Mathematical Programming 26 (1983) 326–344
North-Holland

AN INTERACTIVE WEIGHTED TCHEBYCHEFF PROCEDURE FOR MULTIPLE OBJECTIVE PROGRAMMING

Ralph E. STEUER

College of Business Administration, University of Georgia, Athens, GA 30602, U.S.A.

Eng-Ung CHOO

Department of Business Administration, Simon Fraser University, Burnaby, B.C., Canada

Received 9 November 1981
Revised manuscript received 16 September 1982

The procedure samples the efficient set by computing the nondominated criterion vector that is closest to an ideal criterion vector according to a randomly weighted Tchebycheff metric. Using 'filtering' techniques, maximally dispersed representatives of smaller and smaller subsets of the set of nondominated criterion vectors are presented at each iteration. The procedure has the advantage that it can converge to non-extreme final solutions. Especially suitable for multiple objective linear programming, the procedure is also applicable to integer and nonlinear multiple objective programs.

Key words: Multiple Objective Programming, Interactive Procedures, Efficient Solutions, Tchebycheff Metrics.

1. Introduction

This paper presents an interactive weighted Tchebycheff procedure for solving the multiple objective program

$$\begin{aligned} \max \quad & \{f_1(x) = z_1\}, \\ & \vdots \\ \max \quad & \{f_k(x) = z_k\}, \\ \text{s.t.} \quad & x \in S, \end{aligned}$$

where the f_i may be nonlinear and S may be nonconvex. It is assumed that S is bounded.

With regard to notation, let $Z \subset R^k$ be the set of *all feasible criterion vectors* where Z is the set of images of all $x \in S$ under the f_i. A $\bar{z} \in Z$ is a *nondominated* criterion vector if and only if there does not exist another $z \in Z$ such that $z_i \geq \bar{z}_i$ for all i and $z_i > \bar{z}_i$ for at least one i. Letting $N \subset Z$ denote the set of *all nondominated criterion vectors*, an $\bar{x} \in S$ is an *efficient point* if and only if $\bar{x}$ is an inverse image of a $\bar{z} \in N$. Let $E \subset S$ denote the *efficient set* (the set of all efficient points).

Criterion vector $z^0 \in Z$ is an *optimal criterion vector* if z^0 maximizes the decision-maker's (in practice, mathematically unknown) coordinatewise increasing utility function $U: Z \to R$. It is well known that $z^0 \in N$. If $x^0 \in S$ is an inverse image of z^0, then $x^0 \in E$ and x^0 is an *optimal solution* of the multiple objective program.

The strategy of the interactive procedure is to sample (using different weighted Tchebycheff metrics) a series of progressively smaller subsets of N until we obtain a criterion vector $\bar{z} \in N$ sufficiently close to an optimal criterion vector z^0 to terminate the decision process. By 'sufficiently close' we mean that it is immaterial in the decision-maker's eyes that he has $\bar{z}$ instead of z^0. Then by obtaining an inverse image of $\bar{z}$, we have the *final solution* achieved by the interactive procedure.

The interactive procedure amalgamates the weighted Tchebycheff results of Bowman [1], Choo and Atkins [3], and Choo [2]; the scalarizing function ideas of Wierzbicki [13]; and the 'filtering' techniques of Steuer and Harris [11]. Other related multiple criteria Tchebycheff work has been conducted by Yu [14], Zeleny [15] and Ecker and Shoemaker [5] in connection with various concepts of *compromise solutions*.

Section 2 introduces the concepts of the *ideal criterion vector* and the *augmented weighted Tchebycheff metric*. Section 3 presents weighted Tchebycheff theory for the discrete feasible region case. Theory is extended to problems with polyhedral and nonconvex-continuous feasible regions in Section 4. The algorithmic specification of the interactive procedure is given in Section 5 along with a *convergence factor* discussion in Section 6. Section 7 provides a numerical illustration and Section 8 ends the paper with concluding remarks.

2. The ideal criterion vector and weighted Tchebycheff metrics

Let $z^* \in R^k$ be the *ideal criterion vector* such that

$$z_i^* = \max\{f_i(x) \mid x \in S\} + \epsilon_i$$

where a given $\epsilon_i \geq 0$ ($\epsilon_i = 0$ is permissible) unless

(i) there is more than one nondominated criterion vector that maximizes the ith objective, or

(ii) the only nondominated criterion vector that maximizes the ith objective also maximizes one of the other objectives,

in which case the ϵ_i must be strictly positive. In practice, when an $\epsilon_i > 0$, it is set to a small, but computationally significant, positive scalar value.

With $\lambda \in \bar{\Lambda} = \{\lambda \in R^k \mid \lambda_i \geq 0, \Sigma_{i=1}^k \lambda_i = 1\}$, let

$$\|z^* - z\|_\infty^\lambda = \max_{i=1,\dots,k} \{\lambda_i |z_i^* - z_i|\}$$

define the *weighted Tchebycheff metric* and

$$|||z^* - z|||_\infty^\lambda = \|z^* - z\|_\infty^\lambda + \rho\, e^T(z^* - z)$$

define the *augmented weighted Tchebycheff metric* for measuring the distance between any $z \in Z$ and the ideal criterion vector z^* where e^T is the sum vector of ones. The term 'augmented' is applied because of the presence of the term $\rho\, e^T(z^* - z)$ where ρ is a sufficiently small positive scalar that is explained in Section 3.

Let $\bar{z} \in Z$. Considering the $z \leq z^*$ portion of the $\|z^* - \bar{z}\|_\infty^{\bar{\lambda}}$ or $|||z^* - \bar{z}|||_\infty^{\bar{\lambda}}$ isoquant, $\bar{z}$ is said to *define* the isoquant if and only if

$$\bar{\lambda}_i = \begin{cases} \dfrac{1}{(z_i^* - \bar{z}_i)} \left[\displaystyle\sum_{i=1}^{k} \dfrac{1}{(z_i^* - \bar{z}_i)} \right]^{-1} & \text{if } \bar{z}_i \neq z_i^* \text{ for all } i, \\ 1 & \text{if } \bar{z}_i = z_i^*, \\ 0 & \text{if } \bar{z}_i \neq z_i^* \text{ but } \exists_j \ni \bar{z}_j = z_j^*. \end{cases}$$

If (i) $\bar{z}$ is a defining criterion vector and (ii) $\bar{z}$ is strictly less than z^* (i.e., $\bar{z} < z^*$), $\bar{z}$ is said to be the *vertex* of the isoquant.[1] In Fig. 1(a) we have the isoquant of the weighted Tchebycheff metric defined by $\bar{z}$, and in Fig. 1(b) we have the isoquant of the augmented weighted Tchebycheff metric defined by $\bar{z}$ where

$$\theta_1 = \tan^{-1}\left(\frac{\rho}{1 - \lambda_1 + \rho}\right)$$

and

$$\theta_2 = \tan^{-1}\left(\frac{\rho}{1 - \lambda_2 + \rho}\right).$$

[1] If $\bar{z} \leq z^*$, $\bar{z} \not< z^*$ is a defining criterion vector, it is not unique as a defining vector and the associated isoquant does not have a vertex. For example, see [12, Example 6.2(b)].

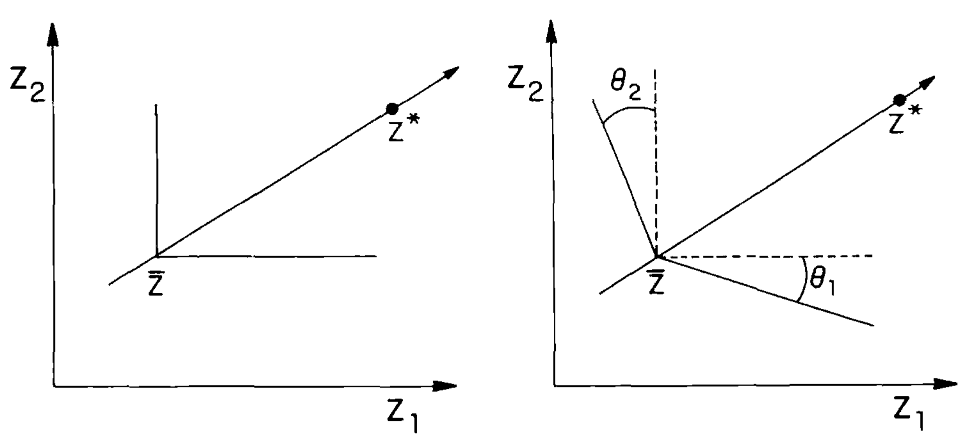

Fig. 1. Isoquants of weighted and augmented weighted Tchebycheff metrics.

Let $\bar{z} < z^*$ and let $\bar{u} \in R^k$ denote the direction from the $\bar{z}$ to z^*. The relationship between $\bar{u}$ and the weighting vector $\bar{\lambda}$ of either Tchebycheff metric having $\bar{z}$ as the defining point of an isoquant is

$$\bar{u} = (1/\bar{\lambda}_1, 1/\bar{\lambda}_2, \ldots, 1/\bar{\lambda}_k)$$

where

$$\bar{\lambda}_i = \frac{1}{(z_i^* - \bar{z}_i)} \left[\sum_{i=1}^{k} \frac{1}{(z_i^* - \bar{z}_i)} \right]^{-1}, \quad 1 \le i \le k.$$

Also, we note that the weighted Tchebycheff metric is the special case of the augmented weighted Tchebycheff metric when $\rho = 0$.

Assuming that $(z^* - \epsilon) \not\in Z$ (since $(z^* - \epsilon) \in Z$ is the trivial case), the interactive weighted Tchebycheff procedure, with details provided in subsequent sections, operates as follows:

1st iteration: After computing the ideal criterion vector z^*, and 1st iteration begins by forming a widely dispersed group of λ weighting vectors to sample the entire set of nondominated criterion vectors N. The sample is developed by determining the $z \in N$ closest to z^* for each of the λ's according to the augmented weighted Tchebycheff metric. As the 1st iteration selection, the decision-maker is asked to identify the most preferred criterion vector $z^{(1)}$ in the sample.

2nd iteration: To begin the 2nd iteration, a second group of λ's, but this time more concentrated than the first, is formed to sample more carefully the nondominated criterion vectors in a neighborhood of $z^{(1)}$. Developing the second sample of nondominated criterion vectors by the same means used in the 1st iteration, the decision-maker is now asked to make his 2nd iteration criterion vector selection $z^{(2)}$.

3rd iteration: To begin the 3rd iteration, an even more concentrated group of λ's is formed to sample an even smaller neighborhood of nondominated criterion vectors but this time centered about $z^{(2)}$. Developing the third sample of nondominated criterion vectors by the same means used in the first two iterations, the decision-maker is asked to make his 3rd iteration criterion vector selection $z^{(3)}$.

The procedure continues iterating in this fashion until a sufficiently acceptable solution to the problem is obtained. The procedure has been designed to solve linear, integer and nonlinear multiple objective programming problems without any restrictions on the shape of the decision-maker's utility function except that it be coordinatewise increasing in criterion space. Consequently, a given problem may have several local optima. However, because the repeated sampling of N lends a 'multiple starting point' property to the algorithm, the likelihood that the algorithm will converge to a local optimum that is not a global optimum is believed to be small.

3. Tchebycheff theory for the discrete case

Assume S is discrete. Then Z is discrete. To determine a $z \in Z$ that is closest to the ideal criterion vector according to the weighted Tchebycheff metric, we employ the *weighted Tchebycheff program*

$$\begin{aligned} \min \quad & \{\alpha\}, \\ \text{s.t.} \quad & \alpha \geq \lambda_i(z_i^* - z_i), \quad 1 \leq i \leq k, \\ & f_i(x) = z_i, \qquad\quad 1 \leq i \leq k, \\ & x \in S. \end{aligned}$$

Thus, by finding the minimal $\alpha \in R$, we find the smallest (in a subset sense) set

$$\Phi(\alpha) = \{z \in R^k \mid z_i \in [z_i^* - (\alpha/\lambda_i), +\infty) \quad \text{when } \lambda_i > 0\}$$

in the nested family of sets $\{\Phi(\alpha)\}_{\alpha \geq 0}$ that intersects Z.

Although the weighted Tchebycheff program may have more than one minimal solution, Theorem 3.1 assures us that amongst the $z \in Z$ tied for being closest to z^* according to the weighted Tchebycheff metric there is at least one nondominated criterion vector. Known in the linear case [4], Theorem 3.1 shows the result to be true in the discrete case.

Theorem 3.1. *Let Z be finite and let $M = \{z \in Z \mid (x, z, \alpha)$ is a minimal solution to the weighted Tchebycheff program$\}$. Then there exists a $\bar{z} \in M$ such that $\bar{z} \in N$.*

Proof. Since Z is finite, $N \neq \emptyset$. Let $\tilde{\alpha}$ be the minimum value of the weighted Tchebycheff program. Suppose there does not exist a $z \in M$ such that $z \in N$. Let $\hat{z}$ be a member of M that is not dominated by another member of M. If $\hat{z} \notin N$, then there exists a $\bar{z} \in N$ such that $\bar{z} \geq \hat{z}$, $\bar{z} \neq \hat{z}$. But with $\tilde{\alpha}$ optimal, this implies that $\bar{z} \in M$ which is a contradiction. Thus there exists a $\bar{z} \in M$ such that $\bar{z} \in N$.

Let

(a) Z be finite, $z^p \in N$ and $z^q \in Z$ such that $z^q \neq z^q$,

(b)
$$\lambda_i^p = \begin{cases} \dfrac{1}{(z_i^* - z_i^p)} \left[\displaystyle\sum_{i=1}^{k} \dfrac{1}{(z_i^* - z_i^p)} \right]^{-1} & \text{if } z_i^p \neq z_i^* \text{ for all } i, \\ 1 & \text{if } z_i^p = z_i^*, \\ 0 & \text{if } z_i^p \neq z_i^* \text{ but } \exists_j \ni z_j^p = z_j^*, \end{cases}$$

(c) α_{pp} be the minimal objective function value of

$$\begin{aligned} \min \quad & \{\alpha\}, \\ \text{s.t.} \quad & \alpha \geq \lambda_i^p(z_i^* - z_i^p), \quad 1 \leq i \leq k, \end{aligned}$$

(d) α_{pq} be the minimal objective function value of

$$\begin{aligned} &\min \quad \{\alpha\}, \\ &\text{s.t.} \quad \alpha \geq \lambda_i^p(z_i^* - z_i^q), \quad 1 \leq i \leq k. \end{aligned}$$

Lemma 3.2. *Assume* (a), (b) *and* (c). *Then there does not exist a* $z^q \neq z^p$ *such that* $z^q \in Z$ *lies in*

$$\Phi(\alpha_{pp}) = \{z \in R^k \mid z_i \in [z_i^* - (\alpha_{pp}/\lambda_i^p), +\infty) \quad \text{when } \lambda_i^p > 0\}.$$

Proof. *Case* 1: $z_i^p \neq z_i^*$ *for all i*. Substituting for λ_i^p in each of the k constraints

$$\alpha \geq \lambda_i^p(z_i^* - z_i^p), \quad 1 \leq i \leq k$$

we have

$$\alpha \geq \left[\sum_{i=1}^{k} \frac{1}{(z_i^* - z_i^p)}\right]^{-1}.$$

Thus

$$\alpha_{pp} = \lambda_i^p(z_i^* - z_i^p), \quad 1 \leq i \leq k$$

or

$$z_i^p = z_i^* - \frac{\alpha_{pp}}{\lambda_i^p}, \quad 1 \leq i \leq k.$$

Since z^p is nondominated, there does not exist a $q \neq p$ such that $z^q \in \Phi(\alpha_{pp})$.

Case 2: $\exists j \ni z_j^p = z_j^*$. With there being only one j such that $z_j^p = z_j^*$, $\lambda_i^p = 0$ for all $i \neq j$. Thus $\alpha_{pp} = 0$ and

$$\Phi(\alpha_{pp}) = \{z \in R^k \mid z_j \in [z_j^p, +\infty)\}.$$

Since z^p is the member of Z for which the jth component is greater than or equal to z_j^*, there does not exist a $q \neq p$ such that $z^q \in \Phi(\alpha_{pp})$.

Lemma 3.3. *Assume* (a), (b), (c) *and* (d). *Then* $\alpha_{pp} < \alpha_{pq}$ *for all* $z^q \in Z$, $q \neq p$.

Proof. Since z^p is nondominated, by Lemma 3.2 no other $z^g \in Z$ lies in

$$\Phi(\alpha_{pp}) = \{z \in R^k \mid z_i \in [z_i^* - (\alpha_{pp}/\lambda_i^p), +\infty) \text{ when } \lambda_i^p > 0\}.$$

Thus, all $z^q \in Z$, $q \neq p$ lie in supersets of $\Phi(\alpha_{pp})$. Hence $\alpha_{pp} < \alpha_{pq}$ for all $z^q \in Z$, $q \neq p$.

Theorem 3.4. *Let Z be finite, $I_Z = \{i \mid z^i \in Z\}$, $I_N = \{i \mid z^i \in N\}$, and $z^p \in N$. Then z^p uniquely minimizes*

$$\begin{aligned} \min \quad & \{\alpha + \rho_p\, e^T(z^* - z)\}, \\ \text{s.t.} \quad & \alpha \geq \lambda_i^p(z_i^* - z_i), \quad 1 \leq i \leq k, \\ & f_i(x) = z_i, \qquad\quad 1 \leq i \leq k, \\ & x \in S, \end{aligned} \tag{3.5}$$

where

$$\rho_p = \frac{1}{2} \min_{q \in I_Z - \{p\}} \left\{ \frac{\alpha_{pq} - \alpha_{pp}}{e^T(z^q - z^p)} \,\Big|\, e^T(z^q - z^p) > 0 \right\} \tag{3.6}$$

and the λ_i^p are as specified in (b).

Proof. By Lemma 3.3 and the construction of ρ_p in (3.6), $\rho_p > 0$. Suppose $z^q \in Z$, $z^q \neq z^p$, minimizes (3.5). Then a lower bound for the minimal value of the objective function is $\alpha_{pq} + \rho_p\, e^T(z^* - z^q)$. The minimality of z^p is preserved, however, if

$$\alpha_{pp} + \rho_p\, e^T(z^* - z^p) < \alpha_{pq} + \rho_p\, e^T(z^* - z^q).$$

Since by Lemma 3.3 $\alpha_{pp} < \alpha_{pq}$, the optimality of z^p can be thwarted only when $e^T(z^* - z^q) < e^T(z^* - z^p)$, that is, when $e^T(z^q - z^p) > 0$. To assure the optimality of z^p we must have

$$\rho_p\, e^T(z^q - z^p) < \alpha_{pq} - \alpha_{pp}$$

for all $q \in I_Z - \{p\}$. This means that we must have

$$\rho_p < \frac{\alpha_{pq} - \alpha_{pp}}{e^T(z^q - z^p)}$$

whenever $e^T(z^q - z^p) > 0$ for all $q \in I_Z - \{p\}$. Thus it suffices for ρ_p to be defined as in (3.6) for z^p to uniquely minimize (3.5) with the weights defined as in (b).

Theorem 3.7. *Let N be finite and let*

$$\rho = \frac{1}{2} \min_{i \in I_N} \left[\min_{j \in I_Z - \{i\}} \left\{ \frac{\alpha_{ij} - \alpha_{ii}}{e^T(z^j - z^i)} \,\Big|\, e^T(z^j - z^i) > 0 \right\} \right]. \tag{3.8}$$

Then $z^p \in N \Leftrightarrow$ there exists a $\lambda \in \bar{\Lambda} = \{\lambda \in R^k \mid \lambda_i \geq 0,\ \Sigma_{i=1}^k \lambda_i = 1\}$ such that z^p minimizes the augmented weighted Tchebycheff program

$$\begin{aligned} \min \quad & \{\alpha + \rho\, e^T(z^* - z)\}, \\ \text{s.t.} \quad & \alpha \geq \lambda_i(z_i^* - z_i), \quad 1 \leq i \leq k, \\ & f_i(x) = z_i, \qquad\quad 1 \leq i \leq k, \\ & x \in S. \end{aligned}$$

Proof. $\Rightarrow$ Follows from Theorem 3.4 when the λ_i are defined as the λ_i^p in (b).

$\Leftarrow$ Suppose $z^p \notin N$ minimizes the augmented weighted Tchebycheff program for some $\bar{\lambda} \in \bar{\Lambda}$. With z^p dominated, there exists a z^q such that $z^q \geq z^p$, $z^q \neq z^p$. Since $(z^* - z^q) \leq (z^* - z^p)$ and $\rho\, e^T(z^* - z^q) < \rho\, e^T(z^* - z^p)$, z^q would have a smaller objective function value than z^p. However, this contradicts the minimality of z^p, thus $z^p \in N$.

Corollary 3.9. *Same as Theorem 3.4 except that ρ as in (3.8) is substituted for ρ_p. Then each $z^p \in N$ has a $\bar{\lambda} \in \bar{\Lambda}$ such that z^p uniquely minimizes the augmented weighted Tchebycheff program.*

Proof. Obvious since $0 < \rho \leq \rho_p$.

Therefore, in the discrete case, not only is every criterion vector returned by the augmented weighted Tchebycheff program nondominated, but each nondominated criterion vector is *uniquely computable*. By uniquely computable we mean that each criterion vector is computable even if the software applied to solve the augmented weighted Tchebycheff program stops as soon as a minimal solution is detected (rather than going on to compute 'all alternative optima'). We also observe that the augmented weighted Tchebycheff method is able to compute *unsupported* nondominated criterion vectors such as z^1, z^2 and z^3 in Fig. 2.

4. For polyhedral and nonconvex-continuous feasible regions

For the polyhedral case (e.g., when $S = \{x \in R^n \mid Ax \lesseqqgtr b,\ x \geq 0\}$) let $I_{Z_v} = \{i \mid z^i$ is the image of an extreme point of $S\}$ and $I_{N_v} = \{i \in I_{Z_v} \mid z^i$ is nondominated$\}$. Then we have Corollaries 4.1 and 4.4.

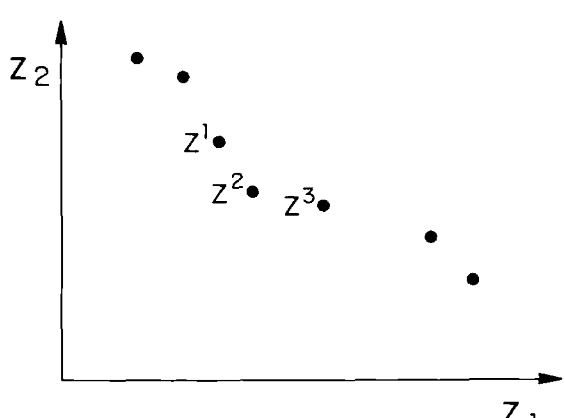

Fig. 2. Illustration of unsupported nondominated criterion vectors.

Corollary 4.1. *Let I_{Z_t} and I_{N_t} be defined as above, and let*

$$\rho = \frac{1}{2} \min_{i \in I_{N_t}} \left[\min_{j \in I_{Z_t} - \{i\}} \left\{ \frac{\alpha_{ij} - \alpha_{ii}}{\mathrm{e}^{\mathrm{T}}(z^j - z^i)} \mid \mathrm{e}^{\mathrm{T}}(z^j - z^i) > 0 \right\} \right]. \tag{4.2}$$

Then $\bar{z} \in N$ uniquely minimizes the augmented weighted Tchebycheff program

$$\begin{aligned} \min \quad & \{\alpha + \rho\, \mathrm{e}^{\mathrm{T}}(z^* - z)\}, \\ \text{s.t.} \quad & \alpha \geq \bar{\lambda}_i (z_i^* - z_i), && 1 \leq i \leq k, \\ & f_i(x) = z_i, && 1 \leq i \leq k, \\ & x \in S \end{aligned}$$

where

$$\bar{\lambda}_i = \begin{cases} \dfrac{1}{(z_i^* - \bar{z}_i)} \left[\displaystyle\sum_{i=1}^{k} \frac{1}{(z_i^* - \bar{z}_i)} \right]^{-1} & \text{if } \bar{z}_i \neq z_i^* \text{ for all } i, \\ 1 & \text{if } \bar{z}_i = z_i^*, \\ 0 & \text{if } \bar{z}_i \neq z_i^* \text{ but } \exists j \ni \bar{z}_j = z_j^*. \end{cases} \tag{4.3}$$

Proof. Follows from the piecewise linearities of the isoquants of the augmented weighted Tchebycheff metric, Theorem 3.4, and the polyhedral properties of S.

Corollary 4.4. *Let ρ be defined as in (4.2). Then $\bar{z} \in N \Leftrightarrow$ there exists a $\lambda \in \bar{\Lambda}$ such that $\bar{z}$ minimizes the augmented weighted Tchebycheff program*

$$\begin{aligned} \min \quad & \{\alpha + \rho\, \mathrm{e}^{\mathrm{T}}(z^* - z)\}, \\ \text{s.t.} \quad & \alpha \geq \lambda_i (z_i^* - z_i), && 1 \leq i \leq k, \\ & f_i(x) = z_i, && 1 \leq i \leq k, \\ & x \in S. \end{aligned}$$

Proof: Follows from the piecewise linearities of the isoquants of the augmented weighted Tchebycheff metric, Theorem 3.7, and the polyhedral properties of S.

Thus, with Corollaries 4.1 and 4.4 taken together, we have the same result in the polyhedral case as we did in the discrete case. That is, all criterion vectors returned by the augmented weighted Tchebycheff program are nondominated and all nondominated criterion vectors are uniquely computable.

To illustrate the nonconvex-continuous feasible region case, consider Fig. 3. In Fig. 3, N is the set of criterion vectors from z^1 to z^2 to z^3. Although $z^2 \in N$, z^2 is *improperly* nondominated (in the Geoffrion sense [7]). That is, $z(\bar{x})$ is *improperly nondominated* if $z(\bar{x}) \in N$ and there does not exist an $\pi > 0$ such that,

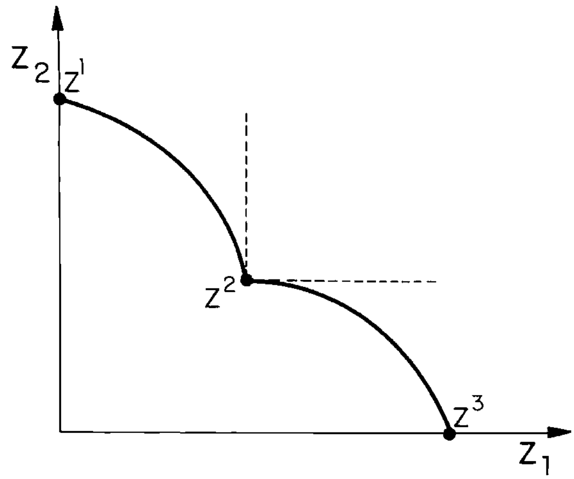

Fig. 3. Illustration of an improperly nondominated criterion vector.

for each i, we have

$$\frac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leq \pi$$

for some j such that $f_j(x) < f_j(\bar{x})$ whenever $x \in S$ and $f_i(x) > f_i(\bar{x})$.

Note that in Fig. 3, for any $\rho > 0$, the augmented weighted Tchebycheff approach cannot compute members of N in a neighborhood of z^2 (e.g., a right-angle is needed at the vertex of the isoquant to compute z^2).

To utilize a weighted Tchebycheff approach but with nonconvex-continuous feasible regions, consider Theorems 4.5 and 4.6.

Theorem 4.5. *Let $\bar{z} \in N$. Then there exists a $\bar{\lambda} \in \bar{\Lambda}$ such that $\bar{z}$ uniquely minimizes the lexicographic weighted Tchebycheff program*

$$\begin{aligned} \min \quad & \{P_1\alpha + P_2\, e^{\mathrm{T}}(z^* - z)\}, & \\ \text{s.t.} \quad & \alpha \geq \bar{\lambda}_i(z_i^* - z_i), & 1 \leq i \leq k, \\ & f_i(x) = z_i, & 1 \leq i \leq k, \\ & x \in S & \end{aligned}$$

where P_1 and P_2 are pre-emptive priority factors (see for instance Lee [9] *or Ignizio* [8]*) such that $P_1 <<< P_2$.*

Proof. Let $\bar{\alpha}$ minimize the weighted Tchebycheff program with the $\bar{\lambda}_i$ as defined in (4.3). Consider the set

$$\Phi(\bar{\alpha}) = \{z \in R^k \mid z_i \in [z_i^* - (\bar{\alpha}/\bar{\lambda}_i), +\infty) \quad \text{when } \bar{\lambda}_i > 0\}.$$

Clearly $\bar{z} \in \Phi(\bar{\alpha})$. From the proof of Lemma 3.2 (which also holds when Z is continuous), there does not exist another $z \in N$ such that $z \in \Phi(\bar{\alpha})$. This means that the associated weighted Tchebycheff program has a unique solution. Since the first stage of the lexicographic weighted Tchebycheff program is the weighted Tchebycheff program, $\bar{z} \in N$ uniquely minimizes the lexicographic weighted Tchebycheff program for $\bar{\lambda}$ as defined in (4.3).

Theorem 4.6. *$\bar{z} \in N \Leftrightarrow$ there exists a $\lambda \in \bar{\Lambda}$ such that $\bar{z}$ minimizes the lexicographic weighted Tchebycheff program*

$$\begin{aligned} \min \quad & \{P_1\alpha + P_2\, e^{\mathrm{T}}(z^* - z)\}, & \\ \text{s.t.} \quad & \alpha \geq \lambda_i(z_i^* - z_i), & 1 \leq i \leq k, \\ & f_i(x) = z_i, & 1 \leq i \leq k, \\ & x \in S. & \end{aligned}$$

Proof. $\Rightarrow$ Follows from Theorem 4.5 when the λ_i are defined as the $\bar{\lambda}_i$ in (4.3).

$\Leftarrow$ Suppose $\bar{z} \notin N$ minimizes the lexicographic weighted Tchebycheff program. Then there exists a $\hat{z} \in Z$ such that $\hat{z}_i \geq \bar{z}_i$ for all i and $\hat{z}_i > \bar{z}_i$ for at least one i. Then $e^{\mathrm{T}}(z^* - \hat{z}) < e^{\mathrm{T}}(z^* - \bar{z})$ which contradicts the minimality of $\bar{z}$ and the theorem is proved.

With regard to the lexicographic weighted Tchebycheff program, if the first-stage minimization of α does not yield a criterion vector that is nondominated, the second-stage minimization of $e^{\mathrm{T}}(z^* - z)$ is guaranteed to move us to one that is.

As with the discrete and polyhedral cases, every nondominated criterion vector in the nonconvex-continuous case is uniquely computable and all criterion vectors returned by the lexicographic weighted Tchebycheff program are nondominated.

Observe that the lexicographic weighted Tchebycheff program can be used with *any* feasible region. Thus it can be used in lieu of the augmented weighted Tchebycheff program if so desired.

In addition to its generalized applicability, the advantage of the lexicographic weighted Tchebycheff program is that it is not necessary to estimate a sufficiently small ρ. However, each lexicographic program requires two stages of optimization which may be undesirable in terms of CPU time requirements in integer and nonlinear cases.

5. Interactive procedure

In [11], 'filtering' procedures are described for computing maximally dispersed representatives of a set. In this paper, the procedures are employed to sequentially sample the nondominated set N. The purpose is to sample a sequence of successively smaller subsets of N until we obtain a criterion vector that is close enough to an optimal criterion vector to terminate the search process. In an interactive framework, each sample of N is evaluated to provide information for the preparation of the next sample.

Let

$p \sim$ the *sample size* (the size of the sample that is developed at each iteration),

$t \sim$ the *number of iterations* (the number of subsets of N to be sampled),

$r \sim$ the *convergence factor* whose purpose is to sequentially reduce weighting vector space.

The determination of values for p, t and r constitutes the subject of Section 6. With r functioning as in Step 12, the interactive procedure is as follows.

Step 1. Solve for the ideal criterion vector $z^* \in R^k$.

Step 2. Let $\bar{\Lambda}^{(1)} = \{\lambda \in R^k \mid \lambda_i \in [0, 1], \Sigma_{i=1}^k \lambda_i = 1\}$.

Step 3. Let $h = 0$.

Step 4. Let $h = h + 1$.

Step 5. Randomly generate, for instance, $100 \times k$ weighting vectors from $\bar{\Lambda}^{(h)}$.

Step 6. Filter the randomly generated λ-vectors of Step 5 to obtain, for instance, $2 \times p$ maximally dispersed representatives.

Step 7. For each of the $2p$ λ-vectors of Step 6, solve the augmented (or lexicographic) weighted Tchebycheff program.

Step 8. Filter the resulting criterion vectors to obtain a sample of p nondominated criterion vectors.

Step 9. Let $z^{(h)}$ designate the criterion vector selected by the decision-maker as the most preferred from the sample of Step 8.

Step 10. Let $\lambda^{(h)}$ designate the λ-vector that has $z^{(h)}$ as the defining point of an augmented weighted Tchebycheff isoquant (since this may be different from the λ-vector that produced $z^{(h)}$ in Step 7) where

$$\lambda_i^{(h)} = \begin{cases} \dfrac{1}{(z_i^* - z_i^{(h)})} \left[\displaystyle\sum_{i=1}^{k} \frac{1}{(z_i^* - z_i^{(h)})} \right]^{-1} & \text{if } z_i^{(h)} \neq z_i^* \text{ for all } i, \\ 1 & \text{if } z_i^{(h)} = z_i^*, \\ 0 & \text{if } z_i^{(h)} \neq z_i^* \text{ but } \exists j \ni z_j^{(h)} = z_j^*. \end{cases}$$

Step 11. If $h < t$, go to Step 12. If $h = t$, go to Step 14.

Step 12. With $\lambda^{(h)}$ being the λ-vector computed in Step 10, define

$$\bar{\Lambda}^{(h+1)} = \left\{ \lambda \in R^k \mid \lambda_i \in [l_i, \mu_i], \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

where

$$[l_i, \mu_i] = \begin{cases} [0, r^h] & \text{if } \lambda_i^{(h)} - \frac{1}{2}r^h \le 0, \\ [1 - r^h, 1] & \text{if } \lambda_i^{(h)} + \frac{1}{2}r^h \ge 1, \\ [\lambda_i^{(h)} - \frac{1}{2}r^h, \lambda_i^{(h)} + \frac{1}{2}r^h] & \text{otherwise,} \end{cases}$$

in which r^h is r raised to the hth power.

Step 13. Go to Step 4.

Step 14. Compute the inverse image of the decision-maker's final criterion vector selection. Stop.

The sample size p (the larger, the better) is negotiated between the analyst and decision maker.

It is recognized that just because we have a widely dispersed group of λ-weighting vectors in Step 6, it will not always follow that the criterion vectors produced in Step 7 are uniformly distributed over N. This is why more than p Tchebycheff programs are solved in Step 7 so that the most redundant of the excess criterion vectors can be discarded by the secondary filtering in Step 8.

The number of weighting vectors to be generated in Step 5, and the number of Tchebycheff programs to be solved in Step 6 are matters of judgment. The numbers selected represent a tradeoff between the degree of uniformity desired in the eventual sample and computer time availability. The numbers $100 \times k$ and $2 \times p$ are suggested as good 'rules-of-thumb'.

With regard to the convergence factor, the smaller r, the faster the rate of reduction of $\bar{\Lambda}$. However, the faster the rate of reduction, the smaller the decision-maker's latitude for making errors and altering his aspirations during the interactive process. Thus the convergence factor r (the solution procedure's mid-course correction ability) and the intended number of iterations t are interrelated. How to choose a well-performing r is described in Section 6. For instance, after five iterations with $r = 0.60$, the $[l_i, \mu_i]$ band about each weight is tightened to 0.0776. This should provide a sufficiently close approximation of the decision-maker's final criterion vector, and if not, either r can be made smaller or the number of iterations t extended.

6. Convergence factors

Let H be a k-dimensional unit hypercube. Suppose we obtain p samples from H and that each sample characterizes a subset hypercube whose volume is greater than or equal to $1/p$. Letting r be the subset hypercube's length on a side,

$$r^k \ge \frac{1}{p} \Rightarrow r \ge \sqrt[k]{1/p}.$$

Employing r as the convergence factor, suppose we wish to iterate until the $[l_i, \mu_i]$ interval width is less than or equal to $1/q$, believing that a final solution can be obtained from the sample of that iteration. Where t is the number of iterations

$$r^{t-1} \leq \frac{1}{q} \Rightarrow r \leq \sqrt[t-1]{1/q}.$$

Hence we have

$$\sqrt[k]{1/p} \leq r \leq \sqrt[t-1]{1/q}. \tag{6.1}$$

Although $\bar{\Lambda}$ is not a hypercube, (6.1) is used as a guide in selecting p, t and r to calibrate the interactive Tchebycheff algorithm. Experience indicates that it normally suffices for q to be between $\frac{1}{2}k$ and $\frac{3}{2}k$ (where k is the number of objectives) and p should be no smaller than k. As seen in (6.1), the number of iterations required decreases as sample size p increases.

7. Computer example

Consider the MOLP

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8		RHS
	−1		−1	1	2	−1	6		max	
Objs		5				2		4	max	
		4		6	2	3	4	−1	max	
			8	6	1	4	−3	7	≤	6
	−1			1		4		3	≤	7
	3		3	5		3	1		≤	7
			−2	7	−2		−2	6	≤	10
s.t.	3	4	−2	1			−1	−1	≤	10
		7	2		7		4		≤	5
	−2	−1	1				−3	4	≤	6
	6	−3		3			6	4	≤	6

To illustrate the interactive procedure, let us use the following 'artificial' utility function to simulate decision-maker judgment.

$$U = z_1^3(1 + z_2) + z_3.$$

For this utility function, the efficient extreme point of greatest utility[2] is (0, $1.1111E-01$, 0, 0, 0, $1.7500E+00$, $1.0556E+00$, 0). Its criterion vector is (4.583333, 4.055556, 9.916667) whose $U = 496.67474$. The optimal point,

[2] The efficient extreme point of greatest utility was computed by using ADBASE [10] to generate all 14 efficient extreme points. The utility function was then applied to the criterion vectors of the 14 efficient extreme points to determine the one with the largest U value.

however, is not extreme. As computed by the Ecker and Kupferschmid code [6], it is (1.4150$E-07$, 1.1111$E-01$, 2.3101$E-07$, 4.4456$E-08$, 6.9258$E-07$, 1.0122$E+00$, 1.0556$E+00$, 3.5847$E-06$). Its associated criterion vector is (5.321081, 2.580048, 7.703370) whose $U = 547.07575$.

With $q = 4$ and $p = 6$, with reference to (6.1) let us set $r = 0.6$ and $t = 4$. Now we apply the interactive procedure outlined in Section 5 to obtain an approximation of the optimal point. For generating weighting vectors and filtering, the LAMBDA and FILTER programs from the ADBASE package [10] are used.

(1) (Step 1) Using ADBASE (or any LP code because the example of this section is linear), each objective is individually maximized to obtain the ideal criterion vector $z^* = (6.333333, 7.000000, 11.490909)$ with all ϵ_i set to zero.

(2) (Step 5) Using LAMBDA, 300 weighting vectors are randomly generated from $\bar{\Lambda}^{(1)}$.

(3) (Step 6) Using FILTER, the 300 weighting vectors are reduced to 12 dispersed representatives.

(4) (Step 7) Using ADBASE (or any LP code in this example), the 12 associated augmented weighted Tchebycheff problems are solved.

(5) (Step 8) Using FILTER, the 12 resulting criterion vectors are filtered to obtain the sample of 6 nondominated criterion vectors of Table 1.

Table 1
Nondominated criterion vectors of the 1st iteration

Vector	z_1	z_2	z_3
1-1	1.569959	5.481774	9.782729
1-2	−0.003318	6.999052	2.011850
1-3	4.206830	4.528909	10.118020
1-6	3.135186	6.261198	3.797733
1-9	5.019167	1.650694	9.923330
1-10	6.002582	1.217058	5.658920

(6) (Step 9) Applying the artificial utility function, criterion vector 1-10 is selected with $U = 485.16195$.

(7) (Step 10) Computing the λ-vector that has 1-10 as the defining point of an augmented weighted Tchebycheff isoquant, we have

$$\lambda^{(1)} = \begin{bmatrix} 0.8977 \\ 0.0514 \\ 0.0509 \end{bmatrix}.$$

(8) (Step 11) With $r^1 = 0.6000$, the $[l_i, \mu_i]$ bounds about $\lambda^{(1)}$ are

$$[l_1, \mu_1] = [0.4000, 1.0000],$$

$$[l_2, \mu_2] = [0.0000, 0.6000],$$

$$[l_3, \mu_3] = [0.0000, 0.6000].$$

(9) (Step 5) Using LAMBDA, 300 weighting vectors are randomly generated from $\bar{\Lambda}^{(2)}$.

(10) (Steps 6, 7 and 8) Using FILTER, the 300 weighting vectors are reduced to 12 and ADBASE is applied to solve the 12 associated augmented weighted Tchebycheff programs. Using FILTER to reduce the 12 resulting criterion vectors, we have the sample of 6 nondominated criterion vectors of Table 2.

Table 2
Nondominated criterion vectors of 2nd iteration

Vector	z_1	z_2	z_3
1-10	6.002582	1.217058	5.658920
2-1	4.504218	4.188966	9.777982
2-2	5.450003	2.322217	7.316659
2-5	4.968017	3.286188	8.762616
2-6	4.883667	1.763610	10.465330
2-8	5.218083	1.484931	9.127668
2-9	6.178412	0.684557	5.286352

(11) (Step 9) With reference to the criterion vector selection of the 1st iteration, the artificial utility function selects criterion vector 2-2 as an improvement with $U = 545.11347$.

(12) (Step 10) Computing the λ-vector that has 2-2 as the defining point of an augmented weighted Tchebycheff isoquant, we have

$$\lambda^{(2)} = \begin{bmatrix} 0.7141 \\ 0.1348 \\ 0.1511 \end{bmatrix}.$$

(13) (Step 11) With $r^2 = 0.3600$, the $[l_i, \mu_i]$ bounds about $\lambda^{(2)}$ are

$$[l_1, \mu_1] = [0.5341, 0.8941],$$
$$[l_2, \mu_2] = [0.0000, 0.3600],$$
$$[l_3, \mu_3] = [0.0000, 0.3600].$$

(14) (Step 5) Using LAMBDA, 300 weighting vectors are randomly generated from $\bar{\Lambda}^{(3)}$.

(15) (Steps 6, 7 and 8) Using FILTER, the 300 weighting vectors are reduced to 12 and ADBASE is applied to solve the 12 associated augmented weighted Tchebycheff programs. Using FILTER to reduce the 12 resulting criterion vectors, we have the sample of 6 nondominated criterion vectors of Table 3.

Table 3
Nondominated criterion vectors of the 3rd iteration

Vector	z_1	z_2	z_3
2-2	5.450003	2.322217	7.316659
3-1	4.942677	3.338669	8.838637
3-2	4.564350	1.410587	8.386281
3-8	5.462019	1.281651	8.151925
3-9	5.470064	2.282094	7.256475
3-10	5.845769	0.961859	6.616926
3-11	5.206253	1.690161	9.007526

(16) (Step 9) Applying the artificial utility function to the new criterion vectors in Table 3, 3-9 is best with $U = 544.44687$. Since this does not represent an improvement we stick with 2-2.

(17) (Step 11) With $r^3 = 0.2160$ and

$$\lambda^{(3)} = \begin{bmatrix} 0.7141 \\ 0.1348 \\ 0.1511 \end{bmatrix}$$

the $[l_i, \mu_i]$ bounds about $\lambda^{(3)}$ are

$$[l_1, \mu_1] = [0.6061, 0.8221],$$
$$[l_2, \mu_2] = [0.0268, 0.2428],$$
$$[l_3, \mu_3] = [0.0431, 0.2591].$$

(18) (Step 5) Using LAMBDA, 300 weighting vectors are randomly generated from $\bar{\Lambda}^{(4)}$.

(19) (Steps 6, 7 and 8) These steps yield Table 4.

Table 4
Nondominated criterion vectors of the 4th iteration

Vector	z_1	z_2	z_3
2-2	5.450003	2.322217	7.316659
4-1	5.164556	2.893110	8.172998
4-2	5.042612	1.331157	8.389553
4-5	5.533965	1.221696	7.864139
4-6	5.385717	2.450788	7.509515
4-9	4.987327	3.247567	8.704685
4-12	5.735848	1.750527	6.459123

(20) (Step 9) With reference to criterion vector 2-2, the artificial utility function selects criterion vector 4-6 as an improvement with $U = 546.58411$.

(21) (Step 13) The inverse image of criterion vector 4-6 is (0, $1.1111E-01$, 0, 0, 0, $9.4762E-01$, $1.0556E+00$, 0).

Clearly, the method can continue iterating with increased degrees of resolution. However, as in this example, acceptable solutions are usually obtained within k iterations.

8. Concluding remarks

The interactive weighted Tchebycheff procedure has been designed to be a practical and useful method for solving multiple objective programming problems. To multiple objective linear programming, the advantage of the interactive procedure is that it can converge to non-extreme final solutions. In other words, rather than being restricted to searching only the efficient extreme points, the procedure is able to explore the interior of efficient facets as necessary. In addition, with suitable software, the procedure is applicable to integer and nonlinear multiple objective programs.

Acknowledgement

The authors thank Carl Bossuyt, University of Kentucky graduate student, for his computer assistance.

References

[1] V.J. Bowman, "On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objectives", *Lecture Notes in Economics and Mathematical Systems* 135 (1980) 76–85.

[2] E.U. Choo, "Multicriteria linear fractional programming", Ph.D. dissertation, University of British Columbia (Vancouver, B.C., 1980).

[3] E.U. Choo and D.R. Atkins, "An interactive algorithm for multicriteria programming", *Computers and Operations Research* 7 (1980) 81–87.

[4] W. Dinkelbach and W. Dürr, "Effizienzaussagen bei Ersatzprogrammen zum Vektormaximumproblem", in: R. Henn, H.P. Künzi and H. Schubert, eds., *Operations Research Verfahren* XII (Verlag Anton Hain, Meisenheim, 1972) pp. 117–123.

[5] J.G. Ecker and N.E. Shoemaker, "Selecting subsets from the set of nondominated vectors in multiple objective linear programming", *SIAM Journal on Control and Optimization* 19 (1981) 505–515.

[6] J.G. Ecker and M. Kupferschmid, "An ellipsoid algorithm for convex programming", Working Paper, Department of Mathematical Sciences, Rensselaer Polytechnic Institute (Troy, NY, 1982).

[7] A.M. Geoffrion, "Proper efficiency and the theory of vector maximization", *Journal of Mathematical Analysis and Applications* 22 (1968) 618–630.

[8] J.P. Ignizio, *Linear programming in single and multiple objective systems* (Prentice-Hall, Englewood Cliffs, NJ, 1982).

[9] S.M. Lee, *Goal programming for decision analysis* (Auerbach Publishers, Philadelphia, PA, 1972).
[10] R.E. Steuer, "Operating manual for the ADBASE multiple objective linear programming computer package (release: 8/80)", College of Business Administration, University of Georgia (Athens, GA, 1980).
[11] R.E. Steuer and F.W. Harris, "Intra-set point generation and filtering in decision and criterion space", *Computers and Operations Research* 7 (1980) 41–53.
[12] R.E. Steuer, "On sampling the efficient set using weighted Tchebycheff metrics", Proceedings of the Task Force Meeting on Multiobjective and Stochastic Optimization, International Institute for Applied Systems Analysis (Laxenburg, Austria, 1982).
[13] A.P. Wierzbicki, "The use of reference objectives in multiobjective optimization", *Lecture Notes in Economics and Mathematical Systems* 177 (1980) 468–486.
[14] P.L. Yu, "A class of solutions for group decision problems", *Management Science* 19 (1973) 936–946.
[15] M. Zeleny, "Compromise programming", in: J.L. Cochrane and M. Zeleny, eds., *Multiple criteria decision making* (University of South Carolina Press, Columbia, SC, 1973) pp. 262–301.