical() $p \geqslant 2$
5. MOA.Chalmet() $p = 2$
6. MOA.KirlikSayin() $p \geqslant 3$
7. MOA.DominguezRios() $p \geqslant 3$
8. MOA.TambyVanderpooten() $p \geqslant 3$

+ optimization attributes coming with a given algorithm

Algorithms currently available (v1.4.3)

1. MOA.Lexicographic() [default] $p \geqslant 2$
2. MOA.Dichotomy() $p = 2$
3. MOA.EpsilonConstraint() $p = 2$
4. MOA.Hierarchical() $p \geqslant 2$
5. MOA.Chalmet() $p = 2$
6. MOA.KirlikSayin() $p \geqslant 3$
7. MOA.DominguezRios() $p \geqslant 3$
8. MOA.TambyVanderpooten() $p \geqslant 3$

+ optimization attributes coming with a given algorithm

Zoom on the ϵ-constraint algorithm

Algorithm:

- `EpsilonConstraint()`

  Y.V. Haimes, L.S. Lasdon, D.A. Wismer (1971). On a bicriterion formation of the problems of integrated system identification and system optimization. *IEEE Transactions on Systems, Man and Cybernetics*, Volume SMC-1, Issue 3, Pages 296-297.

Attributes:

- `MOA.EpsilonConstraintStep()`

  algorithm uses this value as the ϵ by which it partitions the first-objective's space. The default is 1, so that for a pure integer program this algorithm will enumerate Y_N.

- `MOA.SolutionLimit()`

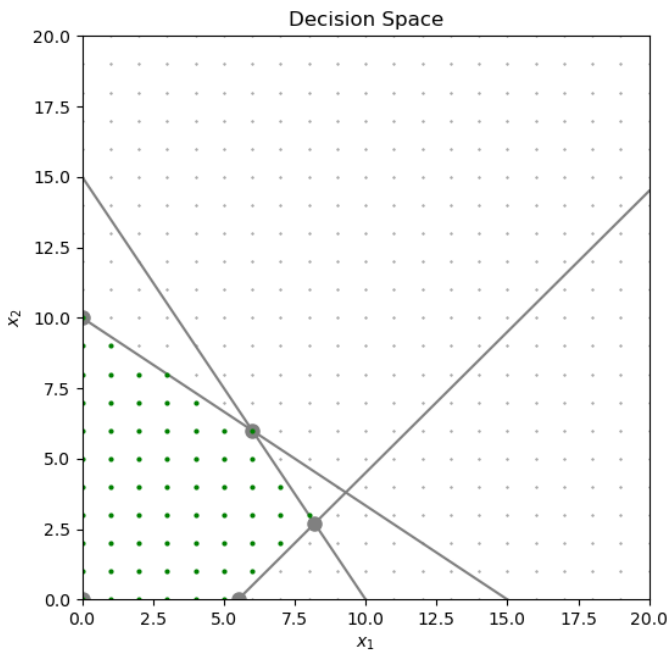
  if this attribute is set then, instead of using the `MOA.EpsilonConstraintStep`, with a slight abuse of notation, `EpsilonConstraint` divides the range of the first-objective's domain in objective space by `SolutionLimit` to obtain the ϵ to use when iterating. Thus, at most `SolutionLimit` solutions are returned.

Example (1/4)

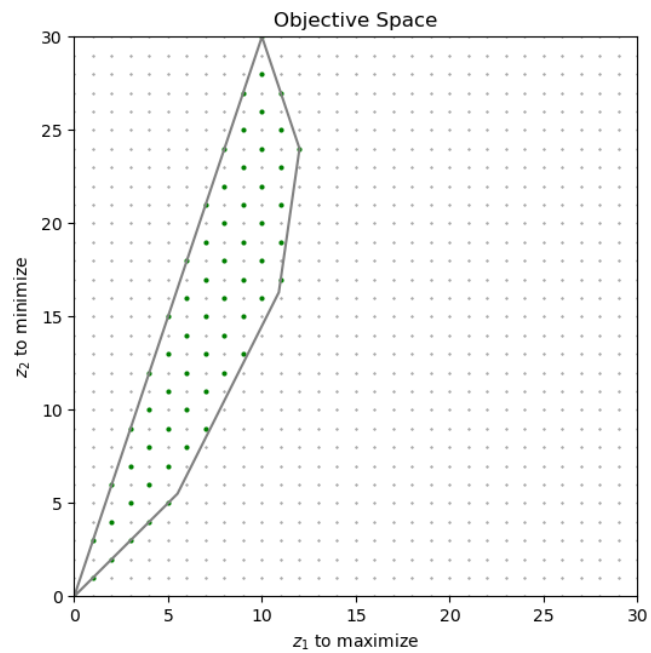
Compute X_E and Y_N for the following MOP:

$$
\begin{array}{lcrcrcr}
\max z_1 & = & x_1 & + & x_2 & & \\
\min z_2 & = & x_1 & + & 3x_2 & & \\
s.t. & & 2x_1 & + & 3x_2 & \leqslant & 30 \\
 & & 3x_1 & + & 2x_2 & \leqslant & 30 \\
 & & x_1 & - & x_2 & \leqslant & 5.5 \\
 & & x_1 & , & x_2 & \in & \mathbb{N}
\end{array}
$$

Decision space:



Objective space:



Example (2/4)

Coding the MOO problem with JuMP

```
julia> using JuMP, HiGHS
julia> import MultiObjectiveAlgorithms as MOA
```

```
julia> model = Model( )
julia> @variable(model, x1⩾0, Int)
julia> @variable(model, x2⩾0, Int)
julia> @expression(model, fct1, x1 + x2)                    # to maximize
julia> @expression(model, fct2, x1 + 3 * x2)                # to minimize
julia> @objective(model, Max, [fct1, (-1) * fct2])
julia> @constraint(model, 2*x1 + 3*x2 ⩽ 30)
julia> @constraint(model, 3*x1 + 2*x2 ⩽ 30)
julia> @constraint(model, x1 - x2 ⩽ 5.5)
```

Example (3/4)

Setup the MIP solver: e.g. `HiGHS`

```
julia> set_optimizer(model,()->MOA.Optimizer(HiGHS.Optimizer))
```

Setup the algorithm: ϵ-constraint; step=1 (default value)

```
julia> set_attribute(model, MOA.Algorithm(), MOA.EpsilonConstraint())
```

Optimize the MOO problem

```
julia> optimize!(model)
```

Example (3/4)

Setup the MIP solver: e.g. `HiGHS`

```
julia> set_optimizer(model,()->MOA.Optimizer(HiGHS.Optimizer))
```

Setup the algorithm: ϵ-constraint; step=1 (default value)

```
julia> set_attribute(model, MOA.Algorithm(), MOA.EpsilonConstraint())
```

Optimize the MOO problem

```
julia> optimize!(model)
```

Example (3/4)

Setup the MIP solver: e.g. `HiGHS`

```
julia> set_optimizer(model,()->MOA.Optimizer(HiGHS.Optimizer))
```

Setup the algorithm: ϵ-constraint; step=1 (default value)

```
julia> set_attribute(model, MOA.Algorithm(), MOA.EpsilonConstraint())
```

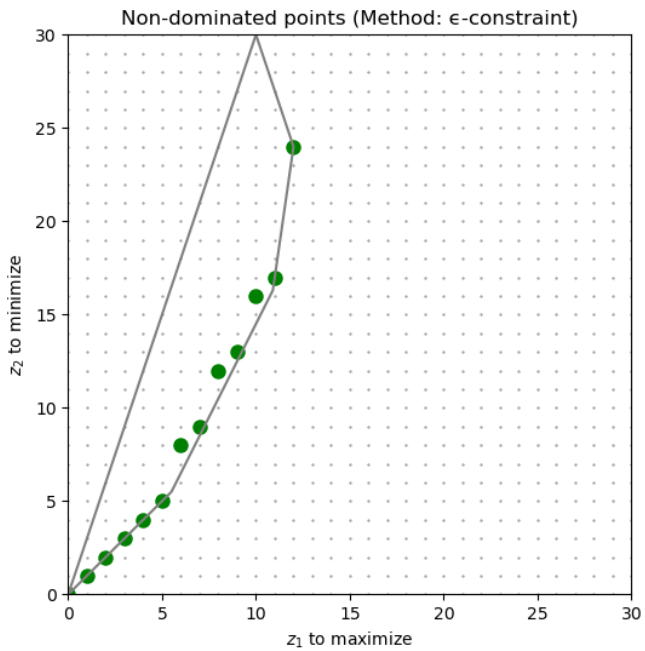
Optimize the MOO problem

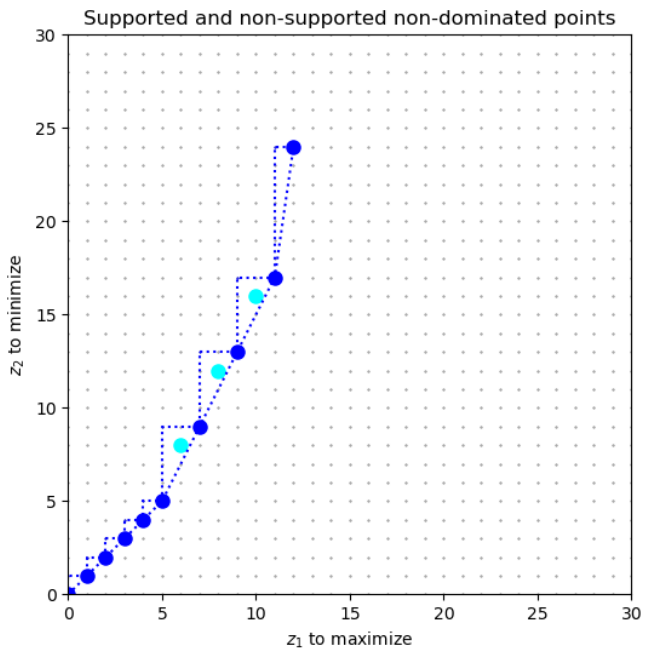
```
julia> optimize!(model)
```

Example (4/4)

Get X_E and Y_N

```
julia> for i in 1:result_count(model)
julia>     z1_opt = objective_value(model; result = i)[1]
julia>     z2_opt = -1 * objective_value(model; result = i)[2]
julia>     x1_opt = value(x1; result = i)
julia>     x2_opt = value(x2; result = i)
julia> end
```





↳ https://github.com/xgandibleux/EURO2025/blob/main/2.examples%20jl/2IPex.ipynb

From
an explicit to an implicit model
with JuMP

Example (1/2)

$$
\begin{array}{lcccccc}
\max z_1 & = & x_1 & + & x_2 & & \\
\min z_2 & = & x_1 & + & 3x_2 & & \\
s.t. & & 2x_1 & + & 3x_2 & \leqslant & 30 \\
 & & 3x_1 & + & 2x_2 & \leqslant & 30 \\
 & & x_1 & - & x_2 & \leqslant & 5.5 \\
 & & x_1 & , & x_2 & \in & \mathbb{N}
\end{array}
\quad \rightarrow \quad
\begin{array}{lcccc}
\max z_k = & \sum_{j=1}^{n} c_j^k x_j & & & k = 1, \ldots, p \\
s.t & \sum_{j=1}^{n} a_{ij} x_j & \leqslant & b_i & i = 1, \ldots, m \\
 & x_j & \in & \mathbb{N} & j = 1, \ldots, n
\end{array}
$$

```
julia> c1 = [1,1]; c2 = -1 * [1,3]; c = vcat(c1',c2')
julia> a = [2 3; 3 2; 1 -1]
julia> b = [30, 30, 5.5]
julia> m,n = size(a)
julia> p = size(c,1)
```

```
julia> md = Model( )
julia> @variable(md, x[1:n]⩾0, Int)
julia> @expression(md, z[k=1:p], sum(c[k,j]*x[j] for j=1:n))
julia> @objective(md, Max, [z[k] for k=1:p])
julia> @constraint(md, ct[i=1:m], sum(a[i,j]*x[j] for j=1:n) ⩽ b[i])
```

Example (1/2)

$$
\begin{array}{lcrcrcl}
\max z_1 & = & x_1 & + & x_2 & & \\
\min z_2 & = & x_1 & + & 3x_2 & & \\
s.t. & & 2x_1 & + & 3x_2 & \leqslant & 30 \\
 & & 3x_1 & + & 2x_2 & \leqslant & 30 \\
 & & x_1 & - & x_2 & \leqslant & 5.5 \\
 & & x_1 & , & x_2 & \in & \mathbb{N}
\end{array}
\quad \rightarrow \quad
\begin{array}{lcrcll}
\max z_k = & & \sum_{j=1}^{n} c_j^k x_j & & & k = 1, \ldots, p \\
s.t & & \sum_{j=1}^{n} a_{ij} x_j & \leqslant & b_i & i = 1, \ldots, m \\
 & & x_j & \in & \mathbb{N} & j = 1, \ldots, n
\end{array}
$$

```
julia> c1 = [1,1]; c2 = -1 * [1,3]; c = vcat(c1',c2')
julia> a = [2 3; 3 2; 1 -1]
julia> b = [30, 30, 5.5]
julia> m,n = size(a)
julia> p = size(c,1)
```

```
julia> md = Model( )
julia> @variable(md, x[1:n]⩾0, Int)
julia> @expression(md, z[k=1:p], sum(c[k,j]*x[j] for j=1:n))
julia> @objective(md, Max, [z[k] for k=1:p])
julia> @constraint(md, ct[i=1:m], sum(a[i,j]*x[j] for j=1:n) ⩽ b[i])
```

Example (1/2)

$$
\begin{array}{lcrcrcl}
\max z_1 & = & x_1 & + & x_2 & & \\
\min z_2 & = & x_1 & + & 3x_2 & & \\
s.t. & & 2x_1 & + & 3x_2 & \leqslant & 30 \\
 & & 3x_1 & + & 2x_2 & \leqslant & 30 \\
 & & x_1 & - & x_2 & \leqslant & 5.5 \\
 & & x_1 & , & x_2 & \in & \mathbb{N}
\end{array}
\quad \rightarrow \quad
\begin{array}{lcll}
\max z_k = & \sum_{j=1}^{n} c_j^k x_j & & k = 1, \ldots, p \\
s.t & \sum_{j=1}^{n} a_{ij} x_j & \leqslant b_i & i = 1, \ldots, m \\
 & x_j & \in \mathbb{N} & j = 1, \ldots, n
\end{array}
$$

```
julia> c1 = [1,1]; c2 = -1 * [1,3]; c = vcat(c1',c2')
julia> a = [2 3; 3 2; 1 -1]
julia> b = [30, 30, 5.5]
julia> m,n = size(a)
julia> p = size(c,1)
```

```
julia> md = Model( )
julia> @variable(md, x[1:n]⩾0, Int)
julia> @expression(md, z[k=1:p], sum(c[k,j]*x[j] for j=1:n))
julia> @objective(md, Max, [z[k] for k=1:p])
julia> @constraint(md, ct[i=1:m], sum(a[i,j]*x[j] for j=1:n) ⩽ b[i])
```

Example (1/2)

$$
\begin{array}{lcrcrcl}
\max z_1 & = & x_1 & + & x_2 & & \\
\min z_2 & = & x_1 & + & 3x_2 & & \\
s.t. & & 2x_1 & + & 3x_2 & \leqslant & 30 \\
 & & 3x_1 & + & 2x_2 & \leqslant & 30 \\
 & & x_1 & - & x_2 & \leqslant & 5.5 \\
 & & x_1 & , & x_2 & \in & \mathbb{N}
\end{array}
\quad \rightarrow \quad
\begin{array}{lcrcll}
\max z_k = & & \sum_{j=1}^{n} c_j^k x_j & & & k = 1, \ldots, p \\
s.t & & \sum_{j=1}^{n} a_{ij} x_j & \leqslant & b_i & i = 1, \ldots, m \\
 & & x_j & \in & \mathbb{N} & j = 1, \ldots, n
\end{array}
$$

```
julia> c1 = [1,1]; c2 = -1 * [1,3]; c = vcat(c1',c2')
julia> a = [2 3; 3 2; 1 -1]
julia> b = [30, 30, 5.5]
julia> m,n = size(a)
julia> p = size(c,1)
```

```
julia> md = Model( )
julia> @variable(md, x[1:n]⩾0, Int)
julia> @expression(md, z[k=1:p], sum(c[k,j]*x[j] for j=1:n))
julia> @objective(md, Max, [z[k] for k=1:p])
julia> @constraint(md, ct[i=1:m], sum(a[i,j]*x[j] for j=1:n) ⩽ b[i])
```

Example (1/3)

$$
\begin{array}{lcrcrcl}
\max z_1 & = & x_1 & + & x_2 & & \\
\min z_2 & = & x_1 & + & 3x_2 & & \\
s.t. & & 2x_1 & + & 3x_2 & \leqslant & 30 \\
 & & 3x_1 & + & 2x_2 & \leqslant & 30 \\
 & & x_1 & - & x_2 & \leqslant & 5.5 \\
 & & x_1 & , & x_2 & \in & \mathbb{N}
\end{array}
\quad \rightarrow \quad
\begin{array}{lcrcll}
\max z_k = & & \sum_{j=1}^{n} c_j^k x_j & & & k = 1, \ldots, p \\
s.t & & \sum_{j=1}^{n} a_{ij} x_j & \leqslant & b_i & i = 1, \ldots, m \\
 & & x_j & \in & \mathbb{N} & j = 1, \ldots, n
\end{array}
$$

```
julia> c1 = [1,1]; c2 = -1 * [1,3]; c = vcat(c1',c2')
julia> a = [2 3; 3 2; 1 -1]
julia> b = [30, 30, 5.5]
julia> m,n = size(a)
julia> p = size(c,1)
```

```
julia> md = Model( )
julia> @variable(md, x[1:n]⩾0, Int)
julia> @expression(md, z[k=1:p], sum(c[k,j]*x[j] for j=1:n))
julia> @objective(md, Max, [z[k] for k=1:p])
julia> @constraint(md, ct[i=1:m], sum(a[i,j]*x[j] for j=1:n) ⩽ b[i])
```

Example (2/3)

Get X_E and Y_N

```
julia> for i in 1:result_count(model)
julia>     z_opt = [objective_value(md; result = i)[k] for k=1:p]
julia>     x_opt = value.(x; result = i)
julia> end
```

Example (3/3)

Plot Y_N

```
julia> using Plots
julia> Plots.scatter(
julia>  [value(z[1]; result = i) for i in 1:result_count(md)],
julia>  [-1 * value(z[2]; result = i) for i in 1:result_count(md)];
julia>  xlabel = "objective 1",
julia>  ylabel = "objective 2",
julia>  title = "objective space",
julia>  legend = false,
julia>  xlims = (0,25),
julia>  ylims = (0,25),
julia>  aspect_ratio=:equal,
julia> )
```

↳ https://github.com/xgandibleux/EURO2025/blob/main/2.examples%20jl/2IPim.ipynb

More...

Oscar Dowson, Xavier Gandibleux, Gokhan Kof. From vOptGeneric.jl to MultiObjectiveAlgorithms.jl *JuliaDays 2023*, 4-6 October 2023. Paris (France).

JuMP:
`https://jump.dev/`

MultiObjectiveAlgorithms:
`https://github.com/jump-dev/MultiObjectiveAlgorithms.jl`

Examples:
`https://jump.dev/JuMP.jl/stable/tutorials/linear/multi_objective_examples/`

Roadmap

Working on it:

- making the 3-obj algorithms reliable
  ↳ working on a set of test instances
  ↳ `https://github.com/xgandibleux/EURO2025/blob/main/2.examples%20jl/3IP.ipynb`

Missing parts:

- solving MOLP with a **multi-objective simplex algorithm**
  ↳ working on HiGHS2LP, a parametric variant of HiGHS
- solving MOMILP with a **branch-and-xx algorithm**
  ↳ works in progress in the literature
- **detecting problem-specific structure** in a presolve
  ↳ solving MOCO with specific algorithms

Roadmap

Working on it:

- making the 3-obj algorithms reliable
  ↳ working on a set of test instances
  ↳ `https://github.com/xgandibleux/EURO2025/blob/main/2.examples%20jl/3IP.ipynb`

Missing parts:

- solving MOLP with a **multi-objective simplex algorithm**
  ↳ working on HiGHS2LP, a parametric variant of HiGHS
- solving MOMILP with a **branch-and-xx algorithm**
  ↳ works in progress in the literature
- **detecting problem-specific structure** in a presolve
  ↳ solving MOCO with specific algorithms

Practice

1. Installing the packages MultiObjectiveAlgorithms and Plots
2. Coding and solving a MOIP problem
   ↳ `https://github.com/xgandibleux/EURO2025/blob/main/3.practice/Part3.ipynb`

Call for participation

European conference on the Julia programming language
October 2-3, 2025 — Paris (France)
`https://juliacon.org/`

Keynote Speakers:

Laura Grigori (EPFL, Switzerland)
Ivet Galabova (HiGHS, Scotland)
Tim Besard (JuliaHub, Belgium)

Venue: