

MATH103 Combinatorics Notes

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A Recurrence Relations

A1 Intro

Remark. Let there be a set $\{1, 2, \dots, n\}$. The number of subsets of it is 2^n since for each number, we could say “include” or “exclude”.

Example 1. Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

(Scratch work begins)

First consider $n = 0$. Then the only *good* subset is \emptyset .

Now consider $n = 1$, both $\emptyset, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\emptyset, 1, 2, 12$. The set 12 is not good.

Similarly, we have $f(3) = 5, f(5) = 8$.

(Scratch ends here)

We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

← notation simplified for fast typing

A2 Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

Remark. Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$.

← Textbook

← Preferred!

Example 2. Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbs one step per stride, sometimes two steps per stride. In how many ways can she do this?

← It is the same recurrence as A1 but with init conditions shifted:
 $f(n) = F_{n+1} = f_{n+2}$.

Table 1: Table of the sequence in two notations

n	0	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21	34
f_n	0	1	1	2	3	5	8	13	21

(Scratch work begins) Let $S(n)$ be the number of ways climbing n steps.

- $S(1) = 1$ • — •
- $S(2) = 2$ • — • — •
• ——— •
- $S(3) = 3$ • — • — • — •
• ——— • — •
• — • ——— •
- $S(4) = 5$ • — • — • — • — •
• ——— • — • — •
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• — • — • ——— •
• ——— • ——— •

Conjecture: maybe Fibonnacci?

(Scratch ends here)

Proof. Consider the set of ways she can cover n steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n - 1$ steps. Thus, this gives us $S(n - 1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n - 2$ steps. Thus, this gives us $S(n - 2)$ ways.

Therefore, we conclude that $S(n) = S(n - 1) + S(n - 2)$. We account the initial conditions and conclude the closed form:

$$S(n) = F_n = f_{n+1}$$

for all n . Since Prof Rad climbs 47 steps, we get $S(47) = 4807526976$. □

A3 Simplex numbers

Definition 1. Two-dimensional triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- ...



1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Theorem 1. $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

First proof. We prove by induction.

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.

Inductive hypothesis: Suppose proved formula for up to $n = k$.

Inductive step: Consider $n = k + 1$.

$$\begin{aligned}
 T_2(k+1) &= 1 + \cdots + k + (k+1) \\
 &= T_2(k) + k + 1 \\
 &= \frac{k(k+1)}{2} + k + 1 \\
 &= \frac{k^2 + k + 2(k+1)}{2} \\
 &= \frac{k^2 + 3k + 2}{2} \\
 &= \frac{(k+1)(k+2)}{2} \\
 &= \frac{(k+1)((k+1)+1)}{2}
 \end{aligned}$$

□

Proof by Gauss. Observe:

$$\begin{aligned}
 T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\
 &= n + (n-1) + \cdots + 2 + 1
 \end{aligned}$$

← Not as good of a proof: we must know what we are proving in the first place!

← Better proof: concluding the formula without knowing it first!

Therefore, we **add** the two rows:

$$\begin{aligned} 2T_2(n) &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_n \\ &= n(n+1) \\ \therefore T_2(n) &= \frac{1}{2}n(n+1) \end{aligned}$$

□

Definition 2. Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

$$\bullet T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$$

Definition 3. Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$

B Ramsey Theory

Invented by Frank Ramsey in 1930. We would need:

- Graph Theory
- Pigeonhole Principle
- Quantifiers
- Counterexamples

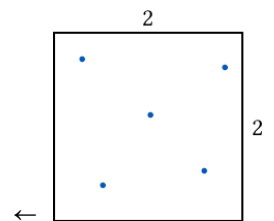
B1 Pigeonhole principle

Theorem 2 (Dirichlet's Pigeonhole Principle). If you put $n + 1$ pigeons in n pigeonholes, then (at least) one pigeonhole will contain (at least) two pigeons.

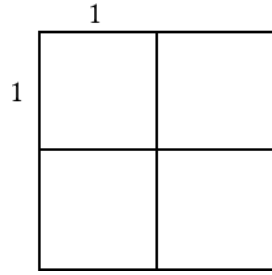
Proof omitted.

□

Example 3. Given 5 points in a square of side length 2, show that there must exist two points whose mutual distance is $\leq \sqrt{2}$.



Proof. Divide square into 4 smaller squares. We now have 4 pigeonholes and 5 dots:



These two points in the same pigeonhole have distance $\leq \sqrt{1^2 + 1^2} = \sqrt{2}$. \square

Example 4. There exists two people in NYC who have exactly the same number of hairs on their head.

Example 5. There are 30 people at a party talking with each other. Afterwards, there will be two people who talked with the same number of people.

Proof. If we put a person who talked to i people into box i , we get 30 boxes; however, we cannot have someone who talked to 0 people and someone who talked to 29 people at the same time! Hence, we combine the box 0 and box 29, and only one of which could be the case.

Now we have 29 boxes and 30 people. By pigeonhole principle, there must be two people who talked with the same amount of people. \square

Theorem 3 (Strong Pigeonhole Principle). Given pigeonholes $1, 2, \dots, n$ with capacities c_1, c_2, \dots, c_n where $c_i \geq 0$; if we have at least $c_1 + c_2 + \dots + c_n + 1$ pigeons in these pigeonholes, then at least one pigeonhole overflows.

Proof. Suppose BWOC that no pigeonhole overflows. Then for all $i = 1, 2, \dots, n$, we have the number of pigeons in $i \leq c_i$.

We add up and get inequalities:

$$\text{total \# pigeons} \leq c_1 + c_2 + \dots + c_n$$

Contradiction! \square

Example 6. There are five people supporting two teams. Then at least one team is supported by 3 people.

Proof. Assume BWOC that the two teams only have two supporters. Let $c_1 = c_2 = 2$. However, by SPP, $5 \geq 2 + 2 + 1$, hence one pigeonhole overflows. Therefore, one team must have > 2 supporters. \square

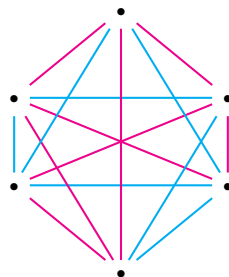
B2 First Ramsey Theorem

There are 6 people taking a class. Then:

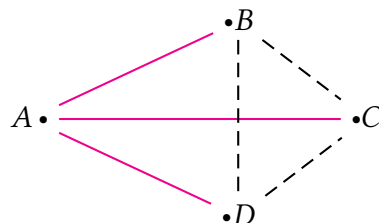
either there exists 3 people such that each pair of them have previously taken a class together,
or (inclusive) there exists 3 people such that no two have taken a class together.

Theorem 4. If we have 6 vertices and we draw all edges between them (a K_6 graph), then for every possible way of coloring the edges **red** and **blue**, there must exist a **monochromatic** triangle.

← K_6 stands for *complete graph on 6 vertices*. It has 15 edges.



Proof. Pick any vertex and call it A . It has 5 edges colored **red** and **blue**. By the SPP, there exists at least 3 edges of the same color. WLOG let these three edges be **red** and call the other three vertices B, C, D .

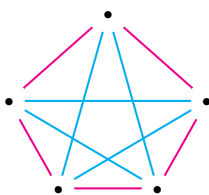


- If BC is **red**, then ABC is a **red** triangle.
- If CD is **red**, then ACD is a **red** triangle.
- If BD is **red**, then ABD is a **red** triangle.
- If none of the above has happened, then BC, CD, BD are all **blue**, meaning that BCD is a **blue** triangle!

□

Theorem 5. If there are 5 instead of 6 vertices, then the above coloring prediction cannot be made with certainty.

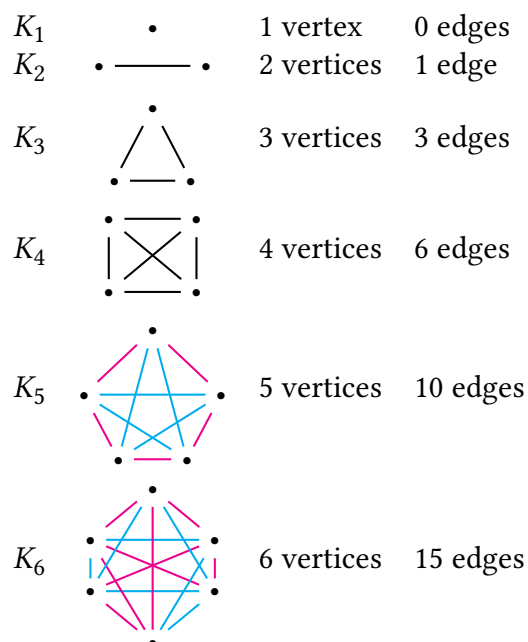
Counterexample.



□

B3 $K_p \rightarrow K_q, K_r$

In graphy theory, K_n is the **complete** graph on n vertices.



Remark. Note that K_n has $1 + 2 + 3 + \dots + (n - 1) = \frac{n(n-1)}{2}$ edges, hence is the $n - 1$ -th triangular number.

Ramsey Theory uses the following language convention: the expression

$$K_p \rightarrow K_q, K_r$$

represents a statement with the following meaning:

Definition 4. If the edges of K_p are colored red/blue, then it necessarily follows that either the K_p contains a red K_q , or K_p contains a blue K_r (or possibly both).

We want to know whether this statement is true for a given triple of (p, q, r) .

Example 7. We proved in B2 that $K_6 \rightarrow K_3, K_3$.

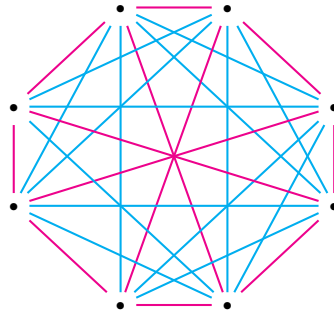
Non-example 8. We also showed that $K_5 \rightarrow K_3, K_3$ is false by exhibiting a coloring of K_5 that does not have a red or blue triangle (counterexample).

← write $K_5 \not\rightarrow K_3, K_3$

Example 9. It is known that $K_{18} \rightarrow K_4, K_4$ and $K_{17} \not\rightarrow K_4, K_4$.

Example 10. Also, $K_9 \rightarrow K_3, K_4$ and $K_8 \not\rightarrow K_3, K_4$.

← Here we have to decide in advance which color goes with the K_3 and which goes with the K_4 due to asymmetry.



This K_8 has no red triangle and no blue K_4 .

Theorem 6 (Ramsey). Let q, r be positive integers. Then there always exists a positive integer p such that

$$K_p \rightarrow K_q, K_r$$

is true.

We would see the following tabel giving us values of p that work.

Define a function $N(q, r)$ recursively:

← $q, r \in \mathbb{Z}^+$

- Base case: $N(1, r) = N(q, 1) = 1$
- Recurrence: $N(q, r) = N(q - 1, r) + N(q, r - 1)$ if $q, r \geq 2$.

We compute the value of $N(q, r)$ for:

← They do look like simplex numbers!

$q \backslash r$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126
6	1	6	21	56	126	252

We would want to prove that $K_{N(q,r)} \rightarrow K_q, K_r$ for all $q, r \geq 1$.

Proof. By induction.

Base case: If $q = r = 1$, then $N = 1$, we need to show that $K_1 \rightarrow K_1, K - r$
and $K_1 \rightarrow K_q, K_1$

for all q, r .

That is, suppose K_1 has its edges colored red/blue, then there exists a red K_1 or a blue K_r , and *vice versa*.

Since there are no edges, this is vacuously true.

Inductive step: We will show that if we are given that A, B are numbers such that

$$K_A \rightarrow K_{q-1}, K_r$$

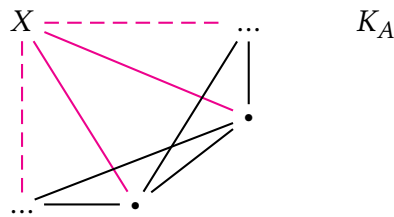
$$\text{and } K_B \rightarrow K_q, K_{r-1}$$

are true, then $K_{A+B} \rightarrow K_q, K_r$.

Consider K_{A+B} colored red and blue. We will show that it has a red K_q or a blue K_r .

Pick a vertex and call it X . It would have $A + B - 1$ edges in total. We claim that X either has at least A red edges, or at least B blue edges. This is indeed true, because if not, the number of red edges would be $\leq A - 1$ and the number of blue edges would be $\leq B - 1$ and so the total number of edges would be $\leq A + B - 2 < A + B - 1$, which is a contradiction.

Now, if X has a red claw of size A :



From our inductive hypothesis $K_A \rightarrow K_{q-1}, K_r$, we must have

either red K_{q-1} , in which case we combine with the vertex X and the red claw to get at least one red K_q .

or blue K_r , in which case we are done.

Similarly, if X has a blue claw of size B , then we make the same argument.

Hence, we know that $K_{A+B} \rightarrow K_q, K_r$ is true whenever $K_A \rightarrow K_{q-1}, K_r$ and $K_B \rightarrow K_q, K_{r-1}$.

□

← This works because as we fill out the table above, each new number we write in will work because it's the sum of the left and above numbers and they both work.

← The neighbouring vertices connected by black edges form K_A , yet to be colored.

B4 Ramsey Numbers

Recall: Let m, n be positive integers. We know that there are numbers $p \in \mathbb{N}$ such that $K_p \rightarrow K_m, K_n$.

Remark. If p works, then so does any $q \geq p$ as K_q would contain copies of K_p .

So the question becomes, if we have $K_p \rightarrow K_m, K_n$, is p the **smallest** such number?

Definition 5. The **Ramsey number** $r(m, n)$ is the smallest such number.

Example 11. We know $K_6 \rightarrow K_3, K_3$ but $K_5 \not\rightarrow K_3, K_3$, so $r(3, 3) = 6$.

Example 12. Mathematicians have proved that

$$\begin{aligned} K_{48} &\rightarrow K_5, K_5 \\ K_{42} &\not\rightarrow K_5, K_5 \end{aligned}$$

so we have $43 \leq r(5, 5) \leq 48$.

Remark. In general,

$$\begin{aligned} K_N \rightarrow K_m, K_n &\iff r(m, n) \leq N \\ K_{N-1} \not\rightarrow K_m, K_n &\iff r(m, n) \geq N \end{aligned}$$

Need both to get the precise value of $r(m, n)$.

Proposition 7. Properties of Ramsey numbers:

- (a) $r(3, 3) = 6$ ← proven in **B2**
- (b) $r(m, n) = r(n, m)$ ← symmetry
- (c) $r(1, n) = 1$ ← $K_1 \rightarrow K_1, K_n$,
 $K_0 \not\rightarrow K_1, K_n$
- (d) $r(2, n) = n$
- (e) $r(m, n) \leq r(m-1, n) + r(m, n-1)$ for all $m, n \geq 2$

Proof for (d). Claim: $K_2 \rightarrow K_2, K_n, K_{n-1} \not\rightarrow K_2, K_n$.

Color K_n . If all edges are blue then we have a **blue** K_n . Else we have some red edges, so we have some **red** K_2 .

Now color K_{n-1} all blue: we realize that we don't have any red K_2 , but we don't have a blue K_n either! □

Proof for (e). Let $A = r(m-1, n)$, $B = r(m, n-1)$. We have shown that if $K_A \rightarrow K_{m-1}, K_n$ and $K_B \rightarrow K_m, K_{n-1}$, then $K_{A+B} \rightarrow K_m, K_n$. Hence $r(m, n) \leq A + B$. □

Known facts:

$$\begin{aligned} r(2, 2) &= 2 \\ r(3, 3) &= 6 \\ r(4, 4) &= 18 \\ 43 &\leq r(5, 5) \leq 48 \\ 102 &\leq r(6, 6) \leq 165 \end{aligned}$$

B5 A lower bound for $r(m, n)$

Our [table of \$N\(m, n\)\$](#) gave us upper bounds for $r(m, n)$. Specifically,

$$r(m, n) \leq N(m, n) = \frac{(n + m - 2)!}{(n - 1)!(m - 1)!} = \binom{m + n - 2}{m - 1}$$

What about lower bound?

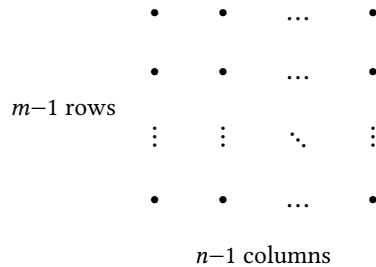
Theorem 8.

$$r(m, n) \geq (m - 1)(n - 1) + 1$$

if and only if $K_{(m-1)(n-1)} \not\rightarrow K_m, K_n$

Proof. We prove this by exhibiting a coloring of $K_{(m-1)(n-1)}$ that has no red K_m , no blue K_n .

Place vertices in grid:



Coloring rule of edges: If two vertices are in the same row, color the edges [blue](#). If two vertices are in the same column, color the edges [red](#). Every other edge arbitrary.

Claim: there exists no [red](#) K_m .

Consider the m vertices of such a K_m . There are $m - 1$ rows. Pigeonhole principle ensures that some vertices must be in the same row. But that edge must be [blue](#)! So this is not a red K_m . Similarly, there are no blue K_n . \square

Thus, we get: $(m - 1)(n - 1) + 1 \leq r(m, n) \leq \frac{(n + m - 2)!}{(n - 1)!(m - 1)!} = \binom{m + n - 2}{m - 1}$.

Observe there is still a huge gap between the bounds. Could we get better?

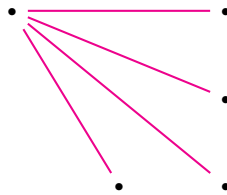
B6 The “parity” improvement

Our methods have shown that $K_{10} \rightarrow K_3, K_4$. But it is actually true that $K_9 \rightarrow K_3, K_4$. Why?

Proof. Given K_9 colored red or blue. We seek a **red** K_3 or a **blue** K_4 .

That is to say that if we ever see a **red 4-claw**, then we are done!

← See [this](#) argument.



In addition, if we ever see a **blue 6-claw**, then we are also done because $K_6 \rightarrow K_3, K_3$ and we either have a **red** K_3 or a **blue** K_3 , which would have to combine with the other vertex to get a blue K_4 .

Now suppose we neither have a red 4-claw nor a blue 6-claw. This implies that each vertex has ≤ 3 **red** edges, and ≤ 5 blue edges. However, in a K_9 , each vertex only has 8 edges, so they must exactly each have 3 red edges and 5 blue edges. Does this exist? We realize that to make this happen, we have:

- 9 vertices
- Each vertex has 3 red edges
- Every edge belongs to two vertices

Hence, we need to have exactly $\frac{3 \times 9}{2} = 13.5$ red edges, but this cannot happen because we need a whole number of edges! Thus, it is not possible that we neither have a red 4-claw nor a blue 6-claw. \square

Lemma 9 (Ramsey inductive step improved by parity). Suppose

← Also seen [here](#).

$$K_A \rightarrow K_{q-1}, K_r$$

$$\text{and } K_B \rightarrow K_q, K_{r-1}$$

are true, then $K_{A+B} \rightarrow K_q, K_r$.

In addition, if A, B are **both even numbers**, then $K_{A+B-1} \rightarrow K_q, K_r$.

B7 Variations

More colors!

For example:

$$K_p \rightarrow K_a, K_b, K_c$$

(given K_p colored red, blue, green, it must contain a red K_a , or a blue K_b , or a green K_c .)

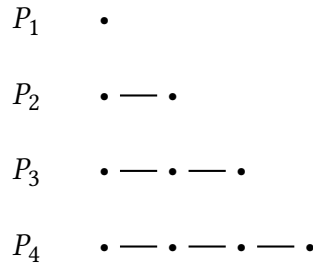
Example 13. It is known that $K_{17} \rightarrow K_3, K_3, K_3$.

Proof sketch. Pick a vertex which has 16 edges. We observe $16 \div 3 = 5\frac{1}{3} \implies$ at least one color occurs 6 times (i.e. we can see a red/blue/green 6-claws). \square

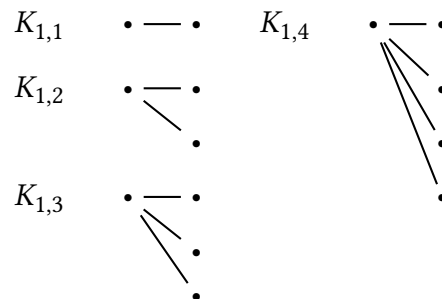
Remark. $r(a, b, c)$ is the smallest number that works for the above.

Other graphs

Paths:



Claws:



Example 14. Show that $r(K_{1,3}, K_{1,3}) = 6$.

Proof. We know $K_6 \rightarrow K_3, K_3$. Pick a vertex that has 5 neighbors. By the strong pigeonhole principle, we must have three edges of the same color \implies either red or blue $K_{1,3}$. \square