MATH135 Complex Analysis Notes

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Contents

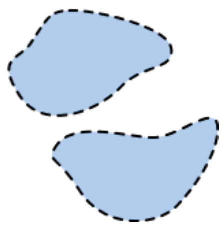
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of $\mathbb{C}$.

- A region without "holes" is simply connected.

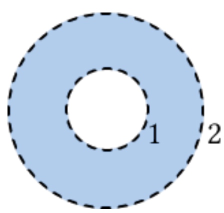
Non-example 1. This is not a region (not connected):



Example 2. $\mathbb{C}$ is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

← this $z \to z_0$ could be from **any** directions!

2. If f is differentiable at every point in Ω, we say f is **analytic** on Ω.

← Means that existence of 1st derivative implies the existence of ∞th derivative! & has Taylor expansion.

3. If f is analytic on $\mathbb{C}$, then f is **entire**.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on $\mathbb{C}$ except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \to z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z \to z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if $z \to z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z \to z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. □

Proposition 1. Let f be differentiable at z_0. Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the "tangent-line approximation" equivalent in $\mathbb{C}$!

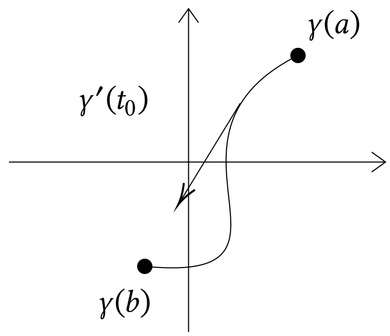
← The RHS is a **linear** function!

In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is locally orientation-preserving and could be approximated by a linear function.

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

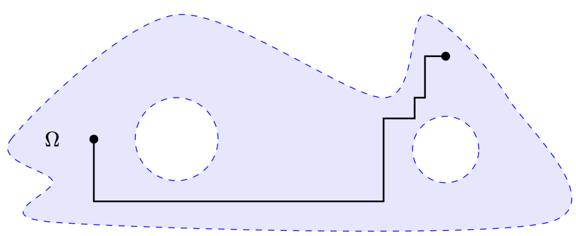
Curves, paths

Definition 3. A **curve** in $\mathbb{C}$ is a function $\gamma : [a,b] \to \mathbb{C}, a, b \in \mathbb{R}$.



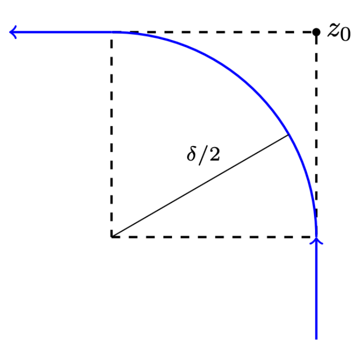
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The "Boxy-path" Theorem)**.** A nonempty open set Ω in $\mathbb{C}$ is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω, each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a "**boxy path**".

Remark. There is also always a **smooth path**. That is:

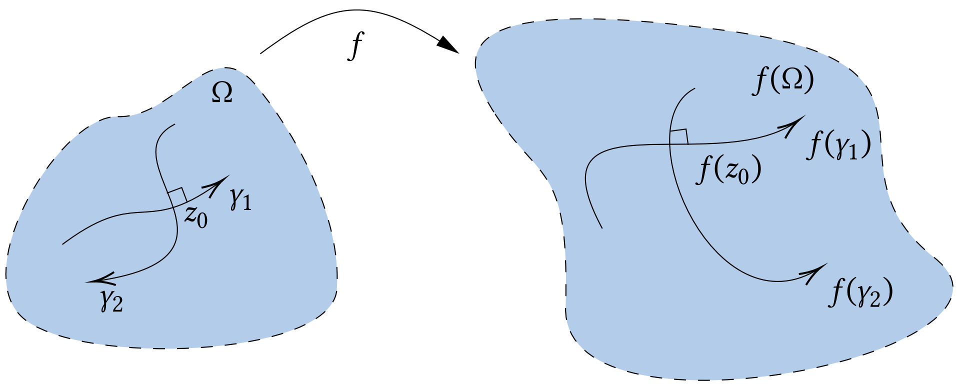


Theorem 3 ("Smooth-path"). A nonempty open set Ω in $\mathbb{C}$ is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See lecture 2 notes. $\square$

Conformality

Let f be an analytic complex function on Ω.



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ. Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(\zeta_0)$ also with angle θ.

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall: $f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$.

Now we write each function with complex variables as $f(z) = u(z) + i\,v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i\,v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r\to 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r\to 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \to 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations)**.**

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \to \mathbb{C}$ is analytic and $f' = 0$ on Ω, then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω. By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω. Hence, $\mathbf{u}, \mathbf{v}$ are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω. $\square$

Orientation-preserving as shown by Jacobian

Let $f : \Omega \to \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 = \bar{f}' \cdot f &= (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right) && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0, implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales $\mathbb{R}$ by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

← $\Delta u = 0$ characterizes steady-state solutions to heat equations on Ω.

Definition 5. Real-valued functions $u : \Omega \to \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω.

Example 8. $u = x^2 - y^2, v = 2xy$.

← Check it!

Remark. Harmonic conjugates are unique up to translation (± constants).

Remark. If u is harmonic on Ω, it does NOT have to have a harmonic conjugate on Ω.

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2}\log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2 + y^2)\right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

← Review quotient rule!

$$
\begin{aligned}
u_{xx} &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2}
\end{aligned}
$$

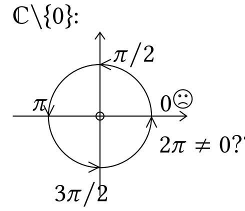
Symmetrically, we find

$$
u_{yy} = \frac{x^2 - y^2}{(x^2+y^2)^2}
$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. □

Now, can we find a harmonic conjugate for the aforementioned u?

← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C}\backslash\{0\}$:



To be resolved later!

We could use the two Cauchy-Riemann Equations. One of them:

$$
\begin{aligned}
v_y &= u_x \\
&= \frac{x}{x^2+y^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v(x,y) &= \int v_y dy + C(x) \qquad \text{unknown function of } x \\
&= \arctan\left(\frac{y}{x}\right) + C(x)
\end{aligned}
$$

Then, we use the second one:

$$
\begin{aligned}
\frac{y}{x^2+y^2} = u_y = -v_x &= -\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right) \\
&= \frac{y}{x^2+y^2} - C'(x) \implies C'(x) = 0
\end{aligned}
$$

Hence, a good harmonic conjugate candidate seems to be

$$
v(x,y) = \arctan\left(\frac{y}{x}\right) + C
$$

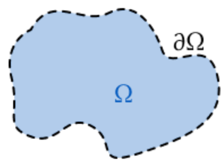
where C is a constant. WLOG, let $C = 0$. Then $v(x,y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$
v(z) = \arg(z)
$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

Physics analogies of harmonic functions

Example 10. Let $T(x,y,t)$ be the temperature at (x,y) at time t of a thermally conductive plate in $\mathbb{C}$. Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \to \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x,y)$ be a harmonic function on Ω.

1. U cannot have a *local* maximum in Ω.
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$.
3. U cannot be locally constant without being globally constant.

← Ω^- denotes the closure of Ω

Theorem 8 (Maximum principle)**.** Let Ω be a bounded region in $\mathbb{C}$ and let $f : \Omega^- \to \mathbb{C}$ be analytic on Ω and continuous on Ω^-.

1. If $|f|$ achieves a local max in Ω, then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations)**.**

$$f(z) = \frac{az+b}{cz+d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C}\backslash\{\frac{-d}{c}\}$ and **comformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C}\backslash\{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$\begin{aligned} f(z) = f(w) \implies \frac{az+b}{cz+d} &= \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

Definition 8 (The extended plane)**.** We set the following convention:

$$\begin{aligned} f(\frac{-d}{c}) &= \infty \\ f(\infty) &= \frac{a}{c} \end{aligned}$$

with this, f is a **bijection** from $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can *associate* $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

Remark. $M_{f\circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

Theorem 9. A Möbius transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with three fixed points in $\widehat{\mathbb{C}}$ is the **identity map** $\mathrm{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transformation.
   ← think about that!
   (a) If $f(z) = z$, we are done since we get the identity!
   (b) Otherwise the function only has one fixed point at ∞.
2. If ∞ is not a fixed point, then $c \neq 0$. Solve:

$$\begin{aligned} f(z) + z \Leftrightarrow \frac{az+b}{cz+d} &= z \\ az + b &= cz^2 + dz \\ cz^2 + (d-a)z - b &= 0 \end{aligned}$$

   is a quadratic which has at most two (distinct) solutions in $\mathbb{C}$. Hence, this transformation fixes at most two points.

□

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so we are done.
   ← since linear transformations preserve circles and lines
2. Now suppose $c \neq 0$. Then

$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c} \end{aligned}$$

   which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c}$$

   We now only need to show that inversion preserves circles.

Let a circle in $\mathbb{R}^2$ be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.
Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle! ← check this!

□

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

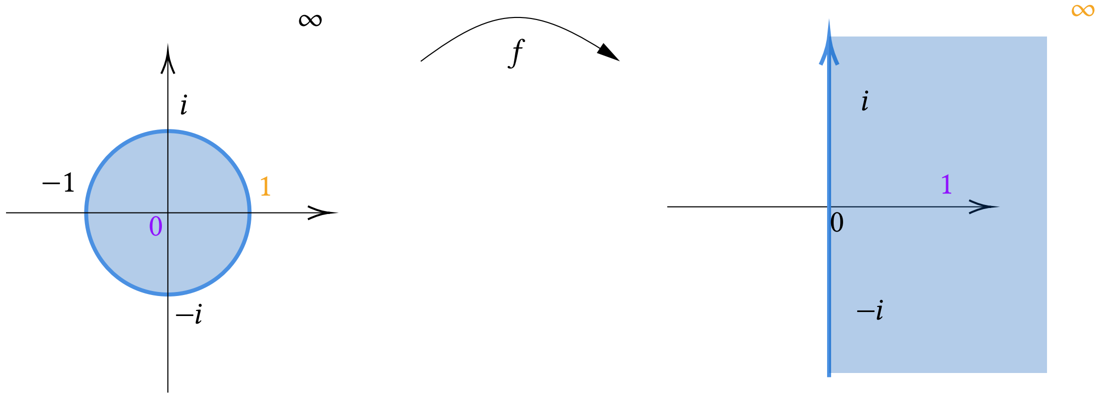
We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

$$\begin{aligned} z_1 &\xmapsto{\phi} 0 \xmapsto{\psi^{-1}} w_1 \\ z_2 &\xmapsto{\phi} 1 \xmapsto{\psi^{-1}} w_2 \\ z_3 &\xmapsto{\phi} \infty \xmapsto{\psi^{-1}} w_3 \end{aligned}$$

and we could simply let $f = \psi^{-1} \circ \phi$. □

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$ where $S_N = a_1 + \cdots + a_N$.

← S_N is the N-th partial sum.

Divergence test

Definition 10 (Divergence test)**.** A pair of contrapositives:

← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\lim_{n\to\infty} a_n \neq 0$ (*including the case where the limit doesn't exist*) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \ldots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞.

← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = \lim_{N\to\infty} S - S_N = 0$.

← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion)**.** $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left| \sum_{n=j-1}^{k} a_n \right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test)**.** Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\to \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.

← do an improper integral!

Moreover, $\int_1^N f(x)\,\mathrm{d}x \le a_1 + \cdots + a_N \le a_1 + \int_1^N f(x)\,\mathrm{d}x$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then ← $a_n = \frac{1}{n}$

$$\ln N \le 1 + \frac{1}{2} + \cdots + \frac{1}{N} \le 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

Theorem 14. The "p-series" $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1$$

Remark. Euler figured out:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979): ← still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202\ldots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. ← Good
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. ← BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series ← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges to $\ln 2$. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection, then

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

Theorem 17 (Riemann Rearrangement Theorem)**.** If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\begin{aligned}\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See notes.

Uniform convergence

Definition 14. A sequence of functions $f_n : X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \to \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!

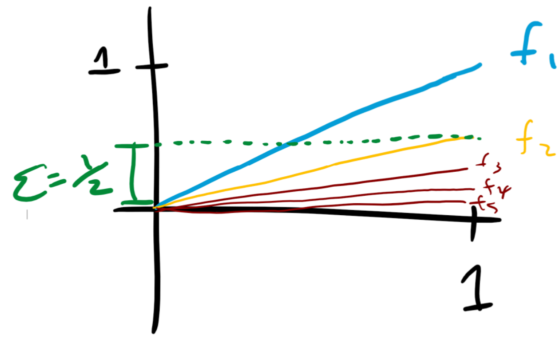


FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \to \mathbb{C}$ are continuous and converges uniformly on X to $f : X \to \mathbb{C}$, then f is continuous on X. In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.

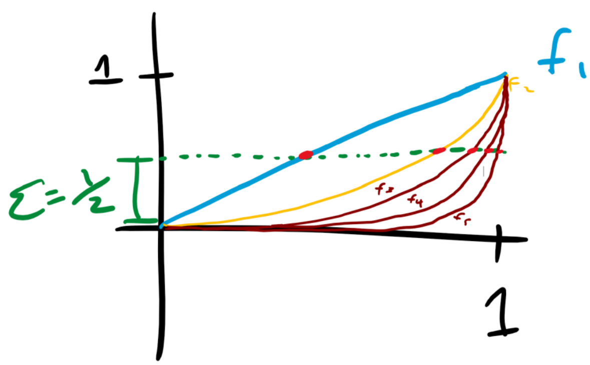


FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a,b] \to \mathbb{C}$ are continuous and converge uniformly on $[a,b]$ to f, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

← Integrals work with uniform convergence

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1,1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test)**.** Let $f_n : X \to \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X.

Power series

Definition 15. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. The a_n is the *coefficient* and z_0 is the *center*.

Convergence of geometric series

Theorem 21. The *geometric series* ($a_n = 1, z_0 = 0$) $\sum_{n=0}^{\infty} z^n$ converges absolutely to $\frac{1}{1-z}$ if $|z| < 1$, and it diverges otherwise.

Moreover, for each $r \in [0,1[$, the convergence is **uniform** on $|z| \le r$.

Proof. If $|z| \ge 1$, then $z^n \not\to 0$, so by the test of divergence, the series diverges.

Now suppose $|z| < 1$. Then

← The fact that we can find a formula for this sum is quite rare!

$$\begin{aligned}\sum_{n=0}^{\infty} z^n &= \lim_{N\to\infty} \sum_{n=0}^{N-1} z^n \\ &= \lim_{N\to\infty} (1 + z + z^2 + \cdots + z^{N-1}) \\ &= \lim_{N\to\infty} \frac{1-z^N}{1-z} \\ &= \frac{1}{1-z} && \text{since } |z| < 1\end{aligned}$$

Which gives us point-wise convergence. Then, for any r such that $|z| \le r < 1$, we have

$$\sum_{n=0}^{\infty} |z^n| \le \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

Hence, by the Weierstrass M-test, the series converges *absolutely and uniformly* on $|z| \le r$. ☐

Remark. Moral of the story:

- The *radius of convergence* $R = 1$ has the property that the series converges on $|z| < R$, and diverges if $|z| > R$.
- The series converges *uniformly* on $|z| \le r < 1$ but not on $|z| < 1$ itself. Why? Let $r = 1$; we need be able to get $N \in \mathbb{N}$ such that for all $n \ge N$, we have $\left|\frac{1-z^N}{1-z} - \frac{1}{1-z}\right| < 1$ for all $|z| < 1$. However, this is not gonna work: as $z \to 1$, observe that this is going to eventually exceed 1.

- The limit function $\dfrac{1}{1-z}$ is **analytic** on $\mathbb{C}\backslash\{1\}$. But the geometric series represents this function only on $|z| < 1$. In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.

  ← the limit function is well-defined way beyond the $\mathbb{D}$!

- The limit function $\dfrac{1}{1-z}$ is cool if $z \neq 1$, but as long as $|z| = 1$ (**even** if $z \neq 1$), the geometric series diverges!

  ← in the complex number sense!

Radius of convergence

Definition 16. The **limit superior** (lim sup) of a sequence of nonnegative real numbers x_n is the largest *limit point* of the x_n:

← limits of a subsequence of x_n

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0} \sup_{m\geq n} x_m$$

If the sequence is unbounded, the lim sup would be ∞.

← the RHS as in real analysis

Example 16. If x_n is the sequence $0, 1, 0, 1, \ldots$ then $\limsup_{n\to\infty} x_n = 1$.

Example 17. If x_n is the sequence $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots$, then $\limsup_{n\to\infty} x_n = 0$.

Remark. If x_n are nonnegative, then

- $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
- $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$

Theorem 22 (Cauchy-Hadamard)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Define $R \in [0, \infty]$ by

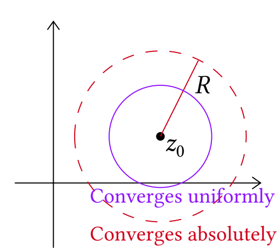
← interpret $\frac{1}{0} = \infty$

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Then the R is the *radius of convergence.*

(a) On $|z - z_0| < R$, the series converges **absolutely**. For each $r \in [0, R[$, the convergence is **uniform** on $|z - z_0| \leq r$.

←



(b) If $|z - z_0| > R$ then the series diverges. For $|z - z_0| = R$ anything could happen!

Example 18. We claim that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has an infinite radius of convergence $R = \infty$. To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \to 0$$

This is because $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \dots \cdot n}$, and in $n!$, there are at least $\frac{1}{2}$ terms that are $> \frac{n}{2}$.
Thus, $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{1/2} \to \infty$.

So $R = \infty$ and we are done ☺. We have that $\exp(z)$ has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k}}_{\text{binomial theorem}} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\
&= \exp(z+w)
\end{aligned}$$

Now define $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Term-by-term differentiation of power series

Lemma 23. $n^{\frac{1}{n}} \to 1$

Proof 1. $e^{\log(n^{\frac{1}{n}})} = e^{\frac{\log n}{n}} \to e^0 = 1$ by l'Hopital. So $n^{\frac{1}{n}} \to 1$. □

Proof 2 (better). Write $n^{\frac{1}{n}} = 1 + \delta_n$ where $\delta_n \geq 0$. The binomial theorem says:

$$\begin{aligned}
n &= (1+\delta_n)^n \\
&= \sum_{k=0}^{\infty} \binom{n}{k} \delta_n^k \cdot 1^{n-k} \\
&= 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \dots
\end{aligned}$$

$$\geq 1 + \frac{n(n-1)}{2}\delta_n^2$$

Therefore, $n - 1 \geq \frac{n(n-1)}{2}\delta_n^2$ and we get $\frac{2}{n} \geq \delta_n^2 \geq 0$ hence $\delta_n \to 0$.

Hence $n^{\frac{1}{n}} \to 1$. $\square$

Theorem 24. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence R, then

$$f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$$

for $|z - z_0| < R$. Moreover, the new series also has a radius of convergence R.

Proof. WLOG $R > 0$ and $z_0 = 0$.

← we just translate it; also $R = 0$ isn't that meaningful

For $|z| < R$ we write:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

← just splitting the function into two parts

and the 'new series'

$$g(z) = \sum_{n=0}^{\infty} na_n z^{n-1} = \lim_{N\to\infty} S_N'(z)$$

We first prove that the radius of convergence for g is the same as f. By Cauchy-Hadamard:

$$\begin{aligned} \frac{1}{R_g} &= \limsup_{n\to\infty} \sqrt[n]{n|a_n|} \\ &= \limsup_{n\to\infty} (n^{\frac{1}{n}})\sqrt[n]{|a_n|} && \text{by the previous lemma,} \\ &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \frac{1}{R} \end{aligned}$$

Thus, $R_g = R$ by Cauchy-Hadamard.

Next, we need to show that $f' = g$ with $|z| < R$.

Fix $0 \leq |w| < R$ and $\varepsilon > 0$. We want a $\delta > 0$ such that whenever $|z - w| < \delta$, we have $\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \varepsilon$.

← just saying that the derivative at any w gets close to $g(w)$

We rewrite:

$$\begin{aligned}\left|\frac{f(z)-f(w)}{z-w}-g(w)\right| &= \left|\frac{[S_N(z)+R_N(z)]-[S_N(w)+R_N(w)]}{z-w}-g(w)\right| \\ &= \left|\frac{S_N(z)-S_N(w)}{z-w}+\frac{R_N(z)-R_N(w)}{z-w}+S_N'(w)-S_N'(w)-g(w)\right| \\ &\leq |S_N'(w)-g(w)|+\left|\frac{R_N(z)-R_N(w)}{z-w}\right|+\left|\frac{S_N(z)-S_N(w)}{z-w}-S_N'(w)\right|\end{aligned}$$

- **1st term**: by def of g and $g(z)=\lim_{N\to\infty}S_N'(z)$, we can always find some $N_1\in\mathbb{N}$ such that any $N\geq N_1$ gives us $|S_N'(w)-g(w)|<\frac{\varepsilon}{3}$.
- **2nd term**: since $|w|<R$, there is an r such that $|w|<r<R$.
For $|z|<r$, we have

← work on a smaller disk

$$\begin{aligned}\left|\frac{R_N(z)-R_N(w)}{z-w}\right| &= \frac{1}{|z-w|}\left|\sum_{n=N}^{\infty}a_nz^n=-\sum_{n=N}^{\infty}a_nw^n\right| \\ &\leq \sum_{n=N}^{\infty}|a_n|\left|\frac{z^n-w^n}{z-w}\right| \\ &= \sum_{n=N}^{\infty}|a_n|\left|\frac{z^n}{z}\cdot\frac{1-\frac{w^n}{z^n}}{1-\frac{w}{z}}\right| && \text{by geometric sequence} \\ &= \sum_{n=N}^{\infty}|a_n|\left|\frac{z^n}{z}\cdot\left(1+\left(\frac{w}{z}\right)+\left(\frac{w}{z}\right)^2+\cdots+\left(\frac{w}{z}\right)^{n-1}\right)\right| \\ &= \sum_{n=N}^{\infty}|a_n|\left|z^{n-1}+z^{n-2}w+\cdots+zw^{n-2}+w^{n-1}\right| \\ &\leq \sum_{n=N}^{\infty}|a_n|\cdot n\cdot r^{n-1}\text{by } |z|,|w|<r<R\end{aligned}$$

Thus, there exists an $N_2\in\mathbb{N}$ such that any $N\geq N_2$ gives us

$$\left|\frac{R_N(z)-R_N(w)}{z-w}\right|<\frac{\varepsilon}{3}$$

- **3rd term**: let $N=\max\{N_1,N_2\}$. The definition of $S_N'(w)$ provides $\gamma>0$ such that if $|z-w|<\gamma$, then we have $\left|\frac{S_N(z)-S_N(w)}{z-w}-S_N'(w)\right|<\frac{\varepsilon}{3}$.

← review def of derivatives!

Now if $0<\delta<\min\{\gamma,r-|w|\}$, then the 3 terms above are all $<\frac{\varepsilon}{3}$. Hence, $\left|\frac{f(z)-f(w)}{z-w}-g(w)\right|<\varepsilon$ holds for this δ. □

Corollary 25. A power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ with $R > 0$ is infinitely differentiable on $|z - z_0| < R$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

are the coefficients of the terms of the power series.

← prove by keep taking derivatives!

Corollary 26. Power series expansions are unique. That is, if $r > 0$ and

← because there is a unique formula for coeffs.

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

on $|z - z_0| < r$, then $a_n = b_n$ for $n \geq 0$.

Remark. Recall that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has a radius of convergence ∞ (it's an *entire* function). Now, if we differentiate it term-by-term:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\,z}\exp(z) &= \frac{\mathrm{d}}{\mathrm{d}\,z}\sum_{n=0}^{\infty}\frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty}\frac{z^{n-1}}{(n-1)!} && \text{let } k = n-1 \\
&= \sum_{k=0}^{\infty}\frac{z^k}{k!} \\
&= \exp(z)
\end{aligned}$$

Thus, the derivative of $\exp(z)$ is itself! Moreover, $\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Remark. We claim that $\exp(z) = e^z$.

Since $e^z e^{c-z}$ is a constant for all constant c, z, we have

$$\frac{\mathrm{d}}{\mathrm{d}\,z}(e^z e^{c-z}) = 0$$

to recover the constant $e^z e^{c-z}$, we let $z = 0$, giving us

$$e^z e^{c-z} = e^c$$

which is the addition formula!

Therefore,

$$\begin{aligned}
\exp(n) &= \exp(1 + 1 + \cdots + 1) \\
&= exp(1)^n \\
&= e^n
\end{aligned}$$

Now that we have derived e, we could use it to derive sin and cos:

Definition 17.

$$\begin{aligned}\cos(z) &= \frac{e^{iz}+e^{-iz}}{2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\end{aligned}$$

$$\begin{aligned}\sin(z) &= \frac{e^{iz}-e^{-iz}}{2i} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}\end{aligned}$$

We observe that we have the following property:

- Radius of convergence $R = \infty$
- $(\cos z)' = -\sin z, (\sin z)' = \cos z$
- $\cos x = \operatorname{Re}(e^{ix}), \sin x = \operatorname{Im} e^{ix}$ for all $x \in \mathbb{R}$
- $\cos(-z) = \cos z, \sin(-z) = -\sin z$
- $\cosh x = \frac{e^x+e^{-x}}{2}$ so $\cosh(ix) = \cos x$
- $e^{iz} = \cos z + i \sin z$
- $$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\ &= \frac{1}{4}(e^{2iz}+2+e^{-2iz}) - \frac{1}{4}(e^{2iz}-2+e^{-2iz}) \\ &= 1 \qquad \forall z \in \mathbb{C}\end{aligned}$$
- $$\begin{aligned}\cos^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 \\ &= \frac{1}{4}(e^{2iz}+2+e^{-2iz}) \\ &= \frac{1}{2} + \frac{e^{2iz}+e^{-2iz}}{4} \\ &= \frac{1}{2}(1+\cos 2z)\end{aligned}$$
- If $x \in \mathbb{R}$ then $\cos x, \sin x$ are real. We get $|\sin x|, |\cos x| \leq 1$.

Definition 18. $f : \mathbb{C} \to \mathbb{C}$ is **periodic** with a *period* ω if $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$.

Theorem 27. There exists a positive real number π such that:

(a) $\cos z, \sin z$ have period 2π

(b) e^z is periodic with period $2\pi i$

(c) π is the area of the unit circle

Proof. By Euler's formula, it suffices to consider e^{iz} only. If ω is a period of e^{iz}, then

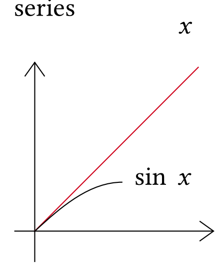
$$e^{iz} = e^{i(z+\omega)} = e^{iz}e^{i\omega}$$

which only happens if $e^{i\omega} = 1$. Conversely, if $e^{i\omega} = 1$, then $e^{i(z+\omega)} = e^{iz}$.

Hence, ω is a period of e^{iz} *if and only if* $e^{iw} = 1$. $\square$

Proposition 28. $\sin x \leq x$ for all $x \geq 0$.

← This is the first term in the power series



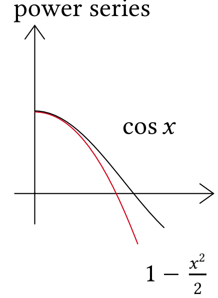
Proof. Since $|\cos t| \leq 1$,

$$\begin{aligned} x - \sin x &= (x - \sin x) - (0 - \sin 0) \\ &= \int_0^x \underbrace{1 - \cos t}_{\geq 0} \, dt \qquad \text{by FTC} \\ &\geq 0 \end{aligned}$$

$\square$

Proposition 29. In addition, $\cos x \geq 1 - \frac{x^2}{2}$ for $x \geq 0$.

← These are the first 2 terms in the power series



Proof. The previous prop gives:

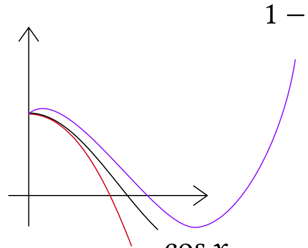
$$\begin{aligned} \cos x - 1 &= \cos x - \cos 0 \\ &= \int_0^x -\sin t \, dt \\ &\geq \int_0^x -t \, dt \\ &= \frac{-x^2}{2} \end{aligned}$$

$\square$

Proposition 30. Furthermore, for $x \geq 0$:

- $\sin x \geq x^3 - \frac{x^3}{6}$
- $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

Proposition 31. There exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$.



Proof. By the previous prop, we have $\cos\sqrt{3} \le 1 - \frac{\sqrt{3}^2}{2} + \frac{\sqrt{3}^4}{24} = \frac{1}{8} < 0$. Moreover, $\cos 0 = 1 > 0$, by IVT, there exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$. □

Proposition 32. $\omega_0 = 4x_0$ is a period of e^{iz}.

Proof. Since $\cos x_0 = 0$, we have $\sin x_0 = \pm 1$. Then $e^{ix_0} = \pm i$. We have $(\pm i)^4 = 1$, so $e^{4ix_0} = 1 = e^0$, so $\omega_0 = 4x_0$ is a period of e^{iz}. □

Proposition 33. ω_0 is the *smallest* positive period of e^{iz}.

Proposition 34. All periods of e^{iz} are integer multiples of $2\pi = 4x_0$.

Proof. Define $\pi = 2x_0$. The area of unit circle is

$$\begin{aligned} 4\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}\,x &= 4\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\,\mathrm{d}\,\theta \\ &= 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta)\,\mathrm{d}\,\theta \\ &= \pi \end{aligned}$$

□