MATH135 Complex Analysis Notes

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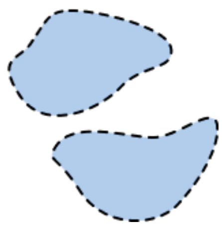
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of $\mathbb{C}$.

- A region without "holes" is simply connected.

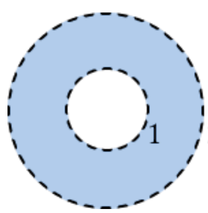
Non-example 1. This is not a region (not connected):



Example 2. $\mathbb{C}$ is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

← this $z \to z_0$ could be from **any** directions!

2. If f is differentiable at every point in Ω, we say f is **analytic** on Ω.

← Means that existence of 1st derivative implies the existence of ∞th derivative! & has Taylor expansion.

3. If f is analytic on $\mathbb{C}$, then f is **entire**.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on $\mathbb{C}$ except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \to z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z\to z_0} = \frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{z-z_0}{z-z_0} = 1.$$

Else if $z \to z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z\to z_0} = \frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{-(z-z_0)}{z-z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. ☐

Proposition 1. Let f be differentiable at z_0. Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$|f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| < \varepsilon|z - z_0|$$
$$|f(z_0) + f'(z_0)(z - z_0) - f(z)| < \varepsilon|z - z_0|$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the "tangent-line approximation" equivalent in $\mathbb{C}$!

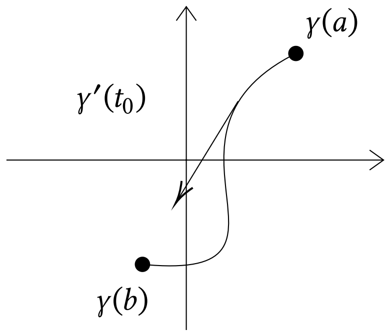
← The RHS is a **linear** function!

In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is locally orientation-preserving and could be approximated by a linear function.

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

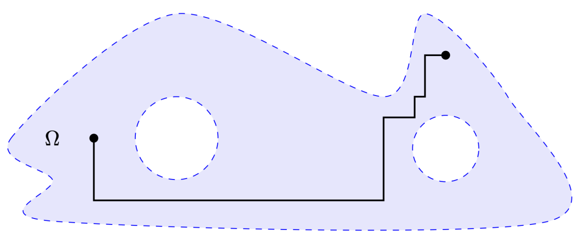
Curves, paths

Definition 3. A **curve** in $\mathbb{C}$ is a function $\gamma : [a,b] \to \mathbb{C}$, $a, b \in \mathbb{R}$.



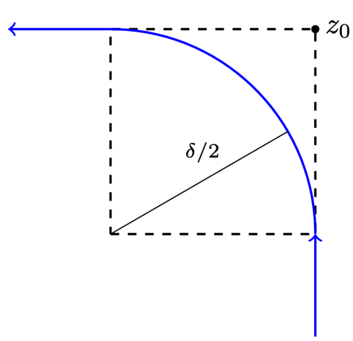
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The "Boxy-path" Theorem)**.** A nonempty open set Ω in $\mathbb{C}$ is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω, each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a "**boxy path**".

Remark. There is also always a **smooth path**. That is:

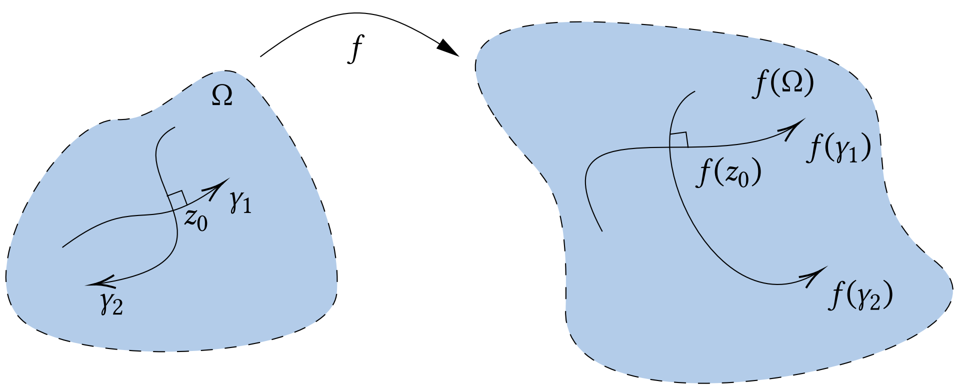


Theorem 3 ("Smooth-path")**.** A nonempty open set Ω in $\mathbb{C}$ is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See lecture 2 notes. □

Conformality

Let f be an analytic complex function on Ω.



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ. Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(\zeta_0)$ also with angle θ.

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall: $f'(z_0) = \lim_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$.

Now we write each function with complex variables as $f(z) = u(z) + i\,v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i\,v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r\to 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r\to 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \to 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations)**.**

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \to \mathbb{C}$ is analytic and $f' = 0$ on Ω, then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω. By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω. Hence, $\mathbf{u}, \mathbf{v}$ are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω. □

Orientation-preserving as shown by Jacobian

Let $f : \Omega \to \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 = \bar{f}' \cdot f &= (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right) && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0, implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales $\mathbb{R}$ by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

← $\Delta u = 0$ characterizes steady-state solutions to heat equations on Ω.

Definition 5. Real-valued functions $u : \Omega \to \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω.

Example 8. $u = x^2 - y^2, v = 2xy$.

← Check it!

Remark. Harmonic conjugates are unique up to translation (± constants).

Remark. If u is harmonic on Ω, it does NOT have to have a harmonic conjugate on Ω.

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2}\log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2 + y^2)\right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

← Review quotient rule!

$$\begin{aligned} u_{xx} &= \frac{(x^2+y^2)-x(2x)}{(x^2+y^2)^2} \\ &= \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

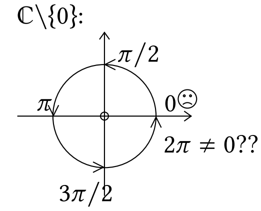
Symmetrically, we find

$$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. □

Now, can we find a harmonic conjugate for the aforementioned u?

← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C}\backslash\{0\}$:



To be resolved later!

We could use the two Cauchy-Riemann Equations. One of them:

$$\begin{aligned} v_y &= u_x \\ &= \frac{x}{x^2+y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v(x,y) &= \int v_y dy + C(x) \qquad \text{unknown function of } x \\ &= \arctan\left(\frac{y}{x}\right) + C(x) \end{aligned}$$

Then, we use the second one:

$$\begin{aligned} \frac{y}{x^2+y^2} = u_y = -v_x &= -\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right) \\ &= \frac{y}{x^2+y^2} - C'(x) \implies C'(x) = 0 \end{aligned}$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x,y) = \arctan\left(\frac{y}{x}\right) + C$$

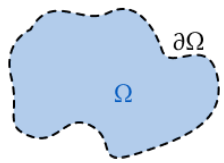
where C is a constant. WLOG, let $C = 0$. Then $v(x,y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$v(z) = \arg(z)$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in $\mathbb{C}$. Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \to \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω.

1. U cannot have a *local* maximum in Ω.
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$. ← Ω^- denotes the closure of Ω
3. U cannot be locally constant without being globally constant.

Theorem 8 (Maximum principle)**.** Let Ω be a bounded region in $\mathbb{C}$ and let $f : \Omega^- \to \mathbb{C}$ be analytic on Ω and continuous on Ω^-.

1. If $|f|$ achieves a local max in Ω, then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations)**.**

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C}\backslash\{\frac{-d}{c}\}$ and **comformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C}\backslash\{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$\begin{aligned} f(z) = f(w) \implies \frac{az+b}{cz+d} &= \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

Definition 8 (The extended plane)**.** We set the following convention:

$$f(\frac{-d}{c}) = \infty$$
$$f(\infty) = \frac{a}{c}$$

with this, f is a **bijection** from $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can *associate* $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

Remark. $M_{f \circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

Theorem 9. A Möbius transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with three fixed points in $\widehat{\mathbb{C}}$ is the **identity map** $\mathrm{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transformation.
   (a) If $f(z) = z$, we are done since we get the identity!
   (b) Otherwise the function only has one fixed point at ∞.
2. If ∞ is not a fixed point, then $c \neq 0$. Solve:
$$\begin{aligned} f(z) + z \Leftrightarrow \frac{az+b}{cz+d} &= z \\ az + b &= cz^2 + dz \\ cz^2 + (d-a)z - b &= 0 \end{aligned}$$
is a quadratic which has at most two (distinct) solutions in $\mathbb{C}$. Hence, this transformation fixes at most two points.

← think about that!

$\square$

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so we are done.
2. Now suppose $c \neq 0$. Then
$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c} \end{aligned}$$
which is a composition of affine, inversion and affine:
$$z \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c}$$
We now only need to show that inversion preserves circles.

← since linear transformations preserve circles and lines

Let a circle in $\mathbb{R}^2$ be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.
Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle! ← check this!

□

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

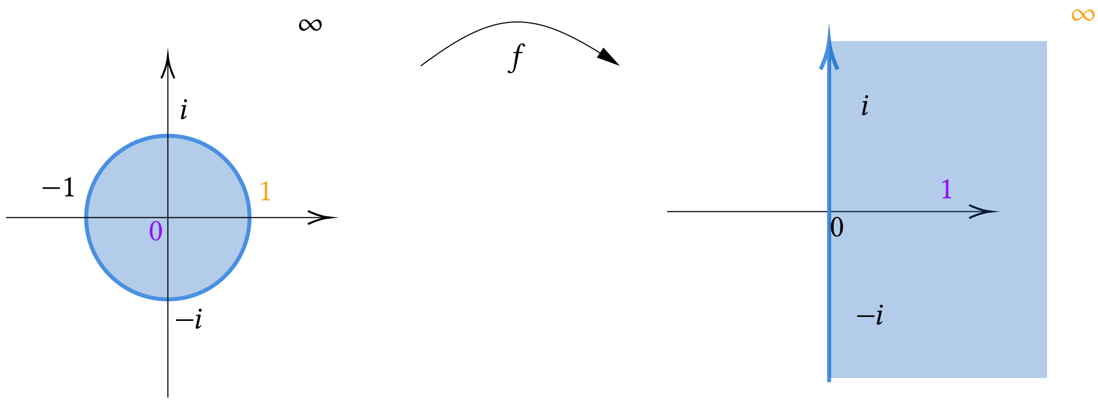
We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

$$\begin{aligned} z_1 &\xmapsto{\phi} 0 \xmapsto{\psi^{-1}} w_1 \\ z_2 &\xmapsto{\phi} 1 \xmapsto{\psi^{-1}} w_2 \\ z_3 &\xmapsto{\phi} \infty \xmapsto{\psi^{-1}} w_3 \end{aligned}$$

and we could simply let $f = \psi^{-1} \circ \phi$. □

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$ where $S_N = a_1 + \dots + a_N$.

← S_N is the N-th partial sum.

Divergence test

Definition 10 (Divergence test)**.** A pair of contrapositives:

← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\lim_{n\to\infty} a_n \neq 0$ (*including the case where the limit doesn't exist*) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \dots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞.

← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = \lim_{N\to\infty} S - S_N = 0$.

← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion)**.** $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left|\sum_{n=j-1}^{k} a_n\right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test)**.** Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\to \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x)\,dx$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.

← do an improper integral!

Moreover, $\int_1^N f(x)\,\mathrm{d}x \le a_1 + \cdots + a_N \le a_1 + \int_1^N f(x)\,\mathrm{d}x$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then ← $a_n = \frac{1}{n}$

$$\ln N \le 1 + \frac{1}{2} + \cdots + \frac{1}{N} \le \ln N$$

Theorem 14. The "p-series" $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1$$

Remark. Euler figured out:

$$\zeta(2) = \frac{\pi^2}{6}$$
$$\zeta(4) = \frac{\pi^4}{90}$$
$$\zeta(6) = \frac{\pi^6}{945}$$
$$\vdots$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979): ← still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202\ldots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. ← Good
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. ← BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

converges to $\ln 2$. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

Theorem 17 (Riemann Rearrangement Theorem)**.** If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\begin{aligned}\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See notes.

Uniform convergence

Definition 14. A sequence of functions $f_n : X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \to \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!

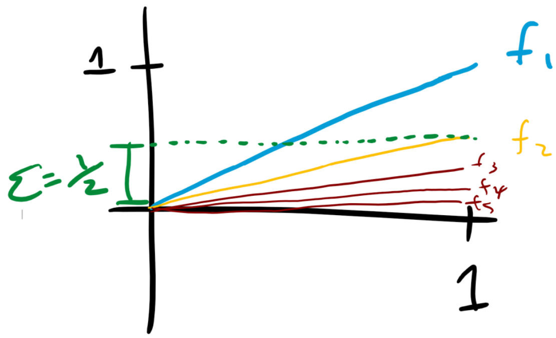


FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \to \mathbb{C}$ are continuous and converges uniformly on X to $f : X \to \mathbb{C}$, then f is continuous on X. In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.

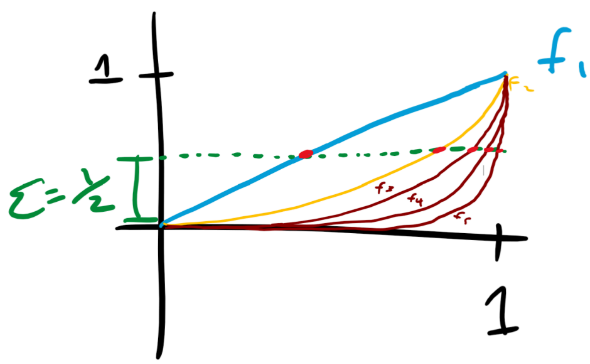


FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a, b] \to \mathbb{C}$ are continuous and converge uniformly on $[a, b]$ to f, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

← Integrals work with uniform convergence

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1, 1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test)**.** Let $f_n : X \to \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X.