MATH172 Galois Theory Notes

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Contents

Rings! Or why $x^2 - 2$ has roots.

Definition 1. A **ring** is a set R together with associative binary *operations* $+$ and $\times$ s.t.:

- $(R, +)$ is an **abelian** group with identity 0
- There exists $1 \in R$ s.t. $r \times 1 = 1 \times r = r$

- $r(s + t) = rs + rt$ and $(s + t)r = sr + tr$ $\qquad \forall s, r, t \in R$

Proposition 1. $0 \times 1 = 0$ (in fact, $0 \times r = 0 \; \forall \, r \in R$)

Proof. Try it! $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Definition 2. If $\times$ is commutative, then R is a commutative ring.

Non-example 1. $\mathbb{N}$ is not a ring.

Example 2. $\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$ are all rings;

- $\mathbb{Z}/n\mathbb{Z}$ is a finite ring
- $M_n(\mathbb{R})$, the set of $n \times n$ real matrices, is a **noncommutative** ring
- Polynomial ring: $\mathbb{Q}[x] = \{a_0 + a_1 x + \cdots + a_n x^n \mid a_i \in \mathbb{Q}\}$ is a commutative ring

- $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$ is a commutative ring

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Phase I plan:
$$ID \supsetneq UFD \supsetneq PID \supsetneq ED \supsetneq Fields$$

Definition 3. Suppose R is a ring and $a, b \in R$ with $ab = 0$ but $a, b \neq 0$; then a, b are called **zero divisors**.

Example 3.

- In $\mathbb{Z}/6\mathbb{Z}$, $\bar{4} \times \bar{3} = \bar{0}$
- In $M_2(\mathbb{R})$, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Definition 4. A commutative ring without zero divisors is called an integral domain (ID)

Why do we want ID? **Cancellation properties**.

- If R is an ID, $a, b, c \in R$, $a \neq 0$ and $ab = ac$, then

$$ab - ac = 0 \implies a(b - c) = 0 \implies b - c = 0 \implies b = c$$

Definition 5. Suppose R is an ID. An element $a \in R$ is called a **unit** if $a \neq 0$ and there exists $b \in R$ s.t. $ab = 1$.

← notation: $b = a^{-1}$

An element $r \in R$ is called **irreducible** if $r \neq 0$, r is NOT a unit, and whenever $r = ab$ for some $a, b \in R$ then a or b must be a unit.

- If r and s are irreducibles with $r = us$, then r and s are called **associates**.

Example 4.

- All "prime integers" are irreducibles in $\mathbb{Z}$;
- $2, 3, 1 + \sqrt{-5}, 1 - \sqrt{-5}$ are irreducibles in $\mathbb{Z}[\sqrt{-5}]$.
  - Note: $2 \times 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 6$ says that 6 can be factored in more than one way. This means that $\mathbb{Z}[\sqrt{-5}]$ is NOT an UFD.

Definition 6. An integral domain R is called a <u>unique factorization domain</u> (UFD) if each nonzero, nonunit $a \in R$ can be written as a product of irreducibles **in a unique way** up to associates.

If a is a nonzero, nonunit element of UFD R and $a = r_1 r_2 \ldots r_m = s_1 \ldots s_n$ where r_i, s_j are irreducible, then after reordering $r_i = u_i s_i$ for any i and units u_i, and $m = n$.

← After reordering, there are the same amounts of factors and all factors are the same up to units.

Definition 7. Suppose R is a comm ring. A subset $I \subseteq R$ is called an **ideal** if $(I, +) \leq (R, +)$ and $ir, ri \in I$ for all $i \in I$ and for all $\boxed{r \in R}$.

Why do we want ideals? Such that R/I is a well-defined ring.

Example 5. $\{0\}$ and R are ideals of R.

Example 6. If R is commutative and $a \in R$, then $(a) = \{ar \mid r \in R\}$ is called the **principal ideal** generated by a.

← Prove this (be convinced)! Also known as aR.

Definition 8. A **principal ideal domain** is an integral domain where all ideals are principal ideals.

Example 7. The only ideals of $(\mathbb{Z}, +)$ are of the form $n\mathbb{Z} = (n)$.

← Ideals generated by n

Non-example 8. $\mathbb{Z}[x]$ is a UFD but NOT a PID because the ideal $(2, x) = \{2r + xs \mid r, s \in \mathbb{Z}[x]\}$ is not principal.

← Observe that $(2, x)$ is an ideal made of polynomials with even constant terms. This cannot be principal, since if we only have 2 and not x, we do not have nonzero polynomials with zero const terms.

Lemma 2. If $I \subseteq R$ is an ideal and $1 \in I$, then $I = R$.

Proof. Try it!

Proposition 3. If $I \subseteq R$ is an ideal containing a unit of R then $I = R$.

Proof. If $u \in I$ is a unit then $u^{-1} \in R$, so $uu^{-1} = 1 \in I$. Then the result follows from Lemma 2. $\qquad\square$

Definition 9. A **field** is a commutative ring whose each nonzero element is a *unit*.

Corollary 4. If R is an ID whose ideals are (0) and R, then R is a **field**.

← The converse is also true. **The only ideals in a field are 0 and the field.**

Proof. Suppose $a \in R\backslash\{0\}$ and consider (a). Since $a \in (a)$, $(a) = R$. Hence, we must have that $1 \in (a)$, which means $1 = ar$ for some $r \in R$. $\qquad\square$

Definition 10. Suppose R is an integral domain. A *proper* ideal $P \subsetneq R$ is called **prime** of whenever $ab \in P$ for some $a, b \in R$, then a or $b \in P$.

Non-example 9. (6) is not a prime ideal of $\mathbb{Z}$ since $2 \times 3 \in (6)$ but neither $2, 3 \notin (6)$.

Non-example 10. (2) is not a prime ideal of $\mathbb{Z}[\sqrt{-5}]$ since $6 \in (2)$, but we observe that $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ while $1 \pm \sqrt{-5} \notin (2)$.

← Observe that in $\mathbb{Z}[\sqrt{-5}]$, we have $6 = (1 + \sqrt{-5})(1 - \sqrt{-5}) = 2 \times 3$, so it is not a UFD!

Example 11. (2) is a prime ideal of $\mathbb{Z}$.

Definition 11. A proper ideal $M \subsetneq R$ is called **maximal** if whenever $I \subseteq R$ such that $M \subseteq I \subseteq R$ is an ideal containing M, then either $I = M$ or $I = R$.

Proposition 5. Every proper ideal is contained in **a** maximal ideal.

← This might not be unique in non-local rings.

Proof. Axiom of choice. $\qquad\square$

Proposition 6. Suppose R is a commutative ring.

- (0) is prime *if and only if* R is an integral domain.
- (0) is maximal *if and only if* R is a field.

← By def of prime, if $ab = 0$, then either $a = 0$ or $b = 0$, which means there are NO zero divisors.

(The following is kind of on a tangent)

Definition 12. A commutative ring R with unity is called **Noetherian** if, whenever $I_1 \subseteq I_2 \subseteq \ldots$ is an ascending sequence of (proper) ideals of R, there exists an $n > 0$ such that $I_n = I_{n+1} = \ldots$ are the same ideals thereafter.

← The chain stops ascending!

Theorem 7. R is Noetherian *if and only if* all ideals of R are finitely generated.

Corollary 8. All Principal Ideal Domains are Noetherian.

← Since all ideals are generated by 1 elt.

(Tangent ends here)

Definition 13. Suppose R is a commutative ring with $1 \neq 0$ and $I \subseteq R$ is an ideal. Then the **quotient ring** of R by I is the set

$$R/I = \{r + I \mid r \in R\}$$

with addition and multiplication defined representative-wise.

Remark. The **coset criterion** of ideals: let I be an ideal; the cosets $r + I$, $s + I$ are the same *if and only if $r - s \in I$*.

Example 12.

- In $\mathbb{Z}/(6)$ aka. $\mathbb{Z}/6\mathbb{Z}$, we have $2 + (6) = \{\dots, -10, -4, 2, 8, 14, \dots\} = 26 + (6)$ due to $2 - 26 \in (6)$;

- In $\mathbb{Q}[x]/(x^2 - 2)$, we have

$$\{3x^2 - 47x + 1 + q(x)(x^2 - 2) \mid q(x) \in \mathbb{Q}[x]\} = \{-47x + 7 + q(x)(x^2 - 2) \mid q(x) \in \mathbb{Q}[x]\}$$

    due to $3x^2 - 47x + 1 - (-47x + 7) \in (x^2 - 2)$.

Remark. Let I be an ideal of R. Then $\underline{(I, +) \trianglelefteq (R, +)}$.

Definition 14. R/I is a group under $(r + I) + (s + I) = (r + s) + I$ and the operation $+$ is well-defined. We also define that $(r + I)(s + I) = (rs) + I$. We claim that multiplication in R/I is also well-defined.

Proof. Let $r_1 + I = r_2 + I$ and $s_1 + I = s_2 + I$. By coset criterion, $r_1 - r_2 = i$, $s_1 - s_2 = j$ for some $i, j \in I$. Hence $r_1 s_1 = (r_2 + i)(s_2 + j) = r_2 s_2 + i s_2 + j r_2 + ij$ where the latter three terms are all in the ideal I. Thus, $(r_1 s_1) + I = (r_2 s_2) + I$. $\qquad\square$

From R, R/I inherits nice properties:

- $0 + I = 0_{R/I}$
- $1 + I = 1_{R/I}$
- Multiplication is commutative and distributive over addition in R/I, so it is also a comm. ring with identity.

Definition 15. A function $\varphi : R \to S$ between rings is called a **ring homomorphism** if the following are satisfied:

- $\varphi(r_1 + r_2) = \varphi(r_1) + \varphi(r_2)$
- $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$

Theorem 9. First ring isomorphism theorem

If $\varphi : R \mapsto S$ is a ring homomorphism, then $R/\ker(\varphi) \cong \varphi(R)$.

← Observe that kernels are ideals! And ideals are kernels of some homomorphism too.

Example 13. If R is a ring and I is an ideal, then $\pi : R \to R/I$ where $r \mapsto r + I$ is a surjective homomorphism where $\ker(\pi) = I$. This is the *canonical projection* onto R/I.

Corollary 10. If I is a maximal ideal, then R/I is a field.

← The if and only if version comes in Proposition 14.

Recall Proposition 6. We now have a stronger statement:

Proposition 11. Suppose R is a commutative ring & $P \subseteq R$ is an ideal. Then R/P is an integral domain *if and only if* P is prime.

Proof. R/P is an integral domain *if and only if* whenever $(a+P)(b+P) = 0_{R/P}$ then one of $a + P$ or $b + P$ must already be $0_{R/P}$. This happens *if and only if* whenever $ab + P = P$ then $a + P$ or $b + P$ in P, which happens *if and only if* whenever $ab \in P$ then one of $a, b \in P$, which is the definition of a prime ideal. $\square$

Example 14. The map $\varphi : \mathbb{Z}[x] \to \mathbb{Z}$ where $p(x) \mapsto p(0)$ is a surjective ring homomorphism with $\ker(\varphi) = (x)$. By the the First Isomorphism Theorem 9, $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$. As such, we conclude that (x) is a prime ideal since $\mathbb{Z}$ is an integral domain.

← btw, $(x) \subsetneq (x, 2)$. the latter is the set of polynomials whose <u>constant term is even</u>, so it is also a proper ideal of $\mathbb{Z}[x]$. This is an excellent example where Prime $\not\Rightarrow$ Maximal.

Lemma 12. Suppose R is a comm. ring with $M \subseteq R$ being an ideal.
There is a bijective correspondence between the ideals of R/M and the ideals of R containing M.

Proof. Consider the projection $\pi : R \to R/M$ where $r \mapsto r + M$. It is enough to show:

← To see why this is okay, see Homework 2 Sec. 7.3 P. 24

$$\pi(\pi^{-1}(J)) = J \qquad \text{for all ideals } J \subseteq R/M, \text{ and}$$
$$\pi^{-1}(\pi(I)) = I \qquad \text{for all ideals } M \subseteq I \subseteq R$$

To prove the first statement, observe that, if J is an ideal of R/M, then $\pi^{-1}(J) = \{r \in R \mid r + M \in J\}$ and so

$$\pi(\pi^{-1}(J)) = \{\pi(r) \in R \mid r + M \in J\} = \{r + M \mid r + M \in J\} = J$$

Next, to prove the second statement, first suppose $M \subseteq I \subseteq R$ is an ideal. Let $a \in I$. Then $a + M \in \{\alpha + M \mid \alpha \in I\} = \pi(I)$. This implies that $a \in \pi^{-1}(\pi(I))$, and so $I \subseteq \pi^{-1}(\pi(I))$.

Conversely, suppose $r \in \pi^{-1}(\pi(I))$. This is the same as saying $\pi(r) = r + M \in \pi(I) = \{\alpha + M \mid \alpha \in I\}$. Hence, for any $r \in \pi^{-1}(\pi(I))$, there exists some $a \in I$ such that $r + M = a + M$. Thus, $r - a \in M \subseteq I$ by coset conditions. Since $a \in I$, we have $a + (r - a) \in I$, meaning that $r \in I$ for any $r \in \pi^{-1}(\pi(I))$. This means that $\underline{\pi^{-1}(\pi(I)) \subseteq I}$.

Hence, $\underline{I = \pi^{-1}(\pi(I))}$.

Consequently, for any ideals $J \subseteq R/M$, we know that $\pi^{-1}(J) \subseteq R$ is an ideal containing M. And if $M \subseteq I \subseteq R$ is an ideal, we know $\pi(I) \subseteq R/M$ is an ideal. Since $\pi(\pi^{-1}(J)) = J$ and $I = \pi^{-1}(\pi(I))$ for any I, J, the correspondence is a bijection. $\qquad\square$

← Think about why this contains M!

Proposition 13. Suppose R is a comm. ring with an identity and $I \subseteq R$ is an ideal. Then R/I is a field *if and only if* I is maximal.

Proof. If I is maximal, then there are no other proper ideals strictly containing I. Hence, by Lemma 14, we have that R/I only have ideals (0) and R/I itself. This happens *if and only if* R/I is a field. $\qquad\square$

Corollary 14. If R is a commutative ring with identity and $M \subseteq R$ is maximal, then M is prime.

← Hence <u>maximal implies prime</u>, but prime does not necessarily implies maximal.

Proof. Maximal $\implies$ quotient is a field $\implies$ quotient is an ID $\implies$ prime. $\quad\square$

Definition 16. An integral domain R is an **Euclidean domain** if there exists a norm $N : R \to \mathbb{Z}_{\geq 0}$ with $N(0) = 0$ such that for all $a, b \in R$ with $b \neq 0$, there exists $q, r \in R$ for which

$$a = bq + r$$

with $N(r) < N(b)$ or $r = 0$.

Example 15. $\mathbb{Z}$ is a ED with $N(a) = |a|$.

Example 16. $\mathbb{Q}[x]$ is a ED with $N(p(x)) = \deg(p(x))$.

Example 17. Every field F is a ED with $N(a) = 0 \; \forall \, a \in F$.

← Because in a field everything divides!

Non-example 18. $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a PID that is not an ED.

← This is one of the only good examples!

Why do we care about Euclidean domains?

Remark. <u>Greatest common divisors</u> exist and are relatively quick to compute.

← Using recursive application of Euclidean algorithm.

Definition 17. If $a, b \in R$, then $\gcd(a, b) = c$ means

1. c divides a and b; that is, $a = cr, b = cs$ for some $r, s \in R$

2. If $c' \in R$ with $c'|a$ and $c'|b$, then it must be true that $c'|c$.

Example 19. Say we want to compute the gcd of 47 and 10.

$$47 = 4 \times 10 + 7$$
$$10 = 1 \times 7 + 3$$
$$7 = 2 \times 3 + \boxed{1} \qquad\qquad \leftarrow \text{circled is } \gcd(47, 10)$$
$$3 = 3 \times 1 \qquad\qquad\qquad \leftarrow \text{final line with no remainders}$$

← This is a much faster algorithm than factoring!

This also works for finding gcds in $\mathbb{Q}[x]$ with polynomials long division and norm $\deg(p(x))$.

Remark. If F is a field, then $F[x]$ is a Euclidean domain.

← Just use long division!

Remark. Euclidean domains are PIDs.

Proof. Suppose R is a ED and $I \subseteq R$ is an idea;. Consider $\{N(a) \mid a \in I\backslash\{0\}\}$. This set has a <u>minimal element</u> by properties of natural numbers (or is an empty set *if and only if* $I = (0)$).

Let $d \in I$ be an element of <u>minimum norm</u> (hence $N(d) \leq N(a)$ for all $a \in I$). We claim that $\underline{(d) = I}$. Proof:

Since $d \in I$, we have $rd \in I$ for any $r \in R$. This implies that $(d) \subseteq I$.

Then let $a \in I$. Since R is a ED, we first assumes that there exists $q, r \in R$, $r \neq 0$ such that $a = qd + r$ and $N(r) < N(d)$. But we notice that $r = a - qd$ must be in I as both $a, qd \in I$, contradicting the minimality of $N(d)$. Thus, it must be that $r = 0$. This implies $a = qd$ and thus $a \in (d)$ for all $a \in I$. <u>Consequently, $I \in (d)$</u>, and therefore $I = (d)$. $\qquad\square$

Definition 18. Suppose R is an integral domain and $p \in R\backslash\{0\}$. Then p is a **prime element** if (p) is a prime ideal.

Proposition 15. An element $p \in R$ is prime *if and only if* whenever $p|ab$ then $p|a$ or $p|b$.

Proof. p is prime means that (p) is a prime ideal. This is true *if and only if* whenever $ab \in (p)$ then $a \in (p)$ or $b \in (p)$. This is the same as saying if $ab = kp$ for some $k \in R$ then $a = lp$ or $b = lp$ for some $l \in R$. This is to say that whenever $p|ab$ then $p|a$ or $p|b$. $\qquad\square$

Proposition 16. In an integral domain, all prime elenments are irreducibles.

Proof. Suppose R is an ID and $p \in R$ is prime. If $p = ab$ for some a, b in R, then, WLOG, $p|a$. That is, $a = pk$ for some $k \in R$. Hence, $p = pkb$. Since in an ID cancellation rule holds, $kb = 1$, meaning that b is a unit. Thus, p is irreducible by definition Definition 5. □

Proposition 17. In PIDs, all *nonzero* prime ideals are maximal.

Proof. Suppose R is a PID and $(p) \subseteq R$ is a prime ideal. If $(p) \subseteq (m) \subseteq R$ is an ideal, then $p \in (p) \subseteq (m)$ hence $p = rm$ for some $r \in R$. Since $p|rm$, we have $p|r$ or $p|m$.

If $p|r$, this implies that $r = pk$ for some $k \in R$. Substituting into $p = rm$, we get $p = pkm$. By cancellation, we get $km = 1$, meaning that m is a unit. Hence, $(m) = R$.

If $p|m$, we have $m = pl$ for some $l \in R$, meaning that $m \in (p)$. Hence, $(m) \subseteq (p)$, but we also defined that $(p) \subseteq (m)$, so $(m) = (p)$.

Therefore, (p) has to be the maximal ideal. □

Proposition 18. In an UFD, irreducible implies prime.

Proof. Let R be a UFD and $p \in R$ be irreducible. Let $a, b \in R$ such that $p|ab$. Hence, $pr = ab$ for some $r \in R$. Since R is a UFD, let $a = q_1 \ldots q_n, b = s_1 \ldots s_m$ be the factorization. Since the factorizations are unique and each of the q_i, s_j are irreducible, if $p|ab$, then p must be an associate with one of the q_i, s_j. Therefore, either $p|a$ or $p|b$, implying prime. □

Example 20. $\mathbb{Q}$ is a field, so $\mathbb{Q}[x]$ is a ED. Since EDs are UFDs, irreducible $\implies$ prime. We see that $x^2 - 2 \in \mathbb{Q}[x]$ is an irreducible element, which means that $(x^2 - 2)$ is a prime ideal, meaning that it is a maximum ideal, meaning that $\mathbb{Q}[x]/(x^2 - 2)$ is a field. We observe that it is a field containing $\mathbb{Q}$ and $(\sqrt{2})$.

← In fact, this is the smallest field containing $\mathbb{Q}$ and $(\sqrt{2})$.

Lemma 19. In a PID, irreducible elements are prime.

Proof. Suppose $p \in R$ is irreducible in the principal ideal domain R. If $p|ab$ for some $a, b \in R$, we want to show that either $p|a$ or $p|b$, hereby showing that p is prime. Hence, we consider the ideal $(a, p) = d$, which is necessarily principal for some $d \in R$. Since $a, p \in (d)$, we have $a = dr$ and $p = ds$ for some $r, s \in R$. As p is irreducible, we get that one of d and s is a unit.
We first assume that s is a unit, in which case $d = ps^{-1}$, and so $a = ps^{-1}r$ implying that $p|a$.
In another case, d is a unit, in which case $(a, p) = (d) = R$ and so $1 = ak + pl$ for some $k, l \in R$. Multiplying by b, we get $b = abk + pbl$. Since $p|ab$, we have $b = abk + pbl = pmk + pbl$ for some $m \in R$. Hence, $b = p(mk + bl)$, meaning that $p|b$.

Therefore, whenever $p|ab$, either $p|a$ or $p|b$. Hence, in a PID, p is prime whenever it is irreducible. □

Proposition 20. PIDs are UFDs.

Proof. Suppose R is a PID and $a \in R$ is nonzero, nonunit. If a is irreducible, we are done. If not, we write $a = p_1 q_1$ for some $p_1, q_1 \in R$ nonunit. If p_1, q_1 are irreducibles, we are done. If not, then WLOG say $q_1 = p_2 q_2$ for some nonunits p_2, q_2. We would like to show that this splitting process terminates.

Observe that $(q_1) \subseteq (q_2)$ since $q_2|q_1$. Hence, the chain of splitting results in the chain of ideals $(q_1) \subseteq (q_2) \subseteq (q_3) \subseteq \dots$.

Now consider the ideal[1] $\bigcup_{i=1}^{\infty} (q_i)$. Since this is a PID, we have $\bigcup_{i=1}^{\infty} (q_i) = (q)$ for some $q \in R$. Since $q \in \bigcup_{i=1}^{\infty} (q_i)$, it is contained in some (q_n) for some $n \geq 1$. This implies that $(q) \subseteq (q_n)$, but we also know that $(q_n) \subseteq (q)$, hence $(q) = (q_n)$. Hence, this process terminates, and there exists an n in this chain such that q_n is irreducible. Therefore, R is a <u>factorization domain</u>.

Now we want to prove the <u>uniqueness</u>. That is, if $p_1 \dots p_n = q_1 \dots q_m$ for irreducibles p_i, q_j and $n \leq m$ WLOG, then we want to show that $m = n$ and that $p_i = u_i q_i$ with units u_i up to reordering for all i. We do so by induction on n.

    (*Base case*) If $p_1 = q_1 \dots q_m$ and p_1 irreducible, then $q_2 \dots q_m$ are all units. Hence, $m = 1$ and $p_1 = q_1$.

    (*Inductive step*) Say we have already proven the statement for $n = k$. Then consider $p_1 p_2 \dots p_{k+1} = q_1 q_2 \dots q_m$. Since R is a PID where irreducible implies prime, p_1 is a prime element dividing the product of primes $q_1 q_2 \dots q_m$, so we say WLOG $p_1|q_1$. This means that $q_1 = u_1 p_1$ for some $u \in R$, but since q_1 is not reducible, it forces u_1 to be a unit. Hence, we apply cancellation on both sides and get $p_2 \dots p_{k+1} = (u_1 q_2) \dots q_m$.

By inductive hypothesis, $m - 1 = k$ and p_i, q_i are associates up to reordering for any i. Hence, the factorization must be <u>unique</u>. □

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[1]The proof that this is an ideal is as follows:

We first prove that $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition. Let $r, s \in \bigcup_{n=1}^{\infty} I_n$, where $r \in I_k$ and $s \in I_{k+i}$ for some $k, i \in \mathbb{N}$. Since $I_1 \subseteq I_2 \subseteq \dots$ are ideals of R, we know that $r \in I_k$ implies that $r \in I_{k+i}$. Thus, $r - s \in I_{k+i}$ due to I_{k+i} being an ideal. As $I_{k+i} \subseteq \bigcup_{n=1}^{\infty} I_n$, we have $r - s \in \bigcup_{n=1}^{\infty} I_n$, which means that $\bigcup_{n=1}^{\infty} I_n$ is closed under additive inverse. Hence, $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition.

Then, we prove that for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, we would have $tr, rt \in \bigcup_{n=1}^{\infty} I_n$. Since $r \in \bigcup_{n=1}^{\infty} I_n$, it must be true that $r \in I_k$ for some $k \in \mathbb{N}$. Hence, $tr, rt \in I_k$ due to I_k being an ideal. Therefore, $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$.

In conclusion, since $\bigcup_{n=1}^{\infty} I_n$ is a subgroup of R under addition with the property that $tr, rt \in \bigcup_{n=1}^{\infty} I_n$ for any $t \in R, r \in \bigcup_{n=1}^{\infty} I_n$, it is an ideal of R. □

We have shown:

$$ID \quad \supsetneq \quad UFD \quad \supsetneq \quad PID \quad \supsetneq \quad ED \quad \supsetneq \quad Fields$$

Non-eg: $\mathbb{Z}/6\mathbb{Z}$ $\qquad$ $\mathbb{Z}[\sqrt{-5}]$ $\qquad$ $\mathbb{Z}[x]$ $\qquad$ $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ $\qquad$ $\mathbb{Z}$

$$\vdash\!\!\!\!\!\!\underline{\qquad\qquad Maximal \implies Prime \implies Irreducible \qquad\qquad}$$
$$\vdash\!\!\!\!\!\!\underline{\qquad\qquad\qquad Prime \iff Irreducible \qquad\qquad\quad}$$
$$\vdash\!\!\!\!\!\!\underline{\qquad\qquad\qquad\quad Prime \iff Maximal \qquad\qquad}$$
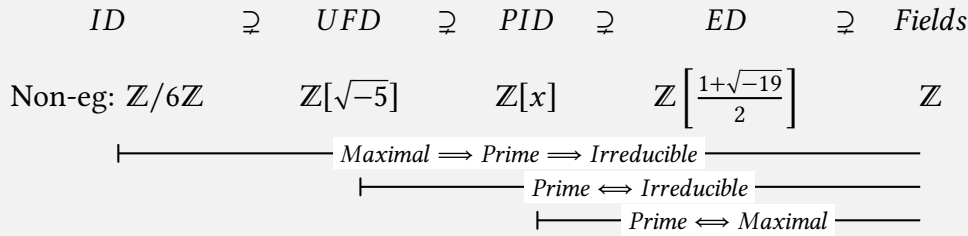
Field extensions

We observe that the polynomial $x^2 - 2 \in \mathbb{Q}[x]$ is irreducible. If we have $x^2 - 2 = p(x)q(x)$ where p, q nonunits, then $\deg(p) + \deg(q) = 2$ and we cannot have any $0+2$ combinations due to constants being units, we only have $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$, but $x \pm \sqrt{2} \notin \mathbb{Q}[x]$!

Since $\mathbb{Q}[x]$ is a UFD, the irreducible element $(x^2 - 2)$ is prime, and since $\mathbb{Q}[x]$ is a PID, $(x^2 - 2)$ is maximal which means that $\mathbb{Q}[x]/(x^2 - 2)$ is a field.

Phase II plan: Field extensions!

Suppose F is a field and $p(x) \in F[x]$ nonzero. Recall that $F[x]$ is a ED with the norm function $\deg(a(x))$ and long division of polynomials. Let $a(x) + (p(x)) \in F[x]/(p(x))$. By the division algorithm, we have $a(x) = p(x)q(x) + r(x)$ for $q(x), r(x) \in F[x]$ and $\deg(r(x)) < \deg(p(x))$ or $r(x)$ is the zero polynomial.

← $a(x)$ is a coset rep

← We can do division algorithm since this is an ED

Now we see that since $a(x) - r(x) \in (p(x))$, they are in the same coset! Hence $a(x) + (p(x)) = r(x) + (p(x))$. We observe that <u>every element of $F[x]/(p(x))$ can be represented by a polynomial of a degree less than $\deg(p(x))$</u>. In other words, if $\deg(p(x)) = n$, then $F[x]/(p(x))$ is of the form

← The expression under the bar functions like $r(x)$! Also note that span is just the set of linear combinations.

$$F[x]/(p(x)) = \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}} \mid a_0, a_1, \ldots, a_{n-1} \in F \right\}$$
$$= \mathrm{Span}_F\{\bar{1}, \bar{x}, \ldots, \overline{x^{n-1}}\}$$

In fact, $F[x]/(p(x))$ is (partly) just a **vector space** over F...

We shall observe that it does not matter if we are using F or $\bar{F}$.
Consider $\varphi : F \hookrightarrow F[x]/(p(x))$ where $a \mapsto \bar{a}$. We observe this is an **injective** map: whenever $\deg(p(x)) = n > 0$, we have $\varphi(a) = \varphi(b)$ *if and only if* $\bar{a} = \bar{b}$, which happens *if and only if* $a - b \in (p(x))$; but the difference of two constants always have deg 0 and cannot be in $(p(x))$ unless it is a straight zero, which tells us that $\bar{a} = \bar{b}$ *if and only if* $a = b$. In other words, $F[x]/(p(x))$ contains an isomorphic copy of F, its field of scalars! Namely, $\boxed{F \cong \varphi(F) = \{\bar{a} \in F[x]/(p(x)) \mid a \in F\}}$.

...Hence, $F[x]/(p(x))$ is a vector space **of dimension** n over the scalar field F that also contains an isomorpic copy of F.

Moreover, if $p(x)$ is irreducible, then $(p(x))$ is prime since this is an ED, and hence, it is also a maximal ideal, meaning that $F[x]/(p(x))$ is a field containing an isomorphic copy of F.

Definition 19. Suppose $F \subseteq K$ are fields. Then K is called a **field extension** of F.

- Notation: K/F or $\begin{smallmatrix} K \\ | \\ F \end{smallmatrix}$ (*the lattice notation*)

The dimension of K as a vector space over F is called the **degree** of the extension.

- Notation: $[K : F]$

But does my field F always have an extension? Here is a systematic way to get extensions:

Example 21. If $p(x) \in F[x]$ is an irreducible polynomial of degree $n \geq 1$ over the field F, then $F[x]/(p(x))$ is a **field extension** of F of degree n.
Furthermore, if $p(x) = a_0 + a_1 x + \cdots + a_n x^n$, then $\bar{x}$ is a **root** of

$$\varphi(p(x)) = \bar{a}_0 + \bar{a}_1 y + \cdots + \bar{a}_n y^n \in (F[x]/(p(x)))[y]$$

because, plugging in $y = \bar{x}$, we get

$$\bar{a}_0 + \bar{a}_1 \bar{x} + \cdots + \bar{a}_n \bar{x}^n = \overline{p(x)} = \bar{0} \in F[x]/(p(x))$$

Hence, the isomorphic copy of the polynomial $p(x)$ has **roots** in the field extension $F[x]/(p(x))$.

So, what the hell is $F[x]/(p(x))$? We have already shown that the field extension $F[x]/(p(x))$ does indeed contain a root of $p(x)$. Now we think about it **the other way around**: if we want to find an extension of F that contains a root of $p(x)$, we would eventually get this one!

Margin notes:

← Why is this not the vector space over $F/(p(x))$ but just F? See the next paragraph.

← all thanks to Euclidean domains!

← Please, this is NOT a quotient. DO NOT CONFUSE THOSE!!

← Since $\varphi(F) \cong F$, and $\varphi(F) \subseteq F[x]/(p(x))$

← We think about modding out by $(p(x))$ as making it equal to zero, which is how we find roots.

Suppose $p(x) \in F[x]$ is irreducible. Let K/F be an extension, and $\alpha \in K$ a root of $p(x)$. Denote by $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α. Consider the map $\varphi : F[x] \to F(\alpha) \subseteq K$ where $q(x) \mapsto q(\alpha)$ is simply the evaluation at α map. We note that $p(x) \in \ker(\varphi) = (d(x))$ since an ED is a PID; this implies that $p(x) = u(x)d(x)$. As $p(x)$ is irreducible, $u(x)$ must be a unit, which means $p(x)$ and $d(x)$ are associates and $\ker(\varphi) = (p(x))$. Therefore,

$$F[x]/(p(x)) = F[x]/\ker(\varphi) \cong \varphi(F[x]) \subseteq F(\alpha)$$

by first isomorphism theorem. However, $F(\alpha) \subseteq K$ the **smallest** subfield of K that contains both F and α, so $\varphi(F[x])$ cannot be smaller than that. Hence, it must be true that $\varphi(F[x]) = F(\alpha)$.

$\leftarrow$ Observe that $\varphi(F[x])$ is a field: $\ker(\varphi)$ is a maximal ideal

Therefore, $\underline{F(\alpha) \text{ is simply } F[x]/(p(x)).}$ $\square$

To summarize so far!

Suppose $p(x) \in F[x]$ is an irreducible polynomial with coefficients in the field F.

- $F[x]/p(x)$ is a **field** containing an isomorphic copy of F in which $\overline{x} = x + (p(x))$ is a **root** of (the image of) $p(y) \in (F[x]/(p(x)))[y]$.

  Example 22. In $\mathbb{Q}[x]/(x^2 - 2)$, we have $x + (x^2 - 2)$ is a root of $y^2 - \overline{2} \in (\mathbb{Q}[x]/(x^2 - 2))[y]$ because

$$
\begin{aligned}
&(x + (x^2 - 2))^2 - (2 + (x^2 - 2)) \\
&= x^2 - 2 + (x^2 - 2) && \text{by coset addition \& multiplication} \\
&= 0 + (x^2 - 2) && \text{since } x^2 - 2 \in (x^2 - 2) \\
&= \overline{0}
\end{aligned}
$$

  Furthermore, if $\deg(p(x)) = n$, then

$$F[x]/(p(x)) = \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}} \mid a_0, a_1, \ldots, a_{n-1} \in F \right\}$$

  is a vector space over F of dimension n.

  Example 23. $\mathbb{Q}[x]/(x^2 - 2) = \{\overline{a}_0 + \overline{a}_1 \overline{x} \mid a_0, a_1 \in \mathbb{Q}\} = \mathrm{Span}_{\mathbb{Q}}\{\overline{1}, \overline{x}\}$

- If K/F is an extension and $\alpha \in K$ is a root of $p(x)$, denote by $F(\alpha)$ **the smallest field containing** F and α.

$\leftarrow$ Read 'F adjoint α'

$$
\begin{array}{c}
K \\
| \\
F(\alpha) \\
| \\
F
\end{array}
$$

Figure 1: Field diagram

Then $F(\alpha) \cong F[x]/(p(x))$, and

$$F(\alpha) = \left\{ \overline{a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}} \mid a_0, a_1, \ldots, a_{n-1} \in F \right\}$$
$$= F[\alpha] \qquad \leftarrow \text{ the polynomial of } \alpha \text{ over } F$$

Example 24. $\mathbb{Q}(\sqrt{2}) = \{a_0 + a_1\sqrt{2} \mid a_0, a_1 \in \mathbb{Q}\} = \mathbb{Q}[\sqrt{2}]$

$\leftarrow$ The eval map
$\varphi : F[x] \to F(\alpha)$
where $f(x) \mapsto f(\alpha)$
has in fact
$\ker(\varphi) = (p(x))$
when α is a root of
$p(x)$.

Irreducibility – a survey

Proposition 21. If $p(x) \in F[x]$, then $\alpha \in F$ is a root *if and only if* $x - \alpha$ divides $p(x)$.

Proof. Write $p(x) = (x - \alpha)q(x) + r(x)$ with $q(x), r(x) \in F[x]$ and $\deg(r(x)) = 0$ or $r(x) = 0$. Then $0 = p(\alpha) = 0 + r(\alpha)$ which forces $r(x) = 0$. □

Corollary 22. A degree-2 or -3 polynomial over a field F is irreducible *if and only if* it has no roots in F.

Proposition 23. Suppose $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ with root $\frac{c}{d}$ written in reduced from (i.e. $\gcd(c, d) = 1$). Then $\boxed{c|a_0 \text{ and } d|a_n}$.

Proof.

$$d^n \cdot p\left(\frac{c}{d}\right) = 0$$
$$0 = (a_0 d^n + a_1 d^{n-1} c + \cdots + a_{n-1} dc^{n-1}) + a_n c^n$$
$$0 = a_0 d^n + (a_1 d^{n-1} c + \cdots + a_{n-1} dc^{n-1} + a_n c^n)$$

Looking at the 2nd line, since d divides all of the ones in the (), it must also divide the last term $a_n c^n$. However, since $\gcd(c, d) = 1$, it forces d to divide a_n.

Similarly, we make the same argument for c and a_0 using the 3rd line. □

Lemma 24. $(R/I)[x] \cong R[x]/(I)$ where $(I) = I[x]$.

Proof. Consider the surjective homomorphism $\pi : R[x] \to (R/I)[x]$. □

Proposition 25 (Eisenstein's Criterion)**.** Suppose $f(x) = \mathbb{1}x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ is a monic polynomial and $p \in \mathbb{Z}$ is a **prime** such that $p \mid a_0, \ldots, a_{n-1}$ but $p^2 \nmid a_0$. Then $\underline{f(x) \text{ is irreducible}}$.

Proof. Assume BWOC that $f(x) = a(x)b(x)$ for some nonunit $a(x), b(x) \in \mathbb{Z}[x]$, then

$$x^n = \bar{f}(x) = \bar{a}(x)\bar{b}(x)$$

in $(\mathbb{Z}/p\mathbb{Z})[x] \cong \mathbb{Z}[x]/p\mathbb{Z}[x]$ since all other terms are divisible by p. Since $\mathbb{Z}/p\mathbb{Z}$ does not contain any zero divisors, $\bar{a}(x), \bar{b}(x)$ must have zero constant terms. Hence $a(x), b(x)$ have constant terms that are multiples of p, so $a(x)b(x)$ have constant term divisible by p^2. This is a contradiction with $p^2 \nmid a_0$. $\square$

Lemma 26 (Gauss' Lemma). *If $p(x) \in \mathbb{Z}[x]$ is reducible in $\mathbb{Q}[x]$, then it is reducible in $\mathbb{Z}[x]$.*

Proof. Suppose $p(x) = a(x)b(x)$ for $a(x), b(x) \in \mathbb{Q}[x]$. Then by multiplying by coefficient denominators, for some $m \in \mathbb{Z}$, we could write $m \cdot p(x) = c(x)d(x)$ for some $c(x), d(x) \in \mathbb{Z}[x]$. Now since $m \in \mathbb{Z}$, we could write $m = q_1 q_2 \ldots q_n$ be a product of irreducibles in $\mathbb{Z}$.

Now in $(\mathbb{Z}/q_1\mathbb{Z})[x] \cong \mathbb{Z}[x]/(q_1\mathbb{Z})[x]$, we observe that $m \cdot p(x) = c(x)d(x) = q_1(q_2 \ldots q_n)p(x)$, meaning that

$$\overline{c(x)}\,\overline{d(x)} = \overline{q_1(q_2 \ldots q_n)p(x)} = \bar{0}$$

Since $(\mathbb{Z}/q_1\mathbb{Z})[x] \cong \mathbb{Z}[x]/(q_1\mathbb{Z})[x]$ is an <u>integral domain</u>, WLOG, $\overline{c(x)} = \bar{0}$ *if and only if* $c(x) \in q_1\mathbb{Z}[x]$, meaning that all coefficients of $c(x)$ are multiples or q_1. Therefore, $\dfrac{1}{q_1}c(x) \in \mathbb{Z}[x]$. $\qquad\qquad$ ← since q_1 is irreducible and hence prime in UFD

Now we repeat the process for all $q_1, q_2, \ldots, q_n$ and we are done. $\square$

Recall that if $F \subseteq K$ are fields, $\alpha \in K$ and $p(x) \in F[x]$ is irreducible with root α, then

$$F[\alpha] = F(\alpha) \cong F[x]/(p(x)) = \overline{\{a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} \mid a_0, a_1 \ldots, a_{n-1} \in F\}}$$

We observe that this has a few implications. For instance, $F(\alpha)$ contains $\dfrac{1}{\alpha}$, meaning that it could also be written as a polynomial of α with coefficients in F (as in $F[\alpha]$)! $\qquad\qquad$ ← since it is a field containing the mult. inverse of α

Definition 20. Suppose K/F is a field extension and $\alpha \in K$. We say that α is **algebraic over** F if there exists $p(x) \in F[x]$ such that $p(\alpha) = 0$. If not, α is **transcendental**.

Definition 21. The extension K/F is an **algebraic extension** if **every** element $\alpha \in K$ is algebraic over F.

Example 25. π is transcendental over $\mathbb{Q}$ but algebraic over $\mathbb{R}$ (since it is a root of $x - \pi$).

Proposition 27. If K/F is a **finite extension** , then it is an algebraic extension.

$\leftarrow$ finite extension just means finite degree $[K : F] < \infty$

Proof. Call $[K : F] = n$ and let $\alpha \in K$. Then the $n + 1$ elements $\{1, \alpha, \alpha^2, \dots, \alpha^n\}$ must be <u>linearly dependent</u>. Hence, by linear algebra, there exist $a_0, a_1, \dots, a_n \in F$ not all zero such that the linear combination $a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n = 0$. Hence, α is a root of $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0$. $\qquad\square$

$\leftarrow$ since $n + 1 > \dim(K/F) = n$

Corollary 28. If K/F is an extension and $\alpha \in K$, then α is algebraic over F *if and only if* $[F(\alpha) : F] < \infty$.

Proof.

($\Longleftarrow$) Follows from prop.

($\Longrightarrow$) If α is algebraic, then there exists an irreducible polynomial $p(x)$ with α as a root and of degree $n < \infty$. Then $F(\alpha) \cong F[x]/(p(x))$ is a n-dimensional vector space over F.
Another perspective: $F(\alpha) = \text{Span}_F\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$.

$\leftarrow$ Review proof of $F(\alpha) \cong F[x]/(p(x))$.

$\qquad\square$

Proposition 29. Suppose K/F is an extension & $\alpha \in K$ is algebraic over F. Then there exists a <u>unique, irreducible, and monic</u> polynomial $m_{\alpha,F}(x) \in F[x]$ that has α as a root.

Remark. We observe that m does depend on the base field F; $m_{\sqrt{2},\mathbb{Q}}(x) = x^2 - 2$, but $m_{\sqrt{2},\mathbb{Q}(\sqrt{2})}(x) = x - \sqrt{2}$.

Proof. Since the subset of $F[x]$ *satisfying α is a root* is nonempty, we can pick one with a **minimal degree**. By multiplying by an element of F if necessary, we can assume WLOG this polynomial is **monic**. Call it $m_{\alpha,F}(x)$.

Assume BWOC that m is the product of two other polynomials of lesser degree such that $m_{\alpha,F}(x) = a(x)b(x)$, then we plug in $0 = m_{\alpha,F}(\alpha) = a(\alpha)b(\alpha)$. Since there are no zero divisors in $F[x]$, WLOG $a(\alpha) = 0$, contradicting the minimality of $m_{\alpha,F}$. Hence $m_{\alpha,F}$ is **irreducible**.

$\leftarrow$ So $F(\alpha) \cong F[x]/(m_{\alpha,F}(x))$

Then, BWOC if $p(x) \in F[x]$ with α as a root and is monic and irreducible, there exist $q(x), r(x) \in F[x]$ such that $p(x) = m_{\alpha,F}(x)q(x) + r(x)$ where $\deg(r) < \deg(m_{\alpha,F})$ or $r(x) = 0$. Then, we observe that $p(\alpha) = 0 = m_{\alpha,F}(\alpha)q(m_{\alpha,F}) + r(\alpha) = 0 + r(\alpha)$. Thus, $r(\alpha) = 0$, so $\deg(r) \geq \deg(m_{\alpha,F})$ unless $r(x) = 0$ by minimality. Hence we must have $r(x) = 0$, so $m_{\alpha,F}|p$. This contradicts the assumption that p is monic and irreducible. Therefore, $m_{\alpha,F}$ is the **only** minimal, monic and irreducible polynomial where α is a root. $\qquad\square$

Definition 22. $m_{\alpha,F}(x)$ is the **minimal** polynomial of α over F.

(The following is kind of on a tangent)

Some exam prep!

- In general, for subrings $R \subseteq S$, we have if $r \in R^\times$, then $r \in S^\times$.
- If we adjoint one root of an irreducible polynomial to a field, the fields are isomorphic no matter which root of that polynomial we adjoint.

(Tangent ends here)

To summarize, if K/F is a field extension and $\alpha \in K$, then α is **algebraic** over F if it is the root of some polynomials in $F[x]$. For each algebraic α, there exists a unique, monic, irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ such that $m(\alpha) = 0$. In that case, the degree of extension $[F(\alpha) : F] = \deg(m_{\alpha,F}(x))$; and, if $p(\alpha) = 0$ for some $p(x) \in F[x]$, then $m_{\alpha,F}|p(x)$. In general, if $[K : F] < \infty$, then K/F is algebraic. Thus, $[F(\alpha) : F] < \infty$ *if and only if* α is algebraic over F.

Proposition 30. If $F \subseteq K \subseteq L$ are fields, then

$$[L : F] = [L : K] \cdot [K : F]$$

$$\leftarrow\ \ _{mn}\left(\begin{array}{c} L \\ |\,n \\ K \\ |\,m \\ F \end{array}\right.$$

Proof. We first see that if $[K : F] = \infty$, then for any $N \in \mathbb{N}$, there exists $\alpha_1, \ldots, \alpha_N \in K$ that are linearly independent over F. In that case, it is certainly true that $\alpha_1, \ldots, \alpha_N \in L$ are linearly independent over F. Thus, $[L : F] = \infty$.

$\leftarrow$ Linear independence implies that whenever $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_N\alpha_N = 0$ for some coefficients $a_1, \ldots, a_N \in F$, then necessarily $a_1 = a_2 = \cdots = a_N = 0$.

If $[L : K] = \infty$, then for any $N \in \mathbb{N}$, there exists $\beta_1, \ldots, \beta_N \in L$ that are linearly independent over K. As a result, it also is linearly independent over F. Hence, $[L : F] = \infty$.

If $[K : F] = m$ and $[L : K] = n$, let $\alpha_1, \ldots, \alpha_m \in K$ be a basis for K over F and $\beta_1, \ldots, \beta_n \in L$ be a basis for L over K.

Claim: $\{\alpha_i\beta_j \mid 1 \le i \le m, 1 \le j \le n\}$ forms a basis for L over F. $\square$

$\leftarrow$ Use linear combinations to prove this claim.

Some nice consequences:

Corollary 31. Suppose K/F is an extension and $\alpha, \beta \in K$ are algebraic over F. Then:

- $F(\alpha, \beta) = (F(\alpha)(\beta))$

  $\leftarrow$ the smallest subfield of K containing F, α, β

- $[F(\alpha, \beta) : F] = [F(\alpha, \beta) : F(\beta)][F(\beta) : F] = \deg\left(m_{\alpha, F(\beta)}(x)\right)\cdot\deg\left(m_{\beta, F(x)}\right)$.
  However, note that the minimal polynomial

$$m_{\alpha, F(\beta)}(x) \mid m_{\alpha, F}(x) \in F(\beta)[x]$$

  $\leftarrow$ since $p(\alpha) = 0 \iff m_{\alpha, F}(x) \mid p(x)$

  so $\deg(m_{\alpha, F(\beta)}(x)) \le \deg(m_{\alpha, F}(x))$. Hence,

$$[F(\alpha, \beta) : F] \le \deg(m_{\alpha, F}(x)) \deg(m_{\beta, F}) < \infty$$

  This means that whenever α, β are algebraic over F, we get that $F(\alpha, \beta)/F$ is an algebraic extension.

- As a result, $\alpha \pm \beta, \alpha\beta, \alpha/\beta$ are all algebraic over F. The algebraic elements hence form a **field**.

Proposition 32. Suppose K/F is an extension. Then $[K : F] < \infty$ *if and only if* $K = F(\alpha_1, \ldots, \alpha_n)$ could be written where $\alpha_1, \ldots, \alpha_n \in K$ are algebraic over F.

In other words, an extension is finite *if and only if* it is generated by adjoining a finite amount of algebraic elements.

Proof.

$(\implies)$ If $[K : F] < \infty$, then suppose $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of K over F. Then $\alpha_1, \ldots, \alpha_n$ are algebraic and every element of K is an F-linear combination of α_is. Hence K must be the smallest field containing F and α_is, which means $K = F(\alpha_1, \ldots, \alpha_n)$.

$(\impliedby)$ We observe that

$$[K : F] = [(F(\alpha_1, \ldots, \alpha_{n-1}))(\alpha_n) : F(\alpha_1, \ldots, \alpha_{n-1})] \cdot \ldots \cdot [F(\alpha_1) : F]$$

$$\le \prod_{i=1}^{n} \deg(m_{\alpha_i, F}(x)) < \infty$$

$\square$

Corollary 33. If L/K and K/F are algebraic extensions, then so is L/F.
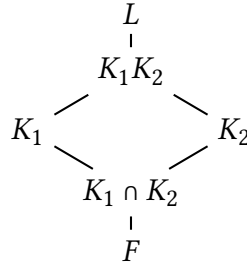
$\leftarrow$ L/K and K/F need not be finite!

Proof. Suppose $\alpha \in L$. Since L/K is algebraic, there exists $p(x) \in K[x]$ such that $p(\alpha) = 0$. Let $\alpha_0, \ldots, \alpha_n \in K$ be the coefficients of $p(x)$, necessarily algebraic over F since K/F algebraic. Therefore,

$$[F(a_0, \ldots, a_n, \alpha) : F] = [(F(a_0, \ldots, a_n))(\alpha) : F(a_0, \ldots, a_n)][F(a_0, \ldots, a_n) : F]$$

Since $p(\alpha) = 0$ has coefficients in $K \supseteq F(a_0, \ldots, a_n)$, we have $[(F(a_0, \ldots, a_n))(\alpha) : F(a_0, \ldots, a_n)] < \infty$. The second term is also clearly $< \infty$. Therefore, $[F(a_0, \ldots, a_n, \alpha) : F] < \infty$, meaning that α is algebraic over F. $\qquad\square$

Definition 23. Suppose L/F is an extension & K_1 and K_2 are intermediate fields. The **composite** field $K_1 K_2$ is the smallest subfield of L containing K_1 and K_2.

$$
\begin{array}{c}
L \\
\mid \\
K_1 K_2 \\
\diagup \quad \diagdown \\
K_1 \qquad K_2 \\
\diagdown \quad \diagup \\
K_1 \cap K_2 \\
\mid \\
F
\end{array}
$$

$\leftarrow$

$$
\begin{array}{c}
L \\
\diagdown \\
\mid \qquad (F(a_0, \ldots, a_n))(\alpha) \\
K \qquad \mid \\
\diagdown \\
\mid \qquad F(a_0, \ldots, a_n) \\
F \diagup
\end{array}
$$

Definition 24. Suppose F is a field and $p(x) \in F[x]$. The **splitting field** of $p(x)$ over F is the smallest field extension of F over which $p(x)$ could be factored into **linear factors**.

$\leftarrow$ Assuming that splitting fields **exist** and are **unique** up to isomorphism.

Remark. If E is the splitting field of $p(x)$ over F then $[E : F] \leq n!$ where $n = \deg(p(x))$.

Remark. Such an extension is called **normal**.

Proposition 34. Splitting fields exist.

Proof outline. By induction on $\deg(p(x))$, whose base case, $\deg(p(x)) = 1$, yields F as a splitting field. More generally, any $p(x)$ has a root α in $F(\alpha) \cong F[x]/(q(x))$ for some irreducible $q(x)$ so $p(x) = (x - \alpha)f(x) \in F(\alpha)[x]$. We observe that $\deg(f(x)) = \deg(p(x)) - 1$. Induction takes care of the rest. $\qquad\square$
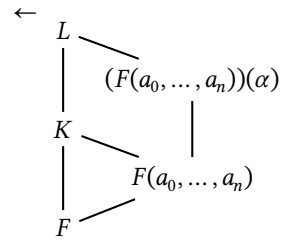
$\leftarrow$ The splitting field of $p(x)$ over F is the same as the splitting field of $f(x)$ over $F(\alpha)$

Remark. K is a splitting field over F *if and only if* every irreducible $p(x) \in F[x]$ that has one root in K has **all** its roots in K.

Non-example 26. $\mathbb{Q}(\sqrt[3]{2})$ over $\mathbb{Q}$ is not such an extension.

Lemma 35. Suppose $\varphi : F_1 \to F_2$ is a field isomorphism, $p_1(x) \in F_1[x]$, and $p_2(x) = "\varphi((p_1(x)))"$ (φ applied to coeffs of $p_1(x)$). Let α_1 be a root of the irreducible factor $q_1(x)$ of $p_1(x)$, and let $q_2(x) = "\varphi(q_1(x))"$ and α_2 be a root of $q_2(x)$. Then there exists an isomorphism $\tau : F_1(\alpha_1) \to F_2(\alpha_2)$ such that $\tau(\alpha_1) = \alpha_2$ and $\tau_{|F_1} = \varphi$ (this means "τ restricted to F_1").

$\leftarrow$ In this way, φ induce a ring isomorphism $F_1[x] \to F_2[x]$.

Proof outline.

$$
F_1(\alpha_1) \xrightarrow{\ \sim\ } F_1[x]/(q_1(x)) \xrightarrow{\ \sim\ } F_2[x]/(q_2(x)) \xrightarrow{\ \sim\ } F_2(\alpha_2)
$$

$$
\alpha_1 \longrightarrow \bar{x} \longrightarrow \bar{x} \longrightarrow \alpha_2
$$

$$
\text{if } a \in F_1, \qquad a \longrightarrow \bar{a} \longrightarrow \overline{\varphi(a)} \longrightarrow \varphi(a)
$$

$\qquad\square$

Proposition 36. Suppose $F_1, F_2, \varphi, p_1(x)$ and $p_2(x)$ are as in Lemma 35. Let $E_1 \& E_2$ be splitting fields of p_1 and p_2 respectively. Then there exists an isomorphism $\sigma : E_1 \to E_2$ such that $\sigma_{|F_1} = \varphi$.

← if we set $F_1 = F_2$ and $p_1 = p_2$, we get corollary: splitting fields are unique up to isomorphism.

Proof. Proceed by induction on $\deg(p_1(x))$. For the base case, if $\deg(p_1(x)) = 1$, then $E_1 = F_1$ and $\sigma = \varphi$.

Assume the result is true for all polynomials of fixed degree $k \geq 1$ and suppose $\deg(p_1(x)) = k+1$. Let α_1 be a root of $p_1(x)$ and α_2 be a root of the φ-corresponding irreducible factor of $p_2(x)$. By Lemma 35, φ can be extended to $\tau : F_1(\alpha_1) \to F_2(\alpha_2)$ such that $\tau_{|F_1} = \varphi$.

In $(F_1(\alpha_1))[x]$, we can factor out $p_1(x) = (x-\alpha_1)g_1(x)$, and in $(F_2(\alpha_2))[x]$ we factor $p_2(x) = (x - \alpha_2)g_2(x)$ with $g_2(x) = \tau(g_1(x))$. We observe that E_1 and E_2 are the splitting fields of g_1 and g_2 over $F_1(\alpha_1)$ and $F_2(\alpha_2)$!

←

$\sigma : \quad E_1 \xrightarrow{\sim} E_2$
$\tau : \quad F_1(\alpha_1) \xrightarrow{} F_2(\alpha_2)$
$\varphi : \quad F_1 \xrightarrow{\sim} F_2$

By inductive hypothesis, τ could be extended to σ and $\sigma_{|F_1(\alpha)} = \tau$ and $\sigma_{|F_1} = \varphi$. $\square$

Corollary 37. Splitting fields are unique.

Proof. Set $F_1 = F_2$, $\varphi =$ id, $p_1(x) = p_2(x)$. $\square$

(The following is kind of on a tangent)

Homework hint: the proof of existence & uniqueness of splitting fields relied on inductive arguments where we adjoin one root at a time. This is the same as saying $E = F(\alpha_1, \ldots, \alpha_n)$ but this tends to overlook isomorphic ways to adjoin roots. In this context, it is convenient to start by considering a specific K containing F and all roots of $p(x)$. In that case, $E = F(\alpha_1, \ldots, \alpha_n)$ becomes more rigorous.

(Tangent ends here)

Definition 25. A polynomial is called **separable** if it doesn't have repeated roots.

Definition 26. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The **formal derivative** of $f(x)$ is the polynomial

$$D_x f(x) = f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$$

← First note that a poly of degree n over a field has exactly n roots.

From this definition, we can check that the usual differential rules hold.

Lemma 38. Suppose F is a field, $f(x)$ is a polynomial in $F[x]$, and E/F a field extension containing a root α of $f(x)$. Then α is a repeated root of $f(x)$ *if and only if* α is a root of the formal derivatve $f'(x)$.

Proof. If α is a repeated root of $f(x)$ then $f(x) = (x-\alpha)^2 g(x)$ for some $g(x) \in E[x]$. In that case, $f'(x) = 2(x-\alpha)g(x) + (x-\alpha)^2 g'(x)$ and so $f'(\alpha) = 0$.

Conversely, if $f'(\alpha) = 0$, then differentiating $f(x) = (x-\alpha)h(x)$ (where $h(x) \in E[x]$) and plugging $x = \alpha$ yields $0 = f'(\alpha) = h(\alpha) + (\alpha - \alpha)h'(\alpha) = h(\alpha)$. This is saying that $h(x) = (x-\alpha)g(x)$ for some $g(x) \in E[x]$. □

Lemma 39. If $f(x) \in F[x]$ is **irreducible and not separable**, then $f'(x) = 0$.

Proof. If $f(x)$ is not separable, we know that there is at least one repeated root. We call it α. Then let $m_{\alpha,F}(x)$ be the minimal polynomial of α over F and we have $f(x) = c \cdot m_{\alpha,F}(x)$ for some constant $c \in F$. Therefore, $\deg(f(x)) = \deg(m_{\alpha,F}(x))$. However, by the previous lemma, $f'(\alpha) = 0$ must also exist since α is a repeated root. If $f'(x) \neq 0$, then we found a polynomial with degree less than the minimal polynomial that has α as a root, which cannot happen. Therefore, $f'(x) = 0$. □

If $f(x)$ is not constant and $f'(x) = 0$, then $\mathrm{char}(F) = p > 0$ and $f(x) = g(x^p)$.

← All powers in $f(x)$ are multiples of $p(x)$.

Proposition 40. If $\mathrm{char}(F) = 0$, or $|F| < \infty$ and $\mathrm{char}(F) = p$, then every irreducible polynomial in $F[x]$ is separable.

Proof. For the case of char=0, it follows from 39.

For the case of char>0, suppose F is a finite field of p^n elements[2]. Then the map $F \to F$ where $\alpha \mapsto \alpha^p$ is a field isomorphism. Hence, every element of F is a pth power.

← Use binomial theorem.

← Such fields are called **perfect**.

Now suppose BWOC $f(x) = \sum_{i=0}^{n} a_i x^i \in F[x]$ is an irreducible but not separable polynomial. Therefore, $f'(x) = 0$ must be true. This happens *if and only if* $f(x) = \sum_{j=0}^{m} a_{jp} x^{jp}$, that is, the x in all terms are of pth degree. However, we know that all elements $a_{jp} \in F$ are already the pth powers of sth else $(b_{jp})^p = a_{jp}$, so

$$f(x) = \sum_{j=0}^{m} (b_{jp}^p) x^{jp}$$

---

[2]See Section 27

and by reverse Binomial Theorem, we get

$$f(x) = \sum_{j=0}^{m}(b_{jp}^{p})x^{jp} = \left(\sum_{j=0}^{m} b_{jp}x^{j}\right)^{p}$$

is not irreducible!   □

Non-example 27. Let $F = \mathbb{F}_p(t) = \left\{\frac{f(t)}{p(t)} \;\middle|\; f(t), g(t) \in \mathbb{F}_p[t], g(t) \neq 0\right\}.$

← This is a field of char>0 but is infinite.

Then $p(x) = x^p - t$ is not separable (but it is irreducible). This can be seen if we suppose α is a root of $p(x)$ (so $\alpha = t$). Then, in $F(\alpha)[x]$, we have $p(x) = x^p - t = x^p - \alpha^p = (x-\alpha)^p$, which tells us $p(x)$ is not separable.

← The coefficients of $p(x)$ are ratios of polys in $\mathbb{F}_p(t)$.

(The following is kind of on a tangent)

Prime fields

Suppose R is a commutative ring with identity. The map $\mathbb{Z} \to R$ where $n \mapsto \pm(\underbrace{1_R + 1_R + \cdots + 1_R}_{|n| \text{ times}})$ ($-$ if $n < 0$) is a ring homomorphism with kernel $n\mathbb{Z}$ where $n = \text{char}(R)$. So:

← check it!

- if char $(R) = 0$, then R contains $\mathbb{Z}$;
- if char $(R) = n > 0$, then R contains $\mathbb{Z}/n\mathbb{Z}$.

If F is a field, then:

- if char $(F) = 0$, then F conatins $\mathbb{Q}$;
- if char $(F) = p > 0$, then p prime and F contains $\mathbb{Z}/p\mathbb{Z}$.

← That is, F is an extension of $\mathbb{Z}/p\mathbb{Z}$!

In other words, every field is an extension of $\mathbb{Q}$ or $\mathbb{F}_p$. Moreover, a finite field is a *finite* extension of $\mathbb{F}_p$: if $[F : \mathbb{F}_p] = n$, then $|F| = p^n$.

← All elts of F are $a_0 + a_1x_1 + \cdots + a_nx_n$ where $a_i \in \mathbb{F}_p$, so we have p^n choices.

In addition, $|F - \{0\}| = p^n - 1 \implies$ if $\alpha \in F - \{0\}$ then $\alpha^{p^n-1} = 1$, implying that if $\alpha \in F$, then $\alpha^{p^n} = \alpha$, meaning that α is a root of $x^{p^n} - x \in F[x]$. Therefore, F is the splitting field of $x^{p^n} - x$. But splitting fields are unique, so we conclude that there is only one unique finite field for each order.

← By Lagrange's Theorem

(Tangent ends here)

Definition 27. An algebraic extension K/F is called (algebraically) **separable** if $m_{\alpha,F}(x)$ is separable for **all** $\alpha \in K$.

Galois Theory

Definition 28. A finite extension K/F is called **Galois** if K/F is <u>normal</u> and <u>separable</u>.

← normal just means it is a splitting field of something

Definition 29. If K/F is an extension, then the **automorphism group** of K/F is defined as

$$\text{Aut}(K/F) = \{\sigma \in \text{Aut}(K) \mid \sigma(a) = a \; \forall \, a \in F\}$$

That is, all the automorphisms of K that also fix the field F.

> **Galois theory** is concerned with the <u>study of roots of polynomials</u> by way of <u>automorphisms of splitting fields</u> (of separable polynomials).
> In particular, we are interested in what
>
> $$\text{Aut}(K) = \{\sigma \,:\, K \to K \text{ isomorphisms}\}$$
>
> (a group under composition) is. Naturally, such groups are *finite*.

← review MATH171 finite groups!

Last time, we showed that $K \supseteq \begin{cases} \mathbb{Q} & \text{if char}(K) = 0 \\ \mathbb{F}_p & \text{if char}(K) = p \end{cases}$, since $\sigma(1) = \sigma(1^2) = (\sigma(1))^2$ implies that $\sigma(1) = 1$ must always be true! Hence, $\sigma(n) = n$ must be true in char 0 fields, or $\sigma(\bar{n}) = \bar{n}$ if char>0 for all $n \in \mathbb{Z}$. Therefore, σ **fixes** the prime subfield $\mathbb{Q}$ or $\mathbb{F}_p$.

Remark. Why does Definition 29 have to fix the field F? Because we've shown that $\text{Aut}(K) = \begin{cases} \text{Aut}(K/\mathbb{Q}) & \text{if char}(K) = 0 \\ \text{Aut}(K/\mathbb{F}_p) & \text{if char}(K) = p \end{cases}$.

Lemma 41. If K/F is an extension, $\alpha \in K$ is algebraic over F and $\sigma \in \text{Aut}(K/F)$, then $\sigma(\alpha)$ is a root of $m_{\alpha,F}(x)$.

Proof. Observe that $m_{\alpha,F}(\alpha) = 0 = \sigma(0) = \sigma(m_{\alpha,F}(\alpha))$. Hence, since $\sigma(\alpha) = \alpha$ for all $\alpha \in F$,

$$\sigma(a_0 + a_1\alpha + \cdots + a_n\alpha^n) = a_0 + a_1\sigma(\alpha) + \cdots + a_n\sigma(\alpha)^n = m_{\alpha,F}(\sigma(\alpha))$$

☐

So if $f(x) \in F[x]$, then every $\sigma \in \text{Aut}(K/F)$ **permutes** the roots of $f(x)$ that lie in K. It would be nice if the roots of $f(x)$ all lived in K. This is why we consider K the splitting field of some polynomial over F.

← Recall: if irreducible $f(x)$ has one root in such K, then all roots lie in K.

Proposition 42. If K is the splitting field of some polynomial $f(x)$ over F (so $[K : F] < \infty$), then $|\text{Aut}(K/F)| \le [K : F]$, with equality if $f(x)$ is separable.

Proof. We will prove a more general statement by induction. If $\sigma : F_1 \to F_2$ is an isomorphism, $f_1(x) \in F_1[x]$ and $f_2(x) = \sigma(f_1(x)) \in F_2[x]$ and E_1 and E_2 are the splitting fields of f_1 and f_2 over F_1 and F_2 respectively. Then we would like to show that there are at most $[E_1 : F_1]$ isomorphisms $\tau : E_1 \to E_2$ such that $\tau|_{F_1} = \sigma$ with equality if f_1 separable.

← The prop above fixes $F_1 = F_2$ and σ being identity.

> *Base case.* If $[E_1 : F_1] = [E_2 : F_2] = 1$, then $E_1 = F_1$, $E_2 = F_2$ and $\tau = \sigma$ is our only choice.
>
> *Inductive step.* Suppose we've proven the result for all extensions of degree $< n$ for some $n \ge 2$. Now consider $[E_1 : F_1] = [E_2 : F_2] = n$. Pick $\alpha \in E_1 \backslash F_1$ and let $\beta \in E_2$ be any root of $\sigma(m_{\alpha, F_1}(x))$. Then σ could be extended to $\rho : F_1(\alpha) \to F_2(\beta)$ such that $\rho(\alpha) = \beta$ and $\rho|_{F_1} = \sigma$. Observe that $[F_1(\alpha) : F_1] = \deg(m_{\alpha, F_1}(x))$. Moreover, the number of extensions of σ to ρ equals the number of *distinct* roots of $\sigma(m_{\alpha, F_1}(x))$. Thus, the number of extensions of σ to $F_1(\alpha)$ is *at most* the degree of $m_{\alpha, F_1}(x)$ which is $[F_1(\alpha) : F_1]$ with equality if $m_{\alpha, F_1}(x)$ is separable. Since $[E_1 : F_1] = [E_2 : F_2] = n$, we have $[E_1 : F_1(\alpha)] < n$, by inductive hypothesis, there are at most $[E_1 : F_1(\alpha)]$ ways of extending ρ to $\tau : E_1 \to E_2$. Hence,

←

$$\tau : \quad E_1 \xrightarrow{\sim} E_2$$

$$\rho : \quad F_1(\alpha) \xrightarrow{} F_2(\beta)$$

$$\sigma : \quad F_1 \xrightarrow{\sim} F_2$$

$$|\{\text{extensions of } \sigma \text{ to } \tau\}| = |\{\text{extensions of } \sigma \text{ to } \rho\}| \, |\{\text{extensions of } \rho \text{ to } \tau\}|$$
$$\le [F_1(\alpha) : F_1][E_1 : F_1(\alpha)]$$
$$= [E_1 : F_1]$$

Looking at the case $F_1 = F_2$, $E_1 = E_2$, $\sigma = \text{id}$, we get our result.   ☐

Definition 30. If K/F is a normal extension, then the extension is **Galois** if $[K : F] = |\text{Aut}(K/F)|$.

Remark. Notation: if K/F is Galois, then use $\text{Gal}(K/F)$ for $\text{Aut}(K/F)$.

Fixed Fields and Automorphism Groups

Definition 31. Suppose K/F is a field extension. If subgroup $H \le \text{Aut}(K/F)$, then the **fixed field** of H is given by $K_H = \{\alpha \in K \mid \sigma(\alpha) = \alpha \text{ for all } \sigma \in H\}$.

← Observe that K_H is indeed a field (the sum, products etc. are also fixed by σ); moreover, it is an intermediate

Remark. Also, observe that if $F \subseteq E \subseteq K$, then $\text{Aut}(K/E) \leq \text{Aut}(K/F)$.

Lemma 43. Suppose K/F is an extension. Then:

(1) If $H_1, H_2 \leq \text{Aut}(K/F)$ with $H_1 \leq H_2$, then $K_{H_2} \subseteq K_{H_1}$.

(2) If $F \subseteq E_1 \subseteq E_2 \subseteq K$ are two intermediate extensions, then $\text{Aut}(K/E_2) \leq \text{Aut}(K/E_1) \leq \text{Aut}(K/F)$.

Example 28. $\mathbb{Q}$ is an intermediate extensions of $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$. Then $\text{Aut}(\mathbb{Q}/\mathbb{Q}) = \{\mathbb{1}\}$. We further observe that since automorphisms permute roots, $\text{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) = \{\mathbb{1}\}$. Hence $\text{Aut}(\mathbb{Q}/\mathbb{Q}) = \text{Aut}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ and so the fixed field by $\mathbb{Q}(\sqrt[3]{2})$ is given by $\mathbb{Q}(\sqrt[3]{2})_{\text{Aut}(\mathbb{Q}/\mathbb{Q})} = \mathbb{Q}(\sqrt[3]{2})$. We note that $\mathbb{Q}(\sqrt[3]{2})$ is not Galois!

$\leftarrow$ Roots of $x^3 - 2$ are $\sqrt[3]{2}, \zeta_3\sqrt[3]{2}, \zeta_3^2\sqrt[3]{2}$, so two roots are not in $\mathbb{Q}(\sqrt[3]{2})$, and so $\sqrt[3]{2}$ could be only mapped to itself.

Theorem 44 (The Fundamental Theorem of Galois Theory)**.** If K/F is a (finite) Galois extension, then the maps $H \mapsto K_H$ and $E \mapsto \text{Aut}(K/E)$ gives an *inclusion-reversing* **bijection** between the subgroups of $\text{Aut}(K/F)$ and the intermediate extensions of K/F.

$\leftarrow$ We see if K/F is Galois, then K/E is also Galois as if K is the splitting field of some poly in F, then it's certainly true for E.
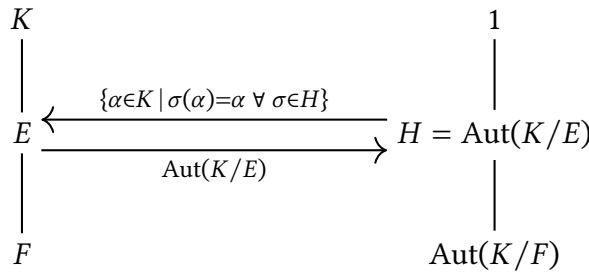
Furthermore, $[K : E] = |\text{Aut}(K/E)|$, and

$$[E : F] = |\text{Aut}(K/F) : \text{Aut}(K/E)| = |\text{Aut}(K/F)|/|\text{Aut}(K/E)|$$

Moreover, E/F is Galois *if and only if* $\text{Aut}(K/E)$ is a **normal subgroup** of $\text{Aut}(K/F)$, in which case

$$\text{Aut}(E/F) = \text{Aut}(K/F)/\text{Aut}(K/E)$$

In other words,

$$
\begin{array}{ccc}
K & & 1 \\
| & & | \\
E & \xleftarrow{\{\alpha \in K \,|\, \sigma(\alpha)=\alpha \,\forall\, \sigma \in H\}} & H = \text{Aut}(K/E) \\
 & \xrightarrow{\text{Aut}(K/E)} & \\
| & & | \\
F & & \text{Aut}(K/F)
\end{array}
$$

where H is a subgroup of $\text{Aut}(K/F)$ that fixes the field E, an extension of F contained in K. If $H \trianglelefteq \text{Aut}(K/F)$, then E/F is normal.
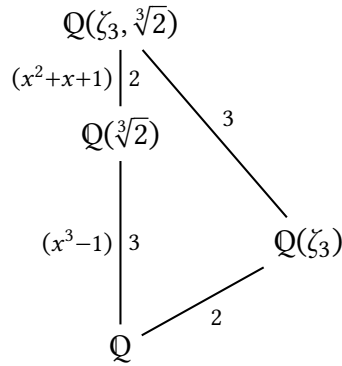
Example 29. Consider $\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}$, the splitting field extension of $x^3 - 2$.[3]

$\leftarrow$ $|\zeta_3| = 3$, a primitive 3rd root of unity.

---

[3]Suppose ζ is a primitive nth root of unity; then so is ζ^k *if and only if* the $\gcd(k,n) = 1$, i.e. k, n are relatively prime.
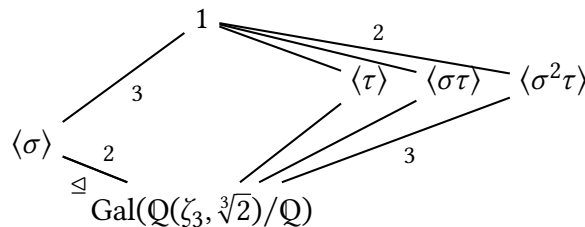
Let $\sigma, \tau \in \mathrm{Aut}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ be given by:

$$\sigma : \begin{cases} \sqrt[3]{2} \mapsto \zeta_3 \sqrt[3]{2} \\ \zeta_3 \mapsto \zeta_3 \end{cases} \qquad \tau : \begin{cases} \sqrt[3]{2} \mapsto \sqrt[3]{2} \\ \zeta_3 \mapsto \zeta_3^2 \end{cases}$$
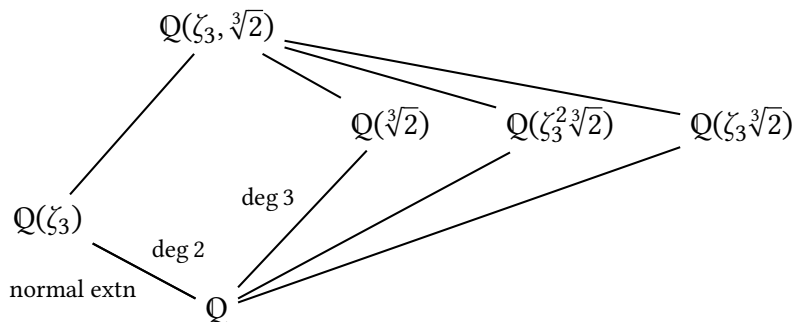
← Isomorphisms preserve order, so they **must** take an nth root of unity to another nth root of unity!

So $\langle \sigma, \tau \mid \sigma^3 = \tau^2 = id, \sigma\tau = \tau\sigma^2 \rangle \cong S_3$.

Since this is a subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ that has the same finite order of 6, this must just be $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q})$ itself; hence, $\mathrm{Gal}(\mathbb{Q}(\zeta_3, \sqrt[3]{2})/\mathbb{Q}) \cong S_3$.

Now we look at the subgroups of S_3 (in reverse):

And then think about the **fixed field** of each subgroup correspondingly:

← Remember normal extn means 'is splitting field', i.e. 'one root -> all roots'

Note that the normal extension corresponds to the normal subgroup!

← $\mathbb{Q}(\sqrt[3]{2})$ is not Galois!

Proposition 45 (Primitive Element Theorem). If K/F is a **finite Galois** extension, then $K = F(\alpha)$ for some $\alpha \in K$.

Definition 32. $K = F(\alpha)$ is a **simple** extension of F and α is a primitive element.

Proof. We first assume $|F| = \infty$.

Recall that K/F is finite *if and only if* $K = F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ where α_i is algebraic over F. We will proceed by induction on n, whose base case $n = 1$ gives a simple extension $F(\alpha_1)/F$.

Recursive case: assume that for some $k \geq 1$ we have $F(\alpha_1, \ldots, \alpha_k)$ being a simple extension $F(\alpha)$. Let K/F be Galois and $K = F(\alpha_1, \ldots, \alpha_{k-1}, \alpha, \beta)$.

Let $E = F(\alpha_1, \ldots, \alpha_{k-1})$ and consider the intermediate family of extensions $\{E(\alpha + t\beta) \mid t \in F\}$. Since $|\mathrm{Gal}(K/F)| < \infty$ as we are talking about finite Galois extensions, there are finitely many distinct such extensions, so $E(\alpha + t_1\beta) = E(\alpha + t_2\beta)$ for some $t_1 \neq t_2$.

Now we see that $\alpha + t_1\beta$ and $\alpha + t_2\beta$ must be in the same field $E(\alpha + t_1\beta)$. Hence, $(\alpha + t_1\beta) - (\alpha + t_2\beta)$ are in the field, so $(t_1 - t_2)^{-1}((\alpha + t_1\beta) - (\alpha + t_2\beta)) = \beta$ is also in the field. Similarly, $\alpha \in E(\alpha + t_1\beta)$. Therefore, $K = E(\alpha, \beta) = E(\alpha + t_1\beta) = F(\alpha_1, \ldots, \alpha_{k-1}, \alpha + t_1\beta)$, which has k elements adjoined and is therefore simple. $\square$

Remark. Above is true for char 0 fields even without the 'Galois' hypothesis.

Proof outline. Since K is a finite extension of F with $[K : F] < \infty$, then there must be $K = F(\alpha_1, \ldots, \alpha_n)$ for some algebraic $\alpha_1, \ldots, \alpha_n \in K$.

Let E be the splitting field of $\prod_{i=1}^{n} m_{\alpha_i, F}(x)$. We call E the **Galois closure** of K over F. Now, we have $E \supset K \supset F$ and E/F is Galois. Thus, there are finitely many intermediate fields between K and F. We can then use a similar proof for Proposition 45. $\square$

Cyclotomic fields

Definition 33. Suppose K is a field. An element $z \in K$ is called an n-th root of unity if $z^n = 1$; and z is a **primitive** n-th root of unity if $z^k \neq 1$ for any $1 \leq k \leq n-1$.

Remark. z is a **primitive** n-th root of unity *if and only if* $|z| = n$ in $K^\times = K \backslash \{0\}$.

Remark. z is an n-th root of unity *if and only if* z is a root of $x^n - 1$.

Lemma 46. If K is a field containing one primitive n-th root of unity, then K contains exactly n roots of unity, exactly $\varphi(n)$ of which are primitive.

Remark. Recall:

- $z^n = 1$ *if and only if* $|z|$ divides n.
- $|z^m| = \frac{|z|}{\gcd(m,|z|)}$.
- $|z^m| = |z|$ *if and only if* $(m,|z|) = 1$.

Example 30. If $|z| = 10$, then $|z^6| = 5 = \frac{10}{(6,10)}$

Proof. If $z \in K$ is a primitive n-th root of unity, then every element of $\langle z \rangle$ is an n-th root of unity, and so a root of $x^n - 1$; but $|\langle z \rangle| = |z| = n$, so the subgroup $\langle z \rangle$ generated by z must consist of all of the n roots of $x^n - 1$. Furthermore, z^m is also a primitive n-th root of unity *if and only if* $(m, n) = 1$; thus, there are exactly $\varphi(n)$ such elements in $\langle z \rangle$. $\qquad\square$

In $\mathbb{C}$, we have $e^{i\frac{2\pi}{n}}$ is a primitive n-th root of unity. Suppose $\zeta_n \in \mathbb{C}$ is a primitive n-th root of unity.

Definition 34. The **cyclotomic polynomial** is given by

$$\Phi_n(x) = \prod_{\substack{0 \le k < n \\ (n,k)=1}} (x - \zeta_n^k)$$

Properties:

- $x^n - 1 = \prod_{0 \le k < n}(x - \zeta_n^k) = \prod_{d|n} \prod_{\substack{0 \le k < n \\ (n,k)=d}} (x - \zeta_n^k) = \prod_{d|n} \Phi_d(x)$
- $\deg(\Phi_n(x)) = \varphi(n)$
- $n = \sum_{d|n} \varphi(d)$

    Example 31. $6 = \varphi(1) + \varphi(2) + \varphi(3) + \varphi(6)$

(The following is kind of on a tangent)

Remark. If K is a finite field, then $K^\times = \langle z \rangle$.

Proof. Since K is finite, it must be an extension of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ for some prime p; so $|K^\times| = p^n - 1$ since $|K| = p^n$. For $d | p^n - 1$, let $\mathcal{O}_d = \{z \in K^\times \mid |z| = d\}$. Observe that $K^\times = \bigcup_{d|p^n-1} \mathcal{O}_d$ is a disjoint union, and so

$$|K^\times| = \sum_{d|p^n-1} |\mathcal{O}_d| = p^n - 1 = \sum_{d|p^n-1} \varphi(d)$$

which implies that $|\mathcal{O}_d| = \varphi(d)$ for all $d | p^n - 1$. Hence, in particular, $\mathcal{O}_{p^n-1}$ is nonempty, so any $z \in \mathcal{O}_{p^n-1}$ generates $K^\times$. $\qquad\square$

Remark. This implies that the Primitive Element Theorem (see Proposition 45) is also true for finite fields.

(Tangent ends here)

Remark. $\Phi_n(x)$ is irreducible over $\mathbb{Q}$, and so it is the minimal polynomial $m_{\zeta_n, \mathbb{Q}}(x)$.

$$\sigma : K \to K$$
$$\leftarrow \quad F(\alpha) \to F(\beta)$$
$$\text{id} : F \to F$$

Lemma 47. Suppose F is a field where all irreducible polynomials are separable. Let $p(x) \in F[x]$ irreducible and split completely in a Galois extension K/F, and let α, β be two roots of $p(x)$. Then there exists $\sigma \in \text{Gal}(K/F)$ such that $\sigma(\alpha) = \beta$.

Definition 35. The extension $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is called the n-th **cyclotomic extension** of $\mathbb{Q}$.

Remark. All other primitive n-th roots of unity are of the form ζ_n^a w/ $(a, n) = 1$, so $\mathbb{Q}(\zeta_n)$ is the splitting field of $\Phi_n(x)$ over $\mathbb{Q}$, so the extension is Galois.

If $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, then σ is completely determined by $\sigma(\zeta_n)$. But $\sigma(\zeta_n)$ must be another primitive n-th root of unity, so $\sigma(\zeta_n)$ must be ζ_n^a for some $0 < a < n$ with $(a, n) = 1$.

Moreover, by Lemma 47, each a corresponds to a Galois automorphism σ; in fact, the map $\text{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times$ where $\sigma \mapsto \bar{a}$ is a **group isomorphism**.

Definition 36. A Galois extension K/F is abelian if $\text{Gal}(K/F)$ is abelian.

$\leftarrow$ All finite field extensions are abelian

Corollary 48. $\mathbb{Q}(\zeta_n)/\mathbb{Q}$ is abelian.

Theorem 49 (Kronecker-Wober)**.** If $K/\mathbb{Q}$ is abelian, then $K \subseteq \mathbb{Q}(\zeta_n)$ for some n.

Proofs of FToGT

- The Galois correspondence is inclusion reversing: $H_1 \leq H_2$ *if and only if* $K_{H_1} \supseteq K_{H_2}$.

  $\leftarrow$ Just try it!

- If K/F is Galois and E is an intermediate extension, then K/E is also Galois.

  $\leftarrow$ Anything in F is also in E!

- The Galois correspondence is a bijection.

Proof. ($\implies$) Suppose E is an intermediate extension of K/F.

$$
\begin{array}{cccc}
K & 1 & K & 1 \\
| & | & | & | \\
 & & K_H \xrightarrow{g} \text{Gal}(K/K_H) & \\
 & \xrightarrow{h} & {\scriptstyle WTS:1} & \\
E \xrightarrow{g} \text{Gal}(K/E) = H & & E & \text{Gal}(K/E) \\
| & | & | & | \\
F & \text{Gal}(K/F) & F & \text{Gal}(K/F)
\end{array}
$$

Suppose BWOC that there exists $\alpha \in K_H \backslash E$. Thus, $m_{\alpha,E}(x)$ has degree
>1. Suppose $\beta \in K$ is a different root of $m_{\alpha,E}(x)$. Hence, we can find
an isomorphism $\sigma \in H$ such that $\alpha \mapsto \beta \neq \alpha$.

← Since K is Galois, if $\alpha \in K$ then all roots in K.

$$
\begin{array}{ccc}
K & & K \\
| & & | \\
E(\alpha) & \xrightarrow{\sigma} & E(\beta) \\
| & & | \\
E & & E
\end{array}
$$

This contradicts the definition of K_H as it is no longer fixed by an element of H. Hence, we have $[K_H : E] = 1$ and so we get the correspondence.

($\Longleftarrow$) Now suppose $H \leq G$. We have:

$$
\begin{array}{ccc}
1 & & K = E(\alpha) \\
| & & | \\
H & \xmapsto{\;h\;} & K_H = E \\
| & & | \\
G & & F
\end{array}
$$

for some $\alpha \in K$ by primitive element theorem. So $|H| \leq |\mathrm{Gal}(K/E)| = [K : E] = \deg(m_{\alpha,E}(x))$.

← In this note, PET is proven through Galois correspondence, but D&F offers a different proof.

Consider $f(x) = \prod_{\alpha_i \in A}(x - \alpha_i)$, where $A = \{\sigma(\alpha) \mid \sigma \in H\}$. We note that $\deg(f(x)) \leq |H|$. Furthermore, if $\sigma \in H$, then $\sigma(f(x)) = \prod_{\alpha_i \in A}(x - \sigma(\alpha_i)) = \prod_{\alpha_i \in A}(x - \alpha_i) = f(x)$ since σ just permutes A. So $f(x) \in K_H[x] = E[x]$ and $f(\alpha) = 0$, meaning that $m_{\alpha,E} \mid f(x) \implies \deg(m_{\alpha,E}(x)) \leq \deg(f(x)) \leq |H|$. Therefore:

$$
|H| \leq |\mathrm{Gal}(K/E)| = [K : E] = \deg(m_{\alpha,E}(x)) \leq \deg(f(x)) \leq |H|
$$

Thus, $|H| = |Gal(K/E)|$, implying $H = \mathrm{Gal}(K/E)$ since they are finite groups containing each other. In other words, $g(h(H)) = H$ *if and only if h is injective and g is surjective.*

$\square$

- If E is an intermediate extension of K/F, then E/F is Galois *if and only if* $\mathrm{Gal}(K/E) \trianglelefteq \mathrm{Gal}(K/F)$.

Proof. Recall [4] that $\varphi \in \mathrm{Gal}(K/F)$ induces a group isomorphism $\mathrm{Gal}(K/E) \xrightarrow{\sim} \mathrm{Gal}(K/\varphi(E))$ where $\sigma \mapsto \varphi\sigma\varphi^{-1}$. Hence E/F is Galois *if and only if* E is normal over F *if and only if*

---

[4]We first prove that the map $\sigma \to \varphi\sigma\varphi^{-1}$ is indeed a map between $\mathrm{Aut}(K/F) \to \mathrm{Aut}(K'/F')$. We know that φ is an isomorphism from K to K', so $\varphi^{-1} : K' \to K$ mapping F' to F; next, σ is an isomorphism $K \to K$ fixing F, and $\varphi : K \to K'$ mapping F to F', so we conclude that $\varphi\sigma\varphi^{-1}$ is an isomorphism from K' to itself fixing F' precisely. Hence, it is an automorphism in the group $\mathrm{Aut}(K'/F')$.

Next we prove that this map is a group homomorphism. Let the map $\sigma \to \varphi\sigma\varphi^{-1}$ be ϕ. We begin

> finish proof

□

Radical extensions and solvable groups

Example 32. Suppose K is the splitting field of $x^4 - 2$ over $\mathbb{Q}$. Then $K = \mathbb{Q}(\sqrt[4]{2}, i)$.

← since $\zeta_4 = i$

Let $\sigma : \begin{cases} \sqrt[4]{2} \mapsto i\sqrt[4]{2} \\ i \mapsto i \end{cases}$ and $\sigma : \begin{cases} \sqrt[4]{2} \mapsto \sqrt[4]{2} \\ i \mapsto -i \end{cases}$.

We look at the subgroup generated by σ, τ:

$$\langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \sigma\tau = \tau\sigma^3 \rangle \cong D_8$$

← (not showing all intermed extns)

Since $[K/\mathbb{Q}] = 8$, we have found the Galois group of $K/\mathbb{Q}$.

Definition 37. A group G is **solvable** if there exists a subgroup sequence $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$ such that every G_i/G_{i-1} is abelian (cyclic).

← Not saying that all the G_i is normal in G.

Proposition 50. If G is a solvable group, then there exists a subgroup sequence $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$ in G such that H_i/H_{i-1} is cyclic.

Proof. We'll proceed by induction on $|G|$. If $|G| = 1, 2, 3$, they are cyclic because prime order groups are cyclic.

Suppose now that the result is true for all G with $|G| \leq k$ for some $k \geq 3$. If G is solvable and $|G| = k+1$, then there exists $\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_{r-1} \trianglelefteq G_r = G$. Let H be a proper normal subgroup of G containing G_{r-1} that is maximal in the sense that there are no more intermediate normal subgroup.

---

by checking that $\phi(\sigma_1 \sigma_2) = \phi(\sigma_1)\phi(\sigma_2)$:

$$\begin{aligned}
\phi(\sigma_1 \sigma_2) &= \varphi\sigma_1\sigma_2\varphi^{-1} \\
&= \varphi\sigma_1 \circ \mathrm{id} \circ \sigma_2\varphi^{-1} \\
&= \varphi\sigma_1(\varphi^{-1}\varphi)\sigma_2\varphi^{-1} \\
&= (\varphi\sigma_1\varphi^{-1})(\varphi\sigma_2\varphi^{-1}) \\
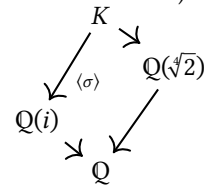&= \phi(\sigma_1)\phi(\sigma_2)
\end{aligned}$$

Hence it is indeed a group homomorphism. Finally, we observe that it is bijective. Since all $\varphi\sigma\varphi^{-1}$ must correspond to a σ, it is surjective; in addition, if $\varphi\sigma_1\varphi^{-1} = \varphi\sigma_2\varphi^{-1}$, then we have:

$$\begin{aligned}
\varphi\sigma_1\varphi^{-1} &= \varphi\sigma_2\varphi^{-1} \\
\varphi^{-1}\varphi\sigma_1\varphi^{-1}\varphi &= \varphi^{-1}\varphi\sigma_2\varphi^{-1}\varphi \\
\sigma_1 &= \sigma_2
\end{aligned}$$

Therefore, we conclude that the map $\sigma \to \varphi\sigma\varphi^{-1}$ defines a group isomorphism $\mathrm{Aut}(K/F) \to \mathrm{Aut}(K'/F')$. □

⬛ finish this

☐

Remark. Recall that A_5 is a simple and nonabelian group. Therefore, it is not solvable.

Since subgroups and quotient groups of solvable groups are solvable, we get:

Corollary 51. S_5 is not solvable.

Why do we care? Suppose F is a field containing all n-th roots of unity.

Lemma 52. $F(\sqrt[n]{a})/F$ is a cyclic Galois extension for all $a \in F$.

← Gal$(F(\sqrt[n]{a})/F)$ is a cyclic group, $\sqrt[n]{a}$ is any root of $x^n - a$

Proof. The roots of $x^n - a$ are $\zeta_n^i \sqrt[n]{a}$ for $0 \leq i \leq n-1$, so $F(\sqrt[n]{a})$ is the splitting field of $x^n - a$ over F. Then $F(\sqrt[n]{a})/F$ is Galois. Consider the map $\mathrm{Gal}(F(\sqrt[n]{a})/F) \to \langle \zeta_n \rangle$ with the element $\sqrt[n]{a} \mapsto \zeta_n^i$ mapped to ζ_n^i. This map is an injective group homomorphism. This tells us that $\mathrm{Gal}(F(\sqrt[n]{a})/F)$ is isomorphic to a subgroup of the cyclic group $\langle \zeta_n \rangle$. Hence, it must be cyclic. □

← check it!

Lemma 53. If K/F is a cyclic Galois extension of degree n, then $K = F(\sqrt[n]{a})$ for some $a \in F$.

← Converse of the previous lemma :)

Proof. Write $\mathrm{Gal}(K/F) = \langle \sigma \rangle$ as we said it is cyclic. For $\alpha \in K$, define $\ell(\alpha) = \alpha + \zeta_n \sigma(\alpha) + \cdots + \zeta_n^{n-1} \sigma^{n-1}(\alpha)$. Observe that $\sigma(\ell(\alpha)) = \zeta_n \sigma(\alpha) + \cdots + \zeta_n^{n-1} \sigma^{n-1}(\alpha) + \zeta_n^n \sigma^n(\alpha)$. But the order of σ is n, so $\sigma^n(\alpha) = \alpha$. Hence, $\sigma(\ell(\alpha)) = \zeta_n^{-1} \ell(\alpha)$.

Now we observe that when $\ell(\alpha) \neq 0$[5], we have $\sigma^i(\ell(\alpha)) = (\zeta_n^{-1})^i \ell(\alpha)$ is $\ell(\alpha)$ *if and only if* $n|i$. Hence, $\ell(\alpha)$ is fixed only by the identity automorphism, and so $\ell(\alpha)$ is not in any proper subfield of K. That is, $K = F(\ell(\alpha))$. Moreover, $\sigma(\ell(\alpha)^n) = \left(\zeta_n^{-1} \ell(\alpha) \right)^n$ is $\ell(\alpha)^n$ *if and only if* $\ell(\alpha) \in F$. Setting $a = \ell(\alpha)^n$ (hence $\sqrt[n]{a} = \ell(\alpha)$) completes the proof. □

← fundamental theorem!

Definition 38. Suppose F is a field, and α is algebraic over F. Then α is **solvable by radicals** if $\alpha \in K \supseteq F$ such that there exists a sequence of intermediate extensions $F = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_r = K$ for which $K_{i+1} = K_i(\sqrt[n]{a_i})$ for some $a_i \in K_i$ for all $0 \leq i \leq r-1$. Such a K is called a **root** (or radical, or solvable) extension of F.

Proposition 54. If K/F is solvable, then there exists a field L containing K for which L/F is a solvable Galois extension.

---

[5]Such element always exists due to Linear Independence of Characters (not included in this note).

Theorem 55. A Galois extension K/F is solvable *if and only if* the Galois group K/F is solvable.

Remark. Observe $p(x) = x^5 - 4x - 2$ has splitting field with Galois group isomorphic to S_5, so the quintic is not solvable.

Suppose K/F is a Galois, solvable extension. Hence, we could write $F = K_0 \subseteq \cdots \subseteq K_r = K$ with $K_{i+1} = K_i(\sqrt[n_i]{a_i})$ for some $a_i \in K_i$ for all $1 \le i < r$ and F contains all the requisite roots of unity. Then, for each i, we have K/K_i is Galois by FToGT. By the lemmas above, K_{i+1}/K_i is a cyclic Galois extension. By FToGT, $G_{i+1} \trianglelefteq G_i$; moreover, $\mathrm{Gal}(K_{i+1}/K_i) \cong \mathrm{Gal}(K/K_i)/\mathrm{Gal}(K/K_{i+1}) = G_i/G_{i+1}$, a cyclic group for all i.

←

Therefore, $G = \mathrm{Gal}(K/F)$ us a solvable group.

Proposition 56. If K/F is a solvable extension, then there exists $L \supseteq K$ such that L/F is a solvable Galois extension with solvable Galois group.

← That is, you can add enough things until the extension is Galois!

Proof outline. Since K/F is solvable, there exists $F = K_0 \subseteq \cdots \subseteq K_r = K$ with $K_{i+1} = K_i(\sqrt[n_i]{a_i})$ for some $a_i \in K_i$, $n_i \in \mathbb{N}$ for all $1 \le i < r$.
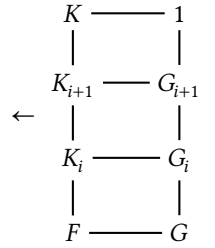
Main idea: let L be the splitting field of $\left(\prod_{i=0}^{r-1} m_{\sqrt[n_i]{a_i}, F}(x) \right) \left(x^{\prod_{i=0}^{r-1} n_i} - 1 \right)$. Call $N = \prod_{i=0}^{r-1} n_i$. Observe that ζ_N^{N/n_i} has an order n_i.

We would like to have:

← This diagram is to zoom in from K_j to K_{j+1}.
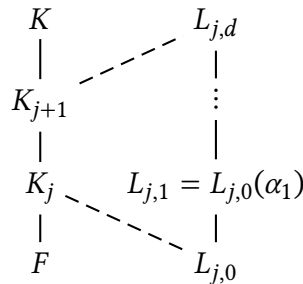
where $K_j = F(\sqrt[n]{a_0}, \ldots, \sqrt[n]{a_{j-1}})$ and $L_{j,0} = (F(\zeta_N))(\sqrt[n]{a_0}$ & all Gal conj., $\ldots, \sqrt[n]{a_{j-1}}$ & all Gal conj.) such that we eventually get a Galois extension after adjoining all Ls onto F.

Still to show that $L_{j,k+1} = L_{j,k}(\alpha_{k+1})$. $\square$