A primal matheuristic for multi-objective binary linear optimization problems

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   - Conic constraints on objective space $\mathcal{Y}$
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Multi-objective 0-1 linear optimization $p - 01LP$

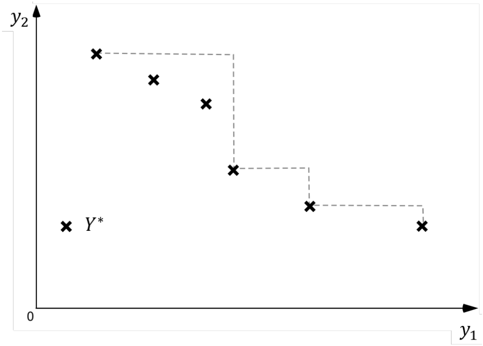
$$\begin{aligned} \min y(x) &= Cx \\ \text{subject to } Ax &\geq b \\ x &\in \{0,1\}^n \end{aligned}$$

where

- $x \in \{0,1\}^n$, the vector of n binary variables $x_j,\ j \in N = \{1, \ldots, n\}$
- $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$, the m constraints $A_i x \geq b_i,\ i = 1, \ldots, m$
- $C \in \mathbb{R}^{p \times n}$, the p linear objective functions, $C_k x, k = 1, \ldots, p$
- $\mathcal{X} := \{x \in \{0,1\}^n \mid Ax \geq b\} \subseteq \mathbb{R}^n$, the feasible decision space
- $\mathcal{Y} := \{y(x) \in \mathbb{R}^p \mid x \in \mathcal{X}\} \subseteq \mathbb{R}^p$, the outcome set

Nondominated Points ($\mathcal{Y}^*$) and Efficient Solutions ($\mathcal{X}^*$)

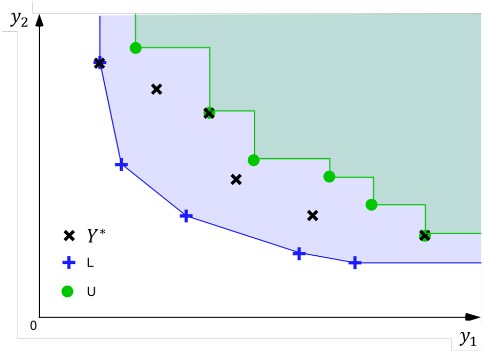
- $\mathcal{Y}^* = \{y^* \in \mathcal{Y} : \nexists\, y \neq y^* \in \mathcal{Y} \mid y_k \leq y_k^*,\ \forall k = 1, ..., p\}$
- $\mathcal{X}^* = \{x^* \in \mathcal{X} : y(x^*) \in \mathcal{Y}^*\}$



Lower ($\mathcal{L}$) and Upper Bound ($\mathcal{U}$) sets

An upper bound set for $Y' \subset \mathcal{Y}^*$ is a subset $\mathcal{U} \subset \mathbb{R}^p$ such that

- For each $y \in Y'$ there is some $u \in \mathcal{U}$ such that $y \leq u$
- There is no pair $(y, u) \in Y' \times \mathcal{U}$ such that u dominates y



Matthias Ehrgott and Xavier Gandibleux
Bounds and Bound Sets for Biobjective Combinatorial Optimization Problems
Lecture Notes in Economics and Mathematical Systems, (2001)

Matthias Ehrgott and Xavier Gandibleux
Bound sets for biobjective combinatorial optimization problems
Computers & Operations Research, (2007)

Multi-Objective Primal Math-Heuristic

To compute an upper bound set $\mathcal{U}$ well representative of $\mathcal{Y}^*$, and not necessary a close approximation of all $y \in \mathcal{Y}^*$

The main characteristics:

- Generic designed for any p-01LP with $p \geq 2$
- Easy to implement only MIP solver is needed

The proposed matheuristic, inspired by Feasibility Pump Heuristic for $1-01$LP problem, is called Gravity Machine

Feasibility Pump Heuristic for $p - 01LP$ problem

- Introduced by Fischetti, Glover and Lodi, for $p = 1$
- Integrated in Gurobi, CPLEX, and GLPK solvers, for $p = 1$
- Adapted by Pal and Charkhgard, for $p \geq 2$

Matteo Fischetti, Fred Glover, and Andrea Lodi
The feasibility pump
Mathematical Programming (2005)

Aritra Pal and Hadi Charkhgard
A Feasibility Pump and Local Search Based Heuristic for Bi-Objective Pure Integer Linear Programming
INFORMS Journal on Computing (2019)

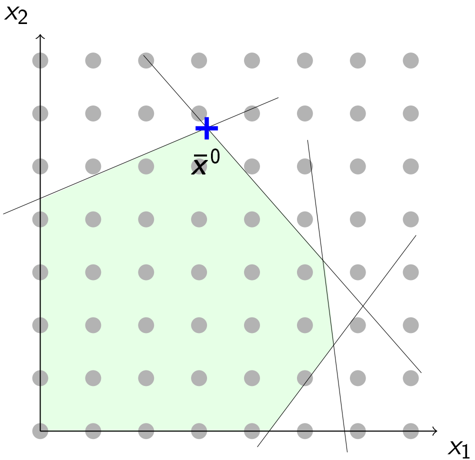
Xavier: Ajouter une référence!

Feasibility Pump Heuristic for $1-01LP$ problem

$\overline{\mathcal{X}} := \{x \in [0,1]^n \mid Ax \geq b\}$, the LP-relaxation space

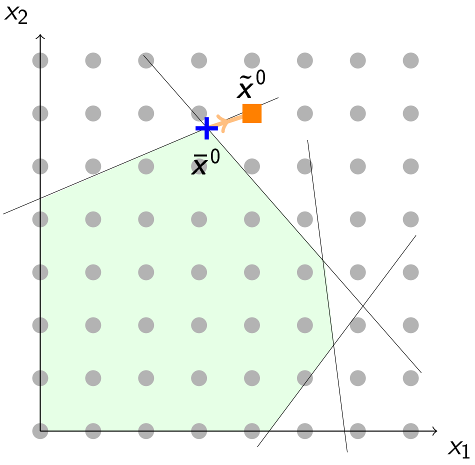
- Initial solution: $\overline{x}^0 \in Argmin\{cx : x \in \overline{\mathcal{X}}\}$
- At each iteration t, two sequences of solutions:
  - Binary $\tilde{x}^t \in \{0,1\}^n$: $\tilde{x}^t = Round(\overline{x}^t)$
  - Fractional $\overline{x}^t \in [0,1]^n$: $\overline{x}^t = Project(\tilde{x}^{t-1})$
- Stopping: $\overline{x}^t \in \{0,1\}^n$ or $\tilde{x}^t \in \mathcal{X}$ or timeout limit
- **Cycling:** $\tilde{x}^t = Pertub(\tilde{x}^t)$

Feasibility Pump Heuristic for $1-01LP$ problem



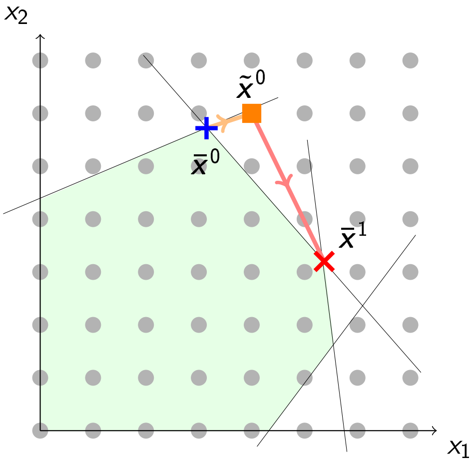
1. $\bar{x}^0$ =An optimal LP solution

Feasibility Pump Heuristic for $1-01LP$ problem



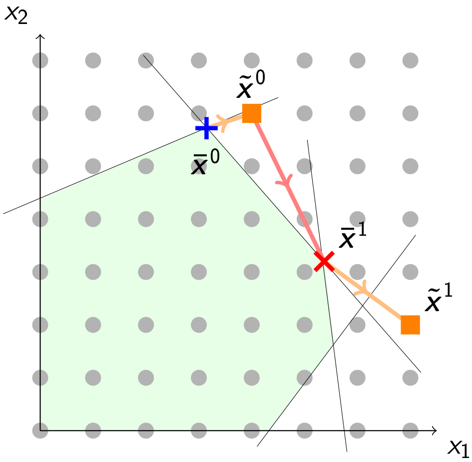
1. $\bar{x}^0$ =An optimal LP solution
2. $\tilde{x}^0 = Round(\bar{x}^0)$

Feasibility Pump Heuristic for $1 - 01LP$ problem



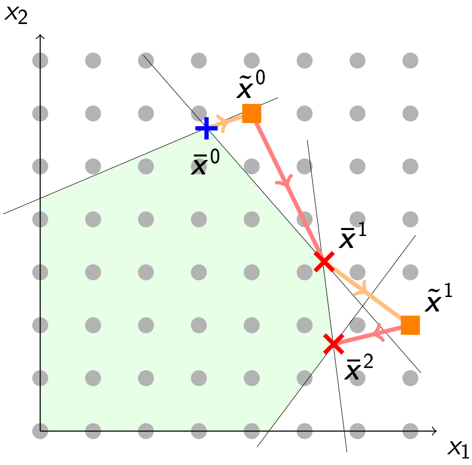
1. $\bar{x}^0$ =An optimal LP solution
2. $\tilde{x}^0 = Round(\bar{x}^0)$
3. $\bar{x}^1 = Project(\tilde{x}^0)$

Feasibility Pump Heuristic for $1-01LP$ problem



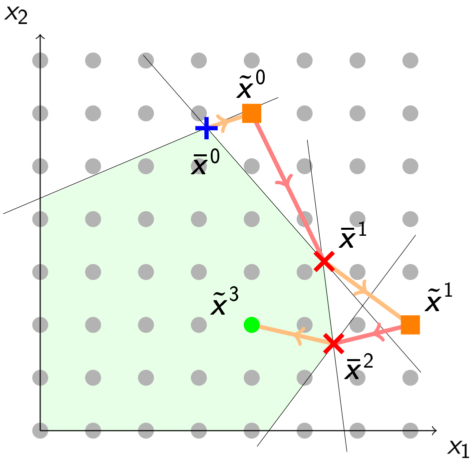
1. $\bar{x}^0$ =An optimal LP solution
2. $\tilde{x}^0 = Round(\bar{x}^0)$
3. $\bar{x}^1 = Project(\tilde{x}^0)$
4. $\tilde{x}^1 = Round(\bar{x}^1)$

Feasibility Pump Heuristic for $1 - 01LP$ problem



1. $\bar{x}^0$ =An optimal LP solution
2. $\tilde{x}^0 = Round(\bar{x}^0)$
3. $\bar{x}^1 = Project(\tilde{x}^0)$
4. $\tilde{x}^1 = Round(\bar{x}^1)$
5. $\bar{x}^2 = Project(\tilde{x}^1)$

Feasibility Pump Heuristic for $1 - 01LP$ problem



1. $\bar{x}^0$ =An optimal LP solution
2. $\tilde{x}^0 = Round(\bar{x}^0)$
3. $\bar{x}^1 = Project(\tilde{x}^0)$
4. $\tilde{x}^1 = Round(\bar{x}^1)$
5. $\bar{x}^2 = Project(\tilde{x}^1)$
6. $\tilde{x}^3 = Round(\bar{x}^2)$

Gravity Machine Matheuristic for $p - 01LP$ problem

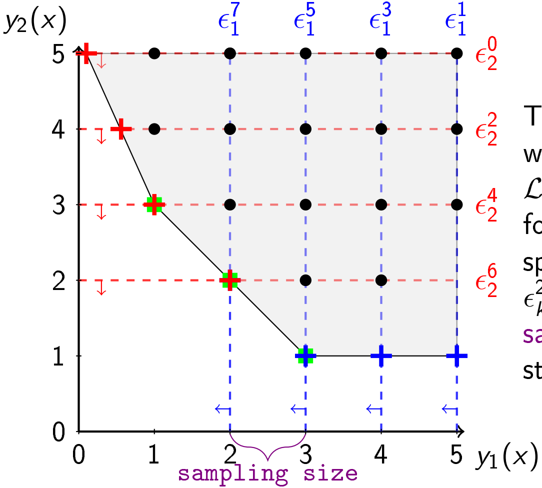
Algorithm 1: Outline of Gravity Machine

Data: $\mathcal{D}, n_{\mathcal{L}}$, *TimeOut*
Result: $\mathcal{L}, \mathcal{U}$

begin
  $(\mathcal{L}, F) \leftarrow Generator(\mathcal{D}, n_{\mathcal{L}})$; /* Initial LP-solutions */
  forall $y(\bar{x}) \in \mathcal{L}, \bar{x} \notin F$ **do**
    Stop $\leftarrow$ false;
    while not Stop **do**
      $(\tilde{x}, H, cycle) \leftarrow Round(\bar{x}, H)$; /* Rounding */
      $(\bar{x}, F) \leftarrow Project(\tilde{x})$; /* LP-Solution */
      Stop $\leftarrow \overline{x}^t \in \{0,1\}^n$ or $\tilde{x}^t \in \mathcal{X}$ or $\neg TimeOut$; /* Stopping */
      if not Stop and cycle **then**
        $\tilde{x} \leftarrow Perturb(\tilde{x})$; /* Perturbing */
      end
    end
  end
  $\mathcal{U} \leftarrow$ NonDominated(F); /* In $\mathcal{O}(nlog(n))$ by Kung, Luccio and Preparata (1975) */
end

Generator $\rightsquigarrow$ ϵ-constraint LP method

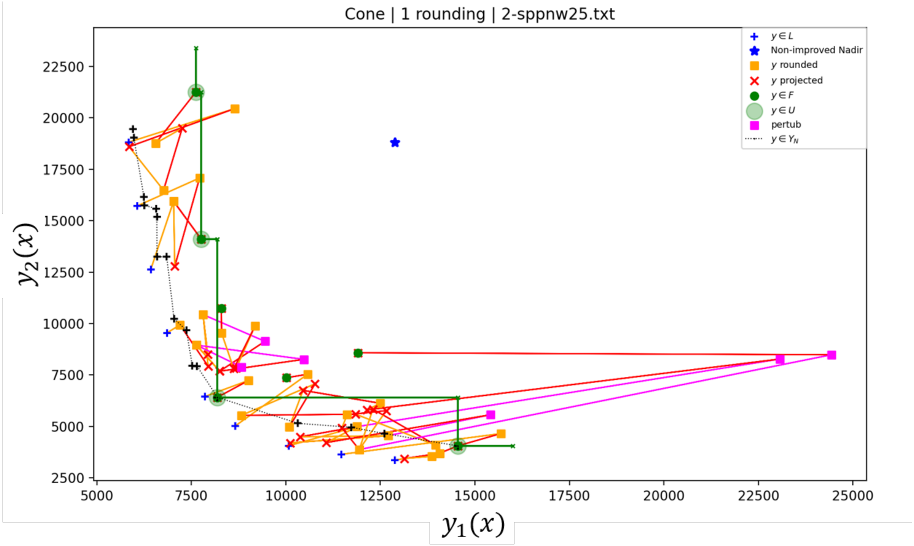
A set of generators $(\bar{x}^k, \bar{y}^k)$ with $\bar{y}^k := y(\bar{x}^k)$



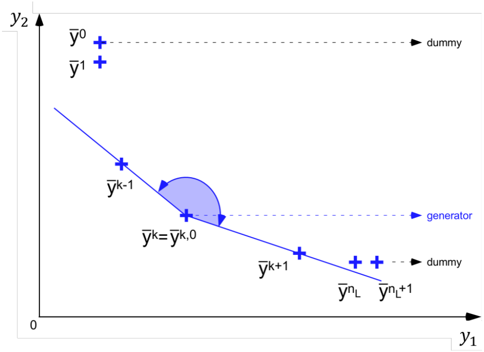
The set of $+$ and $+$ forms a well-distributed lower bound set $\mathcal{L}$ of $\mathcal{Y}^*$. It can be noticed that for each fixed objective k the space between each ϵ_k^{2t+1} and ϵ_k^{2t+3} is less or equal than the sampling size for an arbitrary step t.

Gravity Machine Matheuristic for $2-01LP$ problem

Inner operations in Gravity Machine

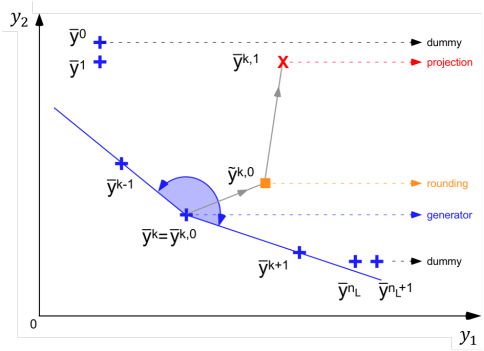


Conic constraints on objective space $\mathcal{Y}$



Hard-constraints over $\bar{y}^{k,t}$, $t = 0$

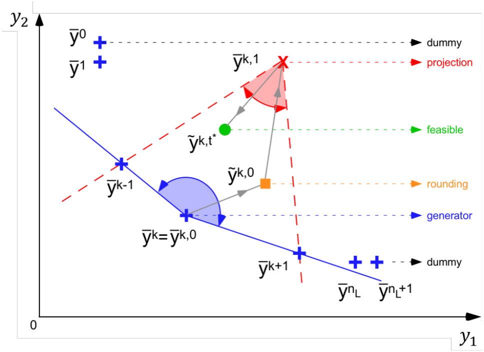
Conic constraints on objective space $\mathcal{Y}$



$$(\tilde{x}^{k,t}, H, cycle) \leftarrow Round(\bar{x}^{k,t},\ k,\ H)$$

$$\bar{x}^{k,t+1} := argmin\{\Delta(x, \tilde{x}^{k,t}), x \in \overline{\mathcal{X}}\}, t \geq 0$$

Conic constraints on objective space $\mathcal{Y}$



Soft-constraints over $\bar{y}^{k,t}$, $t > 0$

Discret constraints on decision space $\mathcal{X}$: MIP Relaxation $\mathcal{X}_J$

- Introduced independently by Glover, and Hanafi & Wilbaut in 2006
- Same variables are imposed to be discret

Let $J \subseteq N$, a MIP relaxation relative to subset J is

$$\mathcal{X}_J := \left\{ x \in \{0,1\}^{|J|} \times [0,1]^{|N|-|J|} \mid Ax \geq b \right\}$$

- LP relaxation: $\mathcal{X}_\emptyset = \overline{\mathcal{X}}$
- Inclusion: $J' \subset J \Rightarrow \mathcal{X}_{J'} \subset \mathcal{X}_J$
- Original problem: $\mathcal{X}_N = \mathcal{X}$

The more $|J|$ increases, the more difficult to solve the problem over $\mathcal{X}_J$

Selection of the discrete subset $J \subset N$

Let $\overline{x}$ an optimal LP solution of

$$min\{f(x) : x \in \overline{\mathcal{X}}\}$$

and $\tilde{x} = Round(\overline{x})$, where

- Generator: $f(x) = \lambda Cx, \lambda \in \mathbb{R}^p$
- Projection: $f(x) = \Delta(\tilde{x}, x) = \sum_{j \in N^0(\tilde{x})} x_j + \sum_{j \in N^1(\tilde{x})} (1 - x_j)$

$$J \subset N^*(\overline{x})$$

with

- $N^0(x) = \{j \in N : x_j = 0\}$
- $N^1(x) = \{j \in N : x_j = 1\}$
- $N^*(x) = \{j \in N : x_j \notin \{0,1\}\}$

Selection of the discrete subset $J \subset N^*(\overline{x})$

We introduce a **priority order** and **two parameters** α and τ to select fractional variables:

- We consider the closest to $\frac{1}{2}$ value variables
- $\boldsymbol{\tau}$ first variables are set binary
- $\lfloor \boldsymbol{\alpha} \cdot (|\boldsymbol{N}^*(\overline{\boldsymbol{x}})| - \boldsymbol{\tau}) \rfloor$ remaining variables are also set binary

Pseudo-cut constraints on decision space $\mathcal{X}$

Let be x' a binary vector in $\{0,1\}^n$, the following inequality

$$\Delta(x', x) = \sum_{j \in N^0(x')} x_j + \sum_{j \in N^1(x')} (1 - x_j) \geq \delta(= 1)$$

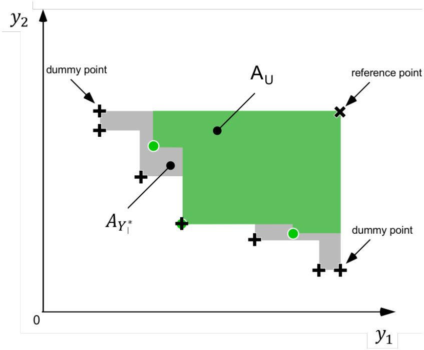
cuts off solution x' without cutting off any other solution in $\{0,1\}^n$

- Called canonical cut constraint by Balas and Jeroslow (1972) for $\delta = 1$
- Used in Local branching by Fischetti and Lodi (2003) for $\delta \geq 1$
- Used in convergent heuristic for the 0-1MIP by Hanafi and Wilbaut (2011) for $\delta \geq 1$

Environment and Instances

- Algorithm coded in Julia language
- Algebraic modeling language JuMP
- Open source GLPK for MIP solver
- Intel(R) Core(TM) i7 processor at 2,20 GHz with 16 Go of RAM
- 44 instances transformed from the OR-library with $p = 2$
- $n_{\mathcal{L}} = 30$

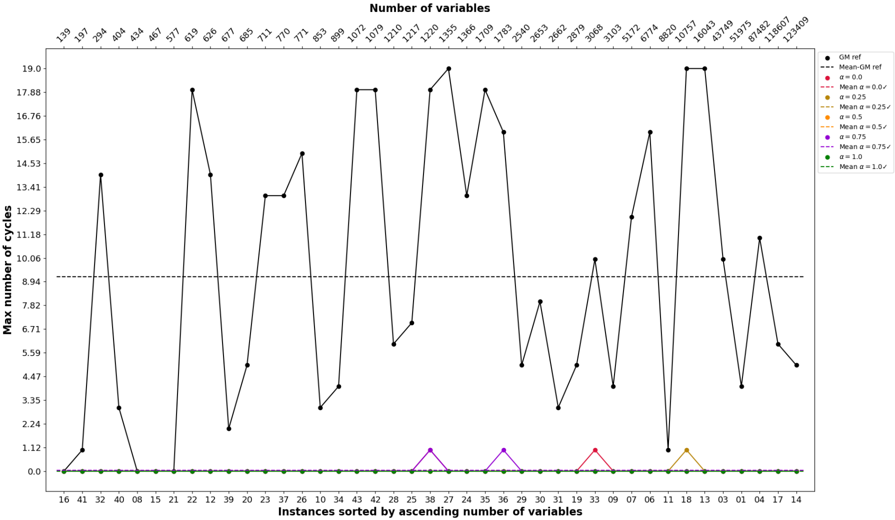
Quality measure $\rightsquigarrow$ The Hypervolume Indicator



$$r = \frac{A_{\mathcal{U}}}{A_{\mathcal{Y}^*}}$$

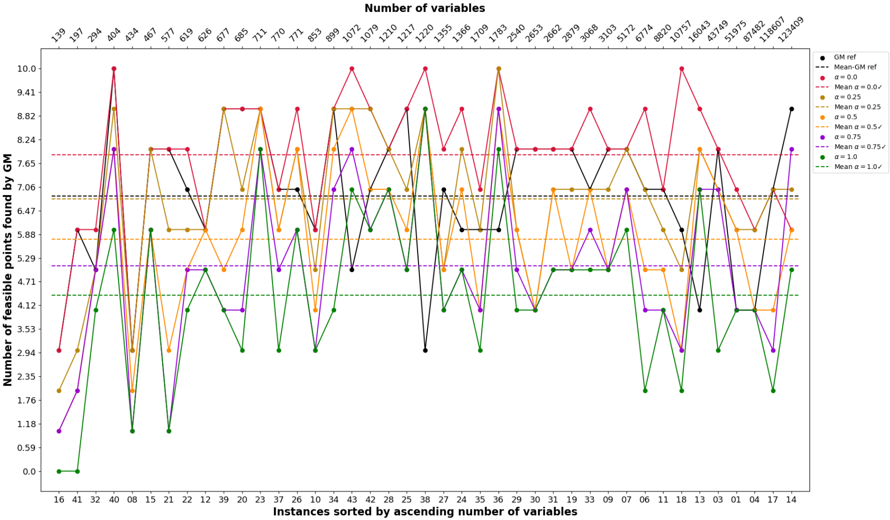
Number of cycles

Number of cycles with MILP-Projection and different level for α in MILP-Relaxation



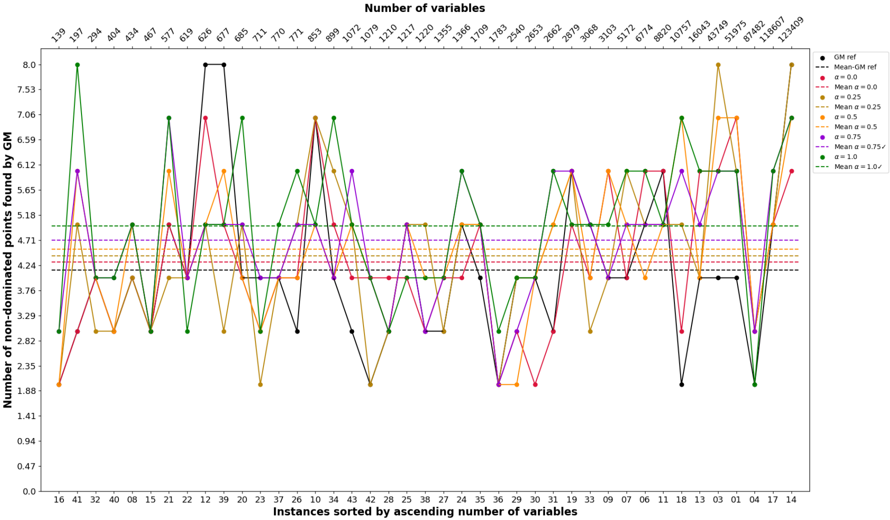
Number of feasible solutions found by GM

Number of feasible point found by GM (%) with MILP-Projection and different level for α in MILP-Relaxation



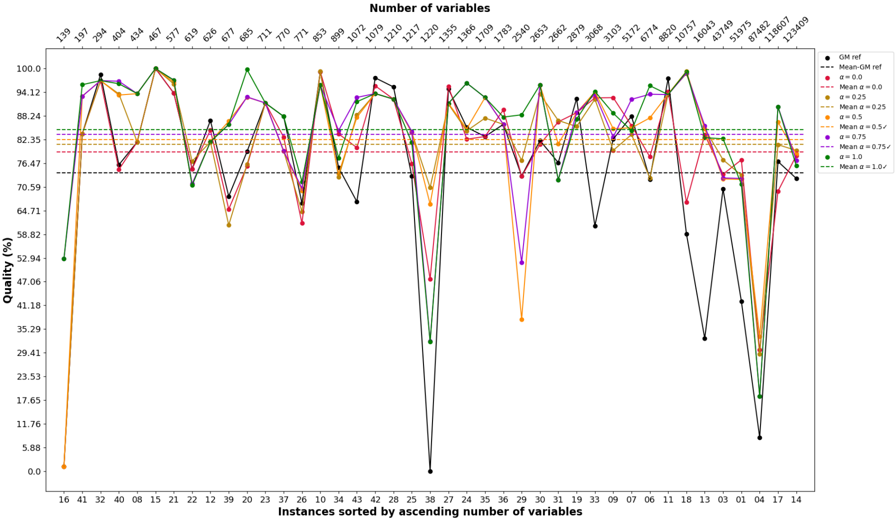
Number of Non-dominated found by GM

Number of non-dominated found by GM (%) with MILP-Projection and different level for α in MILP-Relaxation



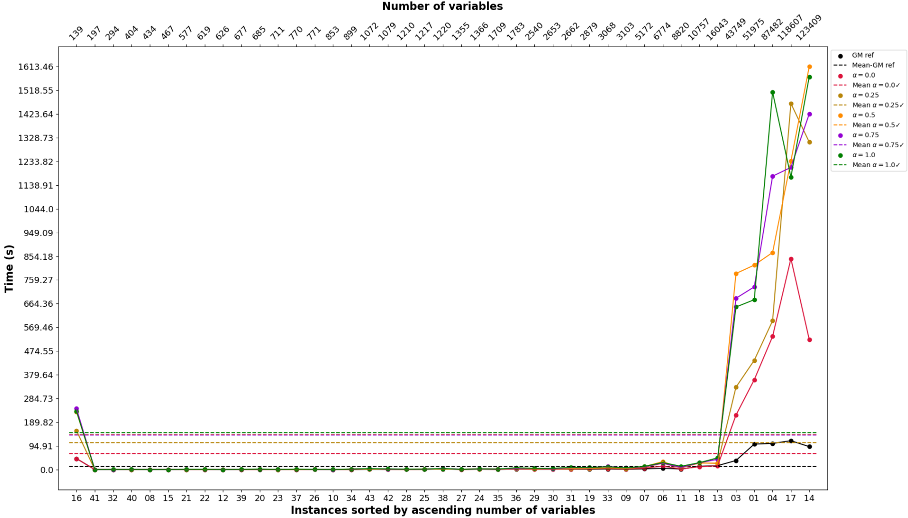
Quality

Quality (%) with MILP-Projection and different level for α in MILP-Relaxation



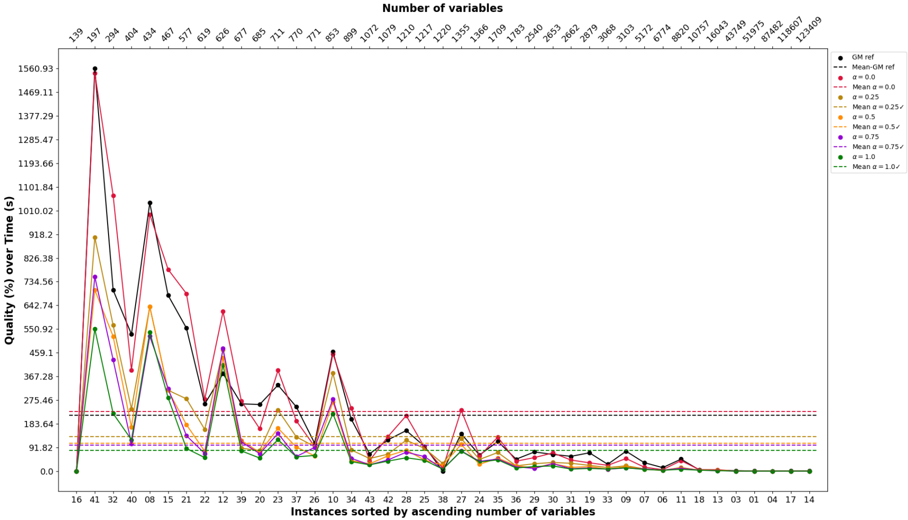
Time

Time with MILP-Projection and different level for α in MILP-Relaxation



Quality/Time

Quality/Time with MILP-Projection and different level for α in MILP-Relaxation



Conclusions

- Easy to explain/understand/implement ✓
- Few parameters ✓
- Few cycles ✓
- Good quality of solutions ✓
- Solutions well distributed along side $\mathcal{Y}_N$ $\sim$
- Good ratio *quality/time* ×