UE Assignment where both links are used and costs are equal.

Find minimum demand, d, for which this assignment is valid

$(x_1, x_2 \geq 0)$

$$c_1(x_1) = c_2(x_2) \Rightarrow 1 + \frac{1}{x_1} = 3 + x_2$$

$$\frac{1}{x_1} = 2 + x_2$$

$$1 = 2x_1 + x_2 x_1$$

reminder: $x_1 + x_2 = d$

$\therefore x_1 = d - x_2$

$x_2 = d - x_1$

Solving for x_1 $(x_2 = d - x_1)$

$$1 = 2x_1 + x_1(d - x_1)$$

$$1 = 2x_1 + dx_1 - x_1^2$$

$$x_1^2 - (2+d)x_1 + 1 = 0$$

Subbing into:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where:

$a = 1$, $b = -(2+d)$, $c = 1$

$$x_1 = \frac{-(2+d) \pm \sqrt{(-(2+d))^2 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{2+d \pm \sqrt{d^2 + 4d}}{2}$$

Solving for x_2 $(x_1 = d - x_2)$

$$1 = 2(d - x_2) + x_2(d - x_2)$$

$$1 = 2d - 2x_2 + x_2 d - x_2^2$$

$$x_2^2 + 2x_2 - x_2 d - 2d + 1 = 0$$

$$x_2^2 + x_2(2 - d) + (1 - 2d) = 0$$

Subbing into

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where:

$a = 1$, $b = (2 - d)$, $c = (1 - 2d)$

$$x_2 = \frac{-(2-d) \pm \sqrt{(2-d)^2 - 4 \times 1 \times (1 - 2d)}}{2 \times 1}$$

$$= \frac{-2 + d \pm \sqrt{d^2 - 4d + 4 - 4 + 8d}}{2}$$

Reminder: $x_1 \geq 0$ to be valid solution

$$\frac{2+d\pm\sqrt{d^2+4d}}{2} \geq 0$$

$$\frac{d\pm\sqrt{d^2+4d}}{2} \geq -1$$

$$d\pm\sqrt{d^2+4d} \geq -2$$

For: $d+\sqrt{d^2+4d} \geq -2$

$$\sqrt{d^2+4d} \geq -2-d$$

$$d^2+4d \geq (-2-d)^2$$

$$\cancel{d^2}+\cancel{4d} \geq \cancel{d^2}+\cancel{4d}+4$$

$0 \geq 4$ [Invalid]

For: $d-\sqrt{d^2+4d} \geq -2$

$$2+d \geq \sqrt{d^2+4d}$$

$$(2+d)^2 \geq d^2+4d$$

$$4+\cancel{4d}+\cancel{d^2} \geq \cancel{d^2}+\cancel{4d}$$

$4 \geq 0$ [Not useful]

Reminder $x_2 \geq 0$ to be a valid solution

$$\frac{-2+d\pm\sqrt{d^2+4d}}{2} \geq 0$$

$$\frac{d\pm\sqrt{d^2+4d}}{2} \geq 1$$

$$d\pm\sqrt{d^2+4d} \geq 2$$

For: $d+\sqrt{d^2+4d} \geq 2$

$$\sqrt{d^2+4d} \geq 2-d$$

$$d^2+4d \geq (2-d)^2$$

$$\cancel{d^2}+4d \geq \cancel{d^2}-4d+4$$

$$8d \geq 4$$

[Valid solution] $\underline{d \geq 0.5}$ A

For: $d-\sqrt{d^2+4d} \geq 2$

$$d-2 \geq \sqrt{d^2+4d}$$

$$(d-2)^2 \geq d^2+4d$$

$$\cancel{d^2}-4d+4 \geq \cancel{d^2}+4d$$

$$4 \geq 8d$$

[Valid solution] $\underline{d \leq 0.5}$ B

Therefore using solution A $d \geq 0.5$ and B $d \leq 0.5$, d must be 0.5 to satisfy both solutions

$$\boxed{d = 0.5}$$

$c_1 = 1 + 1/x_1$

$c_2 = 3 + x_2$

sanity check using $d = 0.5$:

$$x_1 = \frac{2+d\pm\sqrt{d^2+4d}}{2} = 2 \text{ or } 0.5$$

$$x_2 = \frac{-2+d\pm\sqrt{d^2+4d}}{2} = 0 \text{ or } -1.5$$

$c_1(2) = 1.5$

$c_1(0.5) = 3$

$c_2(0) = 3$

invalid $c_2(-1.5) = 1.5$

→ Two pairs

∴ valid value of minimum d