

MATH135 Complex Analysis Notes

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January 25, 2024

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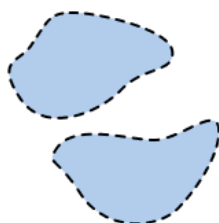
Regions, differentiability, analyticity

Regions

Definition 1. A **region** is a nonempty, connected, open subset of \mathbb{C} .

- A region without “holes” is simply connected.

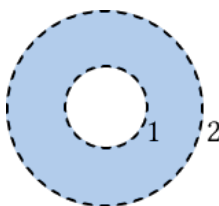
Non-example 1. This is not a region (not connected):



Example 2. \mathbb{C} is a region.

Example 3. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk is a region.

Example 4. $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$, the annulus region is a region that is not *simply-connected*:



Complex derivatives and analyticity

Definition 2. Let Ω be a region. Let $z_0 \in \Omega$ and $f : \Omega \rightarrow \mathbb{C}$ be a function.

1. Complex function f is **differentiable** at z_0 if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

2. If f is differentiable at every point in Ω , we say f is **analytic** on Ω .
3. If f is analytic on \mathbb{C} , then f is **entire**.

← this $z \rightarrow z_0$ could be from **any** directions!

← Means that existence of 1st derivative implies the existence of ∞ th derivative! & has Taylor expansion.

← Usual calculus rules work here :)

Example 5. Polynomials are entire functions.

Example 6. Rational functions are analytic on \mathbb{C} except where the denominator vanishes.

Non-example 7. $f(z) = \bar{z}$ is NOT analytic **anywhere**!

Proof. Let $z_0 \in \mathbb{C}$. Then $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$.

If $z \rightarrow z_0$ horizontally, then $z - z_0 \in \mathbb{R}$, meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if $z \rightarrow z_0$ vertically, then $\overline{z - z_0} = -(z - z_0)$, meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that $1 \neq -1$, thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere. \square

Proposition 1. Let f be differentiable at z_0 . Then, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that **whenever** $0 < |z - z_0| < \delta$, **we have** $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$.

Remark. Now consider multiplying $|z - z_0|$ on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near z_0 (when the distance $< \varepsilon$),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the “tangent-line approximation” equivalent in \mathbb{C} !

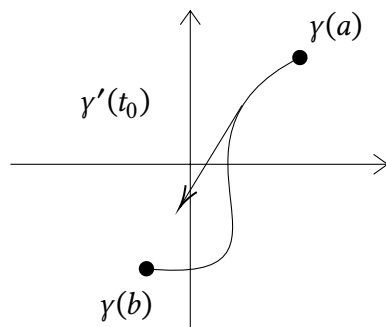
In addition, $f(z_0) + f'(z_0)(z - z_0)$ means to take $z - z_0$, rotate and dilate by $f'(z_0)$, then translate by $f(z_0)$. If $f'(z_0) \neq 0$, this function is locally orientation-preserving and could be approximated by a linear function.

← The RHS is a **linear** function!

← This explains why $z \mapsto \bar{z}$ is NOT analytic anywhere: it is orientation-reversing.

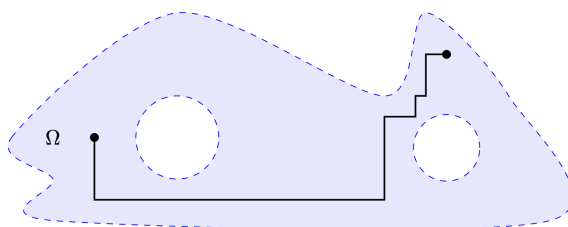
Curves, paths

Definition 3. A **curve** in \mathbb{C} is a function $\gamma : [a, b] \rightarrow \mathbb{C}, a, b \in \mathbb{R}$.



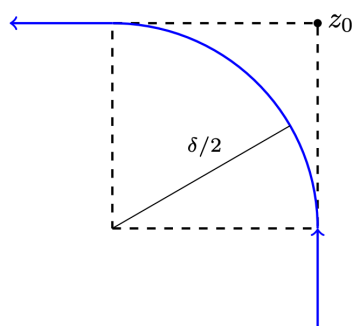
Definition 4. Parameterize $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$. Then $\gamma'(t_0) = (x'(t_0), y'(t_0))$ is a **tangent vector** to the curve at $\gamma(t_0)$ (assume $\gamma'(t_0) \neq \mathbf{0}$, aka. γ is regular at $\gamma(t_0)$.)

Theorem 2 (The “Boxy-path” Theorem). A nonempty open set Ω in \mathbb{C} is connected *if and only if* each pair of distinct points in Ω can be joined by a sequence of line segments lying in Ω , each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region Ω there exists a “**boxy path**”.

Remark. There is also always a **smooth path**. That is:

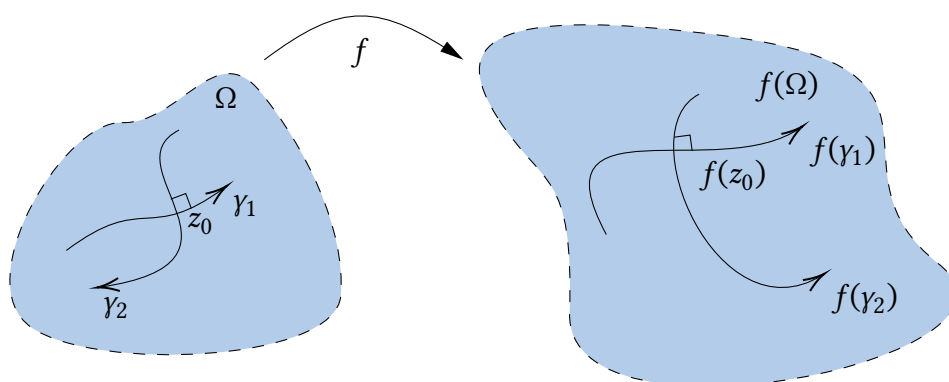


Theorem 3 (“Smooth-path”). A nonempty open set Ω in \mathbb{C} is connected if and only if each pair of distinct points in Ω can be joined by a continuously differentiable curve in Ω that is regular at every point.

Proof. See [lecture 2 notes](#). □

Conformality

Let f be an analytic complex function on Ω .

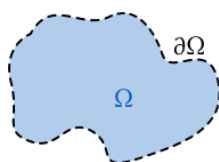


Let $z_0 \in \Omega$ such that $f'(z_0) \neq 0$. Let γ_1, γ_2 be two curves that pass through z_0 intersecting with an angle θ . Then $f(\gamma_1), f(\gamma_2)$ are two curves in $f(\Omega)$ passing through $f(z_0)$ also with angle θ .

Therefore, f is **conformal**!

Physics analogies of harmonic functions

Example 8. Let $T(x, y, t)$ be the temperature at (x, y) at time t of a thermally conductive plate in \mathbb{C} . Assume the plate gives rise to a **bounded** region Ω (with boundary denoted $\partial\Omega$). Temperature on $\partial\Omega$ is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where α is a constant.

We think the system tends towards a thermal equilibrium as $t \rightarrow \infty$. At equilibrium, $\frac{\partial T}{\partial t}$ is **zero**. Hence, at equilibrium, $\Delta T = T_{xx} + T_{yy} = 0$.

Idea: Harmonic function behave like equilibrium temperature distributions!

Proposition 4. Let $U(x, y)$ be a harmonic function on Ω .

1. U cannot have a *local* maximum in Ω .
2. The absolute maximum of U on Ω^- occurs on $\partial\Omega$.
3. U cannot be locally constant without being globally constant.

← Ω^- denotes the closure of Ω

Theorem 5 (Maximum principle). Let Ω be a bounded region in \mathbb{C} and let $f : \Omega^- \rightarrow \mathbb{C}$ be analytic on Ω and continuous on Ω^- .

1. If $|f|$ achieves a local max in Ω , then f is constant.
2. The global max of $|f|$ on Ω^- is attained on $\partial\Omega$.