MATH135 Complex Analysis Notes

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Contents

Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a <u>nonempty, connected, open subset</u> of $\mathbb{C}$.

- A region without "holes" is <u>simply connected</u>.

Non-example 1. This is not a region (not connected):
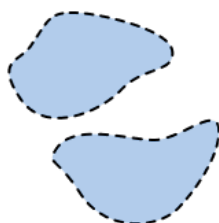
Example 2. $\mathbb{C}$ is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:
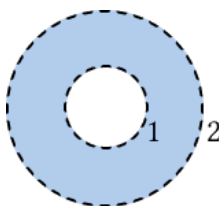
Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

   exists.

   ← this $z \to z_0$ could be from **any** directions!

2. If f is differentiable at every point in Ω, we say f is **analytic** on Ω.

3. If f is analytic on $\mathbb{C}$, then f is **entire**.

   ← Means that existence of 1st derivative implies the existence of ∞th derivative! & has Taylor expansion.

   ← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on $\mathbb{C}$ except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \to z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z \to z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if $z \to z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z \to z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. $\qquad\square$

Proposition 1. Let f be differentiable at z_0. Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$|f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| < \varepsilon|z - z_0|$$
$$|f(z_0) + f'(z_0)(z - z_0) - f(z)| < \varepsilon|z - z_0|$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the "tangent-line approximation" equivalent in $\mathbb{C}$!

$\leftarrow$ The RHS is a **linear** function!

In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is <u>locally orientation-preserving</u> and could be approximated by a linear function.

$\leftarrow$ This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

Curves, paths

Definition 3. A **curve** in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}, a, b \in \mathbb{R}$.
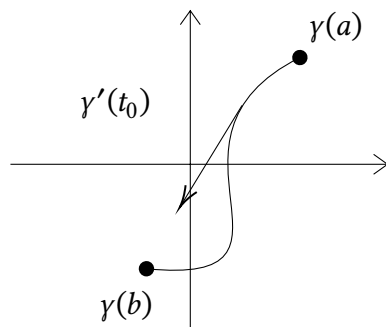
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq 0$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The "Boxy-path" Theorem). A nonempty open set Ω in $\mathbb{C}$ is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω, each of which is parallel to either to the real or imaginary axis.
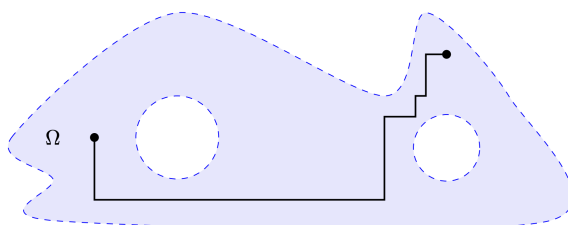
In other words, between any 2 points in a region Ω there exists a "**boxy path**".

Remark. There is also always a **smooth path**. That is:
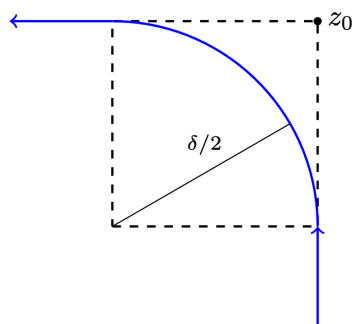
Theorem 3 ("Smooth-path"). A nonempty open set Ω in $\mathbb{C}$ is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See lecture 2 notes.   □

Conformality

Let f be an analytic complex function on Ω.
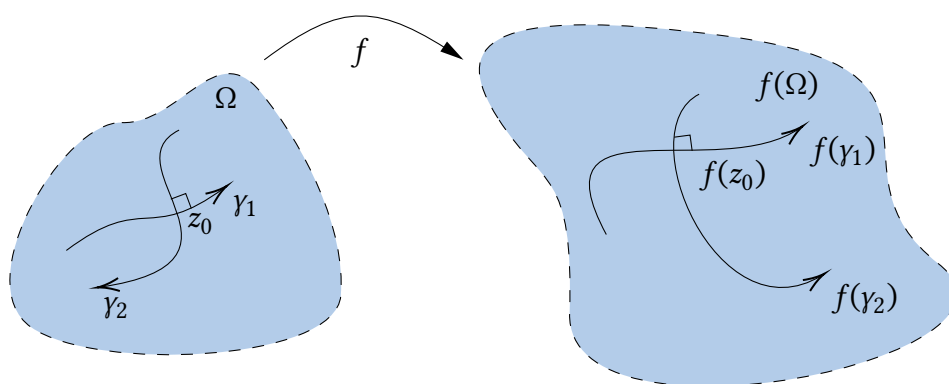
Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ. Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(\zeta_0)$ also with angle θ.

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall: $f'(z_0) = \lim\limits_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$.

Now we write each function with complex variables as $f(z) = u(z) + i\,v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i\,v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$f'(z_0) = \lim_{r \to 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r \to 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r}$$
$$= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0)$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \to 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations).

$$u_x = v_y$$
$$u_y = -v_x$$

Corollary 5. If $f : \Omega \to \mathbb{C}$ is analytic and $f' = 0$ on Ω, then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω. By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω. Hence, $\mathbf{u}, \mathbf{v}$ are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω. $\qquad\square$

Orientation-preserving as shown by Jacobian

Let $f : \Omega \to \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned}
|f'|^2 = \bar{f}' \cdot f &= (u_x - iv_x)(u_x + iv_x) \\
&= u_x^2 + v_x^2 \\
&= u_x u_x + v_x v_x & \text{and by Cauchy-Riemann,} \\
&= u_x v_y - u_y v_x \\
&= \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \right) & \text{the Jacobian of } f!
\end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0, implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0

2. f scales $\mathbb{R}$ by $|f'(z_0)|^2$ near z_0

3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

$\leftarrow$ $\Delta u = 0$ characterizes steady-state solutions to heat equations on Ω.

Definition 5. Real-valued functions $u : \Omega \to \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω.

Example 8. $u = x^2 - y^2, v = 2xy$.

$\leftarrow$ Check it!

Remark. Harmonic conjugates are unique up to translation ($\pm$ constants).

Remark. If u is harmonic on Ω, it does NOT have to have a harmonic conjugate on Ω.

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2}\log(x^2 + y^2)$.

Then,

$$\begin{aligned}
u_x &= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2 + y^2)\right) \\
&= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2}
\end{aligned}$$

Hence,

$$u_{xx} = \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2}$$
$$= \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. □

Now, can we <u>find a harmonic conjugate for the aforementioned u</u>?

We could use the two Cauchy-Riemann Equations. One of them:

$$v_y = u_x$$
$$= \frac{x}{x^2 + y^2}$$

Therefore,

$$v(x, y) = \int v_y dy + C(x) \qquad \text{unknown function of } x$$
$$= \arctan\left(\frac{y}{x}\right) + C(x)$$

← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C}\backslash\{0\}$:

[diagram of circle with angles $\pi/2$, π, 0 ☺, $2\pi \neq 0??$, $3\pi/2$]

To be resolved later!

Then, we use the second one:

$$\frac{y}{x^2 + y^2} = u_y = -v_x = -\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right)$$
$$= \frac{y}{x^2 + y^2} - C'(x) \implies C'(x) = 0$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x, y) = \arctan\left(\frac{y}{x}\right) + C$$

where C is a constant. WLOG, let $C = 0$. Then $v(x, y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$v(z) = \arg(z)$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!
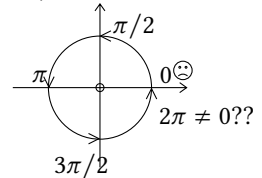
Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in $\mathbb{C}$. Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).
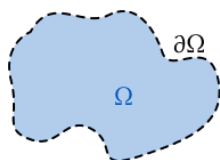
Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha\Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \to \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω.

1. U cannot have a *local* maximum in Ω.
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$. $\leftarrow$ Ω^- denotes the closure of Ω
3. U cannot be locally constant without being globally constant.

Theorem 8 (Maximum principle). Let Ω be a bounded region in $\mathbb{C}$ and let $f : \Omega^- \to \mathbb{C}$ be analytic on Ω and continuous on Ω^-.

1. If $|f|$ achieves a local max in Ω, then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations).

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C}\backslash\{\frac{-d}{c}\}$ and **comformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C}\backslash\{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$f(z) = f(w) \implies \frac{az+b}{cz+d} = \frac{aw+b}{cw+d}$$
$$(az+b)(cw+d) = (cz+d)(aw+b)$$
$$aczw + bcw + adz + bd = aczw + adw + bcz + bd$$
$$(ad-bc)z = (ad-bc)w$$
$$z = w$$

$\square$

Definition 8 (The extended plane). We set the following convention:

$$f(\frac{-d}{c}) = \infty$$
$$f(\infty) = \frac{a}{c}$$

with this, f is a **bijection** from $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can *associate* $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ with the matrix

← this association is not a bijection: it's only so up to scaling

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Remark. $M_{f \circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw - b}{-cw + a}$$

Theorem 9. A Möbius transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with three fixed points in $\widehat{\mathbb{C}}$ is the **identity map** $\text{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transforma-
   tion.

   $\leftarrow$ think about that!

   (a) If $f(z) = z$, we are done since we get the identity!

   (b) Otherwise the function only has one fixed point at ∞.

2. If ∞ is not a fixed point, then $c \neq 0$. Solve:

$$f(z) + z \Leftrightarrow \frac{az + b}{cz + d} = z$$
$$az + b = cz^2 + dz$$
$$cz^2 + (d - a)z - b = 0$$

is a quadratic which has at most two (distinct) solutions in $\mathbb{C}$. Hence, this transformation fixes at most two points.

$\square$

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so
   we are done.

   $\leftarrow$ since linear transformations preserve circles and lines

2. Now suppose $c \neq 0$. Then

$$f(z) = \frac{a}{d}z + \frac{b}{d}$$
$$= \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d}$$
$$= \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c}$$

which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz + d} \mapsto \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c}$$

We now only need to show that inversion preserves circles.

Let a circle in $\mathbb{R}^2$ be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.

Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle!   ← check this!

$\square$

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that

satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

$$z_1 \xmapsto{\phi} 0 \xmapsto{\psi^{-1}} w_1$$
$$z_2 \xmapsto{\phi} 1 \xmapsto{\psi^{-1}} w_2$$
$$z_3 \xmapsto{\phi} \infty \xmapsto{\psi^{-1}} w_3$$

and we could simply let $f = \psi^{-1} \circ \phi$. $\square$

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$f(0) = 1$$
$$f(-1) = 0$$
$$f(1) = \infty$$
$$f(i) = i$$
$$f(-i) = -i$$
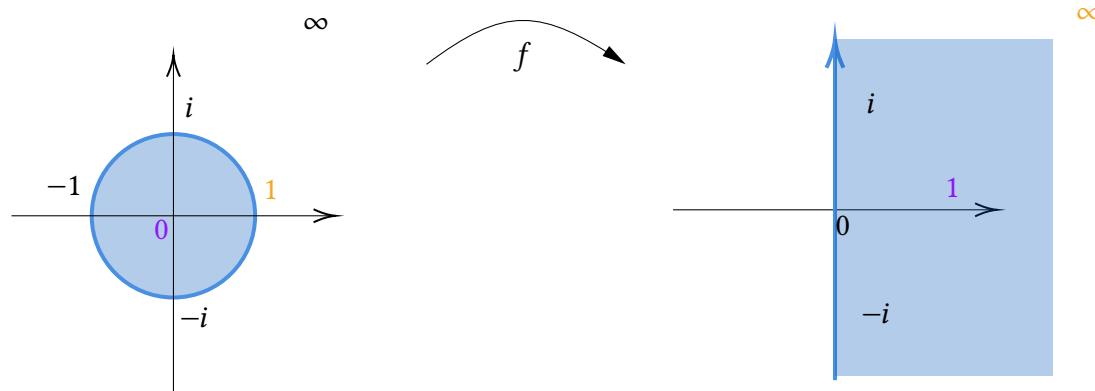
Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$ where $S_N = a_1 + \cdots + a_N$.   ← S_N is the N-th partial sum.

Divergence test

Definition 10 (Divergence test). A pair of contrapositives: ← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

2. If $\lim_{n\to\infty} a_n \neq 0$ (*including the case where the limit doesn't exist*) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \ldots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞.   ← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = \lim_{N\to\infty} S - S_N = 0$.   ← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion). $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left| \sum_{n=j-1}^{k} a_n \right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test). Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\to \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x)\, \mathrm{d}x$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.   ← do an improper integral!

Moreover, $\int_1^N f(x)\,dx \le a_1 + \cdots + a_N \le a_1 + \int_1^N f(x)\,dx$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then

<div style="text-align:right">← $a_n = \frac{1}{n}$</div>

$$\ln N \le 1 + \frac{1}{2} + \cdots + \frac{1}{N} \le 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

Theorem 14. The "p-series" $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad\qquad \text{for } \mathrm{Re}(s) > 1$$

Remark. Euler figured out:

$$\zeta(2) = \frac{\pi^2}{6}$$
$$\zeta(4) = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{\pi^6}{945}$$
$$\vdots$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979):

<div style="text-align:right">← still an open question in mathematics</div>

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202\ldots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. ← Good
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. ← BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series

<div style="text-align:right">← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!</div>

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges to ln 2. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

Theorem 17 (Riemann Rearrangement Theorem). If $\sum_{n=1}^{\infty} a_n$ is a <u>conditionally convergent</u> series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{j=0}^{\infty} b_j \right) = (a_0 + a_1 + \dots)(b_0 + b_1 + \dots)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots$$

$$= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See notes.

Uniform convergence

Definition 14. A sequence of functions $f_n : X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \to \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!
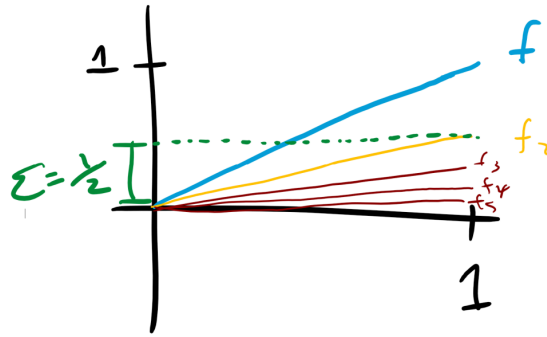
FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \to \mathbb{C}$ are continuous and converges uniformly on X to $f : X \to \mathbb{C}$, then f is continuous on X. In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n \to \infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.
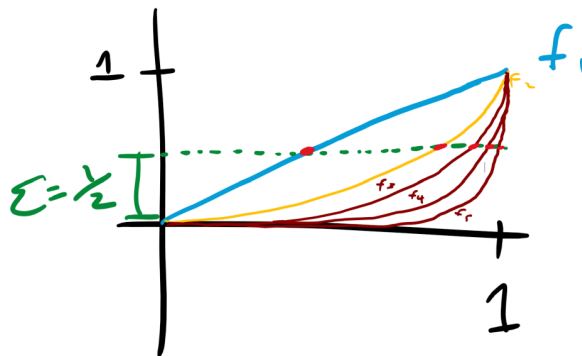
FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a, b] \to \mathbb{C}$ are continuous and converge uniformly on $[a, b]$ to f, then

← Integrals work with uniform convergence

$$\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1, 1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test)**.** Let $f_n : X \to \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X.

Power series

Definition 15. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. The a_n is the *coefficient* and z_0 is the *center*.

Convergence of geometric series

Theorem 21. The *geometric series* $(a_n = 1, z_0 = 0)$ $\sum_{n=0}^{\infty} z^n$ converges absolutely to $\frac{1}{1-z}$ if $|z| < 1$, and it diverges otherwise.

Moreover, for each $r \in [0, 1[$, the convergence is **uniform** on $|z| \leq r$.

Proof. If $|z| \geq 1$, then $z^n \not\to 0$, so by the test of divergence, the series diverges.

Now suppose $|z| < 1$. Then

$$\sum_{n=0}^{\infty} z^n = \lim_{N \to \infty} \sum_{n=0}^{N-1} z^n$$

$$= \lim_{N \to \infty} (1 + z + z^2 + \cdots + z^{N-1})$$

$$= \lim_{N \to \infty} \frac{1 - z^N}{1 - z}$$

$$= \frac{1}{1 - z} \qquad\qquad \text{since } |z| < 1$$

$\leftarrow$ The fact that we can find a formula for this sum is quite rare!

Which gives us point-wise convergence. Then, for any r such that $|z| \leq r < 1$, we have

$$\sum_{n=0}^{\infty} |z^n| \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} < \infty$$

Hence, by the Weierstrass M-test, the series converges *absolutely and uniformly* on $|z| \leq r$. $\square$

Remark. Moral of the story:

- The *radius of convergence $R = 1$* has the property that the series converges on $|z| < R$, and diverges if $|z| > R$.

- The series converges *uniformly* on $|z| \leq r < 1$ but not on $|z| < 1$ itself. Why? Let $r = 1$; we need be able to get $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|\frac{1-z^N}{1-z} - \frac{1}{1-z}\right| < 1$ for all $|z| < 1$. However, this is not gonna work: as $z \to 1$, observe that this is going to eventually exceed 1.

- The limit function $\dfrac{1}{1-z}$ is **analytic** on $\mathbb{C}\backslash\{1\}$. But the geometric series represents this function only on $|z| < 1$. In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.

               ← the limit function is well-defined way beyond the $\mathbb{D}$!

- The limit function $\dfrac{1}{1-z}$ is cool if $z \neq 1$, but as long as $|z| = 1$ (**even** if $z \neq 1$), the geometric series diverges!

               ← in the complex number sense!

Radius of convergence

Definition 16. The **limit superior** (lim sup) of a sequence of nonnegative real numbers x_n is the largest *limit point* of the x_n:

               ← limits of a subsequence of x_n

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0} \sup_{m\geq n} x_m$$

If the sequence is unbounded, the lim sup would be ∞.

               ← the RHS as in real analysis

Example 16. If x_n is the sequence $0, 1, 0, 1, \ldots$ then $\limsup_{n\to\infty} x_n = 1$.

Example 17. If x_n is the sequence $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots$, then $\limsup_{n\to\infty} x_n = 0$.

Remark. If x_n are nonnegative, then

- $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
- $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$

Theorem 22 (Cauchy-Hadamard). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Define $R \in [0, \infty]$ by

               ← interpret $\frac{1}{0} = \infty$

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Then the R is the *radius of convergence*.

(a) On $|z - z_0| < R$, the series converges **absolutely**. For each $r \in [0, R[$, the convergence is **uniform** on $|z - z_0| \leq r$.

(b) If $|z - z_0| > R$ then the series diverges. For $|z - z_0| = R$ anything could happen!

Converges uniformly

Converges absolutely

Example 18. We claim that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has an infinite radius of convergence $R = \infty$. To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \to 0$$
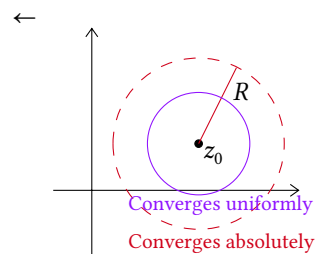
This is because $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \cdots \cdot n}$, and in $n!$, there are at least $\frac{1}{2}$ terms that are $> \frac{n}{2}$.

Thus, $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2} \right)^{\frac{n}{2}} \right)^{\frac{1}{n}} = \left(\frac{n}{2} \right)^{1/2} \to \infty$.

So $R = \infty$ and we are done ☺. We have that $\exp(z)$ has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$
\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!} \right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k}}_{\text{binomial theorem}} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n \\
&= \exp(z+w)
\end{aligned}
$$

Now define $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Term-by-term differentiation of power series

Lemma 23. $n^{\frac{1}{n}} \to 1$

Proof 1. $e^{\log(n^{\frac{1}{n}})} = e^{\frac{\log n}{n}} \to e^0 = 1$ by l'Hopital. So $n^{\frac{1}{n}} \to 1$. $\square$

Proof 2 (better). Write $n^{\frac{1}{n}} = 1 + \delta_n$ where $\delta_n \geq 0$. The binomial theorem says:

$$
\begin{aligned}
n &= (1 + \delta_n)^n \\
&= \sum_{k=0}^{\infty} \binom{n}{k} \delta_n^k \cdot 1^{n-k} \\
&= 1 + n\delta_n + \frac{n(n-1)}{2} \delta_n^2 + \dots
\end{aligned}
$$

$$\geq 1 + \frac{n(n-1)}{2}\delta_n^2$$

Therefore, $n - 1 \geq \frac{n(n-1)}{2}\delta_n^2$ and we get $\frac{2}{n} \geq \delta_n^2 \geq 0$ hence $\delta_n \to 0$.

Hence $n^{\frac{1}{n}} \to 1$.   □

Theorem 24. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence R, then

$$f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$$

for $|z - z_0| < R$. Moreover, the new series also has a radius of convergence R.

Proof. WLOG $R > 0$ and $z_0 = 0$.   ← we just translate it; also $R = 0$ isn't that meaningful

For $|z| < R$ we write: ← just splitting the function into two parts

$$f(z) = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

and the 'new series'

$$g(z) = \sum_{n=0}^{\infty} na_n z^{n-1} = \lim_{N \to \infty} S_N'(z)$$

We first prove that the radius of convergence for g is the same as f. By Cauchy-Hadamard:

$$\frac{1}{R_g} = \limsup_{n \to \infty} \sqrt[n]{n|a_n|}$$

$$= \limsup_{n \to \infty} (n^{\frac{1}{n}})\sqrt[n]{|a_n|} \qquad \text{by the previous lemma,}$$

$$= \limsup_{n \to \infty} \sqrt[n]{|a_n|}$$

$$= \frac{1}{R}$$

Thus, $R_g = R$ by Cauchy-Hadamard.

Next, we need to show that $f' = g$ with $|z| < R$.

Fix $0 \leq |w| < R$ and $\varepsilon > 0$. We want a $\delta > 0$ such that whenever $|z - w| < \delta$, we have $\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \varepsilon$.   ← just saying that the derivative at any w gets close to $g(w)$

We rewrite:

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| = \left| \frac{[S_N(z) + R_N(z)] - [S_N(w) + R_N(w)]}{z - w} - g(w) \right|$$

$$= \left| \frac{S_N(z) - S_N(w)}{z - w} + \frac{R_N(z) - R_N(w)}{z - w} + S_N'(w) - S_N'(w) - g(w) \right|$$

$$\leq \left| S_N'(w) - g(w) \right| + \left| \frac{R_N(z) - R_N(w)}{z - w} \right| + \left| \frac{S_N(z) - S_N(w)}{z - w} - S_N'(w) \right|$$

- **1st term**: by def of g and $g(z) = \lim_{N \to \infty} S_N'(z)$, we can always find some $N_1 \in \mathbb{N}$ such that any $N \geq N_1$ gives us $\left| S_N'(w) - g(w) \right| < \frac{\varepsilon}{3}$.

- **2nd term**: since $|w| < R$, there is an r such that $|w| < r < R$. $\leftarrow$ work on a smaller
  For $|z| < r$, we have   disk

$$\left| \frac{R_N(z) - R_N(w)}{z - w} \right| = \frac{1}{|z - w|} \left| \sum_{n=N}^{\infty} a_n z^n = - \sum_{n=N}^{\infty} a_n w^n \right|$$

$$\leq \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n - w^n}{z - w} \right|$$

$$= \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n}{z} \cdot \frac{1 - \frac{w^n}{z^n}}{1 - \frac{w}{z}} \right| \qquad \text{by geometric sequence}$$

$$= \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n}{z} \cdot \left(1 + \left(\frac{w}{z} \right) + \left(\frac{w}{z} \right)^2 + \cdots + \left(\frac{w}{z} \right)^{n-1} \right) \right|$$

$$= \sum_{n=N}^{\infty} |a_n| \left| z^{n-1} + z^{n-2} w + \cdots + z w^{n-2} + w^{n-1} \right|$$

$$\leq \sum_{n=N}^{\infty} |a_n| \cdot n \cdot r^{n-1} \quad \text{by } |z|, |w| < r < R$$

Thus, there exists an $N_2 \in \mathbb{N}$ such that any $N \geq N_2$ gives us

$$\left| \frac{R_N(z) - R_N(w)}{z - w} \right| < \frac{\varepsilon}{3}$$

- **3rd term**: let $N = \max\{N_1, N_2\}$. The definition of $S_N'(w)$ provides $\gamma > 0$ $\leftarrow$ review def of
  such that if $|z - w| < \gamma$, then we have $\left| \frac{S_N(z) - S_N(w)}{z - w} - S_N'(w) \right| < \frac{\varepsilon}{3}$.   derivatives!

Now if $0 < \delta < \min\{\gamma, r - |w|\}$, then the 3 terms above are all $< \frac{\varepsilon}{3}$. Hence, $\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \varepsilon$ holds for this δ. $\square$

Corollary 25. A power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with $R > 0$ is infinitely differentiable on $|z - z_0| < R$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

are the coefficients of the terms of the power series.

$\leftarrow$ prove by keep taking derivatives!

Corollary 26. Power series expansions are unique. That is, if $r > 0$ and

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

$\leftarrow$ because there is a unique formula for coeffs.

on $|z - z_0| < r$, then $a_n = b_n$ for $n \geq 0$.

Remark. Recall that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has a radius of convergence ∞ (it's an *entire* function). Now, if we differentiate it term-by-term:

$$\frac{d}{dz} \exp(z) = \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{z^{n-1}}{(n-1)!} \qquad \text{let } k = n - 1$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$= \exp(z)$$

Thus, the derivative of $\exp(z)$ is itself! Moreover, $\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Remark. We claim that $\exp(z) = e^z$.

Since $e^z e^{c-z}$ is a constant for all constant c, z, we have

$$\frac{d}{dz}(e^z e^{c-z}) = 0$$

to recover the constant $e^z e^{c-z}$, we let $z = 0$, giving us

$$e^z e^{c-z} = e^c$$

which is the addition formula!

Therefore,

$$\exp(n) = \exp(1 + 1 + \cdots + 1)$$
$$= exp(1)^n$$
$$= e^n$$

Elementary functions

Now that we have derived e, we could use it to derive sin and cos:

Definition 17.

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

We observe that we have the following property:

- Radius of convergence $R = \infty$
- $(\cos z)' = -\sin z, (\sin z)' = \cos z$
- $\cos x = \text{Re}\,(e^{ix}), \sin x = \text{Im}\,e^{ix}$ for all $x \in \mathbb{R}$
- $\cos(-z) = \cos z, \sin(-z) = -\sin z$
- $\cosh x = \frac{e^x + e^{-x}}{2}$ so $\cosh(ix) = \cos x$
- $e^{iz} = \cos z + i \sin z$

-

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$
$$= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz})$$
$$= 1 \qquad \forall z \in \mathbb{C}$$

-

$$\cos^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2$$
$$= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz})$$
$$= \frac{1}{2} + \frac{e^{2iz} + e^{-2iz}}{4}$$
$$= \frac{1}{2}(1 + \cos 2z)$$

- If $x \in \mathbb{R}$ then $\cos x, \sin x$ are real. We get $|\sin x|, |\cos x| \leq 1$.

Definition 18. $f : \mathbb{C} \to \mathbb{C}$ is **periodic** with a *period* ω if $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}$.

Theorem 27. There exists a positive real number π such that:

(a) $\cos z, \sin z$ have period 2π

(b) e^z is periodic with period $2\pi i$

(c) π is the area of the unit circle

Proof. By Euler's formula, it suffices to consider e^{iz} only. If ω is a period of e^{iz}, then

$$e^{iz} = e^{i(z+\omega)} = e^{iz} e^{i\omega}$$

which only happens if $e^{i\omega} = 1$. Conversely, if $e^{i\omega} = 1$, then $e^{i(z+\omega)} = e^{iz}$.

Hence, ω is a period of e^{iz} *if and only if* $e^{iw} = 1$. $\qquad\square$

Proposition 28. $\sin x \leq x$ for all $x \geq 0$.

← This is the first term in the power series

x

Proof. Since $|\cos t| \leq 1$,

$$x - \sin x = (x - \sin x) - (0 - \sin 0)$$
$$= \int_0^x \underbrace{1 - \cos t}_{\geq 0} \, dt \qquad \text{by FTC}$$
$$\geq 0$$

$\sin x$

$\qquad\square$
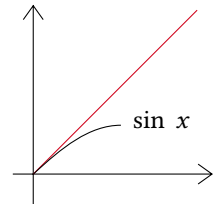
Proposition 29. In addition, $\cos x \geq 1 - \frac{x^2}{2}$ for $x \geq 0$.

← These are the first 2 terms in the power series

Proof. The previous prop gives:

$$\cos x - 1 = \cos x - \cos 0$$
$$= \int_0^x -\sin t \, dt$$
$$\geq \int_0^x -t \, dt$$
$$= \frac{-x^2}{2}$$

$\cos x$

$1 - \frac{x^2}{2}$

$\qquad\square$
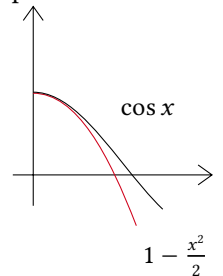
Proposition 30. Furthermore, for $x \geq 0$:

- $\sin x \geq x^3 - \frac{x^3}{6}$

- $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

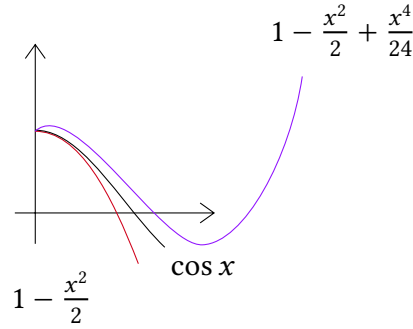**Proposition 31.** There exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$.

Proof. By the previous prop, we have $\cos \sqrt{3} \leq 1 - \frac{\sqrt{3}^2}{2} + \frac{\sqrt{3}^4}{24} = \frac{1}{8} < 0$. Moreover, $\cos 0 = 1 > 0$, by IVT, there exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$. $\qquad\square$

Proposition 32. $\omega_0 = 4x_0$ is a period of e^{iz}.

Proof. Since $\cos x_0 = 0$, we have $\sin x_0 = \pm 1$. Then $e^{ix_0} = \pm i$. We have $(\pm i)^4 = 1$, so $e^{4ix_0} = 1 = e^0$, so $\omega_0 = 4x_0$ is a period of e^{iz}. $\qquad\square$

Proposition 33. ω_0 is the *smallest* positive period of e^{iz}.

Proposition 34. All periods of e^{iz} are integer multiples of $2\pi = 4x_0$.

Proof. Define $\pi = 2x_0$. The area of unit circle is

$$4 \int_0^1 \sqrt{1-x^2} \, dx = 4 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \theta} \, d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$$

$$= \pi$$

$\square$

Complex logarithm

We know: $e^0 = 1, e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718\ldots$

Since $\frac{d}{dx}e^x = e^x$, it is positive. If $x > 0$, we conclude that e^x is strictly increasing!
As $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1 + x$, so $\lim_{x\to\infty} e^x = \infty$,

Therefore, e^x is a **bijection** from $\mathbb{R}$ to $(0, \infty)$. This means it has an inverse that is a bijection from $(0, \infty)$ to $\mathbb{R}$.
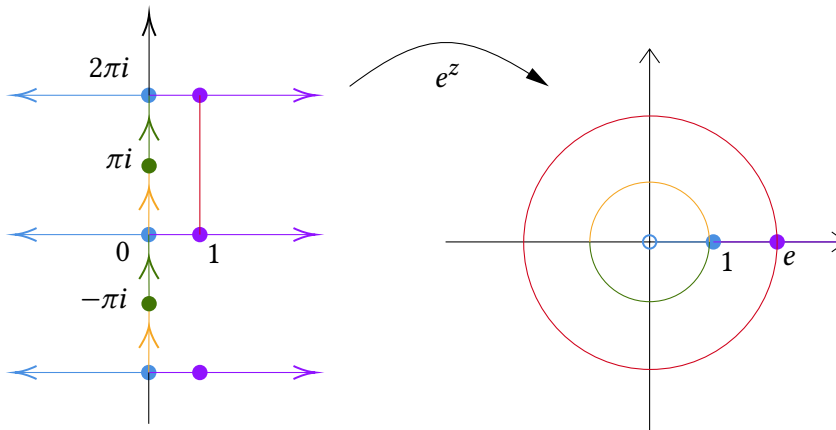
Definition 19. $\ln x$ is the inverse of e^x for $x \in (0, +\infty)$.

Now what about the complex case? Let $z \neq 0$ and $z = re^{i\theta}$ where $r = |z| > 0$ and $\theta = \arg z \in \mathbb{R}$.

$\leftarrow$ cf. trig properties

$\leftarrow$ Only determined up to addition of multiples of 2π

Hence, $z = re^{i\theta} = e^{\ln r}e^{i\theta} = e^{\ln r + i\theta}$. However, the θ is ambiguous to addition of multiples of 2π!

Definition 20. If $z \neq 0$, a **logarithm** of z is a $w \in \mathbb{C}$ such that $e^w = z$.

We could graph the function e^z with $z \in \mathbb{C}$:

Definition 21. If Ω is a region in $\mathbb{C}$, then a continuous $l : \Omega \to \mathbb{C}$ is a **branch** of the logarithm if $e^{l(z)} = z$ for all $z \in \Omega$.

$\leftarrow$ note $0 \notin \Omega$

Example 19. If $\Omega = \mathbb{C}\backslash(-\infty, 0]$ such that $\theta \in (-\pi, \pi)$, a logarithm could be defined on it. This is the **principal branch** of the logarithm.

$\leftarrow$ See graphed Riemann surface

Remark. Suppose $l(z)$ is a branch of the logarithm and l is analytic, then:

$$e^{l(z)} = z \implies \frac{d}{dz}e^{l(z)} = l'(z)e^{l(z)} = 1$$

Since $e^{l(z)} = z$, we conclude $l'(z) = \frac{1}{z}$.

Complex power

Definition 22. If $z \neq 0$, *define* $z^a = e^{a \log z}$.   ← NOT well-defined!

Remark. The definition of complex powers should coincide with the old one:
$z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n} = r^n e^{in\theta}$.

Check:

$$z^n = e^{n \log z} = e^{n(\ln r + i\theta + i2\pi k)}$$
$$= e^{n \ln r} e^{in\theta} \underbrace{e^{i2\pi nk}}_{=1}$$
$$= r^n e^{in\theta}$$

is true for any $k \in \mathbb{Z}$.

How about n-th roots?

$$z^{\frac{1}{n}} = e^{\frac{1}{n} \log z}$$
$$= e^{\frac{1}{n}(\ln r + i\theta + i2\pi k)}$$
$$= e^{\frac{1}{n} \ln r} e^{\frac{i\theta}{n}} \underbrace{e^{\frac{i2\pi k}{n}}}_{n \text{ distinct}}$$
$$= r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)}$$

Riemann surface

We still have a problem: $\ln z$ is still not a function on $\mathbb{C}$! The branch depends on the arbitrary choice of domain. What shall we do to make it not dependent on a choice?

Answer: let $\ln$ not live on the complex plane, but infinitely many copies of the slit plane $\mathbb{C} \setminus (-\infty, 0]$, each one being glued to the next along the slit $(-\infty, 0]$.

Example 20. $z^{1/2}$ would live on a surface:
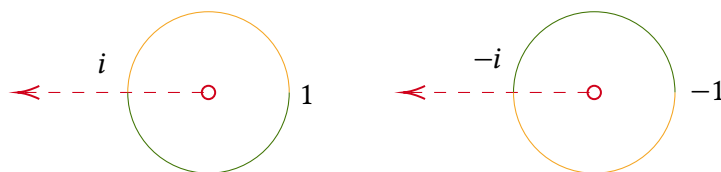
Cauchy's theorem and its consequences

Complex integration

Definition 23. A complex-valued function $\gamma : [a, b] \to \mathbb{C}$ is **piecewise** continuous if γ is continuous at all but *finitely many* points of $[a, b]$ and γ has one-sided limits that are *finite* at each point (of discontinuity).
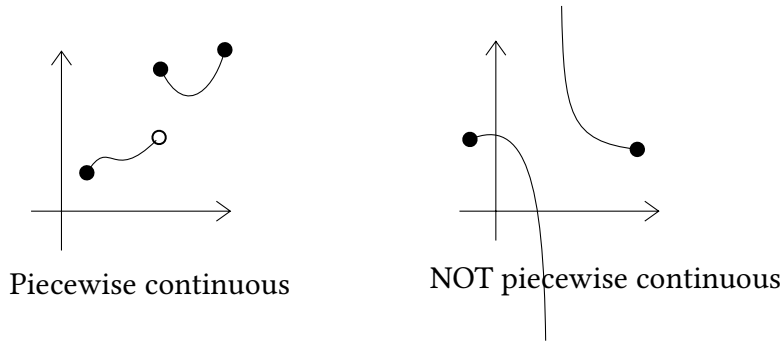

Piecewise continuous   NOT piecewise continuous

If γ is piecewise continuous, then $\int_a^b \operatorname{Re} \gamma(t) \, dt$ and $\int_a^b \operatorname{Im} \gamma(t) \, dt$ exist. Then we define **complex integration**:

$$\int_a^b \gamma(t) \, dt = \int_a^b \operatorname{Re} \gamma(t) \, dt + i \cdot \int_a^b \operatorname{Im} \gamma(t) \, dt$$

That is,

$$\operatorname{Re}\left(\int_a^b \gamma(t) \, dt\right) = \int_a^b \operatorname{Re} \gamma(t) \, dt$$

$$\operatorname{Im}\left(\int_a^b \gamma(t) \, dt\right) = \int_a^b \operatorname{Im} \gamma(t) \, dt$$

In addition, if γ_1, γ_2 are both $[a, b] \to \mathbb{C}$ and piecewise cont., and $c_1, c_2 \in \mathbb{C}$, then

$$\int_a^b (c_1\gamma_1(t) + c_2\gamma_2(t)) \, dt = c_1 \int_a^b \gamma_1(t) \, dt + c_2 \int_a^b \gamma_2(t) \, dt$$

Proposition 35 (Triangle inequality). If $\gamma : [a, b] \to \mathbb{C}$ is piecewise continuous, then

$$\left|\int_a^b \gamma(t) \, dt\right| \le \int_a^b |\gamma(t)| \, dt$$

Proof. WLOG assume $\int_a^b \gamma(t)\,dt \neq 0$. Define $\lambda = \dfrac{\left|\int_a^b \gamma(t)\,dt\right|}{\int_a^b \gamma(t)\,dt}$ and note $|\lambda| = 1$.

Thus,

$$
\begin{aligned}
\left|\int_a^b \gamma(t)\,dt\right| &= \lambda \int_a^b \gamma(t)\,dt \\
&= \int_a^b \lambda\gamma(t)\,dt && \text{because LHS is } \in \mathbb{R} \\
&= \operatorname{Re}\int_a^b \lambda\gamma(t)\,dt \\
&\leq \int_a^b |\lambda\gamma(t)|\,dt && \because \operatorname{Re} z \leq |z| \\
&= \int_a^b |\gamma(t)|\,dt && \because |\lambda| = 1
\end{aligned}
$$

$\square$

Complex differentiability

Definition 24. $\gamma : [a,b] \to \mathbb{C}$ is **differentiable** at $t \in [a,b]$ if $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are differentiable (in the sense of real variables). We define

$$\gamma'(t) = (\operatorname{Re}\gamma)'(t) + i \cdot (\operatorname{Im}\gamma)'(t)$$

Definition 25. $\gamma : [a,b] \to \mathbb{C}$ is **piecewise** C^1 if:

$\leftarrow$ C^1 is one-time differentiable

(a) γ is continuous on $[a,b]$.

(b) γ is differentiable at all but finitely many points of $[a,b]$.

(c) γ' is continuous at each point where it exists.

(d) γ' has finite one-sided limits at every point of discontinuity.

Fundamental theorem of calculus, complex edition

If $\gamma : [a,b] \to \mathbb{C}$ is piecewise C^1, then:

$$\int_a^b \gamma'(t)\,dt = \gamma(b) - \gamma(a)$$

Definition 26. If γ is C^1, then the arclength of γ is:

$$L(\gamma) = \int_a^b |\gamma'(t)| \, dt$$

Definition 27. If $\gamma : [a,b] \to \Omega$ is piecewise C^1 and $f : \Omega \to \mathbb{C}$ is continuous, then

$$\int_\gamma f(z) \, dz = \int_a^b f(\gamma(t))\gamma'(t) \, dt$$

where $z = \gamma(t)$ and $dz = \gamma'(t) \, dt$

$\gamma'(t)$ is instantaneous velocity, so its absolute value is the speed

We have **linearity** w.r.t. f:

$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z)) \, dz = c_1 \int_\gamma f_1(z) \, dz + c_2 \int_\gamma f_2(z) \, dz$$

Remark. Arclength is independent from parameterization.

Proof. Let $\gamma : [a,b] \to \Omega$ be piecewise C^1. Let $\alpha : [c,d] \to [a,b]$ is an increasing, piecewise C^1 surjection such that $\alpha(c) = a, \alpha(d) = b$. Then $\phi = \gamma \circ \alpha : [c,d] \to \Omega$ is also piecewise C^1. Hence, by substituting $s = \alpha(t), ds = \alpha'(t) \, dt$:

$$\int_\phi f(z) \, dz = \int_c^d f(\phi(t))\phi'(t) \, dt$$

$$= \int_c^d f(\gamma(\alpha(t)))\gamma'(\alpha(t))\alpha'(t) \, dt$$

$$= \int_a^b f(\gamma(s))\gamma'(s) \, ds$$

$$= \int_\gamma f(z) \, dz$$

$\square$

An important estimate

Let f be continuous. Since $\int_\gamma f(z) \, dz = \int_a^b f(\gamma(t))\gamma'(t) \, dt$, we observe:

$$\left| \int_\gamma f(z) \, dz \right| = \left| \int_a^b f(\gamma(t))\gamma'(t) \, dt \right|$$

$$\leq \int_a^b |f(\gamma(t))| \, |\gamma'(t)| \, dt$$
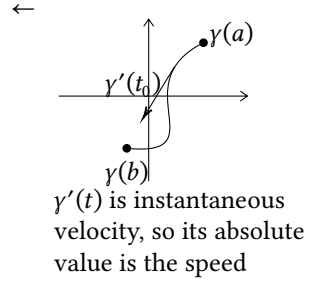
$$\leq \max_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|\, dt$$

$$= \max_{z\in\gamma} |f(z)| \cdot L(\gamma)$$

Definition 28. If $\gamma : [a,b] \to \mathbb{C}$, the reverse of γ is $(-\gamma) : [-b, -a] \to \mathbb{C}$ defined by $(-\gamma)(t) = \gamma(-t)$. Hence,

← going around the track backwards
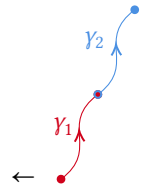
$$\int_{-\gamma} f(z)\, dz = -\int_\gamma f(z)\, dz$$

Remark. We can also break up the curve and integral the two parts separately:

$$\int_\gamma f(z)\, dz = \int_{\gamma_1} f(z)\, dz + \int_{\gamma_2} f(z)\, dz$$

Fundamental theorem of calculus for contour integrals

If $\gamma : [a,b] \to \mathbb{C}$ is piecewise C^1, and $f : \Omega \to \mathbb{C}$ is analytic , then

← Assuming f' continuous, which we would prove later

$$\int_\gamma f'(z)\, dz = f(\gamma(b)) - f(\gamma(a))$$

If $\gamma(a) = \gamma(b)$, then $\int_\gamma f'(z)\, dz = 0$.

Proof.

$$\int_\gamma f'(z)\, dz = \int_a^b f'(\gamma(t))\gamma'(t)\, dt$$

$$= \int_a^b (f \circ \gamma)'(t)\, dt \qquad \text{chain rule}$$

$$= f(\gamma(b)) - f(\gamma(a))$$

$\square$

Example 21. Let γ be a circle of radius R centered at z_0: $\gamma(t) = z_0 + Re^{it}, t \in [0, 2\pi]$. We would like to find $\int_\gamma (z - z_0)^n\, dz$.

If $n \neq -1$, then $\left(\frac{(z-z_0)^{n+1}}{n+1}\right)' = (z - z_0)^n$. Thus,

$$\int_\gamma (z - z_0)^n\, dz = \int_\gamma \left(\frac{(z - z_0)^{n+1}}{n + 1}\right)'\, dz = 0$$

by FTC.

If $n = -1$,

$$\int_\gamma (z - z_0)^n \, dz = \int_\gamma \frac{1}{z - z_0} \, dz = \int_0^{2\pi} i \, dt = 2\pi i$$
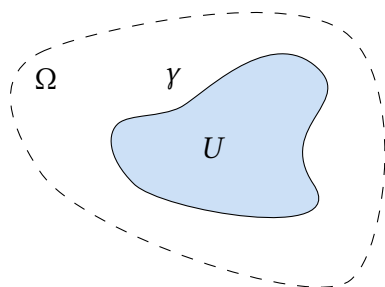
Cauchy's theorem

Take 1

Theorem 36 (Cauchy's). Let Ω be a region in $\mathbb{C}$ containing a *simple* piecewise C^1 *closed* curve γ and its interior.

$\leftarrow$ does not self-intersect

$\leftarrow$ holes not allowed in the interior

If $f : \Omega \to \mathbb{C}$ is analytic, then $\int_\gamma f(z) \, dz = 0$.

"Proof". Let U be the union of γ and its interior. Let $f = u + iv$ as usual, write $dz = dx + i \, dy$:

$$\int_\gamma f(z) \, dz = \int_\gamma (u + iv)(dx + i \, dy)$$

$$= \int_\gamma u \, dx - v \, dy + i \int_\gamma v \, dx + u \, dy$$

$$= \int\int_U (-v_x - u_y) \, dx \, dy + i \int\int_U (u_x - v_y) \, dx \, dy \quad \text{by Green's thm}$$

$$= 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \text{by Cauchy-Riemann}$$

$\square$

However, this 'proof' heavily relies on the fact that u, v are C^1 and that the partial derivatives are continuous. This assumes f' is continuous, but we aren't sure about that yet!

$\leftarrow$ See Goursat's Lemma

Take 2: deformation version

Theorem 37 (Cauchy's). Let γ_1, γ_2 be piecewise C^1 curves in a region Ω with the same start and end points. If γ_1 can be continuously deformed to γ_2 without ever passing outside of Ω, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$$

By the *previous* statement of Cauchy's theorem (in Theorem 36), we observe that $\int_{\gamma_1-\gamma_2} f(z)\,dz = 0$, so this one falls out.

Non-example 22. The γ_1, γ_2 in the picture below cannot be continuously deformed into each other!
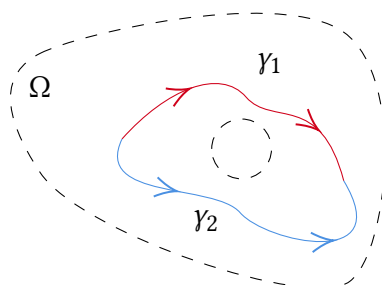
Fresnel integrals

Consider:

$$\int_0^\infty \sin(t^2)\,dt \qquad \text{and} \qquad \int_0^\infty \cos(t^2)\,dt$$

aka.

$$\lim_{R\to\infty} \int_0^R \sin(t^2)\,dt \qquad \text{and} \qquad \lim_{R\to\infty} \int_0^R \cos(t^2)\,dt$$

It's not obvious that these integrals converge!
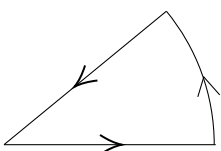
Solution: PIZZA! Let γ be the 'sum' of all 3 curves as shown. Let $R \to \infty$. Then, by Cauchy's theorem, $\int_\gamma e^{iz^2}\,dz = 0$.

(Scratch work begins)

Remark. We don't know how to write out the antiderivative of $f(z) = e^{iz^2}$ but we can use series!

$$f(z) = e^{iz^2}$$

$$= \sum_{n=0}^{\infty} \frac{(iz^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{i^n z^{2n}}{n!}$$

And so

$$F(z) = \sum_{n=0}^{\infty} \frac{i^n z^{2n+1}}{(2n+1)n!}$$

(Scratch ends here)

Now we return to the integral. Strategy:

$$0 = \int_\gamma e^{iz^2}\, d = \underbrace{\int_{\gamma_1} e^{iz^2}\, dz}_{I_1(R)} + \underbrace{\int_{\gamma_2} e^{iz^2}\, dz}_{I_2(R)} + \underbrace{\int_{\gamma_3} e^{iz^2}\, dz}_{I_3(R)}$$
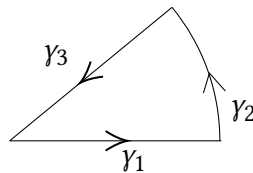
Evaluate $I_1(R)$: We observe that z is real for this one. Parameterize $z = t$ where t is a real variable.

$$I_1(R) = \int_{\gamma_1} e^{it^2}\, dt$$

$$= \int_0^R \cos(t^2)\, dt + i \cdot \int_0^R \sin(t^2)\, dt$$

Hence, $\lim_{R\to\infty} I_1(R) = \int_0^\infty \cos(t^2)\, dt + i \cdot \int_0^\infty \sin(t^2)\, dt$.

Evaluate $I_2(R)$:

Parameterize γ_2 as $z = Re^{i\theta}$ where $\theta \in [0, \frac{\pi}{4}]$. Hence, $\mathrm{d}z = iRe^{i\theta}\,\mathrm{d}\theta$. Then:

$$
\begin{aligned}
|I_2(R)| &= \left| \int_{\gamma_2} e^{i\theta^2}\,\mathrm{d}\theta \right| \\
&= \left| \int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} iRe^{i\theta}\,\mathrm{d}\theta \right| \\
&= \left| R \int_0^{\frac{\pi}{4}} e^{iR^2 e^{i2\theta}} e^{i\theta}\,\mathrm{d}\theta \right| \\
&\leq R \int_0^{\frac{\pi}{4}} \left| e^{iR^2 e^{i2\theta}} \right|\,\mathrm{d}\theta && \text{by tri. ineq.} \\
&\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta}\,\mathrm{d}\theta && \text{since when } x, y \in \mathbb{R},\ \left| e^{x+iy} \right| = e^x \\
&\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4\theta}{\pi}}\,\mathrm{d}\theta && \text{since when } x \in [0, \frac{\pi}{2}],\ \frac{2}{\pi}x \leq \sin x \\
&= \frac{-R\pi}{R^2 4} e^{-R\frac{4\theta}{\pi}} \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}
$$

Thus, $\lim_{R\to\infty} I_2(R) = 0$. :)

Evaluate $I_3(R)$:

$$
\begin{aligned}
I_3(R) &= \int_{\gamma_3} e^{iz^2}\,\mathrm{d}z \\
&= \int_R^0 e^{i(e^{i\frac{\pi}{4}}t)^2} e^{i\frac{\pi}{4}}\,\mathrm{d}t \\
&= -e^{i\frac{\pi}{4}} \int_0^R e^{-t^2}\,\mathrm{d}t \\
\lim_{R\to\infty} I_3(R) &= -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \right) \int_0^\infty e^{-t^2}\,\mathrm{d}t \quad \text{by Gaussian integral, } \int_0^\infty e^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2} \\
&= -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}
\end{aligned}
$$

Therefore, we see $I_1(R) + I_2(R) + I_3(R) = 0$ where $\lim_{R\to\infty} I_1(R) = \int_0^\infty \cos(t^2)\,\mathrm{d}t + i \cdot \int_0^\infty \sin(t^2)\,\mathrm{d}t$, $I_2(R) \to 0$ and $I_3(R) = -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}$. Hence, we would be able to conclude that

$$
\int_0^\infty \sin(t^2)\,\mathrm{d}t = \sqrt{\frac{\pi}{8}} \qquad \text{and} \qquad \int_0^\infty \cos(t^2)\,\mathrm{d}t = \sqrt{\frac{\pi}{8}}
$$

Goursat's lemma

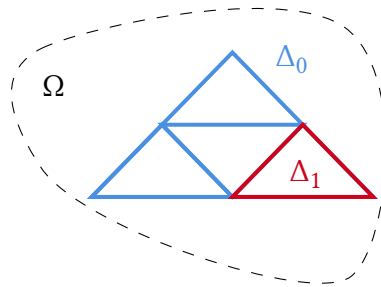This lemma patches the hole that we have to assume f' continuous in Cauchy's theorem!

Lemma 38 (Goursat's). If $f : \Omega \to \mathbb{C}$ is analytic and Δ is a triangle in Ω whose interior lies inside Ω, then $\int_\Delta f(z)\,\mathrm{d}z = 0$.

← Does not assume f' continuous!

Proof. WLOG orient $\Delta_0 = \Delta$ counterclockwise. Bisect sides of Δ_0 and construct smaller triangles Δ_{0j} where $j = 1, 2, 3, 4$. Then,

$$I = \int_{\Delta_0} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\Delta_{0j}} f(z)\,\mathrm{d}z$$

By triangle inequality,

$$|I| \leq \sum_{j=1}^{4} \left| \int_{\Delta_{0j}} f(z)\,\mathrm{d}z \right|$$

Thus, there exists $j \in \{1, 2, 3, 4\}$ such that

$$\frac{|I|}{4} \leq \left| \int_{\Delta_{0j}} f(z)\,\mathrm{d}z \right|$$

For this j, define $\Delta_1 = \Delta_{0j}$.

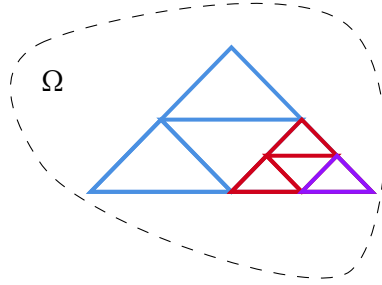We disect Δ_1 again into smaller triangles Δ_{1j} where $j = 1, 2, 3, 4$. Then,

$$I = \int_{\Delta_1} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\Delta_{1j}} f(z)\,\mathrm{d}z$$

Again, by triangle inequality, there is a $j \in \{1, 2, 3, 4\}$ such that

$$\frac{|I|}{4^2} \leq \frac{1}{4} \left| \int_{\Delta_1} f(z)\,\mathrm{d}z \right| \leq \left| \int_{\Delta_{1j}} f(z)\,\mathrm{d}z \right|$$

For this j, define $\Delta_2 = \Delta_{1j}$.

...continue in this manner to get nested triangles Δ_n such that

$$\frac{|I|}{4^{n+1}} \leq \frac{1}{4}\left|\int_{\Delta_n} f(z)\,\mathrm{d}z\right| \leq \left|\int_{\Delta_{nj}} f(z)\,\mathrm{d}z\right|$$

for all $n \geq 0$.

Now let $\ell = L(\Delta_0)$ denote perimeter of the original triangle (blue).
Then $L(\Delta_n) = \frac{\ell}{2^n}$. $\leftarrow$ Perimeter of Δ_n

Let K_n denote the triangle Δ_n union with its interior such that K_n is closed (in fact, compact!). Let $\zeta_n \in K_n$ for $n \geq 0$. Then there is $N \in \mathbb{N}$, such that for all $m, n \geq N$ we have $|\zeta_m - \zeta_n| \leq \operatorname{diam}(K_N) \leq \frac{\ell}{2^N}$. Thus, ζ_n as a sequence is Cauchy.

Let $z_0 = \lim_{n\to\infty} \zeta_n$, note $z_0 \in \bigcap_{n=0}^{\infty} K_n$ and $z_0 \in \Omega$. Since f is analytic at z_0, given $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \frac{\varepsilon}{\ell^2}$$

Now consider multiplying $|z - z_0|$ on both sides:

$$|f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| < \frac{\varepsilon}{\ell^2}|z - z_0|$$

$$|f(z_0) + f'(z_0)(z - z_0) - f(z)| < \frac{\varepsilon}{\ell^2}|z - z_0|$$

Since $f(z_0) + f'(z_0)(z - z_0)$ is **linear**, it has an antiderivative on $\mathbb{C}$. Thus,

$$\int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0)\,\mathrm{d}z = 0$$

by FTC! Now pick n large enough so that $|z - z_0 < \delta$ for all $z \in \Delta_n$. Thus,

$$|I| \leq 4^n\left|\int_{\Delta_n} f(z)\,\mathrm{d}z\right|$$
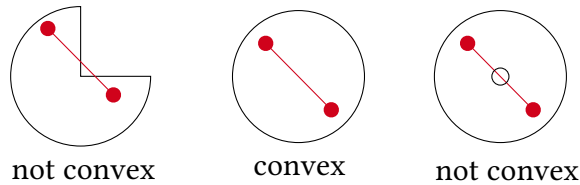
$$= 4^n \left| \int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0) - f(z) \right|$$

$$\le 4^n \frac{\varepsilon}{\ell^2} |z - z_0| \frac{\ell}{2^n} \qquad \text{by tri. ineq. and } \left| \int_\gamma g(z) \, dz \right| \le \sup_{z \in \gamma} |g(z)| \cdot L(\gamma)$$

$$< \frac{4^n \varepsilon}{\ell 2^n} \cdot \frac{\ell}{2^n}$$

$$= \varepsilon$$

$\square$

Local antiderivative

Theorem 39. If Ω is convex and $f : \Omega \to \mathbb{C}$ is analytic, then f has an antiderivative on Ω.

Remark. Line segments don't exit the region in convex shapes:

not convex   convex   not convex

Proof. Fix $w \in \Omega$ and define:

$$F(z) = \int_{[w,z]} f(\zeta) \, d\zeta$$

for $z \in \Omega$.

$\leftarrow$ $[w, z]$ is the line segment from w to z.

This is well-defined if Ω is convex.

Now we want to show that F' is f. That is equivalent to showing that for all $\varepsilon > 0, z_0 \in \Omega$, there exists $\delta > 0$ s.t. whenever $|z - z_0| < \delta$, we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \varepsilon$$

Let $z_0 \in \Omega$ be given and $\varepsilon > 0$. Goursat says integrals around the triangle is 0, so

we suppose $z \in \Omega \backslash \{z_0, w\}$ and get a triangle:
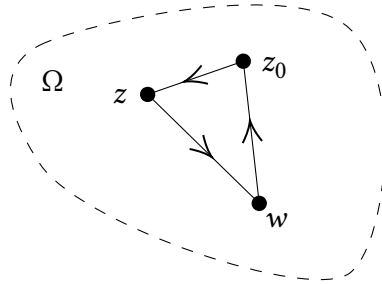
and we know that

$$\underbrace{\int_{[w,z_0]} f(\zeta)\, \mathrm{d}\zeta}_{F(z_0)} + \int_{[z_0,z]} f(\zeta)\, \mathrm{d}\zeta + \underbrace{\int_{[z,w]} f(\zeta)\, \mathrm{d}\zeta}_{-F(z)} = 0$$

So $F(z) - F(z_0) = \int_{[z_0,z]} f(\zeta)\, \mathrm{d}\zeta$. Thus,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,z]} (f(\zeta) - f(z_0))\, \mathrm{d}\zeta$$

Since f is analytic at z_0, it is continuous there. Given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have $|f(z) - f(z_0)| < \varepsilon$.

Therefore, whenever $|z - z_0| < \delta$, we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \frac{\varepsilon}{|z - z_0|} L([z_0, z])$$
$$= \frac{\varepsilon}{|z - z_0|} |z - z_0|$$
$$= \varepsilon$$

$\leftarrow$ still by $\left| \int_\gamma g(z)\, \mathrm{d}z \right| \leq \sup_{z \in \gamma} |g(z)| \cdot L(\gamma)$

$\square$

Cauchy's theorem, Take 3

Cauchy's theorem for convex regions

Theorem 40. If Ω is convex, $f : \Omega \to \mathbb{C}$ analytic and γ is a piecewise C^1 curve in Ω, then $\int_\gamma f(z)\, \mathrm{d}z = 0$.

$\leftarrow$ Since Ω is convex, the interior of γ lies inside Ω.

Proof. Previous theorem says f has an antiderivative F on Ω. Thus,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma F'(z)\,\mathrm{d}z = 0$$

by FTC! □

Cauchy's integral formula

Cauchy's integral formula for a circle

Theorem 41. If f is analytic on a region Ω that contains the circle γ and its interior, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

for all z inside of γ.   ← this Ω doesn't
                                                                      need to be convex

Proof. Let $r > 0$ be small enough so that the closed ball $B_r(z)^-$ is in the interior of γ. Let $C_r(z) = \{\zeta \in \mathbb{C} : |\zeta - z| = r\}$ traversed clockwise.
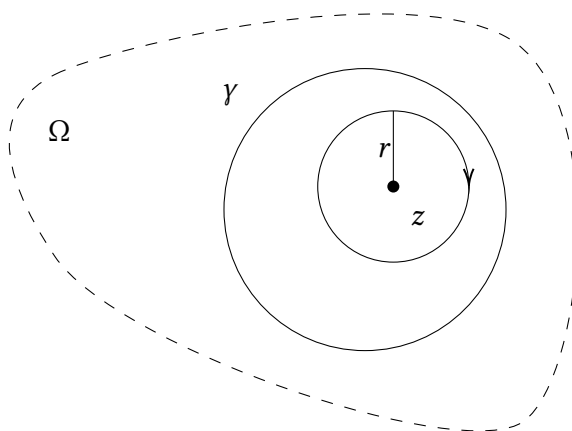
Construct $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as pictured:

Cauchy's theorem for convex regions says $\int_{\gamma_i} \frac{f(\zeta)\,d\zeta}{\zeta - z} = 0$ for all $i = 1, 2, 3, 4$.

Hence,

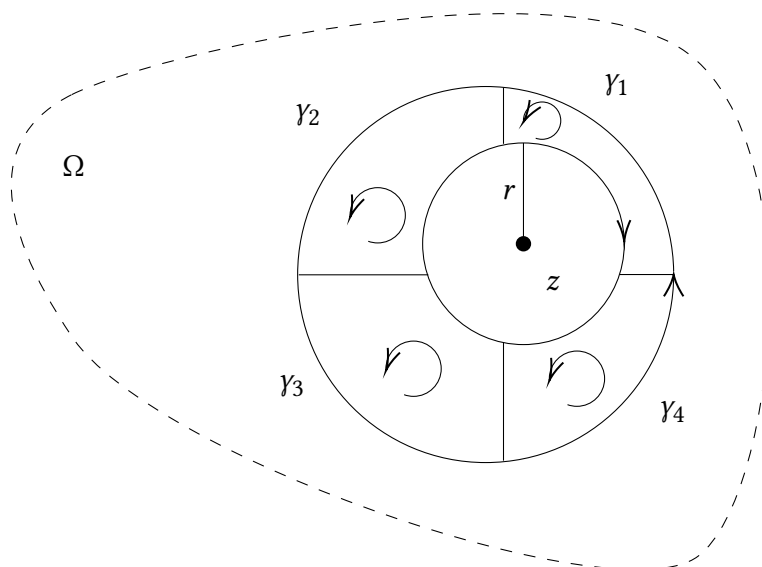$$0 = \sum_{j=1}^{4} \int_{\gamma_j} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z} - \int_{C_r(z)} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

And thus:

$$\int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z} = \int_{C_r(z)} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

for all $r > 0$ that is *sufficiently* small.

Therefore:

$$\left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z} - f(z) \cdot 1 \right| = \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z} - f(z) \cdot \left(\frac{1}{2\pi i} \int_{C_r(z)} \frac{d\zeta}{\zeta - z} \right) \right|$$

$$= \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta)\,d\zeta}{\zeta - z} - f(z) \cdot \left(\frac{1}{2\pi i} \int_{C_r(z)} \frac{d\zeta}{\zeta - z} \right) \right|$$

$$= \lim_{r \to 0^+} \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \right|$$

$$\leq \lim_{r \to 0^+} \max_{|\zeta - z| = r} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \cdot r$$

$$= 0$$

← by HW6 Ex5, or Thm12 Lect 11

$\square$

Mean value properties

Corollary **42** (Mean value property for analytic functions). If f analytic on an open set Ω which contains $B_r(z)^-$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt$$

Proof. Apply Theorem 41 with $\zeta = z + re^{it}$ and $d\zeta = ire^{it} \, dt, t \in [0, 2\pi]$ and get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) \, d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(z + re^{it})ire^{it} \, dt}{z + re^{it} - z} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt \end{aligned}$$

$\square$

Remark. There is a mean value property for harmonic functions!

Existence of power series expansions

Theorem **43.** If $f : \Omega \to \mathbb{C}$ is analytic and $z_0 \in \Omega$ then f has a power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

that converges **locally uniformly** on the disk

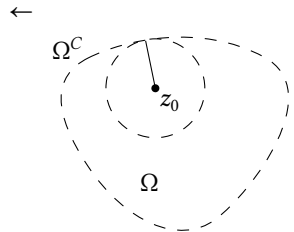$$|z - z_0| < \text{dist}(z_0, \Omega^C) = \inf_{w \in \Omega^C} |z_0 - w|$$

when Ω^C is nonempty.

Moreover, the radius of convergence is the radius of the largest open disk centered at z_0 upon which f could be analytically continued.

Proof. Let $r < \text{dist}(z_0, \Omega^C)$ and $|z - z_0| \leq \rho < r$. Then

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) \, d\zeta}{\zeta - z}$$

for all $|z - z_0| < \rho$.

As a function of ζ, the series

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

and so by geometric series formula:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \qquad \text{for } |z - z_0| \leq \rho$$

converges uniformly on $|\zeta - z_0| = r$ by the Weierstrass M-test with $M_n = \left|\frac{z - z_0}{\zeta - z_0}\right|^n \leq \left(\frac{\rho}{r}\right)^n$.

Thus,

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

$$= \frac{1}{2\pi i} \int_{C_r(z_0)} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \cdot f(\zeta)\,d\zeta$$

$$= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

And so we have our $\frac{f^{(n)}(z_0)}{n!} = a_n$ in the highlighted part above. $\qquad\square$

Remark. Consequently, we also get Cauchy's theorem of derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,d\zeta}{(\zeta - z_0)^{n+1}}$$

Example 23. What is the radius of convergence for the power series of

$$f(x) = \frac{e^{\sin x} + e^{-x^2} + x^2 + 7x^3}{\cos x}$$

centered at $x_0 = 2$?

The theorem guarantees the existence of the power series, and the RoC would simply be the radius of which f could be analytically continued. We observe that $f(x)$ cannot be defined when $\cos x = 0$, i.e. $x = \frac{\pi}{2}$. Hence, the radius of convergence is just $2 - \frac{\pi}{2}$ – no need to compute *any* derivatives or coefficients!

So now we have this result for computing the derivatives and integrals around a circle $C_r(z_0)$. Can we extend this to other closed curves of any shapes?
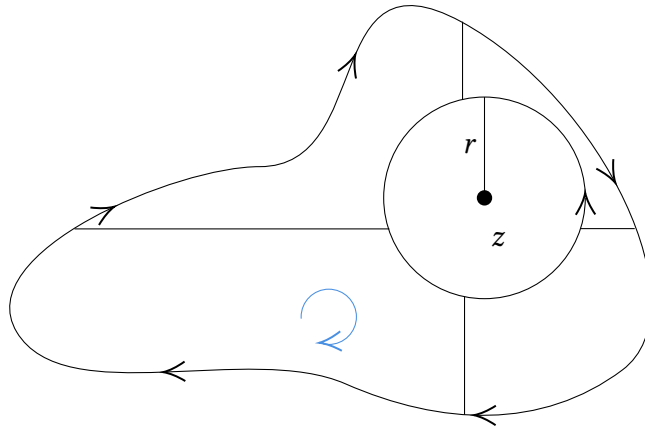
Same techniques! Hence,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{n+1}}$$

on any such closed curve γ.

Liouville's theorem

Theorem 44 (Liouville's). A bounded entire function is constant. $\leftarrow$ analytic on $\mathbb{C}$

Proof. Suppose f is entire and $|f(z)| \le M$ is bounded by M for all $z \in \mathbb{C}$. Then

$$f'(z) = \frac{1!}{2\pi i} \int_{C_R(z)} \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta$$

by Cauchy's integral formula. Hence, $|f'(z)| \le \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}$ by the upper bound. Since f is entire, there is no limit in what R could be, so we let $R \to \infty$ and observe that $|f'(z)| = 0$ for all $z \in \mathbb{C}$. Hence, f' is identically 0, and so f is constant. $\qquad\qquad\square$

Non-example 24. We know $|\cos x| \le 1$ for all $x \in \mathbb{R}$, but $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ is **not** bounded on $\mathbb{C}$. In fact, $\cos(-ix) = \frac{e^x+e^{-x}}{2}$ is unbounded for real x, so $\cos x$ is not bounded on the imaginary axis. Hence, we can't use Liouville's theorem here!

Fundamental theorem of algebra

Theorem 45 (FToA). Every **nonconstant** complex polynomial has a zero in $\mathbb{C}$.

← recall "$\mathbb{C}$ is an algebraically closed field"

Proof. Suppose towards a contradiction that p is a **nonconstant** polynomial over $\mathbb{C}$ with no zeros in $\mathbb{C}$. Then $f = \frac{1}{p}$ is an entire function because we never divide by 0. Recall HW2 Ex2 showed that $\lim_{|z|\to\infty} p(z) = \infty$. That is, for any $M > 0$, there exists $R > 0$ such that whenever $|z| > R$, we have $|p(z)| > M$.

Thus, $\lim_{|z|\to\infty} f(z) = \lim_{|z|\to\infty} \frac{1}{p(z)} = 0$. In particular, we can find a $R > 0$ such that whenever $|z| > R$, we have $|f(z)| < 1$ is bounded outside of the circle $|z| = R$. Since the closed disk $|z| \leq R$ is compact and f is continuous, f is bounded inside this closed disk $|z| \leq R$.

← Extreme value theorem

Hence, f is a bounded entire function, meaning that it is constant by Liouville's theorem, and hence p is **constant** too. This cause a contradiction. $\square$

Zeros of analytic functions

Recall the analytic functions are infinitely differentiable.

Suppose $f : \Omega \to \mathbb{C}$ is analytic and $f(z_0) = 0$ for some $z_0 \in \Omega$, and f is not identically 0 on an open neighbourhood of z_0. Then

$$f(z) = \sum_{j=n}^{\infty} a_j(z - z_0)^j$$

for some $n \geq 1$ such that $a_n \neq 0$. Hence:

← the lowest power term that has a nonzero coefficient, and also n is the order of the zero z_0.

$$f(z) = \sum_{j=n}^{\infty} a_j(z - z_0)^j$$
$$= (z - z_0)^n \sum_{j=n}^{\infty} a_j(z - z_0)^{j-n}$$
$$= (z - z_0)^n \sum_{k=0}^{\infty} a_{n+k}(z - z_0)^k$$

let $g(z) = \sum_{k=0}^{\infty} a_{n+k}(z - z_0)^k$. Observe that g is analytic and $g(z_0) = a_n \neq 0$. This and the continuity of g at z_0 ensures that g is nonzero on some open disk $|z - z_0| < \delta$. Therefore, $f(z) = (z - z_0)^n g(z)$ does not vanish on $0 < |z - z_0| < \delta$.

Remark. The zeros of f are **isolated** in Ω. That is, we can't have a sequence of zeros of f converging to some $z_0 \in \Omega$, as then we can't find a nonzero disk around z_0!

Theorem 46. If $f : \Omega \to \mathbb{C}$ is analytic and not identically zero, then each zero of f is isolated and has finite order.

Proof. Assume BWOC that the zeros are not isolated.

By definition, Ω is connected. By definition×2, a subset $S \subseteq \Omega$ is **clopen** if it is open and closed as a subset of Ω. In a connected region Ω, only $\emptyset, \Omega$ are clopen.
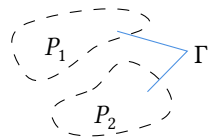
Let $S = \{z \in \Omega : f^{(j)}(z) = 0 \quad \forall j = 0, 1, 2, \dots\}$. If $z_0 \in S$ then f is zero on some open disk centered at z_0. Hence, S is open!

Now suppose w is a limit of a sequence in S. Since f is continuous, $f^{(j)}$ is continuous for all $j \in \mathbb{N}$. This enruses that $f^{(j)}(w) = 0$. Thus, S is closed!

Therefore, S is clopen in Ω, so either S is the empty set or $S = \Omega$. If $S = \Omega$, then f is the zero function, so that cannot happen! Therefore, $S = \emptyset$, and so we don't have a cluster of zeros. $\qquad\square$

← Example of nontrivial clopen subsets:

In this Γ (NOT a region), the clopen subsets are $P_1, P_2, \emptyset, \Gamma$.

Corollary 47. If f is a nonconstant analytic function, its zero set is **countable**. This is because within an open region, we can have at most countably infinite number of disjoint open sets. We let these open sets be $f^{-1}(\{0\})$.

Identity theorem

Definition 29. An **accumulation** point of S is a point that is the limit of a sequence of **distinct** points of S.

accumulation point
& limit point

limit point of the trivial sequence, but not accumulation
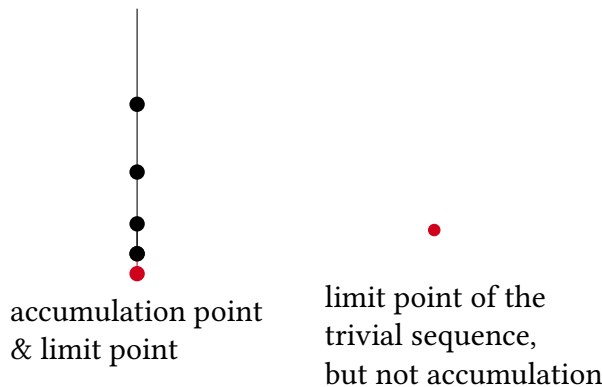
Theorem 48. Let $f, g : \Omega \to \mathbb{C}$ be analytic. If $f = g$ on a subset of Ω that has an accumulation point in Ω, then $f = g$ on the entire Ω.

Proof. If the zero set of $f - g$ has an accumulation point in Ω, then $f - g$ has a zero that is not isolated (no open disk around it since some zeros keep converging to that accumulation point), so $f - g$ is identically zero on Ω. $\qquad\square$

Example 25. There is only one way to extend $\cos x, \sin x, \exp x$ from $\mathbb{R}$ to $\mathbb{C}$ because two entire functions that agree on $\mathbb{R}$ agree on $\mathbb{C}$.

Example 26. Similarly, there is also only one way to get an analytic continuation of the Riemann zeta function to $\text{Re } s > 0$.

Maximum modulus principle

Recall this handwavy physics application here. We now have a more rigorous way to state this!

$\leftarrow$ Not exactly equivalent, though.

Theorem 49 (Maximum modulus principle). Let f be analytic on a region Ω that contains a piecewise C' simple closed curve γ and its interior. Then

$$|f(z)| \leq \max_{\zeta \in \gamma} |f(\zeta)|$$

for all z in the interior of γ.

Proof. Let $M = \max_{\zeta \in \gamma} |f(\zeta)|$. Fix z inside γ. Let L denote the length of γ and let $r = \inf_{\zeta \in \gamma} |z - \zeta|$, which is positive (so z isn't arbitrarily close to γ).

Apply Cauchy's integral formula to the n-th power of f:

$$f(z)^n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)^n \, d\zeta}{\zeta - z}$$

Thus,

$$|f(z)|^n \leq \frac{1}{2\pi} \cdot \frac{M^n}{r} L$$

Now we just take the n-th root everywhere:

$$|f(z)| \leq M \left(\frac{L}{2\pi r}\right)^{1/n}$$

We use arbitrarily large n and get $|f(z)| \leq M$. $\qquad\square$

Schwarz' lemma

Lemma 50 (Schwarz'). Let $f : \mathbb{D} \to \mathbb{D}^-$ be analytic and $f(0) = 0$. Then:

(a) $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

(b) If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $f(z) = \lambda z$ for some λ with $|\lambda| = 1$.

Proof part (a). Since $f(0) = 0$, we have that the constant term of f is 0, and so $f(z) = zg(z)$ for some g analytic on $\mathbb{D}$. Thus,

$$f'(z) = g(z) + zg'(z)$$

and hence $f'(0) = g(0)$. Hence,

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

If $|z| \leq r < 1$, then by the maximum modulus principle,

$$\begin{aligned} |g(z)| &\leq \max_{|\zeta|=r} |g(\zeta)| \\ &= \max_{|\zeta|=r} \left| \frac{f(\zeta)}{\zeta} \right| \\ &\leq \frac{1}{r} \qquad\qquad\qquad \text{since } f : \mathbb{D} \to \mathbb{D}^- \end{aligned}$$

Let $r \to 1^-$ and get $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, which is the (a) part of our result. $\square$

Proof part (b). Maximum modulus principle says the given conditions imply g is constant. The constant λ has absolute value 1, so $\frac{f(z)}{z} = \lambda$ and so $f(z) = \lambda z$. $\square$

Automorphism group of a region

Definition 30. Let Ω be a region in $\mathbb{C}$. We let the automorphism group of the region $\mathrm{Aut}(\Omega)$ be the set of all **bijective analytic functions** from Ω to Ω.

- $\mathrm{Aut}(\Omega)$ contains the identity function $f(z) = z$.

- $\mathrm{Aut}(\Omega)$ is closed under composition.

- $\mathrm{Aut}(\Omega)$ is closed under inverses: if $f : \Omega \to \Omega$ is an analytic bijection, then $f^{-1} : \Omega \to \Omega$ exists and is **analytic**.

$\leftarrow$ And composition is a binary operation with associativity

Remark. There appears to be a 'counterexample':
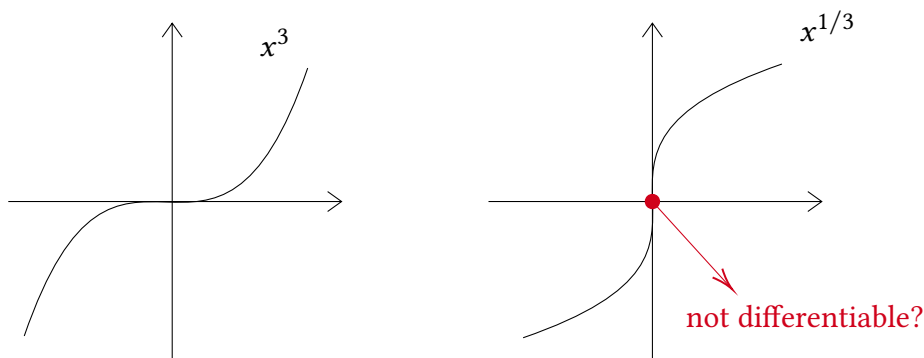
However, this is only true in $\mathbb{R}$. In $\mathbb{C}$, we observe that z^3 is **not** a bijection at 0, so this function is not in the group at all!

Theorem 51 (Riemann Mapping). If Ω is simply connected (no holes), then it could be conformally mapped to a disk.

$\leftarrow$ Except for the entire $\mathbb{C}$, which has only constant functions if bounded (by Liouville thm).
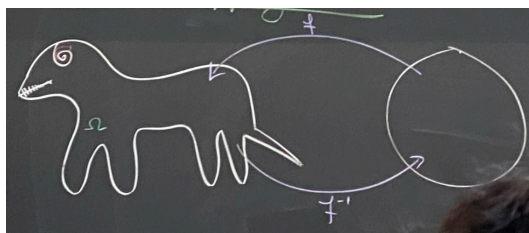
Recall from HW1 that for each $w \in \mathbb{D}$, we have a bijection

$$\phi_w(z) = \frac{-z + w}{-\bar{w}z + 1}$$

from $\mathbb{D}$ to $\mathbb{D}$ and $\phi \circ \phi = \text{id}$. To see this, observe the matrix representation

$$\begin{bmatrix} -1 & w \\ -\bar{w} & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & w \\ -\bar{w} & 1 \end{bmatrix} = \begin{bmatrix} 1 - |w|^2 & 0 \\ 0 & 1 - |w|^2 \end{bmatrix} \sim I$$

Furthermore, note $\phi_w(0) = w$. Suppose $f \in \text{Aut}(\mathbb{D})$, then there is a unique $w \in \mathbb{D}$ such that $f(w) = 0$. Define $g = f \circ \phi_w \in \text{Aut}(\mathbb{D})$. Note that $g(0) = f(\phi_w(0)) = f(w) = 0$. By Schwarz' lemma, we have $|g(z)| \leq |z|$ for all $z \in \mathbb{D}$.

Since $g^{-1} \in \text{Aut}(\mathbb{D})$, we also have $|g^{-1}(z)| \leq |z|$ for all $z \in \mathbb{D}$. Now substitute $g(z)$ for z since it's also in the disk. Hence, $|z| = |g^{-1}(g(z))| \leq |g(z)|$. Therefore, we are forced to conclude that $|z| = |g(z)|$ for ALL $z \in \mathbb{D}$!

Since $|z| = |g(z)|$ for ALL $z \in \mathbb{D}$, Schwarz' lemma says $g(z) = \lambda z$ for some $|\lambda| = 1$. Let $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thus,

$$g(z) = f(\phi_w(z)) = e^{i\theta} z$$

and so

$$e^{i\theta}\phi_w(z) = f(\phi_w(\phi_w(z))) = f(z)$$

since $\phi_w \circ \phi_w = \text{id}$. Therefore, $f(z) = e^{i\theta}\dfrac{w - z}{1 - \bar{w}z}$.

Therefore,

Proposition 52.

$$\text{Aut}(\mathbb{D}) = \left\{ e^{i\theta}\frac{w - z}{1 - \bar{w}z} \ \middle|\ \theta \in [0, 2\pi), w \in \mathbb{D} \right\}$$

Remark. The topological representation of the automorphism group of $\mathbb{D}$ is a 'skinless torus' (collection of open disks revolving from 0 to 2π).

Morera's theorem

Theorem 53 (Morera). If $f : \Omega \to \mathbb{C}$ is continuous and $\int_\gamma f(\zeta)\,d\zeta = 0$ for all _____ γ in Ω, then f is analytic on Ω.

← the blank can be 'rectangles', 'triangles', 'piecewise C^1 closed curves', etc.

Proof see notes.   □

Weierstrass convergence theorem

Let $f_n(z)$ be analytic for every $n \in \mathbb{N}$. We are still not sure that $\sum_{n=1}^{\infty} f_n(z)$ is analytic yet!

Theorem 54 (Weierstrass convergence). If $f_n : \Omega \to \mathbb{C}$ are analytic and f_n converges *locally uniformly* on Ω to the limit function f ($\Leftrightarrow$ uniform convergence on compact sets), then f is analytic and for each fixed m, f_n^m converges to $f^{(m)}$ locally uniformly on Ω and f is infinitely differentiable.

Remark. This is a huge contrast with the *Weierstrass approximation theorem* in real analysis, which says that if $f : [0, 1] \to \mathbb{R}$, then there is a sequence of polynomials p_n such that p_n converges to f uniformly on $[0, 1]$. That is, even the most pathological, nowhere-differentiable functions in $\mathbb{R}$ are a limit of some polynomial sequences! However, in the $\mathbb{C}$ world, the limit of any analytic function is still analytic.

Laurent series & isolated singularities

Sometimes the domain Ω isn't the largest domain where an analytic function can be analytic. So far, we know we can find the largest disk centered at a point in Ω in which a function is analytic and the power series exists there:

← the disk could exceed the bounds of Ω!

$z_0 \in \Omega$

Ω

z_0

f is representable as a power series centered at z_0 (f can be analytically continued here)
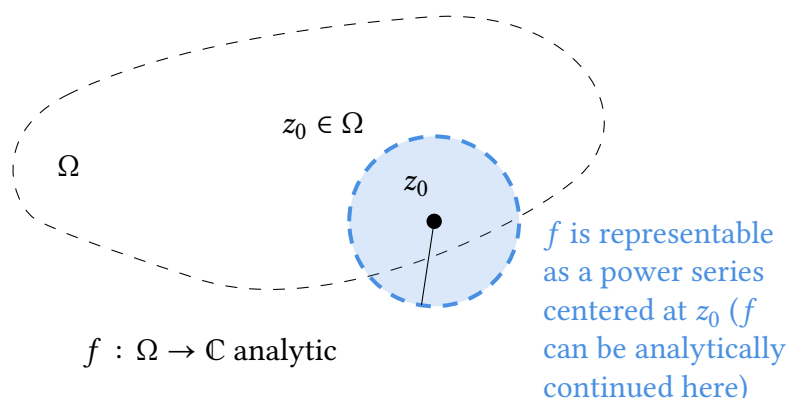
$f : \Omega \to \mathbb{C}$ analytic

Can we do even better than that?

Laurent series

Example 27. Let $f(z) = \frac{1}{z(z-1)}$ analytic on $\mathbb{C}\backslash\{0, 1\}$. We realize that if we restrict $0 < |z| < 1$, then
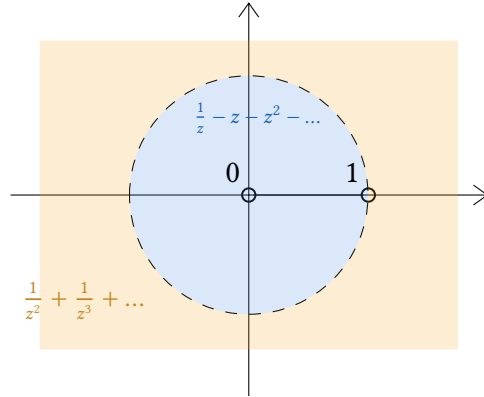
$$f(z) = \frac{-1}{z} \cdot \frac{1}{1-z} = \frac{-1}{z} - 1 - z - z^2 - \ldots$$

is the **Laurent series** of $f(z)$ centered at 0, a point where $f(z)$ isn't even defined!

Example 28. We continue with the previous function. This time, we restrict $|z| > 1$ and express it as:

$$f(z) = \frac{1}{z^2(1 - \frac{1}{z})} = \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

Hence, we now have different series for f in different regions:

Definition 31 (Laurent series). A series in the form $\displaystyle\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is a **Laurent series**. It converges at $z \in \mathbb{C}$ if **both** the <u>analytic part</u> $\displaystyle\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and the <u>principal part</u> $\displaystyle\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converge at z. If this occurs, the Laurent series would be

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

and also converges.

Lemma 55. Recall that if $n \neq -1$, then $\frac{(z-z_0)^{n+1}}{n+1}$ is an antiderivative of $(z-z_0)^n$ on $\mathbb{C}$. **So** if γ is simple closed, then by FTC,
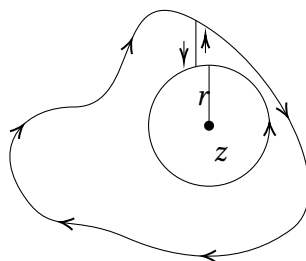
$$\frac{1}{2\pi i}\int_{\gamma}(z-z_0)^n \, dz = 0$$

whenever $n \neq 1$. In addition, by Cauchy's integral formula, $\frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-z_0} = 1$. Therefore, if z_0 is in the interior of a simple closed curve γ, then

$$\frac{1}{2\pi i}\int_{\gamma}(z-z_0)^n \, dz = \begin{cases} 0 & n \neq -1 \\ 1 & n = -1 \end{cases}$$

Proof. We previously know the result above when γ is a circle. We now extend it

to all simple closed curves by a familiar trick as follows:

$\square$

Laurent expansion theorem

Theorem 56 (Laurent expansion). Suppose f is analytic on the annular region $A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$. Then f has a locally uniformly convergent Laurent expansion

$\leftarrow$ $R_1 = 0, R_2 = \infty$ are okay
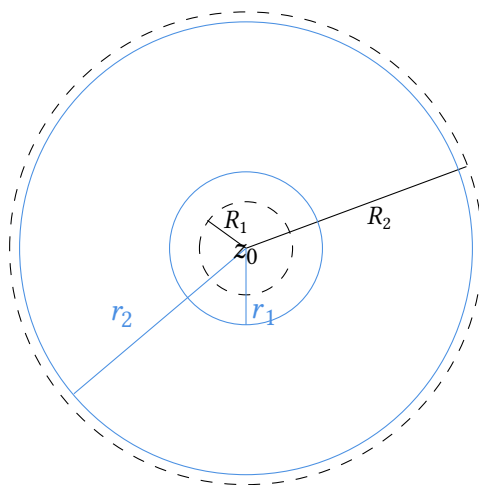
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

on A. Moreover, the Laurent coefficients are

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$$

for any r such that $R_1 < r < R_2$.

Proof gist. For a gist of why this works:

$\leftarrow$ For rigorous proof, see notes.

Cauchy's integral formula reveals that

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

whenever

$$R_1 < r_1 < |z| < r_2 < R_2.$$

These integrals are independent of r_1 and r_2 so long as $r_1 < |z| < r_2$. $\square$

Remark. If $n \geq 0$ and f is analytic on $|z| < R_2$, then we should get that the Taylor series expansion and the Laurent expansion for the same function f to match. They indeed do match by Cauchy's integral formula:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z_0)^{n+1}}$$