

MATH172 Galois Theory Notes

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November 9, 2023

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Rings! Or why $x^2 - 2$ has roots.

Definition 1. A **ring** is a set R together with associative binary *operations* $+$ and \times s.t.:

← map from $R \times R \mapsto R$

- $(R, +)$ is an **abelian** group with identity 0
- There exists $1 \in R$ s.t. $r \times 1 = 1 \times r = r$
- $r(s + t) = rs + rt$ and $(s + t)r = sr + tr \quad \forall s, r, t \in R$

← this is optional

Proposition 1. $0 \times 1 = 0$ (in fact, $0 \times r = 0 \quad \forall r \in R$)

Proof. Try it! □

Definition 2. If \times is commutative, then R is a commutative ring.

Non-example 1. \mathbb{N} is not a ring.

Example 2. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all rings;

- $\mathbb{Z}/n\mathbb{Z}$ is a finite ring
- $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, is a **noncommutative** ring
- Polynomial ring: $\mathbb{Q}[x] = \{a_0 + a_1x + \dots + a_nx^n \mid a_i \in \mathbb{Q}\}$ is a commutative ring
- $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ is a commutative ring

← square brackets just mean "polynomials in..."

Phase I plan:

$$ID \supsetneq UFD \supsetneq PID \supsetneq ED \supsetneq Fields$$

Definition 3. Suppose R is a ring and $a, b \in R$ with $ab = 0$ but $a, b \neq 0$; then a, b are called **zero divisors**.

Example 3.

- In $\mathbb{Z}/6\mathbb{Z}$, $\bar{4} \times \bar{3} = \bar{0}$
- In $M_2(\mathbb{R})$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Definition 4. A commutative ring without zero divisors is called an integral domain (ID)

Why do we want ID? **Cancellation properties.**

- If R is an ID, $a, b, c \in R$, $a \neq 0$ and $ab = ac$, then

$$ab - ac = 0 \implies a(b - c) = 0 \implies b - c = 0 \implies b = c$$

Definition 5. Suppose R is an ID. An element $a \in R$ is called a **unit** if $a \neq 0$ and there exists $b \in R$ s.t. $ab = 1$.

← notation: $b = a^{-1}$

An element $r \in R$ is called **irreducible** if $r \neq 0$, r is NOT a unit, and whenever $r = ab$ for some $a, b \in R$ then a or b must be a unit.

- If r and s are irreducibles with $r = us$, then r and s are called **associates**.

Example 4.

- All “prime integers” are irreducibles in \mathbb{Z} ;
- $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are irreducibles in $\mathbb{Z}[\sqrt{-5}]$.
 - Note: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6$ says that 6 can be factored in more than one way. This means that $\mathbb{Z}[\sqrt{-5}]$ is NOT an UFD.

Definition 6. An integral domain R is called a unique factorization domain (UFD) if each nonzero, nonunit $a \in R$ can be written as a product of irreducibles **in a unique way** up to associates.

If a is a nonzero, nonunit element of UFD R and $a = r_1 r_2 \dots r_m = s_1 \dots s_n$ where r_i, s_j are irreducible, then after reordering $r_i = u_i s_i$ for any i and units u_i , and $m = n$.

← After reordering, there are the same amounts of factors and all factors are the same up to units.

Definition 7. Suppose R is a comm ring. A subset $I \subseteq R$ is called an **ideal** if $(I, +) \leq (R, +)$ and $ir, ri \in I$ for all $i \in I$ and for all $\overline{r \in R}$.

Why do we want ideals? Such that R/I is a well-defined ring.

Example 5. $\{0\}$ and R are ideals of R .

Example 6. If R is commutative and $a \in R$, then $(a) = \{ar \mid r \in R\}$ is called the **principal ideal** generated by a .

← Prove this (be convinced)!
Also known as aR .

Definition 8. A **principal ideal domain** is an integral domain where all ideals are principal ideals.

Example 7. The only ideals of $(\mathbb{Z}, +)$ are of the form $n\mathbb{Z} = (n)$.

← Ideals generated by n

Non-example 8. $\mathbb{Z}[x]$ is a UFD but NOT a PID because the ideal $(2, x) = \{2r + xs \mid r, s \in \mathbb{Z}[x]\}$ is not principal.

← Observe that $(2, x)$ is an ideal made of polynomials with even constant terms. This cannot be principal, since if we only have 2 and not x , we do not have nonzero polynomials with zero const terms.

Lemma 2. If $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.

Proof. Try it!

Proposition 3. If $I \subseteq R$ is an ideal containing a unit of R then $I = R$.

Proof. If $u \in I$ is a unit then $u^{-1} \in R$, so $uu^{-1} = 1 \in I$. Then the result follows from Lemma 2. \square

Definition 9. A **field** is a commutative ring whose each nonzero element is a unit.

Corollary 4. If R is an ID whose ideals are (0) and R , then R is a **field**.

Proof. Suppose $a \in R \setminus \{0\}$ and consider (a) . Since $a \in (a)$, $(a) = R$. Hence, we must have that $1 \in (a)$, which means $1 = ar$ for some $r \in R$. \square

Definition 10. Suppose R is an integral domain. A *proper* ideal $P \subsetneq R$ is called **prime** if whenever $ab \in P$ for some $a, b \in R$, then a or $b \in P$.

Non-example 9. (6) is not a prime ideal of \mathbb{Z} since $2 \times 3 \in (6)$ but neither $2, 3 \in (6)$.

Non-example 10. (2) is not a prime ideal of $\mathbb{Z}[\sqrt{-5}]$ since $6 \in (2)$, but we observe that $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ while $1 \pm \sqrt{-5} \notin (2)$.

Example 11. (2) is a prime ideal of \mathbb{Z} .

Definition 11. A proper ideal $M \subsetneq R$ is called **maximal** if whenever $I \subseteq R$ such that $M \subseteq I \subseteq R$ is an ideal containing M , then either $I = M$ or $I = R$.

Proposition 5. Every proper ideal is contained in a maximal ideal.

Proof. Axiom of choice. \square

Proposition 6. Suppose R is a commutative ring.

- (0) is prime *if and only if* R is an integral domain.
- (0) is maximal *if and only if* R is a field.

(The following is kind of on a tangent)

Definition 12. A commutative ring R with unity is called **Noetherian** if, whenever $I_1 \subseteq I_2 \subseteq \dots$ is an ascending sequence of (proper) ideals of R , there exists an $n > 0$ such that $I_n = I_{n+1} = \dots$ are the same ideals thereafter.

Theorem 7. R is Noetherian *if and only if* all ideals of R are finitely generated.

Corollary 8. All Principal Ideal Domains are Noetherian.

← The converse is also true. **The only ideals in a field are 0 and the field.**

← Observe that in $\mathbb{Z}[\sqrt{-5}]$, we have $6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \times 3$, so it is not a UFD!

← This might not be unique in non-local rings.

← By def of prime, if $ab = 0$, then either $a = 0$ or $b = 0$, which means there are NO zero divisors.

← The chain stops ascending!

← Since all ideals are generated by 1 elt.

(Tangent ends here)

Definition 13. Suppose R is a commutative ring with $1 \neq 0$ and $I \subseteq R$ is an ideal. Then the quotient ring of R by I is the set

$$R/I = \{r + I \mid r \in R\}$$

with addition and multiplication defined representative-wise.

Remark. The **coset criterion** of ideals: let I be an ideal; the cosets $r + I$, $s + I$ are the same *if and only if* $r - s \in I$.

Example 12.

- In $\mathbb{Z}/(6)$ aka. $\mathbb{Z}/6\mathbb{Z}$, we have $2 + (6) = \{\dots, -10, -4, 2, 8, 14, \dots\} = 26 + (6)$ due to $2 - 26 \in (6)$;
- In $\mathbb{Q}[x]/(x^2 - 2)$, we have

$$\{3x^2 - 47x + 1 + q(x)(x^2 - 2) \mid q(x) \in \mathbb{Q}[x]\} = \{-47x + 7 + q(x)(x^2 - 2) \mid q(x) \in \mathbb{Q}[x]\}$$

$$\text{due to } 3x^2 - 47x + 1 - (-47x + 7) \in (x^2 - 2).$$

Remark. Let I be an ideal of R . Then $(I, +) \trianglelefteq (R, +)$.

Definition 14. R/I is a group under $(r + I) + (s + I) = (r + s) + I$ and the operation $+$ is well-defined. We also define that $(r + I)(s + I) = (rs) + I$. We claim that multiplication in R/I is also well-defined.

Proof. Let $r_1 + I = r_2 + I$ and $s_1 + I = s_2 + I$. By coset criterion, $r_1 - r_2 = i$, $s_1 - s_2 = j$ for some $i, j \in I$. Hence $r_1 s_1 = (r_2 + i)(s_2 + j) = r_2 s_2 + i s_2 + j r_2 + ij$ where the latter three terms are all in the ideal I . Thus, $(r_1 s_1) + I = (r_2 s_2) + I$. \square

From R , R/I inherits nice properties:

- $0 + I = 0_{R/I}$
- $1 + I = 1_{R/I}$
- Multiplication is commutative and distributive over addition in R/I , so it is also a comm. ring with identity.

Definition 15. A function $\varphi : R \rightarrow S$ between rings is called a **ring homomorphism** if the following are satisfied:

- $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$
- $\varphi(r_1 r_2) = \varphi(r_1) \varphi(r_2)$

Theorem 9. First ring isomorphism theorem

If $\varphi : R \mapsto S$ is a ring homomorphism, then $R/\ker(\varphi) \cong \varphi(R)$.

Example 13. If R is a ring and I is an ideal, then $\pi : R \rightarrow R/I$ where $r \mapsto r + I$ is a surjective homomorphism where $\ker(\pi) = I$. This is the *canonical projection* onto R/I .

Corollary 10. If I is a maximal ideal, then R/I is a field.

Recall Proposition 6. We now have a stronger statement:

Proposition 11. Suppose R is a commutative ring & $P \subseteq R$ is an ideal. Then R/P is an integral domain *if and only if* P is prime.

Proof. R/P is an integral domain *if and only if* whenever $(a+P)(b+P) = 0_{R/P}$ then one of $a+P$ or $b+P$ must already be $0_{R/P}$. This happens *if and only if* whenever $ab+P = P$ then $a+P$ or $b+P$ in P , which happens *if and only if* whenever $ab \in P$ then one of $a, b \in P$, which is the definition of a prime ideal. \square

Example 14. The map $\varphi : \mathbb{Z}[x] \rightarrow \mathbb{Z}$ where $p(x) \mapsto p(0)$ is a surjective ring homomorphism with $\ker(\varphi) = (x)$. By the First Isomorphism Theorem 9, $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. As such, we conclude that (x) is a prime ideal since \mathbb{Z} is an integral domain.

Lemma 12. Suppose R is a comm. ring with $M \subseteq R$ being an ideal.

There is a bijective correspondence between the ideals of R/M and the ideals of R containing M .

Proof. Consider the projection $\pi : R \rightarrow R/M$ where $r \mapsto r + M$. It is enough to show:

$$\begin{aligned} \pi(\pi^{-1}(J)) &= J && \text{for all ideals } J \subseteq R/M, \text{ and} \\ \pi^{-1}(\pi(I)) &= I && \text{for all ideals } M \subseteq I \subseteq R \end{aligned}$$

To prove the first statement, observe that, if J is an ideal of R/M , then $\pi^{-1}(J) = \{r \in R \mid r + M \in J\}$ and so

$$\pi(\pi^{-1}(J)) = \{\pi(r) \in R/M \mid r + M \in J\} = \{r + M \mid r + M \in J\} = J$$

Next, to prove the second statement, first suppose $M \subseteq I \subseteq R$ is an ideal. Let $a \in I$. Then $a + M \in \{\alpha + M \mid \alpha \in I\} = \pi(I)$. This implies that $a \in \pi^{-1}(\pi(I))$, and so $I \subseteq \pi^{-1}(\pi(I))$.

← Observe that kernels are ideals! And ideals are kernels of some homomorphism too.

← The *if and only if* version comes in Proposition 14.

← btw, $(x) \subseteq (x, 2)$. the latter is the set of polynomials whose constant term is even, so it is also a proper ideal of $\mathbb{Z}[x]$. This is an excellent example where Prime \nRightarrow Maximal.

← To see why this is okay, see Homework 2 Sec. 7.3 P. 24

Conversely, suppose $r \in \pi^{-1}(\pi(I))$. This is the same as saying $\pi(r) = r + M \in \pi(I) = \{\alpha + M \mid \alpha \in I\}$. Hence, for any $r \in \pi^{-1}(\pi(I))$, there exists some $a \in I$ such that $r + M = a + M$. Thus, $r - a \in M \subseteq I$ by coset conditions. Since $a \in I$, we have $a + (r - a) \in I$, meaning that $r \in I$ for any $r \in \pi^{-1}(\pi(I))$. This means that $\pi^{-1}(\pi(I)) \subseteq I$.

Hence, $I = \pi^{-1}(\pi(I))$.

Consequently, for any ideals $J \subseteq R/M$, we know that $\pi^{-1}(J) \subseteq R$ is an ideal containing M . And if $M \subseteq I \subseteq R$ is an ideal, we know $\pi(I) \subseteq R/M$ is an ideal. Since $\pi(\pi^{-1}(J)) = J$ and $I = \pi^{-1}(\pi(I))$ for any I, J , the correspondence is a bijection. \square

← Think about why this contains M !

Proposition 13. Suppose R is a comm. ring with an identity and $I \subseteq R$ is an ideal. Then R/I is a field *if and only if* I is maximal.

Proof. If I is maximal, then there are no other proper ideals strictly containing I . Hence, by Lemma 14, we have that R/I only have ideals (0) and R/I itself. This happens *if and only if* R/I is a field. \square

Corollary 14. If R is a commutative ring with identity and $M \subseteq R$ is maximal, then M is prime.

← Hence maximal implies prime, but prime does not necessarily implies maximal.

Proof. Maximal \implies quotient is a field \implies quotient is an ID \implies prime. \square

Definition 16. An integral domain R is an **Euclidean domain** if there exists a norm $N : R \rightarrow \mathbb{Z}_{\geq 0}$ with $N(0) = 0$ such that for all $a, b \in R$ with $b \neq 0$, there exists $q, r \in R$ for which

$$a = bq + r$$

with $N(r) < N(b)$ or $r = 0$.

Example 15. \mathbb{Z} is a ED with $N(a) = |a|$.

Example 16. $\mathbb{Q}[x]$ is a ED with $N(p(x)) = \deg(p(x))$.

Example 17. Every field F is a ED with $N(a) = 0 \forall a \in F$.

Non-example 18. $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID that is not an ED.

← Because in a field everything divides!

← This is one of the only good examples!

Why do we care about Euclidean domains?

Remark. Greatest common divisors exist and are relatively quick to compute.

← Using recursive application of Euclidean algorithm.

Definition 17. If $a, b \in R$, then $\gcd(a, b) = c$ means

1. c divides a and b ; that is, $a = cr, b = cs$ for some $r, s \in R$
2. If $c' \in R$ with $c'|a$ and $c'|b$, then it must be true that $c'|c$.

← All other common divisors divide the gcd.

Example 19. Say we want to compute the gcd of 47 and 10.

$$47 = 4 \times 10 + 7$$

$$10 = 1 \times 7 + 3$$

$$7 = 2 \times 3 + \textcircled{1}$$

$$3 = 3 \times 1$$

← circled is $\gcd(47, 10)$

← final line with no remainders

← This is a much faster algorithm than factoring!

This also works for finding gcds in $\mathbb{Q}[x]$ with polynomials long division and norm $\deg(p(x))$.

Remark. If F is a field, then $F[x]$ is a Euclidean domain.

← Just use long division!

Remark. Euclidean domains are PIDs.

Proof. Suppose R is a ED and $I \subseteq R$ is an ideal. Consider $\{N(a) \mid a \in I \setminus \{0\}\}$. This set has a minimal element by properties of natural numbers (or is an empty set if and only if $I = (0)$).

Let $d \in I$ be an element of minimum norm (hence $N(d) \leq N(a)$ for all $a \in I$). We claim that $\underline{(d) = I}$. Proof:

Since $d \in I$, we have $rd \in I$ for any $r \in R$. This implies that $(d) \subseteq I$.

Then let $a \in I$. Since R is a ED, we first assume that there exists $q, r \in R, r \neq 0$ such that $a = qd + r$ and $N(r) < N(d)$. But we notice that $r = a - qd$ must be in I as both $a, qd \in I$, contradicting the minimality of $N(d)$. Thus, it must be that $r = 0$. This implies $a = qd$ and thus $a \in (d)$ for all $a \in I$. Consequently, $I \subseteq (d)$, and therefore $I = (d)$. \square

Definition 18. Suppose R is an integral domain and $p \in R \setminus \{0\}$. Then p is a **prime element** if (p) is a prime ideal.

Proposition 15. An element $p \in R$ is prime if and only if whenever $p|ab$ then $p|a$ or $p|b$.

Proof. p is prime means that (p) is a prime ideal. This is true if and only if whenever $ab \in (p)$ then $a \in (p)$ or $b \in (p)$. This is the same as saying if $ab = kp$ for some $k \in R$ then $a = lp$ or $b = lp$ for some $l \in R$. This is to say that whenever $p|ab$ then $p|a$ or $p|b$. \square

Proposition 16. In an integral domain, all prime elements are irreducibles.

Proof. Suppose R is an ID and $p \in R$ is prime. If $p = ab$ for some a, b in R , then, WLOG, $p|a$. That is, $a = pk$ for some $k \in R$. Hence, $p = pkb$. Since in an ID cancellation rule holds, $kb = 1$, meaning that b is a unit. Thus, p is irreducible by definition Definition 5. \square

Proposition 17. In PIDs, all *nonzero* prime ideals are maximal.

Proof. Suppose R is a PID and $(p) \subseteq R$ is a prime ideal. If $(p) \subseteq (m) \subseteq R$ is an ideal, then $p \in (p) \subseteq (m)$ hence $p = rm$ for some $r \in R$. Since $p \nmid rm$, we have $p|r$ or $p|m$.

If $p|r$, this implies that $r = pk$ for some $k \in R$. Substituting into $p = rm$, we get $p = pkm$. By cancellation, we get $km = 1$, meaning that m is a unit. Hence, $(m) = R$.

If $p|m$, we have $m = pl$ for some $l \in R$, meaning that $m \in (p)$. Hence, $(m) \subseteq (p)$, but we also defined that $(p) \subseteq (m)$, so $(m) = (p)$.

Therefore, (p) has to be the maximal ideal. \square

Proposition 18. In an UFD, irreducible implies prime.

Proof. Let R be a UFD and $p \in R$ be irreducible. Let $a, b \in R$ such that $p|ab$. Hence, $pr = ab$ for some $r \in R$. Since R is a UFD, let $a = q_1 \dots q_n, b = s_1 \dots s_m$ be the factorization. Since the factorizations are unique and each of the q_i, s_j are irreducible, if $p|ab$, then p must be an associate with one of the q_i, s_j . Therefore, either $p|a$ or $p|b$, implying prime. \square

Example 20. \mathbb{Q} is a field, so $\mathbb{Q}[x]$ is a ED. Since EDs are UFDs, irreducible \implies prime. We see that $x^2 - 2 \in \mathbb{Q}[x]$ is an irreducible element, which means that $(x^2 - 2)$ is a prime ideal, meaning that it is a maximum ideal, meaning that $\mathbb{Q}[x]/(x^2 - 2)$ is a field. We observe that it is a field containing \mathbb{Q} and $(\sqrt{2})$.

← In fact, this is the smallest field containing \mathbb{Q} and $(\sqrt{2})$.

Lemma 19. In a PID, irreducible elements are prime.

Proof. Suppose $p \in R$ is irreducible in the principal ideal domain R . If $p|ab$ for some $a, b \in R$, we want to show that either $p|a$ or $p|b$, hereby showing that p is prime. Hence, we consider the ideal $(a, p) = d$, which is necessarily principal for some $d \in R$. Since $a, p \in (d)$, we have $a = dr$ and $p = ds$ for some $r, s \in R$. As p is irreducible, we get that one of d and s is a unit.

We first assume that s is a unit, in which case $d = ps^{-1}$, and so $a = ps^{-1}r$ implying that $p|a$.

In another case, d is a unit, in which case $(a, p) = (d) = R$ and so $1 = ak + pl$ for some $k, l \in R$. Multiplying by b , we get $b = abk + pbl$. Since $p|ab$, we have $b = abk + pbl = pmk + pbl$ for some $m \in R$. Hence, $b = p(mk + bl)$, meaning that $p|b$.

Therefore, whenever $p|ab$, either $p|a$ or $p|b$. Hence, in a PID, p is prime whenever it is irreducible. \square

Proposition 20. PIDs are UFDs.

Proof. Suppose R is a PID and $a \in R$ is nonzero, nonunit. If a is irreducible, we are done. If not, we write $a = p_1 q_1$ for some $p_1, q_1 \in R$ nonunit. If p_1, q_1 are irreducibles, we are done. If not, then WLOG say $q_1 = p_2 q_2$ for some nonunits p_2, q_2 . We would like to show that this splitting process terminates.

Observe that $(q_1) \subseteq (q_2)$ since $q_2|q_1$. Hence, the chain of splitting results in the chain of ideals $(q_1) \subseteq (q_2) \subseteq (q_3) \subseteq \dots$.

Now consider the ideal¹ $\bigcup_{i=1}^{\infty} (q_i)$. Since this is a PID, we have $\bigcup_{i=1}^{\infty} (q_i) = (q)$ for some $q \in R$. Since $q \in \bigcup_{i=1}^{\infty} (q_i)$, it is contained in some (q_n) for some $n \geq 1$. This implies that $(q) \subseteq (q_n)$, but we also know that $(q_n) \subseteq (q)$, hence $(q) = (q_n)$. Hence, this process terminates, and there exists an n in this chain such that q_n is irreducible. Therefore, R is a factorization domain.

Now we want to prove the uniqueness. That is, if $p_1 \dots p_n = q_1 \dots q_m$ for irreducibles p_i, q_j and $n \leq m$ WLOG, then we want to show that $m = n$ and that $p_i = u_i q_i$ with units u_i up to reordering for all i . We do so by induction on n .

(Base case) If $p_1 = q_1 \dots q_m$ and p_1 irreducible, then $q_2 \dots q_m$ are all units. Hence, $m = 1$ and $p_1 = q_1$.

(Inductive step) Say we have already proven the statement for $n = k$. Then consider $p_1 p_2 \dots p_{k+1} = q_1 q_2 \dots q_m$. Since R is a PID where irreducible implies prime, p_1 is a prime element dividing the product of primes $q_1 q_2 \dots q_m$, so we say WLOG $p_1|q_1$. This means that $q_1 = u_1 p_1$ for some $u \in R$, but since q_1 is not reducible, it forces u_1 to be a unit. Hence, we apply cancellation on both sides and get $p_2 \dots p_{k+1} = (u_1 q_2) \dots q_m$.

By inductive hypothesis, $m - 1 = k$ and p_i, q_i are associates up to reordering for any i . Hence, the factorization must be unique. \square

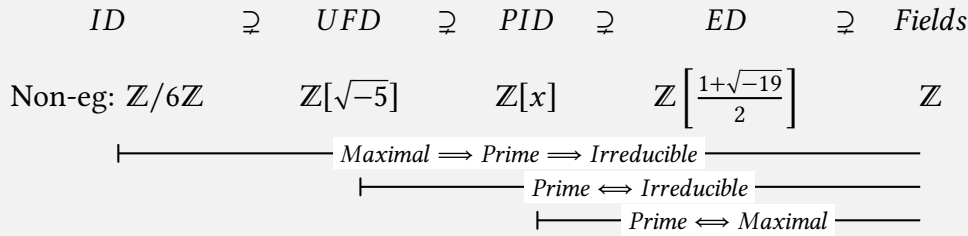
¹The proof that this is an ideal is as follows:

We first prove that $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition. Let $r, s \in \bigcup_{n=1}^{\infty} I_n$, where $r \in I_k$ and $s \in I_{k+i}$ for some $k, i \in \mathbb{N}$. Since $I_1 \subseteq I_2 \subseteq \dots$ are ideals of R , we know that $r \in I_k$ implies that $r \in I_{k+i}$. Thus, $r - s \in I_{k+i}$ due to I_{k+i} being an ideal. As $I_{k+i} \subseteq \bigcup_{n=1}^{\infty} I_n$, we have $r - s \in \bigcup_{n=1}^{\infty} I_n$, which means that $\bigcup_{n=1}^{\infty} I_n$ is closed under additive inverse. Hence, $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition.

Then, we prove that for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, we would have $tr, rt \in \bigcup_{n=1}^{\infty} I_n$. Since $r \in \bigcup_{n=1}^{\infty} I_n$, it must be true that $r \in I_k$ for some $k \in \mathbb{N}$. Hence, $tr, rt \in I_k$ due to I_k being an ideal. Therefore, $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$.

In conclusion, since $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition with the property that $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, it is an ideal of R . \square

We have shown:



Field extensions

We observe that the polynomial $x^2 - 2 \in \mathbb{Q}[x]$ is irreducible. If we have $x^2 - 2 = p(x)q(x)$ where p, q nonunits, then $\deg(p) + \deg(q) = 2$ and we cannot have any $0+2$ combinations due to constants being units, we only have $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, but $x \pm \sqrt{2} \notin \mathbb{Q}[x]$!

Since $\mathbb{Q}[x]$ is a UFD, the irreducible element $(x^2 - 2)$ is prime, and since $\mathbb{Q}[x]$ is a PID, $(x^2 - 2)$ is maximal which means that $\mathbb{Q}[x]/(x^2 - 2)$ is a field.

Phase II plan: Field extensions!

Suppose F is a field and $p(x) \in F[x]$ nonzero. Recall that $F[x]$ is a ED with the norm function $\deg(a(x))$ and long division of polynomials. Let $a(x) + (p(x)) \in F[x]/(p(x))$. By the division algorithm, we have $a(x) = p(x)q(x) + r(x)$ for $q(x), r(x) \in F[x]$ and $\deg(r(x)) < \deg(p(x))$ or $r(x)$ is the zero polynomial.

Now we see that since $a(x) - r(x) \in (p(x))$, they are in the same coset! Hence $a(x) + (p(x)) = r(x) + (p(x))$. We observe that every element of $F[x]/(p(x))$ can be represented by a polynomial of a degree less than $\deg(p(x))$. In other words, if $\deg(p(x)) = n$, then $F[x]/(p(x))$ is of the form

$$\begin{aligned} F[x]/(p(x)) &= \left\{ \overline{a_0 + a_1x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \dots, a_{n-1} \in F \right\} \\ &= \text{Span}_F\{\bar{1}, \bar{x}, \dots, \bar{x}^{n-1}\} \end{aligned}$$

← $a(x)$ is a coset rep
 ← We can do division algorithm since this is an ED

← The expression under the bar functions like $r(x)$! Also note that span is just the set of linear combinations.

In fact, $F[x]/(p(x))$ is (partly) just a **vector space** over F ...

We shall observe that it does not matter if we are using F or \bar{F} .

Consider $\varphi : F \hookrightarrow F[x]/(p(x))$ where $a \mapsto \bar{a}$. We observe this is an **injective** map: whenever $\deg(p(x)) = n > 0$, we have $\varphi(a) = \varphi(b)$ if and only if $\bar{a} = \bar{b}$, which happens if and only if $a - b \in (p(x))$; but the difference of two constants always have $\deg 0$ and cannot be in $(p(x))$ unless it is a straight zero, which tells us that $\bar{a} = \bar{b}$ if and only if $a = b$. In other words, $F[x]/(p(x))$ contains an isomorphic copy of F , its field of scalars! Namely, $\overline{F \cong \varphi(F) = \{\bar{a} \in F[x]/(p(x)) \mid a \in F\}}$.

...Hence, $F[x]/(p(x))$ is a vector space **of dimension n** over the scalar field F that also contains an isomorphic copy of F .

Moreover, if $p(x)$ is irreducible, then $(p(x))$ is prime since this is an ED, and hence, it is also a maximal ideal, meaning that $F[x]/(p(x))$ is a field containing an isomorphic copy of F .

Definition 19. Suppose $F \subseteq K$ are fields. Then K is called a **field extension** of F .

- Notation: K/F or $\begin{smallmatrix} K \\ | \\ F \end{smallmatrix}$ (the lattice notation)

The dimension of K as a vector space over F is called the **degree** of the extension.

- Notation: $[K : F]$

But does my field F always have an extension? Here is a systematic way to get extensions:

Example 21. If $p(x) \in F[x]$ is an irreducible polynomial of degree $n \geq 1$ over the field F , then $F[x]/(p(x))$ is a **field extension** of F of degree n .

Furthermore, if $p(x) = a_0 + a_1x + \dots + a_nx^n$, then \bar{x} is a **root** of

$$\varphi(p(x)) = \bar{a}_0 + \bar{a}_1y + \dots + \bar{a}_ny^n \in (F[x]/(p(x)))[y]$$

because, plugging in $y = \bar{x}$, we get

$$\bar{a}_0 + \bar{a}_1\bar{x} + \dots + \bar{a}_n\bar{x}^n = \overline{p(x)} = \bar{0} \in F[x]/(p(x))$$

Hence, the isomorphic copy of the polynomial $p(x)$ has **roots** in the field extension $F[x]/(p(x))$.

So, what the hell is $F[x]/(p(x))$? We have already shown that the field extension $F[x]/(p(x))$ does indeed contain a root of $p(x)$. Now we think about it **the other way around**: if we want to find an extension of F that contains a root of $p(x)$, we would eventually get this one!

← Why is this not the vector space over $F/(p(x))$ but just F ? See the next paragraph.

← all thanks to Euclidean domains!

← Please, this is NOT a quotient. DO NOT CONFUSE THOSE!!

← Since $\varphi(F) \cong F$, and $\varphi(F) \subseteq F[x]/(p(x))$

← We think about modding out by $(p(x))$ as making it equal to zero, which is how we find roots.

Suppose $p(x) \in F[x]$ is irreducible. Let K/F be an extension, and $\alpha \in K$ a root of $p(x)$. Denote by $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α . Consider the map $\varphi : F[x] \rightarrow F(\alpha) \subseteq K$ where $q(x) \mapsto q(\alpha)$ is simply the evaluation at α map. We note that $p(x) \in \ker(\varphi) = (d(x))$ since an ED is a PID; this implies that $p(x) = u(x)d(x)$. As $p(x)$ is irreducible, $u(x)$ must be a unit, which means $p(x)$ and $d(x)$ are associates and $\ker(\varphi) = (p(x))$. Therefore,

$$F[x]/(p(x)) = F[x]/\ker(\varphi) \cong \varphi(F[x]) \subseteq F(\alpha)$$

by first isomorphism theorem. However, $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α , so $\varphi(F[x])$ cannot be smaller than that. Hence, it must be true that $\varphi(F[x]) = F(\alpha)$.

← Observe that $\varphi(F[x])$ is a field: $\ker(\varphi)$ is a maximal ideal

Therefore, $F(\alpha)$ is simply $F[x]/(p(x))$. □

To summarize so far!

Suppose $p(x) \in F[x]$ is an irreducible polynomial with coefficients in the field F .

- $F[x]/p(x)$ is a **field** containing an isomorphic copy of F in which $\bar{x} = x + (p(x))$ is a **root** of (the image of) $p(y) \in (F[x]/(p(x)))[y]$.

Example 22. In $\mathbb{Q}[x]/(x^2 - 2)$, we have $x + (x^2 - 2)$ is a root of $y^2 - \bar{2} \in (\mathbb{Q}[x]/(x^2 - 2))[y]$ because

$$\begin{aligned} & (x + (x^2 - 2))^2 - (2 + (x^2 - 2)) \\ &= x^2 - 2 + (x^2 - 2) && \text{by coset addition \& multiplication} \\ &= 0 + (x^2 - 2) && \text{since } x^2 - 2 \in (x^2 - 2) \\ &= \bar{0} \end{aligned}$$

Furthermore, if $\deg(p(x)) = n$, then

$$F[x]/(p(x)) = \{ \overline{a_0 + a_1x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \dots, a_{n-1} \in F \}$$

is a vector space over F of dimension n .

Example 23. $\mathbb{Q}[x]/(x^2 - 2) = \{\bar{a}_0 + \bar{a}_1\bar{x} \mid a_0, a_1 \in \mathbb{Q}\} = \text{Span}_{\mathbb{Q}}\{\bar{1}, \bar{x}\}$

- If K/F is an extension and $\alpha \in K$ is a root of $p(x)$, denote by $F(\alpha)$ the **smallest field containing F and α** .

← Read 'F adjoint α '

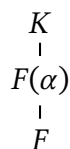


Figure 1: Field diagram

Then $F(\alpha) \cong F[x]/(p(x))$, and

$$\begin{aligned} F(\alpha) &= \left\{ \overline{a_0 + a_1x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \dots, a_{n-1} \in F \right\} \\ &= F[\alpha] \quad \leftarrow \text{the polynomial of } \alpha \text{ over } F \end{aligned}$$

← The eval map
 $\varphi : F[x] \rightarrow F(\alpha)$
 where $f(x) \mapsto f(\alpha)$
 has in fact
 $\ker(\varphi) = (p(x))$
 when α is a root of
 $p(x)$.

Example 24. $\mathbb{Q}(\sqrt{2}) = \{a_0 + a_1\sqrt{2} \mid a_0, a_1 \in \mathbb{Q}\} = \mathbb{Q}[\sqrt{2}]$

Irreducibility – a survey

Proposition 21. If $p(x) \in F[x]$, then $\alpha \in F$ is a root *if and only if* $x - \alpha$ divides $p(x)$.

Proof. Write $p(x) = (x - \alpha)q(x) + r(x)$ with $q(x), r(x) \in F[x]$ and $\deg(r(x)) = 0$ or $r(x) = 0$. Then $0 = p(\alpha) = 0 + r(\alpha)$ which forces $r(x) = 0$. \square

Corollary 22. A degree-2 or -3 polynomial over a field F is irreducible *if and only if* it has no roots in F .

Proposition 23. Suppose $p(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ with root $\frac{c}{d}$ written in reduced form (i.e. $\gcd(c, d) = 1$). Then $\boxed{c|a_0 \text{ and } d|a_n}$.

Proof.

$$\begin{aligned} d^n \cdot p\left(\frac{c}{d}\right) &= 0 \\ 0 &= (a_0d^n + a_1d^{n-1}c + \cdots + a_{n-1}dc^{n-1}) + a_nc^n \\ 0 &= a_0d^n + (a_1d^{n-1}c + \cdots + a_{n-1}dc^{n-1} + a_nc^n) \end{aligned}$$

Looking at the 2nd line, since d divides all of the ones in the $()$, it must also divide the last term a_nc^n . However, since $\gcd(c, d) = 1$, it forces d to divide a_n .

Similarly, we make the same argument for c and a_0 using the 3rd line. \square

Lemma 24. $(R/I)[x] \cong R[x]/(I)$ where $(I) = I[x]$.

Proof. Consider the surjective homomorphism $\pi : R[x] \rightarrow (R/I)[x]$. \square

Proposition 25 (Eisenstein's Criterion). Suppose $f(x) = 1x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x]$ is a monic polynomial and $p \in \mathbb{Z}$ is a **prime** such that $p \mid a_0, \dots, a_{n-1}$ but $p^2 \nmid a_0$. Then $f(x)$ is irreducible.

Proof. Assume BWOC that $f(x) = a(x)b(x)$ for some nonunit $a(x), b(x) \in \mathbb{Z}[x]$, then

$$x^n = \bar{f}(x) = \bar{a}(x)\bar{b}(x)$$

in $(\mathbb{Z}/p\mathbb{Z})[x] \cong \mathbb{Z}[x]/p\mathbb{Z}[x]$ since all other terms are divisible by p . Since $\mathbb{Z}/p\mathbb{Z}$ does not contain any zero divisors, $\bar{a}(x), \bar{b}(x)$ must have zero constant terms. Hence $a(x), b(x)$ have constant terms that are multiples of p , so $a(x)b(x)$ have constant term divisible by p^2 . This is a contradiction with $p^2 \nmid a_0$. \square

Lemma 26 (Gauss' Lemma). If $p(x) \in \mathbb{Z}[x]$ is reducible in $\mathbb{Q}[x]$, then it is reducible in $\mathbb{Z}[x]$.

Proof. Suppose $p(x) = a(x)b(x)$ for $a(x), b(x) \in \mathbb{Q}[x]$. Then by multiplying by coefficient denominators, for some $m \in \mathbb{Z}$, we could write $m \cdot p(x) = c(x)d(x)$ for some $c(x), d(x) \in \mathbb{Z}[x]$. Now since $m \in \mathbb{Z}$, we could write $m = q_1 q_2 \dots q_n$ be a product of irreducibles in \mathbb{Z} .

Now in $(\mathbb{Z}/q_1\mathbb{Z})[x] \cong \mathbb{Z}[x]/(q_1\mathbb{Z})[x]$, we observe that $m \cdot p(x) = c(x)d(x) = q_1(q_2 \dots q_n)p(x)$, meaning that

$$\overline{c(x)} \overline{d(x)} = \overline{q_1(q_2 \dots q_n)p(x)} = \bar{0}$$

Since $(\mathbb{Z}/q_1\mathbb{Z})[x] \cong \mathbb{Z}[x]/(q_1\mathbb{Z})[x]$ is an integral domain, WLOG, $\overline{c(x)} = \bar{0}$ if and only if $c(x) \in q_1\mathbb{Z}[x]$, meaning that all coefficients of $c(x)$ are multiples of q_1 . Therefore, $\frac{1}{q_1}c(x) \in \mathbb{Z}[x]$.

← since q_1 is irreducible and hence prime in UFD

Now we repeat the process for all q_1, q_2, \dots, q_n and we are done. \square

Recall that if $F \subseteq K$ are fields, $\alpha \in K$ and $p(x) \in F[x]$ is irreducible with root α , then

$$F[\alpha] = F(\alpha) \cong F[x]/(p(x)) = \overline{\{a_0 + a_1x + \dots + a_{n-1}x^{n-1} \mid a_0, a_1, \dots, a_{n-1} \in F\}}$$

We observe that this has a few implications. For instance, $F(\alpha)$ contains $\frac{1}{\alpha}$, meaning that it could also be written as a polynomial of α with coefficients in F (as in $F[\alpha]$)!

← since it is a field containing the mult. inverse of α

Definition 20. Suppose K/F is a field extension and $\alpha \in K$. We say that α is **algebraic over** F if there exists $p(x) \in F[x]$ such that $p(\alpha) = 0$. If not, α is **transcendental**.

Definition 21. The extension K/F is an **algebraic extension** if **every** element $\alpha \in K$ is algebraic over F .

Example 25. π is transcendental over \mathbb{Q} but algebraic over \mathbb{R} (since it is a root of $x - \pi$).

Proposition 27. If K/F is a **finite extension**, then it is an algebraic extension.

← finite extension
just means finite
degree
 $[K : F] < \infty$

Proof. Call $[K : F] = n$ and let $\alpha \in K$. Then the $n + 1$ elements $\{1, \alpha, \alpha^2, \dots, \alpha^n\}$ must be linearly dependent. Hence, by linear algebra, there exist $a_0, a_1, \dots, a_n \in F$ not all zero such that the linear combination $a_0 + a_1\alpha + a_2\alpha^2 + \dots + a_n\alpha^n = 0$. Hence, α is a root of $a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0$. \square

← since $n + 1 > \dim(K/F) = n$

Corollary 28. If K/F is an extension and $\alpha \in K$, then α is algebraic over F if and only if $[F(\alpha) : F] < \infty$.

Proof.

(\Leftarrow) Follows from prop.

(\Rightarrow) If α is algebraic, then there exists an irreducible polynomial $p(x)$ with α as a root and of degree $n < \infty$. Then $F(\alpha) \cong F[x]/(p(x))$ is a n -dimensional vector space over F .

Another perspective: $F(\alpha) = \text{Span}_F\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$.

← Review proof of
 $F(\alpha) \cong F[x]/(p(x))$.

\square

Proposition 29. Suppose K/F is an extension & $\alpha \in K$ is algebraic over F . Then there exists a unique, irreducible, and monic polynomial $m_{\alpha,F}(x) \in F[x]$ that has α as a root.

Remark. We observe that m does depend on the base field F ; $m_{\sqrt{2},\mathbb{Q}}(x) = x^2 - 2$, but $m_{\sqrt{2},\mathbb{Q}(\sqrt{2})}(x) = x - \sqrt{2}$.

Proof. Since the subset of $F[x]$ satisfying α is a root is nonempty, we can pick one with a **minimal degree**. By multiplying by an element of F if necessary, we can assume WLOG this polynomial is **monic**. Call it $m_{\alpha,F}(x)$.

Assume BWOC that m is the product of two other polynomials of lesser degree such that $m_{\alpha,F}(x) = a(x)b(x)$, then we plug in $0 = m_{\alpha,F}(\alpha) = a(\alpha)b(\alpha)$. Since there are no zero divisors in $F[x]$, WLOG $a(\alpha) = 0$, contradicting the minimality of $m_{\alpha,F}$. Hence $m_{\alpha,F}$ is **irreducible**.

← So $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$

Then, BWOC if $p(x) \in F[x]$ with α as a root and is monic and irreducible, there exist $q(x), r(x) \in F[x]$ such that $p(x) = m_{\alpha,F}(x)q(x) + r(x)$ where $\deg(r) < \deg(m_{\alpha,F})$ or $r(x) = 0$. Then, we observe that $p(\alpha) = 0 = m_{\alpha,F}(\alpha)q(\alpha) + r(\alpha) = 0 + r(\alpha)$. Thus, $r(\alpha) = 0$, so $\deg(r) \geq \deg(m_{\alpha,F})$ unless $r(x) = 0$ by minimality. Hence we must have $r(x) = 0$, so $m_{\alpha,F} | p$. This contradicts the assumption that p is monic and irreducible. Therefore, $m_{\alpha,F}$ is the **only** minimal, monic and irreducible polynomial where α is a root. \square

Definition 22. $m_{\alpha,F}(x)$ is the **minimal** polynomial of α over F .

(The following is kind of on a tangent)

Some exam prep!

- In general, for subrings $R \subseteq S$, we have if $r \in R^\times$, then $r \in S^\times$.
- If we adjoin one root of an irreducible polynomial to a field, the fields are isomorphic no matter which root of that polynomial we adjoin.

(Tangent ends here)

To summarize, if K/F is a field extension and $\alpha \in K$, then α is **algebraic** over F if it is the root of some polynomials in $F[x]$. For each algebraic α , there exists a unique, monic, irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ such that $m(\alpha) = 0$. In that case, the degree of extension $[F(\alpha) : F] = \deg(m_{\alpha,F}(x))$; and, if $p(\alpha) = 0$ for some $p(x) \in F[x]$, then $m_{\alpha,F} | p(x)$. In general, if $[K : F] < \infty$, then K/F is algebraic. Thus, $[F(\alpha) : F] < \infty$ if and only if α is algebraic over F .

Proposition 30. If $F \subseteq K \subseteq L$ are fields, then

$$[L : F] = [L : K] \cdot [K : F]$$

$$\leftarrow mn \begin{pmatrix} L \\ K \\ F \end{pmatrix} \begin{matrix} n \\ m \end{matrix}$$

Proof. We first see that if $[K : F] = \infty$, then for any $N \in \mathbb{N}$, there exists $\alpha_1, \dots, \alpha_N \in K$ that are linearly independent over F . In that case, it is certainly true that $\alpha_1, \dots, \alpha_N \in L$ are linearly independent over F . Thus, $[L : F] = \infty$.

If $[L : K] = \infty$, then for any $N \in \mathbb{N}$, there exists $\beta_1, \dots, \beta_N \in L$ that are linearly independent over K . As a result, it also is linearly independent over F . Hence, $[L : F] = \infty$.

If $[K : F] = m$ and $[L : K] = n$, let $\alpha_1, \dots, \alpha_m \in K$ be a basis for K over F and $\beta_1, \dots, \beta_n \in L$ be a basis for L over K .

Claim: $\{\alpha_i \beta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ forms a basis for L over F . □

Some nice consequences:

- ← Linear independence implies that whenever $a_1 \alpha_1 + a_2 \alpha_2 + \dots + a_N \alpha_N = 0$ for some coefficients $a_1, \dots, a_N \in F$, then necessarily $a_1 = a_2 = \dots = a_N = 0$.
- ← Use linear combinations to prove this claim.

Corollary 31. Suppose K/F is an extension and $\alpha, \beta \in K$ are algebraic over F . Then:

- $F(\alpha, \beta) = (F(\alpha)(\beta))$
- $[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\beta)][F(\beta) : F] = \deg(m_{\alpha, F(\beta)}(x)) \cdot \deg(m_{\beta, F}(x))$.
However, note that the minimal polynomial

$$m_{\alpha, F(\beta)}(x) \mid m_{\alpha, F}(x) \in F(\beta)[x]$$

so $\deg(m_{\alpha, F(\beta)}(x)) \leq \deg(m_{\alpha, F}(x))$. Hence,

$$[F(\alpha, \beta) : F] \leq \deg(m_{\alpha, F}(x)) \deg(m_{\beta, F}(x)) < \infty$$

This means that whenever α, β are algebraic over F , we get that $F(\alpha, \beta)/F$ is an algebraic extension.

- As a result, $\alpha \pm \beta, \alpha\beta, \alpha/\beta$ are all algebraic over F . The algebraic elements hence form a **field**.

← the smallest subfield of K containing F, α, β
← since $p(\alpha) = 0 \iff m_{\alpha, F}(x) \mid p(x)$

Proposition 32. Suppose K/F is an extension. Then $[K : F] < \infty$ if and only if $K = F(\alpha_1, \dots, \alpha_n)$ could be written where $\alpha_1, \dots, \alpha_n \in K$ are algebraic over F .

In other words, an extension is finite if and only if it is generated by adjoining a finite amount of algebraic elements.

Proof.

(\implies) If $[K : F] < \infty$, then suppose $\{\alpha_1, \dots, \alpha_n\}$ is a basis of K over F . Then $\alpha_1, \dots, \alpha_n$ are algebraic and every element of K is an F -linear combination of α_i s. Hence K must be the smallest field containing F and α_i s, which means $K = F(\alpha_1, \dots, \alpha_n)$.

(\impliedby) We observe that

$$\begin{aligned} [K : F] &= [(F(\alpha_1, \dots, \alpha_{n-1}))(\alpha_n) : F(\alpha_1, \dots, \alpha_{n-1})] \cdot \dots \cdot [F(\alpha_1) : F] \\ &\leq \prod_{i=1}^n \deg(m_{\alpha_i, F}(x)) < \infty \end{aligned}$$

□

Corollary 33. If L/K and K/F are algebraic extensions, then so is L/F .

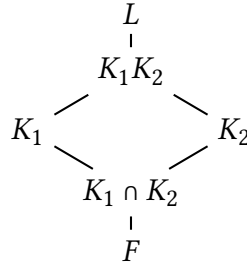
← L/K and K/F need not be finite!

Proof. Suppose $\alpha \in L$. Since L/K is algebraic, there exists $p(x) \in K[x]$ such that $p(\alpha) = 0$. Let $\alpha_0, \dots, \alpha_n \in K$ be the coefficients of $p(x)$, necessarily algebraic over F since K/F algebraic. Therefore,

$$[F(\alpha_0, \dots, \alpha_n, \alpha) : F] = [(F(\alpha_0, \dots, \alpha_n))(\alpha) : F(\alpha_0, \dots, \alpha_n)][F(\alpha_0, \dots, \alpha_n) : F]$$

Since $p(\alpha) = 0$ has coefficients in $K \supseteq F(\alpha_0, \dots, \alpha_n)$, we have $[(F(\alpha_0, \dots, \alpha_n))(\alpha) : F(\alpha_0, \dots, \alpha_n)] < \infty$. The second term is also clearly $< \infty$. Therefore, $[F(\alpha_0, \dots, \alpha_n, \alpha) : F] < \infty$, meaning that α is algebraic over F . \square

Definition 23. Suppose L/F is an extension & K_1 and K_2 are intermediate fields. The **composite** field K_1K_2 is the smallest subfield of L containing K_1 and K_2 .



Definition 24. Suppose F is a field and $p(x) \in F[x]$. The **splitting field** of $p(x)$ over F is the smallest field extension of F over which $p(x)$ could be factored into **linear factors**.

Remark. If E is the splitting field of $p(x)$ over F then $[E : F] \leq n!$ where $n = \deg(p(x))$.

Remark. Such an extension is called **normal**.

Proposition 34. Splitting fields exist.

Proof outline. By induction on $\deg(p(x))$, whose base case, $\deg(p(x)) = 1$, yields F as a splitting field. More generally, any $p(x)$ has a root α in $F(\alpha) \cong F[x]/(q(x))$ for some irreducible $q(x)$ so $p(x) = (x - \alpha)f(x) \in F(\alpha)[x]$. We observe that $\deg(f(x)) = \deg(p(x)) - 1$. Induction takes care of the rest. \square

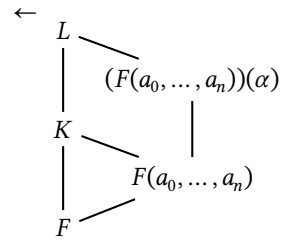
Remark. K is a splitting field over F if and only if every irreducible $p(x) \in F[x]$ that has one root in K has **all** its roots in K .

Non-example 26. $\mathbb{Q}(\sqrt[3]{2})$ over \mathbb{Q} is not such an extension.

Lemma 35. Suppose $\varphi : F_1 \rightarrow F_2$ is a field isomorphism, $p_1(x) \in F_1[x]$, and $p_2(x) = \varphi(p_1(x))$ (φ applied to coeffs of $p_1(x)$). Let α_1 be a root of the irreducible factor $q_1(x)$ of $p_1(x)$, and let $q_2(x) = \varphi(q_1(x))$ and α_2 be a root of $q_2(x)$. Then there exists an isomorphism $\tau : F_1(\alpha_1) \rightarrow F_2(\alpha_2)$ such that $\tau(\alpha_1) = \alpha_2$ and $\tau|_{F_1} = \varphi$ (this means “ τ restricted to F_1 ”).

Proof outline.

$$\begin{array}{ccccccc}
 F_1(\alpha_1) & \xrightarrow{\sim} & F_1[x]/(q_1(x)) & \xrightarrow{\sim} & F_2[x]/(q_2(x)) & \xrightarrow{\sim} & F_2(\alpha_2) \\
 \alpha_1 & \xrightarrow{\quad} & \bar{x} & \xrightarrow{\quad} & \bar{x} & \xrightarrow{\quad} & \alpha_2 \\
 \text{if } a \in F_1, & a & \xrightarrow{\quad} & \bar{a} & \xrightarrow{\quad} & \overline{\varphi(a)} & \xrightarrow{\quad} & \varphi(a)
 \end{array}$$



← Assuming that splitting fields **exist** and are **unique** up to isomorphism.

← The splitting field of $p(x)$ over F is the same as the splitting field of $f(x)$ over $F(\alpha)$

← In this way, φ induce a ring isomorphism $F_1[x] \rightarrow F_2[x]$.

Proposition 36. Suppose $F_1, F_2, \varphi, p_1(x)$ and $p_2(x)$ are as in Lemma 35. Let E_1, E_2 be splitting fields of p_1 and p_2 respectively. Then there exists an isomorphism $\sigma : E_1 \rightarrow E_2$ such that $\sigma|_{F_1} = \varphi$.

← if we set $F_1 = F_2$ and $p_1 = p_2$, we get corollary: splitting fields are unique up to isomorphism.

Proof. Proceed by induction on $\deg(p_1(x))$. For the base case, if $\deg(p_1(x)) = 1$, then $E_1 = F_1$ and $\sigma = \varphi$.

Assume the result is true for all polynomials of fixed degree $k \geq 1$ and suppose $\deg(p_1(x)) = k+1$. Let α_1 be a root of $p_1(x)$ and α_2 be a root of the φ -corresponding irreducible factor of $p_2(x)$. By Lemma 35, φ can be extended to $\tau : F_1(\alpha_1) \rightarrow F_2(\alpha_2)$ such that $\tau|_{F_1} = \varphi$.

In $(F_1(\alpha_1))[x]$, we can factor out $p_1(x) = (x - \alpha_1)g_1(x)$, and in $(F_2(\alpha_2))[x]$ we factor $p_2(x) = (x - \alpha_2)g_2(x)$ with $g_2(x) = \tau(g_1(x))$. We observe that E_1 and E_2 are the splitting fields of g_1 and g_2 over $F_1(\alpha_1)$ and $F_2(\alpha_2)$!

←

$$\begin{array}{ccc} \sigma : & E_1 & \xrightarrow{\sim} E_2 \\ | & \downarrow & \downarrow \\ \tau : & F_1(\alpha_1) & \xrightarrow{\sim} F_2(\alpha_2) \\ | & \downarrow & \downarrow \\ \varphi : & F_1 & \xrightarrow{\sim} F_2 \end{array}$$

By inductive hypothesis, τ could be extended to σ and $\sigma|_{F_1(\alpha_1)} = \tau$ and $\sigma|_{F_1} = \varphi$. \square

Corollary 37. Splitting fields are unique.

Proof. Set $F_1 = F_2, \varphi = \text{id}, p_1(x) = p_2(x)$. \square

(The following is kind of on a tangent)

Homework hint: the proof of existence & uniqueness of splitting fields relied on inductive arguments where we adjoin one root at a time. This is the same as saying $E = F(\alpha_1, \dots, \alpha_n)$ but this tends to overlook isomorphic ways to adjoin roots. In this context, it is convenient to start by considering a specific K containing F and all roots of $p(x)$. In that case, $E = F(\alpha_1, \dots, \alpha_n)$ becomes more rigorous.

(Tangent ends here)

Definition 25. A polynomial is called **separable** if it doesn't have repeated roots.

← First note that a poly of degree n over a field has exactly n roots.

Definition 26. Suppose $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$. The **formal derivative** of $f(x)$ is the polynomial

$$D_x f(x) = f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + a_1$$

From this definition, we can check that the usual differential rules hold.

Lemma 38. Suppose F is a field, $f(x)$ is a polynomial in $F[x]$, and E/F a field extension containing a root α of $f(x)$. Then α is a repeated root of $f(x)$ if and only if α is a root of the formal derivative $f'(x)$.

Proof. If α is a repeated root of $f(x)$ then $f(x) = (x - \alpha)^2 g(x)$ for some $g(x) \in E[x]$. In that case, $f'(x) = 2(x - \alpha)g(x) + (x - \alpha)^2 g'(x)$ and so $f'(\alpha) = 0$.

Conversely, if $f'(\alpha) = 0$, then differentiating $f(x) = (x - \alpha)h(x)$ (where $h(x) \in E[x]$) and plugging $x = \alpha$ yields $0 = f'(\alpha) = h(\alpha) + (\alpha - \alpha)h'(\alpha) = h(\alpha)$. This is saying that $h(x) = (x - \alpha)g(x)$ for some $g(x) \in E[x]$. \square

Lemma 39. If $f(x) \in F[x]$ is **irreducible and not separable**, then $f'(x) = 0$.

finish proof

\square

If $f(x)$ is not constant and $f'(x) = 0$, then $\text{char}(F) = p > 0$ and $f(x) = g(x^p)$.

Proposition 40. If $\text{char}(F) = 0$, or $|F| < \infty$ and $\text{char}(F) = p$, then every irreducible polynomial in $F[x]$ is separable.

← All powers in $f(x)$ are multiples of $p(x)$.

Proof. For the case of $\text{char}=0$, it follows from 39.

For the case of $\text{char}>0$, suppose F is a finite field of p^n elements². Then the map $F \rightarrow F$ where $\alpha \mapsto \alpha^p$ is a field isomorphism. Hence, every element of F is a p^{th} power.

← Use binomial theorem.

← Such fields are called **perfect**.

Now suppose BWOC $f(x) = \sum_{i=0}^n a_i x^i \in F[x]$ is an irreducible but not separable polynomial. Therefore, $f'(x) = 0$ must be true. This happens *if and only if* $f(x) = \sum_{j=0}^m a_{jp} x^{jp}$, that is, the x in all terms are of p^{th} degree. However, we know that all elements $a_{jp} \in F$ are already the p^{th} powers of sth else $(b_{jp})^p = a_{jp}$, so

$$f(x) = \sum_{j=0}^m (b_{jp}^p) x^{jp}$$

and by reverse Binomial Theorem, we get

$$f(x) = \sum_{j=0}^m (b_{jp}^p) x^{jp} = \left(\sum_{j=0}^m b_{jp} x^j \right)^p$$

is not irreducible! \square

²See Section 27

Non-example 27. Let $F = \mathbb{F}_p(t) = \left\{ \frac{f(t)}{p(t)} \mid f(t), g(t) \in \mathbb{F}_p[t], g(t) \neq 0 \right\}$.

← This is a field of char>0 but is infinite.

Then $p(x) = x^p - t$ is not separable (but it is irreducible). This can be seen if we suppose α is a root of $p(x)$ (so $\alpha^p = t$). Then, in $F(\alpha)[x]$, we have $p(x) = x^p - t = x^p - \alpha^p = (x - \alpha)^p$, which tells us $p(x)$ is not separable.

← The coefficients of $p(x)$ are ratios of polys in $\mathbb{F}_p(t)$.

(The following is kind of on a tangent)

Prime fields

Suppose R is a commutative ring with identity. The map $\mathbb{Z} \rightarrow R$ where $n \mapsto \underbrace{\pm(1_R + 1_R + \cdots + 1_R)}_{|n| \text{ times}}$ ($-$ if $n < 0$) is a ring homomorphism with kernel $n\mathbb{Z}$ where $n = \text{char}(R)$. So:

← check it!

- if $\text{char}(R) = 0$, then R contains \mathbb{Z} ;
- if $\text{char}(R) = n > 0$, then R contains $\mathbb{Z}/n\mathbb{Z}$.

If F is a field, then:

- if $\text{char}(F) = 0$, then F contains \mathbb{Q} ;
- if $\text{char}(F) = p > 0$, then p prime and F contains $\mathbb{Z}/p\mathbb{Z}$.

← That is, F is an extension of $\mathbb{Z}/p\mathbb{Z}$!

In other words, every field is an extension of \mathbb{Q} or \mathbb{F}_p . Moreover, a finite field is a finite extension of \mathbb{F}_p : if $[F : \mathbb{F}_p] = n$, then $|F| = p^n$.

In addition, $|F - \{0\}| = p^n - 1 \implies$ if $\alpha \in F - \{0\}$ then $\alpha^{p^n-1} = 1$, implying that if $\alpha \in F$, then $\alpha^{p^n} = \alpha$, meaning that α is a root of $x^{p^n} - x \in F[x]$. Therefore, F is the splitting field of $x^{p^n} - x$. But splitting fields are unique, so we conclude that there is only one unique finite field for each order.

← By Lagrange's Theorem

(Tangent ends here)

Definition 27. An algebraic extension K/F is called (algebraically) **separable** if $m_{\alpha,F}(x)$ is separable for **all** $\alpha \in K$.

Galois Theory

Definition 28. A finite extension K/F is called **Galois** if K/F is normal and separable.

← normal just means it is a splitting field of something

Definition 29. If K/F is an extension, then the **automorphism group** of K/F is defined as

$$\text{Aut}(K/F) = \{\sigma \in \text{Aut}(K) \mid \sigma(a) = a \forall a \in F\}$$

That is, all the automorphisms of K that also fix the field F .

Galois theory is concerned with the study of roots of polynomials by way of automorphisms of splitting fields (of separable polynomials). In particular, we are interested in what

$$\text{Aut}(K) = \{\sigma : K \rightarrow K \text{ isomorphisms}\}$$

(a group under composition) is. Naturally, such groups are *finite*.

← review MATH171 finite groups!

Last time, we showed that $K \supseteq \begin{cases} \mathbb{Q} & \text{if char}(K) = 0 \\ \mathbb{F}_p & \text{if char}(K) = p \end{cases}$, since $\sigma(1) = \sigma(1^2) = (\sigma(1))^2$ implies that $\sigma(1) = 1$ must always be true! Hence, $\sigma(n) = n$ must be true in char 0 fields, or $\sigma(\bar{n}) = \bar{n}$ if char > 0 for all $n \in \mathbb{Z}$. Therefore, σ **fixes** the prime subfield \mathbb{Q} or \mathbb{F}_p .

Remark. Why does Definition 29 have to fix the field F ? Because we've shown that $\text{Aut}(K) = \begin{cases} \text{Aut}(K/\mathbb{Q}) & \text{if char}(K) = 0 \\ \text{Aut}(K/\mathbb{F}_p) & \text{if char}(K) = p \end{cases}$.

Lemma 41. If K/F is an extension, $\alpha \in K$ is algebraic over F and $\sigma \in \text{Aut}(K/F)$, then $\sigma(\alpha)$ is a root of $m_{\alpha,F}(x)$.

Proof. Observe that $m_{\alpha,F}(\alpha) = 0 = \sigma(0) = \sigma(m_{\alpha,F}(\alpha))$. Hence, since $\sigma(\alpha) = \alpha$ for all $\alpha \in F$,

$$\sigma(a_0 + a_1\alpha + \cdots + a_n\alpha^n) = a_0 + a_1\sigma(\alpha) + \cdots + a_n\sigma(\alpha)^n = m_{\alpha,F}(\sigma(\alpha))$$

□

So if $f(x) \in F[x]$, then every $\sigma \in \text{Aut}(K/F)$ **permutes** the roots of $f(x)$ that lie in K . It would be nice if the roots of $f(x)$ all lived in K . This is why we consider K the splitting field of some polynomial over F .

← Recall: if irreducible $f(x)$ has one root in such K , then all roots lie in K .

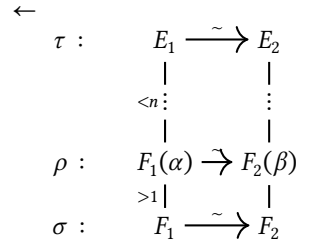
Proposition 42. If K is the splitting field of some polynomial $f(x)$ over F (so $[K : F] < \infty$), then $|\text{Aut}(K/F)| \leq [K : F]$, with equality if $f(x)$ is separable.

Proof. We will prove a more general statement by induction. If $\sigma : F_1 \rightarrow F_2$ is an isomorphism, $f_1(x) \in F_1[x]$ and $f_2(x) = \sigma(f_1(x)) \in F_2[x]$ and E_1 and E_2 are the splitting fields of f_1 and f_2 over F_1 and F_2 respectively. Then we would like to show that there are at most $[E_1 : F_1]$ isomorphisms $\tau : E_1 \rightarrow E_2$ such that $\tau|_{F_1} = \sigma$ with equality if f_1 separable.

← The prop above fixes $F_1 = F_2$ and σ being identity.

Base case. If $[E_1 : F_1] = [E_2 : F_2] = 1$, then $E_1 = F_1$, $E_2 = F_2$ and $\tau = \sigma$ is our only choice.

Inductive step. Suppose we've proven the result for all extensions of degree $< n$ for some $n \geq 2$. Now consider $[E_1 : F_1] = [E_2 : F_2] = n$. Pick $\alpha \in E_1 \setminus F_1$ and let $\beta \in E_2$ be any root of $\sigma(m_{\alpha, F_1}(x))$. Then σ could be extended to $\rho : F_1(\alpha) \rightarrow F_2(\beta)$ such that $\rho(\alpha) = \beta$ and $\rho|_{F_1} = \sigma$. Observe that $[F_1(\alpha) : F_1] = \deg(m_{\alpha, F_1}(x))$. Moreover, the number of extensions of σ to ρ equals the number of *distinct* roots of $\sigma(m_{\alpha, F_1}(x))$. Thus, the number of extensions of σ to $F_1(\alpha)$ is *at most* the degree of $m_{\alpha, F_1}(x)$ which is $[F_1(\alpha) : F_1]$ with equality if $m_{\alpha, F_1}(x)$ is separable. Since $[E_1 : F_1] = [E_2 : F_2] = n$, we have $[E_1 : F_1(\alpha)] < n$, by inductive hypothesis, there are at most $[E_1 : F_1(\alpha)]$ ways of extending ρ to $\tau : E_1 \rightarrow E_2$. Hence,



$$\begin{aligned}
 |\{\text{extensions of } \sigma \text{ to } \tau\}| &= |\{\text{extensions of } \sigma \text{ to } \rho\}| |\{\text{extensions of } \rho \text{ to } \tau\}| \\
 &\leq [F_1(\alpha) : F_1] [E_1 : F_1(\alpha)] \\
 &= [E_1 : F_1]
 \end{aligned}$$

Looking at the case $F_1 = F_2$, $E_1 = E_2$, $\sigma = \text{id}$, we get our result. \square

Definition 30. If K/F is a normal extension, then the extension is **Galois** if $[K : F] = |\text{Aut}(K/F)|$.

Remark. Notation: if K/F is Galois, then use $\text{Gal}(K/F)$ for $\text{Aut}(K/F)$.

Fixed Fields and Automorphism Groups

Definition 31. Suppose K/F is a field extension. If subgroup $H \leq \text{Aut}(K/F)$, then the **fixed field** of H is given by $K_H = \{\alpha \in K \mid \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}$.

Remark. Also, observe that if $F \subseteq E \subseteq K$, then $\text{Aut}(K/E) \leq \text{Aut}(K/F)$.

Lemma 43. Suppose K/F is an extension. Then:

- (1) If $H_1, H_2 \leq \text{Aut}(K/F)$ with $H_1 \leq H_2$, then $K_{H_2} \subseteq K_{H_1}$.

← Observe that K_H is indeed a field (the sum, products etc. are also fixed by σ); moreover, it is an intermediate extension $F \subseteq K_H \subseteq K$.

(2) If $F \subseteq E_1 \subseteq E_2 \subseteq K$ are two intermediate extensions, then $\text{Aut}(K/E_2) \leq \text{Aut}(K/E_1) \leq \text{Aut}(K/F)$.

Example 28. \mathbb{Q} is an intermediate extensions of $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. Then $\text{Aut}(\mathbb{Q}/\mathbb{Q}) = \{1\}$. We further observe that since automorphisms permute roots, $\text{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{1\}$. Hence $\text{Aut}(\mathbb{Q}/\mathbb{Q}) = \text{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ and so the fixed field by $\mathbb{Q}(\sqrt[3]{2})$ is given by $\mathbb{Q}(\sqrt[3]{2})_{\text{Aut}(\mathbb{Q}/\mathbb{Q})} = \mathbb{Q}(\sqrt[3]{2})$. We note that $\mathbb{Q}(\sqrt[3]{2})$ is not Galois!

Theorem 44 (The Fundamental Theorem of Galois Theory). If K/F is a (finite) Galois extension, then the maps $H \mapsto K_H$ and $E \mapsto \text{Aut}(K/E)$ gives an *inclusion-reversing bijection* between the subgroups of $\text{Aut}(K/F)$ and the intermediate extensions of K/F .

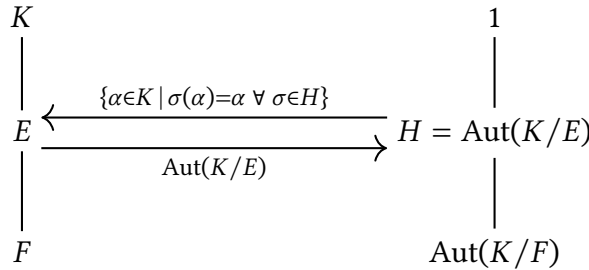
Furthermore, $[K : E] = |\text{Aut}(K/E)|$, and

$$[E : F] = |\text{Aut}(K/F) : \text{Aut}(K/E)| = |\text{Aut}(K/F)| / |\text{Aut}(K/E)|$$

Moreover, E/F is Galois *if and only if* $\text{Aut}(K/E)$ is a **normal subgroup** of $\text{Aut}(K/F)$, in which case

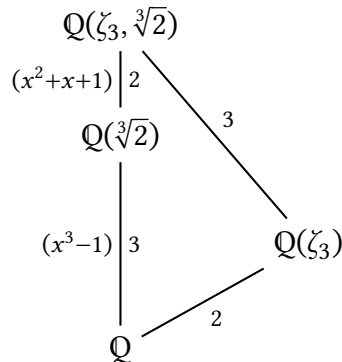
$$\text{Aut}(E/F) = \text{Aut}(K/F) / \text{Aut}(K/E)$$

In other words,



where H is a subgroup of $\text{Aut}(K/F)$ that fixes the field E , an extension of F contained in K . If $H \trianglelefteq \text{Aut}(K/F)$, then E/F is normal.

Example 29. Consider $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$, the splitting field extension of $x^3 - 2$.³



³Suppose ζ is a primitive n th root of unity; then so is ζ^k *if and only if* the $\gcd(k, n) = 1$, i.e. k, n are relatively prime.

← Roots of $x^3 - 2$ are $\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}$, so two roots are not in $\mathbb{Q}(\sqrt[3]{2})$, and so $\sqrt[3]{2}$ could be only mapped to itself.

← We see if K/F is Galois, then K/E is also Galois as if K is the splitting field of some poly in F , then it's certainly true for E .

← $|\zeta_3| = 3$, a primitive 3rd root of unity.

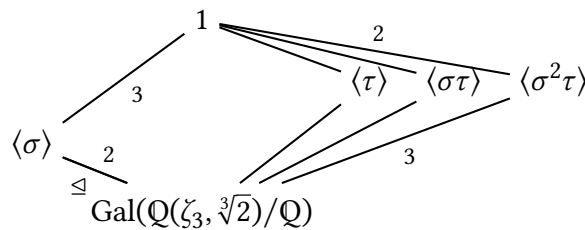
Let $\sigma, \tau \in \text{Aut}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ be given by:

$$\sigma : \begin{cases} \sqrt[3]{2} \mapsto \zeta_3 \sqrt[3]{2} \\ \zeta_3 \mapsto \zeta_3 \end{cases} \quad \tau : \begin{cases} \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \zeta_3 \mapsto \zeta_3^2 \end{cases}$$

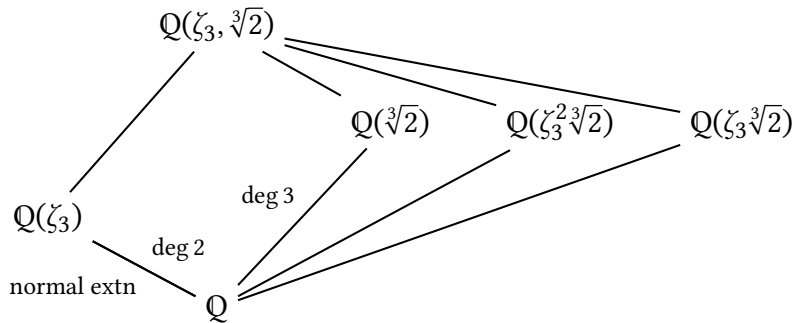
So $\langle \sigma, \tau \mid \sigma^3 = \tau^2 = \text{id}, \sigma\tau = \tau\sigma^2 \rangle \cong S_3$.

Since this is a subgroup of $\text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ that has the same finite order of 6, this must just be $\text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ itself; hence, $\text{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$.

Now we look at the subgroups of S_3 (in reverse):



And then think about the **fixed field** of each subgroup correspondingly:



Note that the normal extension corresponds to the normal subgroup!

Proposition 45 (Primitive Element Theorem). If K/F is a **finite Galois** extension, then $K = F(\alpha)$ for some $\alpha \in K$.

Definition 32. $K = F(\alpha)$ is a **simple** extension of F and α is a primitive element.

Proof. We first assume $|F| = \infty$.

Recall that K/F is finite *if and only if* $K = F(\alpha_1, \alpha_2, \dots, \alpha_n)$ where α_i is algebraic over F . We will proceed by induction on n , whose base case $n = 1$ gives a simple extension $F(\alpha_1)/F$.

← Isomorphisms preserve order, so they **must** take an n th root of unity to another n th root of unity!

← Remember normal extn means 'is splitting field', i.e. 'one root \rightarrow all roots'

← $\mathbb{Q}(\sqrt[3]{2})$ is not Galois!

Recursive case: assume that for some $k \geq 1$ we have $F(\alpha_1, \dots, \alpha_k)$ being a simple extension $F(\alpha)$. Let K/F be Galois and $K = F(\alpha_1, \dots, \alpha_{k-1}, \alpha, \beta)$.

Let $E = F(\alpha_1, \dots, \alpha_{k-1})$ and consider the intermediate family of extensions $\{E(\alpha + t\beta) \mid t \in F\}$. Since $|\text{Gal}(K/F)| < \infty$ as we are talking about finite Galois extensions, there are finitely many distinct such extensions, so $E(\alpha + t_1\beta) = E(\alpha + t_2\beta)$ for some $t_1 \neq t_2$.

← such that $K = E(\alpha, \beta)$

Now we see that $\alpha + t_1\beta$ and $\alpha + t_2\beta$ must be in the same field $E(\alpha + t_1\beta)$. Hence, $(\alpha + t_1\beta) - (\alpha + t_2\beta)$ are in the field, so $(t_1 - t_2)^{-1}((\alpha + t_1\beta) - (\alpha + t_2\beta)) = \beta$ is also in the field. Similarly, $\alpha \in E(\alpha + t_1\beta)$. Therefore, $K = E(\alpha, \beta) = E(\alpha + t_1\beta) = F(\alpha_1, \dots, \alpha_{k-1}, \alpha + t_1\beta)$, which has k elements adjoined and is therefore simple. \square

Remark. Above is true for char 0 fields even without the ‘Galois’ hypothesis.

Proof outline. Since K is a finite extension of F with $[K : F] < \infty$, then there must be $K = F(\alpha_1, \dots, \alpha_n)$ for some algebraic $\alpha_1, \dots, \alpha_n \in K$.

← Idea: if anything is not Galois, we add enough things to it to make it Galois!

Let E be the splitting field of $\prod_{i=1}^n m_{\alpha_i, F}(x)$. We call E the **Galois closure** of K over F . Now, we have $E \supset K \supset F$ and E/F is Galois. Thus, there are finitely many intermediate fields between K and F . We can then use a similar proof for Proposition 45. \square

Cyclotomic fields

Definition 33. Suppose K is a field. An element $z \in K$ is called an n -th root of unity if $z^n = 1$; and z is a **primitive** n -th root of unity if $z^k \neq 1$ for any $1 \leq k \leq n-1$.

Remark. z is a **primitive** n -th root of unity *if and only if* $|z| = n$ in $K^\times = K \setminus \{0\}$.

Remark. z is an n -th root of unity *if and only if* z is a root of $x^n - 1$.

Lemma 46. If K is a field containing one primitive n -th root of unity, then K contains exactly n roots of unity, exactly $\varphi(n)$ of which are primitive.

← $\varphi(n)$ is Euler’s totient function, the count of integers $< n$ that are relatively prime to n .

Remark. Recall:

- $z^n = 1$ *if and only if* $|z|$ divides n .
- $|z^m| = \frac{|z|}{\gcd(m, |z|)}$.
- $|z^m| = |z|$ *if and only if* $(m, |z|) = 1$.

Example 30. If $|z| = 10$, then $|z^6| = 5 = \frac{10}{(6, 10)}$

Proof. If $z \in K$ is a primitive n -th root of unity, then every element of $\langle z \rangle$ is an n -th root of unity, and so a root of $x^n - 1$; but $|\langle z \rangle| = |z| = n$, so the subgroup $\langle z \rangle$ generated by z must consist of all of the n roots of $x^n - 1$. Furthermore, z^m is also a primitive n -th root of unity *if and only if* $(m, n) = 1$; thus, there are exactly $\varphi(n)$ such elements in $\langle z \rangle$. \square

In \mathbb{C} , we have $e^{i\frac{2\pi}{n}}$ is a primitive n -th root of unity. Suppose $\zeta_n \in \mathbb{C}$ is a primitive n -th root of unity.

Definition 34. The **cyclotomic polynomial** is given by

$$\Phi_n(x) = \prod_{\substack{0 \leq k < n \\ (n,k)=1}} (x - \zeta_n^k)$$

Properties:

- $x^n - 1 = \prod_{0 \leq k < n} (x - \zeta_n^k) = \prod_{d|n} \prod_{\substack{0 \leq k < n \\ (n,k)=d}} (x - \zeta_n^k) = \prod_{d|n} \Phi_d(x)$
- $\deg(\Phi_n(x)) = \varphi(n)$
- $n = \sum_{d|n} \varphi(d)$

Example 31. $6 = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(6)$

(The following is kind of on a tangent)

Remark. If K is a finite field, then $K^\times = \langle z \rangle$.

Proof. Since K is finite, it must be an extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime p ; so $|K^\times| = p^n - 1$ since $|K| = p^n$. For $d|p^n - 1$, let $\mathcal{O}_d = \{z \in K^\times \mid |z| = d\}$. Observe that $K^\times = \bigcup_{d|p^n-1} \mathcal{O}_d$ is a disjoint union, and so

$$|K^\times| = \sum_{d|p^n-1} |\mathcal{O}_d| = p^n - 1 = \sum_{d|p^n-1} \varphi(d)$$

which implies that $|\mathcal{O}_d| = \varphi(d)$ for all $d|p^n - 1$. Hence, in particular, \mathcal{O}_{p^n-1} is nonempty, so any $z \in \mathcal{O}_{p^n-1}$ generates K^\times . \square

Remark. This implies that the Primitive Element Theorem (see Proposition 45) is also true for finite fields.

(Tangent ends here)

Remark. $\Phi_n(x)$ is irreducible over \mathbb{Q} , and so it is the minimal polynomial $m_{\zeta_n, \mathbb{Q}}(x)$.

Lemma 47. Suppose F is a field where all irreducible polynomials are separable.

Let $p(x) \in F[x]$ irreducible and split completely in a Galois extension K/F , and let α, β be two roots of $p(x)$. Then there exists $\sigma \in \text{Gal}(K/F)$ such that $\sigma(\alpha) = \beta$.

Definition 35. The extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is called the n -th **cyclotomic extension** of \mathbb{Q} .

Remark. All other primitive n -th roots of unity are of the form ζ_n^a w/ $(a, n) = 1$, so $\mathbb{Q}(\zeta_n)$ is the splitting field of $\Phi_n(x)$ over \mathbb{Q} , so the extension is Galois.

If $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, then σ is completely determined by $\sigma(\zeta_n)$. But $\sigma(\zeta_n)$ must be another primitive n -th root of unity, so $\sigma(\zeta_n)$ must be ζ_n^a for some $0 < a < n$ with $(a, n) = 1$.

Moreover, by Lemma 47, each a corresponds to a Galois automorphism σ ; in fact, the map $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times$ where $\sigma \mapsto \bar{a}$ is a **group isomorphism**.

Definition 36. A Galois extension K/F is abelian if $\text{Gal}(K/F)$ is abelian.

Corollary 48. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is abelian.

Theorem 49 (Kronecker-Wober). If K/\mathbb{Q} is abelian, then $K \subseteq \mathbb{Q}(\zeta_n)$ for some n .

$$\begin{array}{ccc} \sigma : K & \rightarrow & K \\ \downarrow & & \downarrow \\ \leftarrow F(\alpha) & \xrightarrow{\sigma} & F(\beta) \\ \downarrow & & \downarrow \\ \text{id} : F & \rightarrow & F \end{array}$$

← All finite field extensions are abelian

Radical extensions and soluble groups

Example 32. Suppose K is the splitting field of $x^4 - 2$ over \mathbb{Q} . Then $K = \mathbb{Q}(\sqrt[4]{2}, i)$.

Let $\sigma : \begin{cases} \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ i \mapsto i \end{cases}$ and $\tau : \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto -i \end{cases}$.

We look at the subgroup generated by σ, τ :

$$\langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle \cong D_8$$

Since $[K/\mathbb{Q}] = 8$, we have found the Galois group of K/\mathbb{Q} .

← since $\zeta_4 = i$

