MATH103 Combinatorics Notes

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January 24, 2024

Contents

A Recurrence Relations

A1 Intro

Remark. Let there be a set $\{1, 2, \ldots, n\}$. The number of subsets of it is 2^n since for each number, we could say "include" or "exclude".

Example 1. Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

(Scratch work begins)

First consider $n = 0$. Then the only *good* subset is $\varnothing$.

Now consider $n = 1$, both $\varnothing, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\varnothing, 1, 2, 12$. The set 12 is not good. ← notation simplified for fast typing

Similarly, we have $f(3) = 5, f(5) = 8$.

(Scratch ends here)

We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

A2 Fibonnacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

Remark. Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and ← Textbook
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$. ← Preferred!

Example 2. Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbds one step per stride, sometimes two steps per stride. In how many ways can she do this? ← It is the same recurrence as A1 but with init conditions shifted: $f(n) = F_{n+1} = f_{n+2}$.

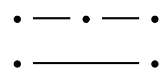
Table 1: Table of the sequence in two notations

n	0	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21	34
f_n	0	1	1	2	3	5	8	13	21

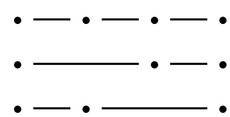
(Scratch work begins) Let $S(n)$ be the number of ways climbing n steps.

- $S(1) = 1$

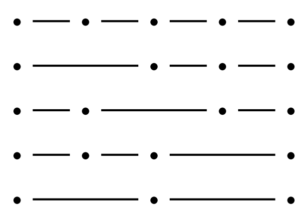
- $S(2) = 2$



- $S(3) = 3$



- $S(4) = 5$



Conjecture: maybe Fibonnacci?

(Scratch ends here)

Proof. Consider the set of ways she can cover n steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n - 1$ steps. Thus, this gives us $S(n - 1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n - 2$ steps. Thus, this gives us $S(n - 2)$ ways.

Therefore, we conclude that $S(n) = S(n - 1) + S(n - 2)$. We account the initial conditions and conclude the closed form:

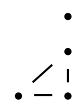
$$S(n) = F_n = f_{n+1}$$

for all n. Since Prof Rad climbs 47 steps, we get $\boxed{S(47) = 4807526976}$. □

A3 Simplex numbers

Definition 1. *Two-dimensional* triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- …



$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

Theorem 1. $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

First proof. We prove by induction.

← Not as good of a proof: we must know what we are proving in the first place!

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.
Inductive hypothesis: Suppose proved formula for up to $n = k$.
Inductive step: Consider $n = k + 1$.

$$\begin{aligned} T_2(k+1) &= 1 + \cdots + k + (k+1) \\ &= T_2(k) + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k^2 + k + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)\left((k+1)+1\right)}{2} \end{aligned}$$

□

Proof by Gauss. Observe:

← Better proof: concluding the formula without knowing it first!

$$\begin{aligned} T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\ &= n + (n-1) + \cdots + 2 + 1 \end{aligned}$$

Therefore, we **add** the two rows:

$$
\begin{aligned}
2T_2(n) &= \underbrace{(n+1)+(n+1)+\cdots+(n+1)}_{n} \\
&= n(n+1) \\
\therefore \quad T_2(n) &= \frac{1}{2}n(n+1)
\end{aligned}
$$

□

Definition 2. Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

- $T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$

Definition 3. Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$

B Ramsey Theory

Invented by Frank Ramsey in 1930. We would need:

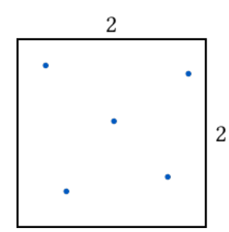
- Graph Theory
- Pigeonhole Principle
- Quantifiers
- Counterexamples

Pigeonhole principle

Theorem 2 (Dirichlet's Pigeonhole Principle)**.** If you put $n+1$ pigeons in n pigeonholes, then (at least) one pigeonhole will contain (at least) two pigeons.

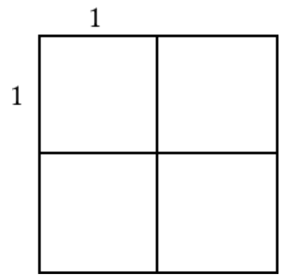
Proof omitted. □

Example 3. Given 5 points in a square of side length 2, show that there must exist two points whose mutual distance is $\leq \sqrt{2}$.



$\leftarrow$

Proof. Divide square into 4 smaller squares. We now have 4 pigeonholes and 5 dots:



These two points in the same pigeonhole have distance $\leq \sqrt{1^2 + 1^2} = \sqrt{2}$. $\square$

Example 4. There exists two people in NYC who have exactly the same number of hairs on their head.

Example 5. There are 30 people at a party talking with each other. Afterwards, there will be two people who talked with the same number of people.

Proof. If we put a person who talked to i people into box i, we get 30 boxes; however, we cannot have someone who talked to 0 people and someone who talked at 29 people at the same time! Hence, we combine the box 0 and box 29, and only one of which could be the case.

Now we have 29 boxes and 30 people. By pigeonhole principle, there must be two people who talked with the same amount of people. $\square$

Theorem 3 (Strong Pigeonhole Principle)**.** Given pigeonholes $1, 2, \dots, n$ with capacities $c_1, c_2, \dots, c_n$ where $c_i \geq 0$; if we have at least $c_1 + c_2 + \cdots + c_n + 1$ pigeons in these pigeonholes, then at least one pigeonhole overflows.

Proof. Suppose BWOC that no pigeonhole overflows. Then for all $i = 1, 2, \dots, n$, we have the number of pigeons in $i \leq c_i$.

We add up and get inequalities:

$$\text{total \# pigeons} \leq c_1 + c_2 + \cdots + c_n$$

Contradiction! $\square$

Example 6. There are five people supporting two teams. Then at least one team is supported by 3 people.

Proof. Assume BWOC that the two teams only have two supporters. Let $c_1 = c_2 = 2$. However, by SPP, $5 \geq 2 + 2 + 1$, hence one pigeonhole overflows. Therefore, one team must have > 2 supporters. $\square$

First Ramsey Theorem

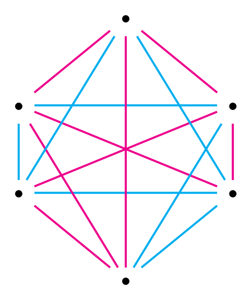
There are 6 people taking a class. Then:

either there exists 3 people such that each pair of them have previously taken a class together,

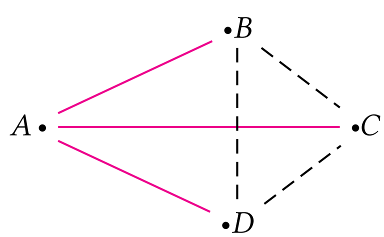
or (inclusive) there exists 3 people such that no two have taken a class together.

Theorem 4. If we have 6 vertices and we draw all edges between them (a K_6 graph), then for every possible way of coloring the edges red and blue, there must exist a **monochromatic** triangle.

← K_6 stands for *complete graph on 6 vertices.* It has 15 edges.



Proof. Pick any vertex and call it A. It has 5 edges colored red and blue. By the SPP, there exists at least 3 edges of the same color. WLOG let these three edges be red and call the other three vertices B, C, D.

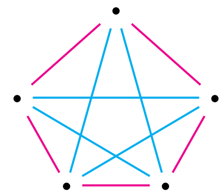


- If BC is red, then ABC is a red triangle.
- If CD is red, then ACD is a red triangle.
- If BD is red, then ABD is a red triangle.
- If none of the above has happened, then BC, CD, BD are all blue, meaning that BCD is a blue triangle!

□

Theorem 5. If there are 5 instead of 6 vertices, then the above coloring prediction cannot be made with certainty.

Counterexample.



□