MATH135 Complex Analysis Notes

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Contents

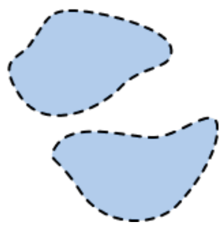
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of $\mathbb{C}$.

- A region without "holes" is simply connected.

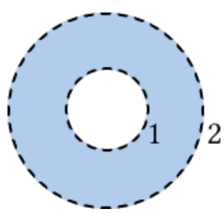
Non-example 1. This is not a region (not connected):



Example 2. $\mathbb{C}$ is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

← this $z \to z_0$ could be from **any** directions!

2. If f is differentiable at every point in Ω, we say f is **analytic** on Ω.

← Means that existence of 1st derivative implies the existence of ∞th derivative! & has Taylor expansion.

3. If f is analytic on $\mathbb{C}$, then f is **entire**.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on $\mathbb{C}$ except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \to z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z\to z_0} = \frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{z-z_0}{z-z_0} = 1.$$

Else if $z \to z_0$ vertically, then $\overline{z-z_0} = -(z-z_0)$, meaning that

$$\lim_{z\to z_0} = \frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{-(z-z_0)}{z-z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. □

Proposition 1. Let f be differentiable at z_0. Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0)\cdot(z-z_0) - f(z) + f(z_0)| &< \varepsilon|z-z_0| \\ |f(z_0) + f'(z_0)(z-z_0) - f(z)| &< \varepsilon|z-z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z-z_0)$$

this is the "tangent-line approximation" equivalent in $\mathbb{C}$!

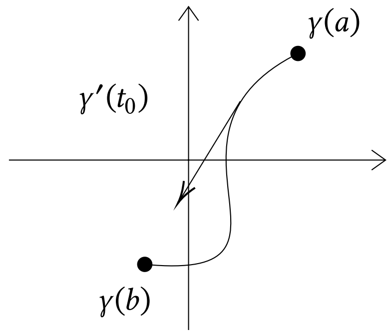
← The RHS is a **linear** function!

In addition, $f(z_0) + f'(z_0)(z-z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is locally orientation-preserving and could be approximated by a linear function.

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

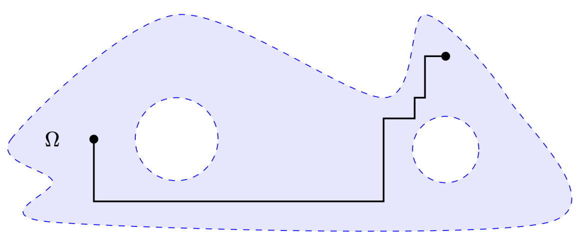
Curves, paths

Definition 3. A **curve** in $\mathbb{C}$ is a function $\gamma : [a,b] \to \mathbb{C}, a, b \in \mathbb{R}$.



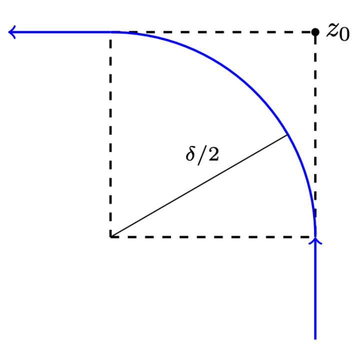
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The "Boxy-path" Theorem)**.** A nonempty open set Ω in $\mathbb{C}$ is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω, each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a "**boxy path**".

Remark. There is also always a **smooth path**. That is:

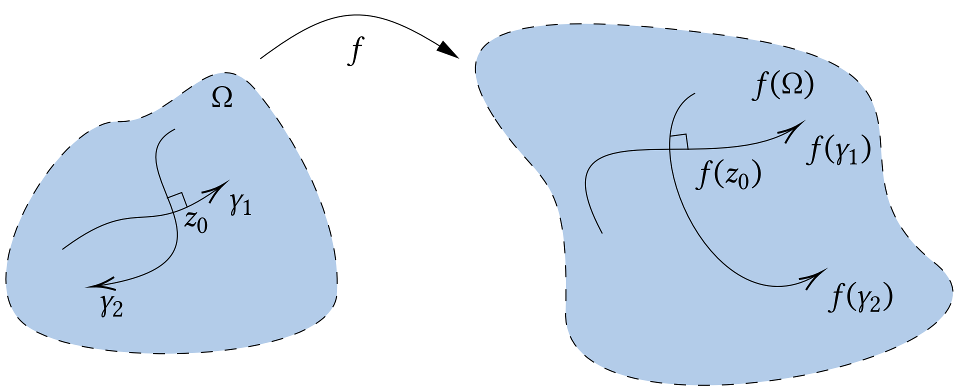


Theorem 3 (“Smooth-path”). A nonempty open set Ω in $\mathbb{C}$ is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See lecture 2 notes. □

Conformality

Let f be an analytic complex function on Ω.



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ. Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(\zeta_0)$ also with angle θ.

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall: $f'(z_0) = \lim\limits_{h\to 0} \dfrac{f(z_0+h) - f(z_0)}{h}$.

Now we write each function with complex variables as $f(z) = u(z) + i\,v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i\,v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r\to 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r\to 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \to 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations)**.**

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \to \mathbb{C}$ is analytic and $f' = 0$ on Ω, then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω. By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω. Hence, $\mathbf{u}, \mathbf{v}$ are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω. $\square$

Orientation-preserving as shown by Jacobian

Let $f : \Omega \to \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 = \bar{f}' \cdot f &= (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right) && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0, implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales $\mathbb{R}$ by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

← $\Delta u = 0$ characterizes steady-state solutions to heat equations on Ω.

Definition 5. Real-valued functions $u : \Omega \to \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω.

Example 8. $u = x^2 - y^2, v = 2xy$.

← Check it!

Remark. Harmonic conjugates are unique up to translation (± constants).

Remark. If u is harmonic on Ω, it does NOT have to have a harmonic conjugate on Ω.

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log|z|$ is harmonic on $\mathbb{C}\backslash\{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2}\log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2 + y^2)\right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

← Review quotient rule!

$$\begin{aligned} u_{xx} &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2+y^2)^2} \end{aligned}$$

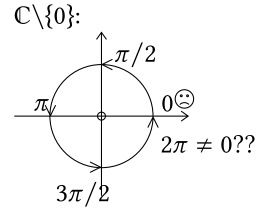
Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. □

Now, can we find a harmonic conjugate for the aforementioned u?

← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C}\backslash\{0\}$:



To be resolved later!

We could use the two Cauchy-Riemann Equations. One of them:

$$\begin{aligned} v_y &= u_x \\ &= \frac{x}{x^2+y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v(x,y) &= \int v_y dy + C(x) && \text{unknown function of } x \\ &= \arctan\left(\frac{y}{x}\right) + C(x) \end{aligned}$$

Then, we use the second one:

$$\begin{aligned} \frac{y}{x^2+y^2} = u_y = -v_x &= -\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right) \\ &= \frac{y}{x^2+y^2} - C'(x) \implies C'(x) = 0 \end{aligned}$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x,y) = \arctan\left(\frac{y}{x}\right) + C$$

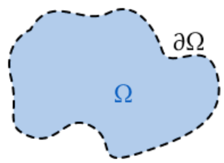
where C is a constant. WLOG, let $C = 0$. Then $v(x,y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$v(z) = \arg(z)$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in $\mathbb{C}$. Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \to \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω.

1. U cannot have a *local* maximum in Ω.
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$. ← Ω^- denotes the closure of Ω
3. U cannot be locally constant without being globally constant.

Theorem 8 (Maximum principle)**.** Let Ω be a bounded region in $\mathbb{C}$ and let $f : \Omega^- \to \mathbb{C}$ be analytic on Ω and continuous on Ω^-.

1. If $|f|$ achieves a local max in Ω, then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations)**.**

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C}\backslash\{\frac{-d}{c}\}$ and **comformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C}\backslash\{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$\begin{aligned} f(z) = f(w) \implies \frac{az+b}{cz+d} &= \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

Definition 8 (The extended plane)**.** We set the following convention:

$$\begin{aligned} f(\frac{-d}{c}) &= \infty \\ f(\infty) &= \frac{a}{c} \end{aligned}$$

with this, f is a **bijection** from $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can *associate* $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

Remark. $M_{f \circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw-b}{-cw+a}$$

Theorem 9. A Möbius transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with three fixed points in $\widehat{\mathbb{C}}$ is the **identity map** $\mathrm{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transformation.
   (a) If $f(z) = z$, we are done since we get the identity!
   (b) Otherwise the function only has one fixed point at ∞.

← think about that!

2. If ∞ is not a fixed point, then $c \neq 0$. Solve:

$$\begin{aligned} f(z) + z \Leftrightarrow \frac{az+b}{cz+d} &= z \\ az + b &= cz^2 + dz \\ cz^2 + (d-a)z - b &= 0 \end{aligned}$$

is a quadratic which has at most two (distinct) solutions in $\mathbb{C}$. Hence, this transformation fixes at most two points.

□

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so we are done.

← since linear transformations preserve circles and lines

2. Now suppose $c \neq 0$. Then

$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c} \end{aligned}$$

which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c}$$

We now only need to show that inversion preserves circles.

Let a circle in $\mathbb{R}^2$ be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.
Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle! ← check this!

□

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

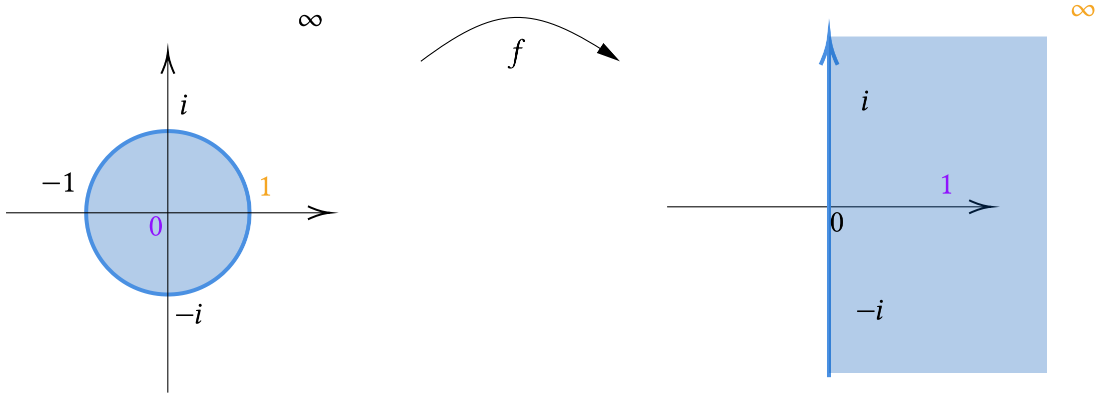
We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

$$\begin{aligned} z_1 &\xmapsto{\phi} 0 \xmapsto{\psi^{-1}} w_1 \\ z_2 &\xmapsto{\phi} 1 \xmapsto{\psi^{-1}} w_2 \\ z_3 &\xmapsto{\phi} \infty \xmapsto{\psi^{-1}} w_3 \end{aligned}$$

and we could simply let $f = \psi^{-1} \circ \phi$. □

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$ where $S_N = a_1 + \cdots + a_N$.

← S_N is the N-th partial sum.

Divergence test

Definition 10 (Divergence test)**.** A pair of contrapositives:

← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\lim_{n\to\infty} a_n \neq 0$ (*including the case where the limit doesn't exist*) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \ldots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞.

← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = \lim_{N\to\infty} S - S_N = 0$.

← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion)**.** $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left|\sum_{n=j-1}^{k} a_n\right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test)**.** Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\to \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x)\, \mathrm{d}x$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.

← do an improper integral!

Moreover, $\int_1^N f(x)\,\mathrm{d}x \le a_1 + \cdots + a_N \le a_1 + \int_1^N f(x)\,\mathrm{d}x$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then

$\leftarrow a_n = \frac{1}{n}$

$$\ln N \le 1 + \frac{1}{2} + \cdots + \frac{1}{N} \le 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

Theorem 14. The "p-series" $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } \operatorname{Re}(s) > 1$$

Remark. Euler figured out:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979):

$\leftarrow$ still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202\ldots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. $\leftarrow$ Good
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. $\leftarrow$ BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series

$\leftarrow$ Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges to $\ln 2$. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection, then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

Theorem 17 (Riemann Rearrangement Theorem)**.** If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\begin{aligned}\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See notes.

Uniform convergence

Definition 14. A sequence of functions $f_n : X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \to \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!

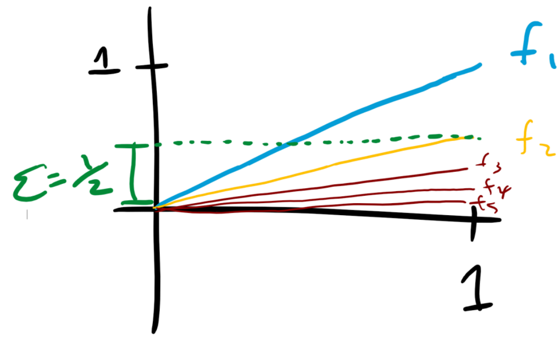


FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \to \mathbb{C}$ are continuous and converges uniformly on X to $f : X \to \mathbb{C}$, then f is continuous on X. In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.

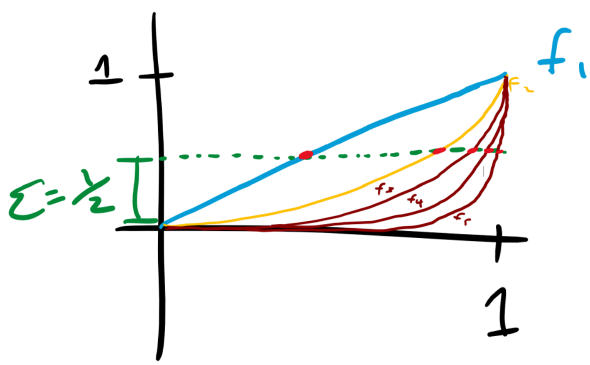


FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a, b] \to \mathbb{C}$ are continuous and converge uniformly on $[a, b]$ to f, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\, \mathrm{d}\, x = \int_a^b f(x)\, \mathrm{d}\, x$$

← Integrals work with uniform convergence

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1, 1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test)**.** Let $f_n : X \to \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X.

Power series

Definition 15. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. The a_n is the *coefficient* and z_0 is the *center.*

Convergence of geometric series

Theorem 21. The *geometric series* ($a_n = 1, z_0 = 0$) $\sum_{n=0}^{\infty} z^n$ converges absolutely to $\frac{1}{1-z}$ if $|z| < 1$, and it diverges otherwise.

Moreover, for each $r \in [0, 1[$, the convergence is **uniform** on $|z| \leq r$.

Proof. If $|z| \geq 1$, then $z^n \not\to 0$, so by the test of divergence, the series diverges.

Now suppose $|z| < 1$. Then

← The fact that we can find a formula for this sum is quite rare!

$$\begin{aligned}\sum_{n=0}^{\infty} z^n &= \lim_{N\to\infty} \sum_{n=0}^{N-1} z^n \\ &= \lim_{N\to\infty} (1 + z + z^2 + \cdots + z^{N-1}) \\ &= \lim_{N\to\infty} \frac{1-z^N}{1-z} \\ &= \frac{1}{1-z} && \text{since } |z| < 1\end{aligned}$$

Which gives us point-wise convergence. Then, for any r such that $|z| \leq r < 1$, we have

$$\sum_{n=0}^{\infty} |z^n| \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

Hence, by the Weierstrass M-test, the series converges *absolutely and uniformly* on $|z| \leq r$. □

Remark. Moral of the story:

- The *radius of convergence* $R = 1$ has the property that the series converges on $|z| < R$, and diverges if $|z| > R$.
- The series converges *uniformly* on $|z| \leq r < 1$ but not on $|z| < 1$ itself. Why? Let $r = 1$; we need be able to get $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|\frac{1-z^N}{1-z} - \frac{1}{1-z}\right| < 1$ for all $|z| < 1$. However, this is not gonna work: as $z \to 1$, observe that this is going to eventually exceed 1.

- The limit function $\dfrac{1}{1-z}$ is **analytic** on $\mathbb{C}\backslash\{1\}$. But the geometric series represents this function only on $|z|<1$. In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.

← the limit function is well-defined way beyond the $\mathbb{D}$!

- The limit function $\dfrac{1}{1-z}$ is cool if $z \neq 1$, but as long as $|z|=1$ (**even** if $z \neq 1$), the geometric series diverges!

← in the complex number sense!

Radius of convergence

Definition 16. The **limit superior** (lim sup) of a sequence of nonnegative real numbers x_n is the largest *limit point* of the x_n:

← limits of a subsequence of x_n

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0}\sup_{m\geq n} x_m$$

If the sequence is unbounded, the lim sup would be ∞.

← the RHS as in real analysis

Example 16. If x_n is the sequence $0,1,0,1,\ldots$ then $\limsup\limits_{n\to\infty} x_n = 1$.

Example 17. If x_n is the sequence $0,1,0,\frac{1}{2},0,\frac{1}{3},\ldots$, then $\limsup\limits_{n\to\infty} x_n = 0$.

Remark. If x_n are nonnegative, then

- $\limsup\limits_{n\to\infty}(a_n+b_n) = \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$
- $\limsup\limits_{n\to\infty}(a_n b_n) \leq (\limsup\limits_{n\to\infty} a_n)(\limsup\limits_{n\to\infty} b_n)$

Theorem 22 (Cauchy-Hadamard)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Define $R\in[0,\infty]$ by

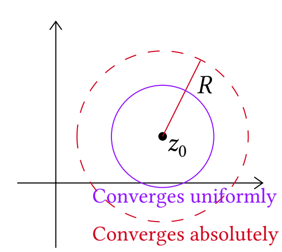
← interpret $\frac{1}{0}=\infty$

$$\frac{1}{R} = \limsup_{n\to\infty}\sqrt[n]{|a_n|}$$

Then the R is the *radius of convergence*.

(a) On $|z-z_0|<R$, the series converges **absolutely**. For each $r\in[0,R[$, the convergence is **uniform** on $|z-z_0|\leq r$.

←



(b) If $|z-z_0|>R$ then the series diverges. For $|z-z_0|=R$ anything could happen!

Example 18. We claim that $\exp(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ has an infinite radius of convergence $R=\infty$. To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \to 0$$

This is because $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \dots \cdot n}$, and in $n!$, there are at least $\frac{1}{2}$ terms that are $> \frac{n}{2}$. Thus, $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{1/2} \to \infty$.

So $R = \infty$ and we are done ☺. We have that $\exp(z)$ has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k}}_{\text{binomial theorem}} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\
&= \exp(z+w)
\end{aligned}$$

Now define $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Term-by-term differentiation of power series

Lemma 23. $n^{\frac{1}{n}} \to 1$

Proof 1. $e^{\log(n^{\frac{1}{n}})} = e^{\frac{\log n}{n}} \to e^0 = 1$ by l'Hopital. So $n^{\frac{1}{n}} \to 1$. □

Proof 2 (better). Write $n^{\frac{1}{n}} = 1 + \delta_n$ where $\delta_n \geq 0$. The binomial theorem says:

$$\begin{aligned}
n &= (1+\delta_n)^n \\
&= \sum_{k=0}^{\infty} \binom{n}{k} \delta_n^k \cdot 1^{n-k} \\
&= 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \dots
\end{aligned}$$

$$\geq 1 + \frac{n(n-1)}{2}\delta_n^2$$

Therefore, $n - 1 \geq \frac{n(n-1)}{2}\delta_n^2$ and we get $\frac{2}{n} \geq \delta_n^2 \geq 0$ hence $\delta_n \to 0$.

Hence $n^{\frac{1}{n}} \to 1$. □

Theorem 24. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence R, then

$$f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$$

for $|z - z_0| < R$. Moreover, the new series also has a radius of convergence R.

Proof. WLOG $R > 0$ and $z_0 = 0$.

← we just translate it; also $R = 0$ isn't that meaningful

For $|z| < R$ we write:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

← just splitting the function into two parts

and the 'new series'

$$g(z) = \sum_{n=0}^{\infty} na_n z^{n-1} = \lim_{N\to\infty} S_N'(z)$$

We first prove that the radius of convergence for g is the same as f. By Cauchy-Hadamard:

$$\begin{aligned}\frac{1}{R_g} &= \limsup_{n\to\infty} \sqrt[n]{n|a_n|} \\ &= \limsup_{n\to\infty} (n^{\frac{1}{n}})\sqrt[n]{|a_n|} && \text{by the previous lemma,} \\ &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \frac{1}{R}\end{aligned}$$

Thus, $R_g = R$ by Cauchy-Hadamard.

Next, we need to show that $f' = g$ with $|z| < R$.

Fix $0 \leq |w| < R$ and $\varepsilon > 0$. We want a $\delta > 0$ such that whenever $|z - w| < \delta$, we have $\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \varepsilon$.

← just saying that the derivative at any w gets close to $g(w)$

We rewrite:

$$\begin{aligned}
\left|\frac{f(z)-f(w)}{z-w}-g(w)\right| &= \left|\frac{[S_N(z)+R_N(z)]-[S_N(w)+R_N(w)]}{z-w}-g(w)\right| \\
&= \left|\frac{S_N(z)-S_N(w)}{z-w}+\frac{R_N(z)-R_N(w)}{z-w}+S'_N(w)-S'_N(w)-g(w)\right| \\
&\leq \left|S'_N(w)-g(w)\right|+\left|\frac{R_N(z)-R_N(w)}{z-w}\right|+\left|\frac{S_N(z)-S_N(w)}{z-w}-S'_N(w)\right|
\end{aligned}$$

- **1st term**: by def of g and $g(z)=\lim_{N\to\infty}S'_N(z)$, we can always find some $N_1\in\mathbb{N}$ such that any $N\geq N_1$ gives us $\left|S'_N(w)-g(w)\right|<\frac{\varepsilon}{3}$.
- **2nd term**: since $|w|<R$, there is an r such that $|w|<r<R$. For $|z|<r$, we have

  ← work on a smaller disk

$$\begin{aligned}
\left|\frac{R_N(z)-R_N(w)}{z-w}\right| &= \frac{1}{|z-w|}\left|\sum_{n=N}^{\infty}a_nz^n=-\sum_{n=N}^{\infty}a_nw^n\right| \\
&\leq \sum_{n=N}^{\infty}|a_n|\left|\frac{z^n-w^n}{z-w}\right| \\
&= \sum_{n=N}^{\infty}|a_n|\left|\frac{z^n}{z}\cdot\frac{1-\frac{w^n}{z^n}}{1-\frac{w}{z}}\right| && \text{by geometric sequence} \\
&= \sum_{n=N}^{\infty}|a_n|\left|\frac{z^n}{z}\cdot\left(1+\left(\frac{w}{z}\right)+\left(\frac{w}{z}\right)^2+\cdots+\left(\frac{w}{z}\right)^{n-1}\right)\right| \\
&= \sum_{n=N}^{\infty}|a_n|\left|z^{n-1}+z^{n-2}w+\cdots+zw^{n-2}+w^{n-1}\right| \\
&\leq \sum_{n=N}^{\infty}|a_n|\cdot n\cdot r^{n-1}\text{by } |z|,|w|<r<R
\end{aligned}$$

  Thus, there exists an $N_2\in\mathbb{N}$ such that any $N\geq N_2$ gives us

$$\left|\frac{R_N(z)-R_N(w)}{z-w}\right|<\frac{\varepsilon}{3}$$

- **3rd term**: let $N=\max\{N_1,N_2\}$. The definition of $S'_N(w)$ provides $\gamma>0$ such that if $|z-w|<\gamma$, then we have $\left|\frac{S_N(z)-S_N(w)}{z-w}-S'_N(w)\right|<\frac{\varepsilon}{3}$.

  ← review def of derivatives!

Now if $0<\delta<\min\{\gamma,r-|w|\}$, then the 3 terms above are all $<\frac{\varepsilon}{3}$. Hence, $\left|\frac{f(z)-f(w)}{z-w}-g(w)\right|<\varepsilon$ holds for this δ. □

Corollary 25. A power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ with $R > 0$ is infinitely differentiable on $|z - z_0| < R$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

are the coefficients of the terms of the power series.

← prove by keep taking derivatives!

Corollary 26. Power series expansions are unique. That is, if $r > 0$ and

← because there is a unique formula for coeffs.

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

on $|z - z_0| < r$, then $a_n = b_n$ for $n \geq 0$.

Remark. Recall that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has a radius of convergence ∞ (it's an *entire* function). Now, if we differentiate it term-by-term:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} \exp(z) &= \frac{\mathrm{d}}{\mathrm{d}z} \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{z^{n-1}}{(n-1)!} && \text{let } k = n - 1 \\
&= \sum_{k=0}^{\infty} \frac{z^k}{k!} \\
&= \exp(z)
\end{aligned}$$

Thus, the derivative of $\exp(z)$ is itself! Moreover, $\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Remark. We claim that $\exp(z) = e^z$.

Since $e^z e^{c-z}$ is a constant for all constant c, z, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(e^z e^{c-z}) = 0$$

to recover the constant $e^z e^{c-z}$, we let $z = 0$, giving us

$$e^z e^{c-z} = e^c$$

which is the addition formula!

Therefore,

$$\begin{aligned}
\exp(n) &= \exp(1 + 1 + \cdots + 1) \\
&= exp(1)^n \\
&= e^n
\end{aligned}$$

Elementary functions

Now that we have derived e, we could use it to derive sin and cos:

Definition 17.

$$\begin{aligned}\cos(z) &= \frac{e^{iz}+e^{-iz}}{2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}\end{aligned}$$

$$\begin{aligned}\sin(z) &= \frac{e^{iz}-e^{-iz}}{2i} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}\end{aligned}$$

We observe that we have the following property:

- Radius of convergence $R = \infty$
- $(\cos z)' = -\sin z, (\sin z)' = \cos z$
- $\cos x = \operatorname{Re}(e^{ix}), \sin x = \operatorname{Im} e^{ix}$ for all $x \in \mathbb{R}$
- $\cos(-z) = \cos z, \sin(-z) = -\sin z$
- $\cosh x = \frac{e^x+e^{-x}}{2}$ so $\cosh(ix) = \cos x$
- $e^{iz} = \cos z + i \sin z$
- $$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 \\ &= \frac{1}{4}(e^{2iz}+2+e^{-2iz}) - \frac{1}{4}(e^{2iz}-2+e^{-2iz}) \\ &= 1 \qquad \forall z \in \mathbb{C}\end{aligned}$$
- $$\begin{aligned}\cos^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 \\ &= \frac{1}{4}(e^{2iz}+2+e^{-2iz}) \\ &= \frac{1}{2} + \frac{e^{2iz}+e^{-2iz}}{4} \\ &= \frac{1}{2}(1+\cos 2z)\end{aligned}$$

- If $x \in \mathbb{R}$ then $\cos x, \sin x$ are real. We get $|\sin x|, |\cos x| \leq 1$.

Definition 18. $f : \mathbb{C} \to \mathbb{C}$ is **periodic** with a *period* ω if $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$.

Theorem 27. There exists a positive real number π such that:

(a) $\cos z, \sin z$ have period 2π

(b) e^z is periodic with period $2\pi i$

(c) π is the area of the unit circle

Proof. By Euler's formula, it suffices to consider e^{iz} only. If ω is a period of e^{iz}, then

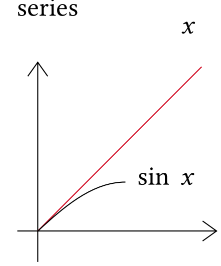
$$e^{iz} = e^{i(z+\omega)} = e^{iz}e^{i\omega}$$

which only happens if $e^{i\omega} = 1$. Conversely, if $e^{i\omega} = 1$, then $e^{i(z+\omega)} = e^{iz}$.

Hence, ω is a period of e^{iz} *if and only if* $e^{iw} = 1$. □

Proposition 28. $\sin x \leq x$ for all $x \geq 0$.

← This is the first term in the power series



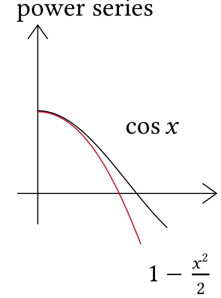
Proof. Since $|\cos t| \leq 1$,

$$\begin{aligned} x - \sin x &= (x - \sin x) - (0 - \sin 0) \\ &= \int_0^x \underbrace{1 - \cos t}_{\geq 0}\, \mathrm{d}t \qquad \text{by FTC} \\ &\geq 0 \end{aligned}$$

□

Proposition 29. In addition, $\cos x \geq 1 - \frac{x^2}{2}$ for $x \geq 0$.

← These are the first 2 terms in the power series



Proof. The previous prop gives:

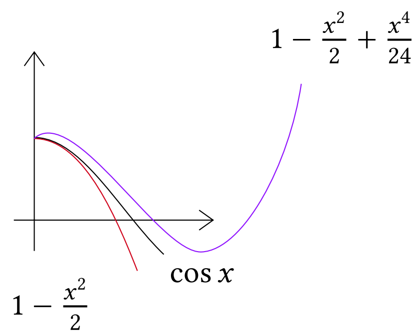
$$\begin{aligned} \cos x - 1 &= \cos x - \cos 0 \\ &= \int_0^x -\sin t\, \mathrm{d}t \\ &\geq \int_0^x -t\, \mathrm{d}t \\ &= \frac{-x^2}{2} \end{aligned}$$

□

Proposition 30. Furthermore, for $x \geq 0$:

- $\sin x \geq x^3 - \frac{x^3}{6}$
- $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

Proposition 31. There exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$.



Proof. By the previous prop, we have $\cos\sqrt{3} \leq 1 - \frac{\sqrt{3}^2}{2} + \frac{\sqrt{3}^4}{24} = \frac{1}{8} < 0$. Moreover, $\cos 0 = 1 > 0$, by IVT, there exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$. □

Proposition 32. $\omega_0 = 4x_0$ is a period of e^{iz}.

Proof. Since $\cos x_0 = 0$, we have $\sin x_0 = \pm 1$. Then $e^{ix_0} = \pm i$. We have $(\pm i)^4 = 1$, so $e^{4ix_0} = 1 = e^0$, so $\omega_0 = 4x_0$ is a period of e^{iz}. □

Proposition 33. ω_0 is the *smallest* positive period of e^{iz}.

Proposition 34. All periods of e^{iz} are integer multiples of $2\pi = 4x_0$.

Proof. Define $\pi = 2x_0$. The area of unit circle is

$$\begin{aligned} 4\int_0^1 \sqrt{1-x^2}\,\mathrm{d}\,x &= 4\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,\mathrm{d}\,\theta \\ &= 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\,\theta \\ &= \pi \end{aligned}$$

□

Complex logarithm

We know: $e^0 = 1, e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718\ldots$

Since $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, it is positive. If $x > 0$, we conclude that e^x is strictly increasing! As $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1 + x$, so $\lim_{x\to\infty} e^x = \infty$,

Therefore, e^x is a **bijection** from $\mathbb{R}$ to $(0, \infty)$. This means it has an inverse that is a bijection from $(0, \infty)$ to $\mathbb{R}$.

Definition 19. $\ln x$ is the inverse of e^x for $x \in (0, +\infty)$.

Now what about the complex case? Let $z \neq 0$ and $z = re^{i\theta}$ where $r = |z| > 0$ and $\theta = \arg z \in \mathbb{R}$.

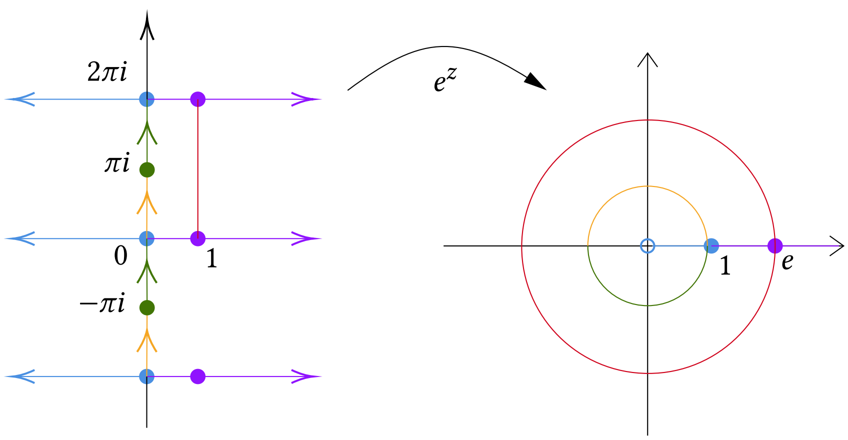
← cf. trig properties

← Only determined up to addition of multiples of 2π

Hence, $z = re^{i\theta} = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta}$. However, the θ is ambiguous to addition of multiples of 2π!

Definition 20. If $z \neq 0$, a **logarithm** of z is a $w \in \mathbb{C}$ such that $e^w = z$.

We could graph the function e^z with $z \in \mathbb{C}$:



Definition 21. If Ω is a region in $\mathbb{C}$, then a continuous $l : \Omega \to \mathbb{C}$ is a **branch** of the logarithm if $e^{l(z)} = z$ for all $z \in \Omega$.

← note $0 \notin \Omega$

Example 19. If $\Omega = \mathbb{C} \backslash (-\infty, 0]$ such that $\theta \in (-\pi, \pi)$, a logarithm could be defined on it. This is the **principal branch** of the logarithm.

← See graphed Riemann surface

Remark. Suppose $l(z)$ is a branch of the logarithm and l is analytic, then:

$$e^{l(z)} = z \implies \frac{\mathrm{d}}{\mathrm{d}z} e^{l(z)} = l'(z) e^{l(z)} = 1$$

Since $e^{l(z)} = z$, we conclude $l'(z) = \frac{1}{z}$.

Complex power

Definition 22. If $z \neq 0$, *define* $z^a = e^{a \log z}$. ← NOT well-defined!

Remark. The definition of complex powers should coincide with the old one: $z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n} = r^n e^{in\theta}$.

Check:

$$\begin{aligned} z^n &= e^{n \log z} = e^{n(\ln r + i\theta + i2\pi k)} \\ &= e^{n \ln r} e^{in\theta} \underbrace{e^{i2\pi nk}}_{=1} \\ &= r^n e^{in\theta} \end{aligned}$$

is true for any $k \in \mathbb{Z}$.

How about n-th roots?

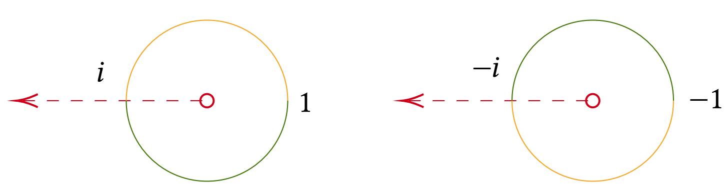
$$\begin{aligned} z^{\frac{1}{n}} &= e^{\frac{1}{n} \log z} \\ &= e^{\frac{1}{n}(\ln r + i\theta + i2\pi k)} \\ &= e^{\frac{1}{n} \ln r} e^{\frac{i\theta}{n}} \underbrace{e^{\frac{i2\pi k}{n}}}_{n \text{ distinct}} \\ &= r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \end{aligned}$$

Riemann surface

We still have a problem: $\ln z$ is still not a function on $\mathbb{C}$! The branch depends on the arbitrary choice of domain. What shall we do to make it not dependent on a choice?

Answer: let ln not live on the complex plane, but infinitely many copies of the slit plane $\mathbb{C} \backslash (-\infty, 0]$, each one being glued to the next along the slit $(-\infty, 0]$.

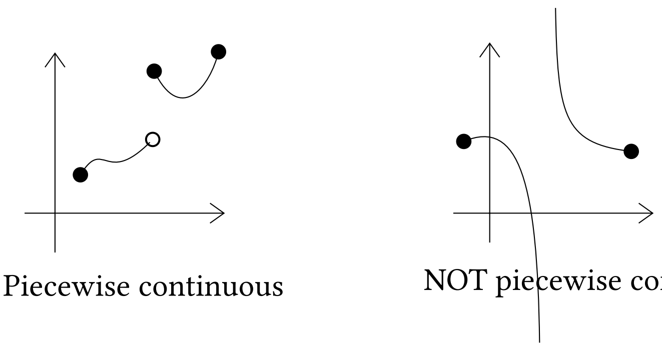
Example 20. $z^{1/2}$ would live on a surface:



Cauchy's theorem and its consequences

Complex integration

Definition 23. A complex-valued function $\gamma : [a,b] \to \mathbb{C}$ is **piecewise** continuous if γ is continuous at all but *finitely many* points of $[a,b]$ and γ has one-sided limits that are *finite* at each point (of discontinuity).



Piecewise continuous NOT piecewise continuous

If γ is piecewise continuous, then $\int_a^b \operatorname{Re}\gamma(t)\,\mathrm{d}t$ and $\int_a^b \operatorname{Im}\gamma(t)\,\mathrm{d}t$ exist. Then we define **complex integration**:

$$\int_a^b \gamma(t)\,\mathrm{d}t = \int_a^b \operatorname{Re}\gamma(t)\,\mathrm{d}t + i \cdot \int_a^b \operatorname{Im}\gamma(t)\,\mathrm{d}t$$

That is,

$$\operatorname{Re}\left(\int_a^b \gamma(t)\,\mathrm{d}t\right) = \int_a^b \operatorname{Re}\gamma(t)\,\mathrm{d}t$$
$$\operatorname{Im}\left(\int_a^b \gamma(t)\,\mathrm{d}t\right) = \int_a^b \operatorname{Im}\gamma(t)\,\mathrm{d}t$$

In addition, if γ_1, γ_2 are both $[a,b] \to \mathbb{C}$ and piecewise cont., and $c_1, c_2 \in \mathbb{C}$, then

$$\int_a^b (c_1\gamma_1(t) + c_2\gamma_2(t))\,\mathrm{d}t = c_1 \int_a^b \gamma_1(t)\,\mathrm{d}t + c_2 \int_a^b \gamma_2(t)\,\mathrm{d}t$$

Proposition 35 (Triangle inequality). If $\gamma : [a,b] \to \mathbb{C}$ is piecewise continuous, then

$$\left|\int_a^b \gamma(t)\,\mathrm{d}t\right| \le \int_a^b |\gamma(t)|\,\mathrm{d}t$$

Proof. WLOG assume $\int_a^b \gamma(t)\,\mathrm{d}t \neq 0$. Define $\lambda = \frac{\left|\int_a^b \gamma(t)\,\mathrm{d}t\right|}{\int_a^b \gamma(t)\,\mathrm{d}t}$ and note $|\lambda| = 1$.

Thus,

$$
\begin{aligned}
\left|\int_a^b \gamma(t)\,\mathrm{d}t\right| &= \lambda \int_a^b \gamma(t)\,\mathrm{d}t \\
&= \int_a^b \lambda\gamma(t)\,\mathrm{d}t && \text{because LHS is } \in \mathbb{R} \\
&= \operatorname{Re} \int_a^b \lambda\gamma(t)\,\mathrm{d}t \\
&\leq \int_a^b |\lambda\gamma(t)|\,\mathrm{d}t && \because \operatorname{Re} z \leq |z| \\
&= \int_a^b |\gamma(t)|\,\mathrm{d}t && \because |\lambda| = 1
\end{aligned}
$$

□

Complex differentiability

Definition 24. $\gamma : [a, b] \to \mathbb{C}$ is **differentiable** at $t \in [a, b]$ if $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are differentiable (in the sense of real variables). We define

$$\gamma'(t) = (\operatorname{Re}\gamma)'(t) + i \cdot (\operatorname{Im}\gamma)'(t)$$

Definition 25. $\gamma : [a, b] \to \mathbb{C}$ is **piecewise C^1** if:

← C^1 is one-time differentiable

(a) γ is continuous on $[a, b]$.
(b) γ is differentiable at all but finitely many points of $[a, b]$.
(c) γ' is continuous at each point where it exists.
(d) γ' has finite one-sided limits at every point of discontinuity.

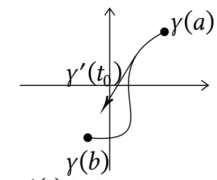
Fundamental theorem of calculus, complex edition

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise C^1, then:

$$\int_a^b \gamma'(t)\,\mathrm{d}t = \gamma(b) - \gamma(a)$$

Definition 26. If γ is C^1, then the arclength of γ is:

$$L(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}\, t$$



$\gamma'(t)$ is instantaneous velocity, so its absolute value is the speed

Definition 27. If $\gamma : [a, b] \to \Omega$ is piecewise C^1 and $f : \Omega \to \mathbb{C}$ is continuous, then

$$\int_\gamma f(z) \, \mathrm{d}\, z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}\, t$$

where $z = \gamma(t)$ and $\mathrm{d}\, z = \gamma'(t) \, \mathrm{d}\, t$

We have **linearity** w.r.t. f:

$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z)) \, \mathrm{d}\, z = c_1 \int_\gamma f_1(z) \, \mathrm{d}\, z + c_2 \int_\gamma f_2(z) \, \mathrm{d}\, z$$

Remark. Arclength is independent from parameterization.

Proof. Let $\gamma : [a, b] \to \Omega$ be piecewise C^1. Let $\alpha : [c, d] \to [a, b]$ is an increasing, piecewise C^1 surjection such that $\alpha(c) = a, \alpha(d) = b$. Then $\phi = \gamma \circ \alpha : [c, d] \to \Omega$ is also piecewise C^1. Hence, by substituting $s = \alpha(t), \mathrm{d}\, s = \alpha'(t) \, \mathrm{d}\, t$:

$$\begin{aligned}
\int_\phi f(z) \, \mathrm{d}\, z &= \int_c^d f(\phi(t))\phi'(t) \, \mathrm{d}\, t \\
&= \int_c^d f(\gamma(\alpha(t)))\gamma'(\alpha(t))\alpha'(t) \, \mathrm{d}\, t \\
&= \int_a^b f(\gamma(s))\gamma'(s) \, \mathrm{d}\, s \\
&= \int_\gamma f(z) \, \mathrm{d}\, z
\end{aligned}$$

□

An important estimate

Let f be continuous. Since $\int_\gamma f(z) \, \mathrm{d}\, z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}\, t$, we observe:

$$\begin{aligned}
\left| \int_\gamma f(z) \, \mathrm{d}\, z \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}\, t \right| \\
&\leq \int_a^b |f(\gamma(t))| \, |\gamma'(t)| \, \mathrm{d}\, t
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)| \, \mathrm{d}\, t \\
&= \max_{z\in\gamma} |f(z)| \cdot L(\gamma)
\end{aligned}
$$

Definition 28. If $\gamma : [a, b] \to \mathbb{C}$, the reverse of γ is $(-\gamma) : [-b, -a] \to \mathbb{C}$ defined by $(-\gamma)(t) = \gamma(-t)$. Hence,

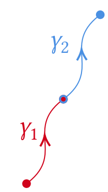
← going around the track backwards

$$\int_{-\gamma} f(z) \, \mathrm{d}\, z = -\int_{\gamma} f(z) \, \mathrm{d}\, z$$

Remark. We can also break up the curve and integral the two parts separately:

$$\int_{\gamma} f(z) \, \mathrm{d}\, z = \int_{\gamma_1} f(z) \, \mathrm{d}\, z + \int_{\gamma_2} f(z) \, \mathrm{d}\, z$$

←



Fundamental theorem of calculus for contour integrals

If $\gamma : [a, b] \to \mathbb{C}$ is piecewise C^1, and $f : \Omega \to \mathbb{C}$ is analytic , then

← Assuming f' continuous, which we would prove later

$$\int_{\gamma} f'(z) \, \mathrm{d}\, z = f(\gamma(b)) - f(\gamma(a))$$

If $\gamma(a) = \gamma(b)$, then $\int_{\gamma} f'(z) \, \mathrm{d}\, z = 0$.

Proof.

$$
\begin{aligned}
\int_{\gamma} f'(z) \, \mathrm{d}\, z &= \int_a^b f'(\gamma(t))\gamma'(t) \, \mathrm{d}\, t \\
&= \int_a^b (f \circ \gamma)'(t) \, \mathrm{d}\, t && \text{chain rule} \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}
$$

□

Example 21. Let γ be a circle of radius R centered at z_0: $\gamma(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$. We would like to find $\int_{\gamma} (z - z_0)^n \, \mathrm{d}\, z$.

If $n \neq -1$, then $\left(\frac{(z-z_0)^{n+1}}{n+1}\right)' = (z - z_0)^n$. Thus,

$$\int_{\gamma} (z - z_0)^n \, \mathrm{d}\, z = \int_{\gamma} \left(\frac{(z - z_0)^{n+1}}{n + 1}\right)' \mathrm{d}\, z = 0$$

by FTC.

If $n = -1$,

$$\int_\gamma (z - z_0)^n \, \mathrm{d}z = \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z = \int_0^{2\pi} i \, \mathrm{d}t = 2\pi i$$

Cauchy's theorem

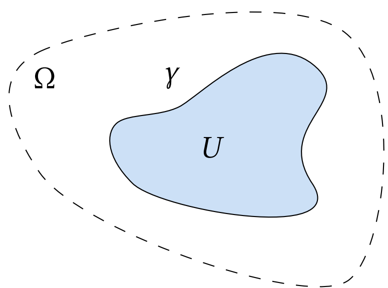
Take 1

Theorem 36 (Cauchy's)**.** Let Ω be a region in $\mathbb{C}$ containing a *simple* piecewise C^1 *closed* curve γ and its interior.

← does not self-intersect

← holes not allowed in the interior

If $f : \Omega \to \mathbb{C}$ is analytic, then $\int_\gamma f(z) \, \mathrm{d}z = 0$.



"Proof". Let U be the union of γ and its interior. Let $f = u + iv$ as usual, write $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$:

$$\begin{aligned}
\int_\gamma f(z) \, \mathrm{d}z &= \int_\gamma (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i \int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y \\
&= \int\int_U (-v_x - u_y)\,\mathrm{d}x\,\mathrm{d}y + i \int\int_U (u_x - v_y)\,\mathrm{d}x\,\mathrm{d}y && \text{by Green's thm} \\
&= 0 && \text{by Cauchy-Riemann}
\end{aligned}$$

□

However, this 'proof' heavily relies on the fact that u, v are C^1 and that the partial derivatives are continuous. This assumes f' is continuous, but we aren't sure about that yet!

← See Goursat's Lemma

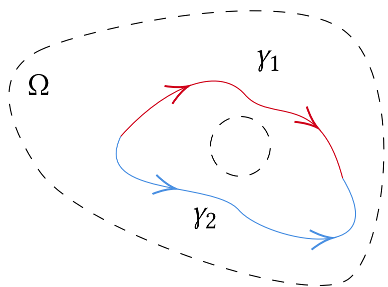
Take 2: deformation version

Theorem 37 (Cauchy's)**.** Let γ_1, γ_2 be piecewise C^1 curves in a region Ω with the same start and end points. If γ_1 can be continuously deformed to γ_2 without ever passing outside of Ω, then

$$\int_{\gamma_1} f(z)\,\mathrm{d}\,z = \int_{\gamma_2} f(z)\,\mathrm{d}\,z$$

By the *previous* statement of Cauchy's theorem (in Theorem 36), we observe that $\int_{\gamma_1 - \gamma_2} f(z)\,\mathrm{d}\,z = 0$, so this one falls out.

Non-example 22. The γ_1, γ_2 in the picture below cannot be continuously deformed into each other!



Fresnel integrals

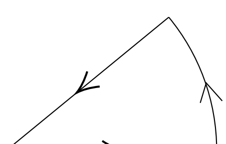
Consider:

$$\int_0^\infty \sin(t^2)\,\mathrm{d}\,t \qquad \text{and} \qquad \int_0^\infty \cos(t^2)\,\mathrm{d}\,t$$

aka.

$$\lim_{R\to\infty} \int_0^R \sin(t^2)\,\mathrm{d}\,t \qquad \text{and} \qquad \lim_{R\to\infty} \int_0^R \cos(t^2)\,\mathrm{d}\,t$$

It's not obvious that these integrals converge!

Solution: PIZZA!  Let γ be the 'sum' of all 3 curves as shown. Let $R \to \infty$. Then, by Cauchy's theorem, $\int_\gamma e^{iz^2}\,\mathrm{d}\,z = 0$.

(Scratch work begins)

Remark. We don't know how to write out the antiderivative of $f(z) = e^{iz^2}$ but we can use series!

$$\begin{aligned} f(z) &= e^{iz^2} \\ &= \sum_{n=0}^{\infty} \frac{(iz^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n z^{2n}}{n!} \end{aligned}$$

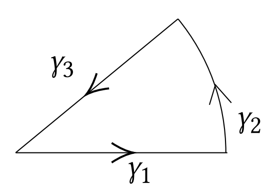
And so

$$F(z) = \sum_{n=0}^{\infty} \frac{i^n z^{2n+1}}{(2n+1)n!}$$

(Scratch ends here)

Now we return to the integral. Strategy:

$$0 = \int_{\gamma} e^{iz^2} \,\mathrm{d} = \underbrace{\int_{\gamma_1} e^{iz^2} \,\mathrm{d}\,z}_{I_1(R)} + \underbrace{\int_{\gamma_2} e^{iz^2} \,\mathrm{d}\,z}_{I_2(R)} + \underbrace{\int_{\gamma_3} e^{iz^2} \,\mathrm{d}\,z}_{I_3(R)}$$



Evaluate $I_1(R)$: We observe that z is real for this one. Parameterize $z = t$ where t is a real variable.

$$\begin{aligned} I_1(R) &= \int_{\gamma_1} e^{it^2} \,\mathrm{d}\,t \\ &= \int_0^R \cos(t^2) \,\mathrm{d}\,t + i \cdot \int_0^R \sin(t^2) \,\mathrm{d}\,t \end{aligned}$$

Hence, $\lim_{R\to\infty} I_1(R) = \int_0^{\infty} \cos(t^2) \,\mathrm{d}\,t + i \cdot \int_0^{\infty} \sin(t^2) \,\mathrm{d}\,t$.

Evaluate $I_2(R)$:

Parameterize γ_2 as $z = Re^{i\theta}$ where $\theta \in [0, \frac{\pi}{4}]$. Hence, $\mathrm{d}\,z = iRe^{i\theta}\,\mathrm{d}\,\theta$. Then:

$$\begin{aligned}
|I_2(R)| &= \left|\int_{\gamma_2} e^{i\theta^2}\,\mathrm{d}\,\theta\right| \\
&= \left|\int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} iRe^{i\theta}\,\mathrm{d}\,\theta\right| \\
&= \left|R\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}} e^{i\theta}\,\mathrm{d}\,\theta\right| \\
&\leq R\int_0^{\frac{\pi}{4}} \left|e^{iR^2e^{i2\theta}}\right|\mathrm{d}\,\theta && \text{by tri. ineq.} \\
&\leq R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\,\mathrm{d}\,\theta && \text{since when } x, y \in \mathbb{R},\ \left|e^{x+iy}\right| = e^x \\
&\leq R\int_0^{\frac{\pi}{4}} e^{-R^2\frac{4\theta}{\pi}}\,\mathrm{d}\,\theta && \text{since when } x \in [0, \frac{\pi}{2}],\ \frac{2}{\pi}x \leq \sin x \\
&= \frac{-R\pi}{R^2 4} e^{-R\frac{4\theta}{\pi}}\Bigg|_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

Thus, $\lim_{R\to\infty} I_2(R) = 0$. :)

Evaluate $I_3(R)$:

$$\begin{aligned}
I_3(R) &= \int_{\gamma_3} e^{iz^2}\,\mathrm{d}\,z \\
&= \int_R^0 e^{i(e^{i\frac{\pi}{4}}t)^2} e^{i\frac{\pi}{4}}\,\mathrm{d}\,t \\
&= -e^{i\frac{\pi}{4}}\int_0^R e^{-t^2}\,\mathrm{d}\,t \\
\lim_{R\to\infty} I_3(R) &= -(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\int_0^\infty e^{-t^2}\,\mathrm{d}\,t \quad \text{by Gaussian integral, } \int_0^\infty e^{-t^2}\,\mathrm{d}\,t = \frac{\sqrt{\pi}}{2} \\
&= -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}
\end{aligned}$$

Therefore, we see $I_1(R) + I_2(R) + I_3(R) = 0$ where $\lim_{R\to\infty} I_1(R) = \int_0^\infty \cos(t^2)\,\mathrm{d}\,t + i\cdot\int_0^\infty \sin(t^2)\,\mathrm{d}\,t$, $I_2(R) \to 0$ and $I_3(R) = -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}$. Hence, we would be able to conclude that

$$\int_0^\infty \sin(t^2)\,\mathrm{d}\,t = \sqrt{\frac{\pi}{8}} \qquad \text{and} \qquad \int_0^\infty \cos(t^2)\,\mathrm{d}\,t = \sqrt{\frac{\pi}{8}}$$

Goursat's lemma

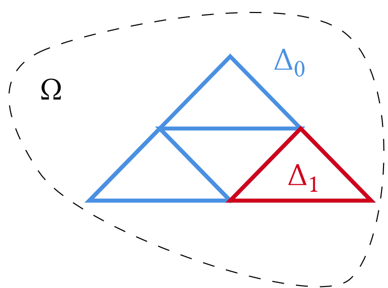
This lemma patches the hole that we have to assume f' continuous in Cauchy's theorem!

Lemma 38 (Goursat's)**.** If $f : \Omega \to \mathbb{C}$ is analytic and Δ is a triangle in Ω whose interior lies inside Ω, then $\int_\Delta f(z)\,\mathrm{d}z = 0$.

← Does not assume f' continuous!

Proof. WLOG orient $\Delta_0 = \Delta$ counterclockwise. Bisect sides of Δ_0 and construct smaller triangles Δ_{0j} where $j = 1, 2, 3, 4$. Then,

$$I = \int_{\Delta_0} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\Delta_{0j}} f(z)\,\mathrm{d}z$$



By triangle inequality,

$$|I| \leq \sum_{j=1}^{4} \left| \int_{\Delta_{0j}} f(z)\,\mathrm{d}z \right|$$

Thus, there exists $j \in \{1, 2, 3, 4\}$ such that

$$\frac{|I|}{4} \leq \left| \int_{\Delta_{0j}} f(z)\,\mathrm{d}z \right|$$

For this j, define $\Delta_1 = \Delta_{0j}$.

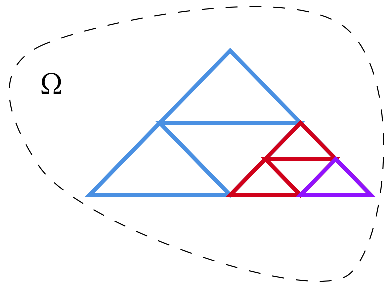
We disect Δ_1 again into smaller triangles Δ_{1j} where $j = 1, 2, 3, 4$. Then,

$$I = \int_{\Delta_1} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\Delta_{1j}} f(z)\,\mathrm{d}z$$

Again, by triangle inequality, there is a $j \in \{1, 2, 3, 4\}$ such that

$$\frac{|I|}{4^2} \leq \frac{1}{4} \left| \int_{\Delta_1} f(z)\,\mathrm{d}z \right| \leq \left| \int_{\Delta_{1j}} f(z)\,\mathrm{d}z \right|$$

For this j, define $\Delta_2 = \Delta_{1j}$.



...continue in this manner to get nested triangles Δ_n such that

$$\frac{|I|}{4^{n+1}} \leq \frac{1}{4}\left|\int_{\Delta_n} f(z)\,\mathrm{d}\,z\right| \leq \left|\int_{\Delta_{nj}} f(z)\,\mathrm{d}\,z\right|$$

for all $n \geq 0$.

Now let $\ell = L(\Delta_0)$ denote perimeter of the original triangle (blue).
Then $L(\Delta_n) = \frac{\ell}{2^n}$. ← Perimeter of Δ_n

Let K_n denote the triangle Δ_n union with its interior such that K_n is closed (in fact, compact!). Let $\zeta_n \in K_n$ for $n \geq 0$. Then there is $N \in \mathbb{N}$, such that for all $m, n \geq N$ we have $|\zeta_m - \zeta_n| \leq \operatorname{diam}(K_N) \leq \frac{\ell}{2^N}$. Thus, ζ_n as a sequence is Cauchy.

Let $z_0 = \lim_{n\to\infty} \zeta_n$, note $z_0 \in \bigcap_{n=0}^{\infty} K_n$ and $z_0 \in \Omega$. Since f is analytic at z_0, given $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \frac{\varepsilon}{\ell^2}$$

Now consider multiplying $|z - z_0|$ on both sides:

$$|f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| < \frac{\varepsilon}{\ell^2}|z - z_0|$$
$$|f(z_0) + f'(z_0)(z - z_0) - f(z)| < \frac{\varepsilon}{\ell^2}|z - z_0|$$

Since $f(z_0) + f'(z_0)(z - z_0)$ is **linear**, it has an antiderivative on $\mathbb{C}$. Thus,

$$\int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0)\,\mathrm{d}\,z = 0$$

by FTC! Now pick n large enough so that $|z - z_0 < \delta$ for all $z \in \Delta_n$. Thus,

$|I| \leq 4^n \left|\int_{\Delta_n} f(z)\,\mathrm{d}\,z\right|$

$$= 4^n \left| \int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0) - f(z) \right|$$

$$\leq 4^n \frac{\varepsilon}{\ell^2} |z - z_0| \frac{\ell}{2^n} \qquad \text{by tri. ineq. and } \left| \int_\gamma g(z)\,\mathrm{d}z \right| \leq \sup_{z \in \gamma} |g(z)| \cdot L(\gamma)$$

$$< \frac{4^n \epsilon}{\ell 2^n} \cdot \frac{\ell}{2^n}$$

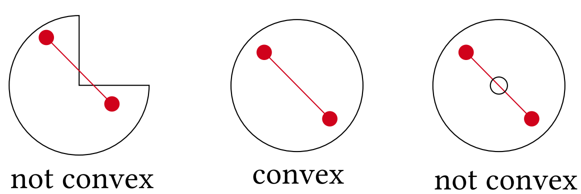
$$= \varepsilon$$

□

Local antiderivative

Theorem 39. If Ω is convex and $f : \Omega \to \mathbb{C}$ is analytic, then f has an antiderivative on Ω.

Remark. Line segments don't exit the region in convex shapes:



Proof. Fix $w \in \Omega$ and define:

$$F(z) = \int_{[w,z]} f(\zeta)\,\mathrm{d}\zeta$$

for $z \in \Omega$.

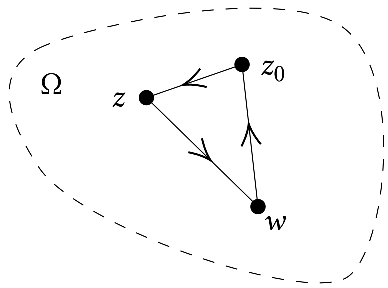
← $[w, z]$ is the line segment from w to z.

This is well-defined if Ω is convex.

Now we want to show that F' is f. That is equivalent to showing that for all $\varepsilon > 0, z_0 \in \Omega$, there exists $\delta > 0$ s.t. whenever $|z - z_0| < \delta$, we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \varepsilon$$

Let $z_0 \in \Omega$ be given and $\varepsilon > 0$. Goursat says integrals around the triangle is 0, so we suppose $z \in \Omega \backslash \{z_0, w\}$ and get a triangle:



and we know that

$$\underbrace{\int_{[w,z_0]} f(\zeta)\,\mathrm{d}\zeta}_{F(z_0)} + \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta + \underbrace{\int_{[z,w]} f(\zeta)\,\mathrm{d}\zeta}_{-F(z)} = 0$$

So $F(z) - F(z_0) = \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta$. Thus,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z_0,z]} (f(\zeta) - f(z_0))\,\mathrm{d}\zeta$$

Since f is analytic at z_0, it is continuous there. Given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have $|f(z) - f(z_0)| < \varepsilon$.

Therefore, whenever $|z - z_0| < \delta$, we have

← still by $\left|\int_\gamma g(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma} |g(z)| \cdot L(\gamma)$

$$\begin{aligned}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &\leq \frac{\varepsilon}{|z - z_0|} L([z_0, z]) \\ &= \frac{\varepsilon}{|z - z_0|}|z - z_0| \\ &= \varepsilon\end{aligned}$$

□

Cauchy's theorem, Take 3

Cauchy's theorem for convex regions

Theorem 40. If Ω is convex, $f : \Omega \to \mathbb{C}$ analytic and γ is a piecewise C^1 curve in Ω, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

← Since Ω is convex, the interior of γ lies inside Ω.

Proof. Previous theorem says f has an antiderivative F on Ω. Thus,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma F'(z)\,\mathrm{d}z = 0$$

by FTC! □

Cauchy's integral formula

Cauchy's integral formula for a circle

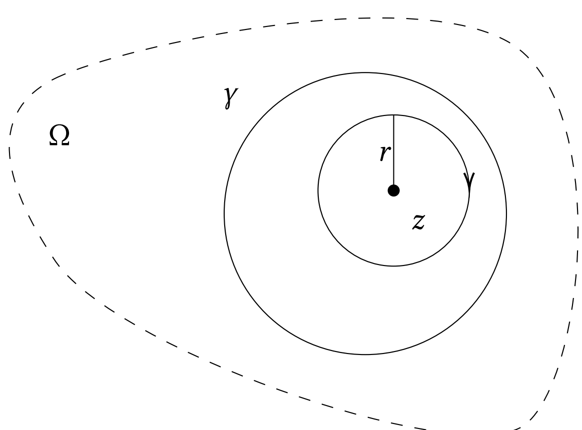
Theorem 41. If f is analytic on a region Ω that contains the circle γ and its interior, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

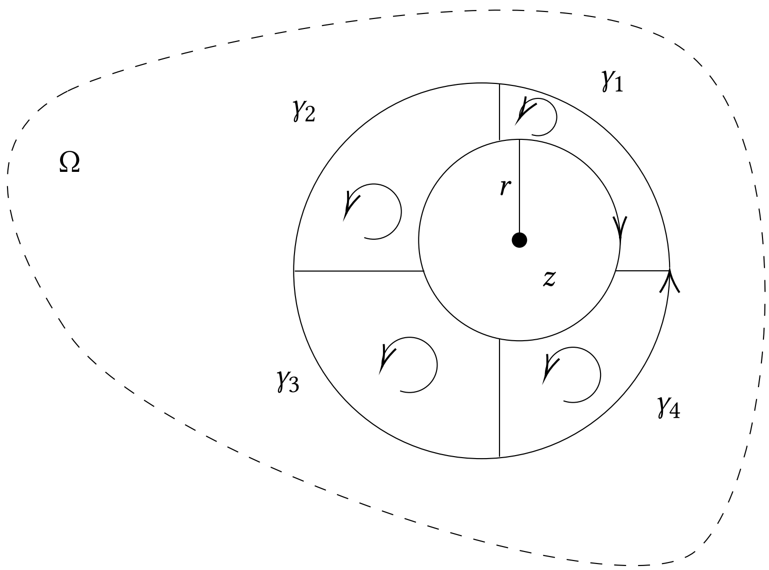
for all z inside of γ.

← this Ω doesn't need to be

Proof. Let $r > 0$ be small enough so that the closed ball $B_r(z)^-$ is in the interior of γ. Let $C_r(z) = \{\zeta \in \mathbb{C} : |\zeta - z| = r\}$ traversed clockwise.



Construct $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as pictured:



Cauchy's theorem for convex regions says $\int_{\gamma_i} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = 0$ for all $i = 1, 2, 3, 4$.

Hence,

$$0 = \sum_{j=1}^{4} \int_{\gamma_j} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

And thus:

$$\int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = \int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

for all $r > 0$ that is *sufficiently* small.

Therefore:

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - f(z) \cdot 1 \right| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - f(z) \cdot \left(\frac{1}{2\pi i} \int_{C_r(z)} \frac{\mathrm{d}\zeta}{\zeta - z} \right) \right| \\
&= \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - f(z) \cdot \left(\frac{1}{2\pi i} \int_{C_r(z)} \frac{\mathrm{d}\zeta}{\zeta - z} \right) \right| \\
&= \lim_{r \to 0^+} \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \right| \\
&\leq \lim_{r \to 0^+} \max_{|\zeta - z| = r} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \cdot r \\
&= 0
\end{aligned}$$

← by HW6 Ex5, or Thm12 Lect 11

□

Mean value properties

Corollary 42 (Mean value property for analytic functions)**.** If f analytic on an open set Ω which contains $B_r(z)^-$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{it})\,\mathrm{d}t$$

Proof. Apply Theorem 41 with $\zeta = z + re^{it}$ and $\mathrm{d}\zeta = ire^{it}\,\mathrm{d}t, t \in [0, 2\pi]$ and get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i}\int_{C_r(z)} \frac{f(z+re^{it})ire^{it}\,\mathrm{d}t}{z + re^{it} - z} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z + re^{it})\,\mathrm{d}t \end{aligned}$$

□

Remark. There is a mean value property for harmonic functions!

Existence of power series expansions

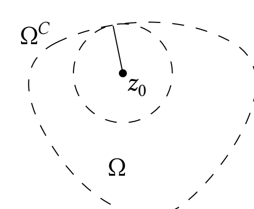
Theorem 43. If $f : \Omega \to \mathbb{C}$ is analytic and $z_0 \in \Omega$ then f has a power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges **locally uniformly** on the disk

$$|z - z_0| < \operatorname{dist}(z_0, \Omega^C) = \inf_{w \in \Omega^C} |z_0 - w|$$

when Ω^C is nonempty. ←



Moreover, the radius of convergence is the radius of the largest open disk centered at z_0 upon which f could be analytically continued.

Proof. Let $r < \operatorname{dist}(z_0, \Omega^C)$ and $|z - z_0| \le \rho < r$. Then

$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

for all $|z - z_0| < \rho$.

As a function of ζ, the series

← geometric series trick!

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

and so by geometric series formula:

$$\begin{aligned}\frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \qquad \text{for } |z - z_0| \leq \rho\end{aligned}$$

converges uniformly on $|\zeta - z_0| = r$ by the Weierstrass M-test with $M_n = \left|\frac{z - z_0}{\zeta - z_0}\right|^n \leq \left(\frac{\rho}{r}\right)^n$.

Thus,

$$\begin{aligned}f(z) &= \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{C_r(z_0)} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \cdot f(\zeta)\,\mathrm{d}\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z_0)^{n+1}}\end{aligned}$$

And so we have our $\frac{f^{(n)}(z_0)}{n!} = a_n$ in the highlighted part above. □

Remark. Consequently, we also get Cauchy's theorem of derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z_0)^{n+1}}$$

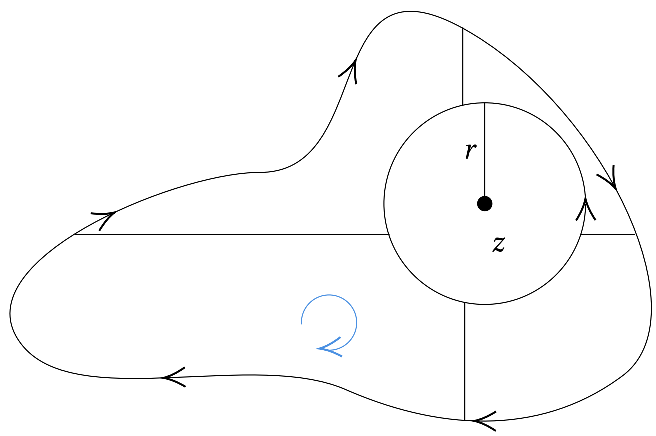
Example 23. What is the radius of convergence for the power series of

$$f(x) = \frac{e^{\sin x} + e^{-x^2} + x^2 + 7x^3}{\cos x}$$

centered at $x_0 = 2$?

The theorem guarantees the existence of the power series, and the RoC would simply be the radius of which f could be analytically continued. We observe that $f(x)$ cannot be defined when $\cos x = 0$, i.e. $x = \frac{\pi}{2}$. Hence, the radius of convergence is just $2 - \frac{\pi}{2}$ – no need to compute *any* derivatives or coefficients!

So now we have this result for computing the derivatives and integrals around a circle $C_r(z_0)$. Can we extend this to other closed curves of any shapes?



Same techniques! Hence,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{n+1}}$$

on any such closed curve γ.