

# MATH135 Complex Analysis Notes

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April 2, 2024

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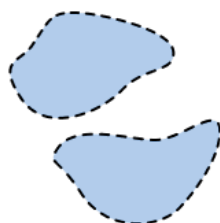
# Regions, differentiability, analyticity

## Regions

**Definition 1.** A **region** is a nonempty, connected, open subset of  $\mathbb{C}$ .

- A region without “holes” is simply connected.

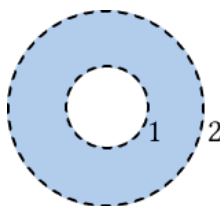
**Non-example 1.** This is not a region (not connected):



**Example 2.**  $\mathbb{C}$  is a region.

**Example 3.**  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ , the open unit disk is a region.

**Example 4.**  $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$ , the annulus region is a region that is not *simply-connected*:



## Complex derivatives and analyticity

**Definition 2.** Let  $\Omega$  be a region. Let  $z_0 \in \Omega$  and  $f : \Omega \rightarrow \mathbb{C}$  be a function.

1. Complex function  $f$  is **differentiable** at  $z_0$  if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

2. If  $f$  is differentiable at every point in  $\Omega$ , we say  $f$  is **analytic** on  $\Omega$ .
3. If  $f$  is analytic on  $\mathbb{C}$ , then  $f$  is **entire**.

← this  $z \rightarrow z_0$  could be from **any** directions!

← Means that existence of 1st derivative implies the existence of  $\infty$ th derivative! & has Taylor expansion.

← Usual calculus rules work here :)

**Example 5.** Polynomials are entire functions.

**Example 6.** Rational functions are analytic on  $\mathbb{C}$  except where the denominator vanishes.

**Non-example 7.**  $f(z) = \bar{z}$  is NOT analytic **anywhere**!

*Proof.* Let  $z_0 \in \mathbb{C}$ . Then  $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$ .

If  $z \rightarrow z_0$  horizontally, then  $z - z_0 \in \mathbb{R}$ , meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if  $z \rightarrow z_0$  vertically, then  $\overline{z - z_0} = -(z - z_0)$ , meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that  $1 \neq -1$ , thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere.  $\square$

**Proposition 1.** Let  $f$  be differentiable at  $z_0$ . Then, for any  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that **whenever**  $0 < |z - z_0| < \delta$ , **we have**  $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$ .

**Remark.** Now consider multiplying  $|z - z_0|$  on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near  $z_0$  (when the distance  $< \varepsilon$ ),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the “tangent-line approximation” equivalent in  $\mathbb{C}$ !

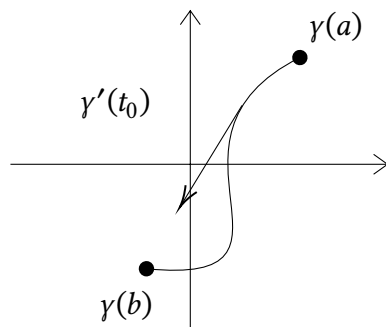
In addition,  $f(z_0) + f'(z_0)(z - z_0)$  means to take  $z - z_0$ , rotate and dilate by  $f'(z_0)$ , then translate by  $f(z_0)$ . If  $f'(z_0) \neq 0$ , this function is locally orientation-preserving and could be approximated by a linear function.

← The RHS is a **linear** function!

← This explains why  $z \mapsto \bar{z}$  is NOT analytic anywhere: it is orientation-reversing.

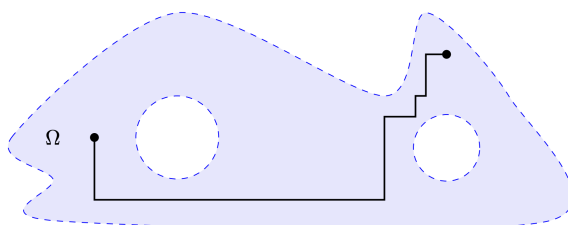
## Curves, paths

**Definition 3.** A **curve** in  $\mathbb{C}$  is a function  $\gamma : [a, b] \rightarrow \mathbb{C}, a, b \in \mathbb{R}$ .



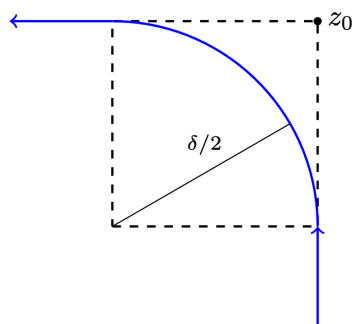
**Definition 4.** Parameterize  $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$ . Then  $\gamma'(t_0) = (x'(t_0), y'(t_0))$  is a **tangent vector** to the curve at  $\gamma(t_0)$  (assume  $\gamma'(t_0) \neq \mathbf{0}$ , aka.  $\gamma$  is regular at  $\gamma(t_0)$ .)

**Theorem 2** (The “Boxy-path” Theorem). A nonempty open set  $\Omega$  in  $\mathbb{C}$  is connected *if and only if* each pair of distinct points in  $\Omega$  can be joined by a sequence of line segments lying in  $\Omega$ , each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region  $\Omega$  there exists a “**boxy path**”.

**Remark.** There is also always a **smooth path**. That is:

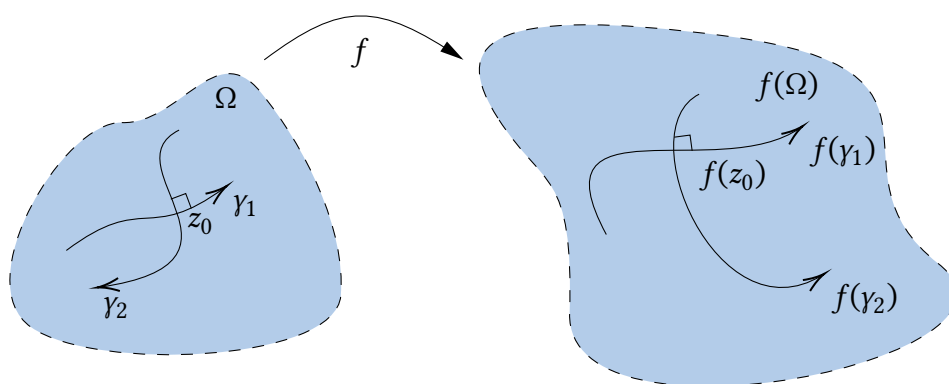


**Theorem 3** (“Smooth-path”). A nonempty open set  $\Omega$  in  $\mathbb{C}$  is connected if and only if each pair of distinct points in  $\Omega$  can be joined by a continuously differentiable curve in  $\Omega$  that is regular at every point.

*Proof.* See [lecture 2 notes](#). □

## Conformality

Let  $f$  be an analytic complex function on  $\Omega$ .



Let  $z_0 \in \Omega$  such that  $f'(z_0) \neq 0$ . Let  $\gamma_1, \gamma_2$  be two curves that pass through  $z_0$  intersecting with an angle  $\theta$ . Then  $f(\gamma_1), f(\gamma_2)$  are two curves in  $f(\Omega)$  passing through  $f(z_0)$  also with angle  $\theta$ .

Therefore,  $f$  is **conformal**!

## Cauchy-Riemann equations, harmonic functions

### Multivariate notion of complex derivatives

Recall: 
$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Now we write each function with complex variables as  $f(z) = u(z) + i v(z)$  where  $u, v$  are real-valued functions.

← meaning their range is real

Since  $\mathbb{C} \cong \mathbb{R}^2$ , we denote every point  $z = (x, y)$ .

Now we let  $f(x, y) = u(x, y) + i v(x, y)$ . We first let the small distance  $h = (r, 0)$  be horizontally approaching 0 with  $r \in \mathbb{R}$ . That is,  $z_0 + h = (x_0 + r, y_0)$ .

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r \rightarrow 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let  $h = ir = (0, r)$  with  $r \rightarrow 0, r \in \mathbb{R}$ , we would get  $f' = v_y - i \cdot u_y$ .

**Remark.** If a derivative exists, the horizontal & the vertical ones should be equal!

**Theorem 4** (Cauchy-Riemann Equations).

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

**Corollary 5.** If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and  $f' = 0$  on  $\Omega$ , then  $f$  is **constant**.

*Proof.* Since  $0 = f' = u_x + i v_x$ , we see that  $u_x = v_x = 0$  on  $\Omega$ . By Cauchy-Riemann,  $v_y = u_y = 0$  is also true on  $\Omega$ . Hence,  $\mathbf{u}, \mathbf{v}$  are constant on either horizontal or vertical segments. By the Boxy Path Theorem,  $f = u + i v$  cannot assume two distinct values in  $\Omega$ .  $\square$

## Orientation-preserving as shown by Jacobian

Let  $f : \Omega \rightarrow \mathbb{C}$  be analytic. Then  $f' = u_x + i v_x$  and hence:

$$\begin{aligned} |f'|^2 &= \bar{f}' \cdot f' = (u_x - i v_x)(u_x + i v_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} && \text{the Jacobian of } f! \end{aligned}$$

Since  $|f'|^2 \geq 0$ , the determinant of the Jacobian is always  $\geq 0$ , implying that  $f$  is always locally orientation-preserving. Moreover,

**Proposition 6.** If  $f'(z_0) \neq 0$ , then  $|f'|^2 > 0$  implies:



1.  $f$  is **injective** near  $z_0$
2.  $f$  scales  $\mathbb{R}$  by  $|f'(z_0)|^2$  near  $z_0$
3.  $f$  preserves orientation near  $z_0$

## The Laplacian, harmonic functions and conjugates

Suppose that  $f = u + iv$  is analytic and  $u, v$  have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function  $u$  is 0!

**Definition 5.** Real-valued functions  $u : \Omega \rightarrow \mathbb{R}$  satisfying that the Laplacian  $\Delta u = u_{xx} + u_{yy}$  is 0 on  $\Omega$  is called **harmonic functions**.

**Definition 6.** A **harmonic conjugate** of  $u$  is a harmonic function  $v : \Omega \rightarrow \mathbb{R}$  such that  $f = u + i \cdot v$  is **analytic** on  $\Omega$ .

**Example 8.**  $u = x^2 - y^2, v = 2xy$ .

**Remark.** Harmonic conjugates are unique up to translation ( $\pm$  constants).

**Remark.** If  $u$  is harmonic on  $\Omega$ , it does NOT have to have a harmonic conjugate on  $\Omega$ .

←  $\Delta u = 0$   
characterizes  
steady-state  
solutions to heat  
equations on  $\Omega$ .

← Check it!

## Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations:  $u_x = v_y$  and  $u_y = -v_x$ .

**Example 9.**  $u(z) = \log |z|$  is harmonic on  $\mathbb{C} \setminus \{0\}$ .

*Proof.* Write  $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2} \log(x^2 + y^2)$ .

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x} \left( \frac{1}{2} \log(x^2 + y^2) \right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} &= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

← Review quotient rule!

Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Hence  $u_{xx} + u_{yy} = 0$ , implying that the function is harmonic.  $\square$

Now, can we find a harmonic conjugate for the aforementioned  $u$ ?

We could use the two Cauchy-Riemann Equations. One of them:

$$\begin{aligned} v_y &= u_x \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v(x, y) &= \int v_y dy + C(x) && \text{unknown function of } x \\ &= \arctan\left(\frac{y}{x}\right) + C(x) \end{aligned}$$

Then, we use the second one:

$$\begin{aligned} \frac{y}{x^2 + y^2} &= u_y = -v_x = -\frac{\partial}{\partial x} \left( \arctan\left(\frac{y}{x}\right) + C(x) \right) \\ &= \frac{y}{x^2 + y^2} - C'(x) \implies C'(x) = 0 \end{aligned}$$

Hence, a good harmonic conjugate candidate seems to be

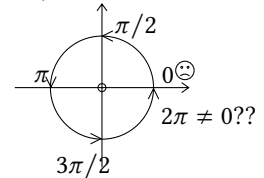
$$v(x, y) = \arctan\left(\frac{y}{x}\right) + C$$

where  $C$  is a constant. WLOG, let  $C = 0$ . Then  $v(x, y) = \arctan\left(\frac{y}{x}\right)$ , meaning that:

$$v(z) = \arg(z)$$

Therefore,  $f(z) = \log|z| + i \cdot \arg(z)$  is analytic!

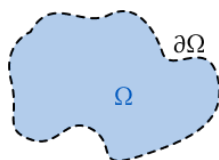
← There is currently a great **CAVEAT** in all of these, because  $v(z) = \arg(z)$  cannot be defined in a continuous manner in all of  $\mathbb{C} \setminus \{0\}$ :



To be resolved later!

## Physics analogies of harmonic functions

**Example 10.** Let  $T(x, y, t)$  be the temperature at  $(x, y)$  at time  $t$  of a thermally conductive plate in  $\mathbb{C}$ . Assume the plate gives rise to a **bounded** region  $\Omega$  (with boundary denoted  $\partial\Omega$ ). Temperature on  $\partial\Omega$  is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where  $\alpha$  is a constant.

We think the system tends towards a thermal equilibrium as  $t \rightarrow \infty$ . At equilibrium,  $\frac{\partial T}{\partial t}$  is **zero**. Hence, at equilibrium,  $\Delta T = T_{xx} + T_{yy} = 0$ .

**Idea:** Harmonic function behave like equilibrium temperature distributions!

**Proposition 7.** Let  $U(x, y)$  be a harmonic function on  $\Omega$ .

1.  $U$  cannot have a *local* maximum in  $\Omega$ .
2. The absolute maximum of  $U$  on  $\Omega^-$  occurs on  $\partial\Omega$ .
3.  $U$  cannot be locally constant without being globally constant.

←  $\Omega^-$  denotes the closure of  $\Omega$

**Theorem 8** (Maximum principle). Let  $\Omega$  be a bounded region in  $\mathbb{C}$  and let  $f : \Omega^- \rightarrow \mathbb{C}$  be analytic on  $\Omega$  and continuous on  $\Omega^-$ .

1. If  $|f|$  achieves a local max in  $\Omega$ , then  $f$  is constant.
2. The global max of  $|f|$  on  $\Omega^-$  is attained on  $\partial\Omega$ .

## Möbius transformations

### Möbius transformations, the extended plane

**Definition 7** (Möbius transformations).

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an  $f$  is **analytic** on  $\mathbb{C} \setminus \{\frac{-d}{c}\}$  and **conformal** there since  $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$  on  $\mathbb{C} \setminus \{\frac{-d}{c}\}$ .

← recall that rational functions are analytic except when the denominator vanishes, i.e.  $cz + d \neq 0$ .

**Remark.** In addition,  $f$  is injective (one-to-one)!

*Proof.*

$$\begin{aligned} f(z) = f(w) &\implies \frac{az+b}{cz+d} = \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

**Definition 8** (The extended plane). We set the following convention:

$$\begin{aligned} f\left(\frac{-d}{c}\right) &= \infty \\ f(\infty) &= \frac{a}{c} \end{aligned}$$

with this,  $f$  is a **bijection** from  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  to itself.

← recall Riemann sphere

## Möbius transformations as matrices

**Remark.** We can associate  $f(z) = \frac{az+b}{cz+d}$  where  $ad-bc \neq 0$  with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

**Remark.**  $M_{f \circ g} = M_f \cdot M_g$

← check this!

**Remark.** The inverse of  $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  is  $M_f^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$  and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

**Theorem 9.** A Möbius transformation  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  with three fixed points in  $\hat{\mathbb{C}}$  is the **identity map**  $\text{id}(z) = z = \frac{z+0}{0z+1}$ .

←  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

*Proof.* Let  $f(z) = \frac{az+b}{cz+d}$  be a Möbius transformation.

1. If  $\infty$  is fixed, then  $c = 0$ . Then  $f(z) = \frac{a}{d}z + \frac{b}{d}$ , which is a **linear** transformation. ← think about that!
  - (a) If  $f(z) = z$ , we are done since we get the identity!
  - (b) Otherwise the function only has one fixed point at  $\infty$ .
2. If  $\infty$  is not a fixed point, then  $c \neq 0$ . Solve:

$$\begin{aligned} f(z) + z &\Leftrightarrow \frac{az + b}{cz + d} = z \\ az + b &= cz^2 + dz \\ cz^2 + (d - a)z - b &= 0 \end{aligned}$$

is a quadratic which has at most two (distinct) solutions in  $\mathbb{C}$ . Hence, this transformation fixes at most two points.

□

## Möbius transformations take circles to circles

**Remark.** Lines can be circles (they are just circles that pass through the point at infinity).

**Theorem 10.** The image of a circle under a Möbius transformation is still a circle.

*Proof.* Let  $f(z) = \frac{az+b}{cz+d}$  be a Möbius transformation.

1. If  $c = 0$ , then  $f(z) = \frac{a}{d}z + \frac{b}{d}$ , which is a **linear/affine** transformation and so we are done.
2. Now suppose  $c \neq 0$ . Then

← since linear transformations preserve circles and lines

$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} \\ &= \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c} \end{aligned}$$

which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz + d} \mapsto \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c}$$

We now only need to show that inversion preserves circles.

Let a circle in  $\mathbb{R}^2$  be  $Ax + By + C(x^2 + y^2) = D$  where  $A, B, C, D \in \mathbb{R}$ . If  $z = x + iy \in \widehat{\mathbb{C}}$ , then  $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$ . Name  $\frac{1}{z} = u + iv$ , note that  $u^2 + v^2 = \frac{1}{x^2+y^2}$ .

Then we note that  $Au - Bv + C = D(u^2 + v^2)$ , which is still a circle!

← check this!

□

**Theorem 11.** Given two triples  $z_1, z_2, z_3$  and  $w_1, w_2, w_3$  of *distinct* points in  $\widehat{\mathbb{C}}$ , then there is always a unique Möbius transformation  $f$  such that  $f(z_i) = w_i$  for all  $i = 1, 2, 3$ .

*Proof.* Claim: the *cross-ratio*  $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$  is a Möbius transformation that satisfies  $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$ .

We can also find another Möbius transformation such that  $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$ . Then:

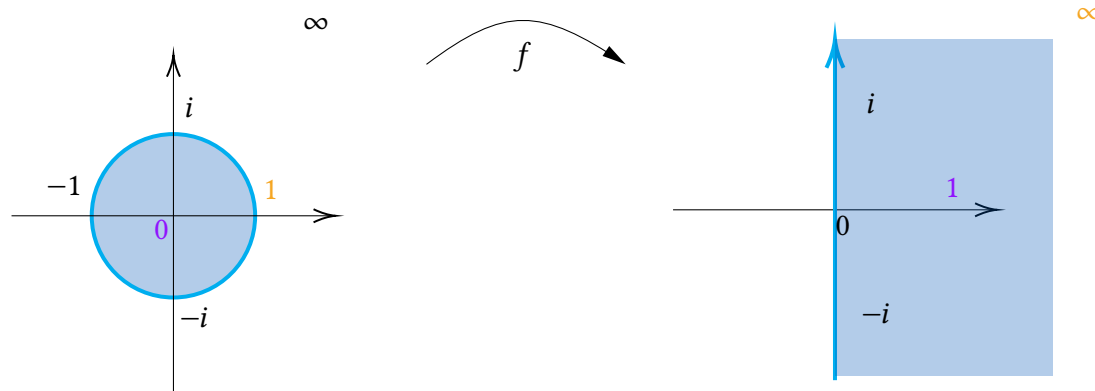
$$\begin{array}{ccc} z_1 & \xrightarrow{\phi} & 0 \xrightarrow{\psi^{-1}} w_1 \\ z_2 & \xrightarrow{\phi} & 1 \xrightarrow{\psi^{-1}} w_2 \\ z_3 & \xrightarrow{\phi} & \infty \xrightarrow{\psi^{-1}} w_3 \end{array}$$

and we could simply let  $f = \psi^{-1} \circ \phi$ .

□

**Example 11.** Let  $f(z) = \frac{z+1}{-z+1}$ . We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



## Recall: infinite series

**Definition 9.**  $\sum_{n=1}^{\infty} a_n$  converges to  $S$  if  $\lim_{N \rightarrow \infty} S_N = S$  where  $S_N = a_1 + \dots + a_N$ .

←  $S_N$  is the  $N$ -th partial sum.

## Divergence test

**Definition 10** (Divergence test). A pair of contrapositives:

← Note it's not an *if and only if* !

1. If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .
2. If  $\lim_{n \rightarrow \infty} a_n \neq 0$  (including the case where the limit doesn't exist) then  $\sum_{n=1}^{\infty} a_n$  diverges.

**Non-example 12.** The harmonic series  $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \dots$  diverges even though  $a_n = \frac{1}{n}$  tends to 0 when  $n$  tends to  $\infty$ .

← diverges, but really **slowly**!

**Theorem 12.** If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} a_n = \lim_{N \rightarrow \infty} S - S_N = 0$ .

← In other words, the tail of a convergent series goes to 0.

**Theorem 13** (Cauchy Criterion).  $\sum_{n=1}^{\infty} a_n$  converges *if and only if* for all  $\varepsilon > 0$ ,

there exists  $N \in \mathbb{N}$  such that  $k > j > N$  implies  $\left| \sum_{n=j}^k a_n \right| = S_k - S_j < \varepsilon$ .

## Integral test

**Definition 11** (Integral test). Define  $a_n = f(n)$  for  $n \in \mathbb{N}$ , where  $f : [1, \infty[ \rightarrow \mathbb{R}$  is (piecewise) continuous, positive and decreasing. Then  $\int_1^{\infty} f(x) dx$  converges *if and only if*  $\sum_{n=1}^{\infty} a_n$  converges.

← do an improper integral!

Moreover,  $\int_1^N f(x) \, dx \leq a_1 + \dots + a_N \leq a_1 + \int_1^N f(x) \, dx$ .

**Example 13.** Apply the above with  $f(x) = \frac{1}{x}$ . Then

←  $a_n = \frac{1}{n}$

$$\ln N \leq 1 + \frac{1}{2} + \dots + \frac{1}{N} \leq 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

**Theorem 14.** The “ $p$ -series”  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges *if and only if*  $p > 1$ .

**Definition 12** (Riemann zeta function).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \operatorname{Re}(s) > 1$$

**Remark.** Euler figured out:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

**Remark.** R. Apéry showed that  $\zeta(3)$  is irrational (1979):

← still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202 \dots$$

but no explicit formula known!

## Absolute convergence

**Definition 13.** A series  $\sum_{n=1}^{\infty} a_n$  is:

1. **absolutely convergent** if  $\sum_{n=1}^{\infty} |a_n|$  converges.
2. **conditionally convergent** if  $\sum_{n=1}^{\infty} a_n$  converges but  $\sum_{n=1}^{\infty} |a_n|$  diverges.

← Good

← BAD

**Theorem 15.** Every absolutely convergent series converges.

**Example 14.** The alternating harmonic series

← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$



converges to  $\ln 2$ . But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge.

**Theorem 16.** An absolutely convergent series may be rearranged without changing its value. That is, if  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  is a bijection, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

**Theorem 17** (Riemann Rearrangement Theorem). If  $\sum_{n=1}^{\infty} a_n$  is a conditionally convergent series of real numbers, then for **any**  $S \in \mathbb{R} \cup \{-\infty, \infty\}$ , there is a bijection  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\sum_{n=1}^{\infty} a_{\phi(n)} = S$ .

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  converge, one might expect that

$$\begin{aligned} \left( \sum_{i=0}^{\infty} a_i \right) \left( \sum_{j=0}^{\infty} b_j \right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^n a_k b_{n-k} \end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See [notes](#).

## Uniform convergence

**Definition 14.** A sequence of functions  $f_n : X \rightarrow \mathbb{C}$  where  $X \subseteq \mathbb{C}$  **converges uniformly** to  $f : X \rightarrow \mathbb{C}$  if for all  $\varepsilon > 0$ , there exists a  $N \in \mathbb{N}$  such that  $n \geq N$  implies  $|f_n(z) - f(z)| < \varepsilon$  for all  $z \in X$ .

← This is MATH131!

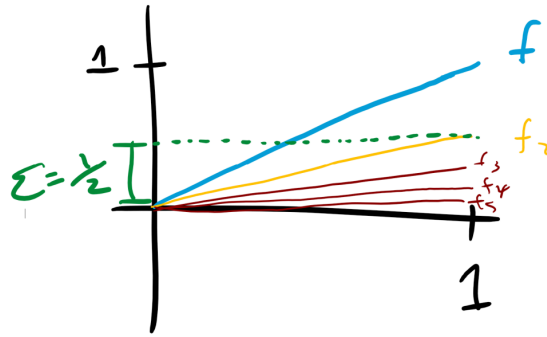


FIGURE 8. Uniform Convergence

**Theorem 18.** If  $f_n : X \rightarrow \mathbb{C}$  are continuous and converges uniformly on  $X$  to  $f : X \rightarrow \mathbb{C}$ , then  $f$  is continuous on  $X$ . In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

**Remark.**  $f_n$  converges to  $f$  pointwise on  $X$  if  $\lim_{n \rightarrow \infty} f_n(z) = f(z)$  for all  $z \in X$ .

← This doesn't say anything about the rate each point converges.

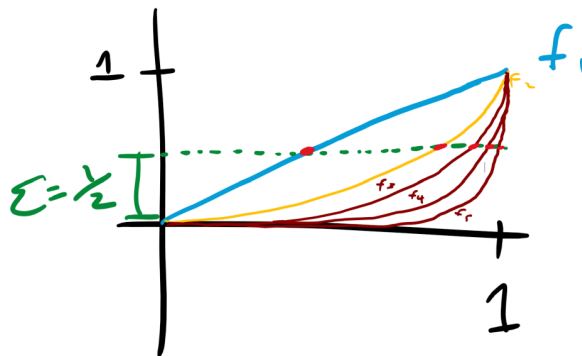


FIGURE 9. Non-uniform convergence

**Theorem 19.** If  $f_n : [a, b] \rightarrow \mathbb{C}$  are continuous and converge uniformly on  $[a, b]$  to  $f$ , then

← Integrals work with uniform convergence

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

**Remark.** Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

**Example 15.**  $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$  on  $[-1, 1]$  converges uniformly to  $f_n(x) = |x|$ . But the limit function is **not** differentiable at  $x = 0$  even though every  $f_n$  were.

**Theorem 20** (Weierstrass M-Test). Let  $f_n : X \rightarrow \mathbb{C}$  satisfy  $|f_n(z)| \leq M_n$  for all  $z \in X$  and  $n \in \mathbb{N}$ . If  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n(z)$  converges both **absolutely** and **uniformly** on  $X$ .

## Power series

**Definition 15.** A **power series** is a series of the form  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ . The  $a_n$  is the *coefficient* and  $z_0$  is the *center*.

## Convergence of geometric series

**Theorem 21.** The *geometric series* ( $a_n = 1, z_0 = 0$ )  $\sum_{n=0}^{\infty} z^n$  converges absolutely to  $\frac{1}{1-z}$  if  $|z| < 1$ , and it diverges otherwise.

Moreover, for each  $r \in [0, 1[$ , the convergence is **uniform** on  $|z| \leq r$ .

*Proof.* If  $|z| \geq 1$ , then  $z^n \not\rightarrow 0$ , so by the test of divergence, the series diverges.

Now suppose  $|z| < 1$ . Then

$$\begin{aligned} \sum_{n=0}^{\infty} z^n &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} z^n \\ &= \lim_{N \rightarrow \infty} (1 + z + z^2 + \cdots + z^{N-1}) \\ &= \lim_{N \rightarrow \infty} \frac{1 - z^N}{1 - z} \\ &= \frac{1}{1 - z} \quad \text{since } |z| < 1 \end{aligned}$$

← The fact that we can find a formula for this sum is quite rare!

Which gives us point-wise convergence. Then, for any  $r$  such that  $|z| \leq r < 1$ , we have

$$\sum_{n=0}^{\infty} |z^n| \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

Hence, by the Weierstrass  $M$ -test, the series converges *absolutely and uniformly* on  $|z| \leq r$ .  $\square$

**Remark.** Moral of the story:

- The *radius of convergence*  $R = 1$  has the property that the series converges on  $|z| < R$ , and diverges if  $|z| > R$ .
- The series converges *uniformly* on  $|z| \leq r < 1$  but not on  $|z| < 1$  itself. Why? Let  $r = 1$ ; we need be able to get  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $\left| \frac{1-z^n}{1-z} - \frac{1}{1-z} \right| < 1$  for all  $|z| < 1$ . However, this is not gonna work: as  $z \rightarrow 1$ , observe that this is going to eventually exceed 1.

- The limit function  $\frac{1}{1-z}$  is **analytic** on  $\mathbb{C} \setminus \{1\}$ . But the geometric series represents this function only on  $|z| < 1$ . In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.
- The limit function  $\frac{1}{1-z}$  is cool if  $z \neq 1$ , but as long as  $|z| = 1$  (**even** if  $z \neq 1$ ), the geometric series diverges!

← the limit function is well-defined way beyond the  $\mathbb{D}$ !

← in the complex number sense!

## Radius of convergence

**Definition 16.** The **limit superior** (lim sup) of a sequence of nonnegative real numbers  $x_n$  is the largest *limit point* of the  $x_n$ :

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \geq 0} \sup_{m \geq n} x_m$$

← limits of a subsequence of  $x_n$

If the sequence is unbounded, the lim sup would be  $\infty$ .

← the RHS as in real analysis

**Example 16.** If  $x_n$  is the sequence  $0, 1, 0, 1, \dots$  then  $\limsup_{n \rightarrow \infty} x_n = 1$ .

**Example 17.** If  $x_n$  is the sequence  $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots$ , then  $\limsup_{n \rightarrow \infty} x_n = 0$ .

**Remark.** If  $x_n$  are nonnegative, then

- $\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$
- $\limsup_{n \rightarrow \infty} (a_n b_n) \leq (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n)$

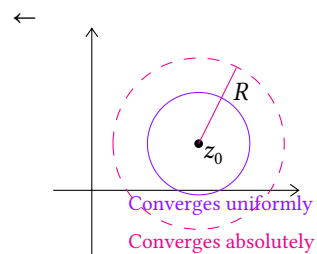
**Theorem 22** (Cauchy-Hadamard). Let  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  be a power series. Define  $R \in [0, \infty]$  by

← interpret  $\frac{1}{0} = \infty$

$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

Then the  $R$  is the *radius of convergence*.

- On  $|z - z_0| < R$ , the series converges **absolutely**. For each  $r \in [0, R[$ , the convergence is **uniform** on  $|z - z_0| \leq r$ .
- If  $|z - z_0| > R$  then the series diverges. **For  $|z - z_0| = R$  anything could happen!**



**Example 18.** We claim that  $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$  has an infinite radius of convergence  $R = \infty$ . To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \rightarrow 0$$

This is because  $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \dots \cdot n}$ , and in  $n!$ , there are at least  $\frac{1}{2}$  terms that are  $> \frac{n}{2}$ .

Thus,  $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{1/2} \rightarrow \infty$ .

So  $R = \infty$  and we are done 😊. We have that  $\exp(z)$  has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$\begin{aligned} \exp(z) \exp(w) &= \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left( \sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!} \right) \\ &= \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!} \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k}}_{\text{binomial theorem}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (z + w)^n \\ &= \exp(z + w) \end{aligned}$$

Now define  $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$ .

## Term-by-term differentiation of power series

**Lemma 23.**  $n^{\frac{1}{n}} \rightarrow 1$

*Proof 1.*  $e^{\log(n^{\frac{1}{n}})} = e^{\frac{\log n}{n}} \rightarrow e^0 = 1$  by l'Hopital. So  $n^{\frac{1}{n}} \rightarrow 1$ . □

*Proof 2 (better).* Write  $n^{\frac{1}{n}} = 1 + \delta_n$  where  $\delta_n \geq 0$ . The binomial theorem says:

$$\begin{aligned} n &= (1 + \delta_n)^n \\ &= \sum_{k=0}^{\infty} \binom{n}{k} \delta_n^k \cdot 1^{n-k} \\ &= 1 + n\delta_n + \frac{n(n-1)}{2} \delta_n^2 + \dots \end{aligned}$$

$$\geq 1 + \frac{n(n-1)}{2} \delta_n^2$$

Therefore,  $n-1 \geq \frac{n(n-1)}{2} \delta_n^2$  and we get  $\frac{2}{n} \geq \delta_n^2 \geq 0$  hence  $\delta_n \rightarrow 0$ .

Hence  $n^{\frac{1}{n}} \rightarrow 1$ . □

**Theorem 24.** If  $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$  has radius of convergence  $R$ , then

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$$

for  $|z-z_0| < R$ . Moreover, the new series also has a radius of convergence  $R$ .

*Proof.* WLOG  $R > 0$  and  $z_0 = 0$ .

For  $|z| < R$  we write:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

and the ‘new series’

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \lim_{N \rightarrow \infty} S'_N(z)$$

We first prove that the radius of convergence for  $g$  is the same as  $f$ . By Cauchy-Hadamard:

$$\begin{aligned} \frac{1}{R_g} &= \limsup_{n \rightarrow \infty} \sqrt[n]{n|a_n|} \\ &= \limsup_{n \rightarrow \infty} (n^{\frac{1}{n}})^n \sqrt[n]{|a_n|} && \text{by the previous lemma,} \\ &= \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \\ &= \frac{1}{R} \end{aligned}$$

Thus,  $R_g = R$  by Cauchy-Hadamard.

Next, we need to show that  $f' = g$  with  $|z| < R$ .

Fix  $0 \leq |w| < R$  and  $\varepsilon > 0$ . We want a  $\delta > 0$  such that whenever  $|z-w| < \delta$ , we have  $\left| \frac{f(z)-f(w)}{z-w} - g(w) \right| < \varepsilon$ .

← we just translate it; also  $R = 0$  isn't that meaningful

← just splitting the function into two parts

← just saying that the derivative at any  $w$  gets close to  $g(w)$

We rewrite:

$$\begin{aligned} \left| \frac{f(z) - f(w)}{z - w} - g(w) \right| &= \left| \frac{[S_N(z) + R_N(z)] - [S_N(w) + R_N(w)]}{z - w} - g(w) \right| \\ &= \left| \frac{S_N(z) - S_N(w)}{z - w} + \frac{R_N(z) - R_N(w)}{z - w} + S'_N(w) - S'_N(w) - g(w) \right| \\ &\leq |S'_N(w) - g(w)| + \left| \frac{R_N(z) - R_N(w)}{z - w} \right| + \left| \frac{S_N(z) - S_N(w)}{z - w} - S'_N(w) \right| \end{aligned}$$

- **1st term:** by def of  $g$  and  $g(z) = \lim_{N \rightarrow \infty} S'_N(z)$ , we can always find some  $N_1 \in \mathbb{N}$  such that any  $N \geq N_1$  gives us  $|S'_N(w) - g(w)| < \frac{\varepsilon}{3}$ .
- **2nd term:** since  $|w| < R$ , there is an  $r$  such that  $|w| < r < R$ .  
For  $|z| < r$ , we have

← work on a smaller disk

$$\begin{aligned} \left| \frac{R_N(z) - R_N(w)}{z - w} \right| &= \frac{1}{|z - w|} \left| \sum_{n=N}^{\infty} a_n z^n - \sum_{n=N}^{\infty} a_n w^n \right| \\ &\leq \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n - w^n}{z - w} \right| \\ &= \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n}{z} \cdot \frac{1 - \frac{w^n}{z^n}}{1 - \frac{w}{z}} \right| \\ &= \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n}{z} \cdot \left( 1 + \left( \frac{w}{z} \right) + \left( \frac{w}{z} \right)^2 + \dots + \left( \frac{w}{z} \right)^{n-1} \right) \right| \\ &= \sum_{n=N}^{\infty} |a_n| |z^{n-1} + z^{n-2}w + \dots + zw^{n-2} + w^{n-1}| \\ &\leq \sum_{n=N}^{\infty} |a_n| \cdot n \cdot r^{n-1} \text{ by } |z|, |w| < r < R \end{aligned}$$

by geometric sequence

Thus, there exists an  $N_2 \in \mathbb{N}$  such that any  $N \geq N_2$  gives us

$$\left| \frac{R_N(z) - R_N(w)}{z - w} \right| < \frac{\varepsilon}{3}$$

- **3rd term:** let  $N = \max\{N_1, N_2\}$ . The definition of  $S'_N(w)$  provides  $\gamma > 0$  such that if  $|z - w| < \gamma$ , then we have  $\left| \frac{S_N(z) - S_N(w)}{z - w} - S'_N(w) \right| < \frac{\varepsilon}{3}$ .

← review def of derivatives!

Now if  $0 < \delta < \min\{\gamma, r - |w|\}$ , then the 3 terms above are all  $< \frac{\varepsilon}{3}$ . Hence,

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \varepsilon \text{ holds for this } \delta. \quad \square$$

**Corollary 25.** A power series  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  with  $R > 0$  is infinitely differentiable on  $|z - z_0| < R$ . Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

are the coefficients of the terms of the power series.

← prove by keep taking derivatives!

**Corollary 26.** Power series expansions are unique. That is, if  $r > 0$  and

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

← because there is a unique formula for coeffs.

on  $|z - z_0| < r$ , then  $a_n = b_n$  for  $n \geq 0$ .

**Remark.** Recall that  $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$  has a radius of convergence  $\infty$  (it's an *entire* function). Now, if we differentiate it term-by-term:

$$\begin{aligned} \frac{d}{dz} \exp(z) &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^{n-1}}{(n-1)!} && \text{let } k = n - 1 \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ &= \exp(z) \end{aligned}$$

Thus, the derivative of  $\exp(z)$  is itself! Moreover,  $\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$ .

**Remark.** We claim that  $\exp(z) = e^z$ .

Since  $e^z e^{c-z}$  is a constant for all constant  $c, z$ , we have

$$\frac{d}{dz} (e^z e^{c-z}) = 0$$

to recover the constant  $e^z e^{c-z}$ , we let  $z = 0$ , giving us

$$e^z e^{c-z} = e^c$$

which is the addition formula!

Therefore,

$$\begin{aligned} \exp(n) &= \exp(1 + 1 + \cdots + 1) \\ &= \exp(1)^n \\ &= e^n \end{aligned}$$



## Elementary functions

Now that we have derived  $e$ , we could use it to derive  $\sin$  and  $\cos$ :

**Definition 17.**

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\end{aligned}$$

$$\begin{aligned}\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}\end{aligned}$$

We observe that we have the following property:

- Radius of convergence  $R = \infty$
- $(\cos z)' = -\sin z$ ,  $(\sin z)' = \cos z$
- $\cos x = \operatorname{Re}(e^{ix})$ ,  $\sin x = \operatorname{Im}(e^{ix})$  for all  $x \in \mathbb{R}$
- $\cos(-z) = \cos z$ ,  $\sin(-z) = -\sin z$
- $\cosh x = \frac{e^x + e^{-x}}{2}$  so  $\cosh(ix) = \cos x$
- $e^{iz} = \cos z + i \sin z$
- 

$$\begin{aligned}\cos^2 z + \sin^2 z &= \left( \frac{e^{iz} + e^{-iz}}{2} \right)^2 + \left( \frac{e^{iz} - e^{-iz}}{2i} \right)^2 \\ &= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) \\ &= 1 \quad \forall z \in \mathbb{C}\end{aligned}$$

•

$$\begin{aligned}\cos^2 z &= \left( \frac{e^{iz} + e^{-iz}}{2} \right)^2 \\ &= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) \\ &= \frac{1}{2} + \frac{e^{2iz} + e^{-2iz}}{4} \\ &= \frac{1}{2}(1 + \cos 2z)\end{aligned}$$

- If  $x \in \mathbb{R}$  then  $\cos x, \sin x$  are real. We get  $|\sin x|, |\cos x| \leq 1$ .

**Definition 18.**  $f : \mathbb{C} \rightarrow \mathbb{C}$  is **periodic** with a *period*  $\omega$  if  $f(z + \omega) = f(z)$  for all  $z \in \mathbb{C}$ .

**Theorem 27.** There exists a positive real number  $\pi$  such that:

- (a)  $\cos z, \sin z$  have period  $2\pi$
- (b)  $e^z$  is periodic with period  $2\pi i$
- (c)  $\pi$  is the area of the unit circle

*Proof.* By Euler's formula, it suffices to consider  $e^{iz}$  only. If  $\omega$  is a period of  $e^{iz}$ , then

$$e^{iz} = e^{i(z+\omega)} = e^{iz} e^{i\omega}$$

which only happens if  $e^{i\omega} = 1$ . Conversely, if  $e^{i\omega} = 1$ , then  $e^{i(z+\omega)} = e^{iz}$ .

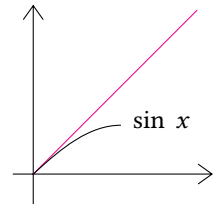
Hence,  $\omega$  is a period of  $e^{iz}$  if and only if  $e^{i\omega} = 1$ . □

**Proposition 28.**  $\sin x \leq x$  for all  $x \geq 0$ .

*Proof.* Since  $|\cos t| \leq 1$ ,

$$\begin{aligned} x - \sin x &= (x - \sin x) - (0 - \sin 0) \\ &= \int_0^x \underbrace{1 - \cos t}_{\geq 0} dt \quad \text{by FTC} \\ &\geq 0 \end{aligned}$$

← This is the first term in the power series



□

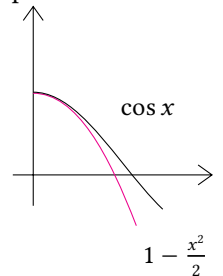
**Proposition 29.** In addition,  $\cos x \geq 1 - \frac{x^2}{2}$  for  $x \geq 0$ .

*Proof.* The previous prop gives:

$$\begin{aligned} \cos x - 1 &= \cos x - \cos 0 \\ &= \int_0^x -\sin t dt \\ &\geq \int_0^x -t dt \\ &= \frac{-x^2}{2} \end{aligned}$$

□

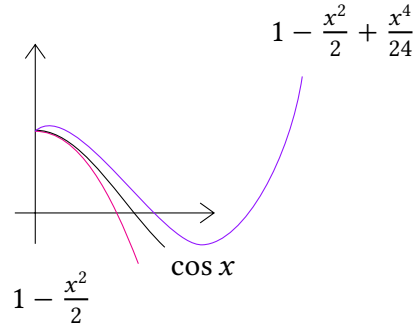
← These are the first 2 terms in the power series



**Proposition 30.** Furthermore, for  $x \geq 0$ :

- $\sin x \geq x^3 - \frac{x^3}{6}$
- $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

**Proposition 31.** There exists  $x_0 \in (0, \sqrt{3})$  such that  $\cos x_0 = 0$ .



*Proof.* By the previous prop, we have  $\cos \sqrt{3} \leq 1 - \frac{\sqrt{3}^2}{2} + \frac{\sqrt{3}^4}{24} = \frac{1}{8} < 0$ . Moreover,  $\cos 0 = 1 > 0$ , by IVT, there exists  $x_0 \in (0, \sqrt{3})$  such that  $\cos x_0 = 0$ .  $\square$

**Proposition 32.**  $\omega_0 = 4x_0$  is a period of  $e^{iz}$ .

*Proof.* Since  $\cos x_0 = 0$ , we have  $\sin x_0 = \pm 1$ . Then  $e^{ix_0} = \pm i$ . We have  $(\pm i)^4 = 1$ , so  $e^{4ix_0} = 1 = e^0$ , so  $\omega_0 = 4x_0$  is a period of  $e^{iz}$ .  $\square$

**Proposition 33.**  $\omega_0$  is the *smallest* positive period of  $e^{iz}$ .

**Proposition 34.** All periods of  $e^{iz}$  are integer multiples of  $2\pi = 4x_0$ .

*Proof.* Define  $\pi = 2x_0$ . The area of unit circle is

$$\begin{aligned} 4 \int_0^1 \sqrt{1-x^2} dx &= 4 \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} d\theta \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \pi \end{aligned}$$

$\square$

## Complex logarithm

We know:  $e^0 = 1, e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718 \dots$

Since  $\frac{d}{dx} e^x = e^x$ , it is positive. If  $x > 0$ , we conclude that  $e^x$  is strictly increasing!  
As  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1 + x$ , so  $\lim_{x \rightarrow \infty} e^x = \infty$ ,

Therefore,  $e^x$  is a **bijection** from  $\mathbb{R}$  to  $(0, \infty)$ . This means it has an inverse that is a bijection from  $(0, \infty)$  to  $\mathbb{R}$ .

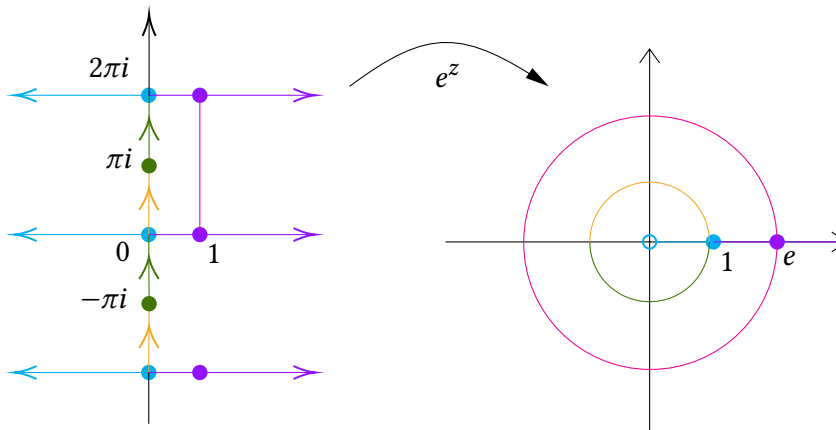
**Definition 19.**  $\ln x$  is the inverse of  $e^x$  for  $x \in (0, +\infty)$ .

Now what about the complex case? Let  $z \neq 0$  and  $z = re^{i\theta}$  where  $r = |z| > 0$  and  $\theta = \arg z \in \mathbb{R}$ .

Hence,  $z = re^{i\theta} = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta}$ . However, the  $\theta$  is ambiguous to addition of multiples of  $2\pi$ !

**Definition 20.** If  $z \neq 0$ , a **logarithm** of  $z$  is a  $w \in \mathbb{C}$  such that  $e^w = z$ .

We could graph the function  $e^z$  with  $z \in \mathbb{C}$ :



**Definition 21.** If  $\Omega$  is a region in  $\mathbb{C}$ , then a continuous  $l : \Omega \rightarrow \mathbb{C}$  is a **branch** of the logarithm if  $e^{l(z)} = z$  for all  $z \in \Omega$ .

**Example 19.** If  $\Omega = \mathbb{C} \setminus (-\infty, 0]$  such that  $\theta \in (-\pi, \pi)$ , a logarithm could be defined on it. This is the **principal branch** of the logarithm.

**Remark.** Suppose  $l(z)$  is a branch of the logarithm and  $l$  is analytic, then:

$$e^{l(z)} = z \implies \frac{d}{dz} e^{l(z)} = l'(z) e^{l(z)} = 1$$

Since  $e^{l(z)} = z$ , we conclude  $l'(z) = \frac{1}{z}$ .

← cf. trig properties

← Only determined up to addition of multiples of  $2\pi$

← note  $0 \notin \Omega$

← See graphed Riemann surface

## Complex power

**Definition 22.** If  $z \neq 0$ , define  $z^a = e^{a \log z}$ .

← NOT well-defined!

**Remark.** The definition of complex powers should coincide with the old one:

$$z^n = \underbrace{z \cdot z \cdot \dots \cdot z}_n = r^n e^{in\theta}.$$

Check:

$$\begin{aligned} z^n &= e^{n \log z} = e^{n(\ln r + i\theta + i2\pi k)} \\ &= e^{n \ln r} e^{in\theta} \underbrace{e^{i2\pi nk}}_{=1} \\ &= r^n e^{in\theta} \end{aligned}$$

is true for any  $k \in \mathbb{Z}$ .

How about  $n$ -th roots?

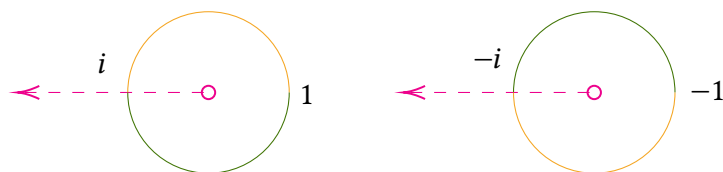
$$\begin{aligned} z^{\frac{1}{n}} &= e^{\frac{1}{n} \log z} \\ &= e^{\frac{1}{n}(\ln r + i\theta + i2\pi k)} \\ &= e^{\frac{1}{n} \ln r} e^{\frac{i\theta}{n}} \underbrace{e^{\frac{i2\pi k}{n}}}_{n \text{ distinct}} \\ &= r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \end{aligned}$$

## Riemann surface

We still have a problem:  $\ln z$  is still not a function on  $\mathbb{C}$ ! The branch depends on the arbitrary choice of domain. What shall we do to make it not dependent on a choice?

Answer: let  $\ln$  not live on the complex plane, but infinitely many copies of the slit plane  $\mathbb{C} \setminus (-\infty, 0]$ , each one being glued to the next along the slit  $(-\infty, 0]$ .

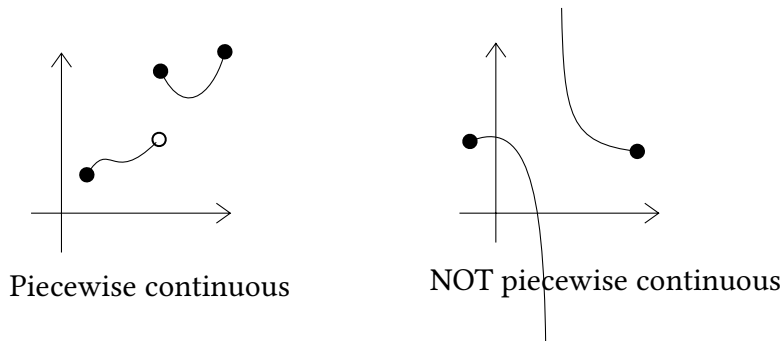
**Example 20.**  $z^{1/2}$  would live on a surface:



# Cauchy's theorem and its consequences

## Complex integration

**Definition 23.** A complex-valued function  $\gamma : [a, b] \rightarrow \mathbb{C}$  is **piecewise continuous** if  $\gamma$  is continuous at all but *finitely many* points of  $[a, b]$  and  $\gamma$  has one-sided limits that are *finite* at each point (of discontinuity).



If  $\gamma$  is piecewise continuous, then  $\int_a^b \operatorname{Re} \gamma(t) dt$  and  $\int_a^b \operatorname{Im} \gamma(t) dt$  exist. Then we define **complex integration**:

$$\int_a^b \gamma(t) dt = \int_a^b \operatorname{Re} \gamma(t) dt + i \cdot \int_a^b \operatorname{Im} \gamma(t) dt$$

That is,

$$\begin{aligned} \operatorname{Re} \left( \int_a^b \gamma(t) dt \right) &= \int_a^b \operatorname{Re} \gamma(t) dt \\ \operatorname{Im} \left( \int_a^b \gamma(t) dt \right) &= \int_a^b \operatorname{Im} \gamma(t) dt \end{aligned}$$

In addition, if  $\gamma_1, \gamma_2$  are both  $[a, b] \rightarrow \mathbb{C}$  and piecewise cont., and  $c_1, c_2 \in \mathbb{C}$ , then

$$\int_a^b (c_1 \gamma_1(t) + c_2 \gamma_2(t)) dt = c_1 \int_a^b \gamma_1(t) dt + c_2 \int_a^b \gamma_2(t) dt$$

**Proposition 35** (Triangle inequality). If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is piecewise continuous, then

$$\left| \int_a^b \gamma(t) dt \right| \leq \int_a^b |\gamma(t)| dt$$

*Proof.* WLOG assume  $\int_a^b \gamma(t) dt \neq 0$ . Define  $\lambda = \frac{\left| \int_a^b \gamma(t) dt \right|}{\int_a^b \gamma(t) dt}$  and note  $|\lambda| = 1$ .

Thus,

$$\begin{aligned} \left| \int_a^b \gamma(t) dt \right| &= \lambda \int_a^b \gamma(t) dt \\ &= \int_a^b \lambda \gamma(t) dt && \text{because LHS is } \in \mathbb{R} \\ &= \operatorname{Re} \int_a^b \lambda \gamma(t) dt \\ &\leq \int_a^b |\lambda \gamma(t)| dt && \because \operatorname{Re} z \leq |z| \\ &= \int_a^b |\gamma(t)| dt && \because |\lambda| = 1 \end{aligned}$$

□

## Complex differentiability

**Definition 24.**  $\gamma : [a, b] \rightarrow \mathbb{C}$  is **differentiable** at  $t \in [a, b]$  if  $\operatorname{Re} \gamma$  and  $\operatorname{Im} \gamma$  are differentiable (in the sense of real variables). We define

$$\gamma'(t) = (\operatorname{Re} \gamma)'(t) + i \cdot (\operatorname{Im} \gamma)'(t)$$

**Definition 25.**  $\gamma : [a, b] \rightarrow \mathbb{C}$  is **piecewise  $C^1$**  if:

←  $C^1$  is one-time differentiable

- (a)  $\gamma$  is continuous on  $[a, b]$ .
- (b)  $\gamma$  is differentiable at all but finitely many points of  $[a, b]$ .
- (c)  $\gamma'$  is continuous at each point where it exists.
- (d)  $\gamma'$  has finite one-sided limits at every point of discontinuity.

## Fundamental theorem of calculus, complex edition

If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is piecewise  $C^1$ , then:

$$\int_a^b \gamma'(t) dt = \gamma(b) - \gamma(a)$$

**Definition 26.** If  $\gamma$  is  $C^1$ , then the arclength of  $\gamma$  is:

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

**Definition 27.** If  $\gamma : [a, b] \rightarrow \Omega$  is piecewise  $C^1$  and  $f : \Omega \rightarrow \mathbb{C}$  is continuous, then

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

where  $z = \gamma(t)$  and  $dz = \gamma'(t) dt$

We have **linearity** w.r.t.  $f$ :

$$\int_{\gamma} (c_1 f_1(z) + c_2 f_2(z)) dz = c_1 \int_{\gamma} f_1(z) dz + c_2 \int_{\gamma} f_2(z) dz$$

**Remark.** Arclength is independent from parameterization.

*Proof.* Let  $\gamma : [a, b] \rightarrow \Omega$  be piecewise  $C^1$ . Let  $\alpha : [c, d] \rightarrow [a, b]$  is an increasing, piecewise  $C^1$  surjection such that  $\alpha(c) = a, \alpha(d) = b$ . Then  $\phi = \gamma \circ \alpha : [c, d] \rightarrow \Omega$  is also piecewise  $C^1$ . Hence, by substituting  $s = \alpha(t), ds = \alpha'(t) dt$ :

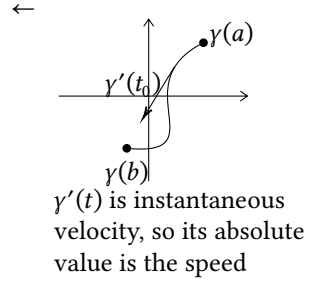
$$\begin{aligned} \int_{\phi} f(z) dz &= \int_c^d f(\phi(t)) \phi'(t) dt \\ &= \int_c^d f(\gamma(\alpha(t))) \gamma'(\alpha(t)) \alpha'(t) dt \\ &= \int_a^b f(\gamma(s)) \gamma'(s) ds \\ &= \int_{\gamma} f(z) dz \end{aligned}$$

□

### An important estimate

Let  $f$  be continuous. Since  $\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$ , we observe:

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &= \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \\ &\leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \end{aligned}$$





$$\begin{aligned} &\leq \max_{t \in [a, b]} |f(\gamma(t))| \int_a^b |\gamma'(t)| dt \\ &= \max_{z \in \gamma} |f(z)| \cdot L(\gamma) \end{aligned}$$

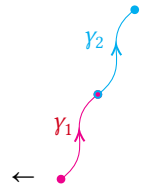
**Definition 28.** If  $\gamma : [a, b] \rightarrow \mathbb{C}$ , the reverse of  $\gamma$  is  $(-\gamma) : [-b, -a] \rightarrow \mathbb{C}$  defined by  $(-\gamma)(t) = \gamma(-t)$ . Hence,

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

**Remark.** We can also break up the curve and integral the two parts separately:

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz$$

← going around the track backwards



## Fundamental theorem of calculus for contour integrals

If  $\gamma : [a, b] \rightarrow \mathbb{C}$  is piecewise  $C^1$ , and  $f : \Omega \rightarrow \mathbb{C}$  is analytic, then

$$\int_{\gamma} f'(z) dz = f(\gamma(b)) - f(\gamma(a))$$

← Assuming  $f'$  continuous, which we would prove later

If  $\gamma(a) = \gamma(b)$ , then  $\int_{\gamma} f'(z) dz = 0$ .

*Proof.*

$$\begin{aligned} \int_{\gamma} f'(z) dz &= \int_a^b f'(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b (f \circ \gamma)'(t) dt && \text{chain rule} \\ &= f(\gamma(b)) - f(\gamma(a)) \end{aligned}$$

□

**Example 21.** Let  $\gamma$  be a circle of radius  $R$  centered at  $z_0$ :  $\gamma(t) = z_0 + Re^{it}$ ,  $t \in [0, 2\pi]$ . We would like to find  $\int_{\gamma} (z - z_0)^n dz$ .

If  $n \neq -1$ , then  $\left( \frac{(z - z_0)^{n+1}}{n+1} \right)' = (z - z_0)^n$ . Thus,

$$\int_{\gamma} (z - z_0)^n dz = \int_{\gamma} \left( \frac{(z - z_0)^{n+1}}{n+1} \right)' dz = 0$$

by FTC.

If  $n = -1$ ,

$$\int_{\gamma} (z - z_0)^n dz = \int_{\gamma} \frac{1}{z - z_0} dz = \int_0^{2\pi} i dt = 2\pi i$$

## Cauchy's theorem

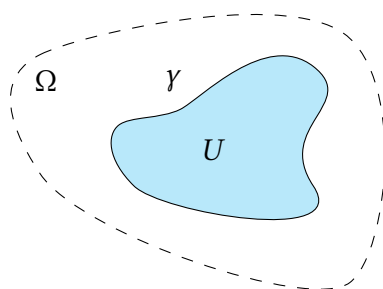
### Take 1

**Theorem 36** (Cauchy's). Let  $\Omega$  be a region in  $\mathbb{C}$  containing a *simple* piecewise  $C^1$  closed curve  $\gamma$  and its interior.

← does not self-intersect

← holes not allowed in the interior

If  $f : \Omega \rightarrow \mathbb{C}$  is analytic, then  $\int_{\gamma} f(z) dz = 0$ .



“Proof”. Let  $U$  be the union of  $\gamma$  and its interior. Let  $f = u + iv$  as usual, write  $dz = dx + i dy$ :

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} (u + iv)(dx + i dy) \\ &= \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy \\ &= \int \int_U (-v_x - u_y) dx dy + i \int \int_U (u_x - v_y) dx dy \quad \text{by Green's thm} \\ &= 0 \quad \text{by Cauchy-Riemann} \end{aligned}$$

□

However, this ‘proof’ heavily relies on the fact that  $u, v$  are  $C^1$  and that the partial derivatives are continuous. This assumes  $f'$  is continuous, but we aren't sure about that yet!

← See [Goursat's Lemma](#)

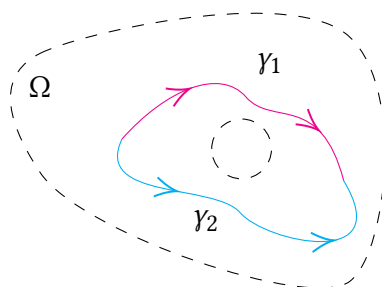
**Take 2: deformation version**

**Theorem 37** (Cauchy's). Let  $\gamma_1, \gamma_2$  be piecewise  $C^1$  curves in a region  $\Omega$  with the same start and end points. If  $\gamma_1$  can be continuously deformed to  $\gamma_2$  without ever passing outside of  $\Omega$ , then

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

By the *previous* statement of Cauchy's theorem (in Theorem 36), we observe that  $\int_{\gamma_1 - \gamma_2} f(z) dz = 0$ , so this one falls out.

**Non-example 22.** The  $\gamma_1, \gamma_2$  in the picture below cannot be continuously deformed into each other!

**Fresnel integrals**

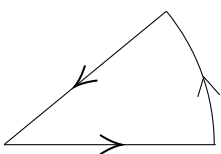
Consider:

$$\int_0^\infty \sin(t^2) dt \quad \text{and} \quad \int_0^\infty \cos(t^2) dt$$

aka.

$$\lim_{R \rightarrow \infty} \int_0^R \sin(t^2) dt \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_0^R \cos(t^2) dt$$

It's not obvious that these integrals converge!



Solution: PIZZA! Let  $\gamma$  be the 'sum' of all 3 curves as shown. Let  $R \rightarrow \infty$ . Then, by Cauchy's theorem,  $\int_\gamma e^{iz^2} dz = 0$ .

(Scratch work begins)

**Remark.** We don't know how to write out the antiderivative of  $f(z) = e^{iz^2}$  but we can use series!

$$\begin{aligned} f(z) &= e^{iz^2} \\ &= \sum_{n=0}^{\infty} \frac{(iz^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n z^{2n}}{n!} \end{aligned}$$

And so

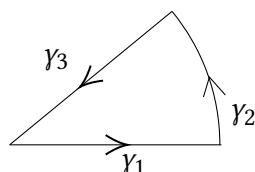
$$F(z) = \sum_{n=0}^{\infty} \frac{i^n z^{2n+1}}{(2n+1)n!}$$

(Scratch ends here)

---

Now we return to the integral. Strategy:

$$0 = \int_{\gamma} e^{iz^2} dz = \underbrace{\int_{\gamma_1} e^{iz^2} dz}_{I_1(R)} + \underbrace{\int_{\gamma_2} e^{iz^2} dz}_{I_2(R)} + \underbrace{\int_{\gamma_3} e^{iz^2} dz}_{I_3(R)}$$



Evaluate  $I_1(R)$ : We observe that  $z$  is real for this one. Parameterize  $z = t$  where  $t$  is a real variable.

$$\begin{aligned} I_1(R) &= \int_{\gamma_1} e^{it^2} dt \\ &= \int_0^R \cos(t^2) dt + i \cdot \int_0^R \sin(t^2) dt \end{aligned}$$

Hence,  $\lim_{R \rightarrow \infty} I_1(R) = \int_0^{\infty} \cos(t^2) dt + i \cdot \int_0^{\infty} \sin(t^2) dt$ .

Evaluate  $I_2(R)$ :

Parameterize  $\gamma_2$  as  $z = Re^{i\theta}$  where  $\theta \in [0, \frac{\pi}{4}]$ . Hence,  $dz = iRe^{i\theta} d\theta$ . Then:

$$\begin{aligned}
 |I_2(R)| &= \left| \int_{\gamma_2} e^{i\theta^2} dz \right| \\
 &= \left| \int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} iRe^{i\theta} d\theta \right| \\
 &= \left| R \int_0^{\frac{\pi}{4}} e^{iR^2 e^{i2\theta}} e^{i\theta} d\theta \right| \\
 &\leq R \int_0^{\frac{\pi}{4}} |e^{iR^2 e^{i2\theta}}| d\theta && \text{by tri. ineq.} \\
 &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} d\theta && \text{since when } x, y \in \mathbb{R}, |e^{x+iy}| = e^x \\
 &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4\theta}{\pi}} d\theta && \text{since when } x \in [0, \frac{\pi}{2}], \frac{2}{\pi}x \leq \sin x \\
 &= \frac{-R\pi}{R^2 4} e^{-R \frac{4\theta}{\pi}} \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} \\
 &\rightarrow 0 \text{ as } R \rightarrow \infty
 \end{aligned}$$

Thus,  $\lim_{R \rightarrow \infty} I_2(R) = 0$ . :)

Evaluate  $I_3(R)$ :

$$\begin{aligned}
 I_3(R) &= \int_{\gamma_3} e^{iz^2} dz \\
 &= \int_R^0 e^{i(e^{i\frac{\pi}{4}}t)^2} e^{i\frac{\pi}{4}} dt \\
 &= -e^{i\frac{\pi}{4}} \int_0^R e^{-t^2} dt \\
 \lim_{R \rightarrow \infty} I_3(R) &= -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \int_0^\infty e^{-t^2} dt \quad \text{by Gaussian integral, } \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \\
 &= -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}
 \end{aligned}$$

Therefore, we see  $I_1(R) + I_2(R) + I_3(R) = 0$  where  $\lim_{R \rightarrow \infty} I_1(R) = \int_0^\infty \cos(t^2) dt + i \cdot \int_0^\infty \sin(t^2) dt$ ,  $I_2(R) \rightarrow 0$  and  $I_3(R) = -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}$ . Hence, we would be able to conclude that

$$\int_0^\infty \sin(t^2) dt = \sqrt{\frac{\pi}{8}} \quad \text{and} \quad \int_0^\infty \cos(t^2) dt = \sqrt{\frac{\pi}{8}}$$

## Goursat's lemma

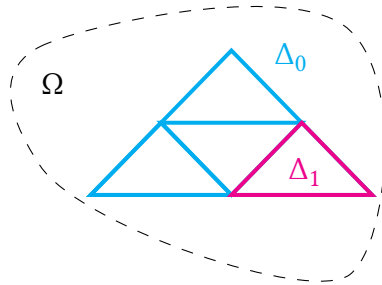
This lemma patches the hole that we have to assume  $f'$  continuous in Cauchy's theorem!

**Lemma 38** (Goursat's). If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and  $\Delta$  is a triangle in  $\Omega$  whose interior lies inside  $\Omega$ , then  $\int_{\Delta} f(z) dz = 0$ .

← Does not assume  $f'$  continuous!

*Proof.* WLOG orient  $\Delta_0 = \Delta$  counterclockwise. Bisect sides of  $\Delta_0$  and construct smaller triangles  $\Delta_{0j}$  where  $j = 1, 2, 3, 4$ . Then,

$$I = \int_{\Delta_0} f(z) dz = \sum_{j=1}^4 \int_{\Delta_{0j}} f(z) dz$$



By triangle inequality,

$$|I| \leq \sum_{j=1}^4 \left| \int_{\Delta_{0j}} f(z) dz \right|$$

Thus, there exists  $j \in \{1, 2, 3, 4\}$  such that

$$\frac{|I|}{4} \leq \left| \int_{\Delta_{0j}} f(z) dz \right|$$

For this  $j$ , define  $\Delta_1 = \Delta_{0j}$ .

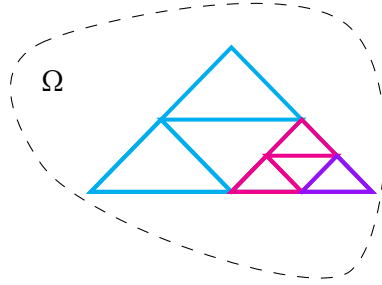
We dissect  $\Delta_1$  again into smaller triangles  $\Delta_{1j}$  where  $j = 1, 2, 3, 4$ . Then,

$$I = \int_{\Delta_1} f(z) dz = \sum_{j=1}^4 \int_{\Delta_{1j}} f(z) dz$$

Again, by triangle inequality, there is a  $j \in \{1, 2, 3, 4\}$  such that

$$\frac{|I|}{4^2} \leq \frac{1}{4} \left| \int_{\Delta_1} f(z) dz \right| \leq \left| \int_{\Delta_{1j}} f(z) dz \right|$$

For this  $j$ , define  $\Delta_2 = \Delta_{1j}$ .



...continue in this manner to get nested triangles  $\Delta_n$  such that

$$\frac{|I|}{4^{n+1}} \leq \frac{1}{4} \left| \int_{\Delta_n} f(z) dz \right| \leq \left| \int_{\Delta_{nj}} f(z) dz \right|$$

for all  $n \geq 0$ .

Now let  $\ell = L(\Delta_0)$  denote perimeter of the original triangle (blue).

Then  $L(\Delta_n) = \frac{\ell}{2^n}$ .

← Perimeter of  $\Delta_n$

Let  $K_n$  denote the triangle  $\Delta_n$  union with its interior such that  $K_n$  is closed (in fact, compact!). Let  $\zeta_n \in K_n$  for  $n \geq 0$ . Then there is  $N \in \mathbb{N}$ , such that for all  $m, n \geq N$  we have  $|\zeta_m - \zeta_n| \leq \text{diam}(K_N) \leq \frac{\ell}{2^N}$ . Thus,  $\zeta_n$  as a sequence is Cauchy.

Let  $z_0 = \lim_{n \rightarrow \infty} \zeta_n$ , note  $z_0 \in \bigcap_{n=0}^{\infty} K_n$  and  $z_0 \in \Omega$ . Since  $f$  is analytic at  $z_0$ , given  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that whenever  $|z - z_0| < \delta$ , we have

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \frac{\varepsilon}{\ell^2}$$

Now consider multiplying  $|z - z_0|$  on both sides:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \frac{\varepsilon}{\ell^2} |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \frac{\varepsilon}{\ell^2} |z - z_0| \end{aligned}$$

Since  $f(z_0) + f'(z_0)(z - z_0)$  is **linear**, it has an antiderivative on  $\mathbb{C}$ . Thus,

$$\int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0) dz = 0$$

by FTC! Now pick  $n$  large enough so that  $|z - z_0| < \delta$  for all  $z \in \Delta_n$ . Thus,

$$|I| \leq 4^n \left| \int_{\Delta_n} f(z) dz \right|$$

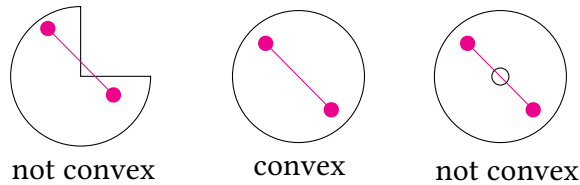
$$\begin{aligned}
 &= 4^n \left| \int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0) - f(z) \right| \\
 &\leq 4^n \frac{\varepsilon}{\ell^2} |z - z_0| \frac{\ell}{2^n} \quad \text{by tri. ineq. and } \left| \int_Y g(z) dz \right| \leq \sup_{z \in Y} |g(z)| \cdot L(Y) \\
 &< \frac{4^n \varepsilon}{\ell 2^n} \cdot \frac{\ell}{2^n} \\
 &= \varepsilon
 \end{aligned}$$

□

### Local antiderivative

**Theorem 39.** If  $\Omega$  is convex and  $f : \Omega \rightarrow \mathbb{C}$  is analytic, then  $f$  has an antiderivative on  $\Omega$ .

**Remark.** Line segments don't exit the region in convex shapes:



*Proof.* Fix  $w \in \Omega$  and define:

$$F(z) = \int_{[w,z]} f(\zeta) d\zeta$$

for  $z \in \Omega$ .

This is well-defined if  $\Omega$  is convex.

←  $[w, z]$  is the line segment from  $w$  to  $z$ .

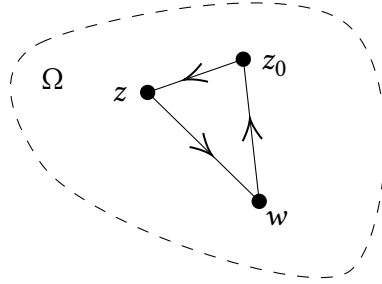
Now we want to show that  $F'$  is  $f$ . That is equivalent to showing that for all  $\varepsilon > 0, z_0 \in \Omega$ , there exists  $\delta > 0$  s.t. whenever  $|z - z_0| < \delta$ , we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \varepsilon$$

Let  $z_0 \in \Omega$  be given and  $\varepsilon > 0$ . Goursat says integrals around the triangle is 0, so



we suppose  $z \in \Omega \setminus \{z_0, w\}$  and get a triangle:



and we know that

$$\underbrace{\int_{[w, z_0]} f(\zeta) d\zeta}_{F(z_0)} + \int_{[z_0, z]} f(\zeta) d\zeta + \underbrace{\int_{[z, w]} f(\zeta) d\zeta}_{-F(z)} = 0$$

So  $F(z) - F(z_0) = \int_{[z_0, z]} f(\zeta) d\zeta$ . Thus,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0, z]} (f(\zeta) - f(z_0)) d\zeta$$

Since  $f$  is analytic at  $z_0$ , it is continuous there. Given  $\varepsilon > 0$ , there exists  $\delta > 0$  such that whenever  $|z - z_0| < \delta$ , we have  $|f(z) - f(z_0)| < \varepsilon$ .

Therefore, whenever  $|z - z_0| < \delta$ , we have

$$\begin{aligned} \left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| &\leq \frac{\varepsilon}{|z - z_0|} L([z_0, z]) \\ &= \frac{\varepsilon}{|z - z_0|} |z - z_0| \\ &= \varepsilon \end{aligned}$$

← still by  
 $\left| \int_{\gamma} g(z) dz \right| \leq \sup_{z \in \gamma} |g(z)| \cdot L(\gamma)$

□

## Cauchy's theorem, Take 3

### Cauchy's theorem for convex regions

**Theorem 40.** If  $\Omega$  is convex,  $f : \Omega \rightarrow \mathbb{C}$  analytic and  $\gamma$  is a piecewise  $C^1$  curve in  $\Omega$ , then  $\int_{\gamma} f(z) dz = 0$ .

← Since  $\Omega$  is convex, the interior of  $\gamma$  lies inside  $\Omega$ .

*Proof.* Previous theorem says  $f$  has an antiderivative  $F$  on  $\Omega$ . Thus,

$$\int_{\gamma} f(z) \, dz = \int_{\gamma} F'(z) \, dz = 0$$

by FTC! □

## Cauchy's integral formula

### Cauchy's integral formula for a circle

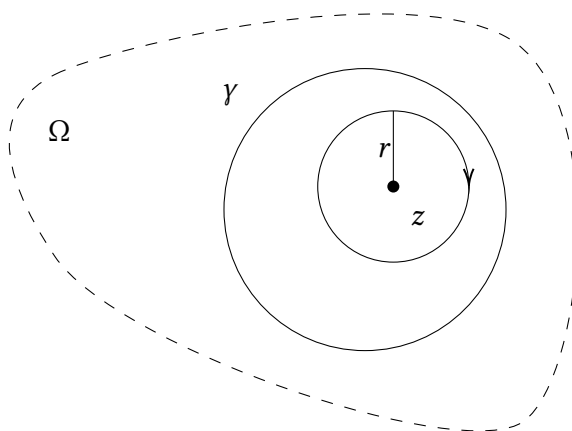
**Theorem 41.** If  $f$  is analytic on a region  $\Omega$  that contains the circle  $\gamma$  and its interior, then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) \, d\zeta}{\zeta - z}$$

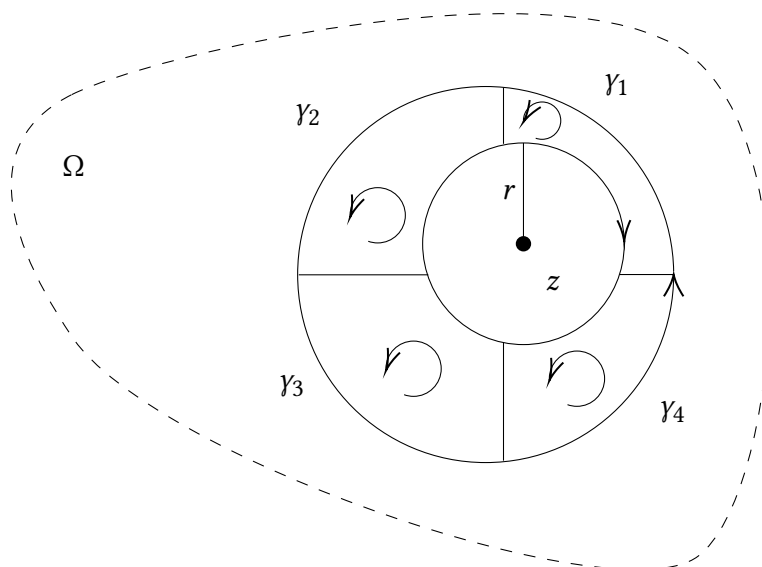
for all  $z$  inside of  $\gamma$ .

← this  $\Omega$  doesn't need to be convex

*Proof.* Let  $r > 0$  be small enough so that the closed ball  $B_r(z)^-$  is in the interior of  $\gamma$ . Let  $C_r(z) = \{\zeta \in \mathbb{C} : |\zeta - z| = r\}$  traversed clockwise.



Construct  $\gamma_1, \gamma_2, \gamma_3, \gamma_4$  as pictured:



Cauchy's theorem for convex regions says  $\int_{\gamma_i} \frac{f(\zeta) d\zeta}{\zeta - z} = 0$  for all  $i = 1, 2, 3, 4$ .

Hence,

$$0 = \sum_{j=1}^4 \int_{Y_j} \frac{f(\zeta) d\zeta}{\zeta - z} = \int_Y \frac{f(\zeta) d\zeta}{\zeta - z} - \int_{C_r(z)} \frac{f(\zeta) d\zeta}{\zeta - z}$$

And thus:

$$\int_Y \frac{f(\zeta) d\zeta}{\zeta - z} = \int_{C_r(z)} \frac{f(\zeta) d\zeta}{\zeta - z}$$

for all  $r > 0$  that is *sufficiently* small.

Therefore:

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} - f(z) \cdot 1 \right| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{\zeta - z} - f(z) \cdot \left( \frac{1}{2\pi i} \int_{C_r(z)} \frac{d\zeta}{\zeta - z} \right) \right| \\ &= \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) d\zeta}{\zeta - z} - f(z) \cdot \left( \frac{1}{2\pi i} \int_{C_r(z)} \frac{d\zeta}{\zeta - z} \right) \right| \\ &= \lim_{r \rightarrow 0^+} \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \right| \\ &\leq \lim_{r \rightarrow 0^+} \max_{|\zeta - z| = r} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \cdot r \\ &= 0 \end{aligned}$$

← by HW6 Ex5, or  
Thm12 Lect 11

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## Mean value properties

**Corollary 42** (Mean value property for analytic functions). If  $f$  analytic on an open set  $\Omega$  which contains  $B_r(z)^-$ , then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) dt$$

*Proof.* Apply Theorem 41 with  $\zeta = z + re^{it}$  and  $d\zeta = ire^{it} dt, t \in [0, 2\pi]$  and get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(z + re^{it}) ire^{it} dt}{z + re^{it} - z} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) dt \end{aligned}$$

□

**Remark.** There is a mean value property for harmonic functions!

## Existence of power series expansions

**Theorem 43.** If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and  $z_0 \in \Omega$  then  $f$  has a power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges **locally uniformly** on the disk

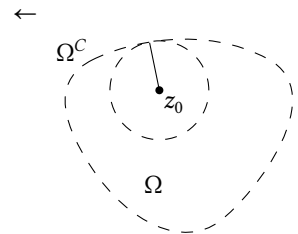
$$|z - z_0| < \text{dist}(z_0, \Omega^C) = \inf_{w \in \Omega^C} |z_0 - w|$$

when  $\Omega^C$  is nonempty.

Moreover, the radius of convergence is the radius of the largest open disk centered at  $z_0$  upon which  $f$  could be analytically continued.

*Proof.* Let  $r < \text{dist}(z_0, \Omega^C)$  and  $|z - z_0| \leq \rho < r$ . Then

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) d\zeta}{\zeta - z}$$



for all  $|z - z_0| < \rho$ .

As a function of  $\zeta$ , the series

← geometric series trick!

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

and so by geometric series formula:

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \end{aligned} \quad \text{for } |z - z_0| \leq \rho$$

converges uniformly on  $|\zeta - z_0| = r$  by the Weierstrass M-test with  $M_n = \left| \frac{z - z_0}{\zeta - z_0} \right|^n \leq \left( \frac{\rho}{r} \right)^n$ .

Thus,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{C_r(z_0)} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \cdot f(\zeta) d\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}} \end{aligned}$$

And so we have our  $\frac{f^{(n)}(z_0)}{n!} = a_n$  in the highlighted part above. □

**Remark.** Consequently, we also get Cauchy's theorem of derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$$

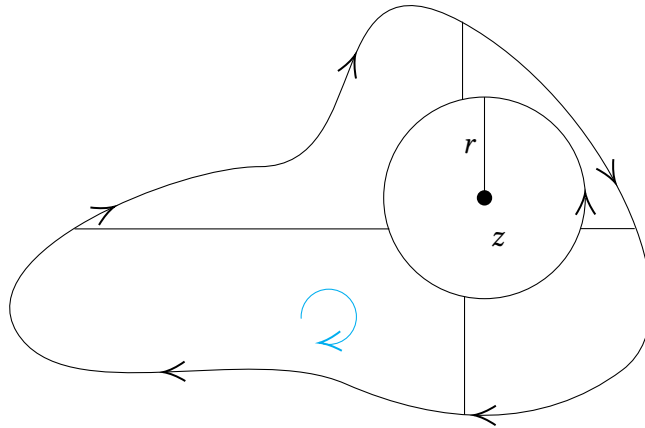
**Example 23.** What is the radius of convergence for the power series of

$$f(x) = \frac{e^{\sin x} + e^{-x^2} + x^2 + 7x^3}{\cos x}$$

centered at  $x_0 = 2$ ?

The theorem guarantees the existence of the power series, and the RoC would simply be the radius of which  $f$  could be analytically continued. We observe that  $f(x)$  cannot be defined when  $\cos x = 0$ , i.e.  $x = \frac{\pi}{2}$ . Hence, the radius of convergence is just  $2 - \frac{\pi}{2}$  – no need to compute *any* derivatives or coefficients!

So now we have this result for computing the derivatives and integrals around a circle  $C_r(z_0)$ . Can we extend this to other closed curves of any shapes?



Same techniques! Hence,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}$$

on any such closed curve  $\gamma$ .

## Liouville's theorem

**Theorem 44** (Liouville's). A bounded entire function is constant.

← analytic on  $\mathbb{C}$

*Proof.* Suppose  $f$  is entire and  $|f(z)| \leq M$  is bounded by  $M$  for all  $z \in \mathbb{C}$ . Then

$$f'(z) = \frac{1!}{2\pi i} \int_{C_R(z)} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

by Cauchy's integral formula. Hence,  $|f'(z)| \leq \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}$  by the upper bound. Since  $f$  is entire, there is no limit in what  $R$  could be, so we let  $R \rightarrow \infty$  and observe that  $|f'(z)| = 0$  for all  $z \in \mathbb{C}$ . Hence,  $f'$  is identically 0, and so  $f$  is constant.  $\square$

**Non-example 24.** We know  $|\cos x| \leq 1$  for all  $x \in \mathbb{R}$ , but  $\cos z = \frac{e^{iz} + e^{-iz}}{2}$  is **not** bounded on  $\mathbb{C}$ . In fact,  $\cos(-ix) = \frac{e^x + e^{-x}}{2}$  is unbounded for real  $x$ , so  $\cos x$  is not bounded on the imaginary axis. Hence, we can't use Liouville's theorem here!

## Fundamental theorem of algebra

**Theorem 45** (FToA). Every **nonconstant** complex polynomial has a zero in  $\mathbb{C}$ .

← recall “ $\mathbb{C}$  is an algebraically closed field”

*Proof.* Suppose towards a contradiction that  $p$  is a **nonconstant** polynomial over  $\mathbb{C}$  with no zeros in  $\mathbb{C}$ . Then  $f = \frac{1}{p}$  is an entire function because we never divide by 0. Recall HW2 Ex2 showed that  $\lim_{|z| \rightarrow \infty} p(z) = \infty$ . That is, for any  $M > 0$ , there exists  $R > 0$  such that whenever  $|z| > R$ , we have  $|p(z)| > M$ .

Thus,  $\lim_{|z| \rightarrow \infty} f(z) = \lim_{|z| \rightarrow \infty} \frac{1}{p(z)} = 0$ . In particular, we can find a  $R > 0$  such that whenever  $|z| > R$ , we have  $|f(z)| < 1$  is bounded outside of the circle  $|z| = R$ . Since the closed disk  $|z| \leq R$  is compact and  $f$  is continuous,  $f$  is bounded inside this closed disk  $|z| \leq R$ .

← Extreme value theorem

Hence,  $f$  is a bounded entire function, meaning that it is constant by Liouville’s theorem, and hence  $p$  is **constant** too. This cause a contradiction.  $\square$

## Zeros of analytic functions

Recall the analytic functions are infinitely differentiable.

Suppose  $f : \Omega \rightarrow \mathbb{C}$  is analytic and  $f(z_0) = 0$  for some  $z_0 \in \Omega$ , and  $f$  is not identically 0 on an open neighbourhood of  $z_0$ . Then

$$f(z) = \sum_{j=n}^{\infty} a_j (z - z_0)^j$$

for some  $n \geq 1$  such that  $a_n \neq 0$ . Hence:

← the lowest power term that has a nonzero coefficient, and also  $n$  is the order of the zero  $z_0$ .

$$\begin{aligned} f(z) &= \sum_{j=n}^{\infty} a_j (z - z_0)^j \\ &= (z - z_0)^n \sum_{j=n}^{\infty} a_j (z - z_0)^{j-n} \\ &= (z - z_0)^n \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k \end{aligned}$$

let  $g(z) = \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k$ . Observe that  $g$  is analytic and  $g(z_0) = a_n \neq 0$ . This and the continuity of  $g$  at  $z_0$  ensures that  $g$  is nonzero on some open disk  $|z - z_0| < \delta$ . Therefore,  $f(z) = (z - z_0)^n g(z)$  does not vanish on  $0 < |z - z_0| < \delta$ .

**Remark.** The zeros of  $f$  are **isolated** in  $\Omega$ . That is, we can't have a sequence of zeros of  $f$  converging to some  $z_0 \in \Omega$ , as then we can't find a nonzero disk around  $z_0$ !

**Theorem 46.** If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and not identically zero, then each zero of  $f$  is isolated and has finite order.

*Proof.* Assume BWOC that the zeros are not isolated.

By definition,  $\Omega$  is connected. By definition $\times 2$ , a subset  $S \subseteq \Omega$  is **clopen** if it is open and closed as a subset of  $\Omega$ . In a connected region  $\Omega$ , only  $\emptyset, \Omega$  are clopen.

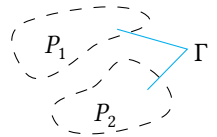
Let  $S = \{z \in \Omega : f^{(j)}(z) = 0 \quad \forall j = 0, 1, 2, \dots\}$ . If  $z_0 \in S$  then  $f$  is zero on some open disk centered at  $z_0$ . Hence,  $S$  is open!

Now suppose  $w$  is a limit of a sequence in  $S$ . Since  $f$  is continuous,  $f^{(j)}$  is continuous for all  $j \in \mathbb{N}$ . This ensures that  $f^{(j)}(w) = 0$ . Thus,  $S$  is closed!

Therefore,  $S$  is clopen in  $\Omega$ , so either  $S$  is the empty set or  $S = \Omega$ . If  $S = \Omega$ , then  $f$  is the zero function, so that cannot happen! Therefore,  $S = \emptyset$ , and so we don't have a cluster of zeros.  $\square$

**Corollary 47.** If  $f$  is a nonconstant analytic function, its zero set is **countable**. This is because within an open region, we can have at most countably infinite number of disjoint open sets. We let these open sets be  $f^{-1}(\{0\})$ .

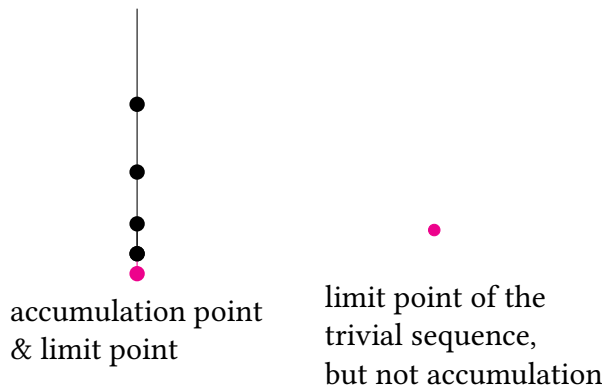
← Example of nontrivial clopen subsets:



In this  $\Gamma$  (NOT a region), the clopen subsets are  $P_1, P_2, \emptyset, \Gamma$ .

## Identity theorem

**Definition 29.** An **accumulation point** of  $S$  is a point that is the limit of a sequence of **distinct** points of  $S$ .



**Theorem 48.** Let  $f, g : \Omega \rightarrow \mathbb{C}$  be analytic. If  $f = g$  on a subset of  $\Omega$  that has an accumulation point in  $\Omega$ , then  $f = g$  on the entire  $\Omega$ .



*Proof.* If the zero set of  $f - g$  has an accumulation point in  $\Omega$ , then  $f - g$  has a zero that is not isolated (no open disk around it since some zeros keep converging to that accumulation point), so  $f - g$  is identically zero on  $\Omega$ .  $\square$

**Example 25.** There is only one way to extend  $\cos x, \sin x, \exp x$  from  $\mathbb{R}$  to  $\mathbb{C}$  because two entire functions that agree on  $\mathbb{R}$  agree on  $\mathbb{C}$ .

**Example 26.** Similarly, there is also only one way to get an analytic continuation of the Riemann zeta function to  $\operatorname{Re} s > 0$ .

## Maximum modulus principle

Recall this handwavy physics application [here](#). We now have a more rigorous way to state this!

← Not exactly equivalent, though.

**Theorem 49** (Maximum modulus principle). Let  $f$  be analytic on a region  $\Omega$  that contains a piecewise  $C'$  simple closed curve  $\gamma$  and its interior. Then

$$|f(z)| \leq \max_{\zeta \in \gamma} |f(\zeta)|$$

for all  $z$  in the interior of  $\gamma$ .

*Proof.* Let  $M = \max_{\zeta \in \gamma} |f(\zeta)|$ . Fix  $z$  inside  $\gamma$ . Let  $L$  denote the length of  $\gamma$  and let  $r = \inf_{\zeta \in \gamma} |z - \zeta|$ , which is positive (so  $z$  isn't arbitrarily close to  $\gamma$ ).

Apply Cauchy's integral formula to the  $n$ -th power of  $f$ :

$$f(z)^n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)^n d\zeta}{\zeta - z}$$

Thus,

$$|f(z)|^n \leq \frac{1}{2\pi} \cdot \frac{M^n}{r} L$$

Now we just take the  $n$ -th root everywhere:

$$|f(z)| \leq M \left( \frac{L}{2\pi r} \right)^{1/n}$$

We use arbitrarily large  $n$  and get  $|f(z)| \leq M$ .  $\square$

## Schwarz' lemma

**Lemma 50** (Schwarz'). Let  $f : \mathbb{D} \rightarrow \mathbb{D}^-$  be analytic and  $f(0) = 0$ . Then:

- (a)  $|f'(0)| \leq 1$  and  $|f(z)| \leq |z|$  for all  $z \in \mathbb{D}$ .  
 (b) If  $|f'(0)| = 1$  or  $|f(z_0)| = |z_0|$  for some  $z_0 \neq 0$ , then  $f(z) = \lambda z$  for some  $\lambda$  with  $|\lambda| = 1$ .

*Proof part (a).* Since  $f(0) = 0$ , we have that the constant term of  $f$  is 0, and so  $f(z) = zg(z)$  for some  $g$  analytic on  $\mathbb{D}$ . Thus,

$$f'(z) = g(z) + zg'(z)$$

and hence  $f'(0) = g(0)$ . Hence,

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

If  $|z| \leq r < 1$ , then by the maximum modulus principle,

$$\begin{aligned} |g(z)| &\leq \max_{|\zeta|=r} |g(\zeta)| \\ &= \max_{|\zeta|=r} \left| \frac{f(\zeta)}{\zeta} \right| \\ &\leq \frac{1}{r} \end{aligned} \quad \text{since } f : \mathbb{D} \rightarrow \mathbb{D}^-$$

Let  $r \rightarrow 1^-$  and get  $|g(z)| \leq 1$  for all  $z \in \mathbb{D}$ , which is the (a) part of our result.  $\square$

*Proof part (b).* Maximum modulus principle says the given conditions imply  $g$  is constant. The constant  $\lambda$  has absolute value 1, so  $\frac{f(z)}{z} = \lambda$  and so  $f(z) = \lambda z$ .  $\square$

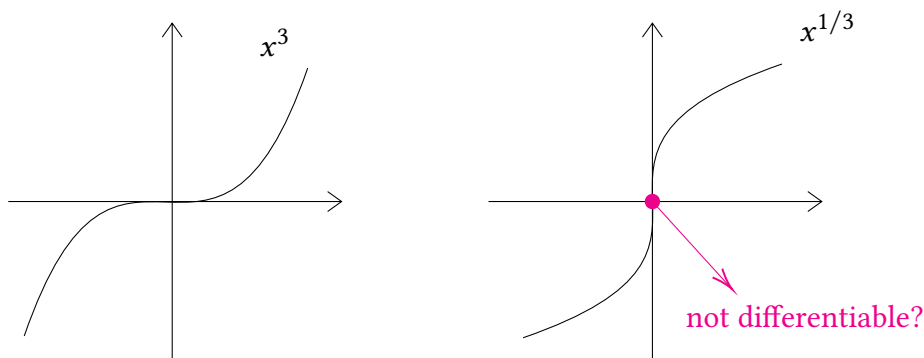
## Automorphism group of a region

**Definition 30.** Let  $\Omega$  be a region in  $\mathbb{C}$ . We let the automorphism group of the region  $\text{Aut}(\Omega)$  be the set of all **bijective analytic functions** from  $\Omega$  to  $\Omega$ .

- $\text{Aut}(\Omega)$  contains the identity function  $f(z) = z$ .
- $\text{Aut}(\Omega)$  is closed under composition.
- $\text{Aut}(\Omega)$  is closed under inverses: if  $f : \Omega \rightarrow \Omega$  is an analytic bijection, then  $f^{-1} : \Omega \rightarrow \Omega$  exists and is **analytic**.

← And composition is a binary operation with associativity

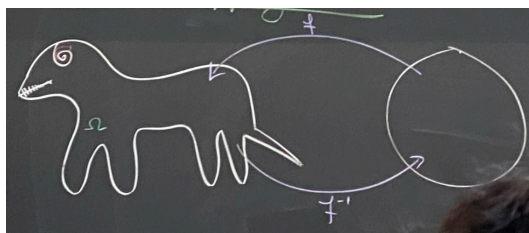
**Remark.** There appears to be a ‘counterexample’:



However, this is only true in  $\mathbb{R}$ . In  $\mathbb{C}$ , we observe that  $z^3$  is **not** a bijection at 0, so this function is not in the group at all!

**Theorem 51** (Riemann Mapping). If  $\Omega$  is simply connected (no holes), then it could be conformally mapped to a disk.

← Except for the entire  $\mathbb{C}$ , which has only constant functions if bounded (by Liouville thm).



Recall from HW1 that for each  $w \in \mathbb{D}$ , we have a bijection

$$\phi_w(z) = \frac{-z + w}{-\bar{w}z + 1}$$

from  $\mathbb{D}$  to  $\mathbb{D}$  and  $\phi \circ \phi = \text{id}$ . To see this, observe the matrix representation

$$\begin{bmatrix} -1 & w \\ -\bar{w} & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & w \\ -\bar{w} & 1 \end{bmatrix} = \begin{bmatrix} 1 - |w|^2 & 0 \\ 0 & 1 - |w|^2 \end{bmatrix} \sim I$$

Furthermore, note  $\phi_w(0) = w$ . Suppose  $f \in \text{Aut}(\mathbb{D})$ , then there is a unique  $w \in \mathbb{D}$  such that  $f(w) = 0$ . Define  $g = f \circ \phi_w \in \text{Aut}(\mathbb{D})$ . Note that  $g(0) = f(\phi_w(0)) = f(w) = 0$ . By Schwarz' lemma, we have  $|g(z)| \leq |z|$  for all  $z \in \mathbb{D}$ .

Since  $g^{-1} \in \text{Aut}(\mathbb{D})$ , we also have  $|g^{-1}(z)| \leq |z|$  for all  $z \in \mathbb{D}$ . Now substitute  $g(z)$  for  $z$  since it's also in the disk. Hence,  $|z| = |g^{-1}(g(z))| \leq |g(z)|$ . Therefore, we are forced to conclude that  $|z| = |g(z)|$  for ALL  $z \in \mathbb{D}$ !

Since  $|z| = |g(z)|$  for ALL  $z \in \mathbb{D}$ , Schwarz' lemma says  $g(z) = \lambda z$  for some  $|\lambda| = 1$ . Let  $\lambda = e^{i\theta}$  for some  $\theta \in \mathbb{R}$ . Thus,

$$g(z) = f(\phi_w(z)) = e^{i\theta} z$$

and so

$$e^{i\theta} \phi_w(z) = f(\phi_w(\phi_w(z))) = f(z)$$

since  $\phi_w \circ \phi_w = \text{id}$ . Therefore,  $f(z) = e^{i\theta} \frac{w - z}{1 - \bar{w}z}$ .

Therefore,

**Proposition 52.**

$$\text{Aut}(\mathbb{D}) = \left\{ e^{i\theta} \frac{w - z}{1 - \bar{w}z} \mid \theta \in [0, 2\pi), w \in \mathbb{D} \right\}$$

**Remark.** The topological representation of the automorphism group of  $\mathbb{D}$  is a 'skinless torus' (collection of open disks revolving from 0 to  $2\pi$ ).

## Morera's theorem

**Theorem 53** (Morera). If  $f : \Omega \rightarrow \mathbb{C}$  is continuous and  $\int_\gamma f(\zeta) d\zeta = 0$  for all \_\_\_\_\_  $\gamma$  in  $\Omega$ , then  $f$  is analytic on  $\Omega$ .

*Proof see notes.*

□

← the blank can be  
'rectangles',  
'triangles',  
'piecewise  $C^1$   
closed curves', etc.

## Weierstrass convergence theorem

Let  $f_n(z)$  be analytic for every  $n \in \mathbb{N}$ . We are still not sure that  $\sum_{n=1}^{\infty} f_n(z)$  is analytic yet!

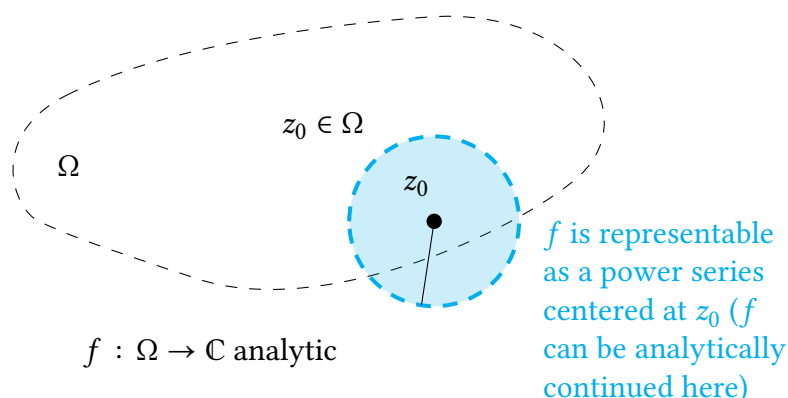
**Theorem 54** (Weierstrass convergence). If  $f_n : \Omega \rightarrow \mathbb{C}$  are analytic and  $f_n$  converges *locally uniformly* on  $\Omega$  to the limit function  $f$  ( $\Leftrightarrow$  **uniform convergence on compact sets**), then  $f$  is analytic and for each fixed  $m$ ,  $f_n^m$  converges to  $f^{(m)}$  locally uniformly on  $\Omega$  and  $f$  is infinitely differentiable.

**Remark.** This is a huge contrast with the *Weierstrass approximation theorem* in real analysis, which says that if  $f : [0, 1] \rightarrow \mathbb{R}$ , then there is a sequence of polynomials  $p_n$  such that  $p_n$  converges to  $f$  uniformly on  $[0, 1]$ . That is, even the most pathological, nowhere-differentiable functions in  $\mathbb{R}$  are a limit of some polynomial sequences! However, in the  $\mathbb{C}$  world, the limit of any analytic function is still analytic.

## Laurent series & isolated singularities

Sometimes the domain  $\Omega$  isn't the largest domain where an analytic function can be analytic. So far, we know we can find the largest disk centered at a point in  $\Omega$  in which a function is analytic and the power series exists there:

← the disk could exceed the bounds of  $\Omega$ !



Can we do even better than that?

## Laurent series

**Example 27.** Let  $f(z) = \frac{1}{z(z-1)}$  analytic on  $\mathbb{C} \setminus \{0, 1\}$ . We realize that if we restrict  $0 < |z| < 1$ , then

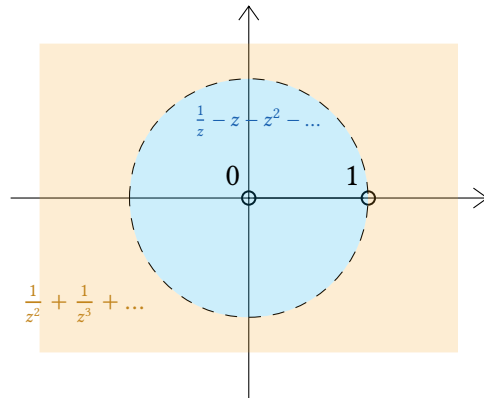
$$f(z) = \frac{-1}{z} \cdot \frac{1}{1-z} = \frac{-1}{z} - 1 - z - z^2 - \dots$$

is the **Laurent series** of  $f(z)$  centered at 0, a point where  $f(z)$  isn't even defined!

**Example 28.** We continue with the previous function. This time, we restrict  $|z| > 1$  and express it as:

$$f(z) = \frac{1}{z^2(1 - \frac{1}{z})} = \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

Hence, we now have different series for  $f$  in different regions:



**Definition 31** (Laurent series). A series in the form  $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$  is a **Laurent series**. It converges at  $z \in \mathbb{C}$  if **both** the analytic part  $\sum_{n=0}^{\infty} a_n(z - z_0)^n$  and the principal part  $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$  converge at  $z$ . If this occurs, the Laurent series would be

$$\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$$

and also converges.

**Lemma 55.** Recall that if  $n \neq -1$ , then  $\frac{(z - z_0)^{n+1}}{n+1}$  is an antiderivative of  $(z - z_0)^n$  on  $\mathbb{C}$ . So if  $\gamma$  is simple closed, then by FTC,

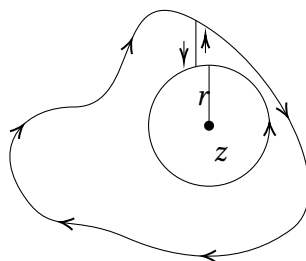
$$\frac{1}{2\pi i} \int_{\gamma} (z - z_0)^n dz = 0$$

whenever  $n \neq -1$ . In addition, by Cauchy's integral formula,  $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = 1$ . Therefore, if  $z_0$  is in the interior of a simple closed curve  $\gamma$ , then

$$\frac{1}{2\pi i} \int_{\gamma} (z - z_0)^n dz = \begin{cases} 0 & n \neq -1 \\ 1 & n = -1 \end{cases}$$

*Proof.* We previously know the result above when  $\gamma$  is a circle. We now extend it

to all simple closed curves by a familiar trick as follows:



□

## Laurent expansion theorem

**Theorem 56** (Laurent expansion). Suppose  $f$  is analytic on the annular region  $A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$ . Then  $f$  has a locally uniformly convergent Laurent expansion

←  $R_1 = 0, R_2 = \infty$  are okay

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

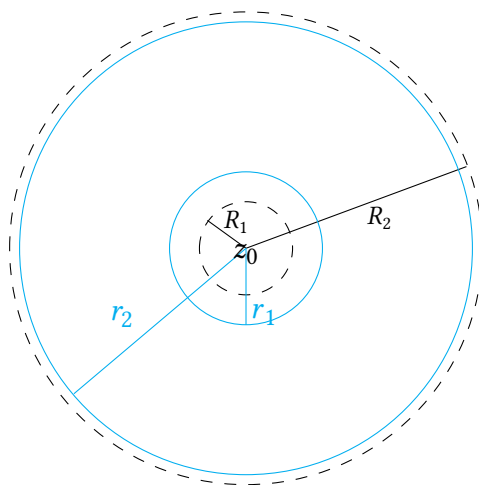
on  $A$ . Moreover, the Laurent coefficients are

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$$

for any  $r$  such that  $R_1 < r < R_2$ .

*Proof gist.* For a gist of why this works:

← For rigorous proof, see [notes](#).



Cauchy's integral formula reveals that

$$f(z) = \frac{1}{2\pi i} \int_{C_{r_2}(z_0)} \frac{f(\zeta) d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{C_{r_1}(z_0)} \frac{f(\zeta) d\zeta}{\zeta - z}$$

whenever

$$R_1 < r_1 < |z| < r_2 < R_2.$$

These integrals are independent of  $r_1$  and  $r_2$  so long as  $r_1 < |z| < r_2$ .  $\square$

**Remark.** If  $n \geq 0$  and  $f$  is analytic on  $|z| < R_2$ , then we should get that the Taylor series expansion and the Laurent expansion for the same function  $f$  to match. They indeed do match by Cauchy's integral formula:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$$

## Isolated singularities

**Definition 32.** If  $f$  is analytic on  $0 < |z - z_0| < R$  (a deleted neighbourhood of  $z_0$ ), then  $z_0$  is an **isolated singularity** of  $f$ .

**Definition 33.** If the principal part of the Laurent expansion for  $f$  at  $z_0$  is 0 (i.e.  $a_{-1} = a_{-2} = \dots = 0$ ), then  $z_0$  is a **removable singularity** of  $f$ . The Laurent expansion for  $f$  at  $z_0$  is simply a power series  $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$  suggests we set  $f(z_0) = a_0$ , in which case  $f$  is analytic at  $z_0$ .

**Example 29.** Observe

$$\begin{aligned} \frac{\sin z}{z} &= \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \end{aligned}$$

We define  $f(0) = 1$ , so  $\frac{\sin z}{z}$  is actually entire!

← This agrees with L'Hôpital's rule.

**Theorem 57.** If  $z_0$  is an isolated singularity of an analytic function  $f$ , then  $z_0$  is removable *if and only if* any of the following hold:

- (a)  $f$  is bounded on some deleted neighbourhood of  $z_0$
- (b)  $\lim_{z \rightarrow z_0} f(z)$  exists
- (c)  $\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$

←  $\infty$  doesn't count

**Remark.** (a) and (b) implies (c), (b) implies (a).



*Proof.* It suffices to show that (c)  $\iff$  removable.

( $\implies$ ) If  $z_0$  is removable, then  $f$  is analytic at  $z_0$ , so all of the above follow.

( $\impliedby$ ) Suppose (c) holds. Then for all  $\varepsilon > 0$ , there exists  $0 \leq r < 1$  such that whenever  $|z - z_0| < 2r$ , we have  $|f(z)(z - z_0)| < \varepsilon$ .

Then, for all  $n \geq 1$ , we have

$$\begin{aligned} |a_{-n}| &= \left| \frac{1}{2\pi i} \int_{C_r(z_0)} f(\zeta)(\zeta - z_0)^{n-1} d\zeta \right| \\ &= \left| \frac{1}{2\pi i} \int_{C_r(z_0)} f(\zeta)(\zeta - z_0)(\zeta - z_0)^{n-2} d\zeta \right| \\ &\leq \frac{1}{2\pi} \cdot \varepsilon r^{n-2} 2\pi r \\ &= \varepsilon r^{n-1} \\ &< \varepsilon \end{aligned}$$

Thus,  $a_{-n} = 0$  for all  $n \geq 1$ . The principal part of the Laurent expansion of  $f$  is zero.

□

**Definition 34.** If the principal part of  $f$  at  $z_0$  is of the form

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)}$$

where  $a_{-n} \neq 0$ , then  $z_0$  is a **pole** of  $f$  of order  $n$ .

**Definition 35.** A pole of order 1 is a **simple pole**.

**Theorem 58.** If  $z_0$  is an isolated singularity of  $f$ , then  $z_0$  is a **pole** of order  $\leq n$  if and only if there is an analytic  $\phi(z)$  on a deleted neighbourhood of  $z_0$  such that

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}$$

This occurs if and only if any of the following hold:

- (a)  $(z - z_0)^n f(z)$  is bounded on some deleted neighbourhood of  $z_0$
- (b)  $\lim_{z \rightarrow z_0} f(z)(z - z_0)^n$  exists
- (c)  $\lim_{z \rightarrow z_0} f(z)(z - z_0)^{n+1} = 0$

**Remark.** We can think of poles and zeros in the following fashion:

$$\begin{array}{ll} f(z) = (z - z_0)^j F(z) & g(z) = \frac{G(z)}{(z - z_0)^k} \\ f \text{ has a zero of order } j \text{ at } z_0 & g \text{ has a pole of order } k \text{ at } z_0 \\ F \text{ doesn't vanish at } z_0 & G \text{ doesn't vanish at } z_0 \end{array} \quad \text{Then:}$$

$$f(z)g(z) = (z - z_0)^{j-k} F(z)G(z)$$

- ←  $n$  must be **finite** such that we can clear the denominator
- ← Return to the full neighbourhood by the trick of removable singularity

- If  $j = k$ ,  $z_0$  is a removable singularity for  $fg$  and is not a zero.
- If  $j > k$ , then  $z_0$  is a zero.
- If  $k > j$ , then  $z_0$  is a pole.

Poles are nice! They could be removed like the denominators of rational functions. However, some other singularities cannot do so.

**Definition 36.** If the principal part of the Laurent series for  $f$  at  $z_0$  is  $\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n}$  where **infinitely** many  $a_{-n} \neq 0$ , then  $z_0$  is an essential singularity of  $f$ .

**Remark.** It is not hard to make an essential singularity: take any entire function with infinite power series. Plug in  $\frac{1}{x}$  instead of  $x$ .

**Example 30.**  $e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{1}{n!x^n}$  has an essential singularity at 0.

**Theorem 59** (Casorati-Weierstrass). Let  $z_0$  be an essential singularity of  $f$ . For **each**  $w \in \mathbb{C}$ , there is a sequence  $z_n (n \geq 1)$  such that  $z_n \rightarrow z_0$  and  $f(z_n) \rightarrow w$ .

← This is wild!  $f$  could almost splatter everywhere near  $z_0$ . There isn't a reasonable value to assign to  $f(z_0)$ .

*Proof.* Suppose towards a contradiction that there exists  $w \in \mathbb{C}$  such that no such  $z_n$  exists. Then there exists  $\varepsilon > 0$  and  $\delta > 0$  such that when  $0 < |z - z_0| < \delta$ , we have  $|f(z) - w| \geq \varepsilon$  (that is,  $f$  is not getting close to  $w$ ). Thus,  $g(z) = \frac{1}{f(z) - w}$  is analytic on  $0 < |z - z_0| < \delta$  and  $|g(z)| \leq \frac{1}{\varepsilon}$  there. The singularity  $z_0$  of  $g$  is therefore removable. Then  $f(z) = w + \frac{1}{g(z)}$ , which is either analytic or has a pole at  $z_0$  (if  $g(z_0) = 0$ ). This causes a contradiction.  $\square$

**Theorem 60** (Great Picard). If  $z_0$  is an essential singularity of  $f$ , then in any deleted neighbourhood of  $z_0$ , we have  $f$  assuming **every** complex value (with at most one exception) **infinitely** many times.

**Example 31.**  $f(z) = e^{\frac{1}{z}}$  has an essential singularity at  $z_0 = 0$ . (Note  $f(z) \neq 0$  is the exceptional value that is never assumed.) Let  $w \neq 0$  and let  $z = \frac{1}{\log w}$  where  $\log w$  is a nonzero logarithm of  $w$ . Then

← so no 1 for  $w$

$$f(z) = e^{\frac{1}{1/\log w}} = e^{\log w} = w$$

**Theorem 61** (Little Picard). If  $f$  is entire and nonconstant, then  $f$  assumes every complex value, with at most one exception.

*Proof.* If  $f$  is a nonconstant polynomial and  $w \in \mathbb{C}$ , then the polynomial  $f(z) - w$  has a zero in  $\mathbb{C}$  by the Fundamental Theorem of Algebra, so  $f$  assumes the value of  $w$ .

If  $f$  is not a polynomial, then  $f(\frac{1}{z})$  has an essential singularity at 0. Then use Great Picard Thm.  $\square$

← Non-polynomial means the Taylor series is infinite

## Residues

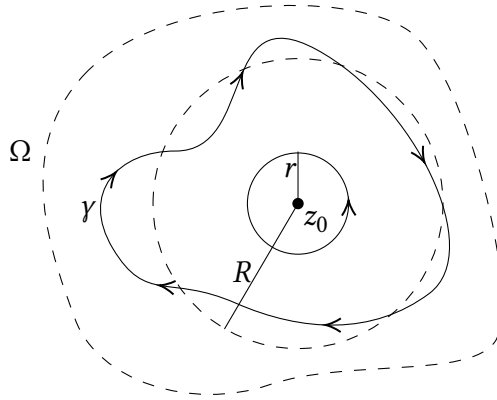
**Definition 37.** Let the Laurent series  $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$  be analytic at  $0 < |z - z_0| < R$ . The coefficient  $a_{-1}$  is the **residue** of  $f$  at  $z_0$ . Notation:

$$\text{Res}(f; z_0) = a_{-1}$$

**Theorem 62** (Residue, simple vers.). Let  $f : \Omega \rightarrow \mathbb{C}$  analytic except on the isolated singularity  $z_0$ . Then:

$$\frac{1}{2\pi i} \int_{\gamma} f(\zeta) d\zeta = \text{Res}(f; z_0)$$

for any simple closed curve  $\gamma$  in  $\Omega$  with  $z_0$  in its interior and whose interior is contained in  $\Omega$ .



*Proof.* The Laurent expansion  $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$  converges locally uniformly on some punctured disk  $0 < |z - z_0| < R$ . If  $r \in (0, R)$  is sufficiently small, then the deformation version of Cauchy's theorem implies

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} f(z) dz &= \frac{1}{2\pi i} \int_{C_r(z_0)} f(z) dz \\ &= \frac{1}{2\pi i} \int_{C_r(z_0)} \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n dz \\ &= \sum_{n=-\infty}^{\infty} a_n \left( \frac{1}{2\pi i} \int_{C_r(z_0)} (z - z_0)^n dz \right) \end{aligned}$$

Observe  $\left( \frac{1}{2\pi i} \int_{C_r(z_0)} (z - z_0)^n dz \right) = 0$  unless  $n = -1$ , in which it's 1. Hence:

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} f(z) dz &= a_{-1} \\ &= \text{Res}(f; z_0) \end{aligned}$$

The interchange of sum and integral is permissible because the Laurent series converges uniformly on  $C_r(z_0)$ .  $\square$

**Lemma 63.** If  $z_0$  is a **simple** pole of  $f$ , then

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z)$$

*Proof.* Near  $z_0$ , we have

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

Thus,  $(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + \dots$  tends to  $a_{-1}$  when  $z \rightarrow z_0$ . So  $a_{-1} = \lim_{z \rightarrow z_0} (z - z_0)f(z)$ .  $\square$

**Remark.** **Cauchy's integral formula** is a special case of the residue formula as we rename the function to introduce a simple pole at  $z_0$ :

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) dz}{z - z_0} &= \text{Res}\left(\frac{f(z)}{z - z_0}; z_0\right) \\ &= \lim_{z \rightarrow z_0} (z - z_0) \frac{f(z)}{z - z_0} \\ &= f(z_0) \end{aligned}$$

**Example 32.** Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^2} dx$$

in which  $a \neq 0$  is real. We assume that  $a > 0$ ; the case  $a < 0$  is similar. Since

$$\left| \frac{\cos ax}{1 + x^2} \right| \leq \frac{1}{1 + x^2}$$

on  $(-\infty, 0]$  and  $[0, \infty)$ , it follows that the improper integral converges by the comparison test.

This allows us to consider the integral from  $-\infty$  to  $\infty$  directly, without having to consider the improper integrals over the positive and negative parts separately. Therefore, write

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^2} dx = \text{Re} \left( \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{iax}}{1 + x^2} dx \right)$$

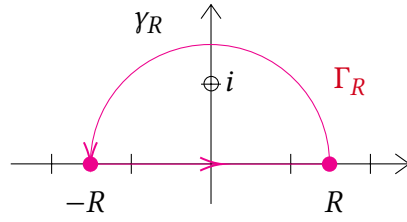
where we let

$$f(z) = \frac{e^{iaz}}{1 + z^2} = \frac{e^{iaz}}{(z - i)(z + i)}$$

which has two simple poles  $z = \pm i$ . We focus on  $i$  first.

← That means the pole is order 1, and the principal part of the Laurent series at that point only has 1 term.

For  $R > 1$  (so that  $i$  is enclosed), let  $\Gamma_R$  denote the semicircular curve obtained by joining  $[-R, R]$  with  $\gamma_R$ , the upper half of the circle  $|z| = R$ :



Since  $i$  is a pole enclosed in  $\Gamma_R$ , the residue theorem implies  $\int_{\Gamma_R} f(z) dz = 2\pi i \operatorname{Res}(f, i)$ . By Lemma 63,

$$\operatorname{Res}(f, i) = \lim_{z \rightarrow i} (z - i)f(z) = \lim_{z \rightarrow i} \frac{e^{iaz}}{(z + i)} = \frac{e^{-a}}{2i}$$

so it follows that

$$\int_{-R}^R \frac{e^{iax}}{1+x^2} dx + \int_{\gamma_R} \frac{e^{iaz}}{1+z^2} dz = 2\pi i \operatorname{Res}(f, i) = \pi e^{-a}$$

We look at  $\int_{\gamma_R} \frac{e^{iaz}}{1+z^2} dz$ . If  $z = x + iy$  is on  $\gamma_R$ , then  $y \geq 0$  and hence (since  $a > 0$ ):

$$\begin{aligned} \left| \int_{\gamma_R} \frac{e^{iaz}}{1+z^2} dz \right| &= \left| \int_{\gamma_R} \frac{e^{iaz}}{1+z^2} dz \right| \\ &\leq \pi R \sup_{z \in \gamma_R} \frac{|e^{iaz}|}{|1+z^2|} \quad \text{by upper bound over length of curve} \\ &\leq \pi R \sup_{x+iy \in \gamma_R} \frac{e^{-ay}}{R^2 - 1} \quad \text{since } |e^{iax}| = 1 \\ &= \frac{\pi R}{R^2 - 1} \end{aligned}$$

which tends to zero as  $R \rightarrow \infty$ . Let  $R \rightarrow \infty$  and get

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2} dx = \pi e^{-a}$$

Thus the real part would be our answer

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2} dx = \pi e^{-a}$$

# Residue theory

## Index, aka. winding number of a curve

**Definition 38.** Let  $\gamma$  be a closed, piecewise  $C^1$  curve and  $z_0 \notin \gamma$ . The **index** (also called the **winding number**) of  $\gamma$  with respect to  $z_0$  is

← Number of counterclockwise loop-arounds

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0}$$

**Remark.** If the curve  $\gamma : [a, b] \rightarrow \mathbb{C}$  is parameterized on  $t$ , and  $\gamma(a) = \gamma(b)$  (closed), then let  $z = \gamma(t)$ ,  $dz = \gamma'(t) dt$ . Then we have

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t) dt}{\gamma(t) - z_0}$$

**Lemma 64.** If  $\gamma$  is a closed curve and  $z_0 \notin \gamma$ , then  $I(\gamma; z_0) \in \mathbb{Z}$ .

*Proof.* Parameterize  $\gamma$  as above using  $s$ . Then

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s) ds}{\gamma(s) - z_0}$$

Define

$$g(t) = \int_a^t \frac{\gamma'(s) ds}{\gamma(s) - z_0}$$

Since  $\gamma$  is piecewise, by FTC, we have

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

for all but finitely many  $t \in [a, b]$ . Thus,

$$\begin{aligned} \frac{d}{dt} (e^{-g(t)}(\gamma(t) - z_0)) &= e^{-g(t)}\gamma'(t) - g'(t)e^{-g(t)}(\gamma(t) - z_0) \\ &= e^{-g(t)}\gamma'(t) - \frac{\gamma'(t)}{\gamma(t) - z_0}e^{-g(t)}(\gamma(t) - z_0) \\ &= 0 \end{aligned}$$

for all  $t$  where  $g'(t)$  exists. Therefore,  $e^{-g(t)}(\gamma(t) - z_0)$  is piecewise constant. But this function is also continuous, so it's constant! Therefore:

$$e^{-g(b)}(\gamma(b) - z_0) = e^{-g(a)}(\gamma(a) - z_0)$$

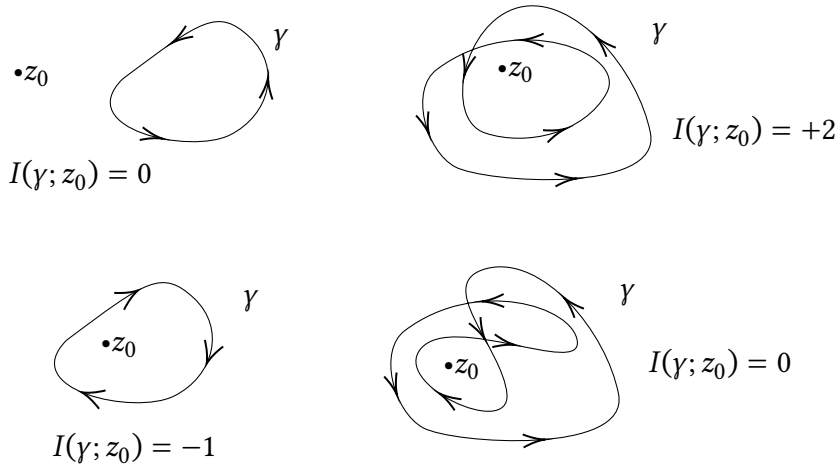
The blue terms are the same since  $\gamma(a) = \gamma(b)$ . Therefore,  $e^{-g(b)} = e^{-g(a)} = e^0 = 1$  since  $g(a) = 0$ .

Hence,  $g(b) = 2\pi in$  for some  $n \in \mathbb{Z}$ . Thus:

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_0} = \frac{1}{2\pi i} g(b) = n \in \mathbb{Z}$$

□

**Remark.** Winding number essentially tracks the change of argument when the curve is traversed.



## Simply connected domains

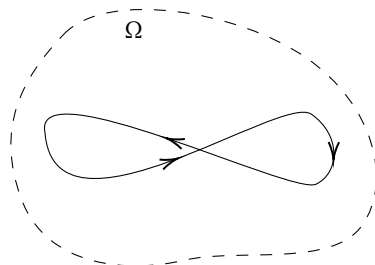
**Definition 39.** A region  $\Omega$  is **simply connected** if it has no holes. In other words:

- (a)  $I(\gamma; z_0) = 0$  for **every** closed curve  $\gamma$  in  $\Omega$  and every  $z_0 \notin \Omega$ .
- (b) Every closed curve  $\gamma$  in  $\Omega$  is **homotopic** to a point in  $\Omega$ .

← homotopic means  
can be continuously  
deformed without  
passing outside  $\Omega$

Recall Theorem 36. We can now extend beyond simple curves:

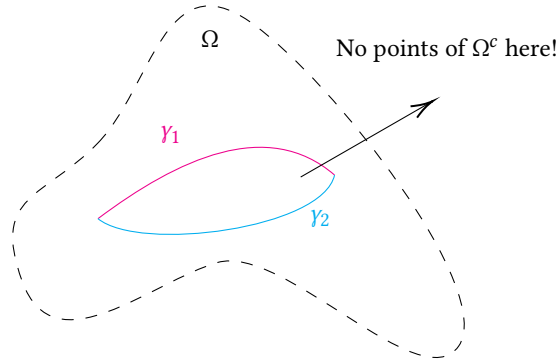
**Theorem 65** (Cauchy's, for simply connected domains). If  $\Omega$  is **simply connected**,  $f$  is analytic on  $\Omega$ , then  $\int_{\gamma} f(z) dz = 0$  for any closed curve  $\gamma$  in  $\Omega$ .



**Theorem 66.** A region  $\Omega$  is **simply connected** if and only if every analytic function  $f : \Omega \rightarrow \mathbb{C}$  has an **antiderivative** on  $\Omega$ .

*Proof.* We proved in Theorem 39 that every analytic function  $f : \Omega \rightarrow \mathbb{C}$  on a *convex*  $\Omega$  has an antiderivative. We adapt the proof.

( $\Rightarrow$ )



Then use Theorem 37.

( $\Leftarrow$ ) Suppose BWO that  $\Omega$  is **not** simply connected. Then there is a  $z_0 \in \Omega^c$  and  $\gamma$  a closed curve in  $\Omega$  such that  $I(\gamma; z_0) \neq 0$ . Then  $f(z) = \frac{1}{z-z_0}$  does not have an antiderivative on  $\Omega$ .

□

**Theorem 67.** If  $\Omega$  is simply connected and  $f : \Omega \rightarrow \mathbb{C}$  is analytic and **never 0**, then there is an analytic  $g : \Omega \rightarrow \mathbb{C}$  such that  $f = e^g$ . That is, it's got a log!

*Proof.* The function  $f'/f$  is analytic on  $\Omega$ , thus it has an antiderivative  $F$  on  $\Omega$ . Since

$$(fe^{-F})' = f'e^{-F} - F'fe^{-F} = f'e^{-F} - f'e^{-F} = 0$$

it follows that  $fe^{-F} = c$  for some constant  $c$ . Thus,  $g = \log c + F$  (we may choose any fixed branch of  $\log c$ ). □

**Theorem 68** (Residue, general case). Let  $\Omega$  be a simply-connected region and let  $z_1, z_2, \dots, z_n \in \Omega$  be distinct. If  $f : \Omega \setminus \{z_1, z_2, \dots, z_n\} \rightarrow \mathbb{C}$  is analytic and  $\gamma$  is a closed curve in  $\Omega$  that passes through no  $z_i$ , then

$$\frac{1}{2\pi i} \int_{\gamma} f(\zeta) d\zeta = \sum_{j=1}^n \text{Res}(f; z_j) I(\gamma, z_j)$$



## The argument principle

Suppose  $f : \Omega \rightarrow \mathbb{C}$  is analytic and has zeros only at  $z_1, z_2, \dots, z_n \in \Omega$  (repeated according to multiplicity). Write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n) g(z)$$

where  $g(z)$  is analytic and nonvanishing on  $\Omega$ . The product formula for derivatives implies

$$f'(z) = (z - z_2)(z - z_3) \cdots (z - z_n) g(z) + (z - z_1)(z - z_3) \cdots (z - z_n) g(z) + \cdots + (z - z_1)(z - z_2) \cdots (z - z_{n-1}) g(z) + (z - z_1)(z - z_2) \cdots (z - z_n) g'(z).$$

Divide by  $f(z)$  and obtain the logarithmic derivative of  $f$ :

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

← Since  $(\log f)' = f'/f$

If  $\gamma$  is a simple closed curve in  $\Omega$  whose interior lies in  $\Omega$  and which contains each  $z_i$  in its interior, then

← a simple closed curve can only envelope a finite amount of zeros!

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz &= \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - z_k} + \frac{1}{2\pi i} \int_{\gamma} \frac{g'(z)}{g(z)} dz \\ &= \sum_{k=1}^n I(\gamma; z_k) + 0 \\ &= \left( \sum_{k=1}^n 1 \right) + 0 \\ &= n \end{aligned}$$

The final integral vanishes by Cauchy's theorem since  $g'/g$  is analytic on  $\Omega$ . Integrating the logarithmic derivative  $f'/f$  of an analytic function  $f$  around a closed curve  $\gamma$  counts the number of zeros of  $f$ , repeated according to multiplicity, inside of  $\gamma$ .

**Theorem 69** (The Argument Principle). Let  $\Omega$  be a region in  $\mathbb{C}$  and let  $\gamma$  be a simple closed curve in  $\Omega$  with its interior in  $\Omega$ . If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and has no zeros on  $\gamma$ , then the number of zeros  $Z_f(\gamma)$  of  $f$ , repeated according to multiplicity, in the interior of  $\gamma$  is finite and is given by

$$Z_f(\gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz$$

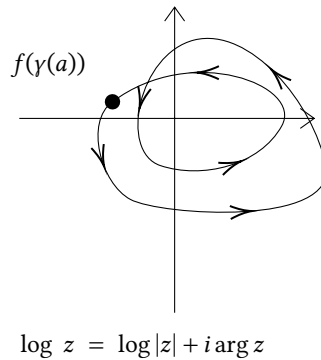
*Proof.* In light of the preceding discussion, we only need to show that  $f$  has only finitely many zeros inside of  $\gamma$ . Let  $G$  denote the union of  $\gamma$  and its interior. Since  $G$  is closed and bounded, it is compact. If  $f$  had infinitely many distinct zeros  $z_n$  inside of  $\gamma$ , these would have an accumulation point in  $G \subseteq \Omega$ . The identity theorem would imply that  $f$  is identically zero on  $\Omega$ , which contradicts the hypothesis that  $f$  does not vanish on  $\gamma$ .  $\square$

**Remark.** Why *argument* principle? Let  $\gamma : [a, b] \rightarrow \mathbb{C}$  be a parametrization and consider the curve  $f \circ \gamma : [a, b] \rightarrow \mathbb{C}$ . The following computation shows that the number of zeros of  $f$  inside  $\gamma$  equals the winding number of  $f \circ \gamma$  with respect to the origin:

$$\begin{aligned} Z_f(\gamma) &= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz \\ &= \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t) dt}{f(\gamma(t))} \\ &= \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{d\zeta}{\zeta - 0} \\ &= I(f \circ \gamma; 0) \end{aligned}$$

in which  $\zeta = f(\gamma(t))$  and  $d\zeta = f'(\gamma(t))\gamma'(t) dt$  by the chain rule.

It allows computers to compute roots with great ease. As soon as we have an error  $< \frac{1}{2}$  we are done.



**Corollary 70** (Root counting formula). If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and  $\gamma$  is a simple closed curve in  $\gamma$  with its interior in  $\Omega$  such that  $f(z) \neq w$  on  $\gamma$ , then the number of roots of  $f(z) = w$  inside  $\gamma$  (with multiplicity) is

$$Z_{f(z)-w}(\gamma) = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - w} dz$$

**Example 33.** Consider the function

$$f(z) = z \left( \frac{z + \frac{1}{2}}{1 + \frac{1}{2}z} \right) \left( \frac{z - \frac{3}{4}}{1 - \frac{3}{4}z} \right) \left( \frac{z - \frac{4i}{5}}{1 + \frac{4i}{5}z} \right)$$

Being a product of disk automorphisms,  $f$  maps  $\mathbb{D} \rightarrow \mathbb{D}$ . It has roots  $0, \frac{-1}{2}, \frac{4i}{5}, \frac{3}{4}$ . We could observe the increment of winding number corresponding to how many times zeros are included.

