Plug-and-Play Methods Provably Converge with Properly Trained Denoisers

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Abstract

Plug-and-play (PnP) is a non-convex framework that integrates modern denoising priors, such as BM3D or deep learning-based denoisers, into ADMM or other proximal algorithms. An advantage of PnP is that one can use pre-trained denoisers when there is not sufficient data for end-to-end training. Although PnP has been recently studied extensively with great empirical success, theoretical analysis addressing even the most basic question of convergence has been insufficient. In this paper, we theoretically establish convergence of PnP-FBS and PnP-ADMM, without using diminishing stepsizes, under a certain Lipschitz condition on the denoisers. We then propose real spectral normalization, a technique for training deep learning-based denoisers to satisfy the proposed Lipschitz condition. Finally, we present experimental results validating the theory.

1. Introduction

Many modern image processing algorithms recover or denoise an image through the optimization problem

$$\underset{x\in\mathbb{R}^d}{\text{minimize}} \quad f(x)+\gamma g(x),$$

where the optimization variable $x\in\mathbb{R}^d$ represents the image, $f(x)$ measures data fidelity, $g(x)$ measures noisiness or complexity of the image, and $\gamma\geq 0$ is a parameter representing the relative importance between f and g. Total variation denoising, inpainting, and compressed sensing fall under this setup. *A priori* knowledge of the image, such as that the image should have small noise, is encoded in $g(x)$. So $g(x)$ is small if x has small noise or complexity. *A posteriori* knowledge of the image, such as noisy or partial measurements of the image, is encoded in $f(x)$. So $f(x)$ is small if x agrees with the measurements.

First-order iterative methods are often used to solve such optimization problems, and ADMM is one such method:

$$\begin{aligned}
x^{k+1} &= \underset{x\in\mathbb{R}^d}{\operatorname{argmin}} \left\{\sigma^2 g(x)+(1/2)\|x-(y^k-u^k)\|^2\right\}\\
y^{k+1} &= \underset{y\in\mathbb{R}^d}{\operatorname{argmin}} \left\{\alpha f(y)+(1/2)\|y-(x^{k+1}+u^k)\|^2\right\}\\
u^{k+1} &= u^k+x^{k+1}-y^{k+1}
\end{aligned}$$

with $\sigma^2=\alpha\gamma$. Given a function h on $\mathbb{R}^d$ and $\alpha>0$, define the proximal operator of h as

$$\mathrm{Prox}_{\alpha h}(z)=\underset{x\in\mathbb{R}^d}{\operatorname{argmin}} \left\{\alpha h(x)+(1/2)\|x-z\|^2\right\},$$

which is well-defined if h is proper, closed, and convex. Now we can equivalently write ADMM as

$$\begin{aligned}
x^{k+1} &= \mathrm{Prox}_{\sigma^2 g}(y^k-u^k)\\
y^{k+1} &= \mathrm{Prox}_{\alpha f}(x^{k+1}+u^k)\\
u^{k+1} &= u^k+x^{k+1}-y^{k+1}.
\end{aligned}$$

We can interpret the subroutine $\mathrm{Prox}_{\sigma^2 g}:\mathbb{R}^d\to\mathbb{R}^d$ as a denoiser, i.e.,

$$\mathrm{Prox}_{\sigma^2 g}: \text{noisy image} \mapsto \text{less noisy image}$$

(For example, if σ is the noise level and $g(x)$ is the total variation (TV) norm, then $\mathrm{Prox}_{\sigma^2 g}$ is the standard Rudin–Osher–Fatemi (ROF) model (Rudin et al., 1992).) We can think of $\mathrm{Prox}_{\alpha f}:\mathbb{R}^d\to\mathbb{R}^d$ as a mapping enforcing consistency with measured data, i.e.,

$$\mathrm{Prox}_{\alpha f}: \text{less consistent} \mapsto \text{more consistent with data}$$

More precisely speaking, for any $x\in\mathbb{R}^d$ we have

$$g(\mathrm{Prox}_{\sigma^2 g}(x))\leq g(x), \qquad f(\mathrm{Prox}_{\alpha f}(x))\leq f(x).$$

However, some state-of-the-art image denoisers with great empirical performance do not originate from optimization problems. Such examples include non-local means (NLM) (Buades et al., 2005), Block-matching and 3D filtering (BM3D) (Dabov et al., 2007), and convolutional neural

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