

# MATH135 Complex Analysis Notes

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January 25, 2024

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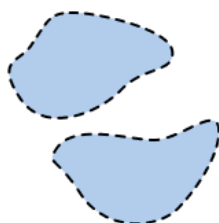
# Regions, differentiability, analyticity

## Regions

**Definition 1.** A **region** is a nonempty, connected, open subset of  $\mathbb{C}$ .

- A region without “holes” is simply connected.

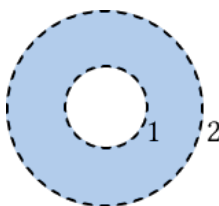
**Non-example 1.** This is not a region (not connected):



**Example 2.**  $\mathbb{C}$  is a region.

**Example 3.**  $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ , the open unit disk is a region.

**Example 4.**  $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$ , the annulus region is a region that is not *simply-connected*:



## Complex derivatives and analyticity

**Definition 2.** Let  $\Omega$  be a region. Let  $z_0 \in \Omega$  and  $f : \Omega \rightarrow \mathbb{C}$  be a function.

1. Complex function  $f$  is **differentiable** at  $z_0$  if

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

2. If  $f$  is differentiable at every point in  $\Omega$ , we say  $f$  is **analytic** on  $\Omega$ .
3. If  $f$  is analytic on  $\mathbb{C}$ , then  $f$  is **entire**.

← this  $z \rightarrow z_0$  could be from **any** directions!

← Means that existence of 1st derivative implies the existence of  $\infty$ th derivative! & has Taylor expansion.

← Usual calculus rules work here :)

**Example 5.** Polynomials are entire functions.

**Example 6.** Rational functions are analytic on  $\mathbb{C}$  except where the denominator vanishes.

**Non-example 7.**  $f(z) = \bar{z}$  is NOT analytic **anywhere**!

*Proof.* Let  $z_0 \in \mathbb{C}$ . Then  $\frac{f(z)-f(z_0)}{z-z_0} = \frac{\bar{z}-\bar{z}_0}{z-z_0}$ .

If  $z \rightarrow z_0$  horizontally, then  $z - z_0 \in \mathbb{R}$ , meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{z - z_0}{z - z_0} = 1.$$

Else if  $z \rightarrow z_0$  vertically, then  $\overline{z - z_0} = -(z - z_0)$ , meaning that

$$\lim_{z \rightarrow z_0} = \frac{f(z) - f(z_0)}{z - z_0} = \frac{\bar{z} - \bar{z}_0}{z - z_0} = \frac{-(z - z_0)}{z - z_0} = -1.$$

We observe that  $1 \neq -1$ , thus, the limit from different directions are not the same. We conclude that the limit does not exist anywhere.  $\square$

**Proposition 1.** Let  $f$  be differentiable at  $z_0$ . Then, for any  $\varepsilon > 0$ , there exists some  $\delta > 0$  such that **whenever**  $0 < |z - z_0| < \delta$ , **we have**  $|f'(z_0) - \frac{f(z)-f(z_0)}{z-z_0}| < \varepsilon$ .

**Remark.** Now consider multiplying  $|z - z_0|$  on both sides of Proposition 1:

$$\begin{aligned} |f'(z_0) \cdot (z - z_0) - f(z) + f(z_0)| &< \varepsilon |z - z_0| \\ |f(z_0) + f'(z_0)(z - z_0) - f(z)| &< \varepsilon |z - z_0| \end{aligned}$$

That is to say, near  $z_0$  (when the distance  $< \varepsilon$ ),

$$f(z) \approx f(z_0) + f'(z_0)(z - z_0)$$

this is the “tangent-line approximation” equivalent in  $\mathbb{C}$ !

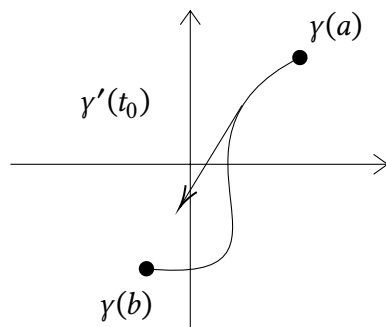
In addition,  $f(z_0) + f'(z_0)(z - z_0)$  means to take  $z - z_0$ , rotate and dilate by  $f'(z_0)$ , then translate by  $f(z_0)$ . If  $f'(z_0) \neq 0$ , this function is locally orientation-preserving and could be approximated by a linear function.

← The RHS is a **linear** function!

← This explains why  $z \mapsto \bar{z}$  is NOT analytic anywhere: it is orientation-reversing.

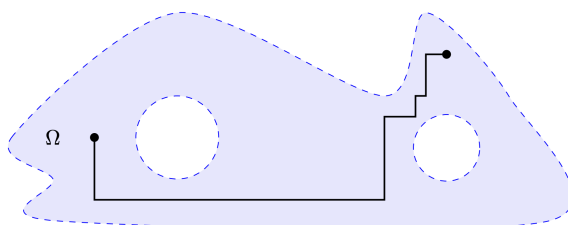
## Curves, paths

**Definition 3.** A **curve** in  $\mathbb{C}$  is a function  $\gamma : [a, b] \rightarrow \mathbb{C}, a, b \in \mathbb{R}$ .



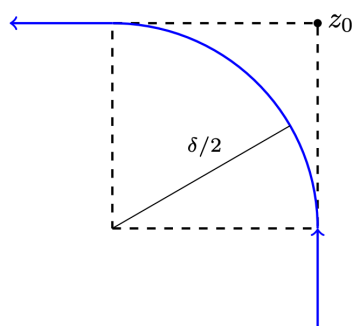
**Definition 4.** Parameterize  $\gamma(t) = (x(t), y(t)) = x(t) + iy(t)$ . Then  $\gamma'(t_0) = (x'(t_0), y'(t_0))$  is a **tangent vector** to the curve at  $\gamma(t_0)$  (assume  $\gamma'(t_0) \neq \mathbf{0}$ , aka.  $\gamma$  is regular at  $\gamma(t_0)$ .)

**Theorem 2** (The “Boxy-path” Theorem). A nonempty open set  $\Omega$  in  $\mathbb{C}$  is connected *if and only if* each pair of distinct points in  $\Omega$  can be joined by a sequence of line segments lying in  $\Omega$ , each of which is parallel to either to the real or imaginary axis.



In other words, between any 2 points in a region  $\Omega$  there exists a “**boxy path**”.

**Remark.** There is also always a **smooth path**. That is:

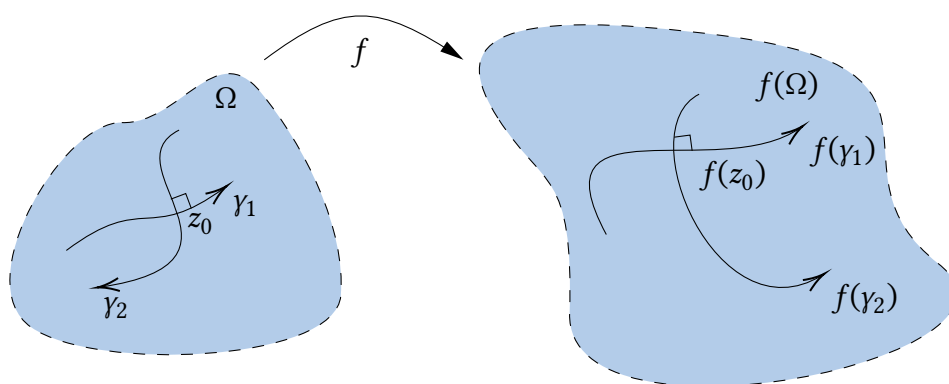


**Theorem 3** (“Smooth-path”). A nonempty open set  $\Omega$  in  $\mathbb{C}$  is connected if and only if each pair of distinct points in  $\Omega$  can be joined by a continuously differentiable curve in  $\Omega$  that is regular at every point.

*Proof.* See [lecture 2 notes](#). □

## Conformality

Let  $f$  be an analytic complex function on  $\Omega$ .



Let  $z_0 \in \Omega$  such that  $f'(z_0) \neq 0$ . Let  $\gamma_1, \gamma_2$  be two curves that pass through  $z_0$  intersecting with an angle  $\theta$ . Then  $f(\gamma_1), f(\gamma_2)$  are two curves in  $f(\Omega)$  passing through  $f(z_0)$  also with angle  $\theta$ .

Therefore,  $f$  is **conformal**!

## Cauchy-Riemann equations, harmonic functions

### Multivariate notion of complex derivatives

Recall: 
$$f'(z_0) = \lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Now we write each function with complex variables as  $f(z) = u(z) + i v(z)$  where  $u, v$  are real-valued functions.

← meaning their range is real

Since  $\mathbb{C} \cong \mathbb{R}^2$ , we denote every point  $z = (x, y)$ .

Now we let  $f(x, y) = u(x, y) + i v(x, y)$ . We first let the small distance  $h = (r, 0)$  be horizontally approaching 0 with  $r \in \mathbb{R}$ . That is,  $z_0 + h = (x_0 + r, y_0)$ .

$$\begin{aligned} f'(z_0) &= \lim_{r \rightarrow 0} \frac{u(x_0 + r, y_0) - u(x_0, y_0)}{r} + i \cdot \lim_{r \rightarrow 0} \frac{v(x_0 + r, y_0) - v(x_0, y_0)}{r} \\ &= u_x(x_0, y_0) + i \cdot v_x(x_0, y_0) \end{aligned}$$

Similarly, if we vertically let  $h = ir = (0, r)$  with  $r \rightarrow 0, r \in \mathbb{R}$ , we would get  $f' = v_y - i \cdot u_y$ .

**Remark.** If a derivative exists, the horizontal & the vertical ones should be equal!

**Theorem 4** (Cauchy-Riemann Equations).

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

**Corollary 5.** If  $f : \Omega \rightarrow \mathbb{C}$  is analytic and  $f' = 0$  on  $\Omega$ , then  $f$  is **constant**.

*Proof.* Since  $0 = f' = u_x + iv_x$ , we see that  $u_x = v_x = 0$  on  $\Omega$ . By Cauchy-Riemann,  $v_y = u_y = 0$  is also true on  $\Omega$ . Hence,  $\mathbf{u}, \mathbf{v}$  are constant on either horizontal or vertical segments. By the Boxy Path Theorem,  $f = u + iv$  cannot assume two distinct values in  $\Omega$ .  $\square$

## Orientation-preserving as shown by Jacobian

Let  $f : \Omega \rightarrow \mathbb{C}$  be analytic. Then  $f' = u_x + iv_x$  and hence:

$$\begin{aligned} |f'|^2 &= \bar{f}' \cdot f' = (u_x - iv_x)(u_x + iv_x) \\ &= u_x^2 + v_x^2 \\ &= u_x u_x + v_x v_x && \text{and by Cauchy-Riemann,} \\ &= u_x v_y - u_y v_x \\ &= \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} && \text{the Jacobian of } f! \end{aligned}$$

Since  $|f'|^2 \geq 0$ , the determinant of the Jacobian is always  $\geq 0$ , implying that  $f$  is always locally orientation-preserving. Moreover,

**Proposition 6.** If  $f'(z_0) \neq 0$ , then  $|f'|^2 > 0$  implies:

1.  $f$  is **injective** near  $z_0$
2.  $f$  scales  $\mathbb{R}$  by  $|f'(z_0)|^2$  near  $z_0$
3.  $f$  preserves orientation near  $z_0$

## The Laplacian, harmonic functions and conjugates

Suppose that  $f = u + iv$  is analytic and  $u, v$  have continuous second partial derivatives. Then:

$$u_{xx} + u_{yy} = \Delta u = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0$$

This means that the Laplacian of this function  $u$  is 0!

**Definition 5.** Real-valued functions  $u : \Omega \rightarrow \mathbb{R}$  satisfying that the Laplacian  $\Delta u = u_{xx} + u_{yy}$  is 0 on  $\Omega$  is called **harmonic functions**.

**Definition 6.** A **harmonic conjugate** of  $u$  is a harmonic function  $v : \Omega \rightarrow \mathbb{R}$  such that  $f = u + i \cdot v$  is **analytic** on  $\Omega$ .

**Example 8.**  $u = x^2 - y^2, v = 2xy$ .

**Remark.** Harmonic conjugates are unique up to translation ( $\pm$  constants).

**Remark.** If  $u$  is harmonic on  $\Omega$ , it does NOT have to have a harmonic conjugate on  $\Omega$ .

←  $\Delta u = 0$   
characterizes  
steady-state  
solutions to heat  
equations on  $\Omega$ .

← Check it!

## Finding a harmonic conjugate

Recall that the real and imaginary parts of an analytic function are **harmonic**, in addition to satisfying the Cauchy-Riemann Equations:  $u_x = v_y$  and  $u_y = -v_x$ .

**Example 9.**  $u(z) = \log |z|$  is harmonic on  $\mathbb{C} \setminus \{0\}$ .

*Proof.* Write  $u(x, y) = \log(\sqrt{x^2 + y^2}) = \frac{1}{2} \log(x^2 + y^2)$ .

Then,

$$\begin{aligned} u_x &= \frac{\partial}{\partial x} \left( \frac{1}{2} \log(x^2 + y^2) \right) \\ &= \frac{1}{2} \cdot \frac{2x}{x^2 + y^2} \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Hence,

$$\begin{aligned} u_{xx} &= \frac{(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} \\ &= \frac{y^2 - x^2}{(x^2 + y^2)^2} \end{aligned}$$

Symmetrically, we find

$$u_{yy} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Hence  $u_{xx} + u_{yy} = 0$ , implying that the function is harmonic.  $\square$

Now, can we find a harmonic conjugate for the aforementioned  $u$ ?

We could use the two Cauchy-Riemann Equations. One of them:

$$\begin{aligned} v_y &= u_x \\ &= \frac{x}{x^2 + y^2} \end{aligned}$$

Therefore,

$$\begin{aligned} v(x, y) &= \int v_y dy + C(x) && \text{unknown function of } x \\ &= \arctan\left(\frac{y}{x}\right) + C(x) \end{aligned}$$

Then, we use the second one:

$$\begin{aligned} \frac{y}{x^2 + y^2} &= u_y = -v_x = -\frac{\partial}{\partial x} \left( \arctan\left(\frac{y}{x}\right) + C(x) \right) \\ &= \frac{y}{x^2 + y^2} - C'(x) \implies C'(x) = 0 \end{aligned}$$

Hence, a good harmonic conjugate candidate seems to be

$$v(x, y) = \arctan\left(\frac{y}{x}\right) + C$$

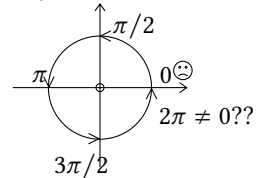
where  $C$  is a constant. WLOG, let  $C = 0$ . Then  $v(x, y) = \arctan\left(\frac{y}{x}\right)$ , meaning that:

$$v(z) = \arg(z)$$

Therefore,  $f(z) = \log|z| + i \cdot \arg(z)$  is analytic!

← Review quotient rule!

← There is currently a great **CAVEAT** in all of these, because  $v(z) = \arg(z)$  cannot be defined in a continuous manner in all of  $\mathbb{C} \setminus \{0\}$ .

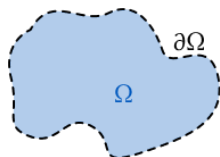


To be resolved later!



## Physics analogies of harmonic functions

**Example 10.** Let  $T(x, y, t)$  be the temperature at  $(x, y)$  at time  $t$  of a thermally conductive plate in  $\mathbb{C}$ . Assume the plate gives rise to a **bounded** region  $\Omega$  (with boundary denoted  $\partial\Omega$ ). Temperature on  $\partial\Omega$  is a fixed function (time-independent).



Now given the heat equation:

$$\frac{\partial T}{\partial t} - \alpha \Delta T = 0$$

where  $\alpha$  is a constant.

We think the system tends towards a thermal equilibrium as  $t \rightarrow \infty$ . At equilibrium,  $\frac{\partial T}{\partial t}$  is **zero**. Hence, at equilibrium,  $\Delta T = T_{xx} + T_{yy} = 0$ .

**Idea:** Harmonic function behave like equilibrium temperature distributions!

**Proposition 7.** Let  $U(x, y)$  be a harmonic function on  $\Omega$ .

1.  $U$  cannot have a *local* maximum in  $\Omega$ .
2. The absolute maximum of  $U$  on  $\Omega^-$  occurs on  $\partial\Omega$ .
3.  $U$  cannot be locally constant without being globally constant.

←  $\Omega^-$  denotes the closure of  $\Omega$

**Theorem 8** (Maximum principle). Let  $\Omega$  be a bounded region in  $\mathbb{C}$  and let  $f : \Omega^- \rightarrow \mathbb{C}$  be analytic on  $\Omega$  and continuous on  $\Omega^-$ .

1. If  $|f|$  achieves a local max in  $\Omega$ , then  $f$  is constant.
2. The global max of  $|f|$  on  $\Omega^-$  is attained on  $\partial\Omega$ .