MATH135 Complex Analysis Notes

Xuehuai He

April 11, 2024

Contents

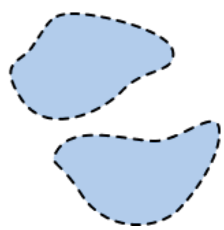
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of $\mathbb{C}$.

- A region without "holes" is simply connected.

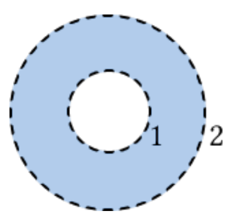
Non-example 1. This is not a region (not connected):



Example 2. $\mathbb{C}$ is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \to \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

← this $z \to z_0$ could be from **any** directions!

2. If f is differentiable at every point in Ω, we say f is **analytic** on Ω.

← Means that existence of 1st derivative implies the existence of ∞th derivative! & has Taylor expansion.

3. If f is analytic on $\mathbb{C}$, then f is **entire**.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on $\mathbb{C}$ except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \to z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z \to z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if $z \to z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z \to z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. □

Proposition 1. Let f be differentiable at z_0. Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the "tangent-line approximation" equivalent in $\mathbb{C}$!

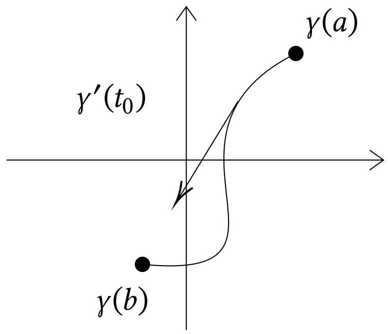
← The RHS is a **linear** function!

In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is <u>locally orientation-preserving</u> and could be approximated by a linear function.

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

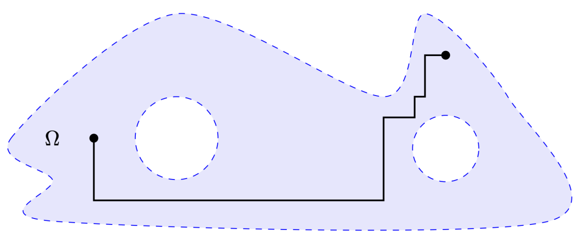
Curves, paths

Definition 3. A **curve** in $\mathbb{C}$ is a function $\gamma : [a, b] \to \mathbb{C}, a, b \in \mathbb{R}$.



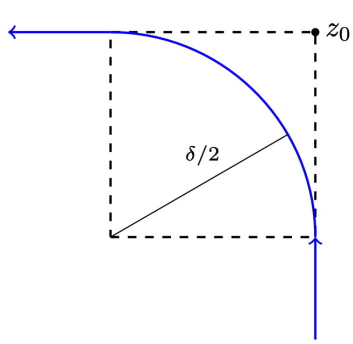
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The "Boxy-path" Theorem)**.** A nonempty open set Ω in $\mathbb{C}$ is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω, each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a "**boxy path**".

Remark. There is also always a **smooth path**. That is:

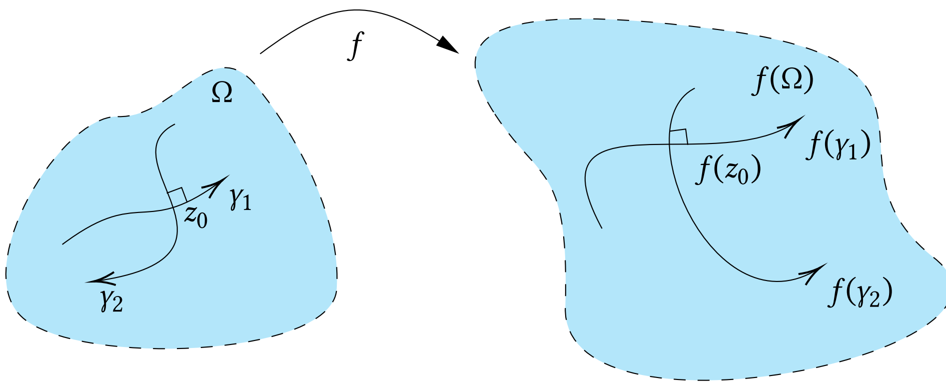


Theorem 3 (“Smooth-path”)**.** A nonempty open set Ω in $\mathbb{C}$ is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See lecture 2 notes. □

Conformality

Let f be an analytic complex function on Ω.



Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ. Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(\zeta_0)$ also with angle θ.

Therefore, f is **conformal**!

Cauchy-Riemann equations, harmonic functions

Multivariate notion of complex derivatives

Recall: $f'(z_0) = \lim\limits_{h \to 0} \dfrac{f(z_0 + h) - f(z_0)}{h}$.

Now we write each function with complex variables as $f(z) = u(z) + i\,v(z)$ where u, v are real-valued functions.

← meaning their range is real

Since $\mathbb{C} \cong \mathbb{R}^2$, we denote every point $z = (x, y)$.

Now we let $f(x, y) = u(x, y) + i\,v(x, y)$. We first let the small distance $h = (r, 0)$ be horizontally approaching 0 with $r \in \mathbb{R}$. That is, $z_0 + h = (x_0 + r, y_0)$.

$$\begin{aligned} f'(z_0) &= \lim_{r \to 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r \to 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let $h = ir = (0, r)$ with $r \to 0, r \in \mathbb{R}$, we would get $f' = v_y - i \cdot u_y$.

Remark. If a derivative exists, the horizontal & the vertical ones should be equal!

Theorem 4 (Cauchy-Riemann Equations)**.**

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

Corollary 5. If $f : \Omega \to \mathbb{C}$ is analytic and $f' = 0$ on Ω, then f is **constant**.

Proof. Since $0 = f' = u_x + iv_x$, we see that $u_x = v_x = 0$ on Ω. By Cauchy-Riemann, $v_y = u_y = 0$ is also true on Ω. Hence, $\mathbf{u}, \mathbf{v}$ are constant on either horizontal or vertical segments. By the Boxy Path Theorem, $f = u + iv$ cannot assume two distinct values in Ω. □

Orientation-preserving as shown by Jacobian

Let $f : \Omega \to \mathbb{C}$ be analytic. Then $f' = u_x + iv_x$ and hence:

$$\begin{aligned} |f'|^2 = \bar{f}' \cdot f &= (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det\left(\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}\right) && \text{the Jacobian of } f! \end{aligned}$$

Since $|f'|^2 \geq 0$, the determinant of the Jacobian is always ≥ 0, implying that f is always locally orientation-preserving. Moreover,

Proposition 6. If $f'(z_0) \neq 0$, then $|f'|^2 > 0$ implies:

1. f is **injective** near z_0
2. f scales $\mathbb{R}$ by $|f'(z_0)|^2$ near z_0
3. f preserves orientation near z_0

The Laplacian, harmonic functions and conjugates

Suppose that $f = u + iv$ is analytic and u, v have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function u is 0!

← $\Delta u = 0$ characterizes steady-state solutions to heat equations on Ω.

Definition 5. Real-valued functions $u : \Omega \to \mathbb{R}$ satisfying that the Laplacian $\Delta u = u_{xx} + u_{yy}$ is 0 on Ω is called **harmonic functions**.

Definition 6. A **harmonic conjugate** of u is a harmonic function $v : \Omega \to \mathbb{R}$ such that $f = u + i \cdot v$ is **analytic** on Ω.

Example 8. $u = x^2 - y^2, v = 2xy$.

← Check it!

Remark. Harmonic conjugates are unique up to translation (± constants).

Remark. If u is harmonic on Ω, it does NOT have to have a harmonic conjugate on Ω.

Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations: $u_x = v_y$ and $u_y = -v_x$.

Example 9. $u(z) = \log|z|$ is harmonic on $\mathbb{C} \backslash \{0\}$.

Proof. Write $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2}\log(x^2 + y^2)$.

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x}\left(\frac{1}{2}\log(x^2 + y^2)\right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

← Review quotient rule!

$$
\begin{aligned}
u_{xx} &= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} \\
&= \frac{y^2 - x^2}{(x^2+y^2)^2}
\end{aligned}
$$

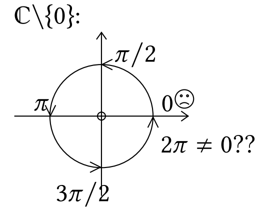
Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

Hence $u_{xx} + u_{yy} = 0$, implying that the function is harmonic. $\square$

Now, can we find a harmonic conjugate for the aforementioned u?

← There is currently a great **CAVEAT** in all of these, because $v(z) = \arg(z)$ cannot be defined in a continuous manner in all of $\mathbb{C} \backslash \{0\}$:



To be resolved later!

We could use the two Cauchy-Riemann Equations. One of them:

$$
\begin{aligned}
v_y &= u_x \\
&= \frac{x}{x^2+y^2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v(x,y) &= \int v_y dy + C(x) \qquad\qquad \text{unknown function of } x \\
&= \arctan\left(\frac{y}{x}\right) + C(x)
\end{aligned}
$$

Then, we use the second one:

$$
\begin{aligned}
\frac{y}{x^2+y^2} = u_y = -v_x &= -\frac{\partial}{\partial x}\left(\arctan\left(\frac{y}{x}\right) + C(x)\right) \\
&= \frac{y}{x^2+y^2} - C'(x) \implies C'(x) = 0
\end{aligned}
$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x,y) = \arctan\left(\frac{y}{x}\right) + C$$

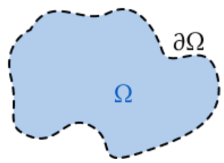
where C is a constant. WLOG, let $C = 0$. Then $v(x,y) = \arctan\left(\frac{y}{x}\right)$, meaning that:

$$v(z) = \arg(z)$$

Therefore, $f(z) = \log|z| + i \cdot \arg(z)$ is analytic!

Physics analogies of harmonic functions

Example 10. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in $\mathbb{C}$. Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \to \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 7. Let $U(x, y)$ be a harmonic function on Ω.

1. U cannot have a *local* maximum in Ω.
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$. ← Ω^- denotes the closure of Ω
3. U cannot be locally constant without being globally constant.

Theorem 8 (Maximum principle)**.** Let Ω be a bounded region in $\mathbb{C}$ and let $f : \Omega^- \to \mathbb{C}$ be analytic on Ω and continuous on Ω^-.

1. If $|f|$ achieves a local max in Ω, then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.

Möbius transformations

Möbius transformations, the extended plane

Definition 7 (Möbius transformations)**.**

$$f(z) = \frac{az + b}{cz + d} \text{ where } ad - bc \neq 0, a, b, c, d \in \mathbb{C}$$

Such an f is **analytic** on $\mathbb{C}\backslash\{\frac{-d}{c}\}$ and **comformal** there since $f'(z) = \frac{ad-bc}{(cz+d)^2} \neq 0$ on $\mathbb{C}\backslash\{\frac{-d}{c}\}$.

← recall that rational functions are analytic except when the denominator vanishes, i.e. $cz + d \neq 0$.

Remark. In addition, f is injective (one-to-one)!

Proof.

$$\begin{aligned} f(z) = f(w) \implies \frac{az+b}{cz+d} &= \frac{aw+b}{cw+d} \\ (az+b)(cw+d) &= (cz+d)(aw+b) \\ aczw + bcw + adz + bd &= aczw + adw + bcz + bd \\ (ad-bc)z &= (ad-bc)w \\ z &= w \end{aligned}$$

□

Definition 8 (The extended plane)**.** We set the following convention:

$$\begin{aligned} f(\frac{-d}{c}) &= \infty \\ f(\infty) &= \frac{a}{c} \end{aligned}$$

with this, f is a **bijection** from $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself.

← recall Riemann sphere

Möbius transformations as matrices

Remark. We can *associate* $f(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ with the matrix

$$M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

← this association is not a bijection: it's only so up to scaling

Remark. $M_{f\circ g} = M_f \cdot M_g$

← check this!

Remark. The inverse of $M_f = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $M_f^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and scaling does not matter, so we could write the **inverse** of such Möbius transformation as:

$$f^{-1}(w) = \frac{dw - b}{-cw + a}$$

Theorem 9. A Möbius transformation $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with three fixed points in $\widehat{\mathbb{C}}$ is the **identity map** $\mathrm{id}(z) = z = \frac{z+0}{0z+1}$.

← $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If ∞ is fixed, then $c = 0$. Then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear** transformation. ← think about that!
   (a) If $f(z) = z$, we are done since we get the identity!
   (b) Otherwise the function only has one fixed point at ∞.
2. If ∞ is not a fixed point, then $c \neq 0$. Solve:

$$\begin{aligned} f(z) + z \Leftrightarrow \frac{az+b}{cz+d} &= z \\ az + b &= cz^2 + dz \\ cz^2 + (d-a)z - b &= 0 \end{aligned}$$

   is a quadratic which has at most two (distinct) solutions in $\mathbb{C}$. Hence, this transformation fixes at most two points.

□

Möbius transformations take circles to circles

Remark. Lines can be circles (they are just circles that pass through the point at infinity).

Theorem 10. The image of a circle under a Möbius transformation is still a circle.

Proof. Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation.

1. If $c = 0$, then $f(z) = \frac{a}{d}z + \frac{b}{d}$, which is a **linear/affine** transformation and so we are done. ← since linear transformations preserve circles and lines
2. Now suppose $c \neq 0$. Then

$$\begin{aligned} f(z) &= \frac{a}{d}z + \frac{b}{d} \\ &= \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c} \end{aligned}$$

   which is a composition of affine, inversion and affine:

$$z \mapsto cz + d \mapsto \frac{1}{cz+d} \mapsto \frac{b - \frac{ad}{c}}{cz+d} + \frac{a}{c}$$

   We now only need to show that inversion preserves circles.

Let a circle in $\mathbb{R}^2$ be $Ax + By + C(x^2 + y^2) = D$ where $A, B, C, D \in \mathbb{R}$. If $z = x + iy \in \widehat{\mathbb{C}}$, then $\frac{1}{z} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$. Name $\frac{1}{z} = u + iv$, note that $u^2 + v^2 = \frac{1}{x^2+y^2}$.
Then we note that $Au - Bv + C = D(u^2 + v^2)$, which is still a circle! ← check this!

□

Theorem 11. Given two triples z_1, z_2, z_3 and w_1, w_2, w_3 of *distinct* points in $\widehat{\mathbb{C}}$, then there is always a unique Möbius transformation f such that $f(z_i) = w_i$ for all $i = 1, 2, 3$.

Proof. Claim: the *cross-ratio* $\phi(z) = \frac{z-z_1}{z-z_3} \cdot \underbrace{\frac{z_2-z_3}{z_2-z_1}}_{\text{const.}}$ is a Möbius transformation that satisfies $\boxed{\phi(z_1) = 0, \phi(z_2) = 1, \phi(z_3) = \infty}$.

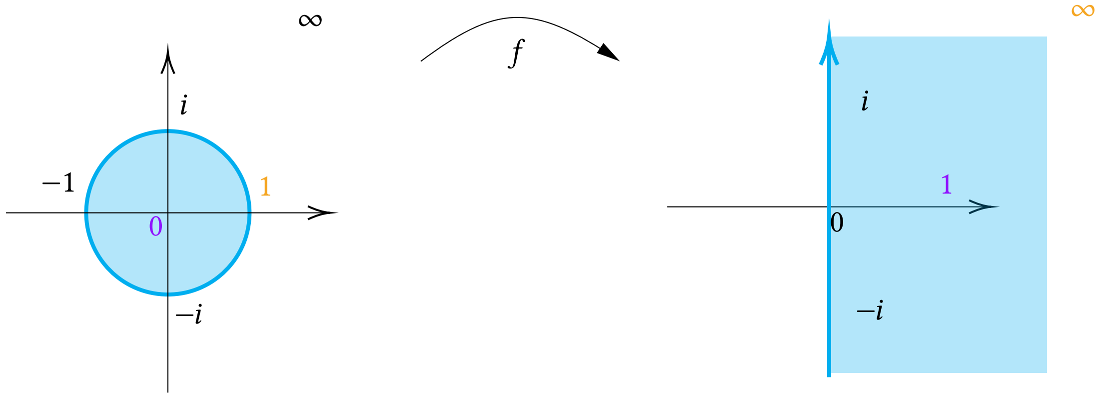
We can also find another Möbius transformation such that $\psi(z_1) = 0, \psi(z_2) = 1, \psi(z_3) = \infty$. Then:

$$\begin{aligned} z_1 &\xmapsto{\phi} 0 \xmapsto{\psi^{-1}} w_1 \\ z_2 &\xmapsto{\phi} 1 \xmapsto{\psi^{-1}} w_2 \\ z_3 &\xmapsto{\phi} \infty \xmapsto{\psi^{-1}} w_3 \end{aligned}$$

and we could simply let $f = \psi^{-1} \circ \phi$. □

Example 11. Let $f(z) = \frac{z+1}{-z+1}$. We compute:

$$\begin{aligned} f(0) &= 1 \\ f(-1) &= 0 \\ f(1) &= \infty \\ f(i) &= i \\ f(-i) &= -i \end{aligned}$$



Recall: infinite series

Definition 9. $\sum_{n=1}^{\infty} a_n$ converges to S if $\lim_{N\to\infty} S_N = S$ where $S_N = a_1 + \dots + a_N$.

← S_N is the N-th partial sum.

Divergence test

Definition 10 (Divergence test)**.** A pair of contrapositives:

← Note it's not an *if and only if* !

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\lim_{n\to\infty} a_n \neq 0$ (*including the case where the limit doesn't exist*) then $\sum_{n=1}^{\infty} a_n$ diverges.

Non-example 12. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \dots$ diverges even though $a_n = \frac{1}{n}$ tends to 0 when n tends to ∞.

← diverges, but really **slowly**!

Theorem 12. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{N\to\infty} \sum_{n=N}^{\infty} a_n = \lim_{N\to\infty} S - S_N = 0$.

← In other words, the tail of a convergent series goes to 0.

Theorem 13 (Cauchy Criterion)**.** $\sum_{n=1}^{\infty} a_n$ converges *if and only if* for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $k > j > N$ implies $\left| \sum_{n=j-1}^{k} a_n \right| = S_k - S_j < \varepsilon$.

Integral test

Definition 11 (Integral test)**.** Define $a_n = f(n)$ for $n \in \mathbb{N}$, where $f : [1, \infty[\to \mathbb{R}$ is (piecewise) continuous, positive and decreasing. Then $\int_1^{\infty} f(x)\, \mathrm{d}x$ converges *if and only if* $\sum_{n=1}^{\infty} a_n$ converges.

← do an improper integral!

Moreover, $\int_1^N f(x)\,\mathrm{d}x \le a_1 + \cdots + a_N \le a_1 + \int_1^N f(x)\,\mathrm{d}x$.

Example 13. Apply the above with $f(x) = \frac{1}{x}$. Then

← $a_n = \frac{1}{n}$

$$\ln N \le 1 + \frac{1}{2} + \cdots + \frac{1}{N} \le 1 + \ln N$$

It is bounded below by a divergent function, so it must be divergent!

Theorem 14. The "p-series" $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges *if and only if* $p > 1$.

Definition 12 (Riemann zeta function)**.**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1$$

Remark. Euler figured out:

$$\begin{aligned} \zeta(2) &= \frac{\pi^2}{6} \\ \zeta(4) &= \frac{\pi^4}{90} \\ \zeta(6) &= \frac{\pi^6}{945} \\ &\vdots \end{aligned}$$

Remark. R. Apéry showed that $\zeta(3)$ is irrational (1979):

← still an open question in mathematics

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202\ldots$$

but no explicit formula known!

Absolute convergence

Definition 13. A series $\sum_{n=1}^{\infty} a_n$ is:

1. **absolutely convergent** if $\sum_{n=1}^{\infty} |a_n|$ converges. ← Good
2. **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges. ← BAD

Theorem 15. Every absolutely convergent series converges.

Example 14. The alternating harmonic series

← Don't re-parenthesize the terms – grouping would change the sequence and thus the partial sums!

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

converges to $\ln 2$. But the convergence is conditional because the absolute value

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

does not converge.

Theorem 16. An absolutely convergent series may be rearranged without changing its value. That is, if $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection, then

← This seems obvious for finite series, but consider how this is extraordinary for infinite series!

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\phi(n)}$$

Theorem 17 (Riemann Rearrangement Theorem)**.** If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers, then for **any** $S \in \mathbb{R} \cup \{-\infty, \infty\}$, there is a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\phi(n)} = S$.

← Meaning we can get it to be equal to whatever we want just by rearranging!

Now if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ converge, one might expect that

$$\begin{aligned}\left(\sum_{i=0}^{\infty} a_i\right)\left(\sum_{j=0}^{\infty} b_j\right) &= (a_0 + a_1 + \dots)(b_0 + b_1 + \dots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \dots \\ &= \sum_{n=0}^{\infty} c_n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}\end{aligned}$$

But this only works if both series are absolutely convergent, in which case the new series is absolutely convergent.

← conditionally convergent doesn't work! See notes.

Uniform convergence

Definition 14. A sequence of functions $f_n : X \to \mathbb{C}$ where $X \subseteq \mathbb{C}$ **converges uniformly** to $f : X \to \mathbb{C}$ if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(z) - f(z)| < \varepsilon$ for all $z \in X$.

← This is MATH131!

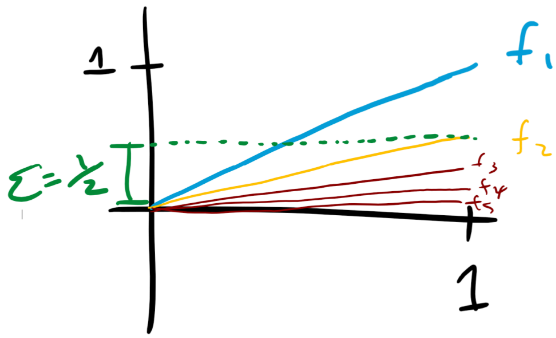


FIGURE 8. Uniform Convergence

Theorem 18. If $f_n : X \to \mathbb{C}$ are continuous and converges uniformly on X to $f : X \to \mathbb{C}$, then f is continuous on X. In other words, the uniform limit of continuous functions is continuous.

← unif. conv. preserves continuity

Remark. f_n converges to f pointwise on X if $\lim_{n\to\infty} f_n(z) = f(z)$ for all $z \in X$.

← This doesn't say anything about the rate each point converges.

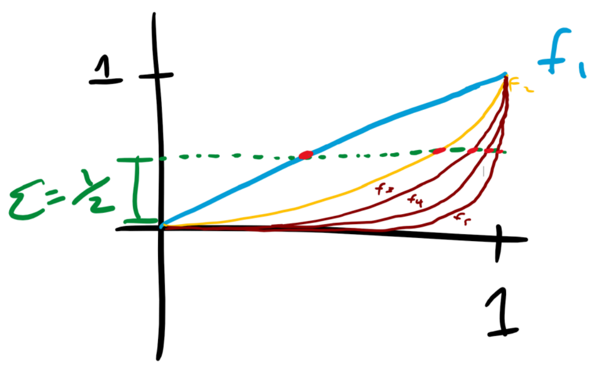


FIGURE 9. Non-uniform convergence

Theorem 19. If $f_n : [a,b] \to \mathbb{C}$ are continuous and converge uniformly on $[a,b]$ to f, then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}\,x = \int_a^b f(x)\,\mathrm{d}\,x$$

← Integrals work with uniform convergence

Remark. Uniform convergence doesn't necessarily preserve differentiability, limit or derivatives!

Example 15. $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ on $[-1,1]$ converges uniformly to $f_n(x) = |x|$. But the limit function is **not** differentiable at $x = 0$ even though every f_n were.

Theorem 20 (Weierstrass M-Test). Let $f_n : X \to \mathbb{C}$ satisfy $|f_n(z)| \leq M_n$ for all $z \in X$ and $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(z)$ converges both **absolutely** and **uniformly** on X.

Power series

Definition 15. A **power series** is a series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. The a_n is the *coefficient* and z_0 is the *center*.

Convergence of geometric series

Theorem 21. The *geometric series* ($a_n = 1, z_0 = 0$) $\sum_{n=0}^{\infty} z^n$ converges absolutely to $\frac{1}{1-z}$ if $|z| < 1$, and it diverges otherwise.

Moreover, for each $r \in [0, 1[$, the convergence is **uniform** on $|z| \leq r$.

Proof. If $|z| \geq 1$, then $z^n \not\to 0$, so by the test of divergence, the series diverges.

Now suppose $|z| < 1$. Then

← The fact that we can find a formula for this sum is quite rare!

$$\begin{aligned}\sum_{n=0}^{\infty} z^n &= \lim_{N\to\infty} \sum_{n=0}^{N-1} z^n \\ &= \lim_{N\to\infty} (1 + z + z^2 + \cdots + z^{N-1}) \\ &= \lim_{N\to\infty} \frac{1-z^N}{1-z} \\ &= \frac{1}{1-z} && \text{since } |z| < 1\end{aligned}$$

Which gives us point-wise convergence. Then, for any r such that $|z| \leq r < 1$, we have

$$\sum_{n=0}^{\infty} |z^n| \leq \sum_{n=0}^{\infty} r^n = \frac{1}{1-r} < \infty$$

Hence, by the Weierstrass M-test, the series converges *absolutely and uniformly* on $|z| \leq r$. □

Remark. Moral of the story:

- The *radius of convergence* $R = 1$ has the property that the series converges on $|z| < R$, and diverges if $|z| > R$.
- The series converges *uniformly* on $|z| \leq r < 1$ but not on $|z| < 1$ itself. Why? Let $r = 1$; we need be able to get $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\left|\frac{1-z^N}{1-z} - \frac{1}{1-z}\right| < 1$ for all $|z| < 1$. However, this is not gonna work: as $z \to 1$, observe that this is going to eventually exceed 1.

- The limit function $\dfrac{1}{1-z}$ is **analytic** on $\mathbb{C}\backslash\{1\}$. But the geometric series represents this function only on $|z| < 1$. In a smaller set, the power series represents the function that might originally be defined on a much larger set. The limit function is the *analytic continuation* of the series.

← the limit function is well-defined way beyond the $\mathbb{D}$!

- The limit function $\dfrac{1}{1-z}$ is cool if $z \neq 1$, but as long as $|z| = 1$ (**even** if $z \neq 1$), the geometric series diverges!

← in the complex number sense!

Radius of convergence

Definition 16. The **limit superior** (lim sup) of a sequence of nonnegative real numbers x_n is the largest *limit point* of the x_n:

← limits of a subsequence of x_n

$$\limsup_{n\to\infty} x_n = \inf_{n\geq 0} \sup_{m\geq n} x_m$$

If the sequence is unbounded, the lim sup would be ∞.

← the RHS as in real analysis

Example 16. If x_n is the sequence $0, 1, 0, 1, \ldots$ then $\limsup_{n\to\infty} x_n = 1$.

Example 17. If x_n is the sequence $0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, \ldots$, then $\limsup_{n\to\infty} x_n = 0$.

Remark. If x_n are nonnegative, then

- $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
- $\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n)(\limsup_{n\to\infty} b_n)$

Theorem 22 (Cauchy-Hadamard)**.** Let $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a power series. Define $R \in [0, \infty]$ by

← interpret $\frac{1}{0} = \infty$

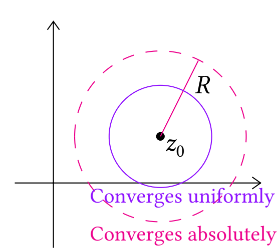
$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$

Then the R is the *radius of convergence*.

(a) On $|z - z_0| < R$, the series converges **absolutely**. For each $r \in [0, R[$, the convergence is **uniform** on $|z - z_0| \leq r$.

(b) If $|z - z_0| > R$ then the series diverges. For $|z - z_0| = R$ anything could happen!

←



Example 18. We claim that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has an infinite radius of convergence $R = \infty$. To check:

$$\sqrt[n]{|a_n|} = \sqrt[n]{\frac{1}{n!}} = \frac{1}{\sqrt[n]{n!}} \to 0$$

This is because $\sqrt[n]{n!} = \sqrt[n]{1 \cdot 2 \cdot \cdots \cdot n}$, and in $n!$, there are at least $\frac{1}{2}$ terms that are $> \frac{n}{2}$.

Thus, $\sqrt[n]{n!} \geq \left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right)^{\frac{1}{n}} = \left(\frac{n}{2}\right)^{1/2} \to \infty$.

So $R = \infty$ and we are done ☺. We have that $\exp(z)$ has absolute convergence on the entire complex plane!

Absolute convergence means that we can multiply term-by-term:

$$\begin{aligned}
\exp(z)\exp(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \cdot \frac{w^{n-k}}{(n-k)!}\right) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} z^k w^{n-k}}_{\text{binomial theorem}} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\
&= \exp(z+w)
\end{aligned}$$

Now define $e = \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$.

Term-by-term differentiation of power series

Lemma 23. $n^{\frac{1}{n}} \to 1$

Proof 1. $e^{\log(n^{\frac{1}{n}})} = e^{\frac{\log n}{n}} \to e^0 = 1$ by l'Hopital. So $n^{\frac{1}{n}} \to 1$. □

Proof 2 (better). Write $n^{\frac{1}{n}} = 1 + \delta_n$ where $\delta_n \geq 0$. The binomial theorem says:

$$\begin{aligned}
n &= (1+\delta_n)^n \\
&= \sum_{k=0}^{\infty} \binom{n}{k} \delta_n^k \cdot 1^{n-k} \\
&= 1 + n\delta_n + \frac{n(n-1)}{2}\delta_n^2 + \dots
\end{aligned}$$

$$\geq 1 + \frac{n(n-1)}{2}\delta_n^2$$

Therefore, $n - 1 \geq \frac{n(n-1)}{2}\delta_n^2$ and we get $\frac{2}{n} \geq \delta_n^2 \geq 0$ hence $\delta_n \to 0$.

Hence $n^{\frac{1}{n}} \to 1$. □

Theorem 24. If $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ has radius of convergence R, then

$$f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1}$$

for $|z - z_0| < R$. Moreover, the new series also has a radius of convergence R.

Proof. WLOG $R > 0$ and $z_0 = 0$.

← we just translate it; also $R = 0$ isn't that meaningful

For $|z| < R$ we write:

$$f(z) = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

← just splitting the function into two parts

and the 'new series'

$$g(z) = \sum_{n=0}^{\infty} na_n z^{n-1} = \lim_{N\to\infty} S_N'(z)$$

We first prove that the radius of convergence for g is the same as f. By Cauchy-Hadamard:

$$\begin{aligned} \frac{1}{R_g} &= \limsup_{n\to\infty} \sqrt[n]{n|a_n|} \\ &= \limsup_{n\to\infty} (n^{\frac{1}{n}})\sqrt[n]{|a_n|} && \text{by the previous lemma,} \\ &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \frac{1}{R} \end{aligned}$$

Thus, $R_g = R$ by Cauchy-Hadamard.

Next, we need to show that $f' = g$ with $|z| < R$.

Fix $0 \leq |w| < R$ and $\varepsilon > 0$. We want a $\delta > 0$ such that whenever $|z - w| < \delta$, we have $\left|\frac{f(z)-f(w)}{z-w} - g(w)\right| < \varepsilon$.

← just saying that the derivative at any w gets close to $g(w)$

We rewrite:

$$\begin{aligned}
\left|\frac{f(z)-f(w)}{z-w}-g(w)\right| &= \left|\frac{[S_N(z)+R_N(z)]-[S_N(w)+R_N(w)]}{z-w}-g(w)\right| \\
&= \left|\frac{S_N(z)-S_N(w)}{z-w}+\frac{R_N(z)-R_N(w)}{z-w}+S_N'(w)-S_N'(w)-g(w)\right| \\
&\le \left|S_N'(w)-g(w)\right| + \left|\frac{R_N(z)-R_N(w)}{z-w}\right| + \left|\frac{S_N(z)-S_N(w)}{z-w}-S_N'(w)\right|
\end{aligned}$$

- **1st term**: by def of g and $g(z)=\lim_{N\to\infty} S_N'(z)$, we can always find some $N_1 \in \mathbb{N}$ such that any $N \ge N_1$ gives us $\left|S_N'(w)-g(w)\right| < \frac{\varepsilon}{3}$.
- **2nd term**: since $|w|<R$, there is an r such that $|w|<r<R$. For $|z|<r$, we have

← work on a smaller disk

$$\begin{aligned}
\left|\frac{R_N(z)-R_N(w)}{z-w}\right| &= \frac{1}{|z-w|}\left|\sum_{n=N}^{\infty} a_n z^n = -\sum_{n=N}^{\infty} a_n w^n\right| \\
&\le \sum_{n=N}^{\infty} |a_n| \left|\frac{z^n-w^n}{z-w}\right| \\
&= \sum_{n=N}^{\infty} |a_n| \left|\frac{z^n}{z}\cdot\frac{1-\frac{w^n}{z^n}}{1-\frac{w}{z}}\right| && \text{by geometric sequence} \\
&= \sum_{n=N}^{\infty} |a_n| \left|\frac{z^n}{z}\cdot\left(1+\left(\frac{w}{z}\right)+\left(\frac{w}{z}\right)^2+\dots+\left(\frac{w}{z}\right)^{n-1}\right)\right| \\
&= \sum_{n=N}^{\infty} |a_n| \left|z^{n-1}+z^{n-2}w+\dots+zw^{n-2}+w^{n-1}\right| \\
&\le \sum_{n=N}^{\infty} |a_n|\cdot n\cdot r^{n-1} \text{by } |z|,|w|<r<R
\end{aligned}$$

Thus, there exists an $N_2 \in \mathbb{N}$ such that any $N \ge N_2$ gives us

$$\left|\frac{R_N(z)-R_N(w)}{z-w}\right| < \frac{\varepsilon}{3}$$

- **3rd term**: let $N=\max\{N_1,N_2\}$. The definition of $S_N'(w)$ provides $\gamma>0$ such that if $|z-w|<\gamma$, then we have $\left|\frac{S_N(z)-S_N(w)}{z-w}-S_N'(w)\right| < \frac{\varepsilon}{3}$.

← review def of derivatives!

Now if $0<\delta<\min\{\gamma, r-|w|\}$, then the 3 terms above are all $<\frac{\varepsilon}{3}$. Hence, $\left|\frac{f(z)-f(w)}{z-w}-g(w)\right| < \varepsilon$ holds for this δ. □

Corollary 25. A power series $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ with $R > 0$ is infinitely differentiable on $|z - z_0| < R$. Moreover,

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

are the coefficients of the terms of the power series.

← prove by keep taking derivatives!

Corollary 26. Power series expansions are unique. That is, if $r > 0$ and

← because there is a unique formula for coeffs.

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n$$

on $|z - z_0| < r$, then $a_n = b_n$ for $n \geq 0$.

Remark. Recall that $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ has a radius of convergence ∞ (it's an *entire* function). Now, if we differentiate it term-by-term:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\exp(z) &= \frac{\mathrm{d}}{\mathrm{d}z}\sum_{n=0}^{\infty}\frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{z^{n-1}}{(n-1)!} && \text{let } k = n-1 \\ &= \sum_{k=0}^{\infty}\frac{z^k}{k!} \\ &= \exp(z)\end{aligned}$$

Thus, the derivative of $\exp(z)$ is itself! Moreover, $\exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.

Remark. We claim that $\exp(z) = e^z$.

Since $e^z e^{c-z}$ is a constant for all constant c, z, we have

$$\frac{\mathrm{d}}{\mathrm{d}z}(e^z e^{c-z}) = 0$$

to recover the constant $e^z e^{c-z}$, we let $z = 0$, giving us

$$e^z e^{c-z} = e^c$$

which is the addition formula!

Therefore,

$$\begin{aligned}\exp(n) &= \exp(1+1+\cdots+1) \\ &= exp(1)^n \\ &= e^n\end{aligned}$$

Elementary functions

Now that we have derived e, we could use it to derive sin and cos:

Definition 17.

$$\begin{aligned}\cos(z) &= \frac{e^{iz}+e^{-iz}}{2}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!}\end{aligned}$$

$$\begin{aligned}\sin(z) &= \frac{e^{iz}-e^{-iz}}{2i}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}\end{aligned}$$

We observe that we have the following property:

- Radius of convergence $R = \infty$
- $(\cos z)' = -\sin z, (\sin z)' = \cos z$
- $\cos x = \operatorname{Re}(e^{ix}), \sin x = \operatorname{Im} e^{ix}$ for all $x \in \mathbb{R}$
- $\cos(-z) = \cos z, \sin(-z) = -\sin z$
- $\cosh x = \frac{e^x+e^{-x}}{2}$ so $\cosh(ix) = \cos x$
- $e^{iz} = \cos z + i\sin z$
-
$$\begin{aligned}\cos^2 z + \sin^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2\\ &= \frac{1}{4}(e^{2iz}+2+e^{-2iz}) - \frac{1}{4}(e^{2iz}-2+e^{-2iz})\\ &= 1 \qquad \forall z \in \mathbb{C}\end{aligned}$$

-
$$\begin{aligned}\cos^2 z &= \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2\\ &= \frac{1}{4}(e^{2iz}+2+e^{-2iz})\\ &= \frac{1}{2}+\frac{e^{2iz}+e^{-2iz}}{4}\\ &= \frac{1}{2}(1+\cos 2z)\end{aligned}$$

- If $x \in \mathbb{R}$ then $\cos x, \sin x$ are real. We get $|\sin x|, |\cos x| \leq 1$.

Definition 18. $f : \mathbb{C} \to \mathbb{C}$ is **periodic** with a *period* ω if $f(z+\omega) = f(z)$ for all $z \in \mathbb{C}$.

Theorem 27. There exists a positive real number π such that:

(a) $\cos z, \sin z$ have period 2π

(b) e^z is periodic with period $2\pi i$

(c) π is the area of the unit circle

Proof. By Euler's formula, it suffices to consider e^{iz} only. If ω is a period of e^{iz}, then

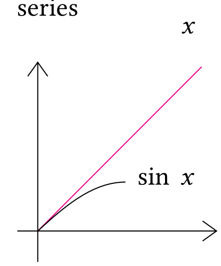
$$e^{iz} = e^{i(z+\omega)} = e^{iz}e^{i\omega}$$

which only happens if $e^{i\omega} = 1$. Conversely, if $e^{i\omega} = 1$, then $e^{i(z+\omega)} = e^{iz}$.

Hence, ω is a period of e^{iz} *if and only if* $e^{iw} = 1$. □

Proposition 28. $\sin x \leq x$ for all $x \geq 0$.

← This is the first term in the power series



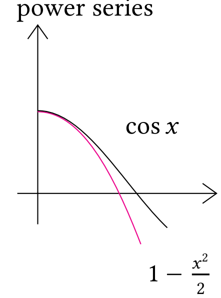
Proof. Since $|\cos t| \leq 1$,

$$\begin{aligned} x - \sin x &= (x - \sin x) - (0 - \sin 0) \\ &= \int_0^x \underbrace{1 - \cos t}_{\geq 0}\, dt \qquad \text{by FTC} \\ &\geq 0 \end{aligned}$$

□

Proposition 29. In addition, $\cos x \geq 1 - \frac{x^2}{2}$ for $x \geq 0$.

← These are the first 2 terms in the power series



Proof. The previous prop gives:

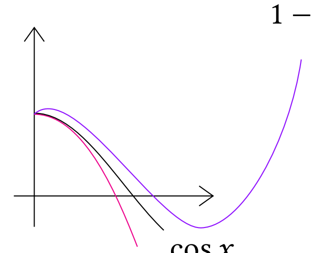
$$\begin{aligned} \cos x - 1 &= \cos x - \cos 0 \\ &= \int_0^x -\sin t\, dt \\ &\geq \int_0^x -t\, dt \\ &= \frac{-x^2}{2} \end{aligned}$$

□

Proposition 30. Furthermore, for $x \geq 0$:

- $\sin x \geq x^3 - \frac{x^3}{6}$
- $\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

Proposition 31. There exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$.



Proof. By the previous prop, we have $\cos \sqrt{3} \leq 1 - \frac{\sqrt{3}^2}{2} + \frac{\sqrt{3}^4}{24} = \frac{1}{8} < 0$. Moreover, $\cos 0 = 1 > 0$, by IVT, there exists $x_0 \in (0, \sqrt{3})$ such that $\cos x_0 = 0$. □

Proposition 32. $\omega_0 = 4x_0$ is a period of e^{iz}.

Proof. Since $\cos x_0 = 0$, we have $\sin x_0 = \pm 1$. Then $e^{ix_0} = \pm i$. We have $(\pm i)^4 = 1$, so $e^{4ix_0} = 1 = e^0$, so $\omega_0 = 4x_0$ is a period of e^{iz}. □

Proposition 33. ω_0 is the *smallest* positive period of e^{iz}.

Proposition 34. All periods of e^{iz} are integer multiples of $2\pi = 4x_0$.

Proof. Define $\pi = 2x_0$. The area of unit circle is

$$\begin{aligned} 4\int_0^1 \sqrt{1 - x^2}\,\mathrm{d}\,x &= 4\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\,\mathrm{d}\,\theta \\ &= 4\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2\theta)\,\mathrm{d}\,\theta \\ &= \pi \end{aligned}$$

□

Complex logarithm

We know: $e^0 = 1, e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718\ldots$

Since $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, it is positive. If $x > 0$, we conclude that e^x is strictly increasing! As $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} > 1 + x$, so $\lim_{x\to\infty} e^x = \infty$,

Therefore, e^x is a **bijection** from $\mathbb{R}$ to $(0, \infty)$. This means it has an inverse that is a bijection from $(0, \infty)$ to $\mathbb{R}$.

Definition 19. $\ln x$ is the inverse of e^x for $x \in (0, +\infty)$.

Now what about the complex case? Let $z \neq 0$ and $z = re^{i\theta}$ where $r = |z| > 0$ and $\theta = \arg z \in \mathbb{R}$.

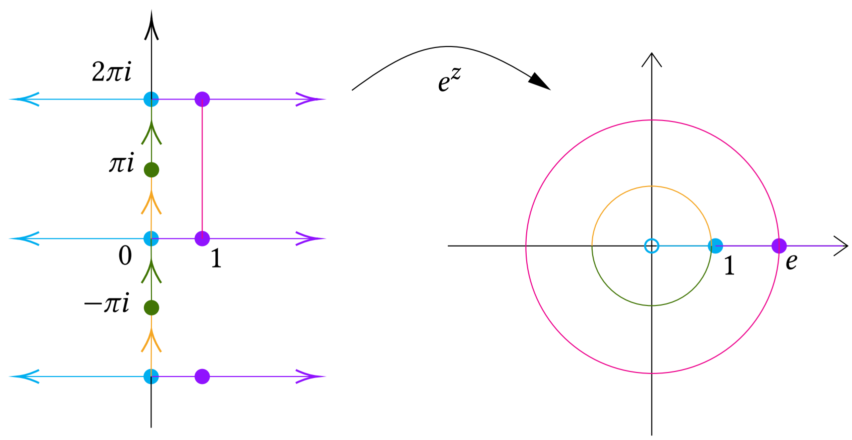
← cf. trig properties

← Only determined up to addition of multiples of 2π

Hence, $z = re^{i\theta} = e^{\ln r} e^{i\theta} = e^{\ln r + i\theta}$. However, the θ is ambiguous to addition of multiples of 2π!

Definition 20. If $z \neq 0$, a **logarithm** of z is a $w \in \mathbb{C}$ such that $e^w = z$.

We could graph the function e^z with $z \in \mathbb{C}$:



Definition 21. If Ω is a region in $\mathbb{C}$, then a continuous $l : \Omega \to \mathbb{C}$ is a **branch** of the logarithm if $e^{l(z)} = z$ for all $z \in \Omega$.

← note $0 \notin \Omega$

Example 19. If $\Omega = \mathbb{C}\backslash(-\infty, 0]$ such that $\theta \in (-\pi, \pi)$, a logarithm could be defined on it. This is the **principal branch** of the logarithm.

← See graphed Riemann surface

Remark. Suppose $l(z)$ is a branch of the logarithm and l is analytic, then:

$$e^{l(z)} = z \implies \frac{\mathrm{d}}{\mathrm{d}z} e^{l(z)} = l'(z) e^{l(z)} = 1$$

Since $e^{l(z)} = z$, we conclude $l'(z) = \frac{1}{z}$.

Complex power

Definition 22. If $z \neq 0$, *define* $z^a = e^{a \log z}$. ← NOT well-defined!

Remark. The definition of complex powers should coincide with the old one: $z^n = \underbrace{z \cdot z \cdot \cdots \cdot z}_{n} = r^n e^{in\theta}$.

Check:

$$\begin{aligned} z^n &= e^{n \log z} = e^{n(\ln r + i\theta + i2\pi k)} \\ &= e^{n \ln r} e^{in\theta} \underbrace{e^{i2\pi nk}}_{=1} \\ &= r^n e^{in\theta} \end{aligned}$$

is true for any $k \in \mathbb{Z}$.

How about n-th roots?

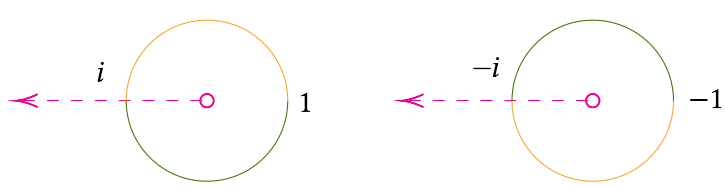
$$\begin{aligned} z^{\frac{1}{n}} &= e^{\frac{1}{n} \log z} \\ &= e^{\frac{1}{n}(\ln r + i\theta + i2\pi k)} \\ &= e^{\frac{1}{n} \ln r} e^{\frac{i\theta}{n}} \underbrace{e^{\frac{i2\pi k}{n}}}_{n \text{ distinct}} \\ &= r^{\frac{1}{n}} e^{i\left(\frac{\theta + 2\pi k}{n}\right)} \end{aligned}$$

Riemann surface

We still have a problem: $\ln z$ is still not a function on $\mathbb{C}$! The branch depends on the arbitrary choice of domain. What shall we do to make it not dependent on a choice?

Answer: let ln not live on the complex plane, but infinitely many copies of the slit plane $\mathbb{C} \backslash (-\infty, 0]$, each one being glued to the next along the slit $(-\infty, 0]$.

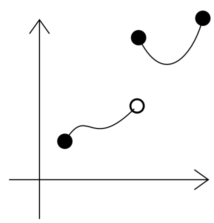
Example 20. $z^{1/2}$ would live on a surface:



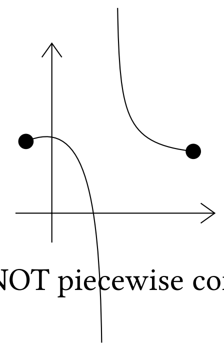
Cauchy's theorem and its consequences

Complex integration

Definition 23. A complex-valued function $\gamma : [a,b] \to \mathbb{C}$ is **piecewise** continuous if γ is continuous at all but *finitely many* points of $[a,b]$ and γ has one-sided limits that are *finite* at each point (of discontinuity).



Piecewise continuous



NOT piecewise continuous

If γ is piecewise continuous, then $\int_a^b \operatorname{Re}\gamma(t)\,\mathrm{d}t$ and $\int_a^b \operatorname{Im}\gamma(t)\,\mathrm{d}t$ exist. Then we define **complex integration**:

$$\int_a^b \gamma(t)\,\mathrm{d}t = \int_a^b \operatorname{Re}\gamma(t)\,\mathrm{d}t + i\cdot\int_a^b \operatorname{Im}\gamma(t)\,\mathrm{d}t$$

That is,

$$\operatorname{Re}\left(\int_a^b \gamma(t)\,\mathrm{d}t\right) = \int_a^b \operatorname{Re}\gamma(t)\,\mathrm{d}t$$
$$\operatorname{Im}\left(\int_a^b \gamma(t)\,\mathrm{d}t\right) = \int_a^b \operatorname{Im}\gamma(t)\,\mathrm{d}t$$

In addition, if γ_1, γ_2 are both $[a,b] \to \mathbb{C}$ and piecewise cont., and $c_1, c_2 \in \mathbb{C}$, then

$$\int_a^b (c_1\gamma_1(t) + c_2\gamma_2(t))\,\mathrm{d}t = c_1\int_a^b \gamma_1(t)\,\mathrm{d}t + c_2\int_a^b \gamma_2(t)\,\mathrm{d}t$$

Proposition 35 (Triangle inequality)**.** If $\gamma : [a,b] \to \mathbb{C}$ is piecewise continuous, then

$$\left|\int_a^b \gamma(t)\,\mathrm{d}t\right| \leq \int_a^b |\gamma(t)|\,\mathrm{d}t$$

Proof. WLOG assume $\int_a^b \gamma(t)\,\mathrm{d}t \neq 0$. Define $\lambda = \frac{\left|\int_a^b \gamma(t)\,\mathrm{d}t\right|}{\int_a^b \gamma(t)\,\mathrm{d}t}$ and note $|\lambda| = 1$.

Thus,

$$\begin{aligned}
\left|\int_a^b \gamma(t)\,\mathrm{d}t\right| &= \lambda \int_a^b \gamma(t)\,\mathrm{d}t \\
&= \int_a^b \lambda\gamma(t)\,\mathrm{d}t && \text{because LHS is } \in \mathbb{R} \\
&= \operatorname{Re} \int_a^b \lambda\gamma(t)\,\mathrm{d}t \\
&\leq \int_a^b |\lambda\gamma(t)|\,\mathrm{d}t && \because \operatorname{Re} z \leq |z| \\
&= \int_a^b |\gamma(t)|\,\mathrm{d}t && \because |\lambda| = 1
\end{aligned}$$

□

Complex differentiability

Definition 24. $\gamma : [a,b] \to \mathbb{C}$ is **differentiable** at $t \in [a,b]$ if $\operatorname{Re}\gamma$ and $\operatorname{Im}\gamma$ are differentiable (in the sense of real variables). We define

$$\gamma'(t) = (\operatorname{Re}\gamma)'(t) + i \cdot (\operatorname{Im}\gamma)'(t)$$

Definition 25. $\gamma : [a,b] \to \mathbb{C}$ is **piecewise C^1** if:

← C^1 is one-time differentiable

(a) γ is continuous on $[a,b]$.
(b) γ is differentiable at all but finitely many points of $[a,b]$.
(c) γ' is continuous at each point where it exists.
(d) γ' has finite one-sided limits at every point of discontinuity.

Fundamental theorem of calculus, complex edition

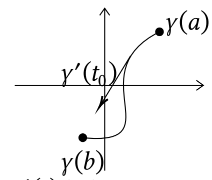
If $\gamma : [a,b] \to \mathbb{C}$ is piecewise C^1, then:

$$\int_a^b \gamma'(t)\,\mathrm{d}t = \gamma(b) - \gamma(a)$$

Definition 26. If γ is C^1, then the arclength of γ is:

$$L(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}\, t$$

$\leftarrow$



$\gamma'(t)$ is instantaneous velocity, so its absolute value is the speed

Definition 27. If $\gamma : [a,b] \to \Omega$ is piecewise C^1 and $f : \Omega \to \mathbb{C}$ is continuous, then

$$\int_\gamma f(z) \, \mathrm{d}\, z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}\, t$$

where $z = \gamma(t)$ and $\mathrm{d}\, z = \gamma'(t) \, \mathrm{d}\, t$

We have **linearity** w.r.t. f:

$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z)) \, \mathrm{d}\, z = c_1 \int_\gamma f_1(z) \, \mathrm{d}\, z + c_2 \int_\gamma f_2(z) \, \mathrm{d}\, z$$

Remark. Arclength is independent from parameterization.

Proof. Let $\gamma : [a,b] \to \Omega$ be piecewise C^1. Let $\alpha : [c,d] \to [a,b]$ is an increasing, piecewise C^1 surjection such that $\alpha(c) = a, \alpha(d) = b$. Then $\phi = \gamma \circ \alpha : [c,d] \to \Omega$ is also piecewise C^1. Hence, by substituting $s = \alpha(t), \mathrm{d}\, s = \alpha'(t) \, \mathrm{d}\, t$:

$$\begin{aligned}
\int_\phi f(z) \, \mathrm{d}\, z &= \int_c^d f(\phi(t))\phi'(t) \, \mathrm{d}\, t \\
&= \int_c^d f(\gamma(\alpha(t)))\gamma'(\alpha(t))\alpha'(t) \, \mathrm{d}\, t \\
&= \int_a^b f(\gamma(s))\gamma'(s) \, \mathrm{d}\, s \\
&= \int_\gamma f(z) \, \mathrm{d}\, z
\end{aligned}$$

$\square$

An important estimate

Let f be continuous. Since $\int_\gamma f(z) \, \mathrm{d}\, z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}\, t$, we observe:

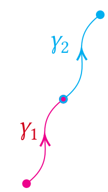
$$\begin{aligned}
\left| \int_\gamma f(z) \, \mathrm{d}\, z \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}\, t \right| \\
&\leq \int_a^b |f(\gamma(t))| \, |\gamma'(t)| \, \mathrm{d}\, t
\end{aligned}$$

$$\begin{aligned}&\le \max_{t\in[a,b]} |f(\gamma(t))| \int_a^b |\gamma'(t)|\,\mathrm{d}t \\ &= \max_{z\in\gamma} |f(z)| \cdot L(\gamma)\end{aligned}$$

Definition 28. If $\gamma : [a,b] \to \mathbb{C}$, the reverse of γ is $(-\gamma) : [-b,-a] \to \mathbb{C}$ defined by $(-\gamma)(t) = \gamma(-t)$. Hence,

← going around the track backwards

$$\int_{-\gamma} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z$$



Remark. We can also break up the curve and integral the two parts separately:

←

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{\gamma_2} f(z)\,\mathrm{d}z$$

Fundamental theorem of calculus for contour integrals

If $\gamma : [a,b] \to \mathbb{C}$ is piecewise C^1, and $f : \Omega \to \mathbb{C}$ is analytic , then

← Assuming f' continuous, which we would prove later

$$\int_\gamma f'(z)\,\mathrm{d}z = f(\gamma(b)) - f(\gamma(a))$$

If $\gamma(a) = \gamma(b)$, then $\int_\gamma f'(z)\,\mathrm{d}z = 0$.

Proof.

$$\begin{aligned}\int_\gamma f'(z)\,\mathrm{d}z &= \int_a^b f'(\gamma(t))\gamma'(t)\,\mathrm{d}t \\ &= \int_a^b (f\circ\gamma)'(t)\,\mathrm{d}t && \text{chain rule} \\ &= f(\gamma(b)) - f(\gamma(a))\end{aligned}$$

□

Example 21. Let γ be a circle of radius R centered at z_0: $\gamma(t) = z_0 + Re^{it}$, $t \in [0, 2\pi]$. We would like to find $\int_\gamma (z-z_0)^n\,\mathrm{d}z$.

If $n \neq -1$, then $\left(\frac{(z-z_0)^{n+1}}{n+1}\right)' = (z-z_0)^n$. Thus,

$$\int_\gamma (z-z_0)^n\,\mathrm{d}z = \int_\gamma \left(\frac{(z-z_0)^{n+1}}{n+1}\right)'\,\mathrm{d}z = 0$$

by FTC.

If $n = -1$,

$$\int_\gamma (z - z_0)^n \,\mathrm{d}z = \int_\gamma \frac{1}{z - z_0} \,\mathrm{d}z = \int_0^{2\pi} i \,\mathrm{d}t = 2\pi i$$

Cauchy's theorem

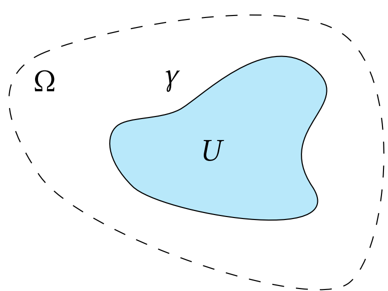
Take 1

Theorem 36 (Cauchy's)**.** Let Ω be a region in $\mathbb{C}$ containing a *simple* piecewise C^1 *closed* curve γ and its interior.

← does not self-intersect

← holes not allowed in the interior

If $f : \Omega \to \mathbb{C}$ is analytic, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.



"Proof". Let U be the union of γ and its interior. Let $f = u + iv$ as usual, write $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$:

$$\begin{aligned}
\int_\gamma f(z)\,\mathrm{d}z &= \int_\gamma (u + iv)(\mathrm{d}x + i\,\mathrm{d}y) \\
&= \int_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\int_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y \\
&= \int\int_U (-v_x - u_y)\,\mathrm{d}x\,\mathrm{d}y + i\int\int_U (u_x - v_y)\,\mathrm{d}x\,\mathrm{d}y && \text{by Green's thm} \\
&= 0 && \text{by Cauchy-Riemann}
\end{aligned}$$

□

However, this 'proof' heavily relies on the fact that u, v are C^1 and that the partial derivatives are continuous. This assumes f' is continuous, but we aren't sure about that yet!

← See Goursat's Lemma

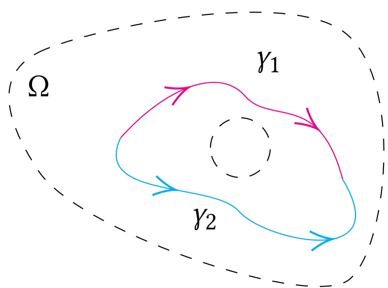
Take 2: deformation version

Theorem 37 (Cauchy's)**.** Let γ_1, γ_2 be piecewise C^1 curves in a region Ω with the same start and end points. If γ_1 can be continuously deformed to γ_2 without ever passing outside of Ω, then

$$\int_{\gamma_1} f(z)\,\mathrm{d}\,z = \int_{\gamma_2} f(z)\,\mathrm{d}\,z$$

By the *previous* statement of Cauchy's theorem (in Theorem 36), we observe that $\int_{\gamma_1-\gamma_2} f(z)\,\mathrm{d}\,z = 0$, so this one falls out.

Non-example 22. The γ_1, γ_2 in the picture below cannot be continuously deformed into each other!



Fresnel integrals

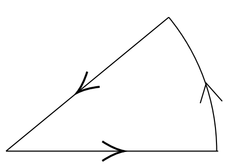
Consider:

$$\int_0^\infty \sin(t^2)\,\mathrm{d}\,t \qquad \text{and} \qquad \int_0^\infty \cos(t^2)\,\mathrm{d}\,t$$

aka.

$$\lim_{R\to\infty}\int_0^R \sin(t^2)\,\mathrm{d}\,t \qquad \text{and} \qquad \lim_{R\to\infty}\int_0^R \cos(t^2)\,\mathrm{d}\,t$$

It's not obvious that these integrals converge!

Solution: PIZZA!  Let γ be the 'sum' of all 3 curves as shown. Let $R \to \infty$. Then, by Cauchy's theorem, $\int_\gamma e^{iz^2}\,\mathrm{d}\,z = 0$.

(Scratch work begins)

Remark. We don't know how to write out the antiderivative of $f(z) = e^{iz^2}$ but we can use series!

$$\begin{aligned} f(z) &= e^{iz^2} \\ &= \sum_{n=0}^{\infty} \frac{(iz^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n z^{2n}}{n!} \end{aligned}$$

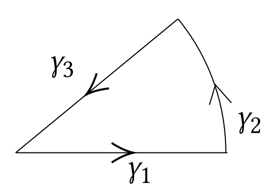
And so

$$F(z) = \sum_{n=0}^{\infty} \frac{i^n z^{2n+1}}{(2n+1)n!}$$

(Scratch ends here)

Now we return to the integral. Strategy:

$$0 = \int_\gamma e^{iz^2}\,\mathrm{d} = \underbrace{\int_{\gamma_1} e^{iz^2}\,\mathrm{d}z}_{I_1(R)} + \underbrace{\int_{\gamma_2} e^{iz^2}\,\mathrm{d}z}_{I_2(R)} + \underbrace{\int_{\gamma_3} e^{iz^2}\,\mathrm{d}z}_{I_3(R)}$$



Evaluate $I_1(R)$: We observe that z is real for this one. Parameterize $z = t$ where t is a real variable.

$$\begin{aligned} I_1(R) &= \int_{\gamma_1} e^{it^2}\,\mathrm{d}t \\ &= \int_0^R \cos(t^2)\,\mathrm{d}t + i \cdot \int_0^R \sin(t^2)\,\mathrm{d}t \end{aligned}$$

Hence, $\lim_{R\to\infty} I_1(R) = \int_0^\infty \cos(t^2)\,\mathrm{d}t + i \cdot \int_0^\infty \sin(t^2)\,\mathrm{d}t$.

Evaluate $I_2(R)$:

Parameterize γ_2 as $z = Re^{i\theta}$ where $\theta \in [0, \frac{\pi}{4}]$. Hence, $\mathrm{d}\,z = iRe^{i\theta}\,\mathrm{d}\,\theta$. Then:

$$\begin{aligned}
|I_2(R)| &= \left|\int_{\gamma_2} e^{i\theta^2}\,\mathrm{d}\,\theta\right| \\
&= \left|\int_0^{\frac{\pi}{4}} e^{i(Re^{i\theta})^2} iRe^{i\theta}\,\mathrm{d}\,\theta\right| \\
&= \left|R\int_0^{\frac{\pi}{4}} e^{iR^2e^{i2\theta}} e^{i\theta}\,\mathrm{d}\,\theta\right| \\
&\le R\int_0^{\frac{\pi}{4}} \left|e^{iR^2e^{i2\theta}}\right|\mathrm{d}\,\theta && \text{by tri. ineq.} \\
&\le R\int_0^{\frac{\pi}{4}} e^{-R^2\sin 2\theta}\,\mathrm{d}\,\theta && \text{since when } x, y \in \mathbb{R},\ \left|e^{x+iy}\right| = e^x \\
&\le R\int_0^{\frac{\pi}{4}} e^{-R^2\frac{4\theta}{\pi}}\,\mathrm{d}\,\theta && \text{since when } x \in [0, \frac{\pi}{2}],\ \frac{2}{\pi}x \le \sin x \\
&= \frac{-R\pi}{R^2 4} e^{-R\frac{4\theta}{\pi}}\Big|_{\theta=0}^{\theta=\frac{\pi}{4}} \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

Thus, $\lim_{R\to\infty} I_2(R) = 0$. :)

Evaluate $I_3(R)$:

$$\begin{aligned}
I_3(R) &= \int_{\gamma_3} e^{iz^2}\,\mathrm{d}\,z \\
&= \int_R^0 e^{i(e^{i\frac{\pi}{4}}t)^2} e^{i\frac{\pi}{4}}\,\mathrm{d}\,t \\
&= -e^{i\frac{\pi}{4}}\int_0^R e^{-t^2}\,\mathrm{d}\,t \\
\lim_{R\to\infty} I_3(R) &= -(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2})\int_0^\infty e^{-t^2}\,\mathrm{d}\,t \quad \text{by Gaussian integral, } \int_0^\infty e^{-t^2}\,\mathrm{d}\,t = \frac{\sqrt{\pi}}{2} \\
&= -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}
\end{aligned}$$

Therefore, we see $I_1(R) + I_2(R) + I_3(R) = 0$ where $\lim_{R\to\infty} I_1(R) = \int_0^\infty \cos(t^2)\,\mathrm{d}\,t + i \cdot \int_0^\infty \sin(t^2)\,\mathrm{d}\,t$, $I_2(R) \to 0$ and $I_3(R) = -\sqrt{\frac{\pi}{8}} - i\sqrt{\frac{\pi}{8}}$. Hence, we would be able to conclude that

$$\int_0^\infty \sin(t^2)\,\mathrm{d}\,t = \sqrt{\frac{\pi}{8}} \qquad \text{and} \qquad \int_0^\infty \cos(t^2)\,\mathrm{d}\,t = \sqrt{\frac{\pi}{8}}$$

Goursat's lemma

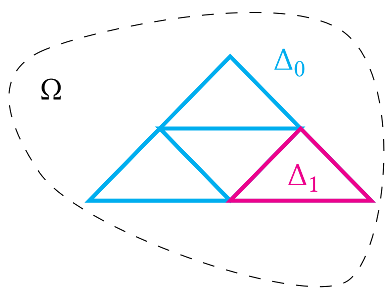
This lemma patches the hole that we have to assume f' continuous in Cauchy's theorem!

Lemma 38 (Goursat's)**.** If $f : \Omega \to \mathbb{C}$ is analytic and Δ is a triangle in Ω whose interior lies inside Ω, then $\int_\Delta f(z)\,\mathrm{d}z = 0$.

← Does not assume f' continuous!

Proof. WLOG orient $\Delta_0 = \Delta$ counterclockwise. Bisect sides of Δ_0 and construct smaller triangles Δ_{0j} where $j = 1, 2, 3, 4$. Then,

$$I = \int_{\Delta_0} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\Delta_{0j}} f(z)\,\mathrm{d}z$$



By triangle inequality,

$$|I| \leq \sum_{j=1}^{4} \left| \int_{\Delta_{0j}} f(z)\,\mathrm{d}z \right|$$

Thus, there exists $j \in \{1, 2, 3, 4\}$ such that

$$\frac{|I|}{4} \leq \left| \int_{\Delta_{0j}} f(z)\,\mathrm{d}z \right|$$

For this j, define $\Delta_1 = \Delta_{0j}$.

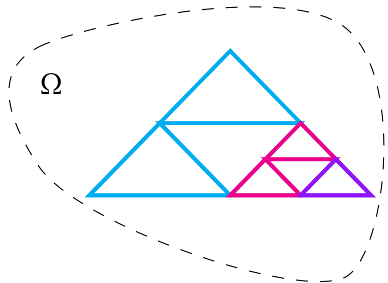
We disect Δ_1 again into smaller triangles Δ_{1j} where $j = 1, 2, 3, 4$. Then,

$$I = \int_{\Delta_1} f(z)\,\mathrm{d}z = \sum_{j=1}^{4} \int_{\Delta_{1j}} f(z)\,\mathrm{d}z$$

Again, by triangle inequality, there is a $j \in \{1, 2, 3, 4\}$ such that

$$\frac{|I|}{4^2} \leq \frac{1}{4} \left| \int_{\Delta_1} f(z)\,\mathrm{d}z \right| \leq \left| \int_{\Delta_{1j}} f(z)\,\mathrm{d}z \right|$$

For this j, define $\Delta_2 = \Delta_{1j}$.



...continue in this manner to get nested triangles Δ_n such that

$$\frac{|I|}{4^{n+1}} \leq \frac{1}{4}\left|\int_{\Delta_n} f(z)\,\mathrm{d}\,z\right| \leq \left|\int_{\Delta_{nj}} f(z)\,\mathrm{d}\,z\right|$$

for all $n \geq 0$.

Now let $\ell = L(\Delta_0)$ denote perimeter of the original triangle (blue).
Then $L(\Delta_n) = \frac{\ell}{2^n}$. ← Perimeter of Δ_n

Let K_n denote the triangle Δ_n union with its interior such that K_n is closed (in fact, compact!). Let $\zeta_n \in K_n$ for $n \geq 0$. Then there is $N \in \mathbb{N}$, such that for all $m, n \geq N$ we have $|\zeta_m - \zeta_n| \leq \operatorname{diam}(K_N) \leq \frac{\ell}{2^N}$. Thus, ζ_n as a sequence is Cauchy.

Let $z_0 = \lim_{n\to\infty} \zeta_n$, note $z_0 \in \bigcap_{n=0}^{\infty} K_n$ and $z_0 \in \Omega$. Since f is analytic at z_0, given $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \frac{\varepsilon}{\ell^2}$$

Now consider multiplying $|z - z_0|$ on both sides:

$$|f'(z_0)\cdot(z - z_0) - f(z) + f(z_0)| < \frac{\varepsilon}{\ell^2}|z - z_0|$$
$$|f(z_0) + f'(z_0)(z - z_0) - f(z)| < \frac{\varepsilon}{\ell^2}|z - z_0|$$

Since $f(z_0) + f'(z_0)(z - z_0)$ is **linear**, it has an antiderivative on $\mathbb{C}$. Thus,

$$\int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0)\,\mathrm{d}\,z = 0$$

by FTC! Now pick n large enough so that $|z - z_0 < \delta$ for all $z \in \Delta_n$. Thus,

$$|I| \leq 4^n \left|\int_{\Delta_n} f(z)\,\mathrm{d}\,z\right|$$

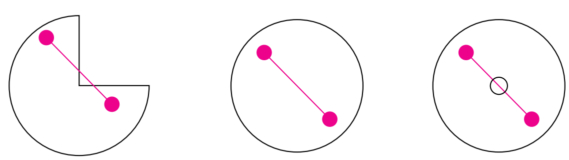
$$
\begin{aligned}
&= 4^n \left| \int_{\Delta_n} f(z_0) + f'(z_0)(z - z_0) - f(z) \right| \\
&\leq 4^n \frac{\varepsilon}{\ell^2} |z - z_0| \frac{\ell}{2^n} && \text{by tri. ineq. and } \left| \int_\gamma g(z)\, \mathrm{d}\, z \right| \leq \sup_{z \in \gamma} |g(z)| \cdot L(\gamma) \\
&< \frac{4^n \varepsilon}{\ell 2^n} \cdot \frac{\ell}{2^n} \\
&= \varepsilon
\end{aligned}
$$

□

Local antiderivative

Theorem 39. If Ω is convex and $f : \Omega \to \mathbb{C}$ is analytic, then f has an antiderivative on Ω.

Remark. Line segments don't exit the region in convex shapes:



not convex convex not convex

Proof. Fix $w \in \Omega$ and define:

$$F(z) = \int_{[w,z]} f(\zeta)\, \mathrm{d}\, \zeta$$

for $z \in \Omega$.

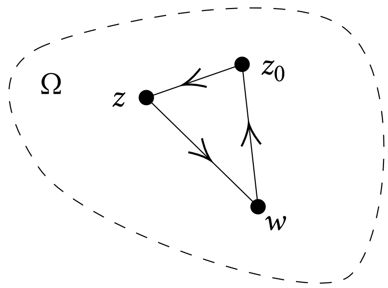
← $[w, z]$ is the line segment from w to z.

This is well-defined if Ω is convex.

Now we want to show that F' is f. That is equivalent to showing that for all $\varepsilon > 0, z_0 \in \Omega$, there exists $\delta > 0$ s.t. whenever $|z - z_0| < \delta$, we have

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| < \varepsilon$$

Let $z_0 \in \Omega$ be given and $\varepsilon > 0$. Goursat says integrals around the triangle is 0, so we suppose $z \in \Omega \backslash \{z_0, w\}$ and get a triangle:



and we know that

$$\underbrace{\int_{[w,z_0]} f(\zeta)\,\mathrm{d}\zeta}_{F(z_0)} + \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta + \underbrace{\int_{[z,w]} f(\zeta)\,\mathrm{d}\zeta}_{-F(z)} = 0$$

So $F(z) - F(z_0) = \int_{[z_0,z]} f(\zeta)\,\mathrm{d}\zeta$. Thus,

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z_0,z]} (f(\zeta) - f(z_0))\,\mathrm{d}\zeta$$

Since f is analytic at z_0, it is continuous there. Given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have $|f(z) - f(z_0)| < \varepsilon$.

Therefore, whenever $|z - z_0| < \delta$, we have

← still by $\left|\int_\gamma g(z)\,\mathrm{d}z\right| \leq \sup_{z\in\gamma} |g(z)| \cdot L(\gamma)$

$$\begin{aligned}\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| &\leq \frac{\varepsilon}{|z - z_0|} L([z_0, z]) \\ &= \frac{\varepsilon}{|z - z_0|}|z - z_0| \\ &= \varepsilon\end{aligned}$$

□

Cauchy's theorem, Take 3

Cauchy's theorem for convex regions

Theorem 40. If Ω is convex, $f : \Omega \to \mathbb{C}$ analytic and γ is a piecewise C^1 curve in Ω, then $\int_\gamma f(z)\,\mathrm{d}z = 0$.

← Since Ω is convex, the interior of γ lies inside Ω.

Proof. Previous theorem says f has an antiderivative F on Ω. Thus,

$$\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma F'(z)\,\mathrm{d}z = 0$$

by FTC! □

Cauchy's integral formula

Cauchy's integral formula for a circle

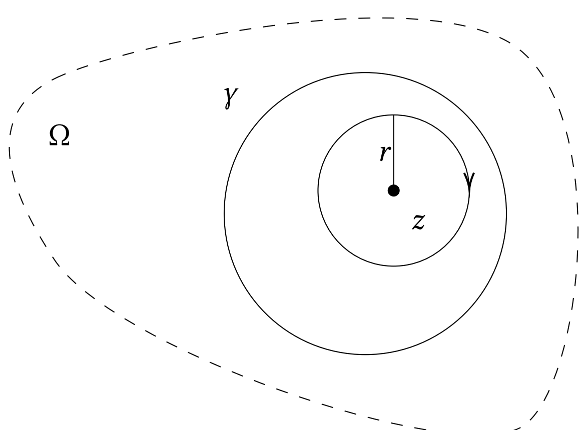
Theorem 41. If f is analytic on a region Ω that contains the circle γ and its interior, then

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

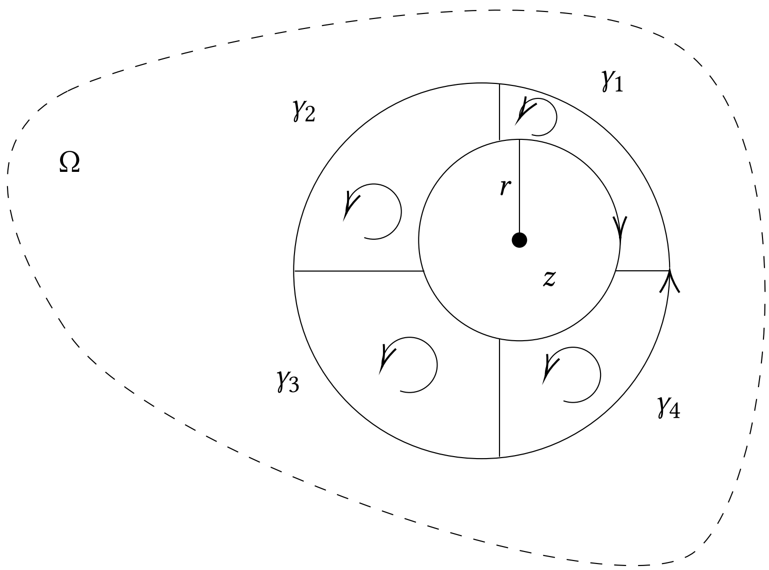
for all z inside of γ.

← this Ω doesn't need to be convex

Proof. Let $r > 0$ be small enough so that the closed ball $B_r(z)^-$ is in the interior of γ. Let $C_r(z) = \{\zeta \in \mathbb{C} : |\zeta - z| = r\}$ traversed clockwise.



Construct $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as pictured:



Cauchy's theorem for convex regions says $\int_{\gamma_i} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = 0$ for all $i = 1, 2, 3, 4$.

Hence,

$$0 = \sum_{j=1}^{4} \int_{\gamma_j} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

And thus:

$$\int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} = \int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

for all $r > 0$ that is *sufficiently* small.

Therefore:

← by HW6 Ex5, or Thm12 Lect 11

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - f(z) \cdot 1 \right| &= \left| \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - f(z) \cdot \left(\frac{1}{2\pi i} \int_{C_r(z)} \frac{\mathrm{d}\zeta}{\zeta - z} \right) \right| \\
&= \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - f(z) \cdot \left(\frac{1}{2\pi i} \int_{C_r(z)} \frac{\mathrm{d}\zeta}{\zeta - z} \right) \right| \\
&= \lim_{r \to 0^+} \left| \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) - f(z)}{\zeta - z} \right| \\
&\le \lim_{r \to 0^+} \max_{|\zeta - z| = r} \left| \frac{f(\zeta) - f(z)}{\zeta - z} \right| \cdot r \\
&= 0
\end{aligned}$$

$\square$

Mean value properties

Corollary 42 (Mean value property for analytic functions)**.** If f analytic on an open set Ω which contains $B_r(z)^-$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt$$

Proof. Apply Theorem 41 with $\zeta = z + re^{it}$ and $d\zeta = ire^{it} \, dt, t \in [0, 2\pi]$ and get

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(\zeta) \, d\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{C_r(z)} \frac{f(z + re^{it}) ire^{it} \, dt}{z + re^{it} - z} \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{it}) \, dt \end{aligned}$$

□

Remark. There is a mean value property for harmonic functions!

Existence of power series expansions

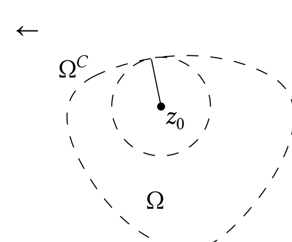
Theorem 43. If $f : \Omega \to \mathbb{C}$ is analytic and $z_0 \in \Omega$ then f has a power series expansions

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

that converges **locally uniformly** on the disk

$$|z - z_0| < \operatorname{dist}(z_0, \Omega^C) = \inf_{w \in \Omega^C} |z_0 - w|$$

when Ω^C is nonempty.



Moreover, the radius of convergence is the radius of the largest open disk centered at z_0 upon which f could be analytically continued.

Proof. Let $r < \operatorname{dist}(z_0, \Omega^C)$ and $|z - z_0| \le \rho < r$. Then

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta) \, d\zeta}{\zeta - z}$$

for all $|z - z_0| < \rho$.

As a function of ζ, the series

← geometric series trick!

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}}$$

and so by geometric series formula:

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0} \right)^n \\ &= \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \qquad \text{for } |z - z_0| \le \rho \end{aligned}$$

converges uniformly on $|\zeta - z_0| = r$ by the Weierstrass M-test with $M_n = \left| \frac{z - z_0}{\zeta - z_0} \right|^n \le \left(\frac{\rho}{r} \right)^n$.

Thus,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\, \mathrm{d}\zeta}{\zeta - z} \\ &= \frac{1}{2\pi i} \int_{C_r(z_0)} \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \cdot f(\zeta)\, \mathrm{d}\zeta \\ &= \sum_{n=0}^{\infty} (z - z_0)^n \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\, \mathrm{d}\zeta}{(\zeta - z_0)^{n+1}} \end{aligned}$$

And so we have our $\frac{f^{(n)}(z_0)}{n!} = a_n$ in the highlighted part above. □

Remark. Consequently, we also get Cauchy's theorem of derivatives:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)\, \mathrm{d}\zeta}{(\zeta - z_0)^{n+1}}$$

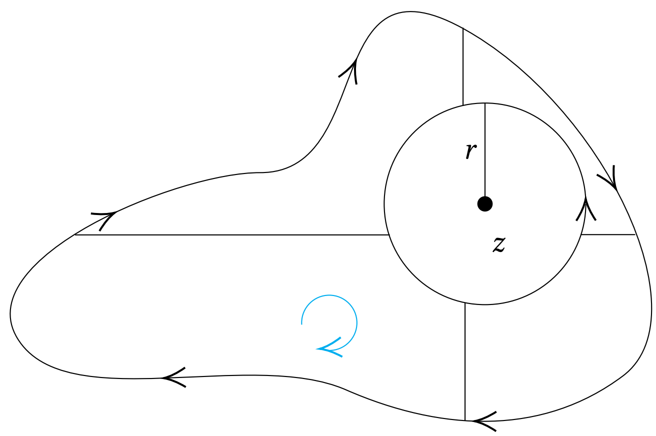
Example 23. What is the radius of convergence for the power series of

$$f(x) = \frac{e^{\sin x} + e^{-x^2} + x^2 + 7x^3}{\cos x}$$

centered at $x_0 = 2$?

The theorem guarantees the existence of the power series, and the RoC would simply be the radius of which f could be analytically continued. We observe that $f(x)$ cannot be defined when $\cos x = 0$, i.e. $x = \frac{\pi}{2}$. Hence, the radius of convergence is just $2 - \frac{\pi}{2}$ – no need to compute *any* derivatives or coefficients!

So now we have this result for computing the derivatives and integrals around a circle $C_r(z_0)$. Can we extend this to other closed curves of any shapes?



Same techniques! Hence,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z)^{n+1}}$$

on any such closed curve γ.

Liouville's theorem

Theorem 44 (Liouville's)**.** A bounded entire function is constant. ← analytic on $\mathbb{C}$

Proof. Suppose f is entire and $|f(z)| \le M$ is bounded by M for all $z \in \mathbb{C}$. Then

$$f'(z) = \frac{1!}{2\pi i} \int_{C_R(z)} \frac{f(\zeta)}{(\zeta - z)^2}\,\mathrm{d}\zeta$$

by Cauchy's integral formula. Hence, $|f'(z)| \le \frac{1}{2\pi} \cdot \frac{M}{R^2} \cdot 2\pi R = \frac{M}{R}$ by the upper bound. Since f is entire, there is no limit in what R could be, so we let $R \to \infty$ and observe that $|f'(z)| = 0$ for all $z \in \mathbb{C}$. Hence, f' is identically 0, and so f is constant. ☐

Non-example 24. We know $|\cos x| \le 1$ for all $x \in \mathbb{R}$, but $\cos z = \frac{e^{iz}+e^{-iz}}{2}$ is **not** bounded on $\mathbb{C}$. In fact, $\cos(-ix) = \frac{e^{x}+e^{-x}}{2}$ is unbounded for real x, so $\cos x$ is not bounded on the imaginary axis. Hence, we can't use Liouville's theorem here!

Fundamental theorem of algebra

Theorem 45 (FToA)**.** Every **nonconstant** complex polynomial has a zero in $\mathbb{C}$.

← recall "$\mathbb{C}$ is an algebraically closed field"

Proof. Suppose towards a contradiction that p is a **nonconstant** polynomial over $\mathbb{C}$ with no zeros in $\mathbb{C}$. Then $f = \frac{1}{p}$ is an entire function because we never divide by 0. Recall HW2 Ex2 showed that $\lim_{|z|\to\infty} p(z) = \infty$. That is, for any $M > 0$, there exists $R > 0$ such that whenever $|z| > R$, we have $|p(z)| > M$.

Thus, $\lim_{|z|\to\infty} f(z) = \lim_{|z|\to\infty} \frac{1}{p(z)} = 0$. In particular, we can find a $R > 0$ such that whenever $|z| > R$, we have $|f(z)| < 1$ is bounded outside of the circle $|z| = R$. Since the closed disk $|z| \leq R$ is compact and f is continuous, f is bounded inside this closed disk $|z| \leq R$.

← Extreme value theorem

Hence, f is a bounded entire function, meaning that it is constant by Liouville's theorem, and hence p is **constant** too. This cause a contradiction. □

Zeros of analytic functions

Recall the analytic functions are infinitely differentiable.

Suppose $f : \Omega \to \mathbb{C}$ is analytic and $f(z_0) = 0$ for some $z_0 \in \Omega$, and f is not identically 0 on an open neighbourhood of z_0. Then

$$f(z) = \sum_{j=n}^{\infty} a_j (z - z_0)^j$$

for some $n \geq 1$ such that $a_n \neq 0$. Hence:

← the lowest power term that has a nonzero coefficient, and also n is the order of the zero z_0.

$$\begin{aligned} f(z) &= \sum_{j=n}^{\infty} a_j (z - z_0)^j \\ &= (z - z_0)^n \sum_{j=n}^{\infty} a_j (z - z_0)^{j-n} \\ &= (z - z_0)^n \sum_{k=0}^{\infty} a_{n+k} (z - z_0)^k \end{aligned}$$

let $g(z) = \sum_{k=0}^{\infty} a_{n+k}(z - z_0)^k$. Observe that g is analytic and $g(z_0) = a_n \neq 0$. This and the continuity of g at z_0 ensures that g is nonzero on some open disk $|z - z_0| < \delta$. Therefore, $f(z) = (z - z_0)^n g(z)$ does not vanish on $0 < |z - z_0| < \delta$.

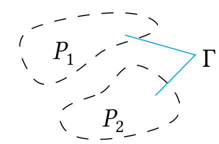
Remark. The zeros of f are **isolated** in Ω. That is, we can't have a sequence of zeros of f converging to some $z_0 \in \Omega$, as then we can't find a nonzero disk around z_0!

Theorem 46. If $f : \Omega \to \mathbb{C}$ is analytic and not identically zero, then each zero of f is isolated and has finite order.

Proof. Assume BWOC that the zeros are not isolated.

By definition, Ω is connected. By definition×2, a subset $S \subseteq \Omega$ is **clopen** if it is open and closed as a subset of Ω. In a connected region Ω, only $\emptyset, \Omega$ are clopen.

← Example of nontrivial clopen subsets:



In this Γ (NOT a region), the clopen subsets are $P_1, P_2, \emptyset, \Gamma$.

Let $S = \{z \in \Omega : f^{(j)}(z) = 0 \quad \forall j = 0, 1, 2, \dots\}$. If $z_0 \in S$ then f is zero on some open disk centered at z_0. Hence, S is open!

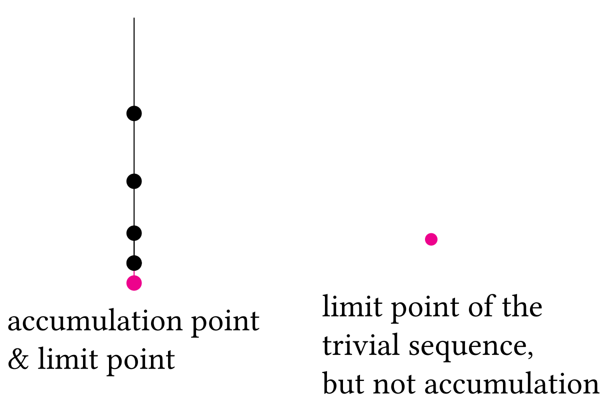
Now suppose w is a limit of a sequence in S. Since f is continuous, $f^{(j)}$ is continuous for all $j \in \mathbb{N}$. This enruses that $f^{(j)}(w) = 0$. Thus, S is closed!

Therefore, S is clopen in Ω, so either S is the empty set or $S = \Omega$. If $S = \Omega$, then f is the zero function, so that cannot happen! Therefore, $S = \emptyset$, and so we don't have a cluster of zeros. ☐

Corollary 47. If f is a nonconstant analytic function, its zero set is **countable**. This is because within an open region, we can have at most countably infinite number of disjoint open sets. We let these open sets be $f^{-1}(\{0\})$.

Identity theorem

Definition 29. An **accumulation** point of S is a point that is the limit of a sequence of **distinct** points of S.



Theorem 48. Let $f, g : \Omega \to \mathbb{C}$ be analytic. If $f = g$ on a subset of Ω that has an accumulation point in Ω, then $f = g$ on the entire Ω.

Proof. If the zero set of $f - g$ has an accumulation point in Ω, then $f - g$ has a zero that is not isolated (no open disk around it since some zeros keep converging to that accumulation point), so $f - g$ is identically zero on Ω. $\square$

Example 25. There is only one way to extend $\cos x, \sin x, \exp x$ from $\mathbb{R}$ to $\mathbb{C}$ because two entire functions that agree on $\mathbb{R}$ agree on $\mathbb{C}$.

Example 26. Similarly, there is also only one way to get an analytic continuation of the Riemann zeta function to $\operatorname{Re} s > 0$.

Maximum modulus principle

Recall this handwavy physics application here. We now have a more rigorous way to state this!

← Not exactly equivalent, though.

Theorem 49 (Maximum modulus principle)**.** Let f be analytic on a region Ω that contains a piecewise C' simple closed curve γ and its interior. Then

$$|f(z)| \leq \max_{\zeta \in \gamma} |f(\zeta)|$$

for all z in the interior of γ.

Proof. Let $M = \max_{\zeta \in \gamma} |f(\zeta)|$. Fix z inside γ. Let L denote the length of γ and let $r = \inf_{\zeta \in \gamma} |z - \zeta$, which is positive (so z isn't arbitrarily close to γ).

Apply Cauchy's integral formula to the n-th power of f:

$$f(z)^n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)^n \,\mathrm{d}\zeta}{\zeta - z}$$

Thus,

$$|f(z)|^n \leq \frac{1}{2\pi} \cdot \frac{M^n}{r} L$$

Now we just take the n-th root everywhere:

$$|f(z)| \leq M \left(\frac{L}{2\pi r}\right)^{1/n}$$

We use arbitrarily large n and get $|f(z)| \leq M$. $\square$

Schwarz' lemma

Lemma 50 (Schwarz')**.** Let $f : \mathbb{D} \to \mathbb{D}^-$ be analytic and $f(0) = 0$. Then:

(a) $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

(b) If $|f'(0)| = 1$ or $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, then $f(z) = \lambda z$ for some λ with $|\lambda| = 1$.

Proof part (a). Since $f(0) = 0$, we have that the constant term of f is 0, and so $f(z) = zg(z)$ for some g analytic on $\mathbb{D}$. Thus,

$$f'(z) = g(z) + zg'(z)$$

and hence $f'(0) = g(0)$. Hence,

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

If $|z| \leq r < 1$, then by the maximum modulus principle,

$$\begin{aligned} |g(z)| &\leq \max_{|\zeta|=r} |g(\zeta)| \\ &= \max_{|\zeta|=r} \left| \frac{f(\zeta)}{\zeta} \right| \\ &\leq \frac{1}{r} && \text{since } f : \mathbb{D} \to \mathbb{D}^- \end{aligned}$$

Let $r \to 1^-$ and get $|g(z)| \leq 1$ for all $z \in \mathbb{D}$, which is the (a) part of our result. □

Proof part (b). Maximum modulus principle says the given conditions imply g is constant. The constant λ has absolute value 1, so $\frac{f(z)}{z} = \lambda$ and so $f(z) = \lambda z$. □

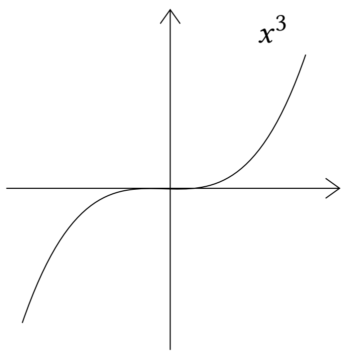
Automorphism group of a region

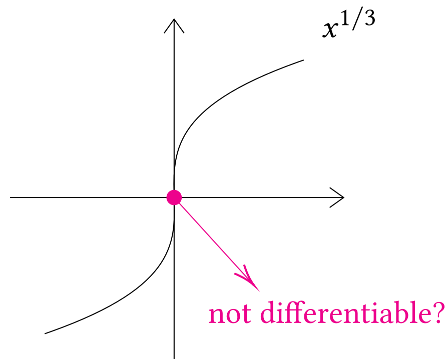
Definition 30. Let Ω be a region in $\mathbb{C}$. We let the automorphism group of the region $\text{Aut}(\Omega)$ be the set of all **bijective analytic functions** from Ω to Ω.

- $\text{Aut}(\Omega)$ contains the identity function $f(z) = z$.
- $\text{Aut}(\Omega)$ is closed under composition.
- $\text{Aut}(\Omega)$ is closed under inverses: if $f : \Omega \to \Omega$ is an analytic bijection, then $f^{-1} : \Omega \to \Omega$ exists and is **analytic**.

← And composition is a binary operation with associativity

Remark. There appears to be a 'counterexample':

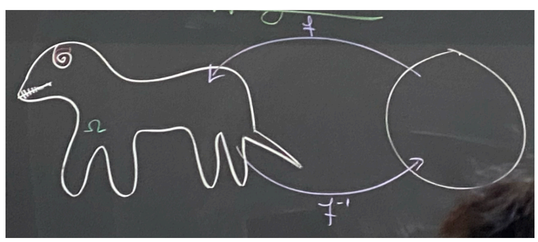




However, this is only true in $\mathbb{R}$. In $\mathbb{C}$, we observe that z^3 is **not** a bijection at 0, so this function is not in the group at all!

Theorem 51 (Riemann Mapping). If Ω is simply connected (no holes), then it could be conformally mapped to a disk.

← Except for the entire $\mathbb{C}$, which has only constant functions if bounded (by Liouville thm).



Recall from HW1 that for each $w \in \mathbb{D}$, we have a bijection

$$\phi_w(z) = \frac{-z + w}{-\bar{w}z + 1}$$

from $\mathbb{D}$ to $\mathbb{D}$ and $\phi \circ \phi = \text{id}$. To see this, observe the matrix representation

$$\begin{bmatrix} -1 & w \\ -\bar{w} & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & w \\ -\bar{w} & 1 \end{bmatrix} = \begin{bmatrix} 1 - |w|^2 & 0 \\ 0 & 1 - |w|^2 \end{bmatrix} \sim I$$

Furthermore, note $\phi_w(0) = w$. Suppose $f \in \text{Aut}(\mathbb{D})$, then there is a unique $w \in \mathbb{D}$ such that $f(w) = 0$. Define $g = f \circ \phi_w \in \text{Aut}(\mathbb{D})$. Note that $g(0) = f(\phi_w(0)) = f(w) = 0$. By Schwarz' lemma, we have $|g(z)| \le |z|$ for all $z \in \mathbb{D}$.

Since $g^{-1} \in \text{Aut}(\mathbb{D})$, we also have $|g^{-1}(z)| \le |z|$ for all $z \in \mathbb{D}$. Now substitute $g(z)$ for z since it's also in the disk. Hence, $|z| = |g^{-1}(g(z))| \le |g(z)|$. Therefore, we are forced to conclude that $|z| = |g(z)|$ for ALL $z \in \mathbb{D}$!

Since $|z| = |g(z)|$ for ALL $z \in \mathbb{D}$, Schwarz' lemma says $g(z) = \lambda z$ for some $|\lambda| = 1$. Let $\lambda = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Thus,

$$g(z) = f(\phi_w(z)) = e^{i\theta} z$$

and so

$$e^{i\theta}\phi_w(z) = f(\phi_w(\phi_w(z))) = f(z)$$

since $\phi_w \circ \phi_w = \mathrm{id}$. Therefore, $f(z) = e^{i\theta}\dfrac{w - z}{1 - \bar{w}z}$.

Therefore,

Proposition 52.

$$\mathrm{Aut}(\mathbb{D}) = \left\{ e^{i\theta}\frac{w-z}{1-\bar{w}z} \;\middle|\; \theta \in [0, 2\pi), w \in \mathbb{D} \right\}$$

Remark. The topological representation of the automorphism group of $\mathbb{D}$ is a 'skinless torus' (collection of open disks revolving from 0 to 2π).

Morera's theorem

Theorem 53 (Morera)**.** If $f : \Omega \to \mathbb{C}$ is continuous and $\int_\gamma f(\zeta)\,\mathrm{d}\zeta = 0$ for all __________ γ in Ω, then f is analytic on Ω.

← the blank can be 'rectangles', 'triangles', 'piecewise C^1 closed curves', etc.

Proof see notes. □

Weierstrass convergence theorem

Let $f_n(z)$ be analytic for every $n \in \mathbb{N}$. We are still not sure that $\sum_{n=1}^{\infty} f_n(z)$ is analytic yet!

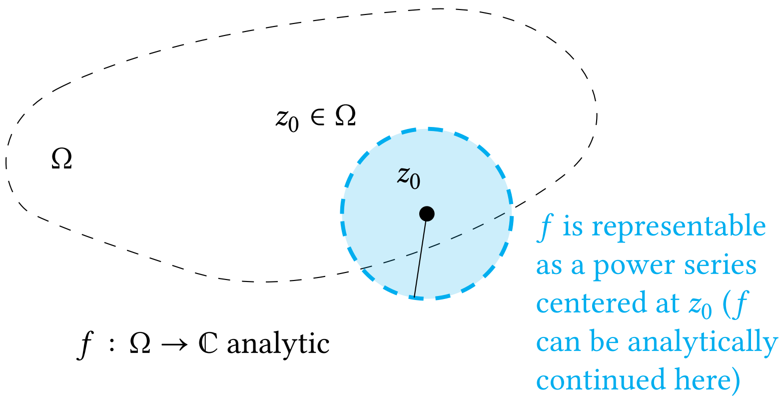
Theorem 54 (Weierstrass convergence)**.** If $f_n : \Omega \to \mathbb{C}$ are analytic and f_n converges *locally uniformly* on Ω to the limit function f ($\Leftrightarrow$ uniform convergence on compact sets), then f is analytic and for each fixed m, f_n^m converges to $f^{(m)}$ locally uniformly on Ω and f is infinitely differentiable.

Remark. This is a huge contrast with the *Weierstrass approximation theorem* in real analysis, which says that if $f : [0, 1] \to \mathbb{R}$, then there is a sequence of polynomials p_n such that p_n converges to f uniformly on $[0, 1]$. That is, even the most pathological, nowhere-differentiable functions in $\mathbb{R}$ are a limit of some polynomial sequences! However, in the $\mathbb{C}$ world, the limit of any analytic function is still analytic.

Laurent series & isolated singularities

Sometimes the domain Ω isn't the largest domain where an analytic function can be analytic. So far, we know we can find the largest disk centered at a point in Ω in which a function is analytic and the power series exists there:

← the disk could exceed the bounds of Ω!



Can we do even better than that?

Laurent series

Example 27. Let $f(z) = \frac{1}{z(z-1)}$ analytic on $\mathbb{C}\backslash\{0, 1\}$. We realize that if we restrict $0 < |z| < 1$, then

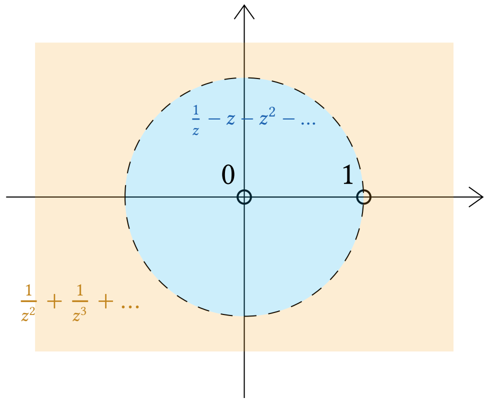
$$f(z) = \frac{-1}{z} \cdot \frac{1}{1-z} = \frac{-1}{z} - 1 - z - z^2 - \ldots$$

is the **Laurent series** of $f(z)$ centered at 0, a point where $f(z)$ isn't even defined!

Example 28. We continue with the previous function. This time, we restrict $|z| > 1$ and express it as:

$$f(z) = \frac{1}{z^2(1 - \frac{1}{z})} = \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

Hence, we now have different series for f in different regions:



Definition 31 (Laurent series)**.** A series in the form $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is a **Laurent series**. It converges at $z \in \mathbb{C}$ if **both** the analytic part $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and the principal part $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ converge at z. If this occurs, the Laurent series would be

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

and also converges.

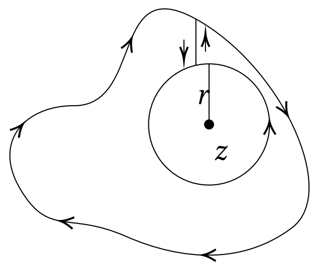
Lemma 55. Recall that if $n \neq -1$, then $\frac{(z-z_0)^{n+1}}{n+1}$ is an antiderivative of $(z-z_0)^n$ on $\mathbb{C}$. **So** if γ is simple closed, then by FTC,

$$\frac{1}{2\pi i}\int_\gamma (z-z_0)^n \,\mathrm{d}z = 0$$

whenever $n \neq 1$. In addition, by Cauchy's integral formula, $\frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-z_0} = 1$. Therefore, if z_0 is in the interior of a simple closed curve γ, then

$$\frac{1}{2\pi i}\int_\gamma (z-z_0)^n \,\mathrm{d}z = \begin{cases} 0 & n \neq -1 \\ 1 & n = -1 \end{cases}$$

Proof. We previously know the result above when γ is a circle. We now extend it to all simple closed curves by a familiar trick as follows:



□

Laurent expansion theorem

Theorem 56 (Laurent expansion)**.** Suppose f is analytic on the annular region $A = \{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$. Then f has a locally uniformly convergent Laurent expansion

← $R_1 = 0, R_2 = \infty$ are okay

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

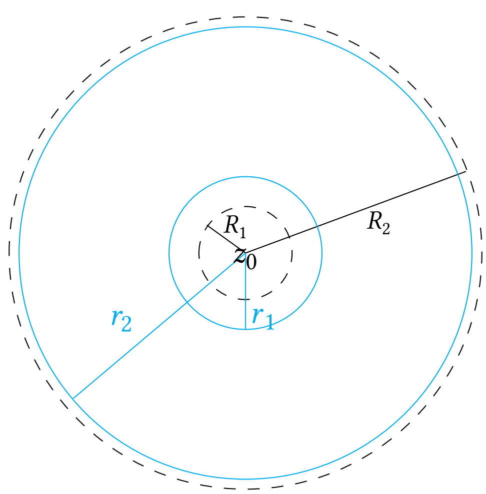
on A. Moreover, the Laurent coefficients are

$$a_n = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$$

for any r such that $R_1 < r < R_2$.

Proof gist. For a gist of why this works:

← For rigorous proof, see notes.



Cauchy's integral formula reveals that

$$f(z) = \frac{1}{2\pi i}\int_{C_{r_2}(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{C_{r_1}(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

whenever

$$R_1 < r_1 < |z| < r_2 < R_2.$$

These integrals are independent of r_1 and r_2 so long as $r_1 < |z| < r_2$. □

Remark. If $n \geq 0$ and f is analytic on $|z| < R_2$, then we should get that the Taylor series expansion and the Laurent expansion for the same function f to match. They indeed do match by Cauchy's integral formula:

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(\zeta)\,\mathrm{d}\zeta}{(\zeta - z_0)^{n+1}}$$

Isolated singularities

Definition 32. If f is analytic on $0 < |z - z_0| < R$ (a deleted neighbourhood of z_0), then z_0 is an **isolated singularity** of f.

Definition 33. If the principal part of the Laurent expansion for f at z_0 is 0 (i.e. $a_{-1} = a_{-2} = \cdots = 0$), then z_0 is a **removable singularity** of f. The Laurent expansion for f at z_0 is simply a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ suggests we set $f(z_0) = a_0$, in which case f is analytic at z_0.

Example 29. Observe

$$\begin{aligned} \frac{\sin z}{z} &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots \end{aligned}$$

We define $f(0) = 1$, so $\frac{\sin z}{z}$ is actually entire!

← This agrees with L'Hôpital's rule.

Theorem 57. If z_0 is an isolated singularity of an analytic function f, then z_0 is removable *if and only if* any of the following hold:

(a) f is bounded on some deleted neighbourhood of z_0

(b) $\lim_{z \to z_0} f(z)$ exists

← ∞ doesn't count

(c) $\lim_{z \to z_0} (z - z_0) f(z) = 0$

Remark. (a) and (b) implies (c), (b) implies (a).

Proof. It suffices to show that (c) $\iff$ removable.

($\implies$) If z_0 is removable, then f is analytic at z_0, so all of the above follow.

($\impliedby$) Suppose (c) holds. Then for all $\varepsilon > 0$, there exists $0 \leq r < 1$ such that whenever $|z - z_0| < 2r$, we have $|f(z)(z - z_0)| < \varepsilon$.
Then, for all $n \geq 1$, we have

$$\begin{aligned}
|a_{-n}| &= \left| \frac{1}{2\pi i} \int_{C_r(z_0)} f(\zeta)(\zeta - z_0)^{n-1} \,\mathrm{d}\zeta \right| \\
&= \left| \frac{1}{2\pi i} \int_{C_r(z_0)} f(\zeta)(\zeta - z_0)(\zeta - z_0)^{n-2} \,\mathrm{d}\zeta \right| \\
&\leq \frac{1}{2\pi} \cdot \varepsilon r^{n-2} 2\pi r \\
&= \varepsilon r^{n-1} \\
&< \varepsilon
\end{aligned}$$

Thus, $a_{-n} = 0$ for all $n \geq 1$. The principal part of the Laurent expansion of f is zero.

□

Definition 34. If the principal part of f at z_0 is of the form

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{(z - z_0)}$$

where $a_{-n} \neq 0$, then z_0 is a **pole** of f of order n.

← n must be **finite** such that we can clear the denominator

Definition 35. A pole of order 1 is a **simple pole**.

Theorem 58. If z_0 is an isolated singularity of f, then z_0 is a **pole** of order $\leq n$ *if and only if* there is an analytic $\phi(z)$ on a deleted neighbourhood of z_0 such that

← Return to the full neighbourhood by the trick of removable singularity

$$f(z) = \frac{\phi(z)}{(z - z_0)^n}$$

This occurs *if and only if* any of the following hold:

(a) $(z - z_0)^n f(z)$ is bounded on some deleted neighbourhood of z_0
(b) $\lim_{z \to z_0} f(z)(z - z_0)^n$ exists
(c) $\lim_{z \to z_0} f(z)(z - z_0)^{n+1} = 0$

Remark. We can think of poles and zeros in the following fashion:

$f(z) = (z - z_0)^j F(z)$	$g(z) = \frac{G(z)}{(z-z_0)^k}$	
f has a zero of order j at z_0	g has a pole of order k at z_0	Then:
F doesn't vanish at z_0	G doesn't vanish at z_0	

$$f(z)g(z) = (z - z_0)^{j-k} F(z)G(z)$$

- If $j = k$, z_0 is a removable singularity for fg and is not a zero.
- If $j > k$, then z_0 is a zero.
- If $k > j$, then z_0 is a pole.

Poles are nice! They could be removed like the denominators of rational functions. However, some other singularities cannot do so.

Definition 36. If the principal part of the Laurent series for f at z_0 is $\sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$ where **infinitely** many $a_{-n} \neq 0$, then z_0 is an essential singularity of f.

Remark. It is not hard to make an essential singularity: take any entire function with infinite power series. Plug in $\frac{1}{x}$ instead of x.

Example 30. $e^{\frac{1}{x}} = \sum_{n=0}^{\infty} \frac{1}{n!z^n}$ has an essential singularity at 0.

Theorem 59 (Casorati-Weierstrass)**.** Let z_0 be an essential singularity of f. For **each** $w \in \mathbb{C}$, there is a sequence $z_n (n \geq 1)$ such that $z_n \to z_0$ and $f(z_n) \to w$.

← This is wild! f could almost splatter everywhere near z_0. There isn't a reasonable value to assign to $f(z_0)$.

Proof. Suppose towards a contradiction that there exists $w \in \mathbb{C}$ such that no such z_n exists. Then there exists $\varepsilon > 0$ and $\delta > 0$ such that when $0 < |z - z_0| < \delta$, we have $|f(z) - w| \geq \varepsilon$ (that is, f is not getting close to w). Thus, $g(z) = \frac{1}{f(z)-w}$ is analytic on $0 < |z - z_0| < \delta$ and $|g(z)| \leq \frac{1}{\varepsilon}$ there. The singularity z_0 of g is therefore removable. Then $f(z) = w + \frac{1}{g(z)}$, which is either analytic or has a pole at z_0 (if $g(z_0) = 0$). This causes a contradiction. □

Theorem 60 (Great Picard)**.** If z_0 is an essential singularity of f, then in any deleted neighbourhood of z_0, we have f assuming **every** complex value (with at most one exception) **infinitely** many times.

Example 31. $f(z) = e^{\frac{1}{z}}$ has an essential singularity at $z_0 = 0$. (Note $f(z) \neq 0$ is the exceptional value that is never assumed.) Let $w \neq 0$ and let $z = \frac{1}{\log w}$ where $\log w$ is a nonzero logarithm of w. Then

← so no 1 for w

$$f(z) = e^{\frac{1}{1/\log w}} = e^{\log w} = w$$

Theorem 61 (Little Picard)**.** If f is entire and nonconstant, then f assumes every complex value, with at most one exception.

Proof. If f is a nonconstant polynomial and $w \in \mathbb{C}$, then the polynomial $f(z) - w$ has a zero in $\mathbb{C}$ by the Fundamental Theorem of Algebra, so f assumes the value of w.

If f is not a polynomial, then $f(\frac{1}{z})$ has an essential singularity at 0. Then use Great Picard Thm. □

← Non-polynomial means the Taylor series is infinite

Residues

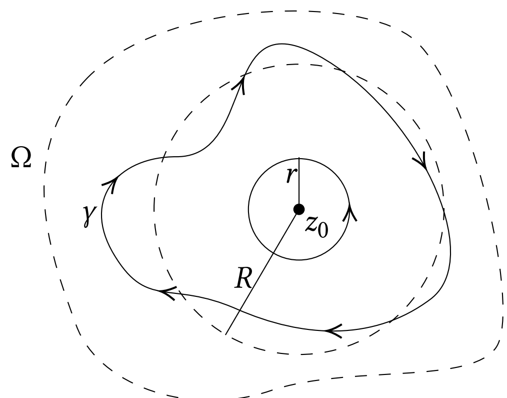
Definition 37. Let the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ be analytic at $0 < |z-z_0| < R$. The coefficient a_{-1} is the **residue** of f at z_0. Notation:

$$\operatorname{Res}(f; z_0) = a_{-1}$$

Theorem 62 (Residue, simple vers.)**.** Let $f : \Omega \to \mathbb{C}$ analytic except on the isolated singularity z_0. Then:

$$\frac{1}{2\pi i}\int_\gamma f(\zeta)\, d\zeta = \operatorname{Res}(f \cdot z_0)$$

for any simple closed curve γ in Ω with z_0 in its interior and whose interior is contained in Ω.



Proof. The Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ converges locally uniformly on some punctured disk $0 < |z-z_0| < R$. If $r \in (0, R)$ is sufficiently small, then the deformation version of Cauchy's theorem implies

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z &= \frac{1}{2\pi i}\int_{C_r(z_0)} f(z)\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n\,\mathrm{d}z \\
&= \sum_{n=-\infty}^{\infty} a_n\left(\frac{1}{2\pi i}\int_{C_r(z_0)} (z-z_0)^n\, dz\right)
\end{aligned}$$

Observe $\left(\frac{1}{2\pi i}\int_{C_r(z_0)} (z-z_0)^n\, dz\right) = 0$ unless $n = -1$, in which it's 1. Hence:

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z &= a_{-1} \\
&= \operatorname{Res}(f; z_0)
\end{aligned}$$

The interchange of sum and integral is permissible because the Laurent series converges uniformly on $C_r(z_0)$. $\square$

Lemma 63. If z_0 is a **simple** pole of f, then

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

← That means the pole is order 1, and the principal part of the Laurent series at that point only has 1 term.

Proof. Near z_0, we have

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$

Thus, $(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + \dots$ tends to a_{-1} when $z \to z_0$. So $a_{-1} = \lim_{z \to z_0}(z - z_0)f(z)$. □

Remark. Cauchy's integral formula is a special case of the residue formula as we rename the function to introduce a simple pole at z_0:

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{f(z)\, \mathrm{d} z}{z - z_0} &= \operatorname{Res}\left(\frac{f(z)}{z - z_0}; z_0\right) \\ &= \lim_{z \to z_0} (z - z_0) \frac{f(z)}{z - z_0} \\ &= f(z_0) \end{aligned}$$

Example 32. Consider the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^2} \, \mathrm{d} x$$

in which $a \neq 0$ is real. We assume that $a > 0$; the case $a < 0$ is similar. Since

$$\left| \frac{\cos ax}{1 + x^2} \right| \leq \frac{1}{1 + x^2}$$

on (-∞,0] and [0,∞), it follows that the improper integral converges by the comparison test.

This allows us to consider the integral from $-\infty$ to ∞ directly, without having to consider the improper integrals over the positive and negative parts separately. Therefore, write

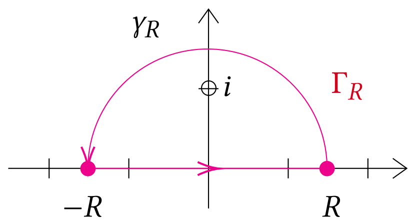
$$\int_{-\infty}^{\infty} \frac{\cos ax}{1 + x^2} dx = \operatorname{Re}\left(\lim_{R \to \infty} \int_{-R}^{R} \frac{e^{iax}}{1 + x^2} dx \right)$$

where we let

$$f(z) = \frac{e^{iaz}}{1 + z^2} = \frac{e^{iaz}}{(z - i)(z + i)}$$

which has two simple poles $z = \pm i$. We focus on i first.

For $R > 1$ (so that i is enclosed), let Γ_R denote the semicircular curve obtained by joining $[-R, R]$ with γ_R, the upper half of the circle $|z| = R$:



Since i is a pole enclosed in Γ_R, the residue theorem implies $\int_{\Gamma_R} f(z)\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, i)$. By Lemma 63,

$$\operatorname{Res}(f, i) = \lim_{z\to i}(z - i)f(z) = \lim_{z\to i}\frac{e^{iaz}}{(z + i)} = \frac{e^{-a}}{2i}$$

so it follows that

$$\int_{-R}^{R}\frac{e^{iax}}{1 + x^2}\,\mathrm{d}x + \int_{\gamma_R}\frac{e^{iaz}}{1 + z^2}\,\mathrm{d}z = 2\pi i \operatorname{Res}(f, i) = \pi e^{-a}$$

We look at $\int_{\gamma_R}\frac{e^{iaz}}{1+z^2}\,\mathrm{d}z$. If $z = x + iy$ is on γ_R, then $y \geq 0$ and hence (since $a > 0$):

$$\begin{aligned}
\left|\int_{\gamma_R}\frac{e^{iaz}}{1 + z^2}\,\mathrm{d}z\right| &= \left|\int_{\gamma_R}\frac{e^{iaz}}{1 + z^2}\,\mathrm{d}z\right| \\
&\leq \pi R \sup_{z\in\gamma_R}\frac{|e^{iaz}|}{|1 + z^2|} && \text{by upper bound over length of curve} \\
&\leq \pi R \sup_{x+iy\in\gamma_R}\frac{e^{-ay}}{R^2 - 1} && \text{since } |e^{iax}| = 1 \\
&= \frac{\pi R}{R^2 - 1}
\end{aligned}$$

which tends to zero as $R \to \infty$. Let $R \to \infty$ and get

$$\int_{-\infty}^{\infty}\frac{e^{iax}}{1 + x^2}\,\mathrm{d}x = \pi e^{-a}$$

Thus the real part would be our answer

$$\int_{-\infty}^{\infty}\frac{\cos ax}{1 + x^2}dx = \pi e^{-a}$$

Residue theory

Index, aka. winding number of a curve

Definition 38. Let γ be a closed, piecewise C^1 curve and $z_0 \notin \gamma$. The **index** (also called the **winding number**) of γ with respect to z_0 is

← Number of counterclockwise loop-arounds

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\, z}{z - z_0}$$

Remark. If the curve $\gamma : [a, b] \to \mathbb{C}$ is parameterized on t, and $\gamma(a) = \gamma(b)$ (closed), then let $z = \gamma(t)$, $\mathrm{d}\, z = \gamma'(t)\, \mathrm{d}\, t$. Then we have

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\, z}{z - z_0} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)\, \mathrm{d}\, t}{\gamma(t) - z_0}$$

Lemma 64. If γ is a closed curve and $z_0 \notin \gamma$, then $I(\gamma; z_0) \in \mathbb{Z}$.

Proof. Parameterize γ as above using s. Then

$$\frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}\, z}{z - z_0} = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(s)\, \mathrm{d}\, s}{\gamma(s) - z_0}$$

Define

$$g(t) = \int_a^t \frac{\gamma'(s)\, \mathrm{d}\, s}{\gamma(s) - z_0}$$

Since γ is piecewise, by FTC, we have

$$g'(t) = \frac{\gamma'(t)}{\gamma(t) - z_0}$$

for all but finitely many $t \in [a, b]$. Thus,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\, t} \left(e^{-g(t)}(\gamma(t) - z_0) \right) &= e^{-g(t)}\gamma'(t) - g'(t) e^{-g(t)}(\gamma(t) - z_0) \\
&= e^{-g(t)}\gamma'(t) - \frac{\gamma'(t)}{\gamma(t) - z_0} e^{-g(t)}(\gamma(t) - z_0) \\
&= 0
\end{aligned}$$

for all t where $g'(t)$ exists. Therefore, $e^{-g(t)}(\gamma(t) - z_0)$ is piecewise constant. But this function is also continuous, so it's constant! Therefore:

$$e^{-g(b)}(\gamma(b) - z_0) = e^{-g(a)}(\gamma(a) - z_0)$$

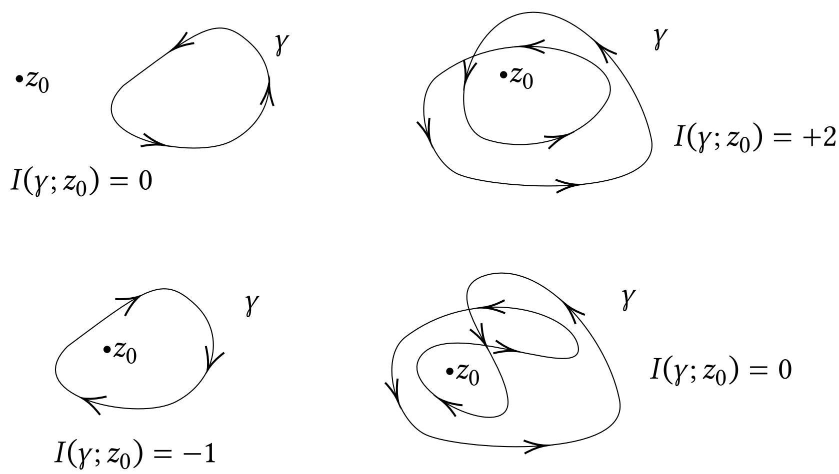
The blue terms are the same since $\gamma(a) = \gamma(b)$. Therefore, $e^{-g(b)} = e^{-g(a)} = e^0 = 1$ since $g(a) = 0$.

Hence, $g(b) = 2\pi i n$ for some $n \in \mathbb{Z}$. Thus:

$$I(\gamma; z_0) = \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0} = \frac{1}{2\pi i} g(b) = n \in \mathbb{Z}$$

□

Remark. Winding number essentially tracks the change of argument when the curve is traversed.



Simply connected domains

Definition 39. A region Ω is **simply connected** if it has no holes. In other words:

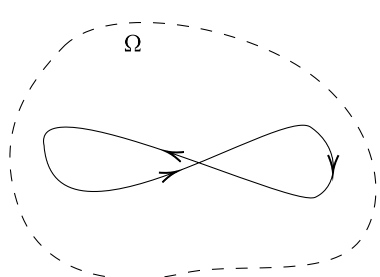
(a) $I(\gamma; z_0) = 0$ for **every** closed curve γ in Ω and every $z_0 \notin \Omega$.

(b) Every closed curve γ in Ω is **homotopic** to a point in Ω.

← homotopic means can be continuously deformed without passing outside Ω

Recall Theorem 36. We can now extend beyond simple curves:

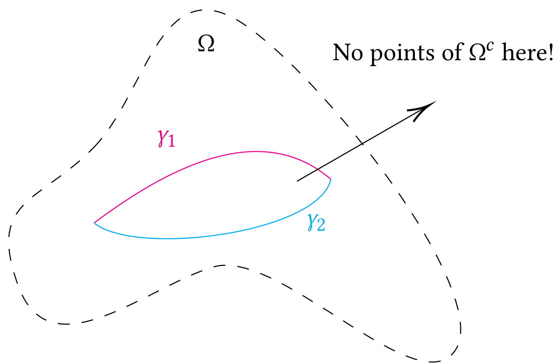
Theorem 65 (Cauchy's, for simply connected domains)**.** If Ω is **simply connected**, f is analytic on Ω, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve γ in Ω.



Theorem 66. A region Ω is **simply connected** *if and only if* every analytic function $f : \Omega \to \mathbb{C}$ has an **antiderivative** on Ω.

Proof. We proved in Theorem 39 that every analytic function $f : \Omega \to \mathbb{C}$ on a *convex* Ω has an antiderivative. We adapt the proof.

($\Longrightarrow$)



Then use Theorem 37.

($\Longleftarrow$) Suppose BWOC that Ω is **not** simply connected. Then there is a $z_0 \in \Omega^c$ and γ a closed curve in Ω such that $I(\gamma; z_0) \neq 0$. Then $f(z) = \frac{1}{z-z_0}$ does not have an antiderivative on Ω.

$\square$

Theorem 67. If Ω is simply connected and $f : \Omega \to \mathbb{C}$ is analytic and **never 0**, then there is an analytic $g : \Omega \to \mathbb{C}$ such that $f = e^g$. That is, it's got a log!

Proof. The function f'/f is analytic on Ω, thus it has an antiderivative F on Ω. Since

$$(fe^{-F})' = f'e^{-F} - F'fe^{-F} = f'e^{-F} - f'e^{-F} = 0$$

it follows that $fe^{-F} = c$ for some constant c. Thus, $g = \log c + F$ (we may choose any fixed branch of $\log c$). $\square$

Theorem 68 (Residue, general case)**.** Let Ω be a simply-connected region and let $z_1, z_2, \ldots, z_n \in \Omega$ be distinct. If $f : \Omega \backslash \{z_1, z_2, \cdots, z_n\} \to \mathbb{C}$ is analytic and γ is a closed curve in Ω that passes through no z_i, then

$$\frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta = \sum_{j=1}^{n} \operatorname{Res}(f; z_j) I(\gamma, z_j)$$

The argument principle

Suppose $f : \Omega \to \mathbb{C}$ is analytic and has zeros only at $z_1, z_2, \ldots, z_n \in \Omega$ (repeated according to multiplicity). Write

$$f(z) = (z - z_1)(z - z_2) \cdots (z - z_n)\, g(z)$$

where $g(z)$ is analytic and nonvanishing on Ω. The product formula for derivatives implies

$$\begin{aligned} f'(z) &= (z - z_2)(z - z_3) \cdots (z - z_n)\, g(z) + (z - z_1)(z - z_3) \cdots (z - z_n)\, g(z) + \cdots \\ &+ (z - z_1)(z - z_2) \cdots (z - z_{n-1})\, g(z) + (z - z_1)(z - z_2) \cdots (z - z_n)\, g'(z). \end{aligned}$$

Divide by $f(z)$ and obtain the logarithmic derivative of f:

← Since $(\log f)' = f'/f$

$$\frac{f'(z)}{f(z)} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_n} + \frac{g'(z)}{g(z)}$$

If γ is a simple closed curve in Ω whose interior lies in Ω and which contains each z_i in its interior, then

← a simple closed curve can only envelope a finite amount of zeros!

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z &= \sum_{k=1}^{n} \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_k} + \frac{1}{2\pi i} \int_\gamma \frac{g'(z)}{g(z)} \, \mathrm{d}z \\ &= \sum_{k=1}^{n} I(\gamma; z_k) + 0 \\ &= \left(\sum_{k=1}^{n} 1 \right) + 0 \\ &= n \end{aligned}$$

The final integral vanishes by Cauchy's theorem since g'/g is analytic on Ω. Integrating the logarithmic derivative f'/f of an analytic function f around a closed curve γ counts the number of zeros of f, repeated according to multiplicity, inside of γ.

Theorem 69 (The Argument Principle). Let Ω be a region in $\mathbb{C}$ and let γ be a simple closed curve in Ω with its interior in Ω. If $f : \Omega \to \mathbb{C}$ is analytic and has no zeros on γ, then the number of zeros $Z_f(\gamma)$ of f, repeated according to multiplicity, in the interior of γ is finite and is given by

$$Z_f(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z$$

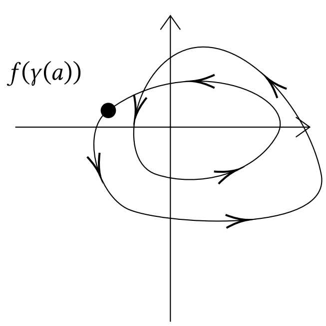
Proof. In light of the preceding discussion, we only need to show that f has only finitely many zeros inside of γ. Let G denote the union of γ and its interior. Since G is closed and bounded, it is compact. If f had infinitely many distinct zeros z_n inside of γ, these would have an accumulation point in $G \subseteq \Omega$. The identity theorem would imply that f is identically zero on Ω, which contradicts the hypothesis that f does not vanish on γ. $\square$

Remark. Why *argument* principle? Let $\gamma : [a, b] \to \mathbb{C}$ be a parametrization and consider the curve $f \circ \gamma : [a, b] \to \mathbb{C}$. The following computation shows that the number of zeros of f inside γ equals the winding number of $f \circ \gamma$ with respect to the origin:

$$\begin{aligned} Z_f(\gamma) &= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z \\ &= \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t) \, \mathrm{d}t}{f(\gamma(t))} \\ &= \frac{1}{2\pi i} \int_{f\circ\gamma} \frac{\mathrm{d}\zeta}{\zeta - 0} \\ &= I(f \circ \gamma; 0) \end{aligned}$$

in which $\zeta = f(\gamma(t))$ and $d\zeta = f'(\gamma(t))\gamma'(t)\,dt$ by the chain rule.

It allows computers to compute roots with great ease. As soon as we have an error $< \frac{1}{2}$ we are done.



$\log z = \log|z| + i \arg z$

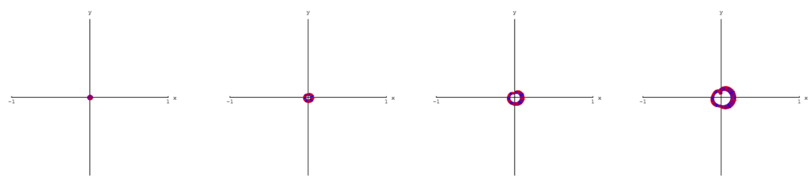
Corollary 70 (Root counting formula)**.** If $f : \Omega \to \mathbb{C}$ is analytic and γ is a simple closed curve in γ with its interior in Ω such that $f(z) \neq w$ on γ, then the number of roots of $f(z) = w$ inside γ (with multiplicity) is

$$Z_{f(z)-w}(\gamma) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - w} \, \mathrm{d}z$$

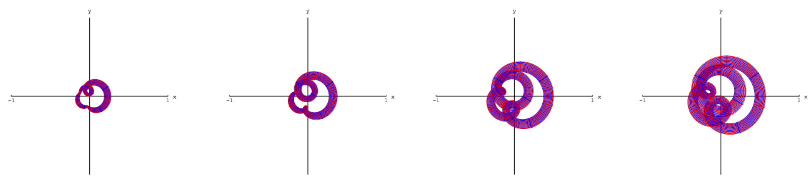
Example 33. Consider the function

$$f(z) = z\left(\frac{z+\frac{1}{2}}{1+\frac{1}{2}z}\right)\left(\frac{z-\frac{3}{4}}{1-\frac{3}{4}z}\right)\left(\frac{z-\frac{4i}{5}}{1+\frac{4i}{5}z}\right)$$

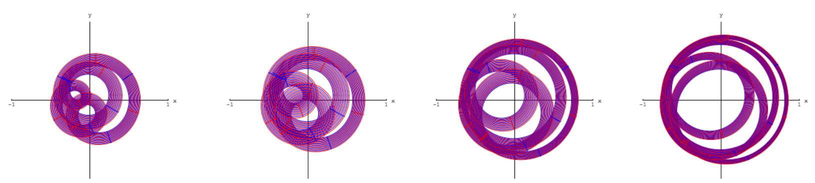
Being a product of disk automorphisms, f maps $\mathbb{D} \to \mathbb{D}$. It has roots $0, \frac{-1}{2}, \frac{4i}{5}, \frac{3}{4}$. We could observe the increment of winding number corresponding to how many times zeros are included.



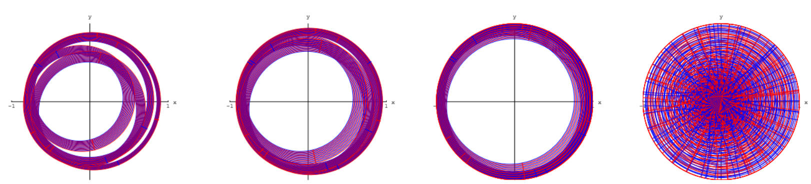
$|z| \leqslant 0.1$ $0.1 \leqslant |z| \leqslant 0.2$ $0.2 \leqslant |z| \leqslant 0.3$ $0.3 \leqslant |z| \leqslant 0.4$



$0.4 \leqslant |z| \leqslant 0.5$ $0.5 \leqslant |z| \leqslant 0.6$ $0.6 \leqslant |z| \leqslant 0.7$ $0.65 \leqslant |z| \leqslant 0.75$



$0.70 \leqslant |z| \leqslant 0.80$ $0.75 \leqslant |z| \leqslant 0.85$ $0.85 \leqslant |z| \leqslant 0.90$ $0.900 \leqslant |z| \leqslant 0.925$



$0.925 \leqslant |z| \leqslant 0.950$ $0.950 \leqslant |z| \leqslant 0.975$ $0.975 \leqslant |z| \leqslant 1$ $0 \leqslant |z| \leqslant 1$

Rouché's theorem

Theorem 71 (Rouché's)**.** Let $f, g : \Omega \to \mathbb{C}$ be analytic on Ω and let γ be a simple closed curve in Ω that is homotopic to a point in Ω. If $|f(z) - g(z)| < |f(z)| + |g(z)|$ on γ, then f, g have the same number of zeros (by multiplicity) inside γ.

← observe that this is a ridiculously lenient hypothesis!

Proof. Note that the hypothesis implies that f, g don't vanish on γ. Therefore, we can divide g on both sides and get $\left|\frac{f}{g}-1\right| < \left|\frac{f}{g}\right| + 1$ on γ. This inequality is violated whenever f/g is a nonpositive real number (≤ 0) on γ.

Thus, f/g maps γ into $\mathbb{C} \setminus (-\infty, 0]$. If $\ell(z)$ is the principal branch of the logarithm, then $\ell\left(\frac{f}{g}\right)$ is defined on γ, and we have the logarithmic derivative

← Recall that the principal branch of the logarithm has domain $\mathbb{C} \setminus (-\infty, 0]$

$$\frac{\mathrm{d}}{\mathrm{d}z}\ell\left(\frac{f}{g}\right) = \frac{(f/g)'}{f/g}$$

on some open set containing γ. The Fundamental Theorem of Calculus and the argument principle imply

$$\begin{aligned}
0 &\overset{FTC}{=} \frac{1}{2\pi i}\int_\gamma \frac{(f(z)/g(z))'}{f(z)/g(z)}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\gamma \left(\frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \cdot \frac{g(z)}{f(z)}\right)\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\gamma \left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right)\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z - \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z \\
&= Z_f(\gamma) - Z_g(\gamma)
\end{aligned}$$

□

Corollary 72 (Weak Rouché's)**.** Let $f, h : \Omega \to \mathbb{C}$ be analytic on Ω and γ be a closed curve in Ω that is homotopic to a point in Ω. If $|h(z)| < |f(z)|$ for all $z \in \gamma$, then f and $f + h$ have the same number of zeros (counted by multiplicity) inside of γ.

← think h perturbs f a little bit

Proof. If $z \in \gamma$, then

$$|(f(z) + h(z)) - f(z)| = |h(z)| < |f(z)| \leq |f(z) + h(z)| + |f(z)|.$$

This is a significant overestimation. Let $f + h$ be the g in Theorem 71 and obtain the result. □

Remark. How to think about Corollary 72? Let $f(z)$ where $z \in \gamma$ be the position of a dog walker in a garden. Let 0 be a tree. Let $f(z) + h(z)$ denote the position of the dog on leash. The fact that $|h| < |f|$ means the leash is shorter than the distance from the walker to the origin. We observe that the dog cannot walk around the tree more times than the owner!

Fundamental Theorem of Algebra

Corollary 73 (FTA)**.** If p is a polynomial of degree $n \geq 1$, then $p(z)$ has exactly n roots in $\mathbb{C}$, counted according to multiplicity.

Proof. If $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ and $a_n \neq 0$, then

← a polynomial is dominated by its leading term

$$\lim_{z \to \infty} \frac{p(z)}{a_n z^n} = 1$$

For sufficiently large $R > 0$,

$$|z| = R \quad \implies \quad \left| \frac{p(z)}{a_n z^n} - 1 \right| < 1$$

and hence

$$|z| = R \quad \implies \quad \left| \frac{p(z)}{f(z)} - \frac{a_n z^n}{g(z)} \right| < \left| \frac{a_n z^n}{g(z)} \right|.$$

Weak Rouche's theorem (Corollary 72) implies that $p(z)$ and $a_n z^n$ have the same number of zeros (namely n), counted according to multiplicities, inside any disk of sufficiently large radius. □

Example 34. Consider the transcendental equation $e^z = 3z^n$, in which n is a positive integer. How many solutions does it have inside the unit circle?

← observe this is hard to solve by non-numerical methods

Let

$$f(z) = e^z - 3z^n \qquad \text{and} \qquad g(z) = -3z^n$$

and note that g has precisely n zeros (counted by multiplicity) in $|z| < 1$.

For $|z| = 1$,

$$|\underbrace{(e^z - 3z^n)}_{f(z)} - \underbrace{(-3z^n)}_{g(z)}| = |e^z| = e^{\operatorname{Re} z} \leq e < 3 = |\underbrace{-3z^n}_{g(z)}| \leq |\underbrace{-3z^n}_{g(z)}| + |\underbrace{e^z - 3z^n}_{f(z)}|$$

Rouché's theorem (Theorem 71) implies that f has exactly n roots inside the unit circle.

Remark. We can also use the argument principle to get $Z_f(\gamma) = I(f \circ \gamma; 0)$ and integrate numerically up to a precision of $1/2$, but Rouché's theorem is certainly more computationally light.

Example 35. Consider

$$f(z) = z^9 - 8z^2 + 5.$$

Since $\deg f = 9$ we do not expect to find its zeros in closed form. However, we can use Rouché's theorem to help locate their general whereabouts.

← cf. Galois theory

Since f has real coefficients and odd degree, the **intermediate value theorem** implies that f has at least one real root. Since f has real coefficients, the non-real roots of f must appear in complex conjugate pairs. Thus, f has an odd number of real roots.

For $|z| = \frac{3}{2}$,

$$\begin{aligned}
|\underbrace{z^9 - 8z^2 + 5}_{f} - \underbrace{z^9}_{g}| &= |8z^2 - 5| \\
&\leq 8(\tfrac{3}{2})^2 + 5 \\
&= 23 \\
&< (\tfrac{3}{2})^9 \quad (\approx 38.44) \\
&= |\underbrace{z^9}_{g}|
\end{aligned}$$

Rouché's theorem implies f has 9 zeros (counted according to multiplicity) in $|z| < \frac{3}{2}$. By FTA, these are all roots of f.

Now we look at smaller regions to gauge the distribution of the roots of f.

For $|z| = 1$,

← this g has 2 roots

$$\begin{aligned}
|\underbrace{z^9 - 8z^2 + 5}_{f} - \underbrace{(-8z^2 + 5)}_{g}| &= |z^9| \\
&= 1 < 3 \leq |\underbrace{-8z^2 + 5}_{g}|
\end{aligned}$$

Rouché's theorem implies f has 2 zeros, counted by multiplicity, in $|z| < 1$.

For $|z| = \frac{1}{2}$,

← this g has 0 roots

$$\begin{aligned}
|\underbrace{z^9 - 8z^2 + 5}_{f} + \underbrace{-5}_{g}| &= |z^9 - 8z^2| \\
&\leq |z|^9 + 8|z|^2 \\
&= \frac{1}{2^9} + 2 \\
&< 5 = |\underbrace{-5}_{g}|
\end{aligned}$$

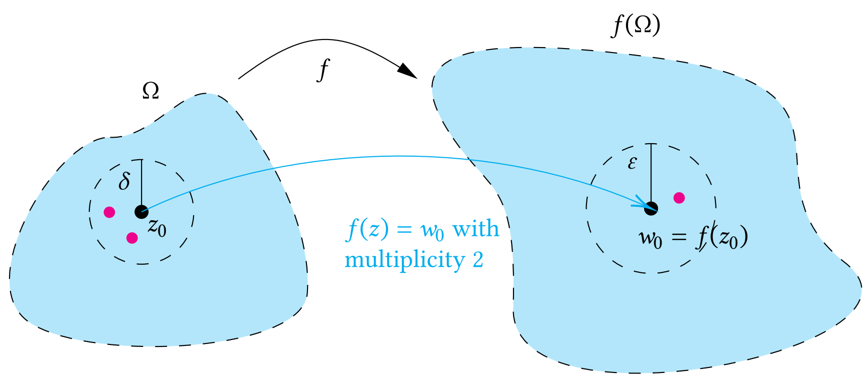
Rouché's theorem implies f has no zeros in $|z| < \frac{1}{2}$.

Local mapping theorem

Theorem 74 (Local mapping)**.** Suppose that $f : \Omega \to \mathbb{C}$ is analytic and nonconstant. Let $z_0 \in \Omega$ and let the value $w_0 = f(z_0)$ be assumed with multiplicity n.

← $f(z) - w_0$ has a zero of order n at z_0

For each sufficiently small $\delta > 0$, there exists $\varepsilon > 0$ such that $0 < |w_0 - w_0| < \varepsilon$ implies that f assumes the value w at exactly n distinct points in $0 < |z - z_0| < \delta$, each with multiplicity one.



Proof. Since the zeros of nonconstant analytic functions are isolated, there is an $r > 0$ such that $B_r(z_0)^-$ is contained in Ω and

$$0 < |z - z_0| \le r \quad \Longrightarrow \quad f(z) \neq w_0 \quad \text{and} \quad f'(z) \neq 0$$

For $0 < \delta < r$,

$$\varepsilon = \min_{|z-z_0|=\delta} |f(z) - w_0| > 0$$

← strictly > 0 because $f(z) \neq w_0$ and circle $|z - z_0| = \delta$ is compact

since the circle $|z - z_0| = \delta$ is compact and $f(z) \neq w_0$ on $|z - z_0| \le r$.

If $0 < |w - w_0| < \varepsilon$ and $|z - z_0| = \delta$, then

$$|\underbrace{(f(z) - w_0)}_{F(z)} - \underbrace{(f(z) - w)}_{G(z)}| = |w - w_0| < \varepsilon \le |\underbrace{f(z) - w_0}_{F(z)}|$$

Rouche's theorem implies that $f - w_0$ and $f - w$ have the same number of zeros in $B_\delta(z_0)$.

By isolated zeros, we know that $f - w_0$ has a zero of order n at z_0 and no other zeros in $B_\delta(z_0)$. Therefore, $f - w$ has exactly n zeros, counted according to multiplicity, in $B_\delta(z_0)$. These zeros must be simple since f' does not vanish on $B_\delta(z_0)$ by isolated zeros. Thus, f assumes the value w_0 at exactly n distinct points in $B_\delta(z_0)$. □

Corollary 75 (Open mapping property)**.** If $f : \Omega \to \mathbb{C}$ is analytic and nonconstant, then if $U \subseteq \Omega$ is open, then $f(U)$ is open.

← i.e. blobs go to blobs

Proof. It suffices to show that $f(\Omega)$ is open since if $U \subseteq \Omega$ is open, we may consider the restriction $f : U \to \mathbb{C}$ instead.

Let $z_0 \in \Omega$, and $w_0 = f(z_0)$. If $\delta > 0$ is sufficiently small, then $B_\delta(z_0) \subseteq \Omega$ and $f(\Omega)$ contains $B_\varepsilon(w_0)$ for some $\varepsilon > 0$. Thus, $f(\Omega)$ is open. □

Theorem 76. If $f : \Omega \to \mathbb{C}$ is analytic and $|f|$ has a local maximum in Ω, then f is constant.

← i.e. local maximum cannot be inside the region and not on the boundary.

Proof. Suppose that $f : \Omega \to \mathbb{C}$ is analytic and nonconstant. If $z_0 \in U \subseteq \Omega$, in which U is open, then $f(U)$ is open and contains $f(z_0)$. Since $f(\Omega)$ contains points of modulus larger than $f(z_0)$, it follows that $|f(z)|$ does not have a local maximum at z_0. □

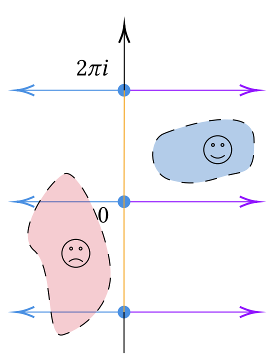
Injectivity

Corollary 77 (Local injectivity)**.** If f is analytic near z_0 and $f'(z_0) \neq 0$, then f is **injective** on some neighborhood of z_0.

Proof ($n = 1$ case of LMT). Let $w_0 = f(z_0)$. If $f'(z_0) \neq 0$, then $f(z) - w_0$ has a zero of order one at z_0.

By the local mapping theorem, for each sufficiently small $\delta > 0$ there exists $\varepsilon > 0$ such that if $0 < |w - w_0| < \varepsilon$, then f assumes the value w at exactly one point in $0 < |z - z_0| < \delta$. □

Example 36. One cannot conclude anything about global injectivity using the preceding results. For example, $f(z) = e^z$ satisfies $f'(z) \neq 0$ for all z, but it is NOT injective on $\mathbb{C}$ since it is $2\pi i$-periodic. It is, however, injective on a small neighborhood of any given point.



Non-example 37. Corollary 77 does not hold for functions of a real variable (if one interprets "analytic" as "differentiable"). Using the definition of the derivative, one can show that

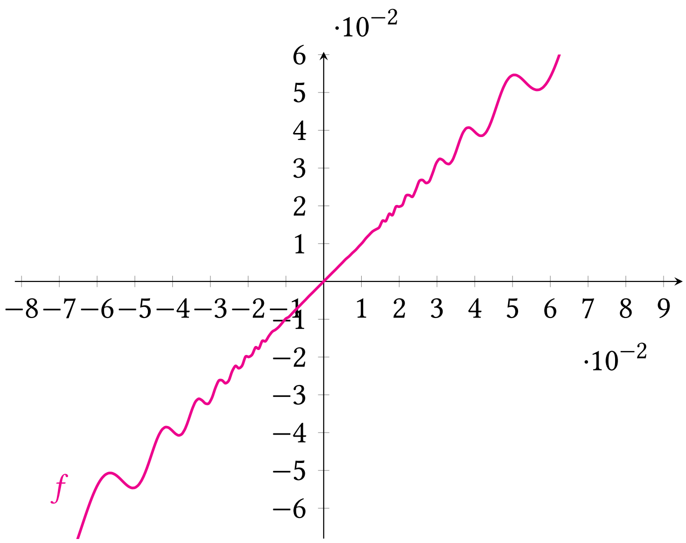
$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x} & x \neq 0, \\ 0 & x = 0, \end{cases}$$

satisfies $f'(0) = 1 > 0$. One might assume that f is injective in some small neighborhood of 0. This turns out to be false (see Figure 1). Indeed, the derivative of f is

$$f'(x) = \begin{cases} 1 + 4x \sin \frac{1}{x} - 2 \cos \frac{1}{x} & x \neq 0, \\ 1 & x = 0, \end{cases}$$

which oscillates between arbitrarily large positive and negative values *infinitely often* as x approaches 0. Thus, f is neither increasing (nor decreasing) on any open interval containing 0. In particular, f is not injective on any neighborhood of 0.

← In complex land, $\sin(1/x)$ has an essential singularity at 0



Corollary 78. If $f : \Omega \to \mathbb{C}$ is injective, then $f'(z) \neq 0$ on Ω.

← i.e. Conformality

Proof. If $f'(z_0) = 0$, then f assumes the value $w_0 = f(z_0)$ at z_0 with multiplicity at least two. The local mapping theorem implies f is not injective on any neighborhood of z_0 since f assumes the value w_0 at at least two distinct points near z_0. □

Summation via residues

Example 38. The function $\sin z$ has a simple zero at $z = 0$. Thus,

← $\sin \pi n = 0$, and the derivative $\cos z$ always has $\cos \pi n \neq 0$.

$$\pi \cot \pi z = \frac{\pi \cos \pi z}{\sin \pi z}$$

has a simple pole at each integer. This property makes the cotangent useful for summing certain infinite series. Let $p(z)$ be a polynomial with no integer zeros. Then

$$f(z) = \frac{\pi \cot \pi z}{p(z)}$$

has an infinite number of simple poles

$$z = 0, \pm 1, \pm 2, \dots$$

from $\cot \pi z$ and a finite number of poles

$$w_1, w_2, \dots, w_r,$$

none of which are integers. Let's find $\operatorname{Res}(f; n), n \in \mathbb{Z}$.

Lemma 79. If g/h has a simple pole at z_0 and $g(z_0) \neq 0$, then

$$\operatorname{Res}\left(\frac{g(z)}{h(z)}; z_0\right) = \frac{g(z_0)}{h'(z_0)}$$

Proof. Since z_0 is a simple pole of g/h, it is a simple zero of h and $h'(z_0) \neq 0$. Thus,

$$\begin{aligned}
\operatorname{Res}\left(\frac{g(z)}{h(z)}; z_0\right) &= \lim_{z \to z_0} (z - z_0) \frac{g(z)}{h(z)} \\
&= \lim_{z \to z_0} g(z) \frac{z - z_0}{h(z) - h(z_0)} \\
&= g(z_0) \lim_{z \to z_0} \frac{z - z_0}{h(z) - h(z_0)} \\
&= \frac{g(z_0)}{h'(z_0)}
\end{aligned}$$

□

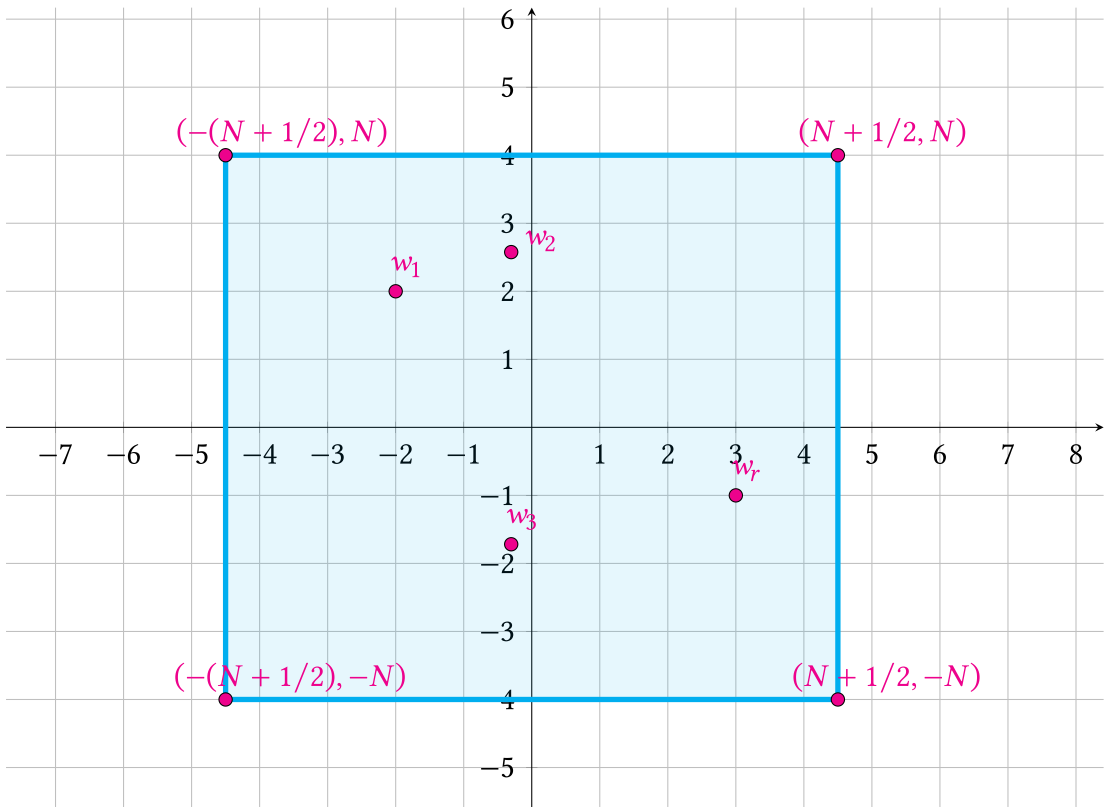
Returning to Example 38: For each $n \in \mathbb{Z}$, the function $f(z)$ has residues

$$\operatorname{Res}(f; n) = \operatorname{Res}\left(\frac{\frac{\pi \cos \pi z}{p(z)}}{\sin \pi z}; n\right) = \frac{\frac{\pi \cos n\pi}{p(n)}}{\pi \cos n\pi} = \frac{1}{p(n)}$$

Let γ_N be the rectangular curve with vertices $\pm(N + \frac{1}{2}) \pm iN$, where $N \in \mathbb{N}$ is so

← we really want to avoid integers because they are poles of f

large that all of $p(z)$'s zeros $w_1, \dots, w_r$ are inside γ_N.



By the residue theorem,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_N}\frac{\pi\cot\pi z}{p(z)}\,\mathrm{d}z &= \sum_{n=-N}^{N}\operatorname{Res}\left(\frac{\pi\cot\pi z}{p(z)}; n\right)+\sum_{j=1}^{r}\operatorname{Res}\left(\frac{\pi\cot\pi z}{p(z)}; w_j\right)\\ &= \sum_{n=-N}^{N}\operatorname{Res}\left(\frac{\pi\cos\pi z/p(z)}{\sin\pi z}; n\right)+\sum_{j=1}^{r}\operatorname{Res}\left(\frac{\pi\cot\pi z}{p(z)}; w_j\right)\\ &= \sum_{n=-N}^{N}\frac{1}{p(n)}+\sum_{j=1}^{r}\operatorname{Res}\left(\frac{\pi\cot\pi z}{p(z)}; w_j\right)\end{aligned}$$

Claim: the integral tends to 0 as $N\to\infty$ if $\deg p \geq 2$. Hence,

$$\sum_{n=-\infty}^{\infty}\frac{1}{p(n)} = -\sum_{j=1}^{r}\operatorname{Res}\left(\frac{\pi\cot\pi z}{p(z)}; w_j\right)$$

Observe that $\deg p(z)\geq 2$ implies that $\sum_{n=0}^{\infty}\frac{1}{p(n)}$ and $\sum_{n=-\infty}^{-1}\frac{1}{p(n)}$ converge by the comparison test with just the leading term.

Lemma 80. There is an $M>0$ such that $|\cot\pi z|\leq M$ on γ_N for all $N\in\mathbb{N}$.

Proof. If $z = x + iy$, in which $x, y \in \mathbb{R}$, then

$$|\cot \pi z| = \left|\frac{e^{i\pi x} + e^{-i\pi z}}{e^{i\pi x} - e^{-i\pi x}}\right| = \left|\frac{1 + e^{-2\pi i x}}{1 - e^{-2\pi i x}}\right| = \left|\frac{1 + e^{-2\pi i x} e^{2\pi y}}{1 - e^{-2\pi i x} e^{2\pi y}}\right|$$

On the **vertical** sides of γ_N, we have $z = \pm(N + \frac{1}{2}) + iy$ where $-N \leq y \leq N$ so that

$$|\cot \pi z| = \left|\frac{1 + e^{\mp 2\pi i (N + \frac{1}{2})} e^{2\pi y}}{1 - e^{\mp 2\pi i (N + \frac{1}{2})} e^{2\pi y}}\right| = \left|\frac{1 - e^{2\pi y}}{1 + e^{2\pi y}}\right|$$

This tends to 1 independently of N as $y \to +\infty$. Thus, there is an M_1 such that $|\cot \pi z| \leq M_1$ on the **vertical** sides of each γ_N. On the **horizontal** sides of γ_N, we have $z = x \pm iN$ where $-(N + \frac{1}{2}) \leq x \leq (N + \frac{1}{2})$. Thus,

$$|\cot \pi z| = \left|\frac{1 + e^{-2\pi i s} e^{\pm 2\pi N}}{1 - e^{-2\pi i s} e^{\pm 2\pi N}}\right| \leq \frac{e^{\pm 2\pi N} + 1}{|e^{\pm 2\pi N} - 1|}$$

which tends to 1 as $N \to \infty$. Consequently, there is an M_2 such that $|\cot \pi z| \leq M_2$ on the vertical sides of each γ_N. Set $M = \max\{M_1, M_2\}$ and conclude that

$$|\cot \pi z| \leq M$$

on each γ_N for $N \in \mathbb{N}$. □

Since $\deg p(z) \geq 2$, there exists a constant C such that

$$\left|\frac{1}{p(z)}\right| \leq \frac{C}{|z|^2}$$

for sufficiently large $|z|$. Thus, for sufficiently large N,

$$\left|\frac{1}{2\pi i} \int_\gamma \frac{\pi \cot \pi z}{p(z)}\, dz\right| \leq M \cdot \frac{C}{N^2} \cdot \underbrace{(2N + 2(N+1))}_{\text{periment of } \gamma_\infty} \leq \frac{5MC}{N},$$

which tends to zero as $N \to \infty$. Consequently, the integral $\frac{1}{2\pi i} \int_{\gamma_N} \frac{\pi \cot \pi z}{p(z)}\, \mathrm{d}\, z$ here tends to zero as $N \to \infty$. This yields this result:

$$\sum_{n=-\infty}^{\infty} \frac{1}{p(n)} = -\sum_{j=1}^{r} \operatorname{Res}\left(\frac{\pi \cot \pi z}{p(z)}; w_j\right)$$

Example 39. Consider the sum

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} + \sum_{n=1}^{\infty} \frac{1}{(-n)^2 + a^2} = \frac{1}{a^2} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

when $a \neq 0$.

Set $p(z) = z^2 + a^2$ which has zeros $w_1 = ia, w_2 = -ia$. This result implies

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = -\left[\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2 + a^2}; ia\right) + \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2 + a^2}; -ia\right)\right]$$

we compute the two residues required:

$$\begin{aligned}
\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2 + a^2}; ia\right) &= \lim_{z \to ia} (z - ia) \cdot \frac{\pi \cot \pi z}{z^2 + a^2} \\
&= \lim_{z \to ia} (z - ia) \cdot \frac{\pi \cot \pi z}{(z - ia)(z + ia)} \\
&= \lim_{z \to ia} \frac{\pi \cot \pi z}{z + ia} \\
&= \frac{\pi \cot \pi ia}{2ia}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2 + a^2}; -ia\right) &= \lim_{z \to -ia} (z - (-ia)) \cdot \frac{\pi \cot \pi z}{z^2 + a^2} \\
&= \lim_{z \to -ia} (z + ia) \cdot \frac{\pi \cot \pi z}{(z - ia)(z + ia)} \\
&= \lim_{z \to -ia} \frac{\pi \cot \pi z}{z - ia} \\
&= \frac{\pi \cot \pi(-ia)}{-2ia} \\
&= \frac{\pi \cot \pi ia}{2ia}
\end{aligned}$$

Putting this together we obtain

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} &= -\left[\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2 + a^2}; ia\right) + \operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2 + a^2}; -ia\right)\right] \\
&= -\left(\frac{\pi \cot \pi ia}{2ia} + \frac{\pi \cot \pi ia}{2ia}\right) \\
&= -\frac{\pi \cot \pi ia}{ia} \\
&= -\frac{\pi}{ia} \cdot \frac{\cos \pi ia}{\sin \pi ia} \\
&= -\frac{\pi}{ia} \cdot \frac{e^{i(\pi ia)} + e^{-i(\pi ia)}}{2} \cdot \frac{2i}{e^{i(\pi ia)} - e^{-i(\pi ia)}} \\
&= -\frac{\pi}{a} \cdot \frac{e^{-\pi a} + e^{\pi a}}{e^{-\pi a} - e^{\pi a}} \\
&= \frac{\pi}{a} \cdot \frac{e^{\pi a} + e^{-\pi a}}{e^{\pi a} - e^{-\pi a}}
\end{aligned}$$

$$= \frac{\pi \coth \pi a}{a}$$

Write

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$$

and observe that

$$\frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi \coth \pi a}{a}$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi a \coth \pi a - 1}{2a^2}$$

We observe that this is actually **analytic** of a on a punctured neighborhood of 0. In fact, we can even show that $a = 0$ is a **removable singularity**.

← plug in $a = 0$, observe that the function is bounded!

Using the Laurent series:

$$\coth z = \frac{1}{z} + \frac{z}{3} - \frac{z^3}{45} + \frac{2z^5}{945} + \cdots$$

we have

$$\begin{aligned}
\frac{\pi a \coth \pi a - 1}{2a^2} &= \frac{\pi a \left(\frac{1}{\pi a} + \frac{\pi a}{3} - \frac{(\pi a)^3}{45} + \cdots\right) - 1}{2a^2} \\
&= \frac{\left(1 + \frac{\pi^2 a^2}{3} - \frac{\pi^4 a^4}{45} + \cdots\right) - 1}{2a^2} \\
&= \frac{\frac{\pi^2 a^2}{3} - \frac{\pi^4 a^4}{45} + \cdots}{2a^2} \\
&= \frac{\pi^2}{6} - \frac{\pi^4 a^2}{90} + \cdots
\end{aligned}$$

Thus, $a = 0$ is a removable singularity and it yields Euler's celebrated formula:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$