Computers & Operations Research 39 (2012) 2929–2943

Contents lists available at SciVerse ScienceDirect

Computers & Operations Research

journal homepage: www.elsevier.com/locate/caor

# An augmented weighted Tchebycheff method with adaptively chosen parameters for discrete bicriteria optimization problems ☆

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ARTICLE INFO

Available online 3 March 2012

*Keywords:*
Augmented weighted Tchebycheff norm
Discrete bicriteria optimization
Adaptive parameter computation

ABSTRACT

The augmented weighted Tchebycheff norm was introduced in the context of multicriteria optimization by Steuer and Choo [21] in order to avoid the generation of weakly nondominated points. It augments a weighted $l_\infty$-norm with an $l_1$-term, multiplied by a "small" parameter $\rho > 0$. However, the appropriate selection of the parameter $\rho$ remained an open question: A too small value of $\rho$ may cause numerical difficulties, while a too large value of $\rho$ may lead to the oversight of some nondominated points.

For discrete bicriteria optimization problems we derive a method for a problem dependent determination of all parameters of the augmented weighted Tchebycheff norm such that all nondominated points can be found and $\rho$ is as large as possible. In a computational study based on randomly generated instances of a bicriteria knapsack problem, the resulting adaptive augmented weighted Tchebycheff method is compared with the lexicographic weighted Tchebycheff method and with the augmented weighted Tchebycheff method with preset parameter values as well as with augmented $\varepsilon$-constraint scalarizations.



## 1. Introduction

The weighted Tchebycheff method and its variations is one of the most common scalarization methods in multiple criteria optimization. It is, for example, frequently used in interactive methods for integer and mixed-integer problems, see Alves and Climaco [2] for a recent review. One of its main advantages can be seen in the fact that, by appropriately varying the weights and/or the reference point, every nondominated point of a general multiple criteria optimization problem can be generated. This explicitly includes non-convex and discrete problems which may have a large percentage of unsupported nondominated points.

A drawback of the weighted Tchebycheff method is, however, that besides nondominated points also weakly nondominated points are generated in general. Since this is often unwanted, Steuer and Choo [21] suggested to add an $l_1$-term, weighted by some parameter $\rho > 0$, to the weighted $l_\infty$-distance between a reference point (usually the ideal or a utopia point) and the feasible set of a given problem. The resulting *augmented weighted Tchebycheff method* combines the advantages of the original (not augmented) approach, namely the potential generation of *every* nondominated point by appropriately modifying the weights and/or the reference point, with the property that weakly nondominated points are avoided.

One numerical difficulty, however, remained: The determination of an appropriate value for the parameter $\rho$ that defines the relative impact of the weighted $l_\infty$- and the $l_1$-term in the overall distance computation. As Ralphs et al. [17] and Steuer [20] pointed out, the value of $\rho$ should on the one hand be chosen large enough to ensure that weakly nondominated points are avoided, while on the other hand $\rho$ has to be selected sufficiently small such that all nondominated points of a given (potentially non-convex) problem can be found. Based on the complete set of nondominated points, Steuer and Choo [21] calculate a bound on $\rho$ such that for all smaller values of $\rho$ all nondominated points can be found. However, if the nondominated set is not known in advance, the determination of this bound is not possible.

Focusing on discrete bicriteria optimization problems, this paper discusses the question on how large a value of $\rho$ can be selected at maximum such that finding all nondominated points of the given problem can still be guaranteed. More precisely, we develop an explicit formula that determines a tight upper bound on the "feasible" values of $\rho$ (where "feasible" is understood in the sense that the determination of all nondominated points of the given problem can be guaranteed) depending on the given problem data and the choice of the reference point. Numerical tests on bicriteria knapsack problems are performed to validate the theoretical results.

### 1.1. Terminology and definitions

Let a multiple criteria optimization problem be given by

$$\begin{aligned} \min \quad & f(x) = [f_1(x), \ldots, f_k(x)]^\top \\ \text{s.t.} \quad & x \in X, \end{aligned} \tag{1}$$

☆ This research was partially supported by the BMBF (Grant 03KLPAK3).

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doi:10.1016/j.cor.2012.02.021

where $k \geq 2$ denotes the number of objective functions, $f_i : X \to \mathbb{R}$, $i=1,\ldots,k$, are the objective functions and $X \neq \emptyset$, $X$ bounded, denotes a discrete feasible set.

We define the notion of optimality for (1) following the Pareto concept. A solution $\bar{x} \in X$ is called *dominated* by a solution $x \in X$ if and only if $f_i(x) \leq f_i(\bar{x})$ for all $i=1,\ldots,k$ and $f_i(x) < f_i(\bar{x})$ for at least one $i \in \{1,\ldots,k\}$. If strict inequality holds for all $k$ components, then $x$ *strictly dominates* $\bar{x}$. If there is no other solution $x \in X$ that dominates $\bar{x}$, then $\bar{x}$ is called an *efficient solution* of (1), or just *efficient*. If there exists no feasible solution that strictly dominates $\bar{x}$, then $\bar{x}$ is called *weakly efficient*. We denote the set of efficient solutions of (1) by $X_E$ and refer to it as the *efficient set*. An efficient solution is called *supported* if it minimizes a weighted sum of the $k$ objectives of (1) involving non-negative weights.

We define $Z := f(X)$ the outcome set which is a set of distinct points in $\mathbb{R}^k$. In order to simplify the notation we equivalently formulate problem (1) in the outcome space as

$$\begin{aligned} \min \quad & z = [z_1,\ldots,z_k]^\top \\ \text{s.t.} \quad & z \in Z. \end{aligned} \tag{2}$$

A point $z \in Z$ is called (*weakly*) *nondominated* if it corresponds to the image of some (weakly) efficient solution $x \in X$. The set of all nondominated points, the *nondominated set,* is denoted by $Z_E$ in the following. Note that the cardinality of $Z_E$ may grow exponentially with the size of the input data, i.e., it is in general an intractable task to completely determine this set (see, e.g., [5]).

A lower bound on the nondominated points of (2) is given by the *ideal point* which we denote by $z^I$. The $i$-th component of the ideal point is defined as the minimum of the $i$-th objective, i.e., $z^I_i := \min\{z_i : z \in Z\}$ for all $i=1,\ldots,k$. A point $z^U$ that strictly dominates $z^I$ is called a *utopia point*. Furthermore, for $S \subset Z_E$, the point $z^L$ whose components are given by $z^L_i := \min\{z_i : z \in S\}$, $i=1,\ldots,k$, is called a *local ideal point* with respect to $S$.

A common technique to solve problems of the form (1) is to iteratively transform the vector valued problem into real valued problems, so-called *scalarizations*. A variety of different scalarization methods exists which differ in their theoretical properties, e.g., with respect to the fact whether the solutions generated by a specific method always correspond to nondominated points of (1) and whether all nondominated points can be generated (see, e.g., [5,13]).

In this article, we study the *Tchebycheff scalarization method* which belongs to the class of *compromise programming methods*, also known as *methods of the global criterion* (see, e.g., [13,23]). Thereby, a feasible point in the criterion space is generated that minimizes the distance to a given reference point with respect to a given metric induced by an appropriate (weighted) $l_p$-norm.

### 1.2. Literature review

In the following we briefly review the literature related to Tchebycheff scalarization methods for general multiple criteria optimization problems.

The general concept of the (weighted) Tchebycheff scalarization approach can already be found in Bowman [3], where the application of the (weighted) Tchebycheff norm is proposed for scalarizations of multi-objective optimization problems. Based on this idea, Steuer and Choo [21] introduced the augmented weighted Tchebycheff program and the lexicographic weighted Tchebycheff program.

From a theoretical point of view, the Tchebycheff method can be seen as a special case of compromise programming [23]. It also has similarity to the reference point method introduced by Wierzbicki [22]. Moreover, it fits in the general concept of oblique norms and gauges, described, e.g., in Klamroth et al. [19] and Schandl et al. [9]. Since the decision maker can easily interpret the reference point and the distance information, the method is frequently used within interactive approaches, see, e.g., Miettinen et al. [14] and Steuer and Choo [21], and Alves and Climaco [2] for a larger survey.

Tchebycheff-type approaches are utilized for continuous as well as for discrete multicriteria optimization problems. Applications of the augmented weighted Tchebycheff method can be found, for example, in Steuer [20]. Luque et al. [11] use the lexicographic approach to solve convex multi-objective programming problems. In Alves and Climaco [1], an augmented Tchebycheff program based on variations of the reference point is used to solve mixed-integer problems. Bozkurt et al. [4] use the weighted Tchebycheff norm to evaluate the quality of solutions generated by heuristics.

Particularly in the context of multiple criteria combinatorial optimization problems, Tchebycheff scalarizations often lead to NP-complete problems (see, e.g., [15]). If the whole nondominated set is to be computed by some iterative, scalarization-based method, Tchebycheff scalarizations are thus often outperformed by other problem-specific approaches. Nevertheless, Tchebycheff scalarizations are also applied in this context due to their general applicability and if, for example, different variants of Tchebycheff-type approaches are to be compared. In Sayin and Kouvelis [18] the lexicographic weighted Tchebycheff method and a variant using the origin as reference point are compared with the help of a bicriteria knapsack problem (aiming at the determination of the entire nondominated set or an approximation of it), and Ralphs et al. [17] use the same problem to evaluate the lexicographic weighted Tchebycheff method and the augmented weighted Tchebycheff method.

An experimental comparison of methods with and without augmentation term for continuous problems can be found in Miettinen et al. [14], where methods with augmentation term significantly outperform equivalent methods without such a term with respect to computational costs.

The use of augmentation terms in order to avoid weakly nondominated points is not limited to Tchebycheff scalarizations. Özpeynirci and Köksalan [16] augment the objective of the classic $\varepsilon$-constraint method by the sum of all other objectives weighted by a scalar. An interval for a suitable choice of the value of this scalar is explicitly derived in the case that all objectives are integer-valued. For the biobjective case, they develop a method to find all nondominated points. Ehrgott and Ruzika [6] present an improved $\varepsilon$-constraint method where, amongst others, slack variables are added to the constraints and the classic objective function is supplemented by a weighted sum of these slack variables. They show that an optimal solution of this augmented problem is properly nondominated if the weights and slack variables are positive. Mavrotas [12] proposes a method called AUGMECON ("augmented $\varepsilon$-constraint method"). He also augments the objective function by a sum of slack variables weighted by a small constant, which is divided by the individual ranges of the objectives obtained from the payoff table.

In most applications, the additional augmentation term, and particularly its weighting factor $\rho$, is merely considered as a technical factor that guarantees nondominance of the determined points, and $\rho$ is set to a small positive constant. However, Kaliszewski [8] points out that the parameter $\rho$ in the augmented weighted Tchebycheff method can also be interpreted as a source of trade-off information: If a bound on the desired trade-off of a nondominated point is given, then appropriate values of the parameters of the augmented weighted Tchebycheff scalarization including $\rho$ can be computed based on this information.

Moreover, Ralphs et al. [17] report for the discrete case that an appropriate choice of the parameter $\rho$ is critical when the complete set of nondominated points has to be generated. Taking $\rho$ too small may cause numerical difficulties because the weight of the augmentation term in the objective function can lose significance with respect to the primary objective. On the other hand, choosing $\rho$ too large may result in the situation that some of the nondominated points are unreachable.

### 1.3. Goals and outline

The main question that we address in this article is how the parameters of the augmented weighted Tchebycheff method have to be chosen such that all possibly existing nondominated points can be found reliably and efficiently, avoiding unnecessary computations whenever possible. As described, for example, in Ralphs et al. [17], this is not automatically guaranteed when the involved parameters are chosen a priori and are, hence, not directly related to the underlying data. Based on a rigorous analysis of the augmented weighted Tchebycheff norm, we derive a tight upper bound on the value of $\rho$ up to which the generation of all nondominated points of a given problem can be ensured. We also address the question whether the derived bound is best possible with respect to all configurations yielding an appropriate parameter choice.

We provide numerical experiments to compare the augmented weighted Tchebycheff method with adaptively chosen parameters to already existing solution approaches that are based on Tchebycheff scalarization methods as well as to an augmented $\varepsilon$-constraint method. Furthermore, we numerically investigate how the choice of the reference point affects the value of $\rho$ and the computational time when the augmented weighted Tchebycheff method with adaptively chosen parameters is applied.

The remainder of this paper is organized as follows. After reviewing different Tchebycheff scalarization methods, we develop explicit formulas for an appropriate parameter choice in the augmented weighted Tchebycheff method in Section 2. Furthermore, we present a generic sequential algorithm that, using the derived formulas for the parameter values, computes the entire set of nondominated points in the discrete, finite case. Linear formulations of Tchebycheff problems are also discussed in this section, as well as the parameter choice for the augmented $\varepsilon$-constraint method. A computational study for the bicriteria knapsack problem can be found in Section 3. Finally, concluding remarks are given in Section 4.

## 2. Parameters of the augmented weighted Tchebycheff norm

In the remainder of this article, we concentrate on discrete bicriteria optimization problems, i.e., on problems with discrete nondominated set $Z_E \subset \mathbb{R}^2$. Furthermore, it is assumed that there exists some (problem dependent) constant $\Delta > 0$ such that $|f_i(x) - f_i(\tilde{x})| \geq \Delta$ holds true for all $i=1,2$ and $x, \tilde{x} \in X$. This assumption is trivially satisfied when, e.g., integer-valued or finite problems are considered. To simplify the following analysis, we restrict ourselves to integer-valued problems and, thus, assume that $\Delta = 1$. However, all results can be easily transferred to the general case, where $\Delta$ is an arbitrary positive scalar. Furthermore, it is assumed that $z \geq 0$ for all $z \in Z$, as every discrete bicriteria optimization problem can easily be brought into this form by adding appropriate constants to the involved objective functions.

In this section, we concentrate on solving problem (1) by means of the augmented weighted Tchebycheff method and present a new solution methodology that is based on adaptively choosing the parameters of the augmented weighted Tchebycheff norm in a problem dependent way. More precisely, we aim at finding appropriate values for the weighting parameters and for $\rho$ such that the generation of the complete nondominated set can be guaranteed, when a generic algorithm is applied that iteratively generates problem scalarizations (and, hence, nondominated points) of problem (1) based on the nondominated points found before.

After a detailed review of different Tchebycheff scalarization methods in Section 2.1 (formulated for general multicriteria problems), we discuss optimal choices of the parameters for the augmented weighted Tchebycheff problem in Section 2.2. In the subsequent section, we present a generic approach that can be used to determine the complete nondominated set of problem (1) under the assumption that the nondominated set is finite. Furthermore, we discuss different linearization techniques for the Tchebycheff problems introduced in Section 2.1 and the augmented $\varepsilon$-constraint scalarization that was implemented for a comparison such that these problems can be solved with a standard LP-solver.

### 2.1. Tchebycheff scalarization methods

For a general multicriteria optimization problem (1), let $w_i > 0$, $i=1,\ldots,k$, with $\sum_{i=1}^k w_i = 1$ be non-negative weights. Then, the *weighted Tchebycheff norm* of a point $z \in \mathbb{R}^k$ is denoted by

$$\|z\|_\infty^w := \max_{i=1,\ldots,k} \{w_i |z_i|\}. \tag{3}$$

If an $l_1$-augmentation term is added to (3), we obtain the *augmented weighted Tchebycheff norm*

$$\|z\|_\rho^w := \|z\|_\infty^w + \rho \|z\|_1, \tag{4}$$

where $\|z\|_1 = |z_1| + \cdots + |z_k|$, $w_i \geq 0$, $i=1,\ldots,k$, $\sum_{i=1}^k w_i = 1$, $\rho \in \mathbb{R}$ and $w_i + \rho > 0$, $i=1,\ldots,k$. The parameter $\rho$ is a non-negative scalar, which is usually chosen as a small positive number.

Applying the (*weighted*) *Tchebycheff scalarization method* to generate nondominated points of problem (1) implies solving a sequence of optimization problems of the form

$$\begin{aligned} \min \quad & \|z-s\|_\infty^w \\ \text{s.t.} \quad & z \in Z, \end{aligned} \tag{5}$$

where $s \in \mathbb{R}^k$ corresponds to an appropriately chosen reference point of the given problem. If not stated otherwise, we assume throughout this paper that $s$ corresponds to the ideal point or to a utopia point. However, we will also use local ideal points in combination with the (weighted) Tchebycheff scalarization method.

From Steuer and Choo [21] we recall that every optimal solution of (5) is at least a weakly nondominated point of the given multicriteria problem, and a nondominated point whenever the solution of (5) is unique. The weights $w_1, \ldots, w_k$ can be used to model the decision makers preferences, i.e., to direct the search for nondominated points towards a preferred region of $Z$, or to iteratively generate the complete nondominated set (or an approximation of it). However, the fact that only weak nondominance can be guaranteed for solutions of (5) is a major drawback of this scalarization method for practical applications.

Two different approaches are frequently used in practice to overcome this shortcoming, both suggested by Steuer and Choo [21]: The *lexicographic weighted Tchebycheff method* consists of two stages. In a first stage, problem (5) is solved to obtain an optimal solution $z^\star$ that is further used in a second stage where the additional problem

$$\begin{aligned} \min \quad & \sum_{i=1}^k z_i \\ \text{s.t.} \quad & z_i \leq z_i^\star, \quad i=1,\ldots,k \\ & z \in Z \end{aligned} \tag{6}$$

is solved to optimality. It is shown in [21] that every optimal solution of (6) is guaranteed to be a nondominated point of (1), and that every nondominated point can be found by selecting appropriate weights in (5). According to the terminology used in Sayin and Kouvelis [18], we also refer to this method as the *two-stage approach* in the following.

The second solution methodology that avoids weakly nondominated points, the so-called *augmented weighted Tchebycheff method*, uses an augmentation of the objective function of (5) by a correction term given by the $l_1$-distance between feasible points $z \in Z$ and the considered reference point $s$. The resulting *augmented weighted Tchebycheff problem* is given by

$$\begin{aligned} \min \quad & \|z-s\|_\rho^w \\ \text{s.t.} \quad & z \in Z. \end{aligned} \tag{7}$$

Note that for $\rho = 0$, problem (7) reduces to problem (5). It is shown in [21] that for any $\rho > 0$ every optimal solution of (7) is a nondominated point of (1). Conversely, in the discrete case, every nondominated point of (1) can be obtained by solving problem (7) with an appropriate choice of the involved parameters.

Based on these results for both strategies, the lexicographic as well as the augmented weighted Tchebycheff approach can be used to iteratively determine the complete set of nondominated points when finiteness of problem (1) is assumed (see Section 2.6 for further details). In this context, the main advantage of the augmented weighted Tchebycheff approach compared to the lexicographic method can be seen in the fact that only one single optimization problem has to be solved to determine one nondominated point, and, thus, it can be expected that less CPU time is required in general (see, e.g., Miettinen et al. [14] for corresponding numerical results). However, no concept for the determination of the parameters (particularly $\rho$) of the augmented weighted Tchebycheff approach has been proposed up to the present such that, for discrete problems, *all* nondominated points can be generated reliably. This question is addressed in the following sections.

### 2.2. Properties of the augmented weighted Tchebycheff norm

Our approach, which will be formulated for discrete bicriteria problems, is motivated by the following observation: Let $z^1, z^2 \in Z$ denote two nondominated points of (1), where $z_1^1 < z_1^2$. Furthermore, let $s$ denote the local ideal point with respect to $\{z^1, z^2\}$ and let

$$B(z^1,z^2) := \{(z_1,z_2)^\top \in Z : z_1 < z_1^2, z_2 < z_2^1\},$$

$$B_\rho^w(z^1,z^2) := \{z \in B(z^1,z^2) : \|z-s\|_\rho^w < \min\{\|z^1-s\|_\rho^w, \|z^2-s\|_\rho^w\}\}.$$

By definition, every nondominated point $z^\star$ satisfying $z_1^\star < z_1^2$ and $z_2^\star < z_2^1$ is an element of $B(z^1,z^2)$. Moreover, every nondominated point $z^\star \in B(z^1,z^2)$ can be generated by solving problem (7), if the parameters $w$ and $\rho$ are chosen appropriately. In particular, the parameters $w$ and $\rho$ can be chosen such that the set $B(z^1,z^2)$ coincides with the set $B_\rho^w(z^1,z^2)$. However, this is not true in general since for arbitrary values of $w$ and $\rho$ it might happen that $\|z^\star - s\|_\rho^w > \min\{\|z^1-s\|_\rho^w, \|z^2-s\|_\rho^w\}$ for some nondominated point $z^\star \in B(z^1,z^2)$, and in this case $z^\star$ is not optimal for (7).

A sufficient condition under which the two sets are equal, i.e. $B(z^1,z^2) = B_\rho^w(z^1,z^2)$, is that for every $z^\star \in Z_E \cap B(z^1,z^2)$ it holds that $\min\{\|z^1-s\|_\rho^w, \|z^2-s\|_\rho^w\} > \|z^\star - s\|_\rho^w$. This condition is certainly satisfied whenever

$$\min\{\|z^1-s\|_\rho^w, \|z^2-s\|_\rho^w\} > \|\bar{z}-s\|_\rho^w \quad \text{with } \bar{z} := (z_1^2 - \Delta, z_2^1 - \Delta)^\top, \tag{8}$$

see Fig. 1 for an illustration. This is the basic idea of our construction. By solving (7) for parameters satisfying condition (8) we find a nondominated point in $B(z^1,z^2)$, if it exists.

We will see in the following that an appropriate choice of the parameters can be derived dependent on the underlying input data and the choice of the reference point $s$. Among the feasible choices for $w_1, w_2$ and $\rho$ satisfying condition (8) we further aim at maximizing the value for $\rho$ in order to avoid numerical difficulties that may arise when too small values for $\rho$ are used.

To ease notation in the following we introduce some simplifications. Recall that we restrict ourselves to the integer-valued case, i.e., $\Delta = 1$, and that we assume that $z_i > 0$ and $z_i - s_i > 0$ for $i = 1,2$ hold for all $z \in Z$. For simplification, we apply the linear mapping $\psi : \mathbb{Z}^2 \to \mathbb{Z}^2$, $z_i \mapsto z_i - s_i$, for $i = 1,2$ to the given problem such that the reference point $s$ coincides with the origin. Let therefore $z^1 := (0,y)^\top$ and $z^2 := (x,0)^\top$ with $x,y \in \mathbb{Z}_+$ in the following. Because of $\Delta = 1$ we may further assume that $x,y > 1$, since if $x = 1$ or $y = 1$, $B(z^1,z^2) \cap Z_E = \emptyset$. In addition, we exclude the case $x = y = 2$, as a contour line passing through the points $(0,2)^\top$, $(1,1)^\top$ and $(2,0)^\top$ implies that $\rho = \infty$. Thus, for $x = y = 2$ we have $B(z^1,z^2) = B_\rho^w(z^1,z^2)$ for every choice of $\rho > 0$.

It is well-known (see, e.g., [20]) that the contour line $\mathcal{L}_\alpha := \{z \in \mathbb{R}^2 : \|z\|_\rho^w = \alpha\}$ of the augmented weighted Tchebycheff norm with respect to a certain level $\alpha > 0$ is piecewise linear for all $\alpha > 0$ and symmetric with respect to the origin. We first focus on augmented weighted Tchebycheff scalarizations such that $\|z^1\|_\rho^w = \|z^2\|_\rho^w = \alpha$ for some level $\alpha > 0$, i.e., we interpret $z^1$ and $z^2$ as the intersection points of $\mathcal{L}_\alpha$ with the two coordinate axes and denote by $z^q$ the inflection point of the contour line in the positive orthant (cf. Fig. 2). Note that since we restrict our search for nondominated points of problem (1) to $B(z^1,z^2)$ and assume $z$ to be non-negative, we can omit the absolute values in (7), which means that we only consider points and contour lines in the positive orthant.

In the remainder of this section we derive a strict upper bound on the value of $\rho$, guaranteeing that (8) is valid for $\Delta = 1$. We further show that the variation of the intersection points of $\mathcal{L}_\alpha$ with the coordinate axes from $z^1, z^2$ to two alternative points $\bar{z}^1, \bar{z}^2$ closer to the reference point does not lead to an increased value of

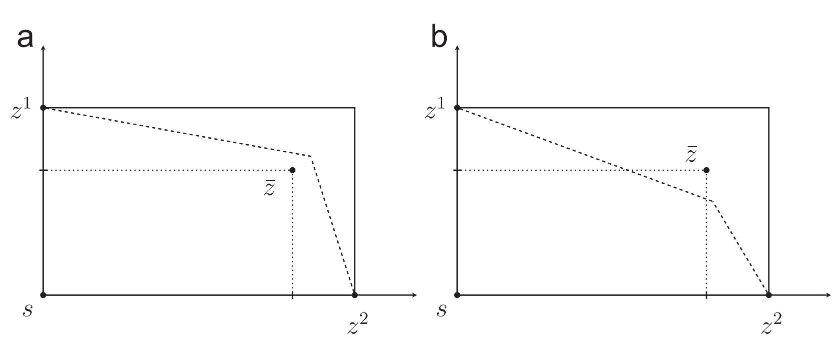


**Fig. 1.** Detail of contour of $\|\cdot\|_\rho^w$ (dashed curve) with parameter choice (a) satisfying and (b) not satisfying condition (8).

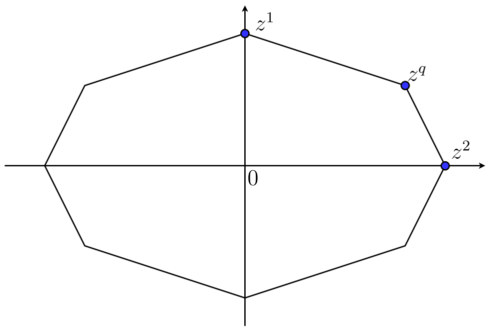


**Fig. 2.** Contour lines of the augmented weighted Tchebycheff norm.

$\rho$. In other words, the computed value of $\rho$ is optimal in the sense that for all choices of points $\bar{z}^1 := (0,y-v)^\top$ and $\bar{z}^2 := (x-u,0)^\top$ with $u,v \in [0,\Delta)$ replacing $z^1,z^2$ in condition (8), $\rho$ becomes maximal for the particular choice $(\bar{z}^1,\bar{z}^2)^\top = (z^1,z^2)^\top$. We partially used Maple$^{TM}$14 for the computations in the following.

As outlined above, we start the analysis with the case that $\|z^1\|_\rho^w = \|z^2\|_\rho^w = \alpha$, i.e., the contour line $\mathcal{L}_\alpha$ intersects the coordinate axes in the two nondominated points $z^1$ and $z^2$ for some level $\alpha > 0$. Before presenting how the parameters $\rho$ and $w_i$ have to be chosen for given and fixed points $z^1$ and $z^2$ in this case, we recall similar results for the weighted Tchebycheff norm that are stated, e.g., in Ralphs et al. [17] and Sayin and Kouvelis [18].

**Lemma 2.1.** *Let $x,y \in \mathbb{R}_{>0}, \alpha \in \mathbb{R}^+$, $z^1 = (0,y)^\top$, $z^2 = (x,0)^\top$, and $w_1,w_2 > 0$ such that $w_1 + w_2 = 1$. Then, the points $z^1$ and $z^2$ lie on the same contour line of a weighted Tchebycheff norm, i.e. $\|z^1\|_\infty^w = \|z^2\|_\infty^w = \alpha$, if and only if*

$$w_1 = \frac{y}{x+y}, \quad w_2 = \frac{x}{x+y} \quad \text{and} \quad \alpha = \frac{xy}{x+y}.$$

Note that the level $\alpha$ is uniquely defined by the coordinates of the points $z^1$ and $z^2$ in the case of a weighted Tchebycheff norm. This property no longer holds true when the augmented weighted Tchebycheff norm is considered. In more detail, for fixed $z^1$ and $z^2$, the weights $w_1,w_2$ and the parameter $\rho$ can be chosen dependent on an appropriately chosen value of $\alpha$.

**Theorem 2.2.** *Let $x,y \in \mathbb{R}_{>0}, \alpha \in \mathbb{R}^+$, $z^1 = (0,y)^\top$, $z^2 = (x,0)^\top$, $\rho \geq 0$ and $w_1,w_2 \geq 0$ with $w_1 + w_2 = 1$ and $w_i + \rho > 0$, $i = 1,2$. Then, the points $z^1$ and $z^2$ lie on the same contour line of an augmented weighted Tchebycheff norm, i.e. $\|z^1\|_\rho^w = \|z^2\|_\rho^w = \alpha$, if and only if it holds that*

$$w_1 = w_1(\alpha) = \frac{\alpha(y-x)+xy}{2xy}, \quad w_2 = w_2(\alpha) = \frac{\alpha(x-y)+xy}{2xy},$$

$$\rho = \rho(\alpha) = \frac{\alpha(x+y)-xy}{2xy},$$

*where*

$$\alpha \in I := \left[\frac{xy}{x+y}, \frac{xy}{\max\{x,y\}-\min\{x,y\}}\right]$$

*for the case that $x \neq y$, and*

$$w_1 = w_2 = \frac{1}{2}, \quad \rho = \rho(\alpha) = \frac{2\alpha - x}{2x}$$

*for all $\alpha \geq x/2$, if $x = y$.*

**Proof.** First, assume that the two points $z^1$ and $z^2$ lie on the same contour line with respect to an appropriate level $\alpha$. This assumption immediately implies that $w_1 = (\alpha - \rho x)/x$ and $w_2 = (\alpha - \rho y)/y$, respectively. As in addition $w_1 + w_2 = 1$, this further shows that

$$\rho = \frac{\alpha(x+y)-xy}{2xy} \tag{9}$$

and thus

$$w_1 = \frac{\alpha(y-x)+xy}{2xy} \quad \text{and} \quad w_2 = \frac{\alpha(x-y)+xy}{2xy}. \tag{10}$$

However, the parameters $w_1,w_2$ and $\rho$ only define a proper weighted augmented Tchebycheff norm if and only if additionally $w_1,w_2 \geq 0, \rho \geq 0$ and $w_i + \rho > 0$, $i = 1,2$, are satisfied. Hence, we have to restrict the value of $\alpha$ such that these conditions hold. A short calculation shows that

$$\rho \geq 0 \Leftrightarrow \alpha \geq \frac{xy}{x+y} \tag{11}$$

which implies the lower bound on the level $\alpha$. Furthermore, it holds that

$$w_1 \geq 0 \Leftrightarrow \begin{cases} \alpha \geq \frac{-xy}{y-x}, & y > x \\ \alpha \in \mathbb{R}^+, & x = y \\ \alpha \leq \frac{xy}{x-y}, & x > y \end{cases} \quad \text{and}$$

$$w_2 \geq 0 \Leftrightarrow \begin{cases} \alpha \leq \frac{xy}{y-x}, & y > x \\ \alpha \in \mathbb{R}^+, & x = y. \\ \alpha \geq \frac{-xy}{x-y}, & x > y \end{cases} \tag{12}$$

As $\alpha \geq 0$ by definition, only $xy/(x-y)$ for $x > y$ and $xy/(y-x)$ for $y > x$ impose a bound on $\alpha$. Hence, we have shown that for $x \neq y$

$$\alpha \leq \frac{xy}{\max\{x,y\}-\min\{x,y\}}$$

has to be valid. If $x=y$, Eq. (12) shows that no upper bound on the value of $\alpha$ is imposed, and we deduce that $w_1 = w_2 = \frac{1}{2}$ and $\rho = (2\alpha - x)/2x$ hold for all $\alpha \geq x/2$ in this case. Elementary calculus shows the converse results stated in the theorem. □

Note that the lower bound on $\alpha$ in Theorem 2.2 refers to the case that $\rho = 0$, i.e., if $\alpha = xy/(x+y)$, a weighted $l_\infty$-norm is considered. For the case that $x=y$, the parameter $\rho$ tends to infinity, when $\alpha \to \infty$ is considered. In this case, the influence of the $l_\infty$-norm vanishes and in the limit the distance is measured by a pure $l_1$-norm. To simplify the following analysis, we concentrate on the two cases that $x = y > 2$ and $x > y \geq 2$. Equivalent results for $y > x \geq 2$ can be obtained by a simple exchange of the variables.

Theorem 2.2 implies another important result for the inflection point $z^q$ of the contour line in the positive orthant, i.e., the point on the contour line where $w_1 z_1^q = w_2 z_2^q$ holds.

**Corollary 2.3.** *Let $w_1,w_2 > 0$ and $\rho \geq 0$ be given as defined in Theorem 2.2. Furthermore, let $z^q$ denote the inflection point of the contour line in the positive orthant. Then, the coordinates of this point are given by*

$$z^q = z^q(\alpha) = \frac{\alpha}{w_1 w_2 + \rho}(w_2,w_1)^\top. \tag{13}$$

**Proof.** As $z^q$ corresponds to the point where $w_1 z_1^q = w_2 z_2^q$ holds, it must be an element of the ray starting from the origin and passing through the point $(1,w_1/w_2)^\top$. Taking further into account that $\|z^q\|_\rho^w = \alpha$ shows that (13) is valid. □

Using the results of Theorem 2.2 and Corollary 2.3, we can derive the coordinates of $z^q$ with respect to $x$, $y$ and $\alpha$. If $x = y > 2$,

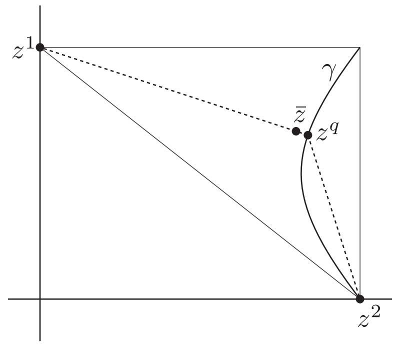


**Fig. 3.** Example for the curve $\gamma$ for $x=5$ and $y=3$. $\gamma$ describes the location of the inflection point $z^q$ for $\alpha \in I=[1.875,7.5]$.

we obtain that

$$z^q(\alpha)=\frac{2\alpha x}{4\alpha-x}(1,1)^\top \tag{14}$$

for all $\alpha \geq x/2$. For the case that $x>y\geq 2$, we obtain that

$$z^q(\alpha)=\frac{2\alpha xy}{4\alpha xy^2-[\alpha(x-y)-xy]^2}\cdot(xy+\alpha(x-y),xy-\alpha(x-y))^\top.$$

To derive an appropriate choice for the parameters we further analyze the curve

$$\gamma:\begin{cases} I\to\mathbb{R}^2,\\ \alpha\mapsto z^q(\alpha)=(f_1(\alpha),f_2(\alpha))^\top\end{cases} \tag{15}$$

with $I:=[xy/(x+y),xy/(x-y)]$ (cf. Theorem 2.2), that is induced by $z^q$ for fixed given integer values $x$ and $y$. For the case that $x=5$ and $y=3$, the curve $\gamma$ is depicted in Fig. 3. In this example, if $\alpha=1.875$, $z^q=(5,3)^\top$ and a pure weighted Tchebycheff norm is considered, while for $\alpha=7.5$ we have that $z^q=z^2$ and the contour line of a weighted $l_1$-norm is obtained.

Stating general properties of $\gamma$, it can be verified by elementary calculus that the mapping $f_2:I\to\mathbb{R}$, i.e., the restriction of $\gamma$ to the second coordinate of $z^q$, defines a continuous, strictly decreasing and, thus, invertible function on $I$ satisfying $f_2(I)=[0,y]$. Furthermore, the functional inverse of $f_2$ is given by

$$f_2^{-1}(t)=xy\cdot\frac{t(x+y)-xy-\sqrt{4t^2xy-4txy^2+x^2y^2}}{t(x-y)^2+2xy^2-2x^2y}$$

with $t\in[0,y]$. To finally obtain an explicit description of the parametric curve $\gamma$, we insert $\alpha=f_2^{-1}(t)$ in $f_1(\alpha)$ and conclude that $\gamma$ can be described in explicit form by the function

$$g:\begin{cases}[0,y]\to\mathbb{R},\\ t\mapsto\dfrac{(R(t)-2tx+3xy)\cdot(t(x+y)-xy-R(t))}{t(2y^2+6xy)-R(t)(x+3y)+x^2y-5xy^2},\end{cases}$$

where $R(t)=\sqrt{4t^2xy-4txy^2+x^2y^2}$. By construction we have that $g(0)=g(y)=x$. Differential calculus further shows that $g$ has a global minimum on $[0,y]$ at $t=y/2$.

Summarizing the discussion above, the formulas for $w_1$, $w_2$ and $\rho$ stated in Theorem 2.2 imply that we can compute parameters of a valid augmented Tchebycheff norm such that $z^1$ and $z^2$ lie on the same contour line. However, our derivations show that under this assumption the coordinates of the inflection point $z^q$ cannot be chosen independently, but that $z^q$ must always be an element of the curve $\gamma$. This implies in particular that, in general, the parameters cannot be set such that (in addition to the fact that the points $z^1$ and $z^2$ lie on the same contour line) the point $\bar{z}=(x-1,y-1)^\top$ is the inflection point of this contour line. This can be seen, for example, in Fig. 3 for the case that $x=5$ and $y=3$, where the point $\bar{z}=(4,2)^\top$ is not an element of $\gamma$.

### 2.3. Parameters guaranteeing completeness

In order to guarantee that no nondominated point $z^\star\in B(z^1,z^2)\cap Z_E$ in the considered region is missed, the parameters of the augmented weighted Tchebycheff norm have to be chosen carefully. Due to the integrality assumption, it is sufficient to require that the point $\bar{z}$ has a smaller level than $\alpha$, because if $\|\bar{z}\|_\rho^w<\alpha$ then $\|z^\star\|_\rho^w<\alpha$ for all $z^\star\in B(z^1,z^2)\cap Z_E$. Since the strict inequality is difficult to handle in this context, we construct the parameters of an augmented weighted Tchebycheff norm such that $\min\{\|z^1\|_\rho^w,\|z^2\|_\rho^w\}\geq\alpha$ and $\|\bar{z}\|_\rho^w\leq\alpha$ is satisfied. However, we have to keep in mind that the case $\|\bar{z}\|_\rho^w=\alpha$ is critical: If $z^1$,$z^2$ and $\bar{z}$ lie on the same level curve, the optimal solution of the associated problem (7) can be any of these three points and, hence, $\bar{z}$ may be missed.

We will proceed with the derivation of the parameters of an augmented weighted Tchebycheff problem such that $\|z^1\|_\rho^w=\|z^2\|_\rho^w=\|\bar{z}\|_\rho^w=\alpha$ holds. This is motivated by the fact that, under the principal assumption that $\|z^1\|_\rho^w=\|z^2\|_\rho^w=\alpha$ and $\|\bar{z}\|_\rho^w\leq\alpha$, this special choice leads to the maximal possible value of $\rho$. The formal proof is given in Lemma 2.4 below. Recall that choosing $\rho$ as large as possible has numerical advantages, cf. the discussion in Section 1.

**Lemma 2.4.** *Let $x>y\geq 2$ and let $\|z^1\|_\rho^w=\|z^2\|_\rho^w=\alpha$ and $\|\bar{z}\|_\rho^w\leq\alpha$ be satisfied. Then, $\rho$ is maximal if $\|\bar{z}\|_\rho^w=\alpha$ holds.*

**Proof.** Using the formulas derived in Theorem 2.2 and assuming that $x>y\geq 2$, elementary calculus shows that

$$\|\bar{z}\|_\rho^w=\alpha\Leftrightarrow\alpha=\frac{xy(x-1)}{xy-y-3x+x^2},$$

where $xy(x-1)/(xy-y-3x+x^2)<xy/(x-y)$. We know from Theorem 2.2 that

$$\rho=\rho(\alpha)=\alpha\cdot\frac{x+y}{2xy}-\frac{1}{2}\quad\text{for }\alpha\in\left[\frac{xy}{x+y},\frac{xy}{x-y}\right)$$

i.e., $\rho$ is a linear function of $\alpha$ for fixed $x$,$y$. As $\rho$ is continuous and strictly increasing, it attains its maximum when $\alpha$ is maximal, i.e., for $\alpha=\|\bar{z}\|_\rho^w$. □

In the following we give a geometrical interpretation of the case $\|z^1\|_\rho^w=\|z^2\|_\rho^w=\|\bar{z}\|_\rho^w=\alpha$ based on the results obtained for the curve $\gamma$.

**Lemma 2.5.** *Let $x>y\geq 2$ and let the points $z^1$, $z^2$ and $\bar{z}$ lie on the same contour line of an augmented weighted Tchebycheff norm with level $\alpha>0$. Then, $\bar{z}$ is an element of the line segment connecting $z^1$ and the inflection point $z^q$.*

**Proof.** We use the straight line connecting the points $z^1$ and $\bar{z}$ that is given by $h(t)=-(1/(x-1))t+y$ to find the intersection point of $h$ with the curve $\gamma$, i.e., the value of $\alpha\in I$ such that the point $z^q=z^q(\alpha)\in\gamma$ also lies on $h$. To prove the result it then suffices to show that the first coordinate of $z^q(\alpha)$ is at least $x-1$. To calculate $z^q(\alpha)$ we have to solve the system of equations

$$f_1(\alpha)=t\wedge f_2(\alpha)=-\frac{t}{x-1}+y$$

in the variables $\alpha$ and $t$. This system has the two solutions

$$(t_1,\alpha_1)=\left(0,\frac{xy}{y-x}\right)\quad\text{and}$$

$$(t_2,\alpha_2)=\left(\frac{xy(x-2)}{x^2y-x-2xy+y}\cdot(x-1),\frac{xy(x-1)}{xy-y-3x+x^2}\right).$$

As it is assumed that $x>y\geq 2$, the solution $(t_1,\alpha_1)$ is not feasible since $\alpha_1<0$ and thus $\alpha_1\notin I$. Thus, since the curve $\gamma$ is continuous and strictly decreasing in $f_2(\alpha)$ for $\alpha\in I$ and connects the points $(x,y)^\top$ and $z^2$, $(t_2,\alpha_2)$ must correspond to a point on $\gamma$ and $\alpha_2\in I$.

We find that for all $x > y \geq 2$ we have that

$$x^2y-x-2xy+y = x(xy-1-2y)+y > x[y(x-2)-1] > 0$$

and further that

$$\frac{t_2}{x-1} = \frac{xy(x-2)}{x^2y-x-2xy+y} = \frac{x^2y-2xy}{x^2y-2xy-(x-y)} > 1.$$

Hence, we can conclude that $t_2 > x-1$. □

Using the value calculated for $\alpha_2$ in Lemma 2.5 in combination with Theorem 2.2 immediately implies the following result.

**Corollary 2.6.** *Let $x > y \geq 2$. Then, the points $z^1$, $z^2$ and $\bar{z}$ lie on the same contour line of an augmented weighted Tchebycheff norm with level $\alpha > 0$ if and only if*

$$w_1 = \frac{xy-y-x}{xy-y-3x+x^2}, \quad w_2 = \frac{x(x-2)}{xy-y-3x+x^2} \quad \text{and} \quad \rho = \frac{x}{xy-y-3x+x^2}.$$

*In this case, $\alpha = xy(x-1)/(xy-y-3x+x^2)$.*

Since $\gamma$ is a continuous curve and strictly decreasing in $f_2(\alpha)$ for all $\alpha \in I$, we finally conclude:

**Theorem 2.7.** *Let $w_1$, $w_2$ and $\rho$ be given as defined in Theorem 2.2.*

1. (a) *Let $x > y \geq 2$. Choosing*
   $\alpha \in \left[\frac{xy}{x+y}, \frac{xy(x-1)}{xy-y-3x+x^2}\right)$
   *implies that*
   $$\|z^1\|_\rho^w = \|z^2\|_\rho^w > \|\bar{z}\|_\rho^w. \quad (16)$$

   (b) *Let $y > x \geq 2$, then (16) is valid whenever*
   $\alpha \in \left[\frac{xy}{x+y}, \frac{xy(y-1)}{xy-x-3y+y^2}\right)$.

2. *Let $x = y > 2$. Then, (16) holds when choosing*
   $\alpha \in \left[\frac{x}{2}, \frac{x(x-1)}{2(x-2)}\right)$.

**Proof.** The first part of the result follows directly from the discussion above. Thus, only the second part of the theorem has to be proven. Since in this case $x = y > 2$, we conclude from (14) that there exists a level $\alpha$ such that $z^q = z^q(\alpha) = (x-1, x-1)^\top$. Elementary calculus shows that this specific level is given by $\alpha = x(x-1)/2(x-2)$. Since $z_2^q(\alpha) = f_2(\alpha) = 2\alpha x/(4\alpha - x)$ is monotonically decreasing for all $\alpha > x/2$, fixing $\alpha < x(x-1)/2(x-2)$ is sufficient to ensure the validity of (16). □

A summary of the derived results for the parameters $w_1, w_2, \rho$ and $\alpha$ of the augmented weighted Tchebycheff norm is given in Table 1.

**Table 1**
Parameter values for an augmented weighted Tchebycheff norm such that the points $z^1$, $z^2$ and $\bar{z}$ lie on the same contour line with level $\alpha$.

| Case | $w_1$ | $w_2$ | $\rho$ | $\alpha$ |
|---|---|---|---|---|
| $x > y \geq 2$ | $\frac{xy-x-y}{xy-y-3x+x^2}$ | $\frac{x(x-2)}{xy-y-3x+x^2}$ | $\frac{x}{xy-y-3x+x^2}$ | $\frac{xy(x-1)}{xy-y-3x+x^2}$ |
| $x = y > 2$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2(x-2)}$ | $\frac{x(x-1)}{2(x-2)}$ |
| $y > x \geq 2$ | $\frac{y(y-2)}{xy-x-3y+y^2}$ | $\frac{xy-x-y}{xy-x-3y+y^2}$ | $\frac{y}{xy-x-3y+y^2}$ | $\frac{xy(y-1)}{xy-x-3y+y^2}$ |

## 2.4. Optimal parameter choice

The general assumption in Sections 2.2 and 2.3 above that restricted the parameter choice for the augmented weighted Tchebycheff norm was that the two points $z^1 = (0,y)^\top$ and $z^2 = (x,0)^\top$ with $x,y \in \mathbb{Z}^+$ lie on the same contour line, and in particular at the intersection of this contour line with the coordinate axes. Since our original goal was to find parameters such that $\rho$ is as large as possible, guaranteeing, however, that all nondominated points in the region $B(z^1,z^2) \cap Z_E$ can be found, the assumption that $\|z^1\|_\rho^w = \|z^2\|_\rho^w = \alpha$ for some level $\alpha$ may be too restrictive. In other words, the integrality of the intersection points of the contour line with the coordinate axes is not necessarily needed to construct an augmented weighted Tchebycheff norm such that $\|\bar{z}\|_\rho^w \leq \min\{\|z^1\|_\rho^w, \|z^2\|_\rho^w\}$. Indeed, as long as $x,y \in \mathbb{Z}^+$, it is sufficient to require that the intersection points of the corresponding contour line with the two coordinate axes are given by $\bar{z}^1 = (0, y-v)^\top$ and $\bar{z}^2 = (x-u, 0)^\top$ with $u,v \in [0,1)$. Note that the cases $u=1$ or $v=1$ are omitted since this would imply that $\rho = 0$. Two examples of possible choices for $u$ and $v$ are depicted in Fig. 4.

Now the question arises whether choosing appropriate values for $u,v \in (0,1)$ and requiring $\|\bar{z}\|_\rho^w \leq \|\bar{z}^1\|_\rho^w = \|\bar{z}^2\|_\rho^w$ leads to a larger value of $\rho$ as compared to the case that $(u,v)=(0,0)$, treated above. Our analysis follows a similar line of arguments as for the case that $(u,v)=(0,0)$. In particular, $x$ is substituted by $x-u$ and $y$ is substituted by $y-v$ in the respective formulas and derivations replacing Theorem 2.2 and the calculation of the point $z^q$.

Applying Lemma 2.4 with $\bar{x} = x-u$ and $\bar{y} = y-v$ instead of $x$ and $y$ implies that it is sufficient to consider only such levels $\alpha$ for which $\|\bar{z}^1\|_\rho^w = \|\bar{z}^2\|_\rho^w = \|\bar{z}\|_\rho^w = \alpha$. For the case that $x \neq y$, the inflection point $\bar{z}^q$ is now given by

$$\bar{z}^q = \bar{z}^q(\alpha,u,v) = \frac{2\alpha\overline{xy}}{4\alpha\overline{xy}^2 - [\alpha(\bar{x}-\bar{y}) - \overline{xy}]^2} \cdot (\overline{xy} + \alpha(\bar{x}-\bar{y}), \overline{xy} - \alpha(\bar{x}-\bar{y}))^\top$$

with

$$\alpha \in \bar{I} = \left[\frac{\overline{xy}}{\bar{x}+\bar{y}}, \frac{\overline{xy}}{\max\{\bar{x},\bar{y}\} - \min\{\bar{x},\bar{y}\}}\right].$$

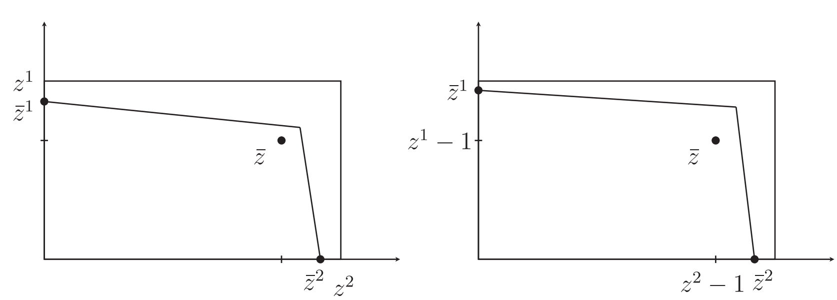


**Fig. 4.** Two examples of augmented weighted Tchebycheff norms with intersection points $\bar{z}^1$ and $\bar{z}^2$ in $(x-1,x]$ and $(y-1,y]$, respectively.

As in Section 2.3, we first restrict our analysis to the case that $x>y\geq 2$. For fixed $x,y\in\mathbb{Z}^+$ we define

$$G:\begin{cases}\bar{I}\times[0,1)\times[0,1)\to\mathbb{R}^2,\\ (\alpha,u,v)\mapsto\bar{z}^q(\alpha,u,v)=(G_1(\alpha,u,v),G_2(\alpha,u,v))^\top.\end{cases}$$

For fixed $u,v\in[0,1)$ we further consider the curve

$$\bar{\gamma}:\begin{cases}\bar{I}\to\mathbb{R}^2,\\ \alpha\mapsto G(\alpha,u,v).\end{cases}$$

As $u$ and $v$ can be chosen arbitrarily from the interval [0,1), there certainly exist appropriate vectors $(\alpha,u,v)$ such that the point $\bar{z}$ is an element of $\bar{\gamma}$. We will characterize these specific values of $u$ and $v$ in the following. Solving the system of equations induced by setting $G(\alpha,u,v)=(x-1,y-1)^\top$, we obtain for $\alpha\in\bar{I}$ the uniquely defined solution

$$(u(\alpha),v(\alpha))=\left(\frac{\alpha(2-y)(x+y-2)+x(y-1)^2}{\alpha(x+y-2)+(y-1)^2},\frac{\alpha(2-x)(x+y-2)+y(x-1)^2}{\alpha(x+y-2)+(x-1)^2}\right).$$

Differential calculus shows that $u$ as a function of $\alpha\in\mathbb{R}^+$ is strictly decreasing and continuous. Hence, $u(\alpha)$ is invertible and its functional inverse can be used in combination with $v(\alpha)$ to deduce the locus of all pairs $(u,v)\in[0,1)^2$ as a function of $u$ such that $\bar{z}^q$ is an element of $\bar{\gamma}$. Elementary calculus shows that this curve is given by

$$\tau:\begin{cases}[0,1)\to\mathbb{R},\\ u\mapsto\dfrac{u(xy-2y+1)+y-x}{u(x-y)+xy-2x+1},\end{cases}\tag{17}$$

where

$$\tau(u)=0\Leftrightarrow u=\frac{x-y}{xy-2y+1}=:u^\star\in(0,1)\quad\text{and}\quad\lim_{u\to1^-}\tau(u)=1.$$

Hence, the considered locus is given as the restriction of the function $\tau$ to the interval $I_u=[u^\star,1)$. Furthermore

$$\frac{\partial\tau(u)}{\partial u}=\frac{(x-1)^2(y-1)^2}{(u(x-y)+xy-2x+1)^2}>0\quad\text{and}$$
$$\frac{\partial^2\tau(u)}{\partial u^2}=\frac{2(x-1)^2(y-1)^2(y-x)}{(u(x-y)+xy-2x+1)^3}<0$$

for all $u\in[0,1)$, and, hence, $\tau$ is strictly increasing and concave. An illustration of the curve $\tau$ for the case that $x=5$ and $y=3$ is given in Fig. 5. These values of $x$ and $y$ imply that $u^\star=0.2$ in this example problem.

In order to analyze for which values of $(u,v)\in[0,1)^2$ the maximal value for $\rho$ is obtained, we consider the subdivision of the square $[0,1)^2$ induced by the curve $\tau$ into the two connected sets

$$\mathcal{S}_1=\{(u,v)\in[0,1)\times[0,1):\ v\geq\tau(u)\},$$

$$\mathcal{S}_2=\{(u,v)\in[0,1)\times[0,1):\ v\leq\tau(u)\}$$

with common boundary $\tau$. Recall that Lemma 2.4 with $\bar{x}=x-u$ and $\bar{y}=y-v$ instead of $x$ and $y$ implies that we only need to consider such levels $\alpha$ for which $\|\bar{z}^1\|^w_\rho=\|\bar{z}^2\|^w_\rho=\|\bar{z}\|^w_\rho=\alpha$.

From the continuity of $G$ and the results obtained for the case that $(u,v)=(0,0)$, we conclude that if $(u,v)\in\mathcal{S}_1$, then the point $\bar{z}$ is an element of the line segment connecting the points $\bar{z}^1$ and $\bar{z}^q(\alpha,u,v)$ for that particular value of $\alpha$. If $(u,v)\in\mathcal{S}_2$, $\bar{z}$ is an element of the line segment connecting $\bar{z}^q(\alpha,u,v)$ and $\bar{z}^2$.

Following the same reasoning as in the case that $(u,v)=(0,0)$, we compute the specific value of $\alpha$, depending on $(u,v)\in\mathcal{S}_1$ or $(u,v)\in\mathcal{S}_2$, to deduce the corresponding value of $\rho$ based on the result of Theorem 2.2. If $(u,v)\in\mathcal{S}_1$, the line segment $h_1$ connecting $\bar{z}^1$ and $\bar{z}$ is given by $h_1(t)=((v-1)/(x-1))t+y-v$. Elementary calculus shows

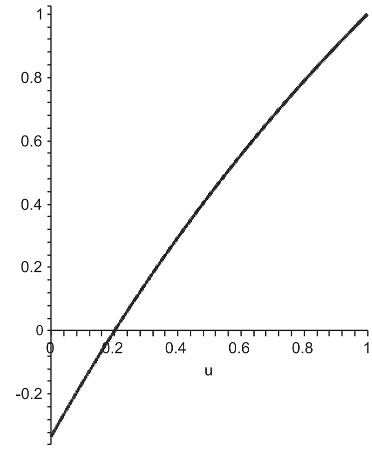


**Fig. 5.** Example for the curve $\tau$ for $x=5$ and $y=3$. For the depicted values of $(u,v)\in[0.2,1)\times[0,1)$, the point $\bar{z}^q$ is an element of the curve $\bar{\gamma}$.

that intersecting $h_1$ with $\bar{\gamma}$ in combination with Theorem 2.2 leads to

$$\alpha^1=\alpha^1(u,v)=\frac{(x-u)(y-v)(x-1)}{u(3-x)+v(1+x)-2uv+xy-y-3x+x^2},\tag{18}$$

$$\rho^1=\rho^1(u,v)=\frac{(x-u)(1-v)}{u(3-x)+v(1+x)-2uv+xy-y-3x+x^2}.\tag{19}$$

Using differential calculus, we obtain that

$$\frac{\partial\rho^1(u,v)}{\partial u}=\frac{(v-1)(y-v)(x-1)}{(u(3-x)+v(1+x)-2uv+xy-y-3x+x^2)^2}<0\quad\forall(u,v)\in\mathcal{S}_1,$$

$$\frac{\partial\rho^1(u,v)}{\partial v}=\frac{(u-x)(x+y-u-1)(x-1)}{(u(3-x)+v(1+x)-2uv+xy-y-3x+x^2)^2}<0\quad\forall(u,v)\in\mathcal{S}_1.$$

This implies that the maximal value of $\rho^1$ is obtained when $(u,v)$ lies at the boundary of $\mathcal{S}_1$:

**Lemma 2.8.** *Let $(u,v)\in\mathcal{S}_1$ and suppose that $\|\bar{z}^1\|^w_{\rho^1}=\|\bar{z}^2\|^w_{\rho^1}=\|\bar{z}\|^w_{\rho^1}=\alpha^1$, where $\alpha^1$ is given as stated in (18). Then, the value of $\rho^1$ in (19) is maximal for $(u,v)=(0,0)$.*

**Proof.** As the closure $\mathrm{cl}(\mathcal{S}_1)$ of the set $\mathcal{S}_1$ is compact and $\rho^1$ is continuous on $\mathrm{cl}(\mathcal{S}_1)$, the maximum of $\rho^1$ within $\mathrm{cl}(\mathcal{S}_1)$ exists. Furthermore, as both partial derivatives of $\rho^1$ with respect to $u$ and $v$ are strictly negative for all $(u,v)\in\mathcal{S}_1$, $\rho^1$ attains its maximum in a boundary point of $\mathrm{cl}(\mathcal{S}_1)$ satisfying $u=0$ or $v=0$. But as the derivatives show that $\rho^1$ is also strictly decreasing in all these points on the boundary, we deduce that the maximum of $\rho^1$ in $\mathcal{S}_1$ is obtained for $(u,v)=(0,0)$. □

Now consider the case that $(u,v)\in\mathcal{S}_2$. The line segment $h_2$ passing through the points $\bar{z}$ and $\bar{z}^2$ is given by $h_2(t)=((y-1)/(u-1))(t+u-x)$. Intersecting $h_2$ with $\bar{\gamma}$ in combination with Theorem 2.2 shows that

$$\alpha^2=\alpha^2(u,v)=\frac{(x-u)(y-v)(y-1)}{u(1+y)+v(3-y)-2uv+xy-x-3y+y^2},\tag{20}$$

$$\rho^2=\rho^2(u,v)=\frac{(y-v)(1-u)}{u(1+y)+v(3-y)-2uv+xy-x-3y+y^2}.\tag{21}$$

We further have that

$$\frac{\partial\rho^2(u,v)}{\partial u}=\frac{(v-y)(x+y-v-1)(y-1)}{(u(1+y)+v(3-y)-2uv+xy-x-3y+y^2)^2}<0\quad\forall(u,v)\in\mathcal{S}_2,$$

$$\frac{\partial \rho^2(u,v)}{\partial v} = \frac{(u-1)(y-1)(x-u)}{(u(1+y)+v(3-y)-2uv+xy-x-3y+y^2)^2} < 0 \quad \forall (u,v) \in \mathcal{S}_2.$$

**Lemma 2.9.** *Let $(u,v) \in \mathcal{S}_2$ and assume that $\|\bar{z}^1\|^w_{\rho^2} = \|\bar{z}^2\|^w_{\rho^2} = \|\bar{z}\|^w_{\rho^2} = \alpha^2$, where $\alpha^2$ is given as stated in (20). Then, the value of $\rho^2$ in (21) becomes maximal for $(u,v) = (u^\star, 0)$, where $u^\star = (x-y)/(xy-2y+1)$.*

**Proof.** The proof follows the same line of argument as the proof of Lemma 2.8, taking additionally into account that the curve $\tau$ is known to be monotonically increasing and concave. □

Combining the results of Lemmas 2.8 and 2.9 completes the analysis for the case that $x > y \geq 2$:

**Theorem 2.10.** *Let $x,y \in \mathbb{Z}^+$ such that $x > y \geq 2$ and let $(u,v) \in [0,1) \times [0,1)$ such that the points $\bar{z}^1 = (0, y-v)^\top$, $\bar{z} = (x-1, y-1)^\top$ and $\bar{z}^2 = (x-u, 0)^\top$ lie on a common contour line of an augmented weighted Tchebycheff norm with parameters $w_1$, $w_2$, $\rho$ and for a certain level $\alpha$. Then, the parameter $\rho$ becomes maximal for $(u,v) = (0,0)$ and can be computed as stated in Table 1.*

**Proof.** According to the results of Lemmas 2.8 and 2.9, only the values of $\rho^1(0,0)$ and $\rho^2(u^\star,0)$ have to be compared. But as the point $(u^\star,0)$ is an element of $\tau$, it is also contained in $\mathcal{S}_1$. Optimality of the point $(u,v) = (0,0) \in \mathcal{S}_1$ and the fact that the partial derivative of $\rho^1$ with respect to $u$ is strictly increasing on the line segment joining $(0,0)$ and $(u^\star,0)$ immediately implies that $\rho^1(0,0) < \rho^2(u^\star,0)$ and thus $(0,0)$ is optimal. □

Note that by symmetry the same result is obtained when $y > x \geq 2$. It remains to discuss the case that $x = y > 2$. Let again $\bar{z}^1 = (0, x-v)^\top$ and $\bar{z}^2 = (x-u, 0)^\top$ with $u,v \in [0,1)$.

First, we consider the case that $u = v$. For this case we deduce from (14) that

$$\bar{z}^q = \bar{z}^q(\alpha,u) = \frac{2\alpha(x-u)}{4\alpha-x+u}(1,1)^\top$$

for all $\alpha \geq (x-u)/2$. As the curve defined by $\bar{z}^q$ corresponds to the line joining the points $(x-u, x-u)^\top$ and $\frac{1}{2}(x-u, x-u)^\top$ when $\alpha \to \infty$ is considered, there must exist a feasible value of $\alpha$ such that the points $\bar{z}$ and $\bar{z}^q$ coincide. This specific value of $\alpha$ can be obtained by intersecting the curve that is induced by $\bar{z}^q(\alpha,u)$ with the line $h_3(t) = ((u-1)/(x-1))t + x - u$ that joins the points $\bar{z}^1$ and $\bar{z}$. We obtain that $\alpha^3 = (x-1)(x-u)/2(x-2+u)$, and, hence, by Theorem 2.2 we have that

$$\rho^3 = \rho^3(u) = \frac{1-u}{2(x-2+u)} \quad \text{and} \quad \frac{\partial \rho^3(u)}{\partial u} = \frac{1-x}{2(x-2+u)^2} < 0$$

for all $u \in [0,1)$. But this implies that $\rho^3$ is maximal for $u = v = 0$, which results in the value of $\rho$ that is stated in Table 1 for $x = y > 2$.

Finally, it remains to discuss the cases $0 \leq v < u < 1$ and $0 \leq u < v < 1$, respectively. By symmetry of the resulting problem, we restrict ourselves to the latter case, i.e., to the case that $\bar{x} > \bar{y} > 2$, where $\bar{x} = x-u$ and $\bar{y} = x-v$. We once more obtain that

$$\bar{z}^q = \bar{z}^q(\alpha,u,v) = \frac{2\alpha\bar{x}\bar{y}}{4\alpha\bar{x}\bar{y}^2 - [\alpha(\bar{x}-\bar{y}) - \bar{x}\bar{y}]^2} \cdot (\bar{x}\bar{y} + \alpha(\bar{x}-\bar{y}), \bar{x}\bar{y} - \alpha(\bar{x}-\bar{y}))^\top$$

with $\alpha \in \bar{I} = [\bar{x}\bar{y}/(\bar{x}+\bar{y}), \bar{x}\bar{y}/(\bar{x}-\bar{y})]$. As $\alpha > 0$, it further follows that

$$\frac{\bar{x}\bar{y} - \alpha(\bar{y}-\bar{x})}{\bar{x}\bar{y} - \alpha(\bar{x}-\bar{y})} > \frac{\bar{x}\bar{y} - \alpha(\bar{x}-\bar{y})}{\bar{x}\bar{y} - \alpha(\bar{x}-\bar{y})} \geq 1,$$

which implies that the curve defined by $\bar{z}^q(\alpha,u,v)$ is always located underneath the line segment joining the origin with the point $(x,x)^\top$. Hence, the value of $\rho^1(u,v)$ can be used to deduce the desired value of $\rho$ such that the points $\bar{z}^1$, $\bar{z}$ and $\bar{z}^2$ lie on a common contour line of an augmented weighted Tchebycheff norm. As $x = y$ by assumption, we obtain that

$$\rho^4 = \rho^4(u,v) = \frac{(x-u)(1-v)}{u(3-x)+v(1+x)-2uv+2x(x-2)}. \tag{22}$$

**Theorem 2.11.** *Let $x,y \in \mathbb{Z}^+$ such that $x = y > 2$ and let $(u,v) \in [0,1) \times [0,1)$ such that the points $\bar{z}^1 = (0, x-v)^\top$, $\bar{z} = (x-1, x-1)^\top$ and $\bar{z}^2 = (x-u, 0)^\top$ lie on a common contour line of an augmented weighted Tchebycheff norm with parameters $w_1$, $w_2$, $\rho$ and for a certain level $\alpha$. Then, the parameter $\rho$ becomes maximal for $(u,v) = (0,0)$ and is given as stated in Table 1.*

**Proof.** We restrict ourselves to the case that $u \leq v$ by symmetry of the problem. If $v = u$, $(v,u) = (0,0)$ is the optimal choice as already shown above. So, assume that $0 \leq u < v < 1$. First, we obtain for the denominator in (22) that

$$u(3-x)+v(1+x)-2uv > u(3-x)+u(1+x)-2uv = 2u(2-v) \geq 0.$$

Combining this result with (22), it follows that

$$\rho^4(u,v) < \frac{(x-u)(1-v)}{2x(x-2)} \leq \frac{x}{2x(x-2)} = \frac{1}{2(x-2)} = \rho,$$

which completes the proof. □

### 2.5. Parameter choice in practice

Combining the results of Theorems 2.10 and 2.11 shows that a maximal value for $\rho$ is attained whenever the coordinates of $z^1$ and $z^2$ are integral. However, as the choice of $\rho$ according to Table 1 implies that the points $z^1$, $\bar{z}$ and $z^2$ lie on a common contour line of an augmented weighted Tchebycheff norm, slightly modified values of the parameters should be selected for practical applications in order to guarantee that the point $\bar{z}$ is found with the corresponding scalarization.

This can be, for example, achieved by constructing a level line passing through the points $(0, y-u)^\top$, $\bar{z} = (x-1, y-1)^\top$ and $(x-u, 0)^\top$ with some $u \in (0,1)$ and, thus, making sure that $\bar{z}$ has a strict lower level than $z^1$ and $z^2$. Note that this is a special case of the construction presented in the previous subsection as we set $v := u$. In order to derive the explicit formulas of the parameters for the case $x > y \geq 2$, we need to know on which part of the level line the point $\bar{z}$ lies. Therefore we consider the function $\tau$ given in (17). It can be easily seen that $\tau$ satisfies $\tau(u) < u$ for all $u \in (0,1)$, so we deduce that $(u,u) \in \mathcal{S}_1$ for all $u \in (0,1)$, see also Fig. 5. This implies that formulas (18) and (19) hold with $v = u$. Using Theorem 2.2 for $x := x-u$ and $y := y-u$ directly yields the weights. The parameters in the other two cases are obtained analogously. They are summarized in Table 2. Note that we obtain the parameters of Table 1 from Table 2 when we set $u = 0$.

### 2.6. Algorithmic framework for augmented weighted Tchebycheff scalarizations

In this section, the application of augmented weighted Tchebycheff scalarizations in the context of a generic algorithm for the generation of the complete set of nondominated points of problem (1) is discussed. We focus on bicriteria problems and assume that the nondominated set has finite cardinality. The algorithmic framework summarized in Algorithm 1 is basically a reimplementation of the algorithm presented in Ralphs et al. [17] which improves Eswaran et al. [7] and Sayin and Kouvelis [18]. By iteratively solving parameterized scalarizations, a representative efficient solution for each nondominated point is computed. In the following, we call one or several parameterized problems that yield a nondominated point a *subproblem*. In our implementation, either a Tchebycheff-type method specified in Section 2.1 or

**Table 2**
Practical choice of parameter values for an augmented weighted Tchebycheff norm such that the points $(0,y-u)^\top$, $(x-1,y-1)^\top$ and $(x-u,0)^\top$ with $u \in (0,1)$ lie on the same contour line.

| Case | $w_1$ | $w_2$ | $\rho$ |
|---|---|---|---|
| $x > y \geq 2$ | $\dfrac{xy-x-y+u(2-u)}{xy-y-3x+x^2+2u(2-u)}$ | $\dfrac{(x-u)(x+u-2)}{xy-y-3x+x^2+2u(2-u)}$ | $\dfrac{(x-u)(1-u)}{xy-y-3x+x^2+2u(2-u)}$ |
| $x = y > 2$ | $\dfrac{1}{2}$ | $\dfrac{1}{2}$ | $\dfrac{(1-u)}{2(x+u-2)}$ |
| $y > x \geq 2$ | $\dfrac{(y-u)(y+u-2)}{xy-x-3y+y^2+2u(2-u)}$ | $\dfrac{xy-x-y+u(2-u)}{xy-x-3y+y^2+2u(2-u)}$ | $\dfrac{(y-u)(1-u)}{xy-x-3y+y^2+2u(2-u)}$ |

an augmented $\varepsilon$-constraint approach is used for solving the subproblems.

**Algorithm 1.** Sequential algorithm for generating the nondominated set.

**Input**: Data of bicriteria discrete optimization problem, specification of scalarization method used to solve subproblems, parameters (optionally).
**Output**: Complete set of nondominated points *ND*.

1: Create empty list *ND*. Compute lexicographic minimal solutions $z^1$ and $z^2$ wrt. $f_1$ and $f_2$, respectively, and ideal point.
2: **if** $z^1 = z^2$ **then**
3:   Set $ND = \{z^1\}$.
4: **else**
5:   Set $ND = \{z^1, z^2\}$.
6:   **while** $\exists$ adjacent pair of points $(z^i, z^{i+1})$ in *ND* not yet investigated **do**
7:     Compute parameters and optionally the new reference point wrt. $z^i$, $z^{i+1}$.
8:     Solve the resulting subproblem and find solution $z$.
9:     **if** $z = z^1$ or $z = z^2$ **then**
10:       Label pair of solutions as investigated
11:     **else**
12:       Insert $z$ as new element between $z^i$ and $z^{i+1}$ in *ND*.
13:     **end if**
14:   **end while**
15: **end if**
16: **return** *ND*.

A list *ND* is maintained in Algorithm 1 to store nondominated points during the course of the algorithm. The list is initialized with the two lexicographic minimal solutions w.r.t. $f_1$ and $f_2$, respectively. All elements which are consecutively added to the list are kept sorted with respect to their first coordinate in ascending order. Furthermore, a flag is assigned to each element which indicates whether the box defined by this and the following element has already been investigated. If not, the set $B(z^i, z^{i+1})$ might contain further nondominated points. In this case, parameters and optionally also a reference point are computed with respect to the points $z^i$ and $z^{i+1}$ by applying the results of Section 2.2.

If the lexicographic weighted Tchebycheff method is used to solve the given subproblem, the values for the involved parameters $w_1$ and $w_2$ are chosen according to the result of Lemma 2.1. The same lemma is used to determine the weights for the classic augmented weighted Tchebycheff method. Note that if this method is applied, the value of $\rho$ is specified at the beginning of the algorithm and remains unchanged for all subproblems. If, in contrast, the augmented weighted Tchebycheff method is applied with adaptively chosen parameters as derived in Sections 2.2–2.5, $w_1$, $w_2$ and $\rho$ are dynamically updated according to the formulas given in Table 2 for some $u \in (0,1)$. Hence, in general, the value of $\rho$ depends on the given subproblem in this case. How the parameters are chosen for the augmented $\varepsilon$-constraint approach will be specified in Section 2.8.

Let $z$ denote the optimal solution of the active subproblem. If $z = z^i$ or $z = z^{i+1}$, the results of Section 2.2 imply that $B(z^i, z^{i+1})$ contains no further nondominated points and, hence, $z^i$ can be labeled as investigated. Otherwise, the nondominated point $z$ induces two new boxes $B(z^i, z)$ and $B(z, z^{i+1})$ that may contain further nondominated points. In this case, $z$ is inserted into *ND* at the respective position to maintain the desired ordering of the elements, and the new subproblem induced by $z^i$ and $z$ is solved in the next iteration. Note that, for simplification, we proceed in ascending order with respect to the first coordinate, i.e., we always choose the first two consecutive entries of *ND* which have not yet been investigated. If an approximation instead of the complete nondominated set shall be generated, the selection of the points for the next subproblem can be easily modified and adapted for this purpose.

This procedure of selecting and solving subproblems is repeated until every element, and, thus, every box between consecutive elements in the list has been investigated. By construction of the algorithm, it is ensured that all nondominated points that can be generated with the respective scalarization method have been found and the list *ND* is returned by the algorithm. The correctness of Algorithm 1 follows from the results of Section 2.2.

**Theorem 2.12.** *Algorithm 1 is correct and returns the complete set of nondominated points of problem* (1), *if it is finite and if an appropriate scalarization method is employed.*

The complexity of Algorithm 1 depends on the number of subproblems that are solved during the course of the algorithm, as well as on the complexity of the respective subproblems. Assuming that the nondominated set has finite cardinality $N$, the determination of the complete nondominated set requires the solution of $N$ different subproblems. In addition, in order to verify that no further nondominated points exist, $N-1$ further subproblems are solved by Algorithm 1. The overall number of subproblems is thus given by $2N-1$ (see also [17]). Since problem (1) may be intractable, it may take an exponential amount of time to generate $Z_E$.

## 2.7. Linear reformulations of Tchebycheff-type problems

We finally discuss several approaches to remodel the different Tchebycheff scalarizations introduced in Section 2.1 and used in Algorithm 1 such that the resulting problem formulations can be solved with a linear solver like, for example, IBM ILOG CPLEX.

Let $z^1, z^2$ be the two consecutive nondominated points selected for further analysis in Step 7 of the algorithm. Motivated by the fact that nondominated points cannot be contained in the set $\{z \in \mathbb{R}^2 \,:\, z = z^1 + \; q \vee z = z^2 + q,\, q \in \mathbb{R}^2_+, q \neq 0\}$, and that nondominated points $\tilde{z}$ with $\tilde{z}_2 > z^1_2$ or $\tilde{z}_1 > z^2_1$ are generated in other subproblems in Algorithm 1, we restricted our consideration in Sections 2.2–2.4 to the set $B(z^1, z^2)$. Consequently, we omitted the absolute values in the formulation of the augmented weighted Tchebycheff problem (7). The restriction to $B(z^1, z^2)$, however, is not explicitly contained in problem (7). Therefore, it needs to be discussed if and how this formulation has to be modified such that the restriction of the search to the set $B(z^1,z^2)$ is contained in the problem formulation. All reformulations discussed in the following are used in the numerical studies presented in Section 3.

First, consider the lexicographic weighted Tchebycheff method. Since either the global ideal point or local ideal points are used as reference points in Algorithm 1, the absolute values in (5) can be omitted. Hence, after replacing the value of the weighted Tchebycheff norm in (5) by an additional variable $\lambda \in \mathbb{R}$, we obtain the linear program

$$\begin{aligned} \min \quad & \lambda \\ \text{s.t.} \quad & \lambda \ge w_i(z_i - s_i), \quad i = 1,2 \\ & z \in Z \end{aligned} \tag{23}$$

in the first stage, and the linear program (6) (with $k=2$) in the second stage of the lexicographic weighted Tchebycheff method.

Different from the weighted Tchebycheff problem, the absolute values in the augmented weighted Tchebycheff method (7) cannot be omitted in combination with local ideal points in the context of Algorithm 1. The reason for this is that dropping the absolute values in (7) may result in the generation of nondominated points that are located outside the box $B(z^1,z^2)$ defined by $z^1$ and $z^2$.

This is illustrated in Fig. 6(a), where in the case of an augmented weighted Tchebycheff scalarization (with the absolute values omitted) the point $z^3 \notin B(z^1,z^2)$ is found whereas the point $z^4 \in B(z^1,z^2)$ is not optimal. Fig. 6(b) shows that this problem does not occur in the case of the weighted Tchebycheff scalarization.

We present two different approaches to overcome this problem. One possibility consists in modeling the absolute values in (7) by introducing two artificial variables $\mu_i \in \mathbb{R}$, $i=1,2$, and four additional constraints (see, e.g., [20]). Thereby we obtain the linear program

$$\begin{aligned} \min \quad & \lambda + \rho(\mu_1 + \mu_2) \\ \text{s.t.} \quad & \lambda \ge w_i \mu_i, \quad i = 1,2, \\ & \mu_i \ge z_i - s_i, \quad i = 1,2, \\ & \mu_i \ge -(z_i - s_i), \quad i = 1,2, \\ & z \in Z. \end{aligned} \tag{24}$$

An alternative approach consists in restricting the search to the desired box by adding either the two constraints

$$z_1 \le z_1^2 \quad \text{and} \quad z_2 \le z_2^1 \tag{25}$$

or

$$z_1 \ge z_1^1 \quad \text{and} \quad z_2 \ge z_2^1 \tag{26}$$

to the relaxed problem

$$\begin{aligned} \min \quad & \lambda + \rho \sum_{j=1}^{2} (z_j - s_j) \\ \text{s.t.} \quad & \lambda \ge w_i(z_j - s_j), \quad i = 1,2 \\ & z \in Z. \end{aligned} \tag{27}$$

Note that if the global ideal point is used as reference point in (7), there cannot exist any nondominated point outside the box defined by two arbitrary nondominated points of problem (1). Thus, in this case, (27) can be directly used in Algorithm 1 without the extension by (25) or (26).

### 2.8. *Parameters of the augmented* ε-constraint *method*

For comparison we have additionally implemented the augmented ε-constraint method since, similar to the augmented weighted Tchebycheff method, the integrality of the considered problems can be exploited to determine appropriate parameters that guarantee the complete determination of the nondominated set. The formulation is based on the work of Özpeynirci and Köksalan [16], slightly modified to adaptively computed augmentation terms in analogy to the augmented weighted Tchebycheff method. We start from the classic ε-constraint formulation in the form $\min\{z_2 : z_1 \le \varepsilon, z \in Z\}$ with $\varepsilon = z_1^2$, i.e., we restrict the first component of the solution $z$ and minimize with respect to the second component. In order to avoid weakly nondominated points, we replace the objective function by a weighted sum including both objectives:

$$\begin{aligned} \min \quad & w z_1 + (1-w) z_2 \\ \text{s.t.} \quad & z_1 \le z_1^2 \\ & z \in Z. \end{aligned} \tag{28}$$

The weight $w \in (0,1)$ is computed such that the points $(s_1, z_2^1)^\top$ and $\bar{z}$ lie on the same level curve, see Fig. 7. A short computation yields $w := 1/(z_1^2 - s_1)$. If the reference point $s$ is a (local) ideal point, we slightly change the values of the weight by a small constant $\eta \in (0,1)$, in order to make sure that any nondominated point in $B(z^1,z^2)$ has a strictly smaller objective function value than $z^1$. This is, for example, achieved by constructing the level curve such that it passes through $(\bar{z}_1, \bar{z}_2 + \eta)^\top$, which results in $w := (1-\eta)/(z_1^2 - s_1 - \eta)$.

## 3. Computational results

To compare the performance of different variations of weighted Tchebycheff and augmented ε-constraint scalarizations and to analyze the effects of adaptively chosen parameters in augmented weighted Tchebycheff scalarizations, we perform numerical tests by applying Algorithm 1 with different scalarized subproblems and using different strategies for the selection of the reference points. Thereby, we are particularly interested in the question which of the presented methods generate the complete nondominated set. Furthermore, the computational times are

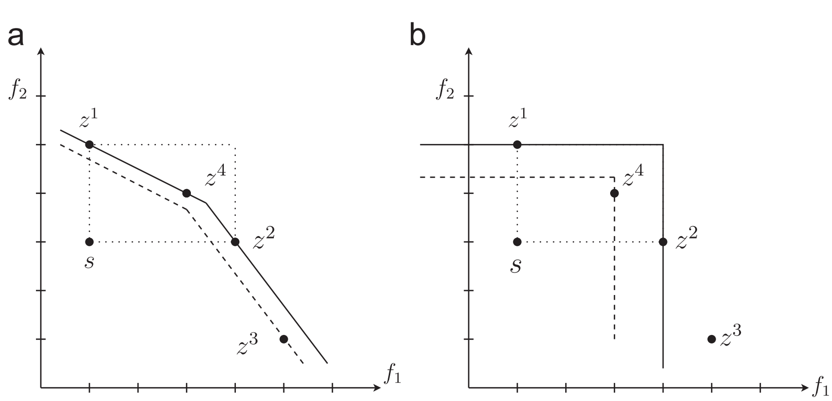


**Fig. 6.** Example in which a nondominated point $z^4$ is (a) missed when solving the augmented weighted Tchebycheff problem with a local ideal point neglecting absolute values and (b) generated when solving a weighted Tchebycheff problem with a local ideal point.

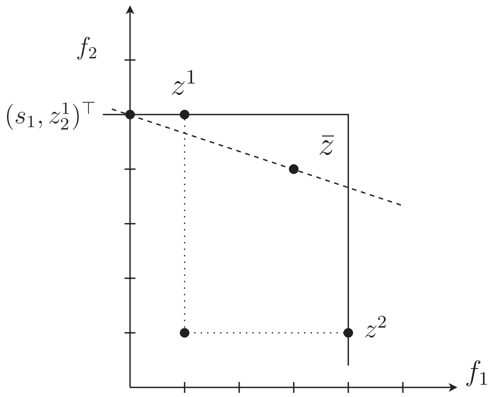


**Fig. 7.** Contour lines of an augmented $\varepsilon$-constraint problem.

compared. Randomly generated instances of bicriteria knapsack problems are used as example problems. In Section 3.1 below, the numerical setup is specified and the used formulation of the bicriteria knapsack problem is formally introduced. The computational results are discussed in Section 3.2.

### 3.1. Computational setup

We consider bicriteria knapsack problems as a well-studied example for an integer-valued, discrete bicriteria optimization problem with non-negative objective values. To be consistent with problem formulation (1), we formulate the problem as a minimization problem with objective coefficients (costs) $c^1, c^2 \in \mathbb{Z}^n_+$, constraint vector (profit) $a \in \mathbb{Z}^n_+$ and minimum profit requirement $b \in \mathbb{Z}_+$:

$$\begin{aligned} \min \quad & f(x) = [c^1x, c^2x] \\ \text{s.t.} \quad & ax \geq b \\ & x_l \in \{0,1\}, \quad l = 1, \ldots, n. \end{aligned} \tag{29}$$

Note that the bicriteria knapsack problem has also been used as a test problem for Tchebycheff methods in Ralphs et al. [17] and Sayin and Kouvelis [18], among others. Similar to these references, we do not aim at outperforming specialized algorithms for the determination of the nondominated set of the bicriteria knapsack problem with respect to computational times. Algorithm 1 is rather used as a general framework to compare the performance of different Tchebycheff scalarizations. For these reasons, a standard LP solver is sufficient for the solution of the respective subproblems. The augmented $\varepsilon$-constraint method shares the same theoretical properties as the augmented weighted Tchebycheff method and is thus included in the comparison. Note, however, that other qualities, for example, related to preference modeling and decision analysis, may be the main arguments in favor of one or the other class of scalarization method.

The test problems are generated using the approach and the code of Sayin and Kouvelis [18]. In this approach, the coefficients $c$ and $a$ are randomly drawn integers from the interval [1,1000], following a uniform distribution. The minimum profit $b$ of a feasible knapsack is set to $b = 0.5 \cdot \sum_{i=1}^n a_i$, rounded to the nearest integer. We consider five different problem sizes $n \in \{50, 75, 100, 125, 150\}$, where $n$ denotes the total number of available items. For every problem size $n$, 30 different instances are evaluated. Note that the instances belonging to one problem size differ in the number of nondominated points.

In the augmented weighted Tchebycheff subproblems, the weights for all methods with a fixed value of $\rho$ are computed as described in Lemma 2.1. When $\rho$ is adaptively chosen, the parameters are determined as specified in Table 2. The scalar $u$ (see Section 2.5) is set to 0.1 by default. In order to use a common parameter for both, the augmented weighted Tchebycheff method and the augmented $\varepsilon$-constraint method (see Section 2.8), we set $\eta := 1-u$ for the augmented weighted Tchebycheff method. Thus, $\eta$ is set to 0.9 by default if not specified differently.

Two series of experiments are presented. In a first test series, we compare the augmented weighted Tchebycheff method with adaptively chosen parameters (AAWT) to the augmented weighted Tchebycheff method with a priori chosen values of $\rho$, where we set $\rho$ to $10^{-2}$,$10^{-3}$ and $10^{-4}$, respectively, as proposed by Steuer [20]. We denote the resulting problems by $P_{0.01}$, $P_{0.001}$ and $P_{0.0001}$, respectively. Furthermore, we compare our adaptive augmented weighted Tchebycheff method to the augmented $\varepsilon$-constraint method ($\varepsilon$-constr). The two-stage or lexicographic weighted Tchebycheff method (TS) is used as reference method, i.e., we compare all methods with TS in terms of the number of determined nondominated points and computational time. Note that we exactly compute the complete nondominated set with the help of the two-stage approach. We confirmed this additionally with the help of a dynamic programming algorithm based on [10]. We also verified that whenever a lower number of points was found by the respective method, it is always a strict subset of the nondominated set.

In a second test series, we investigate the impact of using local ideal points as reference points in the augmented weighted Tchebycheff method. In a first step, we compare the different reformulations of the absolute values described in Section 2.6 with respect to CPU time. In a second step, we test an improved approach that still uses local ideal points but replaces the absolute values only when needed, i.e., whenever nondominated points outside the set $B(z^1,z^2)$ may potentially be generated. Dependent on the parameter $\eta$, we compare CPU time and average values of $\rho$.

For better comparison we always state absolute values for TS and relative deviations from the respective value of TS for all other methods. All figures represent averages over $K := 30$ instances, where the average relative deviations are computed as follows: Let $v_{n,k,M}$ be a considered value (e.g., CPU time) of problem size $n$, instance $k$ and method $M$ different from TS. Then

$$V_{n,M} := \frac{1}{K}\sum_{k=1}^{K}\frac{v_{n,k,M}-v_{n,k,TS}}{v_{n,k,TS}}.$$

We additionally indicate standard deviations whenever useful in order to help to understand the relevance of the observed differences in CPU time or the values of $\rho$. The standard deviations are calculated as

$$\sigma_{n,M} := \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left(\frac{v_{n,k,M}-v_{n,k,TS}}{v_{n,k,TS}}-V_{n,M}\right)^2}$$

for all problem sizes $n$ and methods $M$.

The computational platform for our study is a compute server with 4x Intel Xeon E7540 CPUs (2.0 GHz) and 128 GB of memory. Algorithm 1 is implemented in C++, and as subproblem solver we use CPLEX 11.2.0. We turned off the option of CPLEX to parallelize due to much longer computational times for the small instances.

### 3.2. Results

A direct comparison of the different scalarization approaches used in Ralphs et al. [17] (TS, $P_{0.01}$, $P_{0.001}$, $P_{0.0001}$) and Özpeynirci and Köksalan [16] ($\varepsilon$-constr) with AAWT embedded in an analogous, reimplemented approximation algorithm (see Section 2.6) is given in Tables 3 and 4. Table 3 shows the average number of nondominated points found, and Table 4 shows the average CPU time needed by the respective method.

In Table 3 we observe that the adaptive augmented weighted Tchebycheff method (AAWT) finds the same number of nondominated points as the two-stage method (TS) for all test instances (deviation of 0%). However, this does not hold for the methods in which $\rho$ is fixed a priori: For $\rho = 0.01$ and $\rho = 0.001$, for each problem size, a certain percentage of nondominated points is missed. For example, for $\rho = 0.01$ and $n=50$, on average 6.2% of the nondominated points is missed. When setting $\rho = 0.0001$, for problem sizes $n=50$ and $n=75$, the entire set of nondominated points is found, whereas for larger problem sizes also some nondominated points are missed. This shows the difficulty when using a fixed value for the parameter $\rho$, as already discussed in Section 1.2: Depending on the data of the given problem, a fixed value of $\rho$ may not be appropriate (i.e., too large) for the generation of all nondominated points. In contrast, the complete nondominated set is computed with an adaptive calculation of $\rho$ based on the given problem data.

We note for the $\varepsilon$-constraint method, that, for problem size $n=150$, 0.01% of the nondominated points is missed, which corresponds to one missing solution in one test instance. As the $\varepsilon$-constraint method is formulated as adaptive method, this solution should theoretically have been found and is probably missed for numerical reasons. It could be generated when an additional $\varepsilon$-constraint was set on the first objective. This, in turn, led to much higher computational times of this method.

In Table 4 we show the respective average CPU times and standard deviations. For example, for $n=50$, TS requires on average 1.668 s with a standard deviation of 0.865 s. The adaptive augmented weighted Tchebycheff method requires on average 43.3% less CPU time than the two-stage approach, the respective standard deviation amounts to 3.6%. Analyzing computational time in Table 4 in general, it can be observed that for all problem sizes the methods with augmentation term require more than 43% less CPU time than the two-stage method. This observation reflects the fact that in the latter case, two optimization problems have to be solved for each subproblem and thus around twice the time is needed in general. Note that we did not pass the solution from the first stage to the second stage as it took more time to solve the specific problem in practice.

Comparing computational times among the different methods with augmentation term is only meaningful for those methods that find all nondominated points. We deduce from Tables 3 and 4 that the solution times of all augmented methods which generate the entire set of nondominated points differ only slightly. It seems that the adaptive update of the parameter does not have a negative impact with respect to computational time. Comparing the augmented weighted Tchebycheff method with the augmented $\varepsilon$-constraint method, we state that no method outperforms the other. This is indeed surprising, as the augmented $\varepsilon$-constraint method has a simpler formulation (only one constraint) than the augmented weighted Tchebycheff method (two constraints) and, thus, can be expected to be solved faster.

In the second test series we address the question whether we can obtain better (i.e., larger) values for $\rho$ for AAWT without impairing the CPU times recorded in Table 4. Therefore, we replace the ideal point by local ideal points, individually chosen for each subproblem. As mentioned in Section 2.6, this implies the necessity to take the absolute values in problem (7) into account. In the method $P_{avc}$ we include the absolute values explicitly by introducing additional variables and constraints as specified in (24).

In the methods $P_{ubc}$ and $P_{lbc}$ we omit the absolute values but add upper box constraints (25) and lower box constraints (26), respectively. Since all three methods are equivalent in the sense that they reliably generate all nondominated points, we only report average CPU times in Table 5 and omit the (equal) numbers of determined nondominated points. As before, we show the deviations in CPU time of the three formulations as compared to the two-stage approach. Only method $P_{ubc}$ performs better than TS for all problem sizes. The other two methods even consumed more CPU time than TS for all problem sizes except $P_{lbc}$ for $n=50$. But also the CPU times of $P_{ubc}$, compared to the CPU times of AAWT in Table 4, show a clear impairment (gain with respect to TS is less than 30%). We conclude that the additional variables and constraints seem to make the augmented problem more difficult to solve.

In order to avoid as much as possible of the additional computational burden induced by the linearizations of the absolute values in (7), we additionally implemented an alternative approach, where these reformulations are only used when necessary. Indeed, the absolute values in (7) are only needed if there exists a nondominated point that is located outside the box $B(z^1,z^2)$ and that minimizes the augmented weighted Tchebycheff scalarization with absolute values omitted (cf. Fig. 6(a)). Instead of introducing additional variables and/or constraints in every subproblem, we may also omit the absolute values, i.e. solve problem (27), and check afterwards whether the solution satisfies the box constraints. If the solution lies in the box $B(z^1,z^2)$, we can proceed as usual and turn to the next subproblem. If,

**Table 3**
Relative deviation of the number of nondominated points found by different augmented weighted Tchebycheff methods with respect to TS (absolute values).

| $n$ | TS | AAWT | $P_{0.01}$ | $P_{0.001}$ | $P_{0.0001}$ | $\varepsilon$-Constr |
|---|---|---|---|---|---|---|
| 50 | 46.43 | 0 | −0.0622 | −0.0066 | 0 | 0 |
| 75 | 91.57 | 0 | −0.0695 | −0.0055 | 0 | 0 |
| 100 | 150.60 | 0 | −0.0748 | −0.0051 | −0.0004 | 0 |
| 125 | 225.23 | 0 | −0.0770 | −0.0056 | −0.0005 | 0 |
| 150 | 341.77 | 0 | −0.0695 | −0.0054 | −0.0004 | −0.0001 |

**Table 4**
Absolute CPU time of TS and relative deviation of CPU time (with respect to TS) of different augmented weighted Tchebycheff methods. The second row contains the respective standard deviations.

| $n$ | | TS | AAWT | $P_{0.01}$ | $P_{0.001}$ | $P_{0.0001}$ | $\varepsilon$-Constr |
|---|---|---|---|---|---|---|---|
| 50 | CPU | 1.668 | −0.433 | −0.458 | −0.435 | −0.435 | −0.450 |
| | $\sigma$ | 0.865 | 0.036 | 0.045 | 0.037 | 0.037 | 0.054 |
| 75 | CPU | 5.651 | −0.471 | −0.506 | −0.473 | −0.472 | −0.475 |
| | $\sigma$ | 3.278 | 0.027 | 0.024 | 0.027 | 0.027 | 0.048 |
| 100 | CPU | 13.889 | −0.490 | −0.526 | −0.491 | −0.491 | −0.488 |
| | $\sigma$ | 6.182 | 0.025 | 0.030 | 0.025 | 0.026 | 0.054 |
| 125 | CPU | 30.734 | −0.497 | −0.537 | −0.498 | −0.497 | −0.485 |
| | $\sigma$ | 15.880 | 0.024 | 0.031 | 0.025 | 0.024 | 0.064 |
| 150 | CPU | 67.227 | −0.496 | −0.539 | −0.499 | −0.499 | −0.511 |
| | $\sigma$ | 26.807 | 0.024 | 0.028 | 0.024 | 0.024 | 0.048 |

**Table 5**
Absolute CPU time of TS and relative deviation of CPU time (with respect to TS) of different augmented weighted Tchebycheff methods. Additionally, the respective standard deviations are given.

| $n$ | TS | $\sigma$ | $P_{avc}$ | $\sigma$ | $P_{ubc}$ | $\sigma$ | $P_{lbc}$ | $\sigma$ |
|---|---|---|---|---|---|---|---|---|
| 50 | 1.668 | 0.8654 | 0.073 | 0.1216 | −0.281 | 0.0455 | −0.048 | 0.1268 |
| 75 | 5.651 | 3.2778 | 0.458 | 0.2000 | −0.246 | 0.0476 | 0.301 | 0.1724 |
| 100 | 13.889 | 6.1824 | 0.747 | 0.1939 | −0.225 | 0.0421 | 0.568 | 0.1520 |
| 125 | 30.734 | 15.8798 | 0.951 | 0.2039 | −0.210 | 0.0334 | 0.754 | 0.1707 |
| 150 | 67.227 | 26.8068 | 1.089 | 0.1226 | −0.177 | 0.0270 | 0.850 | 0.1100 |

**Table 6**
Absolute CPU time of TS and relative deviation of CPU time (with respect to TS) of different augmented weighted Tchebycheff methods with different scaling of $\rho$. Additionally, the respective standard deviations are given.

| $n$ | TS | $\sigma$ | $\eta=0.9$ | | | | $\eta=0.1$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | AAWT | $\sigma$ | $P_{comb}$ | $\sigma$ | AAWT | $\sigma$ | $P_{comb}$ | $\sigma$ |
| 50 | 1.668 | 0.865 | −0.433 | 0.036 | −0.416 | 0.036 | −0.432 | 0.041 | −0.446 | 0.039 |
| 75 | 5.651 | 3.278 | −0.471 | 0.027 | −0.430 | 0.035 | −0.471 | 0.027 | −0.479 | 0.026 |
| 100 | 13.889 | 6.182 | −0.490 | 0.025 | −0.423 | 0.031 | −0.490 | 0.026 | −0.494 | 0.025 |
| 125 | 30.734 | 15.880 | −0.497 | 0.024 | −0.427 | 0.028 | −0.496 | 0.023 | −0.501 | 0.024 |
| 150 | 67.227 | 26.807 | −0.496 | 0.024 | −0.408 | 0.027 | −0.498 | 0.025 | −0.495 | 0.024 |

**Table 7**
Comparison of average values of $\rho$ (with standard deviation $\sigma$).

| $n$ | $\eta=0.9$ | | | | $\eta=0.1$ | | | |
|---|---|---|---|---|---|---|---|---|
| | AAWT | $\sigma$ | $P_{comb}$ | $\sigma$ | AAWT | $\sigma$ | $P_{comb}$ | $\sigma$ |
| 50 | **0.00054** | 0.00011 | 0.01040 | 0.00396 | 0.00006 | 0.00001 | **0.00112** | 0.00039 |
| 75 | **0.00039** | 0.00007 | 0.01398 | 0.00394 | 0.00004 | 0.00001 | **0.00150** | 0.00040 |
| 100 | **0.00030** | 0.00004 | 0.01810 | 0.00410 | 0.00003 | 0.00001 | **0.00192** | 0.00039 |
| 125 | **0.00024** | 0.00004 | 0.02033 | 0.00379 | 0.00003 | 0.00001 | **0.00217** | 0.00037 |
| 150 | **0.00019** | 0.00003 | 0.02401 | 0.00360 | 0.00002 | 0.00000 | **0.00254** | 0.00035 |

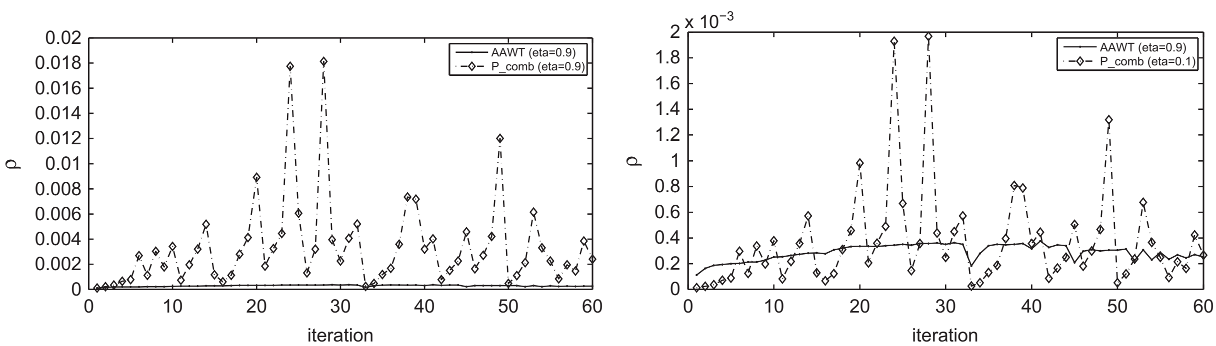


**Fig. 8.** Exemplary development of $\rho$ for one test instance ($n=50$, $N=32$).

however, the solution lies outside the box $B(z^1,z^2)$, we insert the solution (if it is not already contained in *ND*). Then, we repeat the search for nondominated points in the same box, but this time by solving problem $P_{ubc}$, i.e. by explicitly including the box constraints in the problem formulation.

The average CPU times for this alternative approach, denoted by $P_{comb}$, are given in Table 6 in relation to the two-stage approach. For better comparison, Table 6 also contains the data for the AAWT method using the global ideal point. For TS, the global ideal point is used, and $\eta$ is set to $\eta=0.9$ and $\eta=0.1$, respectively.

For $\eta=0.9$, AAWT clearly outperforms $P_{comb}$ for all problem sizes. However, for $\eta=0.1$, $P_{comb}$ performs as good as AAWT. This can be explained as follows: The smaller $\eta$ is, the less likely it is that solutions of (27) lie outside the box $B(z^1,z^2)$, since then the contour of the corresponding augmented weighted Tchebycheff norm is only slightly lifted as compared to the weighted Tchebycheff norm without the augmentation term. Having fewer solutions outside of $B(z^1,z^2)$ implies that fewer of the (computationally more expensive) problems with additional box constraints have to be solved. In the limit, i.e., if no solution lies outside the corresponding box, a computational time similar to AAWT is obtained.

The advantage when using local ideal points instead of the global ideal point is that we can expect that, on average, larger values of $\rho$ in the augmented weighted Tchebycheff subproblems are obtained. This can be observed in Table 7, where values of $\rho$ (averaged over all instances and over all subproblems for each problem size) for AAWT and $P_{comb}$ with $\eta=0.1$ and $\eta=0.9$ are given. It is interesting to note that even for $P_{comb}$ with $\eta=0.1$, larger average values for $\rho$ are obtained than for AAWT with $\eta=0.9$ (the respective columns are highlighted in bold).

Taking into account that the computational times are nearly the same for both variants (see Table 6), this indicates that larger average values for $\rho$, and thus a probably numerically more stable method, are obtained with local ideal points.

Fig. 8 shows an exemplary development of $\rho$ for one selected instance of problem size $n=50$. The value of $\rho$ is plotted for the consecutively solved subproblems. On the left we see that, for all subproblems, larger values of $\rho$ can be achieved when using local ideal points ($P_{comb}$) instead of global ideal points (AAWT) and the same choice of $\eta$ ($=0.9$ for both methods). However, the larger values of $\rho$ imply higher computational times, see Table 6. For both methods shown on the chart on the right, computational times are nearly equal. The values of $\rho$ achieved with the local version ($P_{comb}$) with $\eta=0.1$ are partially higher, partially lower than those of the global version (AAWT) with large $\eta=0.9$, but on average higher, see Table 7.

### 3.3. Summary

We can conclude that the augmented weighted Tchebycheff method with adaptively chosen parameters reliably finds all nondominated points without impairing CPU time with respect to the other presented methods with augmentation terms. The augmented $\varepsilon$-constraint method does not outperform the adaptive augmented weighted Tchebycheff method in general in our test problems. We also tested a variant of the adaptive augmented weighted Tchebycheff method that uses local ideal points. Depending on the parameter $\eta$, we observed a trade-off between CPU time and the average value of $\rho$. When a rather small inclination of the slopes of the level curve ($\eta = 0.1$) is used, similar CPU times as compared to the method using the global ideal point are achieved while getting a larger average value for $\rho$.

## 4. Conclusion

Focusing on bicriteria and discrete problems, we presented explicit formulas to optimally and adaptively determine the parameters defining the augmented weighted Tchebycheff scalarization with respect to the data of a given problem instance. With this approach, numerical difficulties arising from unfavorable parameter values can be avoided, enhancing, for example, iterative generation methods based on augmented weighted Tchebycheff scalarizations. The results of numerical tests validate our theoretical findings from a practical point of view.

Generalizations to multiple criteria problems are under investigation. Moreover, future research should include continuous problems where trade-off information could be used to compute appropriate parameter values. An interesting variation of the augmented weighted Tchebycheff scalarization is obtained when different parameters $\rho_i$, $i=1,\dots,k$, are allowed in the augmentation term. Whether this has a considerable effect on the computational efficiency of corresponding solution methods should be carefully analyzed.

## Acknowledgments

We thank Serpil Sayin for kindly providing us the code to generate the test instances that we used in Section 3 as well as the anonymous reviewers for helpful comments.

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