

MATH135 Complex Analysis Notes

Xuehuai He
February 6, 2024

Contents

Regions, differentiability, analyticity	3
Regions	3
Complex derivatives and analyticity	3
Curves, paths	5
Conformality	6
Cauchy-Riemann equations, harmonic functions	6
Multivariate notion of complex derivatives	6
Cauchy-Riemann Equations	7
Orientation-preserving as shown by Jacobian	7
The Laplacian, harmonic functions and conjugates	8
Harmonic conjugate	8
Finding a harmonic conjugate	8
Physics analogies of harmonic functions	10
Möbius transformations	10
Möbius transformations, the extended plane	10
Möbius transformations as matrices	11
Möbius transformations take circles to circles	12
Recall: infinite series	14
Divergence test	14
Integral test	14
Absolute convergence	15
Riemann Rearrangement Theorem	16
Uniform convergence	16
Uniform convergence preserves continuity	17
Integrals work with uniform convergence	17
Power series	18
Convergence of geometric series	18

Radius of convergence	19
Limit superior	19
Cauchy-Hadamard formula	19

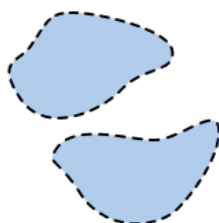
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of \mathbb{C} .

- A region without “holes” is simply connected.

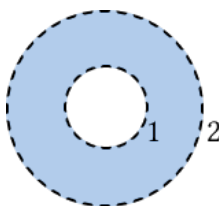
Non-example 1. This is not a region (not connected):



Example 2. \mathbb{C} is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \rightarrow \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

2. If f is differentiable at every point in Ω , we say f is **analytic** on Ω .
3. If f is analytic on \mathbb{C} , then f is **entire**.

← this $z \rightarrow z_0$ could be from **any** directions!

← Means that existence of 1st derivative implies the existence of ∞ th derivative! & has Taylor expansion.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on \mathbb{C} except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \rightarrow z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if $z \rightarrow z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. \square

Proposition 1. Let f be differentiable at z_0 . Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the “tangent-line approximation” equivalent in \mathbb{C} !

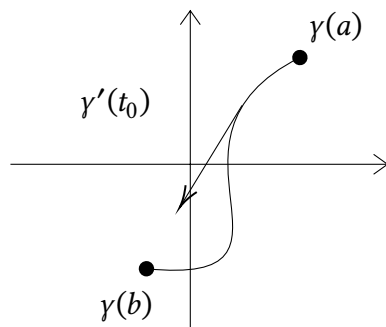
In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is locally orientation-preserving and could be approximated by a linear function.

← The RHS is a **linear** function!

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

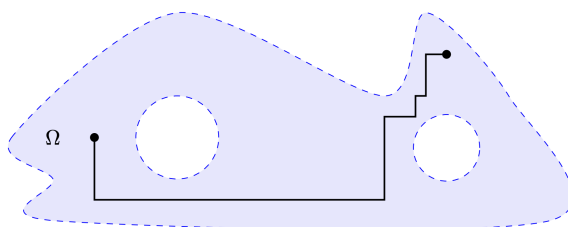
Curves, paths

Definition 3. A **curve** in \mathbb{C} is a function $\gamma : [a, b] \rightarrow \mathbb{C}, a, b \in \mathbb{R}$.



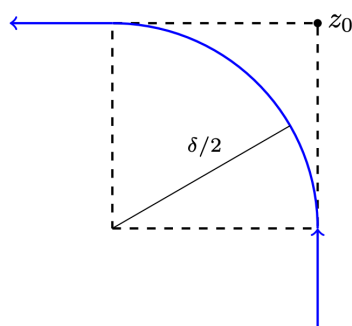
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The “Boxy-path” Theorem). A nonempty open set Ω in \mathbb{C} is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω , each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a “**boxy path**”.

Remark. There is also always a **smooth path**. That is:

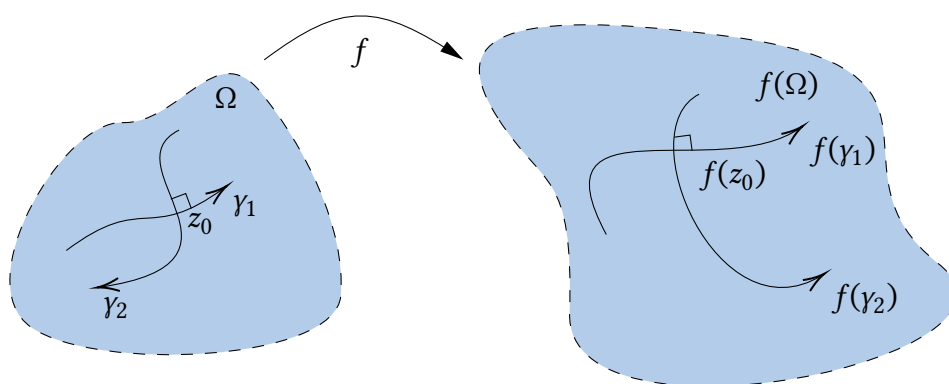


Theorem 3 (“Smooth-path”). A nonempty open set Ω in \mathbb{C} is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See [lecture 2 notes](#). □

Conformality

Let f be an analytic complex function on Ω .



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ . Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(z_0)$ also with angle θ .

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall:
$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Now we write each function with complex variables as $f(z) = u(z) + i v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r \rightarrow 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \rightarrow 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations).

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \rightarrow \mathbb{C}$ is analytic and $f' = 0$ on Ω , then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω . By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω . Hence, \mathbf{u}, \mathbf{v} are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω . \square

Orientation-preserving as shown by Jacobian

Let $f : \Omega \rightarrow \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 &= \bar{f}' \cdot f' = (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0 , implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales \mathbb{R} by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

Definition 5. Real-valued functions $u : \Omega \rightarrow \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \rightarrow \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω .

Example 8. $u = x^2 - y^2, v = 2xy$.

Remark. Harmonic conjugates are unique up to translation (\pm constants).

Remark. If u is harmonic on Ω , it does NOT have to have a harmonic conjugate on Ω .

← $\Delta u = 0$
characterizes
steady-state
solutions to heat
equations on Ω .

← Check it!

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log |z|$ is harmonic on $\mathbb{C} \setminus \{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2} \log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x} \left(\frac{1}{2} \log(x^2 + y^2) \right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} &= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. \square

Now, can we find a harmonic conjugate for the aforementioned u ?

We could use the two Cauchy-Riemann Equations. One of them:

$$\begin{aligned} v_y &= u_x \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v(x, y) &= \int v_y dy + C(x) && \text{unknown function of } x \\ &= \arctan\left(\frac{y}{x}\right) + C(x) \end{aligned}$$

Then, we use the second one:

$$\begin{aligned} \frac{y}{x^2 + y^2} &= u_y = -v_x = -\frac{\partial}{\partial x} \left(\arctan\left(\frac{y}{x}\right) + C(x) \right) \\ &= \frac{y}{x^2 + y^2} - C'(x) \implies C'(x) = 0 \end{aligned}$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x, y) = \arctan\left(\frac{y}{x}\right) + C$$

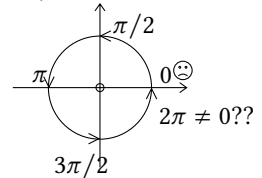
where C is a constant. WLOG, let $C = 0$. Then $v(x, y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$v(z) = \arg(z)$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

← Review quotient rule!

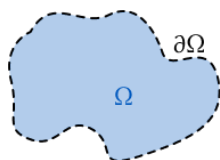
← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C} \setminus \{0\}$:



To be resolved later!

Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in \mathbb{C} . Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \rightarrow \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω .

1. U cannot have a *local* maximum in Ω .
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$.
3. U cannot be locally constant without being globally constant.

← Ω^- denotes the closure of Ω

Theorem 8 (Maximum principle). Let Ω be a bounded region in \mathbb{C} and let $f : \Omega^- \rightarrow \mathbb{C}$ be analytic on Ω and continuous on Ω^- .

1. If $|f|$ achieves a local max in Ω , then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations).

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C} \setminus \{\frac{-d}{c}\}$ and **conformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C} \setminus \{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$\begin{aligned} f(z) = f(w) &\implies \frac{az+b}{cz+d} = \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

Definition 8 (The extended plane). We set the following convention:

$$\begin{aligned} f\left(\frac{-d}{c}\right) &= \infty \\ f(\infty) &= \frac{a}{c} \end{aligned}$$

with this, f is a **bijection** from $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can associate $f(z) = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

Remark. $M_{f \circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

Theorem 9. A Möbius transformation $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with three fixed points in $\hat{\mathbb{C}}$ is the **identity map** $\text{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transformation. ← think about that!
 - (a) If $f(z) = z$, we are done since we get the identity!
 - (b) Otherwise the function only has one fixed point at ∞ .
2. If ∞ is not a fixed point, then $c \neq 0$. Solve:

$$\begin{aligned} f(z) + z &\Leftrightarrow \frac{az + b}{cz + d} = z \\ az + b &= cz^2 + dz \\ cz^2 + (d - a)z - b &= 0 \end{aligned}$$

is a quadratic which has at most two (distinct) solutions in \mathbb{C} . Hence, this transformation fixes at most two points.

□

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so we are done.
2. Now suppose $c \neq 0$. Then

← since linear transformations preserve circles and lines

$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz + d) - \frac{ad}{c} + b}{cz + d} \\ &= \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c} \end{aligned}$$

which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz + d} \mapsto \frac{b - \frac{ad}{c}}{cz + d} + \frac{a}{c}$$

We now only need to show that inversion preserves circles.

Let a circle in \mathbb{R}^2 be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.

Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle!

← check this!

□

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

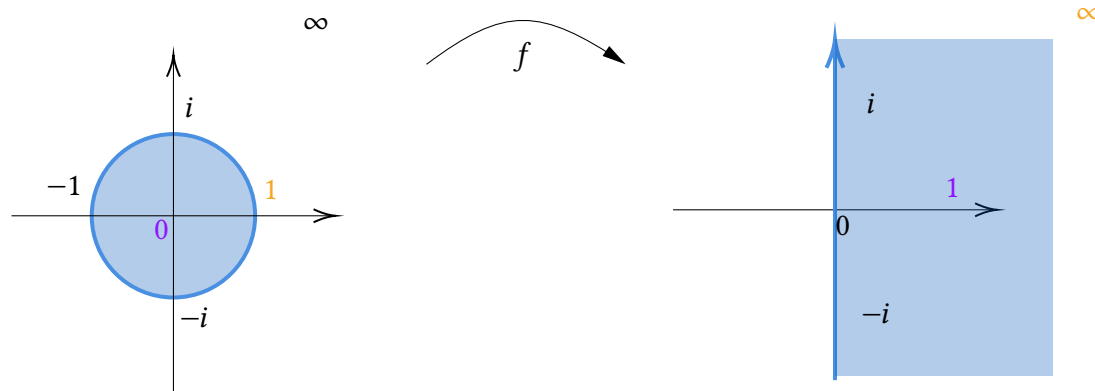
$$\begin{array}{ccc} z_1 & \xrightarrow{\phi} & 0 \xrightarrow{\psi^{-1}} w_1 \\ z_2 & \xrightarrow{\phi} & 1 \xrightarrow{\psi^{-1}} w_2 \\ z_3 & \xrightarrow{\phi} & \infty \xrightarrow{\psi^{-1}} w_3 \end{array}$$

and we could simply let $f = \psi^{-1} \circ \phi$.

□

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N \rightarrow \infty} S_N = S$ where $S_N = a_1 + \dots + a_N$.

← S_N is the N -th partial sum.

Divergence test

Definition 10 (Divergence test). A pair of contrapositives:

← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.
2. If $\lim_{n \rightarrow \infty} a_n \neq 0$ (including the case where the limit doesn't exist) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \dots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞ .

← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N \rightarrow \infty} \sum_{n=N}^{\infty} a_n = \lim_{N \rightarrow \infty} S - S_N = 0$.

← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion). $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$,

there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left| \sum_{n=j}^k a_n \right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test). Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\rightarrow \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x) dx$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.

← do an improper integral!

Moreover, $\int_1^N f(x) dx \leq a_1 + \dots + a_N \leq a_1 + \int_1^N f(x) dx$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then

← $a_n = \frac{1}{n}$

$$\ln N \leq 1 + \frac{1}{2} + \dots + \frac{1}{N} \leq 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

Theorem 14. The “ p -series” $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } \operatorname{Re}(s) > 1$$

Remark. Euler figured out:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979):

← still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202 \dots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges.
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

← Good

← BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series

← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges to $\ln 2$. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

Theorem 17 (Riemann Rearrangement Theorem). If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\begin{aligned} \left(\sum_{i=0}^{\infty} a_i \right) \left(\sum_{j=0}^{\infty} b_j \right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^n a_k b_{n-k} \end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See [notes](#).

Uniform convergence

Definition 14. A sequence of functions $f_n : X \rightarrow \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \rightarrow \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!

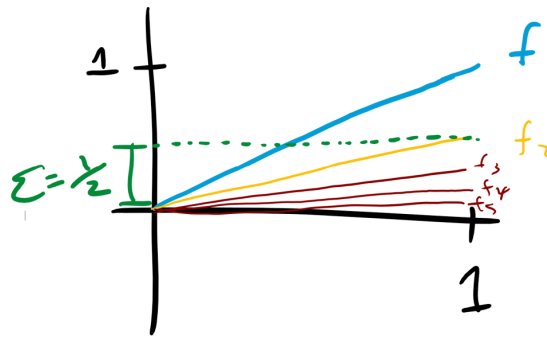


FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \rightarrow \mathbb{C}$ are continuous and converges uniformly on X to $f : X \rightarrow \mathbb{C}$, then f is continuous on X . In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n \rightarrow \infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.

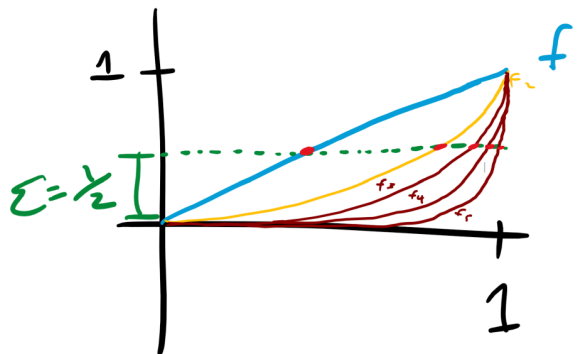


FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a, b] \rightarrow \mathbb{C}$ are continuous and converge uniformly on $[a, b]$ to f , then

← Integrals work with uniform convergence

$$\lim_{n \rightarrow \infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1, 1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test). Let $f_n : X \rightarrow \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X .

Power series

Definition 15. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z - z_0)^n$. The a_n is the *coefficient* and z_0 is the *center*.

Convergence of geometric series

Theorem 21. The *geometric series* ($a_n = 1, z_0 = 0$) $\sum_{n=0}^{\infty} z^n$ converges absolutely to $\frac{1}{1-z}$ if $|z| < 1$, and it diverges otherwise.

Moreover, for each $r \in [0, 1[$, the convergence is **uniform** on $|z| \leq r$.

Proof. If $|z| \geq 1$, then $z^n \not\rightarrow 0$, so by the test of divergence, the series diverges.

Now suppose $|z| < 1$. Then

$$\begin{aligned} \sum_{n=0}^{\infty} z^n &= \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} z^n \\ &= \lim_{N \rightarrow \infty} (1 + z + z^2 + \cdots + z^{N-1}) \\ &= \lim_{N \rightarrow \infty} \frac{1 - z^N}{1 - z} \\ &= \frac{1}{1 - z} \end{aligned} \quad \text{since } |z| < 1$$

← The fact that we can find a formula for this sum is quite rare!

Which gives us point-wise convergence. Then, for any r such that $|z| \leq r < 1$, we have

$$\sum_{n=0}^{\infty} |z^n| \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

Hence, by the Weierstrass M -test, the series converges *absolutely and uniformly* on $|z| \leq r$. \square

Remark. Moral of the story:

- The *radius of convergence* $R = 1$ has the property that the series converges on $|z| < R$, and diverges if $|z| > R$.
- The series converges *uniformly* on $|z| \leq r < 1$ but not on $|z| < 1$ itself. Why? Let $r = 1$; we need be able to get $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left| \frac{1-z^n}{1-z} - \frac{1}{1-z} \right| < 1$ for all $|z| < 1$. However, this is not gonna work: as $z \rightarrow 1$, observe that this is going to eventually exceed 1.

- The limit function $\frac{1}{1-z}$ is **analytic** on $\mathbb{C} \setminus \{1\}$. But the geometric series represents this function only on $|z| < 1$. In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.
- The limit function $\frac{1}{1-z}$ is cool if $z \neq 1$, but as long as $|z| = 1$ (**even** if $z \neq 1$), the geometric series diverges!

← the limit function is well-defined way beyond the \mathbb{D} !

← in the complex number sense!

Radius of convergence

Definition 16. The **limit superior** (lim sup) of a sequence of nonnegative real numbers x_n is the largest *limit point* of the x_n :

$$\limsup_{n \rightarrow \infty} x_n = \inf_{n \geq 0} \sup_{m \geq n} x_m$$

← limits of a subsequence of x_n

If the sequence is unbounded, the lim sup would be ∞ .

← the RHS as in real analysis

Example 16. If x_n is the sequence $0, 1, 0, 1, \dots$ then $\limsup_{n \rightarrow \infty} x_n = 1$.

Example 17. If x_n is the sequence $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \dots$, then $\limsup_{n \rightarrow \infty} x_n = 0$.

Remark. If x_n are nonnegative, then

- $\limsup_{n \rightarrow \infty} (a_n + b_n) = \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n$
- $\limsup_{n \rightarrow \infty} (a_n b_n) \leq (\limsup_{n \rightarrow \infty} a_n)(\limsup_{n \rightarrow \infty} b_n)$

Theorem 22 (Cauchy-Hadamard). Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series. Define $R \in [0, \infty]$ by

← interpret $\frac{1}{0} = \infty$

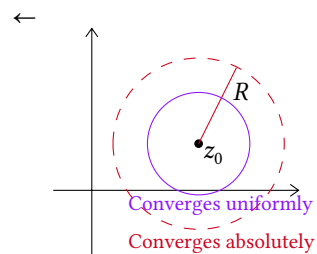
$$\frac{1}{R} = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$$

Then the R is the *radius of convergence*.

- On $|z - z_0| < R$, the series converges **absolutely**. For each $r \in [0, R[$, the convergence is **uniform** on $|z - z_0| \leq r$.
- If $|z - z_0| > R$ then the series diverges. **For $|z - z_0| = R$ anything could happen!**

Example 18. We claim that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has an infinite radius of convergence $R = \infty$. To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \rightarrow 0$$



This is because $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \dots \cdot n}$, and in $n!$, there are at least $\frac{1}{2}$ terms that are $> \frac{n}{2}$.

Thus, $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{1/2} \rightarrow \infty$.

So $R = \infty$ and we are done 😊. We have that $\exp(z)$ has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$\begin{aligned} \exp(z) \exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{w^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!} \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k} \\ &\quad \underbrace{\hspace{10em}}_{\text{binomial theorem}} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} (z + w)^n \\ &= \exp(z + w) \end{aligned}$$

Now define $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.