

MATH135 Complex Analysis Notes

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Contents

Regions, differentiability, analyticity	3
Regions	3
Complex derivatives and analyticity	3
Curves, paths	5
Conformality	6
Cauchy-Riemann equations, harmonic functions	6
Multivariate notion of complex derivatives	6
Cauchy-Riemann Equations	7
Orientation-preserving as shown by Jacobian	7
The Laplacian, harmonic functions and conjugates	8
Harmonic conjugate	8
Finding a harmonic conjugate	8
Physics analogies of harmonic functions	10
Möbius transformations	10
Möbius transformations, the extended plane	10
Möbius transformations as matrices	11
Möbius transformations take circles to circles	12
Recall: infinite series	14
Divergence test	14
Integral test	14
Absolute convergence	15
Riemann Rearrangement Theorem	16
Uniform convergence	16
Uniform convergence preserves continuity	17
Integrals work with uniform convergence	17
Power series	18
Convergence of geometric series	18

Radius of convergence	19
Limit superior	19
Cauchy-Hadamard formula	19
Term-by-term differentiation of power series	20
Derivative series have the same radius of convergence	21
Power series are infinitely differentiable	22
Power series expansions are unique	23
Elementary functions	24
Trig functions in terms of e	24
Periodicity of functions	25
Definition of π	25
Complex logarithm	27
Logarithm	27
Principal branch of the logarithm	27
First derivative of \ln	27
Complex power	28
Riemann surface	28
Cauchy's theorem and its consequences	29
Complex integration	29
Piecewise continuity	29
Triangle inequality	29
Complex differentiability	30
Fundamental theorem of calculus, complex edition	30
An important estimate	31
Fundamental theorem of calculus for contour integrals	32
Cauchy's theorem	33
Take 1	33
Take 2: deformation version	34
Fresnel integrals	34

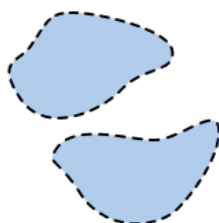
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of \mathbb{C} .

- A region without “holes” is simply connected.

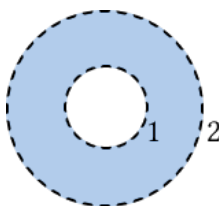
Non-example 1. This is not a region (not connected):



Example 2. \mathbb{C} is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \rightarrow \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

2. If f is differentiable at every point in Ω , we say f is **analytic** on Ω .
3. If f is analytic on \mathbb{C} , then f is **entire**.

← this $z \rightarrow z_0$ could be from **any** directions!

← Means that existence of 1st derivative implies the existence of ∞ th derivative! & has Taylor expansion.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on \mathbb{C} except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \rightarrow z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if $z \rightarrow z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. \square

Proposition 1. Let f be differentiable at z_0 . Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the “tangent-line approximation” equivalent in \mathbb{C} !

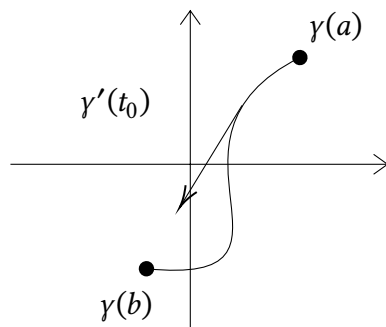
In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is locally orientation-preserving and could be approximated by a linear function.

← The RHS is a **linear** function!

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

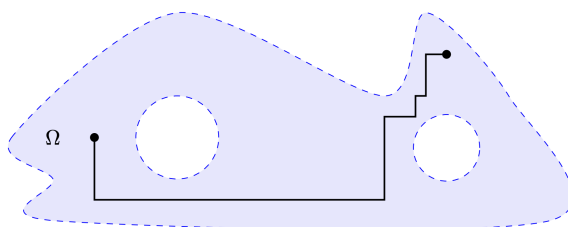
Curves, paths

Definition 3. A **curve** in \mathbb{C} is a function $\gamma : [a, b] \rightarrow \mathbb{C}, a, b \in \mathbb{R}$.



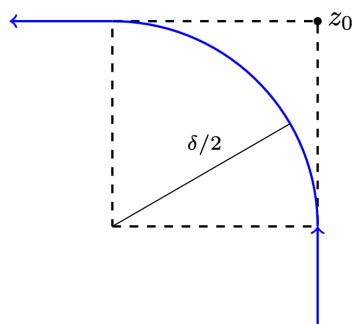
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The “Boxy-path” Theorem). A nonempty open set Ω in \mathbb{C} is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω , each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a “**boxy path**”.

Remark. There is also always a **smooth path**. That is:

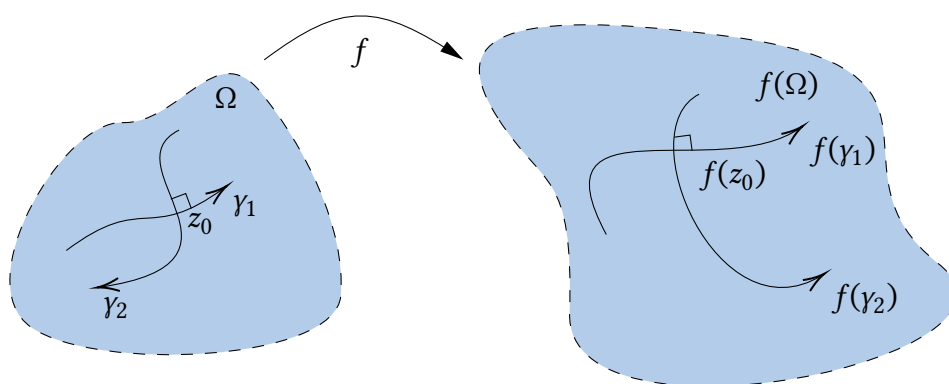


Theorem 3 (“Smooth-path”). A nonempty open set Ω in \mathbb{C} is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See [lecture 2 notes](#). □

Conformality

Let f be an analytic complex function on Ω .



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ . Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(z_0)$ also with angle θ .

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall:
$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Now we write each function with complex variables as $f(z) = u(z) + i v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r \rightarrow 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \rightarrow 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations).

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \rightarrow \mathbb{C}$ is analytic and $f' = 0$ on Ω , then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω . By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω . Hence, \mathbf{u}, \mathbf{v} are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω . \square

Orientation-preserving as shown by Jacobian

Let $f : \Omega \rightarrow \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 &= \bar{f}' \cdot f' = (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0 , implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales \mathbb{R} by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

Definition 5. Real-valued functions $u : \Omega \rightarrow \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \rightarrow \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω .

Example 8. $u = x^2 - y^2, v = 2xy$.

Remark. Harmonic conjugates are unique up to translation (\pm constants).

Remark. If u is harmonic on Ω , it does NOT have to have a harmonic conjugate on Ω .

← $\Delta u = 0$
characterizes
steady-state
solutions to heat
equations on Ω .

← Check it!

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log |z|$ is harmonic on $\mathbb{C} \setminus \{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2} \log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x} \left(\frac{1}{2} \log(x^2 + y^2) \right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} &= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. \square

Now, can we find a harmonic conjugate for the aforementioned u ?

We could use the two Cauchy-Riemann Equations. One of them:

$$\begin{aligned} v_y &= u_x \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v(x, y) &= \int v_y dy + C(x) && \text{unknown function of } x \\ &= \arctan\left(\frac{y}{x}\right) + C(x) \end{aligned}$$

Then, we use the second one:

$$\begin{aligned} \frac{y}{x^2 + y^2} &= u_y = -v_x = -\frac{\partial}{\partial x} \left(\arctan\left(\frac{y}{x}\right) + C(x) \right) \\ &= \frac{y}{x^2 + y^2} - C'(x) \implies C'(x) = 0 \end{aligned}$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x, y) = \arctan\left(\frac{y}{x}\right) + C$$

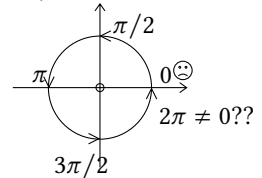
where C is a constant. WLOG, let $C = 0$. Then $v(x, y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$v(z) = \arg(z)$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

← Review quotient rule!

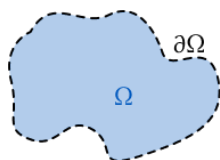
← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C} \setminus \{0\}$:



To be resolved later!

Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in \mathbb{C} . Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \rightarrow \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω .

1. U cannot have a *local* maximum in Ω .
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$.
3. U cannot be locally constant without being globally constant.

← Ω^- denotes the closure of Ω

Theorem 8 (Maximum principle). Let Ω be a bounded region in \mathbb{C} and let $f : \Omega^- \rightarrow \mathbb{C}$ be analytic on Ω and continuous on Ω^- .

1. If $|f|$ achieves a local max in Ω , then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations).

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C} \setminus \{\frac{-d}{c}\}$ and **conformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C} \setminus \{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$\begin{aligned} f(z) = f(w) &\implies \frac{az+b}{cz+d} = \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

Definition 8 (The extended plane). We set the following convention:

$$\begin{aligned} f\left(\frac{-d}{c}\right) &= \infty \\ f(\infty) &= \frac{a}{c} \end{aligned}$$

with this, f is a **bijection** from $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can associate $f(z) = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

Remark. $M_{f \circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

Theorem 9. A Möbius transformation $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with three fixed points in $\hat{\mathbb{C}}$ is the **identity map** $\text{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transformation. ← think about that!
 - (a) If $f(z) = z$, we are done since we get the identity!
 - (b) Otherwise the function only has one fixed point at ∞ .
2. If ∞ is not a fixed point, then $c \neq 0$. Solve:

$$\begin{aligned} f(z) + z &\Leftrightarrow \frac{az + b}{cz + d} = z \\ az + b &= cz^2 + dz \\ cz^2 + (d - a)z - b &= 0 \end{aligned}$$

is a quadratic which has at most two (distinct) solutions in \mathbb{C} . Hence, this transformation fixes at most two points.

□

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so we are done.
2. Now suppose $c \neq 0$. Then

← since linear transformations preserve circles and lines

$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} \\ &= \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c} \end{aligned}$$

which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz + d} \mapsto \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c}$$

We now only need to show that inversion preserves circles.

Let a circle in \mathbb{R}^2 be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.

Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle!

← check this!

□

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

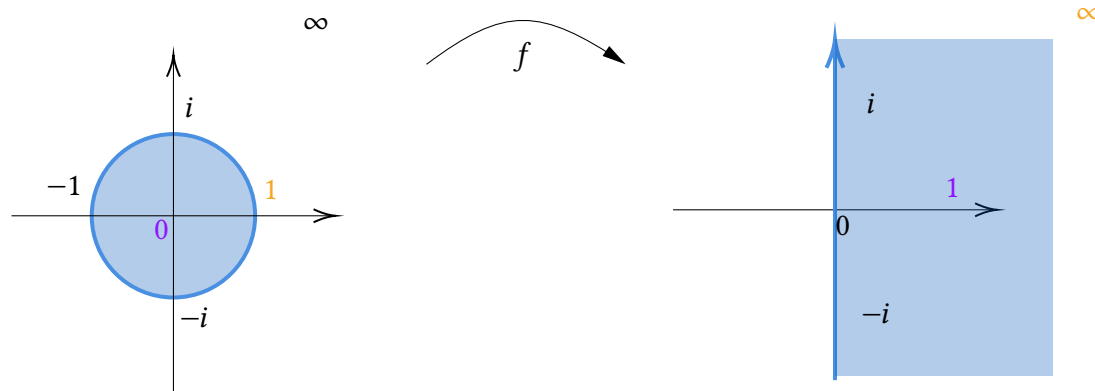
$$\begin{array}{ccccc} z_1 & \xrightarrow{\phi} & 0 & \xrightarrow{\psi^{-1}} & w_1 \\ z_2 & \xrightarrow{\phi} & 1 & \xrightarrow{\psi^{-1}} & w_2 \\ z_3 & \xrightarrow{\phi} & \infty & \xrightarrow{\psi^{-1}} & w_3 \end{array}$$

and we could simply let $f = \psi^{-1} \circ \phi$.

□

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N \rightarrow \infty} S_N = S$ where $S_N = a_1 + \dots + a_N$.

← S_N is the N -th partial sum.

Divergence test

Definition 10 (Divergence test). A pair of contrapositives:

← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.
2. If $\lim_{n \rightarrow \infty} a_n \neq 0$ (including the case where the limit doesn't exist) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \dots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞ .

← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} a_n = \lim_{N \rightarrow \infty} S - S_N = 0$.

← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion). $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$,

there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left| \sum_{n=j}^k a_n \right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test). Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\rightarrow \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x) dx$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.

← do an improper integral!

Moreover, $\int_1^N f(x) \, dx \leq a_1 + \dots + a_N \leq a_1 + \int_1^N f(x) \, dx$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then

← $a_n = \frac{1}{n}$

$$\ln N \leq 1 + \frac{1}{2} + \dots + \frac{1}{N} \leq 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

Theorem 14. The “ p -series” $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \operatorname{Re}(s) > 1$$

Remark. Euler figured out:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979):

← still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202 \dots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

← Good

← BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series

← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges to $\ln 2$. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

Theorem 17 (Riemann Rearrangement Theorem). If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\begin{aligned} \left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{j=0}^{\infty} b_j \right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^n a_k b_{n-k} \end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See [notes](#).

Uniform convergence

Definition 14. A sequence of functions $f_n : X \rightarrow \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \rightarrow \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!

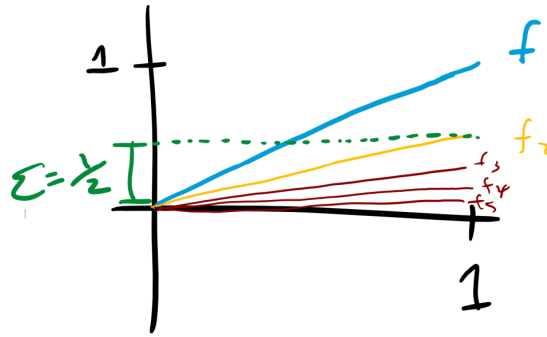


FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \rightarrow \mathbb{C}$ are continuous and converges uniformly on X to $f : X \rightarrow \mathbb{C}$, then f is continuous on X . In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n \rightarrow \infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.

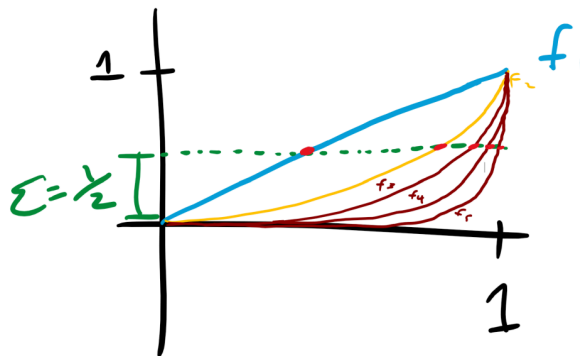


FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a, b] \rightarrow \mathbb{C}$ are continuous and converge uniformly on $[a, b]$ to f , then

← Integrals work with uniform convergence

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx$$

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1, 1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test). Let $f_n : X \rightarrow \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X .

Power series

Definition 15. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. The a_n is the *coefficient* and z_0 is the *center*.

Convergence of geometric series

Theorem 21. The *geometric series* ($a_n = 1, z_0 = 0$) $\sum_{n=0}^{\infty} z^n$ converges absolutely to $\frac{1}{1-z}$ if $|z| < 1$, and it diverges otherwise.

Moreover, for each $r \in [0, 1[$, the convergence is **uniform** on $|z| \leq r$.

Proof. If $|z| \geq 1$, then $z^n \not\rightarrow 0$, so by the test of divergence, the series diverges.

Now suppose $|z| < 1$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} z^n &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} z^n \\ &= \lim_{N \rightarrow \infty} (1 + z + z^2 + \cdots + z^{N-1}) \\ &= \lim_{N \rightarrow \infty} \frac{1 - z^N}{1 - z} \\ &= \frac{1}{1 - z} \end{aligned} \quad \text{since } |z| < 1$$

← The fact that we can find a formula for this sum is quite rare!

Which gives us point-wise convergence. Then, for any r such that $|z| \leq r < 1$, we have

$$\sum_{n=0}^{\infty} |z^n| \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

Hence, by the Weierstrass M -test, the series converges *absolutely and uniformly* on $|z| \leq r$. \square

Remark. Moral of the story:

- The *radius of convergence* $R = 1$ has the property that the series converges on $|z| < R$, and diverges if $|z| > R$.
- The series converges *uniformly* on $|z| \leq r < 1$ but not on $|z| < 1$ itself. Why? Let $r = 1$; we need be able to get $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left| \frac{1-z^n}{1-z} - \frac{1}{1-z} \right| < 1$ for all $|z| < 1$. However, this is not gonna work: as $z \rightarrow 1$, observe that this is going to eventually exceed 1.

- The limit function $\frac{1}{1-z}$ is **analytic** on $\mathbb{C} \setminus \{1\}$. But the geometric series represents this function only on $|z| < 1$. In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.
- The limit function $\frac{1}{1-z}$ is cool if $z \neq 1$, but as long as $|z| = 1$ (**even** if $z \neq 1$), the geometric series diverges!

← the limit function is well-defined way beyond the \mathbb{D} !

← in the complex number sense!

Radius of convergence

Definition 16. The **limit superior** (lim sup) of a sequence of nonnegative real numbers x_n is the largest *limit point* of the x_n :

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \geq 0} \sup_{m \geq n} x_m$$

← limits of a subsequence of x_n

If the sequence is unbounded, the lim sup would be ∞ .

← the RHS as in real analysis

Example 16. If x_n is the sequence $0, 1, 0, 1, \dots$ then $\limsup_{n \rightarrow \infty} x_n = 1$.

Example 17. If x_n is the sequence $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots$, then $\limsup_{n \rightarrow \infty} x_n = 0$.

Remark. If x_n are nonnegative, then

- $\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$
- $\limsup_{n \rightarrow \infty} (a_n b_n) \leq (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n)$

Theorem 22 (Cauchy-Hadamard). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Define $R \in [0, \infty]$ by

← interpret $\frac{1}{0} = \infty$

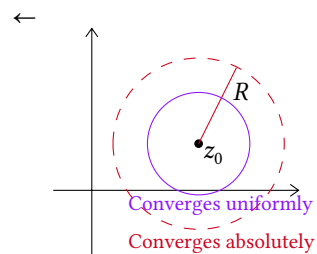
$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

Then the R is the *radius of convergence*.

- On $|z - z_0| < R$, the series converges **absolutely**. For each $r \in [0, R[$, the convergence is **uniform** on $|z - z_0| \leq r$.
- If $|z - z_0| > R$ then the series diverges. **For $|z - z_0| = R$ anything could happen!**

Example 18. We claim that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has an infinite radius of convergence $R = \infty$. To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \rightarrow 0$$



This is because $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \dots \cdot n}$, and in $n!$, there are at least $\frac{1}{2}$ terms that are $> \frac{n}{2}$.

Thus, $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{1/2} \rightarrow \infty$.

So $R = \infty$ and we are done 😊. We have that $\exp(z)$ has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$\begin{aligned} \exp(z) \exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!} \right) \\ &= \sum_{n=0}^{\infty} \underbrace{\frac{1}{n!} \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k}}_{\text{binomial theorem}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (z + w)^n \\ &= \exp(z + w) \end{aligned}$$

Now define $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Term-by-term differentiation of power series

Lemma 23. $n^{\frac{1}{n}} \rightarrow 1$

Proof 1. $e^{\log(n^{\frac{1}{n}})} = e^{\frac{\log n}{n}} \rightarrow e^0 = 1$ by l'Hopital. So $n^{\frac{1}{n}} \rightarrow 1$. □

Proof 2 (better). Write $n^{\frac{1}{n}} = 1 + \delta_n$ where $\delta_n \geq 0$. The binomial theorem says:

$$\begin{aligned} n &= (1 + \delta_n)^n \\ &= \sum_{k=0}^{\infty} \binom{n}{k} \delta_n^k \cdot 1^{n-k} \\ &= 1 + n\delta_n + \frac{n(n-1)}{2} \delta_n^2 + \dots \end{aligned}$$

$$\geq 1 + \frac{n(n-1)}{2} \delta_n^2$$

Therefore, $n-1 \geq \frac{n(n-1)}{2} \delta_n^2$ and we get $\frac{2}{n} \geq \delta_n^2 \geq 0$ hence $\delta_n \rightarrow 0$.

Hence $n^{\frac{1}{n}} \rightarrow 1$. □

Theorem 24. If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence R , then

$$f'(z) = \sum_{n=0}^{\infty} n a_n (z-z_0)^{n-1}$$

for $|z-z_0| < R$. Moreover, the new series also has a radius of convergence R .

Proof. WLOG $R > 0$ and $z_0 = 0$.

For $|z| < R$ we write:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

and the ‘new series’

$$g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1} = \lim_{N \rightarrow \infty} S'_N(z)$$

We first prove that the radius of convergence for g is the same as f . By Cauchy-Hadamard:

$$\begin{aligned} \frac{1}{R_g} &= \limsup_{n \rightarrow \infty} \sqrt[n]{n|a_n|} \\ &= \limsup_{n \rightarrow \infty} (n^{\frac{1}{n}})^n \sqrt[n]{|a_n|} && \text{by the previous lemma,} \\ &= \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} \\ &= \frac{1}{R} \end{aligned}$$

Thus, $R_g = R$ by Cauchy-Hadamard.

Next, we need to show that $f' = g$ with $|z| < R$.

Fix $0 \leq |w| < R$ and $\varepsilon > 0$. We want a $\delta > 0$ such that whenever $|z-w| < \delta$, we have $\left| \frac{f(z)-f(w)}{z-w} - g(w) \right| < \varepsilon$.

← we just translate it; also $R = 0$ isn't that meaningful

← just splitting the function into two parts

← just saying that the derivative at any w gets close to $g(w)$

We rewrite:

$$\begin{aligned} \left| \frac{f(z) - f(w)}{z - w} - g(w) \right| &= \left| \frac{[S_N(z) + R_N(z)] - [S_N(w) + R_N(w)]}{z - w} - g(w) \right| \\ &= \left| \frac{S_N(z) - S_N(w)}{z - w} + \frac{R_N(z) - R_N(w)}{z - w} + S'_N(w) - S'_N(w) - g(w) \right| \\ &\leq |S'_N(w) - g(w)| + \left| \frac{R_N(z) - R_N(w)}{z - w} \right| + \left| \frac{S_N(z) - S_N(w)}{z - w} - S'_N(w) \right| \end{aligned}$$

- **1st term:** by def of g and $g(z) = \lim_{N \rightarrow \infty} S'_N(z)$, we can always find some $N_1 \in \mathbb{N}$ such that any $N \geq N_1$ gives us $|S'_N(w) - g(w)| < \frac{\varepsilon}{3}$.
- **2nd term:** since $|w| < R$, there is an r such that $|w| < r < R$.
For $|z| < r$, we have

← work on a smaller disk

$$\begin{aligned} \left| \frac{R_N(z) - R_N(w)}{z - w} \right| &= \frac{1}{|z - w|} \left| \sum_{n=N}^{\infty} a_n z^n - \sum_{n=N}^{\infty} a_n w^n \right| \\ &\leq \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n - w^n}{z - w} \right| \\ &= \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n}{z} \cdot \frac{1 - \frac{w^n}{z^n}}{1 - \frac{w}{z}} \right| \\ &= \sum_{n=N}^{\infty} |a_n| \left| \frac{z^n}{z} \cdot \left(1 + \left(\frac{w}{z} \right) + \left(\frac{w}{z} \right)^2 + \dots + \left(\frac{w}{z} \right)^{n-1} \right) \right| \\ &= \sum_{n=N}^{\infty} |a_n| |z^{n-1} + z^{n-2}w + \dots + zw^{n-2} + w^{n-1}| \\ &\leq \sum_{n=N}^{\infty} |a_n| \cdot n \cdot r^{n-1} \text{ by } |z|, |w| < r < R \end{aligned}$$

by geometric sequence

Thus, there exists an $N_2 \in \mathbb{N}$ such that any $N \geq N_2$ gives us

$$\left| \frac{R_N(z) - R_N(w)}{z - w} \right| < \frac{\varepsilon}{3}$$

- **3rd term:** let $N = \max\{N_1, N_2\}$. The definition of $S'_N(w)$ provides $\gamma > 0$ such that if $|z - w| < \gamma$, then we have $\left| \frac{S_N(z) - S_N(w)}{z - w} - S'_N(w) \right| < \frac{\varepsilon}{3}$.

← review def of derivatives!

Now if $0 < \delta < \min\{\gamma, r - |w|\}$, then the 3 terms above are all $< \frac{\varepsilon}{3}$. Hence,

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| < \varepsilon \text{ holds for this } \delta. \quad \square$$

Corollary 25. A power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with $R > 0$ is infinitely differentiable on $|z - z_0| < R$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

are the coefficients of the terms of the power series.

← prove by keep taking derivatives!

Corollary 26. Power series expansions are unique. That is, if $r > 0$ and

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

← because there is a unique formula for coeffs.

on $|z - z_0| < r$, then $a_n = b_n$ for $n \geq 0$.

Remark. Recall that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has a radius of convergence ∞ (it's an *entire* function). Now, if we differentiate it term-by-term:

$$\begin{aligned} \frac{d}{dz} \exp(z) &= \frac{d}{dz} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^{n-1}}{(n-1)!} && \text{let } k = n - 1 \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\ &= \exp(z) \end{aligned}$$

Thus, the derivative of $\exp(z)$ is itself! Moreover, $\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Remark. We claim that $\exp(z) = e^z$.

Since $e^z e^{c-z}$ is a constant for all constant c, z , we have

$$\frac{d}{dz} (e^z e^{c-z}) = 0$$

to recover the constant $e^z e^{c-z}$, we let $z = 0$, giving us

$$e^z e^{c-z} = e^c$$

which is the addition formula!

Therefore,

$$\begin{aligned} \exp(n) &= \exp(1 + 1 + \cdots + 1) \\ &= \exp(1)^n \\ &= e^n \end{aligned}$$

Elementary functions

Now that we have derived e , we could use it to derive \sin and \cos :

Definition 17.

$$\begin{aligned}\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\end{aligned}$$

$$\begin{aligned}\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}\end{aligned}$$

We observe that we have the following property:

- Radius of convergence $R = \infty$
- $(\cos z)' = -\sin z$, $(\sin z)' = \cos z$
- $\cos x = \operatorname{Re}(e^{ix})$, $\sin x = \operatorname{Im}(e^{ix})$ for all $x \in \mathbb{R}$
- $\cos(-z) = \cos z$, $\sin(-z) = -\sin z$
- $\cosh x = \frac{e^x + e^{-x}}{2}$ so $\cosh(ix) = \cos x$
- $e^{iz} = \cos z + i \sin z$
-

$$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2} \right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i} \right)^2 \\ &= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) \\ &= 1 \quad \forall z \in \mathbb{C}\end{aligned}$$

•

$$\begin{aligned}\cos^2 z &= \left(\frac{e^{iz} + e^{-iz}}{2} \right)^2 \\ &= \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) \\ &= \frac{1}{2} + \frac{e^{2iz} + e^{-2iz}}{4} \\ &= \frac{1}{2}(1 + \cos 2z)\end{aligned}$$

- If $x \in \mathbb{R}$ then $\cos x, \sin x$ are real. We get $|\sin x|, |\cos x| \leq 1$.

Definition 18. $f : \mathbb{C} \rightarrow \mathbb{C}$ is **periodic** with a *period* ω if $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}$.

Theorem 27. There exists a positive real number π such that:

- (a) $\cos z, \sin z$ have period 2π
- (b) e^z is periodic with period $2\pi i$
- (c) π is the area of the unit circle

Proof. By Euler's formula, it suffices to consider e^{iz} only. If ω is a period of e^{iz} , then

$$e^{iz} = e^{i(z+\omega)} = e^{iz} e^{i\omega}$$

which only happens if $e^{i\omega} = 1$. Conversely, if $e^{i\omega} = 1$, then $e^{i(z+\omega)} = e^{iz}$.

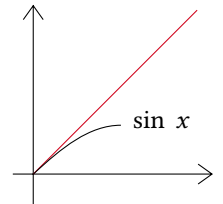
Hence, ω is a period of e^{iz} if and only if $e^{i\omega} = 1$. □

Proposition 28. $\sin x \leq x$ for all $x \geq 0$.

Proof. Since $|\cos t| \leq 1$,

$$\begin{aligned} x - \sin x &= (x - \sin x) - (0 - \sin 0) \\ &= \int_0^x \underbrace{1 - \cos t}_{\geq 0} dt \quad \text{by FTC} \\ &\geq 0 \end{aligned}$$

← This is the first term in the power series



□

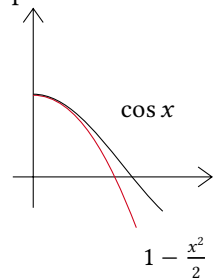
Proposition 29. In addition, $\cos x \geq 1 - \frac{x^2}{2}$ for $x \geq 0$.

Proof. The previous prop gives:

$$\begin{aligned} \cos x - 1 &= \cos x - \cos 0 \\ &= \int_0^x -\sin t dt \\ &\geq \int_0^x -t dt \\ &= \frac{-x^2}{2} \end{aligned}$$

□

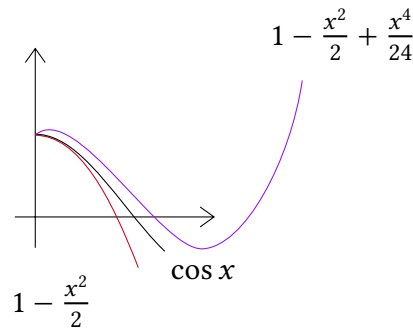
← These are the first 2 terms in the power series



Proposition 30. Furthermore, for $x \geq 0$:

- $\sin x \geq x^3 - \frac{x^3}{6}$
- $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

Proposition 31. There exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$.



Proof. By the previous prop, we have $\cos \sqrt{3} \leq 1 - \frac{\sqrt{3}^2}{2} + \frac{\sqrt{3}^4}{24} = \frac{1}{8} < 0$. Moreover, $\cos 0 = 1 > 0$, by IVT, there exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$. \square

Proposition 32. $\omega_0 = 4x_0$ is a period of e^{iz} .

Proof. Since $\cos x_0 = 0$, we have $\sin x_0 = \pm 1$. Then $e^{ix_0} = \pm i$. We have $(\pm i)^4 = 1$, so $e^{4ix_0} = 1 = e^0$, so $\omega_0 = 4x_0$ is a period of e^{iz} . \square

Proposition 33. ω_0 is the *smallest* positive period of e^{iz} .

Proposition 34. All periods of e^{iz} are integer multiples of $2\pi = 4x_0$.

Proof. Define $\pi = 2x_0$. The area of unit circle is

$$\begin{aligned} 4 \int_0^1 \sqrt{1-x^2} dx &= 4 \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 \theta} d\theta \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 + \cos 2\theta) d\theta \\ &= \pi \end{aligned}$$

\square

Complex logarithm

We know: $e^0 = 1, e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718 \dots$

Since $\frac{d}{dx} e^x = e^x$, it is positive. If $x > 0$, we conclude that e^x is strictly increasing!
As $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1 + x$, so $\lim_{x \rightarrow \infty} e^x = \infty$,

Therefore, e^x is a **bijection** from \mathbb{R} to $(0, \infty)$. This means it has an inverse that is a bijection from $(0, \infty)$ to \mathbb{R} .

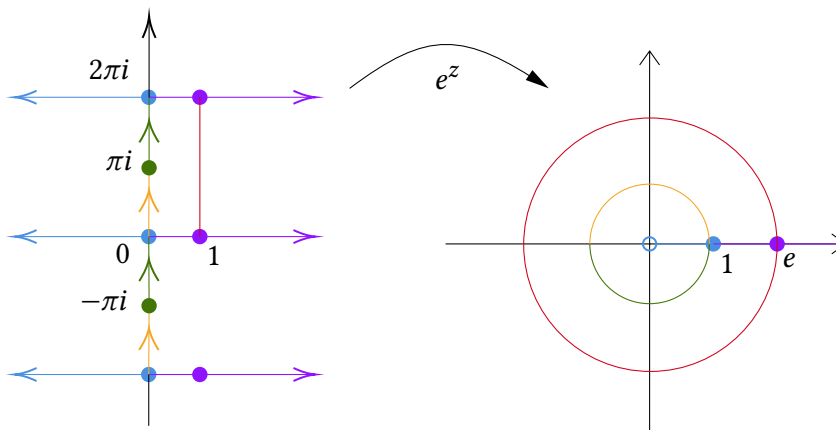
Definition 19. $\ln x$ is the inverse of e^x for $x \in (0, +\infty)$.

Now what about the complex case? Let $z \neq 0$ and $z = re^{i\theta}$ where $r = |z| > 0$ and $\theta = \arg z \in \mathbb{R}$.

Hence, $z = re^{i\theta} = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta}$. However, the θ is ambiguous to addition of multiples of 2π !

Definition 20. If $z \neq 0$, a **logarithm** of z is a $w \in \mathbb{C}$ such that $e^w = z$.

We could graph the function e^z with $z \in \mathbb{C}$:



Definition 21. If Ω is a region in \mathbb{C} , then a continuous $l : \Omega \rightarrow \mathbb{C}$ is a **branch** of the logarithm if $e^{l(z)} = z$ for all $z \in \Omega$.

Example 19. If $\Omega = \mathbb{C} \setminus (-\infty, 0]$ such that $\theta \in (-\pi, \pi)$, a logarithm could be defined on it. This is the **principal branch** of the logarithm.

Remark. Suppose $l(z)$ is a branch of the logarithm and l is analytic, then:

$$e^{l(z)} = z \implies \frac{d}{dz} e^{l(z)} = l'(z) e^{l(z)} = 1$$

Since $e^{l(z)} = z$, we conclude $l'(z) = \frac{1}{z}$.

← cf. trig properties

← Only determined up to addition of multiples of 2π

← note $0 \notin \Omega$

← See graphed Riemann surface

Complex power

Definition 22. If $z \neq 0$, define $z^a = e^{a \log z}$.

← NOT well-defined!

Remark. The definition of complex powers should coincide with the old one:

$$z^n = \underbrace{z \cdot z \cdot \dots \cdot z}_n = r^n e^{in\theta}.$$

Check:

$$\begin{aligned} z^n &= e^{n \log z} = e^{n(\ln r + i\theta + i2\pi k)} \\ &= e^{n \ln r} e^{in\theta} \underbrace{e^{i2\pi nk}}_{=1} \\ &= r^n e^{in\theta} \end{aligned}$$

is true for any $k \in \mathbb{Z}$.

How about n -th roots?

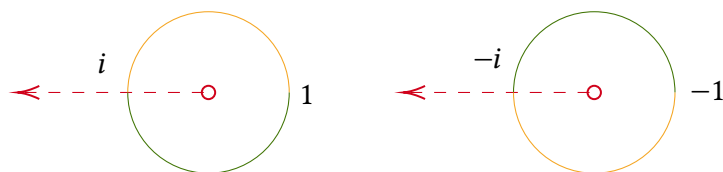
$$\begin{aligned} z^{\frac{1}{n}} &= e^{\frac{1}{n} \log z} \\ &= e^{\frac{1}{n}(\ln r + i\theta + i2\pi k)} \\ &= e^{\frac{1}{n} \ln r} e^{\frac{i\theta}{n}} \underbrace{e^{\frac{i2\pi k}{n}}}_{n \text{ distinct}} \\ &= r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \end{aligned}$$

Riemann surface

We still have a problem: $\ln z$ is still not a function on \mathbb{C} ! The branch depends on the arbitrary choice of domain. What shall we do to make it not dependent on a choice?

Answer: let \ln not live on the complex plane, but infinitely many copies of the slit plane $\mathbb{C} \setminus (-\infty, 0]$, each one being glued to the next along the slit $(-\infty, 0]$.

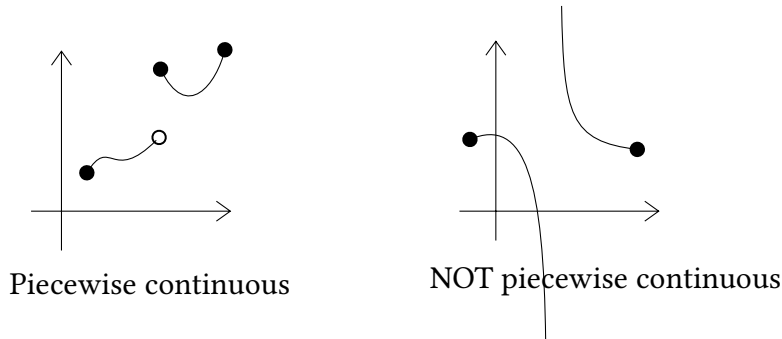
Example 20. $z^{1/2}$ would live on a surface:



Cauchy's theorem and its consequences

Complex integration

Definition 23. A complex-valued function $\gamma : [a, b] \rightarrow \mathbb{C}$ is **piecewise continuous** if γ is continuous at all but *finitely many* points of $[a, b]$ and γ has one-sided limits that are *finite* at each point (of discontinuity).



If γ is piecewise continuous, then $\int_a^b \operatorname{Re} \gamma(t) dt$ and $\int_a^b \operatorname{Im} \gamma(t) dt$ exist. Then we define **complex integration**:

$$\int_a^b \gamma(t) dt = \int_a^b \operatorname{Re} \gamma(t) dt + i \cdot \int_a^b \operatorname{Im} \gamma(t) dt$$

That is,

$$\begin{aligned} \operatorname{Re} \left(\int_a^b \gamma(t) dt \right) &= \int_a^b \operatorname{Re} \gamma(t) dt \\ \operatorname{Im} \left(\int_a^b \gamma(t) dt \right) &= \int_a^b \operatorname{Im} \gamma(t) dt \end{aligned}$$

In addition, if γ_1, γ_2 are both $[a, b] \rightarrow \mathbb{C}$ and piecewise cont., and $c_1, c_2 \in \mathbb{C}$, then

$$\int_a^b (c_1 \gamma_1(t) + c_2 \gamma_2(t)) dt = c_1 \int_a^b \gamma_1(t) dt + c_2 \int_a^b \gamma_2(t) dt$$

Proposition 35 (Triangle inequality). If $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise continuous, then

$$\left| \int_a^b \gamma(t) dt \right| \leq \int_a^b |\gamma(t)| dt$$

Proof. WLOG assume $\int_a^b \gamma(t) dt \neq 0$. Define $\lambda = \frac{\left| \int_a^b \gamma(t) dt \right|}{\int_a^b \gamma(t) dt}$ and note $|\lambda| = 1$.

Thus,

$$\begin{aligned} \left| \int_a^b \gamma(t) dt \right| &= \lambda \int_a^b \gamma(t) dt \\ &= \int_a^b \lambda \gamma(t) dt && \text{because LHS is } \in \mathbb{R} \\ &= \operatorname{Re} \int_a^b \lambda \gamma(t) dt \\ &\leq \int_a^b |\lambda \gamma(t)| dt && \because \operatorname{Re} z \leq |z| \\ &= \int_a^b |\gamma(t)| dt && \because |\lambda| = 1 \end{aligned}$$

□

Complex differentiability

Definition 24. $\gamma : [a, b] \rightarrow \mathbb{C}$ is **differentiable** at $t \in [a, b]$ if $\operatorname{Re} \gamma$ and $\operatorname{Im} \gamma$ are differentiable (in the sense of real variables). We define

$$\gamma'(t) = (\operatorname{Re} \gamma)'(t) + i \cdot (\operatorname{Im} \gamma)'(t)$$

Definition 25. $\gamma : [a, b] \rightarrow \mathbb{C}$ is **piecewise C^1** if:

← C^1 is one-time differentiable

- (a) γ is continuous on $[a, b]$.
- (b) γ is differentiable at all but finitely many points of $[a, b]$.
- (c) γ' is continuous at each point where it exists.
- (d) γ' has finite one-sided limits at every point of discontinuity.

Fundamental theorem of calculus, complex edition

If $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise C^1 , then:

$$\int_a^b \gamma'(t) dt = \gamma(b) - \gamma(a)$$

Definition 26. If γ is C^1 , then the arclength of γ is:

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

Definition 27. If $\gamma : [a, b] \rightarrow \Omega$ is piecewise C^1 and $f : \Omega \rightarrow \mathbb{C}$ is continuous, then

$$\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$$

where $z = \gamma(t)$ and $dz = \gamma'(t) dt$

We have **linearity** w.r.t. f :

$$\int_{\gamma} (c_1 f_1(z) + c_2 f_2(z)) dz = c_1 \int_{\gamma} f_1(z) dz + c_2 \int_{\gamma} f_2(z) dz$$

Remark. Arclength is independent from parameterization.

Proof. Let $\gamma : [a, b] \rightarrow \Omega$ be piecewise C^1 . Let $\alpha : [c, d] \rightarrow [a, b]$ is an increasing, piecewise C^1 surjection such that $\alpha(c) = a, \alpha(d) = b$. Then $\phi = \gamma \circ \alpha : [c, d] \rightarrow \Omega$ is also piecewise C^1 . Hence, by substituting $s = \alpha(t), ds = \alpha'(t) dt$:

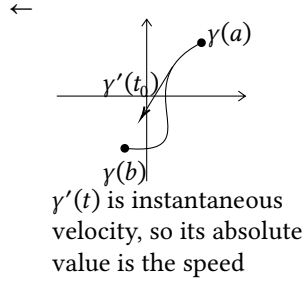
$$\begin{aligned} \int_{\phi} f(z) dz &= \int_c^d f(\phi(t)) \phi'(t) dt \\ &= \int_c^d f(\gamma(\alpha(t))) \gamma'(\alpha(t)) \alpha'(t) dt \\ &= \int_a^b f(\gamma(s)) \gamma'(s) ds \\ &= \int_{\gamma} f(z) dz \end{aligned}$$

□

An important estimate

Let f be continuous. Since $\int_{\gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt$, we observe:

$$\begin{aligned} \left| \int_{\gamma} f(z) dz \right| &= \left| \int_a^b f(\gamma(t)) \gamma'(t) dt \right| \\ &\leq \int_a^b |f(\gamma(t))| |\gamma'(t)| dt \end{aligned}$$



$$\begin{aligned} &\leq \max_{t \in [a, b]} |f(\gamma(t))| \int_a^b |\gamma'(t)| dt \\ &= \max_{z \in \gamma} |f(z)| \cdot L(\gamma) \end{aligned}$$

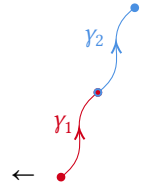
Definition 28. If $\gamma : [a, b] \rightarrow \mathbb{C}$, the reverse of γ is $(-\gamma) : [-b, -a] \rightarrow \mathbb{C}$ defined by $(-\gamma)(t) = \gamma(-t)$. Hence,

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

Remark. We can also break up the curve and integral the two parts separately:

$$\int_{\gamma} f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz$$

← going around the track backwards



Fundamental theorem of calculus for contour integrals

If $\gamma : [a, b] \rightarrow \mathbb{C}$ is piecewise C^1 , and $f : \Omega \rightarrow \mathbb{C}$ is analytic, then

$$\int_{\gamma} f'(z) dz = f(\gamma(b)) - f(\gamma(a))$$

← Assuming f' continuous, which we would prove later

If $\gamma(a) = \gamma(b)$, then $\int_{\gamma} f'(z) dz = 0$.

Proof.

$$\begin{aligned} \int_{\gamma} f'(z) dz &= \int_a^b f'(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b (f \circ \gamma)'(t) dt && \text{chain rule} \\ &= f(\gamma(b)) - f(\gamma(a)) \end{aligned}$$

□

Example 21. Let γ be a circle of radius R centered at z_0 : $\gamma(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$. We would like to find $\int_{\gamma} (z - z_0)^n dz$.

If $n \neq -1$, then $\left(\frac{(z - z_0)^{n+1}}{n+1} \right)' = (z - z_0)^n$. Thus,

$$\int_{\gamma} (z - z_0)^n dz = \int_{\gamma} \left(\frac{(z - z_0)^{n+1}}{n+1} \right)' dz = 0$$

by FTC.

If $n = -1$,

$$\int_{\gamma} (z - z_0)^n dz = \int_{\gamma} \frac{1}{z - z_0} dz = \int_0^{2\pi} i dt = 2\pi i$$

Cauchy's theorem

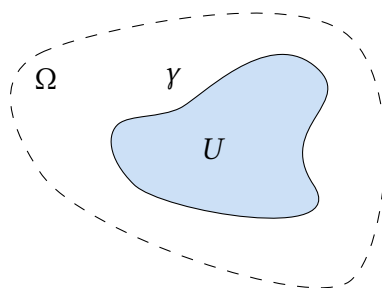
Take 1

Theorem 36 (Cauchy's). Let Ω be a region in \mathbb{C} containing a *simple* piecewise C^1 closed curve γ and its interior.

← does not self-intersect

If $f : \Omega \rightarrow \mathbb{C}$ is analytic, then $\int_{\gamma} f(z) dz = 0$.

← holes not allowed in the interior



“Proof”. Let U be the union of γ and its interior. Let $f = u + iv$ as usual, write $dz = dx + i dy$:

$$\begin{aligned} \int_{\gamma} f(z) dz &= \int_{\gamma} (u + iv)(dx + i dy) \\ &= \int_{\gamma} u dx - v dy + i \int_{\gamma} v dx + u dy \\ &= \int \int_U (-v_x - u_y) dx dy + i \int \int_U (u_x - v_y) dx dy \quad \text{by Green's thm} \\ &= 0 \quad \text{by Cauchy-Riemann} \end{aligned}$$

□

However, this ‘proof’ heavily relies on the fact that u, v are C^1 and that the partial derivatives are continuous. This assumes f' is continuous, but we aren't sure about that yet!

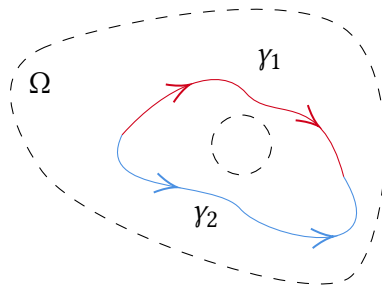
Take 2: deformation version

Theorem 37 (Cauchy's). Let γ_1, γ_2 be piecewise C^1 curves in a region Ω with the same start and end points. If γ_1 can be continuously deformed to γ_2 without ever passing outside of Ω , then

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz$$

By the *previous* statement of Cauchy's theorem (in Theorem 36), we observe that $\int_{\gamma_1 - \gamma_2} f(z) dz = 0$, so this one falls out.

Non-example 22. The γ_1, γ_2 in the picture below cannot be continuously deformed into each other!



Fresnel integrals

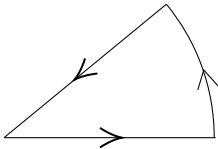
Consider:

$$\int_0^\infty \sin(t^2) dt \quad \text{and} \quad \int_0^\infty \cos(t^2) dt$$

aka.

$$\lim_{R \rightarrow \infty} \int_0^R \sin(t^2) dt \quad \text{and} \quad \lim_{R \rightarrow \infty} \int_0^R \cos(t^2) dt$$

It's not obvious that these integrals converge!

Solution: PIZZA!  Let γ be the 'sum' of all 3 curves as shown. Let $R \rightarrow \infty$. Then, by Cauchy's theorem, $\int_\gamma e^{iz^2} dz = 0$.

(Scratch work begins)

Remark. We don't know how to write out the antiderivative of $f(z) = e^{iz^2}$ but we can use series!

$$\begin{aligned} f(z) &= e^{iz^2} \\ &= \sum_{n=0}^{\infty} \frac{(iz^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n z^{2n}}{n!} \end{aligned}$$

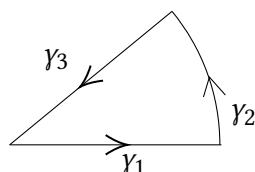
And so

$$F(z) = \sum_{n=0}^{\infty} \frac{i^n z^{2n+1}}{(2n+1)n!}$$

(Scratch ends here)

Now we return to the integral. Strategy:

$$0 = \int_{\gamma} e^{iz^2} dz = \underbrace{\int_{\gamma_1} e^{iz^2} dz}_{I_1(R)} + \underbrace{\int_{\gamma_2} e^{iz^2} dz}_{I_2(R)} + \underbrace{\int_{\gamma_3} e^{iz^2} dz}_{I_3(R)}$$



Evaluate $I_1(R)$: We observe that z is real for this one. Parameterize $z = t$ where t is a real variable.

$$\begin{aligned} I_1(R) &= \int_{\gamma_1} e^{it^2} dt \\ &= \int_0^R \cos(t^2) dt + i \cdot \int_0^R \sin(t^2) dt \end{aligned}$$

Hence, $\lim_{R \rightarrow \infty} I_1(R) = \int_0^{\infty} \cos(t^2) dt + i \cdot \int_0^{\infty} \sin(t^2) dt$.

Evaluate $I_2(R)$:

Parameterize γ_2 as $z = Re^{i\theta}$ where $\theta \in [0, \frac{\pi}{4}]$. Hence, $dz = iRe^{i\theta} d\theta$. Then:

$$\begin{aligned}
 |I_2(R)| &= \left| \int_{\gamma_2} e^{i\theta^2} dz \right| \\
 &= \left| \int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} iRe^{i\theta} d\theta \right| \\
 &= \left| R \int_0^{\frac{\pi}{4}} e^{iR^2 e^{i2\theta}} e^{i\theta} d\theta \right| \\
 &\leq R \int_0^{\frac{\pi}{4}} |e^{iR^2 e^{i2\theta}}| d\theta && \text{by tri. ineq.} \\
 &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \sin 2\theta} d\theta && \text{since when } x, y \in \mathbb{R}, |e^{x+iy}| = e^x \\
 &\leq R \int_0^{\frac{\pi}{4}} e^{-R^2 \frac{4\theta}{\pi}} d\theta && \text{since when } x \in [0, \frac{\pi}{2}], \frac{2}{\pi}x \leq \sin x \\
 &= \frac{-R\pi}{R^2 4} e^{-R \frac{4\theta}{\pi}} \Big|_{\theta=0}^{\theta=\frac{\pi}{4}} \\
 &\rightarrow 0 \text{ as } R \rightarrow \infty
 \end{aligned}$$

Thus, $\lim_{R \rightarrow \infty} I_2(R) = 0$. :)

Evaluate $I_3(R)$:

$$\begin{aligned}
 I_3(R) &= \int_{\gamma_3} e^{iz^2} dz \\
 &= \int_R^0 e^{i(e^{i\frac{\pi}{4}}t)^2} e^{i\frac{\pi}{4}} dt \\
 &= -e^{i\frac{\pi}{4}} \int_0^R e^{-t^2} dt \\
 \lim_{R \rightarrow \infty} I_3(R) &= -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \int_0^\infty e^{-t^2} dt \quad \text{by Gaussian integral, } \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \\
 &= -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}
 \end{aligned}$$

Therefore, we see $I_1(R) + I_2(R) + I_3(R) = 0$ where $\lim_{R \rightarrow \infty} I_1(R) = \int_0^\infty \cos(t^2) dt + i \cdot \int_0^\infty \sin(t^2) dt$, $I_2(R) \rightarrow 0$ and $I_3(R) = -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}$. Hence, we would be able to conclude that

$$\int_0^\infty \sin(t^2) dt = \sqrt{\frac{\pi}{8}} \quad \text{and} \quad \int_0^\infty \cos(t^2) dt = \sqrt{\frac{\pi}{8}}$$