MATH172 Galois Theory Notes

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September 22, 2023

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Rings! Or why $x^2 - 2$ has roots.

Definition 1. A **ring** is a set R together with associative binary *operations* $+$ and $\times$ s.t.:

← map from $R \times R \mapsto R$

- $(R, +)$ is an **abelian** group with identity 0
- There exists $1 \in R$ s.t. $r \times 1 = 1 \times r = r$
- $r(s+t) = rs + rt$ and $(s+t)r = sr + tr \qquad \forall s, r, t \in R$

← this is optional

Proposition 1. $0 \times 1 = 0$ (in fact, $0 \times r = 0 \; \forall \; r \in R$)

Proof. Try it! □

Definition 2. If $\times$ is commutative, then R is a commutative ring.

Non-example 1. $\mathbb{N}$ is not a ring.

Example 2. $\mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R} \subsetneq \mathbb{C}$ are all rings;

- $\mathbb{Z}/n\mathbb{Z}$ is a finite ring
- $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, is a **noncommutative** ring
- Polynomial ring: $\mathbb{Q}[x] = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Q}\}$ is a commutative ring
- $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ is a commutative ring

← square brackets just mean "polynomials in..."

Phase I plan:

$$ID \supsetneq UFD \supsetneq PID \supsetneq ED \supsetneq Fields$$

Definition 3. Suppose R is a ring and $a, b \in R$ with $ab = 0$ but $a, b \neq 0$; then a, b are called **zero divisors**.

Example 3.

- In $\mathbb{Z}/6\mathbb{Z}$, $\bar{4} \times \bar{3} = \bar{0}$
- In $M_2(\mathbb{R})$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Definition 4. A commutative ring without zero divisors is called an integral domain (ID)

Why do we want ID? **Cancellation properties**.

- If R is an ID, $a, b, c \in R$, $a \neq 0$ and $ab = ac$, then

$$ab - ac = 0 \implies a(b - c) = 0 \implies b - c = 0 \implies b = c$$

Definition 5. Suppose R is an ID. An element $a \in R$ is called a **unit** if $a \neq 0$ and there exists $b \in R$ s.t. $ab = 1$.

← notation: $b = a^{-1}$

An element $r \in R$ is called **irreducible** if $r \neq 0$, r is NOT a unit, and whenever $r = ab$ for some $a, b \in R$ then a or b must be a unit.

- If r and s are irreducibles with $r = us$, then r and s are called **associates**.

Example 4.

- All "prime integers" are irreducibles in $\mathbb{Z}$;
- 2,3, $1 + \sqrt{-5}$, $1 - \sqrt{-5}$ are irreducibles in $\mathbb{Z}[\sqrt{-5}]$.
  - Note: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6$ says that 6 can be factored in more than one way. This means that $\mathbb{Z}[\sqrt{-5}]$ is NOT an UFD.

Definition 6. An integral domain R is called a unique factorization domain (UFD) if each nonzero, nonunit $a \in R$ can be written as a product of irreducibles **in a unique way** up to associates.

If a is a nonzero, nonunit element of UFD R and $a = r_1 r_2 \ldots r_m = s_1 \ldots s_n$ where r_i, s_j are irreducible, then after reordering $r_i = u_i s_i$ for any i and units u_i, and $m = n$.

← After reordering, there are the same amounts of factors and all factors are the same up to units.

Definition 7. Suppose R is a comm ring. A subset $I \subseteq R$ is called an **ideal** if $(I, +) \leq (R, +)$ and $ir, ri \in I$ for all $i \in I$ and for all $\boxed{r \in R}$.

Why do we want ideals? Such that R/I is a well-defined ring.

Example 5. $\{0\}$ and R are ideals of R.

Example 6. If R is commutative and $a \in R$, then $(a) = \{ar \mid r \in R\}$ is called the **principal ideal** generated by a.

← Prove this (be convinced)! Also known as aR.

Definition 8. A **principal ideal domain** is an integral domain where all ideals are principal ideals.

Example 7. The only ideals of $(\mathbb{Z}, +)$ are of the form $n\mathbb{Z} = (n)$.

← Ideals generated by n

Non-example 8. $\mathbb{Z}[x]$ is a UFD but NOT a PID because the ideal $(2, x) = \{2r + xs \mid r, s \in \mathbb{Z}[x]\}$ is not principal.

← Observe that $(2, x)$ is an ideal made of polynomials with even constant terms. This cannot be principal, since if we only have 2 and not x, we do not have nonzero polynomials with zero const terms.

Lemma 2. If $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.

Proof. Try it!

Proposition 3. If $I \subseteq R$ is an ideal containing a unit of R then $I = R$.

Proof. If $u \in I$ is a unit then $u^{-1} \in R$, so $uu^{-1} = 1 \in I$. Then the result follows from Lemma 2. □

Definition 9. A **field** is a commutative ring whose <u>each nonzero element is a *unit*</u>.

Corollary 4. If R is an ID whose ideals are (0) and R, then R is a **field**.

← The converse is also true. **The only ideals in a field are 0 and the field.**

Proof. Suppose $a \in R\backslash\{0\}$ and consider (a). Since $a \in (a)$, $(a) = R$. Hence, we must have that $1 \in (a)$, which means $1 = ar$ for some $r \in R$. □

Definition 10. Suppose R is an integral domain. A *proper* ideal $P \subsetneq R$ is called **prime** of whenever $ab \in P$ for some $a, b \in R$, then a or $b \in P$.

Non-example 9. (6) is not a prime ideal of $\mathbb{Z}$ since $2 \times 3 \in (6)$ but neither $2, 3 \notin (6)$.

Non-example 10. (2) is not a prime ideal of $\mathbb{Z}[\sqrt{-5}]$ since $6 \in (2)$, but we observe that $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ while $1 \pm \sqrt{-5} \notin (2)$.

← Observe that in $\mathbb{Z}[\sqrt{-5}]$, we have $6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \times 3$, so it is not a UFD!

Example 11. (2) is a prime ideal of $\mathbb{Z}$.

Definition 11. A proper ideal $M \subsetneq R$ is called **maximal** if whenever $I \subseteq R$ such that $M \subseteq I \subseteq R$ is an ideal containing M, then either $I = M$ or $I = R$.

Proposition 5. Every proper ideal is contained in **a** maximal ideal.

← This might not be unique in non-local rings.

Proof. TBD. □

Proposition 6. Suppose R is a commutative ring.

- (0) is prime *if and only if* R is an integral domain.
- (0) is maximal *if and only if* R is a field.

← By def of prime, if $ab = 0$, then either $a = 0$ or $b = 0$, which means there are NO zero divisors.

(The following is kind of on a tangent)

Definition 12. A commutative ring R with unity is called **Noetherian** if, whenever $I_1 \subseteq I_2 \subseteq \dots$ is an ascending sequence of (proper) ideals of R, there exists an $n > 0$ such that $I_n = I_{n+1} = \dots$ are the same ideals thereafter.

← The chain stops ascending!

Theorem 7. R is Noetherian *if and only if* all ideals of R are finitely generated.

Corollary 8. All Principal Ideal Domains are Noetherian.

← Since all ideals are generated by 1 elt.

(Tangent ends here)

Definition 13. Suppose R is a commutative ring with $1 \neq 0$ and $I \subseteq R$ is an ideal. Then the **quotient ring** of R by I is the set

$$R/I = \{r + I \mid r \in R\}$$

with addition and multiplication defined representative-wise.

Remark. The **coset criterion** of ideals: let I be an ideal; the cosets $r + I$, $s + I$ are the same *if and only if* $r - s \in I$.

Example 12.

- In $\mathbb{Z}/(6)$ aka. $\mathbb{Z}/6\mathbb{Z}$, we have $2 + (6) = \{\dots, -10, -4, 2, 8, 14, \dots\} = 26 + (6)$ due to $2 - 26 \in (6)$;
- In $\mathbb{Q}[x]/(x^2 - 2)$, we have

  $$\{3x^2-47x+1+q(x)(x^2-2) \mid q(x) \in \mathbb{Q}[x]\} = \{-47x+7+q(x)(x^2-2) \mid q(x) \in \mathbb{Q}[x]\}$$

  due to $3x^2 - 47x + 1 - (-47x + 7) \in (x^2 - 2)$.

Remark. Let I be an ideal of R. Then $(I, +) \trianglelefteq (R, +)$.

Definition 14. R/I is a group under $(r+I)+(s+I) = (r+s)+I$ and the operation $+$ is well-defined. We also define that $(r + I)(s + I) = (rs) + I$. We claim that multiplication in R/I is also well-defined.

Proof. Let $r_1 + I = r_2 + I$ and $s_1 + I = s_2 + I$. By coset criterion, $r_1 - r_2 = i$, $s_1 - s_2 = j$ for some $i, j \in I$. Hence $r_1 s_1 = (r_2 + i)(s_2 + j) = r_2 s_2 + i s_2 + j r_2 + ij$ where the latter three terms are all in the ideal I. Thus, $(r_1 s_1) + I = (r_2 s_2) + I$. □

From R, R/I inherits nice properties:

- $0 + I = 0_{R/I}$
- $1 + I = 1_{R/I}$

- Multiplication is commutative and distributive over addition in R/I, so it is also a comm. ring with identity.

Definition 15. A function $\varphi : R \to S$ between rings is called a **ring homomorphism** if the following are satisfied:

- $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$
- $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$

Theorem 9. First ring isomorphism theorem

If $\varphi : R \mapsto S$ is a ring homomorphism, then $R/\ker(\varphi) \cong \varphi(R)$.

← Observe that kernels are ideals! And ideals are kernels of some homomorphism too.

Example 13. If R is a ring and I is an ideal, then $\pi : R \to R/I$ where $r \mapsto r + I$ is a surjective homomorphism where $\ker(\pi) = I$. This is the *canonical projection* onto R/I.

Corollary 10. If I is a maximal ideal, then R/I is a field.

← The *if and only if* version comes in Proposition 14.

Recall Proposition 6. We now have a stronger statement:

Proposition 11. Suppose R is a commutative ring & $P \subseteq R$ is an ideal. Then R/P is an integral domain *if and only if* P is prime.

Proof. R/P is an integral domain *if and only if* whenever $(a+P)(b+P) = 0_{R/P}$ then one of $a + P$ or $b + P$ must already be $0_{R/P}$. This happens *if and only if* whenever $ab + P = P$ then $a + P$ or $b + P$ in P, which happens *if and only if* whenever $ab \in P$ then one of $a, b \in P$, which is the definition of a prime ideal. $\square$

Example 14. The map $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ where $p(x) \mapsto p(0)$ is a surjective ring homomorphism with $\ker(\varphi) = (x)$. By the the First Isomorphism Theorem 9, $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. As such, we conclude that (x) is a prime ideal since $\mathbb{Z}$ is an integral domain.

← btw, $(x) \subsetneq (x, 2)$. the latter is the set of polynomials whose constant term is even, so it is also a proper ideal of $\mathbb{Z}[x]$. This is an excellent example where Prime $\not\Rightarrow$ Maximal.

Lemma 12. Suppose R is a comm. ring with $M \subseteq R$ being an ideal.
There is a bijective correspondence between the ideals of R/M and the ideals of R containing M.

Proof. Consider the projection $\pi : R \to R/M$ where $r \mapsto r + M$. It is enough to show:

← To see why this is okay, see Homework 2 Sec. 7.3 P. 24

$$\begin{aligned} \pi(\pi^{-1}(J)) &= J && \text{for all ideals } J \subseteq R/M, \text{ and} \\ \pi^{-1}(\pi(I)) &= I && \text{for all ideals } M \subseteq I \subseteq R \end{aligned}$$

To prove the first statement, observe that, if J is an ideal of R/M, then $\pi^{-1}(J) = \{r \in R \mid r + M \in J\}$ and so

$$\pi(\pi^{-1}(J)) = \{\pi(r) \in R \mid r + M \in J\} = \{r + M \mid r + M \in J\} = J$$

Next, to prove the second statement, first suppose $M \subseteq I \subseteq R$ is an ideal. Let $a \in I$. Then $a + M \in \{\alpha + M \mid \alpha \in I\} = \pi(I)$. This implies that $a \in \pi^{-1}(\pi(I))$, and so $I \subseteq \pi^{-1}(\pi(I))$.

Conversely, suppose $r \in \pi^{-1}(\pi(I))$. This is the same as saying $\pi(r) = r + M \in \pi(I) = \{\alpha + M \mid \alpha \in I\}$. Hence, for any $r \in \pi^{-1}(\pi(I))$, there exists some $a \in I$ such that $r + M = a + M$. Thus, $r - a \in M \subseteq I$ by coset conditions. Since $a \in I$, we have $a + (r - a) \in I$, meaning that $r \in I$ for any $r \in \pi^{-1}(\pi(I))$. This means that $\pi^{-1}(\pi(I)) \subseteq I$.

Hence, $\underline{I = \pi^{-1}(\pi(I))}$.

Consequently, for any ideals $J \subseteq R/M$, we know that $\pi^{-1}(J) \subseteq R$ is an ideal containing M. And if $M \subseteq I \subseteq R$ is an ideal, we know $\pi(I) \subseteq R/M$ is an ideal. Since $\pi(\pi^{-1}(J)) = J$ and $I = \pi^{-1}(\pi(I))$ for any I, J, the correspondence is a bijection. □

← Think about why this contains M!

Proposition 13. Suppose R is a comm. ring with an identity and $I \subseteq R$ is an ideal. Then R/I is a field *if and only if* I is maximal.

Proof. If I is maximal, then there are no other proper ideals strictly containing I. Hence, by Lemma 14, we have that R/I only have ideals (0) and R/I itself. This happens *if and only if* R/I is a field. □

Corollary 14. If R is a commutative ring with identity and $M \subseteq R$ is maximal, then M is prime.

← Hence maximal implies prime, but prime does not necessarily implies maximal.

Proof. Maximal $\implies$ quotient is a field $\implies$ quotient is an ID $\implies$ prime. □

Definition 16. An integral domain R is an **Euclidean domain** if there exists a norm $N : R \to \mathbb{Z}_{\geq 0}$ with $N(0) = 0$ such that for all $a, b \in R$ with $b \neq 0$, there exists $q, r \in R$ for which

$$a = bq + r$$

with $N(r) < N(b)$ or $r = 0$.

Example 15. $\mathbb{Z}$ is a ED with $N(a) = |a|$.

Example 16. $\mathbb{Q}[x]$ is a ED with $N(p(x)) = \deg(p(x))$.

Example 17. Every field F is a ED with $N(a) = 0 \ \forall \ a \in F$.

← Because in a field everything divides!

Non-example 18. $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID that is not an ED.

← This is one of the only good examples!

Why do we care about Euclidean domains?

Remark. Greatest common divisors exist and are relatively quick to compute.

← Using recursive application of Euclidean algorithm.

Definition 17. If $a, b \in R$, then $\gcd(a, b) = c$ means

1. c divides a and b; that is, $a = cr, b = cs$ for some $r, s \in R$
2. If $c' \in R$ with $c'|a$ and $c'|b$, then it must be true that $c'|c$.

← All other common divisors divide the gcd.

Example 19. Say we want to compute the gcd of 47 and 10.

← This is a much faster algorithm than factoring!

$$\begin{aligned}
47 &= 4 \times 10 + 7 \\
10 &= 1 \times 7 + 3 \\
7 &= 2 \times 3 + 1 && \leftarrow \text{circled is } \gcd(47, 10) \\
3 &= 3 \times 1 && \leftarrow \text{final line with no remainders}
\end{aligned}$$

This also works for finding gcds in $\mathbb{Q}[x]$ with polynomials long division and norm $\deg(p(x))$.

Remark. If F is a field, then $F[x]$ is a Euclidean domain.

← Just use long division!

Remark. Euclidean domains are PIDs.

Proof. Suppose R is a ED and $I \subseteq R$ is an idea;. Consider $\{N(a) \mid a \in I\backslash\{0\}\}$. This set has a minimal element by properties of natural numbers (or is an empty set *if and only if* $I = (0)$).

Let $d \in I$ be an element of minimum norm (hence $N(d) \leq N(a)$ for all $a \in I$). We claim that $(d) = I$. Proof:

Since $d \in I$, we have $rd \in I$ for any $r \in R$. This implies that $(d) \subseteq I$.

Then let $a \in I$. Since R is a ED, we first assumes that there exists $q, r \in R$, $r \neq 0$ such that $a = qd + r$ and $N(r) < N(d)$. But we notice that $r = a - qd$ must be in I as both $a, qd \in I$, contradicting the minimality of $N(d)$. Thus, it must be that $r = 0$. This implies $a = qd$ and thus $a \in (d)$ for all $a \in I$. Consequently, $I \in (d)$, and therefore $I = (d)$. ☐

Definition 18. Suppose R is an integral domain and $p \in R\backslash\{0\}$. Then p is a **prime element** if (p) is a prime ideal.

Proposition 15. An element $p \in R$ is prime *if and only if* whenever $p|ab$ then $p|a$ or $p|b$.

Proof. p is prime means that (p) is a prime ideal. This is true *if and only if* whenever $ab \in (p)$ then $a \in (p)$ or $b \in (p)$. This is the same as saying if $ab = kp$ for some $k \in R$ then $a = lp$ or $b = lp$ for some $l \in R$. This is to say that whenever $p|ab$ then $p|a$ or $p|b$. □

Proposition 16. In an integral domain, all prime elenments are irreducibles.

Proof. Suppose R is an ID and $p \in R$ is prime. If $p = ab$ for some a, b in R, then, WLOG, $p|a$. That is, $a = pk$ for some $k \in R$. Hence, $p = pkb$. Since in an ID cancellation rule holds, $kb = 1$, meaning that b is a unit. Thus, p is irreducible by definition Definition 5. □

Proposition 17. In PIDs, all *nonzero* prime ideals are maximal.

Proof. Suppose R is a PID and $(p) \subsetneq R$ is a prime ideal. If $(p) \subseteq (m) \subseteq R$ is an ideal, then $p \in (p) \subseteq (m)$ hence $p = rm$ for some $r \in R$. Since $p|rm$, we have $p|r$ or $p|m$.

If $p|r$, this implies that $r = pk$ for some $k \in R$. Substituting into $p = rm$, we get $p = pkm$. By cancellation, we get $km = 1$, meaning that m is a unit. Hence, $(m) = R$.

If $p|m$, we have $m = pl$ for some $l \in R$, meaning that $m \in (p)$. Hence, $(m) \subseteq (p)$, but we also defined that $(p) \subseteq (m)$, so $(m) = (p)$.

Therefore, (p) has to be the maximal ideal. □

Proposition 18. In an UFD, irreducible implies prime.

Proof. Let R be a UFD and $p \in R$ be irreducible. Let $a, b \in R$ such that $p|ab$. Hence, $pr = ab$ for some $r \in R$. Since R is a UFD, let $a = q_1 \dots q_n, b = s_1 \dots s_m$ be the factorization. Since the factorizations are unique and each of the q_i, s_j are irreducible, if $p|ab$, then p must be an associate with one of the q_i, s_j. Therefore, either $p|a$ or $p|b$, implying prime. □

Example 20. $\mathbb{Q}$ is a field, so $\mathbb{Q}[x]$ is a ED. Since EDs are UFDs, irreducible $\implies$ prime. We see that $x^2 - 2 \in \mathbb{Q}[x]$ is an irreducible element, which means that $(x^2 - 2)$ is a prime ideal, meaning that it is a maximum ideal, meaning that $\mathbb{Q}[x]/(x^2 - 2)$ is a field. We observe that it is a field containing $\mathbb{Q}$ and $(\sqrt{2})$.

← In fact, this is the smallest field containing $\mathbb{Q}$ and $(\sqrt{2})$.

Lemma 19. In a PID, irreducible elements are prime.

Proof. Suppose $p \in R$ is irreducible in the principal ideal domain R. If $p|ab$ for some $a, b \in R$, we want to show that either $p|a$ or $p|b$, hereby showing that p is prime. Hence, we consider the ideal $(a, p) = d$, which is necessarily principal for some $d \in R$. Since $a, p \in (d)$, we have $a = dr$ and $p = ds$ for some $r, s \in R$. As p is irreducible, we get that one of d and s is a unit.
We first assume that s is a unit, in which case $d = ps^{-1}$, and so $a = ps^{-1}r$ implying that $p|a$.
In another case, d is a unit, in which case $(a, p) = (d) = R$ and so $1 = ak + pl$ for some $k, l \in R$. Multiplying by b, we get $b = abk + pbl$. Since $p|ab$, we have $b = abk + pbl = pmk + pbl$ for some $m \in R$. Hence, $b = p(mk + bl)$, meaning that $p|b$.
Therefore, whenever $p|ab$, either $p|a$ or $p|b$. Hence, in a PID, p is prime whenever it is irreducible. ☐

Proposition 20. PIDs are UFDs.

Proof. Suppose R is a PID and $a \in R$ is nonzero, nonunit. If a is irreducible, we are done. If not, we write $a = p_1q_1$ for some $p_1, q_1 \in R$ nonunit. If p_1, q_1 are irreducibles, we are done. If not, then WLOG say $q_1 = p_2q_2$ for some nonunits p_2, q_2. We would like to show that this splitting process terminates.

Observe that $(q_1) \subseteq (q_2)$ since $q_2|q_1$. Hence, the chain of splitting results in the chain of ideals $(q_1) \subseteq (q_2) \subseteq (q_3) \subseteq \dots$.

Now consider the ideal[1] $\bigcup_{i=1}^{\infty}(q_i)$. Since this is a PID, we have $\bigcup_{i=1}^{\infty}(q_i) = (q)$ for some $q \in R$. Since $q \in \bigcup_{i=1}^{\infty}(q_i)$, it is contained in some (q_n) for some $n \geq 1$. This implies that $(q) \subseteq (q_n)$, but we also know that $(q_n) \subseteq (q)$, hence $(q) = (q_n)$. Hence, this process terminates, and there exists an n in this chain such that q_n is irreducible. Therefore, R is a factorization domain.

Now we want to prove the uniqueness. That is, if $p_1 \dots p_n = q_1 \dots q_m$ for irreducibles p_i, q_j and $n \leq m$ WLOG, then we want to show that $m = n$ and that $p_i = u_iq_i$ with units u_i up to reordering for all i. We do so by induction on n.

[1]The proof that this is an ideal is as follows:
We first prove that $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition. Let $r, s \in \bigcup_{n=1}^{\infty} I_n$, where $r \in I_k$ and $s \in I_{k+i}$ for some $k, i \in \mathbb{N}$. Since $I_1 \subseteq I_2 \subseteq \dots$ are ideals of R, we know that $r \in I_k$ implies that $r \in I_{k+i}$. Thus, $r - s \in I_{k+i}$ due to I_{k+i} being an ideal. As $I_{k+i} \subseteq \bigcup_{n=1}^{\infty} I_n$, we have $r - s \in \bigcup_{n=1}^{\infty} I_n$, which means that $\bigcup_{n=1}^{\infty} I_n$ is closed under additive inverse. Hence, $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition.
Then, we prove that for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, we would have $tr, rt \in \bigcup_{n=1}^{\infty} I_n$. Since $r \in \bigcup_{n=1}^{\infty} I_n$, it must be true that $r \in I_k$ for some $k \in \mathbb{N}$. Hence, $tr, rt \in I_k$ due to I_k being an ideal. Therefore, $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$.
In conclusion, since $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition with the property that $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, it is an ideal of R. ☐

(*Base case*) If $p_1 = q_1 \dots q_m$ and p_1 irreducible, then $q_2 \dots q_m$ are all units. Hence, $m = 1$ and $p_1 = q_1$.

(*Inductive step*) Say we have already proven the statement for $n = k$. Then consider $p_1 p_2 \dots p_{k+1} = q_1 q_2 \dots q_m$. Since R is a PID where irreducible implies prime, p_1 is a prime element dividing the product of primes $q_1 q_2 \dots q_m$, so we say WLOG $p_1 | q_1$. This means that $q_1 = u_1 p_1$ for some $u \in R$, but since q_1 is not reducible, it forces u_1 to be a unit. Hence, we apply cancellation on both sides and get $p_2 \dots p_{k+1} = (u_1 q_2) \dots q_m$.

By inductive hypothesis, $m - 1 = k$ and p_i, q_i are associates up to reordering for any i. Hence, the factorization must be unique. $\square$

We have shown:

ID	$\supsetneq$	*UFD*	$\supsetneq$	*PID*	$\supsetneq$	*ED*	$\supsetneq$	*Fields*
Non-eg: $\mathbb{Z}/6\mathbb{Z}$		$\mathbb{Z}[\sqrt{-5}]$		$\mathbb{Z}[x]$		$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$		$\mathbb{Z}$

Maximal $\Longrightarrow$ *Prime* $\Longrightarrow$ *Irreducible* (from ID onward)

Prime $\Longleftrightarrow$ *Irreducible* (from UFD onward)

Prime $\Longleftrightarrow$ *Maximal* (from PID onward)

Field extensions

We observe that the polynomial $x^2 - 2 \in \mathbb{Q}[x]$ is irreducible. If we have $x^2 - 2 = p(x)q(x)$ where p, q nonunits, then $\deg(p) + \deg(q) = 2$ and we cannot have any $0+2$ combinations due to constants being units, we only have $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, but $x \pm \sqrt{2} \notin \mathbb{Q}[x]$!

Since $\mathbb{Q}[x]$ is a UFD, the irreducible element $(x^2 - 2)$ is prime, and since $\mathbb{Q}[x]$ is a PID, $(x^2 - 2)$ is maximal which means that $\mathbb{Q}[x]/(x^2 - 2)$ is a field.

Phase II plan: Field extensions!

Suppose F is a field and $p(x) \in F[x]$ nonzero. Recall that $F[x]$ is a ED with the norm function $\deg(a(x))$ and long division of polynomials. Let $a(x) + (p(x)) \in F[x]/(p(x))$. By the division algorithm, we have $a(x) = p(x)q(x) + r(x)$ for $q(x), r(x) \in F[x]$ and $\deg(r(x)) < \deg(p(x))$ or $r(x)$ is the zero polynomial.

← $a(x)$ is a coset rep

← We can do division algorithm since this is an ED

Now we see that since $a(x) - r(x) \in (p(x))$, they are in the same coset! Hence $a(x) + (p(x)) = r(x) + (p(x))$. We observe that every element of $F[x]/(p(x))$ can be represented by a polynomial of a degree less than $\deg(p(x))$. In other words, if $\deg(p(x)) = n$, then $F[x]/(p(x))$ is of the form

$$\begin{aligned} F[x]/(p(x)) &= \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \ldots, a_{n-1} \in F \right\} \\ &= \mathrm{Span}_F\{\bar{1}, \bar{x}, \ldots, \overline{x^{n-1}}\} \end{aligned}$$

← The expression under the bar functions like $r(x)$! Also note that span is just the set of linear combinations.

In fact, $F[x]/(p(x))$ is (partly) just a **vector space** over F...

← Why is this not the vector space over $F/(p(x))$ but just F? See the next paragraph.

We shall observe that it does not matter if we are using F or $\bar{F}$.
Consider $\varphi : F \hookrightarrow F[x]/(p(x))$ where $a \mapsto \bar{a}$. We observe this is an **injective** map: whenever $\deg(p(x)) = n > 0$, we have $\varphi(a) = \varphi(b)$ *if and only if* $\bar{a} = \bar{b}$, which happens *if and only if* $a - b \in (p(x))$; but the difference of two constants always have deg 0 and cannot be in $(p(x))$ unless it is a straight zero, which tells us that $\bar{a} = \bar{b}$ *if and only if* $a = b$. In other words, $F[x]/(p(x))$ contains an isomorphic copy of F, its field of scalars! Namely, $\boxed{F \cong \varphi(F) = \{\bar{a} \in F[x]/(p(x)) \mid a \in F\}}$.

...Hence, $F[x]/(p(x))$ is a vector space **of dimension** n over the scalar field F that also contains an isomorpic copy of F.

Moreover, if $p(x)$ is irreducible, then $(p(x))$ is prime since this is an ED, and hence, it is also a maximal ideal, meaning that $F[x]/(p(x))$ is a field containing an isomorphic copy of F.

← all thanks to Euclidean domains!

Definition 19. Suppose $F \subseteq K$ are fields. Then K is called a **field extension** of F.

- Notation: K/F or $\begin{matrix} K \\ | \\ F \end{matrix}$ (*the lattice notation*)

← Please, this is NOT a quotient. DO NOT CONFUSE THOSE!!

The dimension of K as a vector space over F is called the **degree** of the extension.

- Notation: $[K : F]$

But does my field F always have an extension? Here is a systematic way to get extensions:

Example 21. If $p(x) \in F[x]$ is an irreducible polynomial of degree $n \geq 1$ over the field F, then $F[x]/(p(x))$ is a **field extension** of F of degree n.

← Since $\varphi(F) \cong F$, and $\varphi(F) \subseteq F[x]/(p(x))$

Furthermore, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $\bar{x}$ is a **root** of

$$\varphi(p(x)) = \bar{a}_0 + \bar{a}_1 y + \cdots + \bar{a}_n y^n \in (\, F[x]/(p(x))\,)\,[y]$$

because, plugging in $y = \bar{x}$, we get

← We think about modding out by $(p(x))$ as making it equal to zero, which is how we find roots.

$$\bar{a}_0 + \bar{a}_1 \bar{x} + \cdots + \bar{a}_n \bar{x}^n = \overline{p(x)} = \bar{0} \in F[x]/(p(x))$$

Hence, the isomorphic copy of the polynomial $p(x)$ has **roots** in the field extension $F[x]/(p(x))$.

So, what the hell is $F[x]/(p(x))$? We have already shown that the field extension $F[x]/(p(x))$ does indeed contain a root of $p(x)$. Now we think about it **the other way around**: if we want to find an extension of F that contains a root of $p(x)$, we would eventually get this one!

Suppose $p(x) \in F[x]$ is irreducible. Let K/F be an extension, and $\alpha \in K$ a root of $p(x)$. Denote by $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α. Consider the map $\varphi : F[x] \to F(\alpha) \subseteq K$ where $q(x) \mapsto q(\alpha)$ is simply the evaluation at α map. We note that $p(x) \in \ker(\varphi) = (d(x))$ since an ED is a PID; this implies that $p(x) = u(x)d(x)$. As $p(x)$ is irreducible, $u(x)$ must be a unit, which means $p(x)$ and $d(x)$ are associates and $\ker(\varphi) = (p(x))$. Therefore,

$$F[x]/(p(x)) = F[x]/\ker(\varphi) \cong \varphi(F[x]) \subseteq F(\alpha)$$

by first isomorphism theorem. However, $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α, so $\varphi(F[x])$ cannot be smaller than that. Hence, it must be true that $\varphi(F[x]) = F(\alpha)$.

← Observe that $\varphi(F[x])$ is a field: $\ker(\varphi)$ is a maximal ideal

Therefore, $F(\alpha)$ is simply $F[x]/(p(x))$.