COMMUNICATIONS IN ALGEBRA®
2021, VOL. 49, NO. 4, 1717–1724
https://doi.org/10.1080/00927872.2020.1849249



# When is a matrix a sum of involutions or tripotents?

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**ABSTRACT**

We give the necessary and sufficient conditions for an $n\times n$ matrix over an integral domain to be a sum of involutions and, respectively, a sum of tripotents. We determine the integral domains over which every $n\times n$ matrix is a sum of involutions and, respectively, a sum of tripotents. We further determine the commutative reduced rings over which every $n\times n$ matrix is a sum of two tripotents.

**ARTICLE HISTORY**
Received 1 August 2019
Revised 17 October 2020
Communicated by Toma Albu

**KEYWORDS**
Decomposition of matrices; involution; tripotent

**2000 MATHEMATICS SUBJECT CLASSIFICATION**
Primary 15A24; 16S50

## 1. Introduction

Decomposing a matrix into a sum of certain special matrices has been an active topic (see [1–5, 9–11, 13–15]). For instance, Hartwig and Putcha [5] showed that, an $n\times n$ matrix $A$ over a field $F$ of characteristic 0 is a sum of idempotents iff $\mathrm{tr}(A)=k\cdot 1_F,$ where $k\in\mathbb{Z}$ and $k\geq \mathrm{rank}(A),$ and de Seguins Pazzis [4] obtained that an $n\times n$ matrix $A$ over a field $F$ of characteristic $p>0$ is a sum of idempotents iff $\mathrm{tr}(A)\in\mathbb{F}_p,$ the prime subfield of $F$. One of the motivations here is the work by Merino, Paras, and Pelejo in [9], where the authors presented the necessary and sufficient conditions for an $n\times n$ matrix over $K$ (for $K=\mathbb{C},\mathbb{R},\mathbb{Q},\mathbb{Z},\mathbb{Z}_k$) to be a sum of involutions. Recall that an involution (or an involutory element) in a ring is an element whose square equals 1.

In Section 2, as an extension of the work in [9], the necessary and sufficient conditions for an $n\times n$ matrix over an integral domain to be a sum of involutions are obtained, and consequently, the integral domains $R$ are completely determined so that every $n\times n$ matrix over $R$ is a sum of involutions. As a natural common generalization of idempotents and involutions, an element $a$ in a ring is called a tripotent if $a^3=a$. In Section 3, we further give a necessary and sufficient condition for a square matrix over an integral domain to be a sum of tripotents, and determine the integral domains $R$ over which every $n\times n$ matrix is a sum of tripotents. In [6], Hirano and Tominaga showed that, for any nontrivial ring $R$ and any $n\geq 2,$ not every $n\times n$ matrix over $R$ is a sum of two idempotents. In [14], the authors observed that, for any nontrivial ring $R$ and any $n\geq 2,$ not every $n\times n$ matrix over $R$ is a sum of two involutions. In Section 4, we are motivated to consider the question: For which rings $R$, is every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents? We prove that, for a commutative reduced ring $R$, every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents iff $x^5=x$ for all $x\in R$. In particular,

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for an integral domain $R$, every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents iff $R \cong \mathbb{Z}_p$ where $p = 2, 3$ or 5.

In this paper, rings are associative with identity. For a ring $R$, the set of nilpotent elements and the characteristic of $R$ are denoted by $\mathrm{Nil}(R)$ and $\mathrm{ch}(R)$, respectively. We write $\mathbb{Z}_n$ for the ring of integers modulo $n$, $\mathbb{M}_n(R)$ for the ring of $n \times n$ matrices over $R$, $E_{ij}$ for the $n \times n$ matrix which has $(i, j)$-entry as 1 and 0 elsewhere, and $I_n$ for the $n \times n$ identity matrix. The trace of a square matrix $A$ is denoted by $\mathrm{tr}(A)$. A ring $R$ is reduced if $\mathrm{Nil}(R) = 0$.

## 2. Sums of involutions

We start with some useful observations on matrices over a general ring. The first three statements in Lemma 2.1 are from [9].

**Lemma 2.1.** *Let $R$ be a ring with $a \in R$ and $n \geq 2$. The following hold in $\mathbb{M}_n(R)$.*

(1) *If $i \neq j$, then $aE_{ij} = (I + aE_{ij} - 2E_{jj}) + (2E_{jj} - I)$, a sum of two involutions.*
(2) *If $k$ is even, then $kE_{11} = \frac{k}{2}I + \frac{k}{2}(2E_{11} - I)$ where $I$ and $2E_{11} - I$ are involutions.*
(3) *If $n$ is odd, then*

$$E_{11} = \begin{pmatrix} 1 & & & & & \\ & 0 & 1 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & & & \\ & 1 & -1 & & & \\ & 0 & -1 & & & \\ & & & \ddots & & \\ & & & & 1 & -1 \\ & & & & 0 & -1 \end{pmatrix} + \begin{pmatrix} -1 & & & & & \\ & -1 & 0 & & & \\ & -1 & 1 & & & \\ & & & \ddots & & \\ & & & & -1 & 0 \\ & & & & -1 & 1 \end{pmatrix}$$

*is a sum of three involutions.*

(4) *The matrix* $A := \begin{pmatrix} a & 0 & & & \\ 0 & -a & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}$ *is a sum of involutions.*

*Proof.* The statements (1)–(3) are directly verified.

(4) Setting $X := \begin{pmatrix} a-1 & 2a-a^2 & & & \\ 1 & 1-a & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}$, we have

$$A = (I - 2E_{22}) + X - E_{21} + a(a-2)E_{12},$$

where $I - 2E_{22}$ and $X$ are involutions and $-E_{21} + a(a-2)E_{12}$ is a sum of involutions by (1). □

**Proposition 2.2.** *Let $R$ be a ring and $A \in \mathbb{M}_n(R)$ with $\mathrm{tr}(A) = k \cdot 1_R$ for some $k \in \mathbb{Z}$.*

(1) *If $k$ is even, then $A$ is a sum of involutions.*
(2) *If $n$ is odd, then $A$ is a sum of involutions.*

*Proof.* We can assume that $n \geq 2$. Write $A = (a_{ij})$, so $k \cdot 1_R = a_{11} + \cdots + a_{nn}$. Then

$$A = kE_{11} + \sum_{i=2}^{n}(-a_{ii}E_{11} + a_{ii}E_{ii}) + \sum_{i \neq j} a_{ij}E_{ij}.$$

For each $i \geq 2$, $-a_{ii}E_{11} + a_{ii}E_{ii}$ is similar to $-a_{ii}E_{11} + a_{ii}E_{22}$, which is a sum of involutions by Lemma 2.1(4). Hence, $\sum_{i=2}^{n}(-a_{ii}E_{11} + a_{ii}E_{ii})$ is a sum of involutions. Moreover, by Lemma 2.1(1), $\sum_{i \neq j} a_{ij}E_{ij}$ is a sum of involutions. So, to show that $A$ is a sum of involutions, it suffices to show that $kE_{11}$ is a sum of involutions.

(1) If $k$ is even, then $kE_{11}$ is a sum of involutions by Lemma 2.1(2); so (1) holds.
(2) As $n$ is odd, $E_{11}$ is a sum of involutions by Lemma 2.1(3), so $kE_{11}$ is a sum of involutions. Hence, (2) holds. □

The proof of Proposition 2.2 actually implies the following.

**Corollary 2.3.** *Let $R$ be a ring and $n \geq 1$. The following are equivalent.*

(1) *$E_{11} \in \mathbb{M}_n(R)$ is a sum of involutions.*
(2) *Every matrix $A \in \mathbb{M}_n(R)$ with $\operatorname{tr}(A) \in \mathbb{Z} \cdot 1_R$ is a sum of involutions.*

We next focus on matrices over an integral domain.

**Lemma 2.4.** *Let $R$ be an integral domain. If $A^2 = I$ in $\mathbb{M}_n(R)$, then $\operatorname{tr}(A) = k \cdot 1_R$ where $k \in \mathbb{Z}$ with $n \equiv k \pmod 2$.*

*Proof.* Let $Q$ be the field of quotients of $R$. Then, by [9, Proposition 4], $\operatorname{tr}(A) = k \cdot 1_Q$ where $k \in \mathbb{Z}$ with $n \equiv k \pmod 2$. As $1_R = 1_Q, \operatorname{tr}(A) = k \cdot 1_R$. □

**Theorem 2.5.** *Let $R$ be an integral domain and $A \in \mathbb{M}_n(R)$.*

(1) *If $n$ is even, then $A$ is a sum of involutions iff $\operatorname{tr}(A) \in 2\mathbb{Z} \cdot 1_R$.*
(2) *If $n$ is odd, then $A$ is a sum of involutions iff $\operatorname{tr}(A) \in \mathbb{Z} \cdot 1_R$.*

*Proof.* The necessity follows from Lemma 2.4, and the sufficiency is by Proposition 2.2. □

**Corollary 2.6.** *Let $R$ be an integral domain and $n \geq 1$.*

(1) *If $n$ is even, then every matrix in $\mathbb{M}_n(R)$ is a sum of involutions iff $R \cong \mathbb{Z}_p$ where $p > 2$ is a prime.*
(2) *If $n$ is odd, then every matrix in $\mathbb{M}_n(R)$ is a sum of involutions iff $R \cong \mathbb{Z}$ or $R \cong \mathbb{Z}_p$ where $p \geq 2$ is a prime.*

*Proof.* (1) By Theorem 2.5, every matrix in $\mathbb{M}_n(R)$ is a sum of involutions iff $R = 2\mathbb{Z} \cdot 1_R$, iff 2 is a unit of $R$ and $R = \mathbb{Z} \cdot 1_R$, iff 2 is a unit of $R$ and $R$ is an image of $\mathbb{Z}$, iff $R \cong \mathbb{Z}_p$ where $p > 2$ is a prime.

(2) Simillar to (1). □

## 3. Sums as tripotents

As a common generalization of idempotents and involutions, an element $a$ in a ring is called a tripotent if $a^3 = a$. The lemma below is an extension of Lemma 2.4.

**Lemma 3.1.** *Let $R$ be an integral domain. If $A^3 = A \in \mathbb{M}_n(R)$, then $\mathrm{tr}(A) \in \mathbb{Z} \cdot 1_R$.*

*Proof.* We embed the integral domain $R$ into its field of fractions $Q$, and further into the algebraic closure $\bar{Q}$. From equality $A^3 = A$, we deduce that each eigenvalue $\lambda \in \bar{Q}$ of $A$ satisfies $\lambda^3 = \lambda$, hence it must be $\lambda = 0, 1$ or $-1$. Since $\mathrm{tr}(A)$ is the sum of all eigenvalues of $A$ in $\bar{Q}$ counted with multiplicities, it follows that $\mathrm{tr}(A) = k \cdot 1_{\bar{Q}} = k \cdot 1_R$ for some $k \in \mathbb{Z}$. □

**Theorem 3.2.** *Let $R$ be an integral domain and $n \geq 1$. Then $A \in \mathbb{M}_n(R)$ is a sum of tripotents iff $\mathrm{tr}(A) \in \mathbb{Z} \cdot 1_R$.*

*Proof.* The necessity is by Lemma 3.1. For the sufficiency, let $A \in \mathbb{M}_n(R)$ with $\mathrm{tr}(A) = k \cdot 1_R$ for some $k \in \mathbb{Z}$. If $k$ is even, then $A$ is a sum of involutions by Proposition 2.2. If $k$ is odd, then $\mathrm{tr}(A - E_{11}) = (k-1) \cdot 1_R$ with $k - 1$ even, so $A - E_{11}$ is a sum of involutions, and hence $A = E_{11} + (A - E_{11})$ is a sum of an idempotent and involutions. □

**Corollary 3.3.** *Let $R$ be an integral domain and $n \geq 1$. Then every matrix in $\mathbb{M}_n(R)$ is a sum of tripotents iff $R \cong \mathbb{Z}$ or $R \cong \mathbb{Z}_p$ where $p \geq 2$ is a prime.*

**Corollary 3.4.** *Let $n \geq 1$ and $k \geq 2$. Then every matrix in $\mathbb{M}_n(\mathbb{Z}_k)$ is a sum of tripotents.*

*Proof.* Every matrix in $\mathbb{M}_n(\mathbb{Z})$ is a sum of tripotents by Corollary 3.3 and, furthermore, $\mathbb{M}_n(\mathbb{Z}_k)$ is a homomorphic image of $\mathbb{M}_n(\mathbb{Z})$. So every matrix in $\mathbb{M}_n(\mathbb{Z}_k)$ is a sum of tripotents. □

## 4. Sums of two tripotents

The following fact is implict in the proof of [16, Theorem 5.2].

**Lemma 4.1.** [16] *If 3 is a sum of two tripotents in a ring $R$, then $2^3 \cdot 3 \cdot 5 = 0$ in $R$.*

Let $R$ be any ring and $a_0, a_1, \ldots, a_{n-1}$ be elements in $R$. The matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{n-2} \\ 0 & 0 & \cdots & 1 & a_{n-1} \end{pmatrix}$$

is called the companion matrix associated with $a_0, a_1, \ldots, a_{n-1}$.

The following result is well known.

**Lemma 4.2.** [7, p. 192] *Let $K$ be a field and $n \geq 2$. Then every $A \in \mathbb{M}_n(K)$ is similar to its rational canonical form $B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_s \end{pmatrix}$, where $s \geq 1$, $B_i$ is a companion matrix of size $n_i$ and $1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$.*

**Lemma 4.3.** *Let $R$ be a ring and $n \geq 2$. For any idempotent $(n-1) \times (n-1)$ block $E$ and for any $(n-1) \times 1$ block $X$ over $R$, the $n \times n$ matrices*

$$\begin{pmatrix} E & X \\ \mathbf{0} & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -E & X \\ \mathbf{0} & 1 \end{pmatrix}$$

*are tripotent.*

*Proof.* The proof is a straightforward verification. □

**Lemma 4.4.** *Let $R$ be any ring and let $C$ be an $n \times n$ companion matrix associated with $a_0, \ldots, a_{n-1} \in R$. If $a_{n-1} \in \{-2, -1, 0, 1, 2\}$ then $C$ is a sum of two tripotents in $\mathbb{M}_n(R)$.*

*Proof.* The claim is obviously true if $n = 1$, so we are assuming $n \geq 2$. In each of the cases below, we will explicitly decompose $C = E + F$, where $E^3 = E$ is a trivial verification and $F^3 = F$ is a consequence of Lemma 4.3 for each of the cases.

**Case 1:** $n$ is even. If $a_{n-1} = 1$,

$$C = \begin{pmatrix} 1 & 0 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} -1 & & & & & & a_0 \\ & 0 & 0 & & & & a_1 \\ & 1 & -1 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & 0 & 0 & a_{n-3} \\ & & & & 1 & -1 & a_{n-2} \\ & & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = 2$,

$$C = \begin{pmatrix} 0 & 0 & & & \\ 1 & 1 & & & \\ & & \ddots & & \\ & & & 0 & 0 \\ & & & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & & & a_0 \\ & -1 & 0 & & & & a_1 \\ & 1 & 0 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & -1 & 0 & a_{n-3} \\ & & & & 1 & 0 & a_{n-2} \\ & & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = -1$,

$$C = \begin{pmatrix} -1 & 0 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & -1 & 0 \\ & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & & & & & & a_0 \\ & 0 & 0 & & & & a_1 \\ & 1 & 1 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & 0 & 0 & a_{n-3} \\ & & & & 1 & 1 & a_{n-2} \\ & & & & & & -1 \end{pmatrix}.$$

If $a_{n-1} \in \{0, -2\}$,

$$C = \begin{pmatrix} 0 & 0 & & & & & \\ 1 & -1 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 0 & & \\ & & & 1 & -1 & & \\ & & & & & 0 & 0 \\ & & & & & 1 & \pm 1 \end{pmatrix} + \begin{pmatrix} 0 & & & & & & a_0 \\ & 1 & 0 & & & & a_1 \\ & 1 & 0 & & & & a_2 \\ & & & \ddots & & & \vdots \\ & & & & 1 & 0 & a_{n-3} \\ & & & & 1 & 0 & a_{n-2} \\ & & & & & & -1 \end{pmatrix}.$$

**Case 2:** $n \geq 3$ is odd. If $a_{n-1} = 1$,

$$C = \begin{pmatrix} 0 & & & & & \\ & 1 & 0 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \\ & & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & & & & a_0 \\ 1 & -1 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & 0 & 0 & a_{n-3} \\ & & & 1 & -1 & a_{n-2} \\ & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = 2$,

$$C = \begin{pmatrix} 1 & & & & & \\ & 0 & 0 & & & \\ & 1 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 0 \\ & & & & 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 & & & & a_0 \\ 1 & 0 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & -1 & 0 & a_{n-3} \\ & & & 1 & 0 & a_{n-2} \\ & & & & & 1 \end{pmatrix}.$$

If $a_{n-1} = -1$,

$$C = \begin{pmatrix} 0 & & & & & \\ & -1 & 0 & & & \\ & 1 & 0 & & & \\ & & & \ddots & & \\ & & & & -1 & 0 \\ & & & & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & & & & a_0 \\ 1 & 1 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & 0 & 0 & a_{n-3} \\ & & & 1 & 1 & a_{n-2} \\ & & & & & -1 \end{pmatrix}.$$

If $a_{n-1} \in \{0, -2\}$,

$$C = \begin{pmatrix} -1 & & & & & & & \\ & 0 & 0 & & & & & \\ & 1 & -1 & & & & & \\ & & & \ddots & & & & \\ & & & & 0 & 0 & & \\ & & & & 1 & -1 & & \\ & & & & & & 0 & 0 \\ & & & & & & 1 & \pm 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & & & & a_0 \\ 1 & 0 & & & & a_1 \\ & & \ddots & & & \vdots \\ & & & 1 & 0 & a_{n-3} \\ & & & 1 & 0 & a_{n-2} \\ & & & & & -1 \end{pmatrix}.$$

□

**Theorem 4.5.** *Let $R$ be an integral domain and $n \geq 1$. Then every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents iff $R \cong \mathbb{Z}_p$ for $p = 2$, 3 or 5.*

*Proof.* (⇒). If every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents, then $2^3 \cdot 3 \cdot 5 = 0$ in $R$ by Lemma 4.1, so $\mathrm{ch}(R) = p$ where $p=2$, 3 or 5. Let $a \in R$. Then, $aE_{11} = E + F$ where $E^3 = E$ and $F^3 = F$. By Lemma 3.1, $a = \mathrm{tr}(aE_{11}) = \mathrm{tr}(E) + \mathrm{tr}(F) \in \mathbb{Z} \cdot 1_R$. So, $R = \mathbb{Z} \cdot 1_R \cong \mathbb{Z}_p$, with $p=2$, 3 or 5.

(⇐). The sufficiency follows from Lemmas 4.2 and 4.4. □

We next extend Theorem 4.5 from an integral domain to a commutative reduced ring.

**Lemma 4.6.** *Let $R$ be a ring and $n \geq 1$. Then every element of $\mathbb{M}_n(R)$ is a sum of two tripotents iff every element of $\mathbb{M}_n(R/I)$ is a sum of two tripotents for all indecomposable factor rings $R/I$ of $R$.*

*Proof.* The proof is similar to that of [14, Lemma 3.3]. □

Let $R$ be a ring. Note that $x^5 = x$ for all $x \in R$ iff $R = R_1 \oplus R_2 \oplus R_3$, where $R_1$ is zero or a Boolean ring, $R_2$ is zero or a subdirect product of $\mathbb{Z}_3$'s, and $R_3$ is zero or a subdirect product of $\mathbb{Z}_5$'s (see [17, Theorem 2.12]).

**Theorem 4.7.** *Let $R$ be a commutative reduced ring. Every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents iff $x^5 = x$ for all $x \in R$.*

*Proof.* Suppose that every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents. As a reduced ring, $R$ is a subdirect product of integral domains $\{R_\alpha\}$. For each $\alpha$, $\mathbb{M}_n(R_\alpha)$ is a homomorphic image of $\mathbb{M}_n(R)$, so every matrix in $\mathbb{M}_n(R_\alpha)$ is a sum of two tripotents. Thus, $R_\alpha \cong \mathbb{Z}_p$ with $p \in \{2, 3, 5\}$ by Theorem 4.5. So, $x^5 = x$ for all $x \in R_\alpha$ and hence $x^5 = x$ for all $x \in R$.

Suppose that $x^5 = x$ for all $x \in R$. Let $S$ be an indecomposable factor ring of $R$. Then, $x^5 = x$ for all $x \in S$. For any nonzero $a \in S$, $a^4$ is a nonzero idempotent, so $a^4 = 1$ and hence, $a$ is invertible. Thus, $S$ is a field of at most 5 elements, and it is easily seen that $S$ is isomorphic to either $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_5$. So, by Theorem 4.5, every matrix in $\mathbb{M}_n(S)$ is a sum of two tripotents. Thus, by Lemma 4.6, every matrix in $\mathbb{M}_n(R)$ is a sum of two tripotents. □


## Acknowledgments

The authors are indebted to the referee whose valuable comments on rewriting the proof of Lemma 3.1 and integrating several lemmas into Lemma 4.4 have greatly improved the presentation. Part of the work was carried out when Zhou was visiting Beibu Gulf University. He gratefully acknowledges the hospitality from the host institute.

## Funding

This research was supported by Natural Science Foundation of China [11661014, 11961050], Guangxi Natural Science Foundation [2016GXNSFDA380017], and a Discovery Grant [RGPIN-2016-04706] from NSERC of Canada.


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