MATH172 Galois Theory Notes

Xuehuai He

November 12, 2023

Contents

Rings! Or why $x^2 - 2$ has roots.

Definition 1. A **ring** is a set R together with associative binary *operations* $+$ and $\times$ s.t.:

← map from $R \times R \mapsto R$

- $(R, +)$ is an **abelian** group with identity 0
- There exists $1 \in R$ s.t. $r \times 1 = 1 \times r = r$ ← this is optional
- $r(s+t) = rs + rt$ and $(s+t)r = sr + tr \qquad \forall s, r, t \in R$

Proposition 1. $0 \times 1 = 0$ (in fact, $0 \times r = 0 \; \forall \; r \in R$)

Proof. Try it! □

Definition 2. If $\times$ is commutative, then R is a commutative ring.

Non-example 1. $\mathbb{N}$ is not a ring.

Example 2. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all rings;

- $\mathbb{Z}/n\mathbb{Z}$ is a finite ring
- $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, is a **noncommutative** ring
- Polynomial ring: $\mathbb{Q}[x] = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Q}\}$ is a commutative ring ← square brackets just mean "polynomials in..."
- $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ is a commutative ring

---

Phase I plan:

$$ID \supsetneq UFD \supsetneq PID \supsetneq ED \supsetneq Fields$$

Definition 3. Suppose R is a ring and $a, b \in R$ with $ab = 0$ but $a, b \neq 0$; then a, b are called **zero divisors**.

Example 3.

- In $\mathbb{Z}/6\mathbb{Z}$, $\bar{4} \times \bar{3} = \bar{0}$
- In $M_2(\mathbb{R})$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Definition 4. A commutative ring without zero divisors is called an integral domain (ID)

Why do we want ID? **Cancellation properties**.

- If R is an ID, $a, b, c \in R$, $a \neq 0$ and $ab = ac$, then

$$ab - ac = 0 \implies a(b - c) = 0 \implies b - c = 0 \implies b = c$$

Definition 5. Suppose R is an ID. An element $a \in R$ is called a **unit** if $a \neq 0$ and there exists $b \in R$ s.t. $ab = 1$.

← notation: $b = a^{-1}$

An element $r \in R$ is called **irreducible** if $r \neq 0$, r is NOT a unit, and whenever $r = ab$ for some $a, b \in R$ then a or b must be a unit.

- If r and s are irreducibles with $r = us$, then r and s are called **associates**.

Example 4.

- All "prime integers" are irreducibles in $\mathbb{Z}$;
- 2,3, $1 + \sqrt{-5}$, $1 - \sqrt{-5}$ are irreducibles in $\mathbb{Z}[\sqrt{-5}]$.
  - Note: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6$ says that 6 can be factored in more than one way. This means that $\mathbb{Z}[\sqrt{-5}]$ is NOT an UFD.

Definition 6. An integral domain R is called a unique factorization domain (UFD) if each nonzero, nonunit $a \in R$ can be written as a product of irreducibles **in a unique way** up to associates.

If a is a nonzero, nonunit element of UFD R and $a = r_1 r_2 \dots r_m = s_1 \dots s_n$ where r_i, s_j are irreducible, then after reordering $r_i = u_i s_i$ for any i and units u_i, and $m = n$.

← After reordering, there are the same amounts of factors and all factors are the same up to units.

Definition 7. Suppose R is a comm ring. A subset $I \subseteq R$ is called an **ideal** if $(I, +) \leq (R, +)$ and $ir, ri \in I$ for all $i \in I$ and for all $\boxed{r \in R}$.

Why do we want ideals? Such that R/I is a well-defined ring.

Example 5. $\{0\}$ and R are ideals of R.

Example 6. If R is commutative and $a \in R$, then $(a) = \{ar \mid r \in R\}$ is called the **principal ideal** generated by a.

← Prove this (be convinced)! Also known as aR.

Definition 8. A **principal ideal domain** is an integral domain where all ideals are principal ideals.

Example 7. The only ideals of $(\mathbb{Z}, +)$ are of the form $n\mathbb{Z} = (n)$.

← Ideals generated by n

Non-example 8. $\mathbb{Z}[x]$ is a UFD but NOT a PID because the ideal $(2, x) = \{2r + xs \mid r, s \in \mathbb{Z}[x]\}$ is not principal.

← Observe that $(2, x)$ is an ideal made of polynomials with even constant terms. This cannot be principal, since if we only have 2 and not x, we do not have nonzero polynomials with zero const terms.

Lemma 2. If $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.

Proof. Try it!

Proposition 3. If $I \subseteq R$ is an ideal containing a unit of R then $I = R$.

Proof. If $u \in I$ is a unit then $u^{-1} \in R$, so $uu^{-1} = 1 \in I$. Then the result follows from Lemma 2. □

Definition 9. A **field** is a commutative ring whose each nonzero element is a *unit*.

Corollary 4. If R is an ID whose ideals are (0) and R, then R is a **field**.

← The converse is also true. **The only ideals in a field are 0 and the field.**

Proof. Suppose $a \in R \backslash \{0\}$ and consider (a). Since $a \in (a)$, $(a) = R$. Hence, we must have that $1 \in (a)$, which means $1 = ar$ for some $r \in R$. □

Definition 10. Suppose R is an integral domain. A *proper* ideal $P \subsetneq R$ is called **prime** of whenever $ab \in P$ for some $a, b \in R$, then a or $b \in P$.

Non-example 9. (6) is not a prime ideal of $\mathbb{Z}$ since $2 \times 3 \in (6)$ but neither $2, 3 \notin (6)$.

Non-example 10. (2) is not a prime ideal of $\mathbb{Z}[\sqrt{-5}]$ since $6 \in (2)$, but we observe that $6 = (1+\sqrt{-5})(1-\sqrt{-5})$ while $1 \pm \sqrt{-5} \notin (2)$.

← Observe that in $\mathbb{Z}[\sqrt{-5}]$, we have $6 = (1+\sqrt{-5})(1-\sqrt{-5}) = 2 \times 3$, so it is not a UFD!

Example 11. (2) is a prime ideal of $\mathbb{Z}$.

Definition 11. A proper ideal $M \subsetneq R$ is called **maximal** if whenever $I \subseteq R$ such that $M \subseteq I \subseteq R$ is an ideal containing M, then either $I = M$ or $I = R$.

Proposition 5. Every proper ideal is contained in **a** maximal ideal.

← This might not be unique in non-local rings.

Proof. Axiom of choice. □

Proposition 6. Suppose R is a commutative ring.

- (0) is prime *if and only if* R is an integral domain.
- (0) is maximal *if and only if* R is a field.

← By def of prime, if $ab = 0$, then either $a = 0$ or $b = 0$, which means there are NO zero divisors.

(The following is kind of on a tangent)

Definition 12. A commutative ring R with unity is called **Noetherian** if, whenever $I_1 \subseteq I_2 \subseteq \ldots$ is an ascending sequence of (proper) ideals of R, there exists an $n > 0$ such that $I_n = I_{n+1} = \ldots$ are the same ideals thereafter.

← The chain stops ascending!

Theorem 7. R is Noetherian *if and only if* all ideals of R are finitely generated.

Corollary 8. All Principal Ideal Domains are Noetherian.

← Since all ideals are generated by 1 elt.

(Tangent ends here)

Definition 13. Suppose R is a commutative ring with $1 \neq 0$ and $I \subseteq R$ is an ideal. Then the **quotient ring** of R by I is the set

$$R/I = \{r + I \mid r \in R\}$$

with addition and multiplication defined representative-wise.

Remark. The **coset criterion** of ideals: let I be an ideal; the cosets $r + I$, $s + I$ are the same *if and only if* $r - s \in I$.

Example 12.

- In $\mathbb{Z}/(6)$ aka. $\mathbb{Z}/6\mathbb{Z}$, we have $2 + (6) = \{\dots, -10, -4, 2, 8, 14, \dots\} = 26 + (6)$ due to $2 - 26 \in (6)$;
- In $\mathbb{Q}[x]/(x^2 - 2)$, we have

  $$\{3x^2 - 47x + 1 + q(x)(x^2 - 2) \mid q(x) \in \mathbb{Q}[x]\} = \{-47x + 7 + q(x)(x^2 - 2) \mid q(x) \in \mathbb{Q}[x]\}$$

  due to $3x^2 - 47x + 1 - (-47x + 7) \in (x^2 - 2)$.

Remark. Let I be an ideal of R. Then $(I, +) \trianglelefteq (R, +)$.

Definition 14. R/I is a group under $(r + I) + (s + I) = (r + s) + I$ and the operation $+$ is well-defined. We also define that $(r + I)(s + I) = (rs) + I$. We claim that multiplication in R/I is also well-defined.

Proof. Let $r_1 + I = r_2 + I$ and $s_1 + I = s_2 + I$. By coset criterion, $r_1 - r_2 = i$, $s_1 - s_2 = j$ for some $i, j \in I$. Hence $r_1 s_1 = (r_2 + i)(s_2 + j) = r_2 s_2 + i s_2 + j r_2 + ij$ where the latter three terms are all in the ideal I. Thus, $(r_1 s_1) + I = (r_2 s_2) + I$. □

From R, R/I inherits nice properties:

- $0 + I = 0_{R/I}$
- $1 + I = 1_{R/I}$
- Multiplication is commutative and distributive over addition in R/I, so it is also a comm. ring with identity.

Definition 15. A function $\varphi : R \to S$ between rings is called a **ring homomorphism** if the following are satisfied:

- $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$
- $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$

Theorem 9. First ring isomorphism theorem

If $\varphi : R \mapsto S$ is a ring homomorphism, then $R/\ker(\varphi) \cong \varphi(R)$.

← Observe that kernels are ideals! And ideals are kernels of some homomorphism too.

Example 13. If R is a ring and I is an ideal, then $\pi : R \to R/I$ where $r \mapsto r + I$ is a surjective homomorphism where $\ker(\pi) = I$. This is the *canonical projection* onto R/I.

Corollary 10. If I is a maximal ideal, then R/I is a field.

← The *if and only if* version comes in Proposition 14.

Recall Proposition 6. We now have a stronger statement:

Proposition 11. Suppose R is a commutative ring & $P \subsetneq R$ is an ideal. Then R/P is an integral domain *if and only if* P is prime.

Proof. R/P is an integral domain *if and only if* whenever $(a+P)(b+P) = 0_{R/P}$ then one of $a + P$ or $b + P$ must already be $0_{R/P}$. This happens *if and only if* whenever $ab + P = P$ then $a + P$ or $b + P$ in P, which happens *if and only if* whenever $ab \in P$ then one of $a, b \in P$, which is the definition of a prime ideal. □

Example 14. The map $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ where $p(x) \mapsto p(0)$ is a surjective ring homomorphism with $\ker(\varphi) = (x)$. By the the First Isomorphism Theorem 9, $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. As such, we conclude that (x) is a prime ideal since $\mathbb{Z}$ is an integral domain.

← btw, $(x) \subsetneq (x, 2)$. the latter is the set of polynomials whose constant term is even, so it is also a proper ideal of $\mathbb{Z}[x]$. This is an excellent example where Prime $\not\Rightarrow$ Maximal.

Lemma 12. Suppose R is a comm. ring with $M \subsetneq R$ being an ideal.
There is a bijective correspondence between the ideals of R/M and the ideals of R containing M.

Proof. Consider the projection $\pi : R \to R/M$ where $r \mapsto r + M$. It is enough to show:

← To see why this is okay, see Homework 2 Sec. 7.3 P. 24

$$\pi(\pi^{-1}(J)) = J \qquad \text{for all ideals } J \subseteq R/M, \text{ and}$$
$$\pi^{-1}(\pi(I)) = I \qquad \text{for all ideals } M \subseteq I \subseteq R$$

To prove the first statement, observe that, if J is an ideal of R/M, then $\pi^{-1}(J) = \{r \in R \mid r + M \in J\}$ and so

$$\pi(\pi^{-1}(J)) = \{\pi(r) \in R \mid r + M \in J\} = \{r + M \mid r + M \in J\} = J$$

Next, to prove the second statement, first suppose $M \subseteq I \subseteq R$ is an ideal. Let $a \in I$. Then $a + M \in \{\alpha + M \mid \alpha \in I\} = \pi(I)$. This implies that $a \in \pi^{-1}(\pi(I))$, and so $I \subseteq \pi^{-1}(\pi(I))$.

Conversely, suppose $r \in \pi^{-1}(\pi(I))$. This is the same as saying $\pi(r) = r + M \in \pi(I) = \{\alpha + M \mid \alpha \in I\}$. Hence, for any $r \in \pi^{-1}(\pi(I))$, there exists some $a \in I$ such that $r + M = a + M$. Thus, $r - a \in M \subseteq I$ by coset conditions. Since $a \in I$, we have $a + (r - a) \in I$, meaning that $r \in I$ for any $r \in \pi^{-1}(\pi(I))$. This means that $\pi^{-1}(\pi(I)) \subseteq I$.

Hence, $I = \pi^{-1}(\pi(I))$.

Consequently, for any ideals $J \subseteq R/M$, we know that $\pi^{-1}(J) \subseteq R$ is an ideal containing M. And if $M \subseteq I \subseteq R$ is an ideal, we know $\pi(I) \subseteq R/M$ is an ideal. Since $\pi(\pi^{-1}(J)) = J$ and $I = \pi^{-1}(\pi(I))$ for any I, J, the correspondence is a bijection. □

← Think about why this contains M!

Proposition 13. Suppose R is a comm. ring with an identity and $I \subseteq R$ is an ideal. Then R/I is a field *if and only if* I is maximal.

Proof. If I is maximal, then there are no other proper ideals strictly containing I. Hence, by Lemma 14, we have that R/I only have ideals (0) and R/I itself. This happens *if and only if* R/I is a field. □

Corollary 14. If R is a commutative ring with identity and $M \subseteq R$ is maximal, then M is prime.

← Hence maximal implies prime, but prime does not necessarily implies maximal.

Proof. Maximal $\implies$ quotient is a field $\implies$ quotient is an ID $\implies$ prime. □

Definition 16. An integral domain R is an **Euclidean domain** if there exists a norm $N : R \to \mathbb{Z}_{\geq 0}$ with $N(0) = 0$ such that for all $a, b \in R$ with $b \neq 0$, there exists $q, r \in R$ for which

$$a = bq + r$$

with $N(r) < N(b)$ or $r = 0$.

Example 15. $\mathbb{Z}$ is a ED with $N(a) = |a|$.

Example 16. $\mathbb{Q}[x]$ is a ED with $N(p(x)) = \deg(p(x))$.

Example 17. Every field F is a ED with $N(a) = 0 \ \forall \ a \in F$.

← Because in a field everything divides!

Non-example 18. $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID that is not an ED.

← This is one of the only good examples!

Why do we care about Euclidean domains?

Remark. Greatest common divisors exist and are relatively quick to compute.

← Using recursive application of Euclidean algorithm.

Definition 17. If $a, b \in R$, then $\gcd(a, b) = c$ means

1. c divides a and b; that is, $a = cr, b = cs$ for some $r, s \in R$
2. If $c' \in R$ with $c'|a$ and $c'|b$, then it must be true that $c'|c$.

← All other common divisors divide the gcd.

Example 19. Say we want to compute the gcd of 47 and 10.

$$
\begin{aligned}
47 &= 4 \times 10 + 7 \\
10 &= 1 \times 7 + 3 \\
7 &= 2 \times 3 + 1 && \leftarrow \text{circled is } \gcd(47, 10) \\
3 &= 3 \times 1 && \leftarrow \text{final line with no remainders}
\end{aligned}
$$

← This is a much faster algorithm than factoring!

This also works for finding gcds in $\mathbb{Q}[x]$ with polynomials long division and norm $\deg(p(x))$.

Remark. If F is a field, then $F[x]$ is a Euclidean domain.

← Just use long division!

Remark. Euclidean domains are PIDs.

Proof. Suppose R is a ED and $I \subseteq R$ is an idea;. Consider $\{N(a) \mid a \in I \backslash \{0\}\}$. This set has a minimal element by properties of natural numbers (or is an empty set *if and only if* $I = (0)$).

Let $d \in I$ be an element of minimum norm (hence $N(d) \leq N(a)$ for all $a \in I$). We claim that $(d) = I$. Proof:

Since $d \in I$, we have $rd \in I$ for any $r \in R$. This implies that $(d) \subseteq I$.

Then let $a \in I$. Since R is a ED, we first assumes that there exists $q, r \in R$, $r \neq 0$ such that $a = qd + r$ and $N(r) < N(d)$. But we notice that $r = a - qd$ must be in I as both $a, qd \in I$, contradicting the minimality of $N(d)$. Thus, it must be that $r = 0$. This implies $a = qd$ and thus $a \in (d)$ for all $a \in I$. Consequently, $I \in (d)$, and therefore $I = (d)$. □

Definition 18. Suppose R is an integral domain and $p \in R \backslash \{0\}$. Then p is a **prime element** if (p) is a prime ideal.

Proposition 15. An element $p \in R$ is prime *if and only if* whenever $p|ab$ then $p|a$ or $p|b$.

Proof. p is prime means that (p) is a prime ideal. This is true *if and only if* whenever $ab \in (p)$ then $a \in (p)$ or $b \in (p)$. This is the same as saying if $ab = kp$ for some $k \in R$ then $a = lp$ or $b = lp$ for some $l \in R$. This is to say that whenever $p|ab$ then $p|a$ or $p|b$. □

Proposition 16. In an integral domain, all prime elenments are irreducibles.

Proof. Suppose R is an ID and $p \in R$ is prime. If $p = ab$ for some a, b in R, then, WLOG, $p|a$. That is, $a = pk$ for some $k \in R$. Hence, $p = pkb$. Since in an ID cancellation rule holds, $kb = 1$, meaning that b is a unit. Thus, p is irreducible by definition Definition 5. □

Proposition 17. In PIDs, all *nonzero* prime ideals are maximal.

Proof. Suppose R is a PID and $(p) \subsetneq R$ is a prime ideal. If $(p) \subsetneq (m) \subsetneq R$ is an ideal, then $p \in (p) \subsetneq (m)$ hence $p = rm$ for some $r \in R$. Since $p|rm$, we have $p|r$ or $p|m$.

If $p|r$, this implies that $r = pk$ for some $k \in R$. Substituting into $p = rm$, we get $p = pkm$. By cancellation, we get $km = 1$, meaning that m is a unit. Hence, $(m) = R$.

If $p|m$, we have $m = pl$ for some $l \in R$, meaning that $m \in (p)$. Hence, $(m) \subsetneq (p)$, but we also defined that $(p) \subsetneq (m)$, so $(m) = (p)$.

Therefore, (p) has to be the maximal ideal. □

Proposition 18. In an UFD, irreducible implies prime.

Proof. Let R be a UFD and $p \in R$ be irreducible. Let $a, b \in R$ such that $p|ab$. Hence, $pr = ab$ for some $r \in R$. Since R is a UFD, let $a = q_1 \dots q_n, b = s_1 \dots s_m$ be the factorization. Since the factorizations are unique and each of the q_i, s_j are irreducible, if $p|ab$, then p must be an associate with one of the q_i, s_j. Therefore, either $p|a$ or $p|b$, implying prime. □

Example 20. $\mathbb{Q}$ is a field, so $\mathbb{Q}[x]$ is a ED. Since EDs are UFDs, irreducible $\implies$ prime. We see that $x^2 - 2 \in \mathbb{Q}[x]$ is an irreducible element, which means that $(x^2 - 2)$ is a prime ideal, meaning that it is a maximum ideal, meaning that $\mathbb{Q}[x]/(x^2 - 2)$ is a field. We observe that it is a field containing $\mathbb{Q}$ and $(\sqrt{2})$.

← In fact, this is the smallest field containing $\mathbb{Q}$ and $(\sqrt{2})$.

Lemma 19. In a PID, irreducible elements are prime.

Proof. Suppose $p \in R$ is irreducible in the principal ideal domain R. If $p|ab$ for some $a, b \in R$, we want to show that either $p|a$ or $p|b$, hereby showing that p is prime. Hence, we consider the ideal $(a, p) = d$, which is necessarily principal for some $d \in R$. Since $a, p \in (d)$, we have $a = dr$ and $p = ds$ for some $r, s \in R$. As p is irreducible, we get that one of d and s is a unit.
We first assume that s is a unit, in which case $d = ps^{-1}$, and so $a = ps^{-1}r$ implying that $p|a$.
In another case, d is a unit, in which case $(a, p) = (d) = R$ and so $1 = ak + pl$ for some $k, l \in R$. Multiplying by b, we get $b = abk + pbl$. Since $p|ab$, we have $b = abk + pbl = pmk + pbl$ for some $m \in R$. Hence, $b = p(mk + bl)$, meaning that $p|b$.

Therefore, whenever $p|ab$, either $p|a$ or $p|b$. Hence, in a PID, p is prime whenever it is irreducible. $\square$

Proposition 20. PIDs are UFDs.

Proof. Suppose R is a PID and $a \in R$ is nonzero, nonunit. If a is irreducible, we are done. If not, we write $a = p_1 q_1$ for some $p_1, q_1 \in R$ nonunit. If p_1, q_1 are irreducibles, we are done. If not, then WLOG say $q_1 = p_2 q_2$ for some nonunits p_2, q_2. We would like to show that this splitting process terminates.

Observe that $(q_1) \subseteq (q_2)$ since $q_2 | q_1$. Hence, the chain of splitting results in the chain of ideals $(q_1) \subseteq (q_2) \subseteq (q_3) \subseteq \dots$.

Now consider the ideal[1] $\bigcup_{i=1}^{\infty}(q_i)$. Since this is a PID, we have $\bigcup_{i=1}^{\infty}(q_i) = (q)$ for some $q \in R$. Since $q \in \bigcup_{i=1}^{\infty}(q_i)$, it is contained in some (q_n) for some $n \geq 1$. This implies that $(q) \subseteq (q_n)$, but we also know that $(q_n) \subseteq (q)$, hence $(q) = (q_n)$. Hence, this process terminates, and there exists an n in this chain such that q_n is irreducible. Therefore, R is a factorization domain.

Now we want to prove the uniqueness. That is, if $p_1 \dots p_n = q_1 \dots q_m$ for irreducibles p_i, q_j and $n \leq m$ WLOG, then we want to show that $m = n$ and that $p_i = u_i q_i$ with units u_i up to reordering for all i. We do so by induction on n.

(*Base case*) If $p_1 = q_1 \dots q_m$ and p_1 irreducible, then $q_2 \dots q_m$ are all units. Hence, $m = 1$ and $p_1 = q_1$.

(*Inductive step*) Say we have already proven the statement for $n = k$. Then consider $p_1 p_2 \dots p_{k+1} = q_1 q_2 \dots q_m$. Since R is a PID where irreducible implies prime, p_1 is a prime element dividing the product of primes $q_1 q_2 \dots q_m$, so we say WLOG $p_1 | q_1$. This means that $q_1 = u_1 p_1$ for some $u \in R$, but since q_1 is not reducible, it forces u_1 to be a unit. Hence, we apply cancellation on both sides and get $p_2 \dots p_{k+1} = (u_1 q_2) \dots q_m$.

By inductive hypothesis, $m - 1 = k$ and p_i, q_i are associates up to reordering for any i. Hence, the factorization must be unique. $\square$

---

[1]The proof that this is an ideal is as follows:

We first prove that $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition. Let $r, s \in \bigcup_{n=1}^{\infty} I_n$, where $r \in I_k$ and $s \in I_{k+i}$ for some $k, i \in \mathbb{N}$. Since $I_1 \subseteq I_2 \subseteq \dots$ are ideals of R, we know that $r \in I_k$ implies that $r \in I_{k+i}$. Thus, $r - s \in I_{k+i}$ due to I_{k+i} being an ideal. As $I_{k+i} \subseteq \bigcup_{n=1}^{\infty} I_n$, we have $r - s \in \bigcup_{n=1}^{\infty} I_n$, which means that $\bigcup_{n=1}^{\infty} I_n$ is closed under additive inverse. Hence, $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition.

Then, we prove that for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, we would have $tr, rt \in \bigcup_{n=1}^{\infty} I_n$. Since $r \in \bigcup_{n=1}^{\infty} I_n$, it must be true that $r \in I_k$ for some $k \in \mathbb{N}$. Hence, $tr, rt \in I_k$ due to I_k being an ideal. Therefore, $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$.

In conclusion, since $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition with the property that $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, it is an ideal of R. $\square$

We have shown:

ID	$\supsetneq$	UFD	$\supsetneq$	PID	$\supsetneq$	ED	$\supsetneq$	$Fields$
Non-eg: $\mathbb{Z}/6\mathbb{Z}$		$\mathbb{Z}[\sqrt{-5}]$		$\mathbb{Z}[x]$		$\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$		$\mathbb{Z}$

Maximal $\Longrightarrow$ *Prime* $\Longrightarrow$ *Irreducible*

Prime $\Longleftrightarrow$ *Irreducible*

Prime $\Longleftrightarrow$ *Maximal*

Field extensions

We observe that the polynomial $x^2 - 2 \in \mathbb{Q}[x]$ is irreducible. If we have $x^2 - 2 = p(x)q(x)$ where p, q nonunits, then $\deg(p) + \deg(q) = 2$ and we cannot have any $0+2$ combinations due to constants being units, we only have $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, but $x \pm \sqrt{2} \notin \mathbb{Q}[x]$!

Since $\mathbb{Q}[x]$ is a UFD, the irreducible element $(x^2 - 2)$ is prime, and since $\mathbb{Q}[x]$ is a PID, $(x^2 - 2)$ is maximal which means that $\mathbb{Q}[x]/(x^2 - 2)$ is a field.

Phase II plan: Field extensions!

Suppose F is a field and $p(x) \in F[x]$ nonzero. Recall that $F[x]$ is a ED with the norm function $\deg(a(x))$ and long division of polynomials. Let $a(x) + (p(x)) \in F[x]/(p(x))$. By the division algorithm, we have $a(x) = p(x)q(x) + r(x)$ for $q(x), r(x) \in F[x]$ and $\deg(r(x)) < \deg(p(x))$ or $r(x)$ is the zero polynomial.

$\leftarrow$ $a(x)$ is a coset rep

$\leftarrow$ We can do division algorithm since this is an ED

Now we see that since $a(x) - r(x) \in (p(x))$, they are in the same coset! Hence $a(x) + (p(x)) = r(x) + (p(x))$. We observe that <u>every element of $F[x]/(p(x))$ can be represented by a polynomial of a degree less than $\deg(p(x))$</u>. In other words, if $\deg(p(x)) = n$, then $F[x]/(p(x))$ is of the form

$$\begin{aligned} F[x]/(p(x)) &= \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}} \;\middle|\; a_0, a_1, \ldots, a_{n-1} \in F \right\} \\ &= \operatorname{Span}_F\{\bar{1}, \bar{x}, \ldots, \overline{x^{n-1}}\} \end{aligned}$$

$\leftarrow$ The expression under the bar functions like $r(x)$! Also note that span is just the set of linear combinations.

In fact, $F[x]/(p(x))$ is (partly) just a **vector space** over F...

← Why is this not the vector space over $F/(p(x))$ but just F? See the next paragraph.

We shall observe that it does not matter if we are using F or $\bar{F}$.
Consider $\varphi : F \hookrightarrow F[x]/(p(x))$ where $a \mapsto \bar{a}$. We observe this is an **injective** map: whenever $\deg(p(x)) = n > 0$, we have $\varphi(a) = \varphi(b)$ *if and only if* $\bar{a} = \bar{b}$, which happens *if and only if* $a - b \in (p(x))$; but the difference of two constants always have deg 0 and cannot be in $(p(x))$ unless it is a straight zero, which tells us that $\bar{a} = \bar{b}$ *if and only if* $a = b$. In other words, $F[x]/(p(x))$ contains an isomorphic copy of F, its field of scalars! Namely, $\boxed{F \cong \varphi(F) = \{\bar{a} \in F[x]/(p(x)) \mid a \in F\}}$.

...Hence, $F[x]/(p(x))$ is a vector space **of dimension** n over the scalar field F that also contains an isomorpic copy of F.

Moreover, if $p(x)$ is irreducible, then $(p(x))$ is prime since this is an ED, and hence, it is also a maximal ideal, meaning that $F[x]/(p(x))$ is a field containing an isomorphic copy of F.

← all thanks to Euclidean domains!

Definition 19. Suppose $F \subseteq K$ are fields. Then K is called a **field extension** of F.

- Notation: K/F or $\begin{matrix} K \\ | \\ F \end{matrix}$ (*the lattice notation*)

← Please, this is NOT a quotient. DO NOT CONFUSE THOSE!!

The dimension of K as a vector space over F is called the **degree** of the extension.

- Notation: $[K : F]$

But does my field F always have an extension? Here is a systematic way to get extensions:

Example 21. If $p(x) \in F[x]$ is an irreducible polynomial of degree $n \geq 1$ over the field F, then $F[x]/(p(x))$ is a **field extension** of F of degree n.
Furthermore, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $\bar{x}$ is a **root** of

← Since $\varphi(F) \cong F$, and $\varphi(F) \subseteq F[x]/(p(x))$

$$\varphi(p(x)) = \bar{a}_0 + \bar{a}_1 y + \cdots + \bar{a}_n y^n \in (\,F[x]/(p(x))\,)\,[y]$$

because, plugging in $y = \bar{x}$, we get

← We think about modding out by $(p(x))$ as making it equal to zero, which is how we find roots.

$$\bar{a}_0 + \bar{a}_1 \bar{x} + \cdots + \bar{a}_n \bar{x}^n = \overline{p(x)} = \bar{0} \in F[x]/(p(x))$$

Hence, the isomorphic copy of the polynomial $p(x)$ has **roots** in the field extension $F[x]/(p(x))$.

So, what the hell is $F[x]/(p(x))$? We have already shown that the field extension $F[x]/(p(x))$ does indeed contain a root of $p(x)$. Now we think about it **the other way around**: if we want to find an extension of F that contains a root of $p(x)$, we would eventually get this one!

Suppose $p(x) \in F[x]$ is irreducible. Let K/F be an extension, and $\alpha \in K$ a root of $p(x)$. Denote by $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α. Consider the map $\varphi : F[x] \to F(\alpha) \subseteq K$ where $q(x) \mapsto q(\alpha)$ is simply the evaluation at α map. We note that $p(x) \in \ker(\varphi) = (d(x))$ since an ED is a PID; this implies that $p(x) = u(x)d(x)$. As $p(x)$ is irreducible, $u(x)$ must be a unit, which means $p(x)$ and $d(x)$ are associates and $\ker(\varphi) = (p(x))$. Therefore,

$$F[x]/(p(x)) = F[x]/\ker(\varphi) \cong \varphi(F[x]) \subseteq F(\alpha)$$

by first isomorphism theorem. However, $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α, so $\varphi(F[x])$ cannot be smaller than that. Hence, it must be true that $\varphi(F[x]) = F(\alpha)$.

← Observe that $\varphi(F[x])$ is a field: $\ker(\varphi)$ is a maximal ideal

Therefore, $\underline{F(\alpha) \text{ is simply } F[x]/(p(x)).}$ □

To summarize so far!

Suppose $p(x) \in F[x]$ is an irreducible polynomial with coefficients in the field F.

- $F[x]/p(x)$ is a **field** containing an isomorphic copy of F in which $\overline{x} = x + (p(x))$ is a **root** of (the image of) $p(y) \in (F[x]/(p(x)))[y]$.

  Example 22. In $\mathbb{Q}[x]/(x^2-2)$, we have $x + (x^2-2)$ is a root of $y^2 - \overline{2} \in (\mathbb{Q}[x]/(x^2-2))[y]$ because

  $$\begin{aligned} &(x + (x^2-2))^2 - (2 + (x^2-2)) \\ &= x^2 - 2 + (x^2-2) && \text{by coset addition \& multiplication} \\ &= 0 + (x^2-2) && \text{since } x^2-2 \in (x^2-2) \\ &= \bar{0} \end{aligned}$$

  Furthermore, if $\deg(p(x)) = n$, then

  $$F[x]/(p(x)) = \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \dots, a_{n-1} \in F \right\}$$

  is a vector space over F of dimension n.

  Example 23. $\mathbb{Q}[x]/(x^2-2) = \{\bar{a}_0 + \bar{a}_1\bar{x} \mid a_0, a_1 \in \mathbb{Q}\} = \mathrm{Span}_{\mathbb{Q}}\{\bar{1}, \bar{x}\}$

- If K/F is an extension and $\alpha \in K$ is a root of $p(x)$, denote by $F(\alpha)$ **the smallest field containing** F **and** α.

  ← Read 'F adjoint α'

$$\begin{array}{c} K \\ | \\ F(\alpha) \\ | \\ F \end{array}$$

Figure 1: Field diagram

Then $F(\alpha) \cong F[x]/(p(x))$, and

← The eval map $\varphi : F[x] \to F(\alpha)$ where $f(x) \mapsto f(\alpha)$ has in fact $\ker(\varphi) = (p(x))$ when α is a root of $p(x)$.

$$\begin{aligned} F(\alpha) &= \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \dots, a_{n-1} \in F \right\} \\ &= F[\alpha] \qquad \leftarrow \text{the polynomial of } \alpha \text{ over } F \end{aligned}$$

Example 24. $\mathbb{Q}(\sqrt{2}) = \{a_0 + a_1\sqrt{2} \mid a_0, a_1 \in \mathbb{Q}\} = \mathbb{Q}[\sqrt{2}]$

Irreducibility – a survey

Proposition 21. If $p(x) \in F[x]$, then $\alpha \in F$ is a root *if and only if* $x - \alpha$ divides $p(x)$.

Proof. Write $p(x) = (x-\alpha)q(x) + r(x)$ with $q(x), r(x) \in F[x]$ and $\deg(r(x)) = 0$ or $r(x) = 0$. Then $0 = p(\alpha) = 0 + r(\alpha)$ which forces $r(x) = 0$. □

Corollary 22. A degree-2 or -3 polynomial over a field F is irreducible *if and only if* it has no roots in F.

Proposition 23. Suppose $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ with root $\frac{c}{d}$ written in reduced from (i.e. $\gcd(c, d) = 1$). Then $\boxed{c|a_0 \text{ and } d|a_n}$.

Proof.

$$\begin{aligned} d^n \cdot p\left(\frac{c}{d}\right) &= 0 \\ 0 &= (a_0 d^n + a_1 d^{n-1} c + \cdots + a_{n-1} d c^{n-1}) + a_n c^n \\ 0 &= a_0 d^n + (a_1 d^{n-1} c + \cdots + a_{n-1} d c^{n-1} + a_n c^n) \end{aligned}$$

Looking at the 2nd line, since d divides all of the ones in the (), it must also divide the last term $a_n c^n$. However, since $\gcd(c, d) = 1$, it forces d to divide a_n.

Similarly, we make the same argument for c and a_0 using the 3rd line. □

Lemma 24. $(R/I)[x] \cong R[x]/(I)$ where $(I) = I[x]$.

Proof. Consider the surjective homomorphism $\pi : R[x] \to (R/I)[x]$. □

Proposition 25 (Eisenstein's Criterion). Suppose $f(x) = \mathbb{1}x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is a monic polynomial and $p \in \mathbb{Z}$ is a **prime** such that $p \mid a_0, \dots, a_{n-1}$ but $p^2 \nmid a_0$. Then $\underline{f(x) \text{ is irreducible}}$.

Proof. Assume BWOC that $f(x) = a(x)b(x)$ for some nonunit $a(x), b(x) \in \mathbb{Z}[x]$, then

$$x^n = \bar{f}(x) = \bar{a}(x)\bar{b}(x)$$

in $(\mathbb{Z}/p\mathbb{Z})[x] \cong \mathbb{Z}[x]/p\mathbb{Z}[x]$ since all other terms are divisible by p. Since $\mathbb{Z}/p\mathbb{Z}$ does not contain any zero divisors, $\bar{a}(x), \bar{b}(x)$ must have zero constant terms. Hence $a(x), b(x)$ have constant terms that are multiples of p, so $a(x)b(x)$ have constant term divisible by p^2. This is a contradiction with $p^2 \nmid a_0$. □

Lemma 26 (Gauss' Lemma)**.** If $p(x) \in \mathbb{Z}[x]$ is reducible in $\mathbb{Q}[x]$, then it is reducible in $\mathbb{Z}[x]$.

Proof. Suppose $p(x) = a(x)b(x)$ for $a(x), b(x) \in \mathbb{Q}[x]$. Then by multiplying by coefficient denominators, for some $m \in \mathbb{Z}$, we could write $m \cdot p(x) = c(x)d(x)$ for some $c(x), d(x) \in \mathbb{Z}[x]$. Now since $m \in \mathbb{Z}$, we could write $m = q_1 q_2 \dots q_n$ be a product of irreducibles in $\mathbb{Z}$.

Now in $(\mathbb{Z}/q_1\mathbb{Z})[x] \cong \mathbb{Z}[x]/(q_1\mathbb{Z})[x]$, we observe that $m \cdot p(x) = c(x)d(x) = q_1(q_2 \dots q_n)p(x)$, meaning that

$$\overline{c(x)}\,\overline{d(x)} = \overline{q_1(q_2 \dots q_n)p(x)} = \bar{0}$$

Since $(\mathbb{Z}/q_1\mathbb{Z})[x] \cong \mathbb{Z}[x]/(q_1\mathbb{Z})[x]$ is an integral domain, WLOG, $\overline{c(x)} = \bar{0}$ *if and only if* $c(x) \in q_1\mathbb{Z}[x]$, meaning that all coefficients of $c(x)$ are multiples or q_1. Therefore, $\frac{1}{q_1}c(x) \in \mathbb{Z}[x]$.

← since q_1 is irreducible and hence prime in UFD

Now we repeat the process for all $q_1, q_2, \dots, q_n$ and we are done. □

Recall that if $F \subseteq K$ are fields, $\alpha \in K$ and $p(x) \in F[x]$ is irreducible with root α, then

$$F[\alpha] = F(\alpha) \cong F[x]/(p(x)) = \{\overline{a_0 + a_1 x + \dots + a_{n-1}x^{n-1}} \mid a_0, a_1 \dots, a_{n-1} \in F\}$$

We observe that this has a few implications. For instance, $F(\alpha)$ contains $\frac{1}{\alpha}$, meaning that it could also be written as a polynomial of α with coefficients in F (as in $F[\alpha]$)!

← since it is a field containing the mult. inverse of α

Definition 20. Suppose K/F is a field extension and $\alpha \in K$. We say that α is **algebraic over** F if there exists $p(x) \in F[x]$ such that $p(\alpha) = 0$. If not, α is **transcendental**.

Definition 21. The extension K/F is an **algebraic extension** if **every** element $\alpha \in K$ is algebraic over F.

Example 25. π is transcendental over $\mathbb{Q}$ but algebraic over $\mathbb{R}$ (since it is a root of $x - \pi$).

Proposition 27. If K/F is a **finite extension** , then it is an algebraic extension.

← finite extension just means finite degree $[K : F] < \infty$

Proof. Call $[K : F] = n$ and let $\alpha \in K$. Then the $n+1$ elements $\{1, \alpha, \alpha^2, \ldots, \alpha^n\}$ must be linearly dependent. Hence, by linear algebra, there exist $a_0, a_1, \ldots, a_n \in F$ not all zero such that the linear combination $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$. Hence, α is a root of $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$. □

← since $n + 1 > \dim(K/F) = n$

Corollary 28. If K/F is an extension and $\alpha \in K$, then α is algebraic over F *if and only if* $[F(\alpha) : F] < \infty$.

Proof.

- ($\Longleftarrow$) Follows from prop.
- ($\Longrightarrow$) If α is algebraic, then there exists an irreducible polynomial $p(x)$ with α as a root and of degree $n < \infty$. Then $F(\alpha) \cong F[x]/(p(x))$ is a n-dimensional vector space over F.
  Another perspective: $F(\alpha) = \operatorname{Span}_F\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$.

← Review proof of $F(\alpha) \cong F[x]/(p(x))$.

□

Proposition 29. Suppose K/F is an extension & $\alpha \in K$ is algebraic over F. Then there exists a unique, irreducible, and monic polynomial $m_{\alpha,F}(x) \in F[x]$ that has α as a root.

Remark. We observe that m does depend on the base field F; $m_{\sqrt{2},\mathbb{Q}}(x) = x^2 - 2$, but $m_{\sqrt{2},\mathbb{Q}(\sqrt{2})}(x) = x - \sqrt{2}$.

Proof. Since the subset of $F[x]$ *satisfying* α *is a root* is nonempty, we can pick one with a **minimal degree**. By multiplying by an element of F if necessary, we can assume WLOG this polynomial is **monic**. Call it $m_{\alpha,F}(x)$.

Assume BWOC that m is the product of two other polynomials of lesser degree such that $m_{\alpha,F}(x) = a(x)b(x)$, then we plug in $0 = m_{\alpha,F}(\alpha) = a(\alpha)b(\alpha)$. Since there are no zero divisors in $F[x]$, WLOG $a(\alpha) = 0$, contradicting the minimality of $m_{\alpha,F}$. Hence $m_{\alpha,F}$ is **irreducible**.

← So $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$

Then, BWOC if $p(x) \in F[x]$ with α as a root and is monic and irreducible, there exist $q(x), r(x) \in F[x]$ such that $p(x) = m_{\alpha,F}(x)q(x) + r(x)$ where $\deg(r) < \deg(m_{\alpha,F})$ or $r(x) = 0$. Then, we observe that $p(\alpha) = 0 = m_{\alpha,F}(\alpha)q(m_{\alpha,F}) + r(\alpha) = 0 + r(\alpha)$. Thus, $r(\alpha) = 0$, so $\deg(r) \geq \deg(m_{\alpha,F})$ unless $r(x) = 0$ by minimality. Hence we must have $r(x) = 0$, so $m_{\alpha,F} | p$. This contradicts the assumption that p is monic and irreducible. Therefore, $m_{\alpha,F}$ is the **only** minimal, monic and irreducible polynomial where α is a root. □

Definition 22. $m_{\alpha,F}(x)$ is the **minimal** polynomial of α over F.

(The following is kind of on a tangent)

Some exam prep!

- In general, for subrings $R \subseteq S$, we have if $r \in R^\times$, then $r \in S^\times$.
- If we adjoint one root of an irreducible polynomial to a field, the fields are isomorphic no matter which root of that polynomial we adjoint.

(Tangent ends here)

To summarize, if K/F is a field extension and $\alpha \in K$, then α is **algebraic** over F if it is the root of some polynomials in $F[x]$. For each algebraic α, there exists a unique, monic, irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ such that $m(\alpha) = 0$. In that case, the degree of extension $[F(\alpha) : F] = \deg(m_{\alpha,F}(x))$; and, if $p(\alpha) = 0$ for some $p(x) \in F[x]$, then $m_{\alpha,F} | p(x)$. In general, if $[K : F] < \infty$, then K/F is algebraic. Thus, $[F(\alpha) : F] < \infty$ *if and only if* α is algebraic over F.

Proposition 30. If $F \subseteq K \subseteq L$ are fields, then

$$[L : F] = [L : K] \cdot [K : F]$$

← $$\begin{tikzcd} L \arrow[d, dash, "n"] \arrow[dd, dash, bend right, "mn"'] \\ K \arrow[d, dash, "m"] \\ F \end{tikzcd}$$

Proof. We first see that if $[K : F] = \infty$, then for any $N \in \mathbb{N}$, there exists $\alpha_1, \dots, \alpha_N \in K$ that are linearly independent over F. In that case, it is certainly true that $\alpha_1, \dots, \alpha_N \in L$ are linearly independent over F. Thus, $[L : F] = \infty$.

← Linear independence implies that whenever $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_N\alpha_N = 0$ for some coefficients $a_1, \dots, a_N \in F$, then necessarily $a_1 = a_2 = \cdots = a_N = 0$.

If $[L : K] = \infty$, then for any $N \in \mathbb{N}$, there exists $\beta_1, \dots, \beta_N \in L$ that are linearly independent over K. As a result, it also is linearly independent over F. Hence, $[L : F] = \infty$.

If $[K : F] = m$ and $[L : K] = n$, let $\alpha_1, \dots, \alpha_m \in K$ be a basis for K over F and $\beta_1, \dots, \beta_n \in L$ be a basis for L over K.

Claim: $\{\alpha_i\beta_j \mid 1 \le i \le m, 1 \le j \le n\}$ forms a basis for L over F. □

← Use linear combinations to prove this claim.

Some nice consequences:

Corollary 31. Suppose K/F is an extension and $\alpha, \beta \in K$ are algebraic over F. Then:

- $F(\alpha, \beta) = (F(\alpha)(\beta))$
- $[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\beta)][F(\beta) : F] = \deg\left(m_{\alpha,F(\beta)}(x)\right)\cdot\deg\left(m_{\beta,F(x)}\right)$. However, note that the minimal polynomial
$$m_{\alpha,F(\beta)}(x) \mid m_{\alpha,F}(x) \in F(\beta)[x]$$
so $\deg(m_{\alpha,F(\beta)}(x)) \leq \deg(m_{\alpha,F}(x))$. Hence,
$$[F(\alpha, \beta) : F] \leq \deg(m_{\alpha,F}(x))\deg(m_{\beta,F}) < \infty$$
This means that whenever α, β are algebraic over F, we get that $F(\alpha, \beta)/F$ is an algebraic extension.
- As a result, $\alpha \pm \beta, \alpha\beta, \alpha/\beta$ are all algebraic over F. The algebraic elements hence form a **field**.

← the smallest subfield of K containing F, α, β

← since $p(\alpha) = 0 \iff m_{\alpha,F}(x)|p(x)$

Proposition 32. Suppose K/F is an extension. Then $[K : F] < \infty$ *if and only if* $K = F(\alpha_1, \dots, \alpha_n)$ could be written where $\alpha_1, \dots, \alpha_n \in K$ are algebraic over F.

In other words, an extension is finite *if and only if* it is generated by adjoining a finite amount of algebraic elements.

Proof.

($\implies$) If $[K : F] < \infty$, then suppose $\{\alpha_1, \dots, \alpha_n\}$ is a basis of K over F. Then $\alpha_1, \dots, \alpha_n$ are algebraic and every element of K is an F-linear combination of α_is. Hence K must be the smallest field containing F and α_is, which means $K = F(\alpha_1, \dots, \alpha_n)$.

($\impliedby$) We observe that

$$\begin{aligned}[K : F] &= [(F(\alpha_1, \dots, \alpha_{n-1}))(\alpha_n) : F(\alpha_1, \dots, \alpha_{n-1})] \cdot \ldots \cdot [F(\alpha_1) : F] \\ &\leq \prod_{i=1}^{n} \deg(m_{\alpha_i,F}(x)) < \infty\end{aligned}$$

□

Corollary 33. If L/K and K/F are algebraic extensions, then so is L/F.

← L/K and K/F need not be finite!

Proof. Suppose $\alpha \in L$. Since L/K is algebraic, there exists $p(x) \in K[x]$ such that $p(\alpha) = 0$. Let $\alpha_0, \dots, \alpha_n \in K$ be the coefficients of $p(x)$, necessarily algebraic over F since K/F algebraic. Therefore,

$$[F(a_0, \dots, a_n, \alpha) : F] = [(F(a_0, \dots, a_n))(\alpha) : F(a_0, \dots, a_n)][F(a_0, \dots, a_n) : F]$$

Since $p(\alpha) = 0$ has coefficients in $K \supseteq F(\alpha_0, \dots, \alpha_n)$, we have $[(F(a_0, \dots, a_n))(\alpha) : F(a_0, \dots, a_n)] < \infty$. The second term is also clearly $< \infty$. Therefore, $[F(a_0, \dots, a_n, \alpha) : F] < \infty$, meaning that α is algebraic over F. ☐

←
$$\begin{tikzcd}
L \arrow[d, dash] \arrow[dr, dash] & \\
K \arrow[d, dash] \arrow[dr, dash] & (F(a_0, \dots, a_n))(\alpha) \arrow[d, dash] \\
F \arrow[r, dash] & F(a_0, \dots, a_n)
\end{tikzcd}$$

Definition 23. Suppose L/F is an extension & K_1 and K_2 are intermediate fields. The **composite** field K_1K_2 is the smallest subfield of L containing K_1 and K_2.

$$\begin{tikzcd}
 & L \arrow[d, dash] & \\
 & K_1K_2 & \\
K_1 \arrow[ur, dash] & & K_2 \arrow[ul, dash] \\
 & K_1 \cap K_2 \arrow[ul, dash] \arrow[ur, dash] \arrow[d, dash] & \\
 & F &
\end{tikzcd}$$

Definition 24. Suppose F is a field and $p(x) \in F[x]$. The **splitting field** of $p(x)$ over F is the smallest field extension of F over which $p(x)$ could be factored into **linear factors**.

← Assuming that splitting fields **exist** and are **unique** up to isomorphism.

Remark. If E is the splitting field of $p(x)$ over F then $[E : F] \leq n!$ where $n = \deg(p(x))$.

Remark. Such an extension is called **normal**.

Proposition 34. Splitting fields exist.

Proof outline. By induction on $\deg(p(x))$, whose base case, $\deg(p(x)) = 1$, yields F as a splitting field. More generally, any $p(x)$ has a root α in $F(\alpha) \cong F[x]/(q(x))$ for some irreducible $q(x)$ so $p(x) = (x - \alpha)f(x) \in F(\alpha)[x]$. We observe that $\deg(f(x)) = \deg(p(x)) - 1$. Induction takes care of the rest. ☐

← The splitting field of $p(x)$ over F is the same as the splitting field of $f(x)$ over $F(\alpha)$

Remark. K is a splitting field over F *if and only if* every irreducible $p(x) \in F[x]$ that has one root in K has **all** its roots in K.

Non-example 26. $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ is not such an extension.

Lemma 35. Suppose $\varphi : F_1 \to F_2$ is a field isomorphism, $p_1(x) \in F_1[x]$, and $p_2(x) =$" $\varphi((p_1(x)))$" (φ applied to coeffs of $p_1(x)$). Let α_1 be a root of the irreducible factor $q_1(x)$ of $p_1(x)$, and let $q_2(x) =$" $\varphi(q_1(x))$" and α_2 be a root of $q_2(x)$. Then there exists an isomorphism $\tau : F_1(\alpha_1) \to F_2(\alpha_2)$ such that $\tau(\alpha_1) = \alpha_2$ and $\tau_{|F_1} = \varphi$ (this means "τ restricted to F_1").

← In this way, φ induce a ring isomorphism $F_1[x] \to F_2[x]$.

Proof outline.

$$\begin{tikzcd}
 & F_1(\alpha_1) \arrow[r, "\sim"] & F_1[x]/(q_1(x)) \arrow[r, "\sim"] & F_2[x]/(q_2(x)) \arrow[r, "\sim"] & F_2(\alpha_2) \\
 & \alpha_1 \arrow[r] & \bar{x} \arrow[r] & \bar{x} \arrow[r] & \alpha_2 \\
\text{if } a \in F_1, & a \arrow[r] & \bar{a} \arrow[r] & \overline{\varphi(a)} \arrow[r] & \varphi(a)
\end{tikzcd}$$

☐

Proposition 36. Suppose $F_1, F_2, \varphi, p_1(x)$ and $p_2(x)$ are as in Lemma 35. Let $E_1 \& E_2$ be splitting fields of p_1 and p_2 respectively. Then there exists an isomorphism $\sigma : E_1 \to E_2$ such that $\sigma_{|F_1} = \varphi$.

← if we set $F_1 = F_2$ and $p_1 = p_2$, we get corollary: splitting fields are unique up to isomorphism.

Proof. Proceed by induction on $\deg(p_1(x))$. For the base case, if $\deg(p_1(x)) = 1$, then $E_1 = F_1$ and $\sigma = \varphi$.

Assume the result is true for all polynomials of fixed degree $k \geq 1$ and suppose $\deg(p_1(x)) = k+1$. Let α_1 be a root of $p_1(x)$ and α_2 be a root of the φ-corresponding irreducible factor of $p_2(x)$. By Lemma 35, φ can be extended to $\tau : F_1(\alpha_1) \to F_2(\alpha_2)$ such that $\tau_{|F_1} = \varphi$.

In $(F_1(\alpha_1))[x]$, we can factor out $p_1(x) = (x-\alpha_1)g_1(x)$, and in $(F_2(\alpha_2))[x]$ we factor $p_2(x) = (x - \alpha_2)g_2(x)$ with $g_2(x) = \tau(g_1(x))$. We observe that E_1 and E_2 are the splitting fields of g_1 and g_2 over $F_1(\alpha_1)$ and $F_2(\alpha_2)$!

←

$$\begin{tikzcd}
\sigma : \arrow[d, dash] & E_1 \arrow[r, "\sim"] \arrow[d, dash] & E_2 \arrow[d, dash] \\
\tau : \arrow[d, dash] & F_1(\alpha_1) \arrow[r, "\sim"] \arrow[d, dash] & F_2(\alpha_2) \arrow[d, dash] \\
\varphi : & F_1 \arrow[r, "\sim"] & F_2
\end{tikzcd}$$

By inductive hypothesis, τ could be extended to σ and $\sigma_{|F_1(\alpha)} = \tau$ and $\sigma_{|F_1} = \varphi$. □

Corollary 37. Splitting fields are unique.

Proof. Set $F_1 = F_2$, $\varphi = $id, $p_1(x) = p_2(x)$. □

(The following is kind of on a tangent)

Homework hint: the proof of existence & uniqueness of splitting fields relied on inductive arguments where we adjoin one root at a time. This is the same as saying $E = F(\alpha_1, \dots, \alpha_n)$ but this tends to overlook isomorphic ways to adjoin roots. In this context, it is convenient to start by considering a specific K containing F and all roots of $p(x)$. In that case, $E = F(\alpha_1, \dots, \alpha_n)$ becomes more rigorous.

(Tangent ends here)

Definition 25. A polynomial is called **separable** if it doesn't have repeated roots.

← First note that a poly of degree n over a field has exactly n roots.

Definition 26. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The **formal derivative** of $f(x)$ is the polynomial

$$D_x f(x) = f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

From this definition, we can check that the usual differential rules hold.

Lemma 38. Suppose F is a field, $f(x)$ is a polynomial in $F[x]$, and E/F a field extension containing a root α of $f(x)$. Then α is a repeated root of $f(x)$ *if and only if* α is a root of the formal derivatve $f'(x)$.

Proof. If α is a repeated root of $f(x)$ then $f(x) = (x-\alpha)^2 g(x)$ for some $g(x) \in E[x]$. In that case, $f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x)$ and so $f'(\alpha) = 0$.

Conversely, if $f'(\alpha) = 0$, then differentiating $f(x) = (x-\alpha)h(x)$ (where $h(x) \in E[x]$) and plugging $x = \alpha$ yields $0 = f'(\alpha) = h(\alpha) + (\alpha - \alpha)h'(\alpha) = h(\alpha)$. This is saying that $h(x) = (x-\alpha)g(x)$ for some $g(x) \in E[x]$. □

Lemma 39. If $f(x) \in F[x]$ is **irreducible and not separable**, then $f'(x) = 0$.

Proof. If $f(x)$ is not separable, we know that there is at least one repeated root. We call it α. Then let $m_{\alpha,F}(x)$ be the minimal polynomial of α over F and we have $f(x) = c \cdot m_{\alpha,F}(x)$ for some constant $c \in F$. Therefore, $\deg(f(x)) = \deg(m_{\alpha,F}(x))$. However, by the previous lemma, $f'(\alpha) = 0$ must also exist since α is a repeated root. If $f'(x) \neq 0$, then we found a polynomial with degree less than the minimal polynomial that has α as a root, which cannot happen. Therefore, $f'(x) = 0$. □

If $f(x)$ is not constant and $f'(x) = 0$, then $\mathrm{char}(F) = p > 0$ and $f(x) = g(x^p)$.

← All powers in $f(x)$ are multiples of $p(x)$.

Proposition 40. If $\mathrm{char}(F) = 0$, or $|F| < \infty$ and $\mathrm{char}(F) = p$, then every irreducible polynomial in $F[x]$ is separable.

Proof. For the case of char=0, it follows from 39.

For the case of char>0, suppose F is a finite field of p^n elements[2]. Then the map $F \to F$ where $\alpha \mapsto \alpha^p$ is a field isomorphism. Hence, every element of F is a p^{th} power.

← Use binomial theorem.

← Such fields are called **perfect**.

Now suppose BWOC $f(x) = \sum_{i=0}^{n} a_i x^i \in F[x]$ is an irreducible but not separable polynomial. Therefore, $f'(x) = 0$ must be true. This happens *if and only if* $f(x) = \sum_{j=0}^{m} a_{jp} x^{jp}$, that is, the x in all terms are of p^{th} degree. However, we know that all elements $a_{jp} \in F$ are already the p^{th} powers of sth else $(b_{jp})^p = a_{jp}$, so

$$f(x) = \sum_{j=0}^{m} (b_{jp}^p) x^{jp}$$

[2] See Section 27

and by reverse Binomial Theorem, we get

$$f(x) = \sum_{j=0}^{m} (b_{jp}^{p}) x^{jp} = \left(\sum_{j=0}^{m} b_{jp} x^{j} \right)^{p}$$

is not irreducible! □

Non-example 27. Let $F = \mathbb{F}_p(t) = \left\{ \frac{f(t)}{p(t)} \;\middle|\; f(t), g(t) \in \mathbb{F}_p[t], g(t) \neq 0 \right\}$.

← This is a field of char>0 but is infinite.

Then $p(x) = x^p - t$ is not separable (but it is irreducible). This can be seen if we suppose α is a root of $p(x)$ (so $\alpha = t$). Then, in $F(\alpha)[x]$, we have $p(x) = x^p - t = x^p - \alpha^p = (x - \alpha)^p$, which tells us $p(x)$ is not separable.

← The coefficients of $p(x)$ are ratios of polys in $\mathbb{F}_p(t)$.

(The following is kind of on a tangent)

Prime fields

Suppose R is a commutative ring with identity. The map $\mathbb{Z} \to R$ where $n \mapsto \pm(\underbrace{1_R + 1_R + \cdots + 1_R}_{|n| \text{ times}})$ ($-$ if $n < 0$) is a ring homomorphism with kernel $n\mathbb{Z}$ where $n = \text{char}(R)$. So:

← check it!

- if char $(R) = 0$, then R contains $\mathbb{Z}$;
- if char $(R) = n > 0$, then R contains $\mathbb{Z}/n\mathbb{Z}$.

If F is a field, then:

- if char $(F) = 0$, then F conatins $\mathbb{Q}$;
- if char $(F) = p > 0$, then p prime and F contains $\mathbb{Z}/p\mathbb{Z}$.

← That is, F is an extension of $\mathbb{Z}/p\mathbb{Z}$!

In other words, every field is an extension of $\mathbb{Q}$ or $\mathbb{F}_p$. Moreover, a finite field is a *finite* extension of $\mathbb{F}_p$: if $[F : \mathbb{F}_p] = n$, then $|F| = p^n$.

← All elts of F are $a_0 + a_1 x_1 + \cdots + a_n x_n$ where $a_i \in \mathbb{F}_p$, so we have p^n choices.

In addition, $|F - \{0\}| = p^n - 1 \implies$ if $\alpha \in F - \{0\}$ then $\alpha^{p^n - 1} = 1$, implying that if $\alpha \in F$, then $\alpha^{p^n} = \alpha$, meaning that α is a root of $x^{p^n} - x \in F[x]$. Therefore, F is the splitting field of $x^{p^n} - x$. But splitting fields are unique, so we conclude that there is only one unique finite field for each order.

← By Lagrange's Theorem

(Tangent ends here)

Definition 27. An algebraic extension K/F is called (algebraically) **separable** if $m_{\alpha,F}(x)$ is separable for **all** $\alpha \in K$.

Galois Theory

Definition 28. A finite extension K/F is called **Galois** if K/F is normal and separable.

← normal just means it is a splitting field of something

Definition 29. If K/F is an extension, then the **automorphism group** of K/F is defined as

$$\mathrm{Aut}(K/F) = \{\sigma \in \mathrm{Aut}(K) \mid \sigma(a) = a \,\forall\, a \in F\}$$

That is, all the automorphisms of K that also fix the field F.

Galois theory is concerned with the study of roots of polynomials by way of automorphisms of splitting fields (of separable polynomials).
In particular, we are interested in what

$$\mathrm{Aut}(K) = \{\sigma : K \to K \text{ isomorphisms}\}$$

(a group under composition) is. Naturally, such groups are *finite*.

← review MATH171 finite groups!

Last time, we showed that $K \supseteq \begin{Bmatrix} \mathbb{Q} & \text{if char}(K) = 0 \\ \mathbb{F}_p & \text{if char}(K) = p \end{Bmatrix}$, since $\sigma(1) = \sigma(1^2) = (\sigma(1))^2$ implies that $\sigma(1) = 1$ must always be true! Hence, $\sigma(n) = n$ must be true in char 0 fields, or $\sigma(\bar{n}) = \bar{n}$ if char>0 for all $n \in \mathbb{Z}$. Therefore, σ **fixes** the prime subfield $\mathbb{Q}$ or $\mathbb{F}_p$.

Remark. Why does Definition 29 have to fix the field F? Because we've shown that $\mathrm{Aut}(K) = \begin{Bmatrix} \mathrm{Aut}(K/\mathbb{Q}) & \text{if char}(K) = 0 \\ \mathrm{Aut}(K/\mathbb{F}_p) & \text{if char}(K) = p \end{Bmatrix}$.

Lemma 41. If K/F is an extension, $\alpha \in K$ is algebraic over F and $\sigma \in \mathrm{Aut}(K/F)$, then $\sigma(\alpha)$ is a root of $m_{\alpha,F}(x)$.

Proof. Observe that $m_{\alpha,F}(\alpha) = 0 = \sigma(0) = \sigma(m_{\alpha,F}(\alpha))$. Hence, since $\sigma(\alpha) = \alpha$ for all $\alpha \in F$,

$$\sigma(a_0 + a_1\alpha + \cdots + a_n\alpha^n) = a_0 + a_1\sigma(\alpha) + \cdots + a_n\sigma(\alpha)^n = m_{\alpha,F}(\sigma(\alpha))$$

$\square$

So if $f(x) \in F[x]$, then every $\sigma \in \mathrm{Aut}(K/F)$ **permutes** the roots of $f(x)$ that lie in K. It would be nice if the roots of $f(x)$ all lived in K. This is why we consider K the splitting field of some polynomial over F.

← Recall: if irreducible $f(x)$ has one root in such K, then all roots lie in K.

Proposition 42. If K is the splitting field of some polynomial $f(x)$ over F (so $[K : F] < \infty$), then $|\mathrm{Aut}(K/F)| \leq [K : F]$, with equality if $f(x)$ is separable.

Proof. We will prove a more general statement by induction. If $\sigma : F_1 \to F_2$ is an isomorphism, $f_1(x) \in F_1[x]$ and $f_2(x) = \sigma(f_1(x)) \in F_2[x]$ and E_1 and E_2 are the splitting fields of f_1 and f_2 over F_1 and F_2 respectively. Then we would like to show that there are at most $[E_1 : F_1]$ isomorphisms $\tau : E_1 \to E_2$ such that $\tau_{|F_1} = \sigma$ with equality if f_1 separable.

← The prop above fixes $F_1 = F_2$ and σ being identity.

Base case. If $[E_1 : F_1] = [E_2 : F_2] = 1$, then $E_1 = F_1$, $E_2 = F_2$ and $\tau = \sigma$ is our only choice.

←

$$\begin{tikzcd}
\tau : & E_1 \arrow[r, "\sim"] \arrow[d, dash, "<n"'] & E_2 \arrow[d, dash] \\
\rho : & F_1(\alpha) \arrow[r] \arrow[d, dash, ">1"'] & F_2(\beta) \arrow[d, dash] \\
\sigma : & F_1 \arrow[r, "\sim"] & F_2
\end{tikzcd}$$

Inductive step. Suppose we've proven the result for all extensions of degree $< n$ for some $n \geq 2$. Now consider $[E_1 : F_1] = [E_2 : F_2] = n$. Pick $\alpha \in E_1 \backslash F_1$ and let $\beta \in E_2$ be any root of $\sigma(m_{\alpha,F_1}(x))$. Then σ could be extended to $\rho : F_1(\alpha) \to F_2(\beta)$ such that $\rho(\alpha) = \beta$ and $\rho_{|F_1} = \sigma$. Observe that $[F_1(\alpha) : F_1] = \deg(m_{\alpha,F_1}(x))$. Moreover, the number of extensions of σ to ρ equals the number of *distinct* roots of $\sigma(m_{\alpha,F_1}(x))$. Thus, the number of extensions of σ to $F_1(\alpha)$ is *at most* the degree of $m_{\alpha,F_1}(x)$ which is $[F_1(\alpha) : F_1]$ with equality if $m_{\alpha,F_1}(x)$ is separable. Since $[E_1 : F_1] = [E_2 : F_2] = n$, we have $[E_1 : F_1(\alpha)] < n$, by inductive hypothesis, there are at most $[E_1 : F_1(\alpha)]$ ways of extending ρ to $\tau : E_1 \to E_2$. Hence,

$$\begin{aligned}
|\{\text{extensions of } \sigma \text{ to } \tau\}| &= |\{\text{extensions of } \sigma \text{ to } \rho\}||\{\text{extensions of } \rho \text{ to } \tau\}| \\
&\leq [F_1(\alpha) : F_1][E_1 : F_1(\alpha)] \\
&= [E_1 : F_1]
\end{aligned}$$

Looking at the case $F_1 = F_2$, $E_1 = E_2$, $\sigma = \mathrm{id}$, we get our result. $\square$

Definition 30. If K/F is a normal extension, then the extension is **Galois** if $[K : F] = |\mathrm{Aut}(K/F)|$.

Remark. Notation: if K/F is Galois, then use $\mathrm{Gal}(K/F)$ for $\mathrm{Aut}(K/F)$.

Fixed Fields and Automorphism Groups

Definition 31. Suppose K/F is a field extension. If subgroup $H \leq \mathrm{Aut}(K/F)$, then the **fixed field** of H is given by $K_H = \{\alpha \in K \mid \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}$.

← Observe that K_H is indeed a field (the sum, products etc. are also fixed by σ); moreover, it is an intermediate extension

Remark. Also, observe that if $F \subseteq E \subseteq K$, then $\mathrm{Aut}(K/E) \leq \mathrm{Aut}(K/F)$.

Lemma 43. Suppose K/F is an extension. Then:

(1) If $H_1, H_2 \leq \mathrm{Aut}(K/F)$ with $H_1 \leq H_2$, then $K_{H_2} \subseteq K_{H_1}$.

(2) If $F \subseteq E_1 \subseteq E_2 \subseteq K$ are two intermediate extensions, then $\mathrm{Aut}(K/E_2) \leq \mathrm{Aut}(K/E_1) \leq \mathrm{Aut}(K/F)$.

Example 28. $\mathbb{Q}$ is an intermediate extensions of $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. Then $\mathrm{Aut}(\mathbb{Q}/\mathbb{Q}) = \{\mathbb{1}\}$. We further observe that since automorphisms permute roots, $\mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\mathbb{1}\}$. Hence $\mathrm{Aut}(\mathbb{Q}/\mathbb{Q}) = \mathrm{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ and so the fixed field by $\mathbb{Q}(\sqrt[3]{2})$ is given by $\mathbb{Q}(\sqrt[3]{2})_{\mathrm{Aut}(\mathbb{Q}/\mathbb{Q})} = \mathbb{Q}(\sqrt[3]{2})$. We note that $\mathbb{Q}(\sqrt[3]{2})$ is not Galois!

← Roots of $x^3 - 2$ are $\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}$, so two roots are not in $\mathbb{Q}(\sqrt[3]{2})$, and so $\sqrt[3]{2}$ could be only mapped to itself.

Theorem 44 (The Fundamental Theorem of Galois Theory)**.** If K/F is a (finite) Galois extension, then the maps $H \mapsto K_H$ and $E \mapsto \mathrm{Aut}(K/E)$ gives an *inclusion-reversing* **bijection** between the subgroups of $\mathrm{Aut}(K/F)$ and the intermediate extensions of K/F.

← We see if K/F is Galois, then K/E is also Galois as if K is the splitting field of some poly in F, then it's certainly true for E.

Furthermore, $[K : E] = |\mathrm{Aut}(K/E)|$, and

$$[E : F] = |\mathrm{Aut}(K/F) : \mathrm{Aut}(K/E)| = |\mathrm{Aut}(K/F)|/|\mathrm{Aut}(K/E)|$$

Moreover, E/F is Galois *if and only if* $\mathrm{Aut}(K/E)$ is a **normal subgroup** of $\mathrm{Aut}(K/F)$, in which case

$$\mathrm{Aut}(E/F) = \mathrm{Aut}(K/F)/\mathrm{Aut}(K/E)$$

In other words,

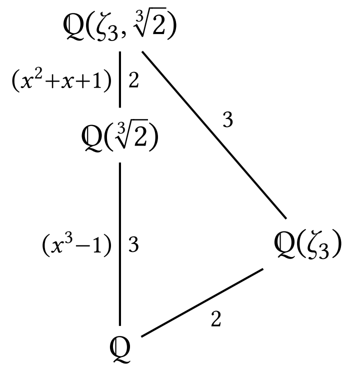
$$\begin{tikzcd}
K \arrow[d, dash] & & 1 \arrow[d, dash] \\
E \arrow[d, dash] \arrow[rr, shift right, "\mathrm{Aut}(K/E)"'] & & H = \mathrm{Aut}(K/E) \arrow[ll, shift right, "\{\alpha\in K \mid \sigma(\alpha)=\alpha\ \forall\ \sigma\in H\}"'] \arrow[d, dash] \\
F & & \mathrm{Aut}(K/F)
\end{tikzcd}$$

where H is a subgroup of $\mathrm{Aut}(K/F)$ that fixes the field E, an extension of F contained in K. If $H \trianglelefteq \mathrm{Aut}(K/F)$, then E/F is normal.

Example 29. Consider $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$, the splitting field extension of $x^3 - 2$.[3]

← $|\zeta_3| = 3$, a primitive 3rd root of unity.

[3]Suppose ζ is a primitive nth root of unity; then so is ζ^k *if and only if* the gcd $(k, n) = 1$, i.e. k, n are relatively prime.



Let $\sigma, \tau \in \mathrm{Aut}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ be given by:

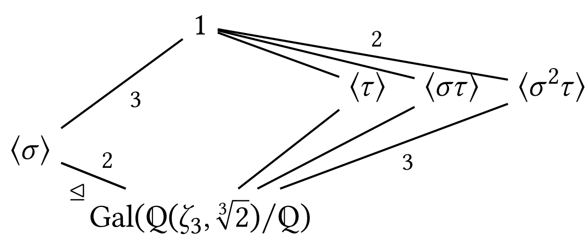
← Isomorphisms preserve order, so they **must** take an nth root of unity to another nth root of unity!

$$\sigma : \begin{cases} \sqrt[3]{2} \mapsto \zeta_3\sqrt[3]{2} \\ \zeta_3 \mapsto \zeta_3 \end{cases} \qquad \tau : \begin{cases} \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \zeta_3 \mapsto \zeta_3^2 \end{cases}$$

So $\langle \sigma, \tau \mid \sigma^3 = \tau^2 = id, \sigma\tau = \tau\sigma^2 \rangle \cong S_3$.

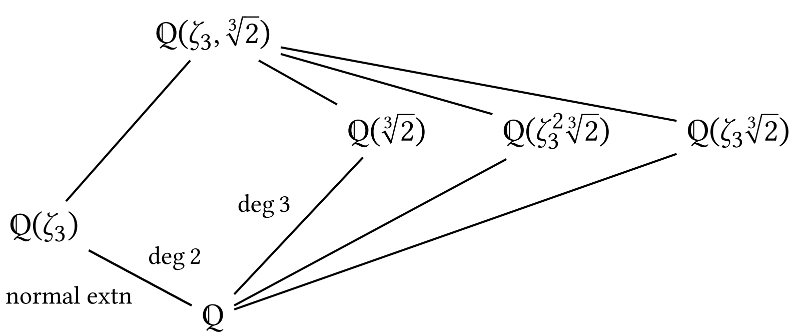
Since this is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ that has the same finite order of 6, this must just be $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ itself; hence, $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$.

Now we look at the subgroups of S_3 (in reverse):



And then think about the **fixed field** of each subgroup correspondingly:

← Remember normal extn means 'is splitting field', i.e. 'one root -> all roots'



Note that the normal extension corresponds to the normal subgroup!

← $\mathbb{Q}(\sqrt[3]{2})$ is not Galois!

Proposition 45 (Primitive Element Theorem)**.** If K/F is a **finite Galois** extension, then $K = F(\alpha)$ for some $\alpha \in K$.

Definition 32. $K = F(\alpha)$ is a **simple** extension of F and α is a primitive element.

Proof. We first assume $|F| = \infty$.

Recall that K/F is finite *if and only if* $K = F(\alpha_1, \alpha_2, \dots, \alpha_n)$ where α_i is algebraic over F. We will proceed by induction on n, whose base case $n = 1$ gives a simple extension $F(\alpha_1)/F$.

Recursive case: assume that for some $k \geq 1$ we have $F(\alpha_1, \dots, \alpha_k)$ being a simple extension $F(\alpha)$. Let K/F be Galois and $K = F(\alpha_1, \dots, \alpha_{k-1}, \alpha, \beta)$.

Let $E = F(\alpha_1, \dots, \alpha_{k-1})$ and consider the intermediate family of extensions $\{E(\alpha + t\beta) \mid t \in F\}$. Since $|\mathrm{Gal}(K/F)| < \infty$ as we are talking about finite Galois extensions, there are finitely many distinct such extensions, so $E(\alpha + t_1\beta) = E(\alpha + t_2\beta)$ for some $t_1 \neq t_2$.

← such that $K = E(\alpha, \beta)$

Now we see that $\alpha + t_1\beta$ and $\alpha + t_2\beta$ must be in the same field $E(\alpha + t_1\beta)$. Hence, $(\alpha + t_1\beta) - (\alpha + t_2\beta)$ are in the field, so $(t_1 - t_2)^{-1}((\alpha + t_1\beta) - (\alpha + t_2\beta)) = \beta$ is also in the field. Similarly, $\alpha \in E(\alpha + t_1\beta)$. Therefore, $K = E(\alpha, \beta) = E(\alpha + t_1\beta) = F(\alpha_1, \dots, \alpha_{k-1}, \alpha + t_1\beta)$, which has k elements adjoined and is therefore simple. $\square$

Remark. Above is true for char 0 fields even without the 'Galois' hypothesis.

Proof outline. Since K is a finite extension of F with $[K : F] < \infty$, then there must be $K = F(\alpha_1, \dots, \alpha_n)$ for some algebraic $\alpha_1, \dots, \alpha_n \in K$.

← Idea: if anything is not Galois, we add enough things to it to make it Galois!

Let E be the splitting field of $\prod_{i=1}^{n} m_{\alpha_i,F}(x)$. We call E the **Galois closure** of K over F. Now, we have $E \supset K \supset F$ and E/F is Galois. Thus, there are finitely many intermediate fields between K and F. We can then use a similar proof for Proposition 45. $\square$

Cyclotomic fields

Definition 33. Suppose K is a field. An element $z \in K$ is called an n-th root of unity if $z^n = 1$; and z is a **primitive** n-th root of unity if $z^k \neq 1$ for any $1 \leq k \leq n-1$.

Remark. z is a **primitive** n-th root of unity *if and only if* $|z| = n$ in $K^\times = K \backslash \{0\}$.

Remark. z is an n-th root of unity *if and only if* z is a root of $x^n - 1$.

Lemma 46. If K is a field containing one primitive n-th root of unity, then K contains exactly n roots of unity, exactly $\varphi(n)$ of which are primitive.

← $\varphi(n)$ is Euler's totient function, the count of integers $< n$ that are relatively prime to n.

Remark. Recall:

- $z^n = 1$ *if and only if* $|z|$ divides n.
- $|z^m| = \frac{|z|}{\gcd(m,|z|)}$.
- $|z^m| = |z|$ *if and only if* $(m, |z|) = 1$.

Example 30. If $|z| = 10$, then $|z^6| = 5 = \frac{10}{(6,10)}$

Proof. If $z \in K$ is a primitive n-th root of unity, then every element of $\langle z \rangle$ is an n-th root of unity, and so a root of $x^n - 1$; but $|\langle z \rangle| = |z| = n$, so the subgroup $\langle z \rangle$ generated by z must consist of all of the n roots of $x^n - 1$. Furthermore, z^m is also a primitive n-th root of unity *if and only if* $(m, n) = 1$; thus, there are exactly $\varphi(n)$ such elements in $\langle z \rangle$. □

In $\mathbb{C}$, we have $e^{i\frac{2\pi}{n}}$ is a primitive n-th root of unity. Suppose $\zeta_n \in \mathbb{C}$ is a primitive n-th root of unity.

Definition 34. The **cyclotomic polynomial** is given by

$$\Phi_n(x) = \prod_{\substack{0 \leq k < n \\ (n,k)=1}} (x - \zeta_n^k)$$

Properties:

- $x^n - 1 = \prod_{0 \leq k < n} (x - \zeta_n^k) = \prod_{d|n} \prod_{\substack{0 \leq k < n \\ (n,k)=d}} (x - \zeta_n^k) = \prod_{d|n} \Phi_d(x)$
- $\deg(\Phi_n(x)) = \varphi(n)$
- $n = \sum_{d|n} \varphi(d)$

  Example 31. $6 = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(6)$

(The following is kind of on a tangent)

Remark. If K is a finite field, then $K^\times = \langle z \rangle$.

Proof. Since K is finite, it must be an extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime p; so $|K^\times| = p^n - 1$ since $|K| = p^n$. For $d | p^n - 1$, let $\mathcal{O}_d = \{z \in K^\times \mid |z| = d\}$. Observe that $K^\times = \bigcup_{d|p^n-1} \mathcal{O}_d$ is a disjoint union, and so

$$|K^\times = \sum_{d|p^n-1} |\mathcal{O}_d| = p^n - 1 = \sum_{d|p^n-1} \varphi(d)$$

which implies that $|\mathcal{O}_d| = \varphi(d)$ for all $d | p^n - 1$. Hence, in particular, $\mathcal{O}_{p^n-1}$ is nonempty, so any $z \in \mathcal{O}_{p^n-1}$ generates $K^\times$. □

Remark. This implies that the Primitive Element Theorem (see Proposition 45) is also true for finite fields.

(Tangent ends here)

Remark. $\Phi_n(x)$ is irreducible over $\mathbb{Q}$, and so it is the minimal polynomial $m_{\zeta_n,\mathbb{Q}}(x)$.

Lemma 47. Suppose F is a field where all irreducible polynomials are separable. Let $p(x) \in F[x]$ irreducible and split completely in a Galois extension K/F, and let α, β be two roots of $p(x)$. Then there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\alpha) = \beta$.

$\leftarrow$
$$\begin{tikzcd}
\sigma : K \arrow[r] & K \\
F(\alpha) \arrow[u, dash, dashed] \arrow[r] & F(\beta) \arrow[u, dash, dashed] \\
\mathrm{id} : F \arrow[u, dash, dashed] \arrow[r] & F \arrow[u, dash, dashed]
\end{tikzcd}$$

Definition 35. The extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is called the n-th **cyclotomic extension** of $\mathbb{Q}$.

Remark. All other primitive n-th roots of unity are of the form ζ_n^a w/ $(a,n) = 1$, so $\mathbb{Q}(\zeta_n)$ is the splitting field of $\Phi_n(x)$ over $\mathbb{Q}$, so the extension is Galois.

If $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, then σ is completely determined by $\sigma(\zeta_n)$. But $\sigma(\zeta_n)$ must be another primitive n-th root of unity, so $\sigma(\zeta_n)$ must be ζ_n^a for some $0 < a < n$ with $(a,n) = 1$.

Moreover, by Lemma 47, each a corresponds to a Galois automorphism σ; in fact, the map $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times$ where $\sigma \mapsto \bar{a}$ is a **group isomorphism**.

Definition 36. A Galois extension K/F is abelian if $\mathrm{Gal}(K/F)$ is abelian.

$\leftarrow$ All finite field extensions are abelian

Corollary 48. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is abelian.

Theorem 49 (Kronecker-Wober)**.** If $K/\mathbb{Q}$ is abelian, then $K \subseteq \mathbb{Q}(\zeta_n)$ for some n.

Radical extensions and soluble groups

Example 32. Suppose K is the splitting field of $x^4 - 2$ over $\mathbb{Q}$. Then $K = \mathbb{Q}(\sqrt[4]{2}, i)$.

$\leftarrow$ since $\zeta_4 = i$

Let $\sigma : \begin{cases} \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ i \mapsto i \end{cases}$ and $\sigma : \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto -i \end{cases}$.

$\leftarrow$ (not showing all intermed extns)

$$\begin{tikzcd}
 & K \arrow[dl, "\langle\sigma\rangle"'] \arrow[dr] & \\
\mathbb{Q}(i) \arrow[dr] & & \mathbb{Q}(\sqrt[4]{2}) \arrow[dl] \\
 & \mathbb{Q} &
\end{tikzcd}$$

We look at the subgroup generated by σ, τ:

$$\langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle \cong D_8$$

Since $[K/\mathbb{Q}] = 8$, we have found the Galois group of $K/\mathbb{Q}$.