MATH135 Complex Analysis Notes

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Contents

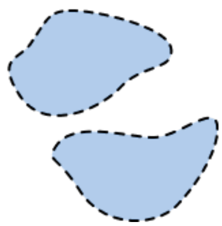
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of $\mathbb{C}$.

- A region without "holes" is simply connected.

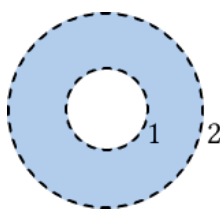
Non-example 1. This is not a region (not connected):



Example 2. $\mathbb{C}$ is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

← this $z \to z_0$ could be from **any** directions!

2. If f is differentiable at every point in Ω, we say f is **analytic** on Ω.

← Means that existence of 1st derivative implies the existence of ∞th derivative! & has Taylor expansion.

3. If f is analytic on $\mathbb{C}$, then f is **entire**.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on $\mathbb{C}$ except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \to z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z\to z_0} = \frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{z-z_0}{z-z_0} = 1.$$

Else if $z \to z_0$ vertically, then $\overline{z-z_0} = -(z-z_0)$, meaning that

$$\lim_{z\to z_0} = \frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{-(z-z_0)}{z-z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. □

Proposition 1. Let f be differentiable at z_0. Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0)\cdot(z-z_0) - f(z) + f(z_0)| &< \varepsilon|z-z_0| \\ |f(z_0) + f'(z_0)(z-z_0) - f(z)| &< \varepsilon|z-z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z-z_0)$$

this is the "tangent-line approximation" equivalent in $\mathbb{C}$!

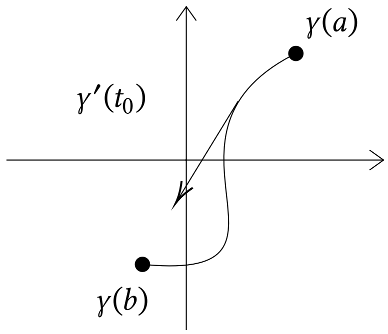
← The RHS is a **linear** function!

In addition, $f(z_0) + f'(z_0)(z-z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is <u>locally orientation-preserving</u> and could be approximated by a linear function.

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

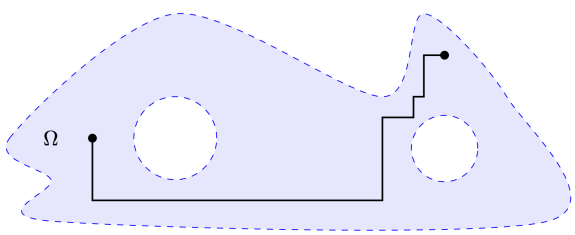
Curves, paths

Definition 3. A **curve** in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}, a, b \in \mathbb{R}$.



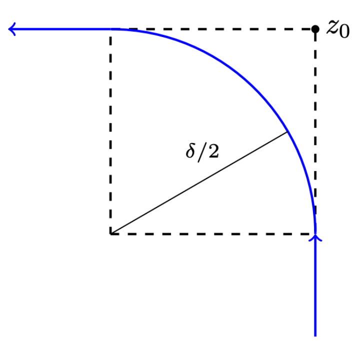
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The "Boxy-path" Theorem)**.** A nonempty open set Ω in $\mathbb{C}$ is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω, each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a "**boxy path**".

Remark. There is also always a **smooth path**. That is:

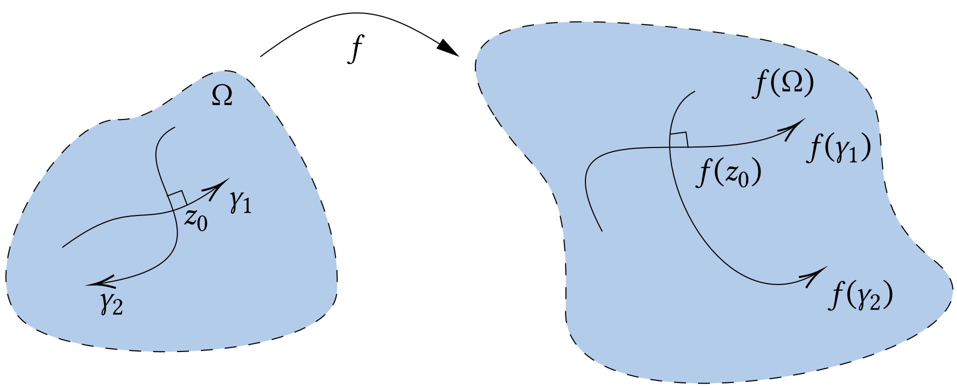


Theorem 3 (“Smooth-path”)**.** A nonempty open set Ω in $\mathbb{C}$ is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See lecture 2 notes. □

Conformality

Let f be an analytic complex function on Ω.



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ. Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(\zeta_0)$ also with angle θ.

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall: $f'(z_0) = \lim\limits_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$.

Now we write each function with complex variables as $f(z) = u(z) + i\,v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i\,v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r\to 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r\to 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \to 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations)**.**

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \to \mathbb{C}$ is analytic and $f' = 0$ on Ω, then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω. By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω. Hence, $\mathbf{u}, \mathbf{v}$ are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω. $\square$

Orientation-preserving as shown by Jacobian

Let $f : \Omega \to \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 = \bar{f}' \cdot f &= (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right) && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0, implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales $\mathbb{R}$ by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

← $\Delta u = 0$ characterizes steady-state solutions to heat equations on Ω.

Definition 5. Real-valued functions $u : \Omega \to \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω.

Example 8. $u = x^2 - y^2, v = 2xy$.

← Check it!

Remark. Harmonic conjugates are unique up to translation (± constants).

Remark. If u is harmonic on Ω, it does NOT have to have a harmonic conjugate on Ω.

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log |z|$ is harmonic on $\mathbb{C}\backslash\{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2} \log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2+y^2)\right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2+y^2} \\ &= \frac{x}{x^2+y^2} \end{aligned}$$

Hence,

← Review quotient rule!

$$
\begin{aligned}
u_{xx} &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2}
\end{aligned}
$$

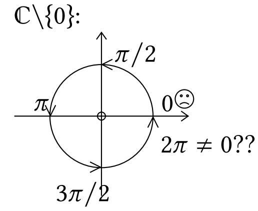
Symmetrically, we find

$$
u_{yy} = \frac{x^2 - y^2}{(x^2+y^2)^2}
$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. □

Now, can we find a harmonic conjugate for the aforementioned u?

← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C}\setminus\{0\}$:



To be resolved later!

We could use the two Cauchy-Riemann Equations. One of them:

$$
\begin{aligned}
v_y &= u_x \\
&= \frac{x}{x^2+y^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v(x,y) &= \int v_y dy + C(x) \qquad \text{unknown function of } x \\
&= \arctan\left(\frac{y}{x}\right) + C(x)
\end{aligned}
$$

Then, we use the second one:

$$
\begin{aligned}
\frac{y}{x^2+y^2} = u_y = -v_x &= -\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right) \\
&= \frac{y}{x^2+y^2} - C'(x) \implies C'(x) = 0
\end{aligned}
$$

Hence, a good harmonic conjugate candidate seems to be

$$
v(x,y) = \arctan\left(\frac{y}{x}\right) + C
$$

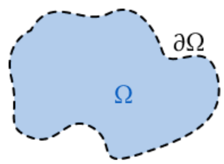
where C is a constant. WLOG, let $C = 0$. Then $v(x,y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$
v(z) = \arg(z)
$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in $\mathbb{C}$. Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \to \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω.

1. U cannot have a *local* maximum in Ω.
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$.
3. U cannot be locally constant without being globally constant.

← Ω^- denotes the closure of Ω

Theorem 8 (Maximum principle)**.** Let Ω be a bounded region in $\mathbb{C}$ and let $f : \Omega^- \to \mathbb{C}$ be analytic on Ω and continuous on Ω^-.

1. If $|f|$ achieves a local max in Ω, then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.