

MATH103 Combinatorics Notes

Xuehuai He

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A Recurrence Relations

A1 Intro

Remark. Let there be a set $\{1, 2, \dots, n\}$. The number of subsets of it is 2^n since for each number, we could say “include” or “exclude”.

Example 1. Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

(Scratch work begins)

First consider $n = 0$. Then the only *good* subset is \emptyset .

Now consider $n = 1$, both $\emptyset, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\emptyset, 1, 2, 12$. The set 12 is not good.

Similarly, we have $f(3) = 5, f(5) = 8$.

(Scratch ends here)

We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

← notation simplified for fast typing

A2 Fibonnacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

Remark. Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$.

← Textbook

← Preferred!

Example 2. Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbds one step per stride, sometimes two steps per stride. In how many ways can she do this?

← It is the same recurrence as A1 but with init conditions shifted:
 $f(n) = F_{n+1} = f_{n+2}$.

Table 1: Table of the sequence in two notations

n	0	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21	34
f_n	0	1	1	2	3	5	8	13	21

(Scratch work begins) Let $S(n)$ be the number of ways climbing n steps.

- $S(1) = 1$ • — •
- $S(2) = 2$ • — • — •
• ————— •
- $S(3) = 3$ • — • — • — •
• ————— • — •
• — • ————— •
- $S(4) = 5$ • — • — • — • — •
• ————— • — • — •
• — • ————— • — •
• — • — • ————— •
• ————— • ————— •

Conjecture: maybe Fibonacci?

(Scratch ends here)

Proof. Consider the set of ways she can cover n steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n - 1$ steps. Thus, this gives us $S(n - 1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n - 2$ steps. Thus, this gives us $S(n - 2)$ ways.

Therefore, we conclude that $S(n) = S(n - 1) + S(n - 2)$. We account the initial conditions and conclude the closed form:

$$S(n) = F_n = f_{n+1}$$

for all n . Since Prof Rad climbs 47 steps, we get $S(47) = 4807526976$. \square

A3 Simplex numbers

Definition 1. Two-dimensional triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- ...



1, 3, 6, 10, 15, 21, 28, 36, 45, 55, ...

Theorem 1. $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

First proof. We prove by induction.

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.

Inductive hypothesis: Suppose proved formula for up to $n = k$.

Inductive step: Consider $n = k + 1$.

$$\begin{aligned}
 T_2(k+1) &= 1 + \cdots + k + (k+1) \\
 &= T_2(k) + k + 1 \\
 &= \frac{k(k+1)}{2} + k + 1 \\
 &= \frac{k^2 + k + 2(k+1)}{2} \\
 &= \frac{k^2 + 3k + 2}{2} \\
 &= \frac{(k+1)(k+2)}{2} \\
 &= \frac{(k+1)((k+1)+1)}{2}
 \end{aligned}$$

□

Proof by Gauss. Observe:

$$\begin{aligned}
 T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\
 &= n + (n-1) + \cdots + 2 + 1
 \end{aligned}$$

← Not as good of a proof: we must know what we are proving in the first place!

← Better proof: concluding the formula without knowing it first!

Therefore, we **add** the two rows:

$$\begin{aligned} 2T_2(n) &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_n \\ &= n(n+1) \\ \therefore T_2(n) &= \frac{1}{2}n(n+1) \end{aligned}$$

□

Definition 2. Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

- $T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$

Definition 3. Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$

Some examples of simplex numbers $T_d(n)$:

$d \backslash n$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	1	3	6	10	15	21	28
3	1	4	10	20	35	56	84
4	1	5	15	35	70	126	210
5	1	6	21	56	126	252	462

B Ramsey Theory

Invented by Frank Ramsey in 1930. We would need:

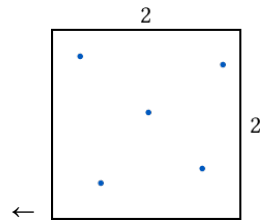
- Graph Theory
- Pigeonhole Principle
- Quantifiers
- Counterexamples

B1 Pigeonhole principle

Theorem 2 (Dirichlet's Pigeonhole Principle). If you put $n + 1$ pigeons in n pigeonholes, then (at least) one pigeonhole will contain (at least) two pigeons.

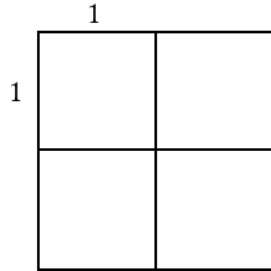
Proof omitted.

□



Example 3. Given 5 points in a square of side length 2, show that there must exist two points whose mutual distance is $\leq \sqrt{2}$.

Proof. Divide square into 4 smaller squares. We now have 4 pigeonholes and 5 dots:



These two points in the same pigeonhole have distance $\leq \sqrt{1^2 + 1^2} = \sqrt{2}$. \square

Example 4. There exists two people in NYC who have exactly the same number of hairs on their head.

Example 5. There are 30 people at a party talking with each other. Afterwards, there will be two people who talked with the same number of people.

Proof. If we put a person who talked to i people into box i , we get 30 boxes; however, we cannot have someone who talked to 0 people and someone who talked to 29 people at the same time! Hence, we combine the box 0 and box 29, and only one of which could be the case.

Now we have 29 boxes and 30 people. By pigeonhole principle, there must be two people who talked with the same amount of people. \square

Theorem 3 (Strong Pigeonhole Principle). Given pigeonholes $1, 2, \dots, n$ with capacities c_1, c_2, \dots, c_n where $c_i \geq 0$; if we have at least $c_1 + c_2 + \dots + c_n + 1$ pigeons in these pigeonholes, then at least one pigeonhole overflows.

Proof. Suppose BWOC that no pigeonhole overflows. Then for all $i = 1, 2, \dots, n$, we have the number of pigeons in $i \leq c_i$.

We add up and get inequalities:

$$\text{total \# pigeons} \leq c_1 + c_2 + \dots + c_n$$

Contradiction! \square

Example 6. There are five people supporting two teams. Then at least one team is supported by 3 people.

Proof. Assume BWOC that the two teams only have two supporters. Let $c_1 = c_2 = 2$. However, by SPP, $5 \geq 2 + 2 + 1$, hence one pigeonhole overflows. Therefore, one team must have > 2 supporters. \square

B2 First Ramsey Theorem

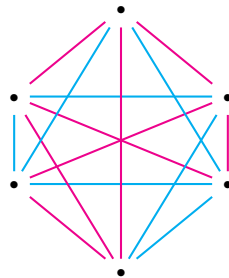
There are 6 people taking a class. Then:

either there exists 3 people such that each pair of them have previously taken a class together,

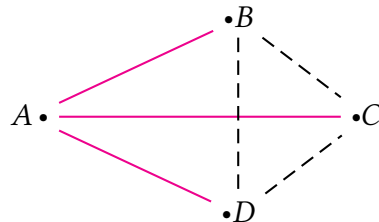
or (inclusive) there exists 3 people such that no two have taken a class together.

Theorem 4. If we have 6 vertices and we draw all edges between them (a K_6 graph), then for every possible way of coloring the edges **red** and **blue**, there must exist a **monochromatic** triangle.

← K_6 stands for complete graph on 6 vertices. It has 15 edges.



Proof. Pick any vertex and call it A . It has 5 edges colored **red** and **blue**. By the SPP, there exists at least 3 edges of the same color. WLOG let these three edges be **red** and call the other three vertices B, C, D .

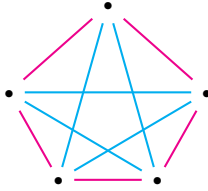


- If BC is **red**, then ABC is a **red** triangle.
- If CD is **red**, then ACD is a **red** triangle.
- If BD is **red**, then ABD is a **red** triangle.
- If none of the above has happened, then BC, CD, BD are all **blue**, meaning that BCD is a **blue** triangle!

□

Theorem 5. If there are 5 instead of 6 vertices, then the above coloring prediction cannot be made with certainty.

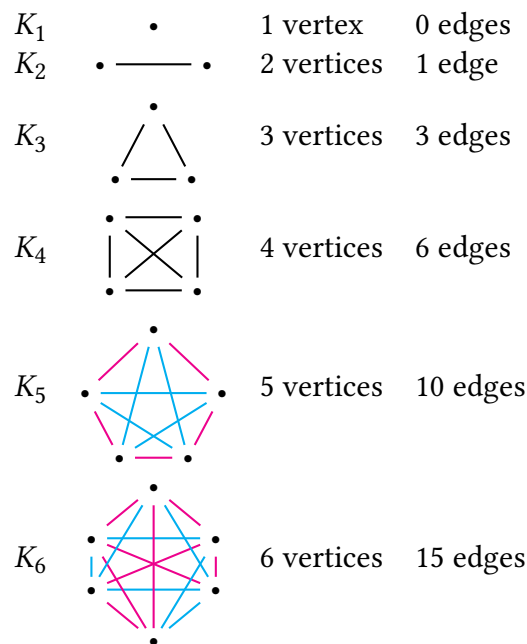
Counterexample.



□

B3 $K_p \rightarrow K_q, K_r$

In graphy theory, K_n is the **complete** graph on n vertices.



Remark. Note that K_n has $1 + 2 + 3 + \cdots + (n - 1) = \frac{n(n-1)}{2}$ edges, hence is the $n - 1$ -th triangular number.

Ramsey Theory uses the following language convention: the expression

$$K_p \rightarrow K_q, K_r$$

represents a statement with the following meaning:

Definition 4. If the edges of K_p are colored red/blue, then it necessarily follows that either the K_p contains a red K_q , or K_p contains a blue K_r (or possibly both).

We want to know whether this statement is true for a given triple of (p, q, r) .

Example 7. We proved in B2 that $K_6 \rightarrow K_3, K_3$.

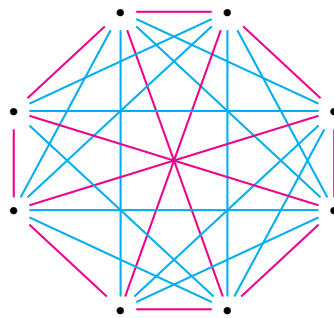
Non-example 8. We also showed that $K_5 \rightarrow K_3, K_3$ is false by exhibiting a coloring of K_5 that does not have a red or blue triangle (counterexample).

← write $K_5 \not\rightarrow K_3, K_3$

Example 9. It is known that $K_{18} \rightarrow K_4, K_4$ and $K_{17} \not\rightarrow K_4, K_4$.

Example 10. Also, $K_9 \rightarrow K_3, K_4$ and $K_8 \not\rightarrow K_3, K_4$.

← Here we have to decide in advance which color goes with the K_3 and which goes with the K_4 due to asymmetry.



This K_8 has no red triangle and no blue K_4 .

Theorem 6 (Ramsey). Let q, r be positive integers. Then there always exists a positive integer p such that

$$K_p \rightarrow K_q, K_r$$

is true.

We would see the following tabel giving us values of p that work.

Define a function $N(q, r)$ recursively:

← $q, r \in \mathbb{Z}^+$

- Base case: $N(1, r) = N(q, 1) = 1$
- Recurrence: $N(q, r) = N(q - 1, r) + N(q, r - 1)$ if $q, r \geq 2$.

We compute the value of $N(q, r)$ for:

← They do look like simplex numbers!

$q \backslash r$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126
6	1	6	21	56	126	252

We would want to prove that $K_{N(q,r)} \rightarrow K_q, K_r$ for all $q, r \geq 1$.

Proof. By induction.

Base case: If $q = r = 1$, then $N = 1$, we need to show that $K_1 \rightarrow K_1, K - r$
and $K_1 \rightarrow K_q, K_1$
for all q, r .

That is, suppose K_1 has its edges colored red/blue, then there exists a red K_1
or a blue K_r , and *vice versa*.

Since there are no edges, this is vacuously true.

Inductive step: We will show that if we are given that A, B are numbers such
that

$$K_A \rightarrow K_{q-1}, K_r$$

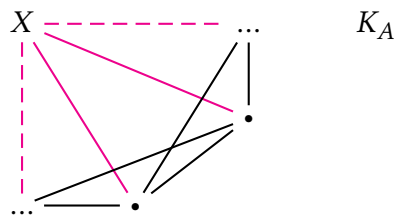
$$\text{and } K_B \rightarrow K_q, K_{r-1}$$

are true, then $K_{A+B} \rightarrow K_q, K_r$.

Consider K_{A+B} colored red and blue. We will show that it has a red K_q or a
blue K_r .

Pick a vertex and call it X . It would have $A + B - 1$ edges in total. We
claim that X either has at least A red edges, or at least B blue edges. This is
indeed true, because if not, the number of red edges would be $\leq A - 1$ and
the number of blue edges would be $\leq B - 1$ and so the total number of edges
would be $\leq A + B - 2 < A + B - 1$, which is a contradiction.

Now, if X has a red claw of size A :



From our inductive hypothesis $K_A \rightarrow K_{q-1}, K_r$, we must have

either red K_{q-1} , in which case we combine with the vertex X and the red
claw to get at least one red K_q .

or blue K_r , in which case we are done.

Similarly, if X has a blue claw of size B , then we make the same argument.

Hence, we know that $K_{A+B} \rightarrow K_q, K_r$ is true whenever $K_A \rightarrow K_{q-1}, K_r$ and
 $K_B \rightarrow K_q, K_{r-1}$.

← This works because
as we fill out the
table above, each
new number we
write in will work
because it's the sum
of the left and
above numbers and
they both work.

← The neighbouring
vertices connected
by black edges form
 K_A , yet to be
colored.

□

B4 Ramsey numbers

Recall: Let m, n be positive integers. We know that there are numbers $p \in \mathbb{N}$ such that $K_p \rightarrow K_m, K_n$.

Remark. If p works, then so does any $q \geq p$ as K_q would contain copies of K_p .

So the question becomes, if we have $K_p \rightarrow K_m, K_n$, is p the **smallest** such number?

Definition 5. The **Ramsey number** $r(m, n)$ is the smallest such number.

Example 11. We know $K_6 \rightarrow K_3, K_3$ but $K_5 \not\rightarrow K_3, K_3$, so $r(3, 3) = 6$.

Example 12. Mathematicians have proved that

$$\begin{aligned} K_{48} &\rightarrow K_5, K_5 \\ K_{42} &\not\rightarrow K_5, K_5 \end{aligned}$$

so we have $43 \leq r(5, 5) \leq 48$.

Remark. In general,

$$\begin{aligned} K_N \rightarrow K_m, K_n &\iff r(m, n) \leq N \\ K_{N-1} \not\rightarrow K_m, K_n &\iff r(m, n) \geq N \end{aligned}$$

Need both to get the precise value of $r(m, n)$.

Proposition 7. Properties of Ramsey numbers:

- | | |
|--|--------------------------------|
| (a) $r(3, 3) = 6$ | ← proven in B2 |
| (b) $r(m, n) = r(n, m)$ | ← symmetry |
| (c) $r(1, n) = 1$ | ← $K_1 \rightarrow K_1, K_n$, |
| (d) $r(2, n) = n$ | $K_0 \not\rightarrow K_1, K_n$ |
| (e) $r(m, n) \leq r(m-1, n) + r(m, n-1)$ for all $m, n \geq 2$ | |

Proof for (d). Claim: $K_2 \rightarrow K_2, K_n, K_{n-1} \not\rightarrow K_2, K_n$.

Color K_n . If all edges are blue then we have a **blue** K_n . Else we have some red edges, so we have some **red** K_2 .

Now color K_{n-1} all blue: we realize that we don't have any red K_2 , but we don't have a blue K_n either! \square

Proof for (e). Let $A = r(m-1, n)$, $B = r(m, n-1)$. We have shown that if $K_A \rightarrow K_{m-1}, K_n$ and $K_B \rightarrow K_m, K_{n-1}$, then $K_{A+B} \rightarrow K_m, K_n$. Hence $r(m, n) \leq A + B$. \square

Known facts:

$$\begin{aligned} r(2, 2) &= 2 \\ r(3, 3) &= 6 \\ r(4, 4) &= 18 \\ 43 &\leq r(5, 5) \leq 48 \\ 102 &\leq r(6, 6) \leq 165 \end{aligned}$$

B5 A lower bound for $r(m, n)$

Our [table of \$N\(m, n\)\$](#) gave us upper bounds for $r(m, n)$. Specifically,

$$r(m, n) \leq N(m, n) = \frac{(n + m - 2)!}{(n - 1)!(m - 1)!} = \binom{m + n - 2}{m - 1}$$

What about lower bound?

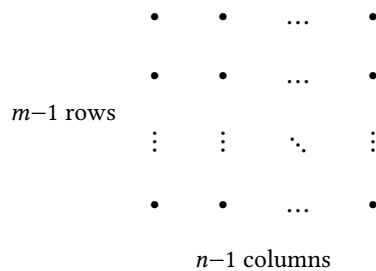
Theorem 8.

$$r(m, n) \geq (m - 1)(n - 1) + 1$$

if and only if $K_{(m-1)(n-1)} \not\rightarrow K_m, K_n$

Proof. We prove this by exhibiting a coloring of $K_{(m-1)(n-1)}$ that has no red K_m , no blue K_n .

Place vertices in grid:



Coloring rule of edges: If two vertices are in the same row, color the edges **blue**. If two vertices are in the same column, color the edges **red**. Every other edge arbitrary.

Claim: there exists no **red** K_m .

Consider the m vertices of such a K_m . There are $m - 1$ rows. Pigeonhole principle ensures that some vertices must be in the same row. But that edge must be **blue**! So this is not a red K_m . Similarly, there are no blue K_n . \square

Thus, we get: $(m-1)(n-1) + 1 \leq r(m, n) \leq \frac{(n+m-2)!}{(n-1)!(m-1)!} = \binom{m+n-2}{m-1}$.

Observe there is still a huge gap between the bounds. Could we get better?

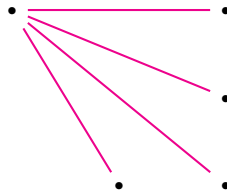
B6 The “parity” improvement

Our methods have shown that $K_{10} \rightarrow K_3, K_4$. But it is actually true that $K_9 \rightarrow K_3, K_4$. Why?

Proof. Given K_9 colored red or blue. We seek a **red K_3** or a **blue K_4** .

That is to say that if we ever see a **red 4-claw**, then we are done!

← See [this](#) argument.



In addition, if we ever see a **blue 6-claw**, then we are also done because $K_6 \rightarrow K_3, K_3$ and we either have a **red K_3** or a **blue K_3** , which would have to combine with the other vertex to get a blue K_4 .

Now suppose we neither have a red 4-claw nor a blue 6-claw. This implies that each vertex has ≤ 3 **red** edges, and ≤ 5 blue edges. However, in a K_9 , each vertex only has 8 edges, so they must exactly each have 3 red edges and 5 blue edges. Does this exist? We realize that to make this happen, we have:

- 9 vertices
- Each vertex has 3 red edges
- Every edge belongs to two vertices

Hence, we need to have exactly $\frac{3 \times 9}{2} = 13.5$ red edges, but this cannot happen because we need a whole number of edges! Thus, it is not possible that we neither have a red 4-claw nor a blue 6-claw. \square

Lemma 9 (Ramsey inductive step improved by parity). Suppose

← Also seen [here](#).

$$K_A \rightarrow K_{q-1}, K_r$$

$$\text{and } K_B \rightarrow K_q, K_{r-1}$$

are true, then $K_{A+B} \rightarrow K_q, K_r$.

In addition, if A, B are **both even numbers**, then $K_{A+B-1} \rightarrow K_q, K_r$.

B7 Variations

More colors!

For example:

$$K_p \rightarrow K_a, K_b, K_c$$

(given K_p colored red, blue, green, it must contain a red K_a , or a blue K_b , or a green K_c .)

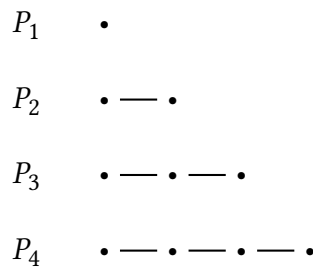
Example 13. It is known that $K_{17} \rightarrow K_3, K_3, K_3$.

Proof sketch. Pick a vertex which has 16 edges. We observe $16 \div 3 = 5\frac{1}{3} \implies$ at least one color occurs 6 times (i.e. we can see a red/blue/green 6-claws). \square

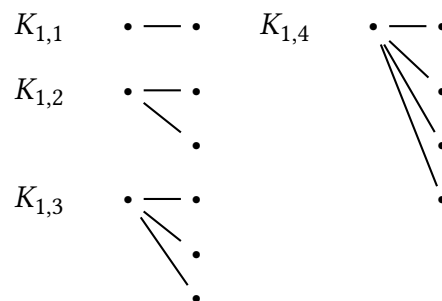
Remark. $r(a, b, c)$ is the smallest number that works for the above.

Other graphs

Paths:



Claws:



Example 14. Show that $r(K_{1,3}, K_{1,3}) = 6$.

Proof. We know $K_6 \rightarrow K_3, K_3$. Pick a vertex that has 5 neighbors. By the strong pigeonhole principle, we must have three edges of the same color \implies either red or blue $K_{1,3}$. \square

C Counting

C1 Three principles

Addition principle

Definition 6. If a set S is *partitioned* into subsets S_1, S_2, \dots, S_n , then the cardinality of S is

$$|S| = |S_1| + \dots + |S_n|.$$

\leftarrow aka. counting by cases

\leftarrow that is, $S = \bigcup S_i$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$

The art lies in:

- making each S_i easy to count, and
- not having too many S_i if there is no formula for $|S_i|$.

Remark (Variations). If the S_i cover S but they overlap, then we have the inequality $|S| < \sum_{i=1}^n |S_i|$ because the overlap implies that we are *overcounting*.

\leftarrow The inclusion/exclusion principle handles overlaps precisely

Example 15. Let S be the set of *good* subsets of $[5] = \{1, 2, 3, 4, 5\}$. We could first try:

\leftarrow *good* meaning no adjacent elements

- S_1 contains the subsets that contain 5
- S_2 contains the subsets that don't contain 5

We have previously shown that $|S_1|$ = number of *good* subsets of $[3]$ and $|S_2|$ = number of *good* subsets of $[4]$.

Alternatively, we could also let T_i be the *good* subsets of $[5]$ with cardinality i . Then

S is partitioned into $T_0 \cup T_1 \cup T_2 \cup T_3$. We count:

$T_0 :$	\emptyset	$ T_0 = 1$
$T_1 :$	1, 2, 3, 4, 5	$ T_1 = 5$
$T_2 :$	13, 14, 15, 24, 25, 35	$ T_2 = 6$
$T_3 :$	135	$ T_3 = 1$

$|S|=13$

Subtraction principle

Definition 7. Let $A \subseteq S$ and A^c be its complement in S . Then A, A^c partition S and $|S| = |A| + |A^c|$. This means that

$$|A| = |S| - |A^c|.$$

Example 16. How many 2-digit numbers have distinct nonzero digits?

Let S be the set of all 2-digit numbers $\{10, 11, \dots, 99\}$ and let A be the subset of those with nonzero distinct digits. We count:

$A^c : 11, 22, \dots, 99$	(distinct fails)
10, 20, \dots , 90	(nonzero fails)

Hence $|A| = |S| - |A^c| = 90 - 18 = 72$.

Multiplication principle

Definition 8. Suppose we have to do two tasks in sequence. We suppose:

- Task 1 has m outcomes
- Task 2 has n outcomes, regardless of how Task 1 was carried out.

← Note: sometimes the 2nd task could depend on the 1st one

Then there are mn ways of carrying out both tasks.

← Similarly for 3 or more tasks in sequence

Example 17. How many 2-digit numbers have distinct nonzero digits?

Let $\textcircled{a} \textcircled{b}$ be the two digits in these numbers. Let Task 1 be selecting digit \textcircled{a} and Task 2 be selecting digit \textcircled{b} .

- Task 1: 9 ways (1,2,...,9)
- Task 2: 8 ways (1,2,...,9 but not same as \textcircled{a})

Hence there are $9 \times 8 = 72$ such numbers.

Tricky example 18. How many **odd** numbers in the range 1000-9999 have distinct digits?

Attempt 1: Let $\textcircled{a} \textcircled{b} \textcircled{c} \textcircled{d}$ be the 4 digits in these numbers and assign them Tasks 1-4. We have:

- Task 1: 9 ways (1-9)
- Task 2: 9 ways (0-9 except \textcircled{a})
- Task 3: 8 ways (0-9 except $\textcircled{a}, \textcircled{b}$)
- **Task 4:** Could be 2 or 3 or 4 or 5 (depending on how many odd digits had already been used)

Hence, the best we can say here is that the answer is between $9 \times 9 \times 8 \times 2$ and $9 \times 9 \times 8 \times 5$.

← BAD! This is a large range!

Attempt 2: Let $\textcircled{a} \textcircled{b} \textcircled{c} \textcircled{d}$ be the 4 digits in these numbers and try the order $\textcircled{d} \textcircled{a} \textcircled{b} \textcircled{c}$ for Tasks 1-4. We have:

- Task 1: 5 ways (1, 3, 5, 7, 9)
- Task 2: 8 ways (1-9 except \textcircled{d})
- Task 3: 8 ways (0-9 except $\textcircled{a}, \textcircled{d}$)
- Task 4: 7 ways (0-9 except $\textcircled{a}, \textcircled{d}, \textcircled{b}$)

Therefore, the number of ways is $5 \times 8 \times 8 \times 7 = 2240$.

Example 19. How many numebrs 0, 1, ... 99999 have exactly one digit 6?

We can assign tasks:

- Task 1: choose a location for 6, giving us 5 ways
- Task 2: assign the remaining digits from left to right, giving us 9^4 ways

Hence there are $5 \times 9^4 = 32805$ such numbers.

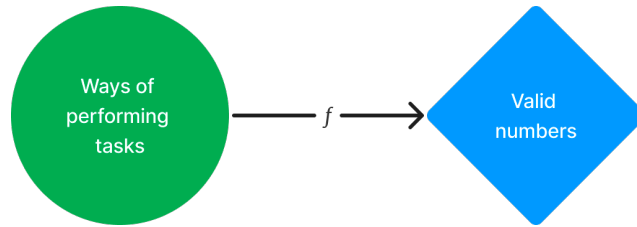
Non-example 20. How many integers 0, 1, ... 99999 have *at least* one digit 6?

Attempt: We can assign tasks:

- Task 1: choose a location for 6, giving us 5 ways
- Task 2: assign the remaining digits from left to right, giving us 10^4 ways

But the answer of 50000 is **wrong**! But why?

The counting process for this problem is corresponding the ways of performing tasks to valid 5-digit numbers:



We have correctly counting the **green** set. However, for this to count the **blue** set, we need f to be **bijective**. That is, every valid number must be obtained in exactly one way. However, in this case, our f is surjective but not injective. For instance, 62516 will be counted *twice*:

- 6___ then 62516; or
- ___6 then 62516.

Hence, 50000 > correct answer!

Correct way: Using the subtract principle to deduct numbers that don't have 6:
 $10^5 - 9^5 = 40951 < 50000$.

C2 Probability

Definition 9.

$$\text{Probability} = \frac{\text{number of favourable cases}}{\text{number of total cases}}$$

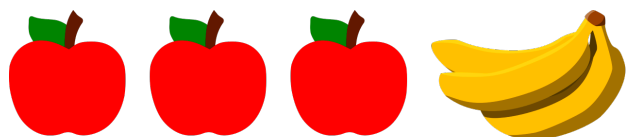
Example 21. Probability of 3 dice rolling the same number: $P = \frac{6}{6^3} = \frac{1}{36}$.

C3 The counting framework

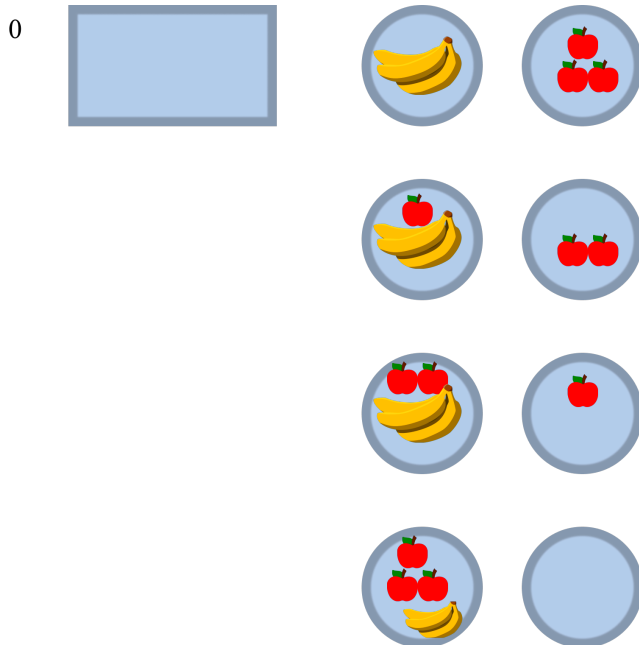
Here is a very general problem:

“How many distinguishable ways to map a multiset S to a multiset T satisfying given constraints?”

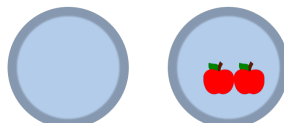
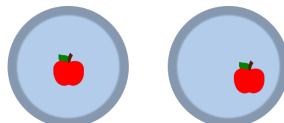
Example 22. There are fruits and plates. Let $S = \{3 \cdot \text{apple}, 1 \cdot \text{banana}\}$ and $T = \{2 \cdot \text{circle}, 1 \cdot \text{rectangle}\}$. How many ways are there to serve fruit on plates such that the rectangular plate has an even number of fruit?



Ans: Organize by number of fruit on rectangle.



2



4



Hence there are 9 ways.

The general counting problem

Let there be multisets

$$S = \{a_1 \cdot 1, a_2 \cdot 2, \dots, a_s \cdot s\}$$

object types $1, 2, \dots, s$

$$T = \{b_1 \cdot U_1, b_2 \cdot U_2, \dots, b_t \cdot U_t\}$$

box types U_1, \dots, U_t

How many distinguishable maps $f : S \rightarrow T$ are there, subject to restrictions on the numbers $u_i = |f^{-1}(U_i)|$?

← The number of items mapped to boxes of type U_i

Remark (Special cases). To recognize which situation applies to the problem:

- By objects:
 - Distinct: $S = \{1, 2, \dots, s\}$
 - Identical: $S = \{s \cdot 1\}$
- By boxes:

- Distinct: $T = \{U_1, \dots, U_t\}$
- Identical: $T = \{t \cdot U_1\}$
- For each case above, we can apply constraints:
 - $0 \leq u_i \leq 1$
 - $0 \leq u_i < \infty$ no constraint
 - $1 \leq u_i < \infty$ nonempty
 - $0 \leq u_i \leq n_i$ for some $n_i \in \mathbb{N}_0$ max. capacity
 - $u_i \in N_i \subseteq \mathbb{N}$

3.3 A Framework for Counting Questions: The Counting Table

95

Table 3.1 Balls and boxes counting problems.

Number of Ways to Put Balls into Boxes				
$S = S_1 = \{1, 2, \dots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls $T = T_1 = \{U_1, U_2, \dots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes Box U_i contains u_i balls				
Conditions on S and $T \rightarrow$ on $u_i \downarrow$	$T = T_1$ distinct $S = S_1$ distinct	$T = T_1$ distinct $S = S_2$ identical	$T = T_2$ identical $S = S_1$ distinct	$T = T_2$ identical $S = S_2$ identical
$0 \leq u_i \leq 1$ Assume $t \geq s$	1	2	3	4
$u_i \geq 0$	5	6	7	8
$u_i \geq 1$	9	10	11	12
for $i = 1, \dots, t$, $0 \leq u_i \leq n_i$, $n_i \in \mathbb{Z}^{\geq 0}$	13	14		
$u_i \in N_i \subset \mathbb{Z}^{\geq 0}$ for $i = 1, \dots, t$	15	16		

Example 23. We have 10 distinct books to be shared between 2 children. Each child needs 2 books to avoid a crisis.

← Table entry (15)

- Distinct objects $\{1, 2, \dots, 10\}$
- Distinct boxes $\{U_1, U_2\}$
- Constraints: $u_i \in [2, \infty[$

Example 24 (Attempt 1). 10 distinct books, 5 of them to be arranged on shelf (order matters)

← Table entry (15)

- $S = \{1, 2, \dots, 10\}$
- $T = \{U_1, U_2, \dots, U_5, U_6\}$ (positions on shelf + extra for unshelved books)
- $u_1 = \dots = u_5 = 1, u_6 = 5$

Example 25 (Attempt 2). 10 distinct books, 5 of them to be arranged on shelf (order matters)

← Table entry (1),
easier!

- $S = \{1, 2, \dots, 5\}$ (numbered stickers to arrange books)
- $T = \{U_1, U_2, \dots, U_{10}\}$ (10 books)
- $0 \leq u_1 \leq 1$ (each book can get 0 or 1 sticker)

Example 26. I have 10 books and will take 5 on holiday.

- Take 1:
 - $S = \{1, 2, \dots, 10\}$
 - $T = \{U_1, U_2\}$
 - $u_1 = u_2 = 5$
- Take 2:
 - $S = \{5 \cdot 1\}$ (identical ‘stickers’ marking on-holiday)
 - $T = \{U_1, U_2, \dots, U_{10}\}$
 - $0 \leq u_1 \leq 1$ (each book can get 0 or 1 sticker)

← Table entry (15)

← Table entry (2)

C4 Permutations of a set

Remark. Recall $[n] = \{1, 2, \dots, n\}$.

Definition 10. Let $0 \leq s \leq t$.

- An **s-permutation** of $[t]$ is an ordered list of s distinct elements of $[t]$.
- A **t-permutation** of $[t]$ is just a **permutation** of $[t]$.

Example 27. 10 books, arrange 5 on shelf: 5-perm of $[10]$

Example 28. 20 athletes, Gold, Silver and Bronze awarded: 3-perm of $[20]$

Theorem 10. The number of s -perms of $[t]$ is

$$\underbrace{t(t-1)\dots(t-s+1)}_{s \text{ terms}}$$

Proof. Select elements of the list one-by-one. We have t ways to pick the first, $t-1$ ways of picking the second, etc. \square

Definition 11 (s -th falling-factorial function). This inspires the following notation

$$(x)_s = x(x-1)\dots(x-s+1) = \frac{t!}{(t-s)!}$$

← There is also a rising $(x)^s = x(x+1)\dots(x+s-1)$

Remark. $(x)_0 = 1, (n)_n = n!$

Example 29. Number of ways to shelve 5 books out of 10 is

$$(10)_5 = 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

C5 Circular permutations

A circular s -perm of $[n]$ is an arrangement of s distinct elements of $[n]$ around a round table. The difference from a non-circular perm is that the orientation of the table does not matter! How many ways to do so?

If there is a head of the table and positions are marked clockwise, then it would be the same as the s -perm $(n)_s$.

However, we consider all ways of marking the ‘head’ of the table to be equivalent, so we divide s upon that. Hence, we get the answer $\frac{1}{s}(n)_s = \frac{n!}{s(n-s)!}$.

C6 Table entries 3,4,5

- $\textcircled{3}$: s distinct objects, t identical boxes, 0 or 1 per box.

$$\# \text{ ways} = \begin{cases} 1 & s \leq t \\ 0 & \text{otherwise} \end{cases}$$

- ④: s identical objects, t identical boxes, 0 or 1 per box.

$$\# \text{ ways} = \begin{cases} 1 & s \leq t \\ 0 & \text{otherwise} \end{cases}$$

- ⑤: s distinct objects, t distinct boxes, no restrictions.

$$\# \text{ ways} = \underbrace{t \times t \times \cdots \times t}_{s \text{ times}} = t^s$$

Remark. $0^0 = 1$ in combinatorics.

← NOT in analysis!!

3.3 A Framework for Counting Questions: The Counting Table 95

Table 3.1 Balls and boxes counting problems.

Number of Ways to Put Balls into Boxes				
$S = S_1 = \{1, 2, \dots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls $T = T_1 = \{U_1, U_2, \dots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes Box U_i contains u_i balls				
Conditions on S and $T \rightarrow$ on $u_i \downarrow$	$T = T_1$ distinct $S = S_1$ distinct	$T = T_1$ distinct $S = S_2$ identical	$T = T_2$ identical $S = S_1$ distinct	$T = T_2$ identical $S = S_2$ identical
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$u_i \geq 0$	5	6	7	8
$u_i \geq 1$	9	10	11	12
for $i = 1, \dots, t$, $0 \leq u_i \leq n_i$, $n_i \in \mathbb{Z}^{>0}$	13	14		
$u_i \in N_i \subset \mathbb{Z}^{\geq 0}$ for $i = 1, \dots, t$	15	16		

C7 Combinations of sets: table entries 2,6,10

If we let $0 \leq s \leq t$, then an s -**combination** of $[t]$ is a subset of size s of $[t]$.

Definition 12. The binomial coefficient $\binom{t}{s}$ is defined (combinatorially) to be the number of s -combinations of $[t]$.

Theorem 11.

$$\binom{t}{s} = \frac{(t)_s}{s!} = \frac{t!}{(t-s)!s!}$$

Proof. Consider the number of s -perms of $[t]$.

We can count it directly: $(t)_s$.

Alternatively, make task 1 ‘selecting a subset of size s ’, which gives us $\binom{t}{s}$ ways. Make task 2 the ways of ordering a subset, which give $s!$ ways. By multiplication principle, the number of s -perms of $[t]$ is $\binom{t}{s}s!$.

We have $(t)_s = \binom{t}{s}s!$, giving us the formula above. \square

Example 30. All of the following are equivalent and satisfy table entry $\textcircled{2}$:

$\leftarrow 0 \leq u_i \leq 1$

- Select s books out of t distinct books;
- Place s stickers on t books, at most one sticker per book;
- Put s identical objects into t distinct boxes, each box can only have at most 1 object.

Example 31. These equivalent problems satisfy table entry $\textcircled{6}$:

$\leftarrow 0 \leq u_i < \infty$

- Solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ where $x_i \geq 0$ are integers.
- Packing a box of 12 bagels of 5 different types of bagels with unlimited supply.
- Number of permutations of 12 objects and 4 ‘drawer dividers’:

$\bullet \bullet \mid \bullet \bullet \bullet \mid \bullet \bullet \mid \bullet \bullet \bullet \bullet \mid \bullet$

Remark. In general, when we count the number of anagrams $s \bullet s$ and $t - 1 \mid s$, there are $s + t - 1$ total symbols. We must select s of the $s + t - 1$ positions to be \bullet . Hence, the number of ways would be

\leftarrow LHS is choose \bullet ,
RHS is choose \mid .

$$\binom{s+t-1}{s} = \binom{s+t-1}{t-1}$$

Example 32. Entry $\textcircled{10}$ is the same as $\textcircled{6}$ except that each box must be nonempty:

- Solutions to $x_1 + \dots + x_s = t$ where $x_i \geq 1$ integers.
- Select s bagels from t types, with each type chosen at least once.
This reduces to the prev problem: select one of each type of bagel first; then, we choose $s - t$ bagels of t types.
- Anagrams with $s \bullet s$ and $t - 1 |$: must avoid $||$ and $|$ at beginning or end. Then we could think about placing $t - 1$ dividers in the $s - 1$ spaces between $\bullet s$, giving us $\binom{s-1}{t-1}$ ways.

C8 Anagrams

Example 33. Find the number of anagrams of the word

COMBINATORIALISTICALLY

First, suppose all 22 letters were distinct. We put subscripts:

C₁O₁M₁B₁I₁N₁A₁T₁O₂R₁A₂L₁I₃S₂T₄C₂A₃L₂L₃Y

Now we have this multiset:

{2C, 2O, M, B, 4I, N, 3A, 2T, R, 3L, S, Y}

We want to find the ways of arranging this!

- Assuming everything is distinct: $22!$ ways.
- Same assumption but with two tasks in multiplication:
 - Task 1: Arrange without subscripts (what we want)
 - Task 2: Add subscripts: the number of ways to add them is

$$2! \times 2! \times 1! \times 1! \times 4! \times 1! \times 3! \times 2! \times 1! \times 3! \times 1! \times 1!$$

Therefore, our answer would be (in multinomial coefficient)

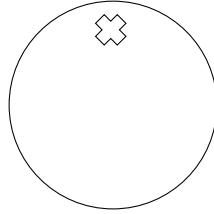
$$\frac{22!}{2! \times 2! \times 1! \times 1! \times 4! \times 1! \times 3! \times 2! \times 1! \times 3! \times 1! \times 1!} = \binom{22}{2, 2, 1, 1, 4, 1, 3, 2, 1, 3, 1, 1}$$

← DO NOT LEAVE
OUT 1s

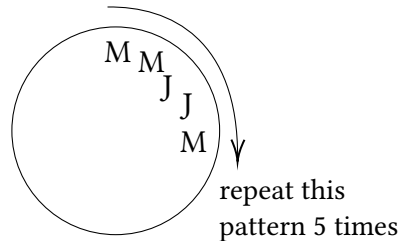
C9 More circular tables

Example 34. An alien conference has 9 Martian hare delegates and 16 Jovian hare delegates, with each type of hare identical. How many distinguishable ways are there to seat them at a circular table?

Method: first consider the ways of arrangement at a marked table.



- We seat 9 Martian hares and fill out the rest with Jovian ones: $\binom{25}{9}$.
- We place all hares (the answer we want) and then mark the table. In the end we should get the same answer of $\binom{25}{9}$.
 - How do we mark the table? There are 25 places that can be marked, but not all of them are distinct! For instance:



then the table only has 5 distinct markings due to rotational symmetry.

Theorem 12. Consider an arrangement on a circular table with n spots. Let R_k be the action of rotating this table by k places. Define:

← F is the *stabilizer* of the group action

$$F = \{k \in \mathbb{Z} \mid R_k \text{ leaves the arrangement unchanged}\}$$

Then we have:

1. F is the set of multiples of some $d \mid n$.
2. A length d pattern would be repeated $\frac{n}{d}$ times (so there are only $\frac{n}{d}$ distinct markings)

Proof. Let d be the smallest positive integer such that $R_d \in F$. Suppose BWOC $k \in F$ but $d \nmid k$. Then let $k = md + r$ with $r \leq d$, and rotation by r would be: $R_k R_{-md}$. Since $k \in F$, this also fixes the arrangement. However, $r < d$ contradicts the fact that d is the smallest positive integer such that $R_d \in F$. \square

← Review Abstract Alg. 1

Back to Example 34: $F = d\mathbb{Z}$ where $d = 1, 5, 25$.

- If $d = 1$, 25 ways of marking the table.

- $d = 5$, 5 ways of marking.
- $d = 25$, 1 way of marking.

Definition 13 (Multichoose notation). Define the ways to select a bag of k items from n different types of item to be $\binom{n}{k} = \binom{k+n-1}{k} = \binom{k+n-1}{n-1}$.

D Binomial Coefficients

We know that $\binom{n}{k}$ is the number of k -subsets of $[n]$, which is $\frac{n!}{k!(n-k)!}$.

D1 Binomial identities

Proposition 13.

$$\binom{n}{k} = \binom{n}{n-k}$$

That is, the number of ways to get k -subsets in $[n]$ is the same as that of $n-k$ -subsets.

Proposition 14.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

This is because the RHS counts **all** subsets of $[n]$, while each summand on the LHS counts the number of k -subsets thereof.

Proposition 15. For $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor} = 2^{n-1}$$

This is because the RHS chooses any subset of $[n-1]$, then makes the subset **even** by putting or not putting the n into it (the n doesn't get to choose).

← $\lfloor a \rfloor$ is the largest integer $\leq a$

Proposition 16 (Binomial recurrence).

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is because the LHS chooses k -subsets of $[n]$, and the RHS splits the case into 1) the subset contains n , which gives us $\binom{n-1}{k-1}$ ways to choose the rest, and 2) the subset doesn't contain n , which gives us $\binom{n-1}{k}$.

This allows us to construct the table:

$n \backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	2	1	0	0
3	1	3	3	1	0
4	1	4	6	4	1

Proposition 17.

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Say there are $2n$ students: n in class G and n in class S. We need to choose n students.

- Method 1: choose n students: $\binom{2n}{n}$
- Method 2: let $k = 0, 1, \dots, n$. Select k S to go and select k G to NOT go. Then we have $\binom{n}{k}$ for each of the process. Hence, the total number of ways is $\sum_{k=0}^n \binom{n}{k}^2$.

D2 Binomial theorem

Theorem 18. Let $n \geq 0$ be an integer. Then:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

← This gives a relationship between the Karaji/Pascal triangle and polynomial algebra.

Proof.

$$(x + y)^n = (x + y)(x + y) \cdots (x + y)$$

A typical term looks like n terms x, y multiplied together, in some amount: $x^{n-k} y^k$.

We get that term by selecting k parentheses to take the y from, and we take x from the rest. This gives $\binom{n}{k}$ ways. \square

Example 35.

$$(x + y)^0 = 1,$$

$$(x + y)^1 = x + y,$$

$$(x + y)^2 = x^2 + 2xy + y^2,$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

$$(x + y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7,$$

...

Example 36.

$$\begin{aligned} 1.01^5 &= \sum_{k=0}^5 \binom{5}{k} 0.01^k \\ &= 1.010510100501 \end{aligned}$$

[illegible]

D3 Further binomial identities

Immediate consequences from the binomial theorem:

Corollary 19. $(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k$

And therefore,

Corollary 20. $2^n = \sum_{k=0}^n \binom{n}{k}, 0^n = \sum_{k=0}^n (-1)^k \binom{n}{k}$

← take $x = 1$ or
 $x = -1$

Theorem 21. Observe:

(a)

$$\sum_{k=1}^n k \binom{n}{k} = 1 \binom{n}{1} + 2 \binom{n}{2} + \cdots + n \binom{n}{n} = n \cdot 2^{n-1}$$

(b)

$$\sum_{k=1}^n k \binom{n}{k} = 1 \binom{n}{1} + 2 \binom{n}{2} + \cdots + n \binom{n}{n} = (n+1)n2^{n-2}$$

(Scratch work begins)

(a)

$$\begin{aligned} &0 \cdot 1 \\ &0 \cdot 1 \quad 1 \cdot 1 &= 1 = 1 \times 1 \\ &0 \cdot 1 \quad 1 \cdot 2 \quad 2 \cdot 1 &= 4 = 2 \times 2 \\ &0 \cdot 1 \quad 1 \cdot 3 \quad 2 \cdot 3 \quad 3 \cdot 1 &= 12 = 3 \times 4 \\ &0 \cdot 1 \quad 1 \cdot 4 \quad 2 \cdot 6 \quad 3 \cdot 4 \quad 4 \cdot 1 &= 32 = 4 \times 8 \end{aligned}$$

(b)

$$\begin{aligned} &0 \cdot 1 \\ &0 \cdot 1 \quad 1 \cdot 1 &= 1 = 1 \times 1 \\ &0 \cdot 1 \quad 1 \cdot 2 \quad 4 \cdot 1 &= 6 = 3 \times 2 \\ &0 \cdot 1 \quad 1 \cdot 3 \quad 4 \cdot 3 \quad 9 \cdot 1 &= 24 = 6 \times 4 \\ &0 \cdot 1 \quad 1 \cdot 4 \quad 4 \cdot 6 \quad 9 \cdot 4 \quad 16 \cdot 1 &= 80 = 10 \times 8 \end{aligned}$$

(Scratch ends here)

Proof (a), method 1. We know $\sum_{k=0}^n \binom{n}{k} x^k \equiv (1+x)^n$. Hence, we can take the derivative of both sides:

$$\sum_{k=1}^n k \binom{n}{k} x^{k-1} \equiv n(1+x)^{n-1}$$

Now let $x = 1$ and obtain $\sum_{k=1}^n k \binom{n}{k} = n \cdot 2^{n-1}$. □

Proof (b), method 1. Similar to proof (a) but we first multiply both sides of the differentiated equation in (a) by x : $\sum_{k=1}^n k \binom{n}{k} x^k \equiv xn(1+x)^{n-1}$. Then we differentiate it again and plug in $x = 1$. □

← check this!

Proof (a), method 2. We let there be n people, with k of them selected to join the elite squad™. One of the person in elite squad™ is given a secret microfiche. In how many ways can this be done?

Way 1: Choose k be the size of the elite squad™, so k could be anything from 1 to n . We need to choose k people among n , giving us $\binom{n}{k}$ ways. Then, we choose one person among the k to give a microfiche. The total number of ways is $\sum_{k=1}^n \binom{n}{k} k$.

Way 2: Assign a microfiche, which can be done in n ways. Decide who else is in the squad: it is either ‘yes’ (in the elite squad™) or ‘no’. Hence, we make a binary decision for $n-1$ people, which gives us 2^{n-1} ways. The total number of ways is $n \cdot 2^{n-1}$.

Both methods give the same answer. □

Proof (b), method 2. We let there be n people, with k of them selected to join the elite squad™. One of the person in elite squad™ is given a homework problem. One of the person (could be the same) in elite squad™ is given an investigation problem. In how many ways can this be done?

Way 1: Choose k be the size of the elite squad™, so k could be anything from 1 to n . We need to choose k people among n , giving us $\binom{n}{k}$ ways. Then, we choose one person among the k to give homework, and choose again to give an investigation. The total number of ways is $\sum_{k=1}^n \binom{n}{k} k^2$.

Way 2: We first consider the case where the homework and investigation go to the same person. Assign them, which can be done in n ways. Decide who else is in the squad: it is either ‘yes’ (in the elite squad™) or ‘no’. Hence, we make a binary decision for $n-1$ people, which gives us 2^{n-1} ways. The total number of ways is $n \cdot 2^{n-1}$.

Next, we consider the case where the homework and investigation go to different people. Assign homework, which can be done in n ways. Then, assign investigation, which can be given to the rest $n - 1$ people. We then decide who else is in the squad, giving us 2^{n-2} ways.

Hence, we have $n \cdot 2^{n-1} + n(n-1) \cdot 2^{n-2} = n(n+1)2^{n-2}$ ways.

Both methods give the same answer. □

D4 Newton's Binomial Theorem

Theorem 22 (Newton). If $x, y \in \mathbb{R}$ and $|\frac{y}{x}| < 1$, let $\alpha > 0$ be real. Then

← not only integers!

$$\begin{aligned}(x+y)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k \\ &= x^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{y}{x}\right)^k\end{aligned}$$

where $\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$ and $(\alpha)_k = \alpha(\alpha-1)\dots(\alpha-k+1)$ is the falling factorial.

Proof. Let $f(z) = (1+z)^\alpha$. Then $f^{(k)}(z) = (\alpha)_k(1+z)^{\alpha-k}$ and so $f^{(k)}(0) = (\alpha)_k$. By the Taylor series of $f(z)$, we get $f(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} z^k$ for all $|z| < 1$. □

Example 37. What is the $\frac{1}{2}$ th row of the Karaji triangle?

k	0	1	2	3	4	5	...
$\binom{\frac{1}{2}}{k}$	1	$\frac{1}{2}$	$\frac{-1}{4}$	$\frac{3}{8}$	$\frac{-15}{16}$	$\frac{105}{32}$	
$\binom{\frac{1}{2}}{k}$	1	$\frac{1}{2}$	$\frac{-1}{8}$	$\frac{1}{16}$	$\frac{-5}{128}$	$\frac{7}{256}$	

Example 38.

$$\begin{aligned}\sqrt{103} &= \sqrt{100+3} \\ &= 10 \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{3}{100}\right)^k \\ &= 10 \cdot \left(1 + \frac{15}{1000} - \frac{1125}{10^7} + \dots\right) \\ &\simeq 10.148875\end{aligned}$$

Actual answer: 10.14889...

D5 Simplex numbers

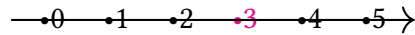
Recall the simplex numbers $T_d(n)$:

$d \backslash n$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	1	3	6	10	15	21	28
3	1	4	10	20	35	56	84
4	1	5	15	35	70	126	210
5	1	6	21	56	126	252	462

Observe that these are the same as binomial coefficients and just organised differently!

Consider the nonnegative integer lattice in $d + 1$ dimensions: $\mathbb{N}^{d+1} \subseteq \mathbb{R}^{d+1}$. What are the points (a_0, a_1, \dots, a_d) such that $a_0 + a_1 + \dots + a_d = k$ for a given $k \geq 0$? How many are there?

Example 39. When $k = 3$ and $d = 0$:



In general, the number of points on the plane satisfying $x_0 + x_1 + \dots + x_d = k - 1$ is equal to $T_d(k)$. From our multichoose argument, this is $T_d(k) = \binom{d+k-1}{d} = \binom{k}{d}$.

E Catalan numbers

E1 Examples

Example 40. How many sequences a_1, a_2, \dots, a_n are there, where n of the terms are $+1$ and n are -1 , such that the partial sums are all nonnegative? That is, $a_1 + a_2 + \dots + a_k \geq 0$ for all k ?

The problem concerns a multiset: n objects of type 1 and n of type 2. We want to count the number of permutations such that the number of type 1 occurring in the first k letters is \geq the number of type 2 occurring in the first k letters for all $k = 0, 1, \dots, 2n$.

Definition 14. Let C_n be the number of permutations as above. This is the n -th **Catalan number**.

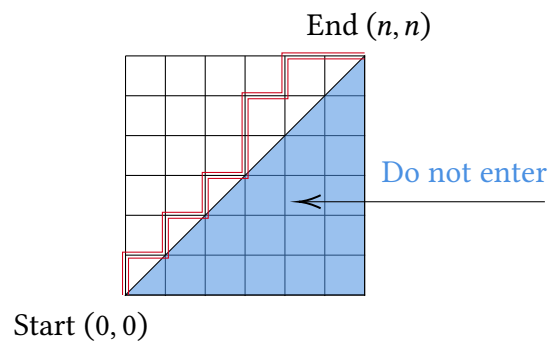
We can calculate a few:

n	C_n	$\binom{2n}{n}$	$C_n / \binom{2n}{n}$
0	1	1	1
1	1	2	1/2
2	2	6	1/3
3	5	20	1/4
4	14	70	1/5

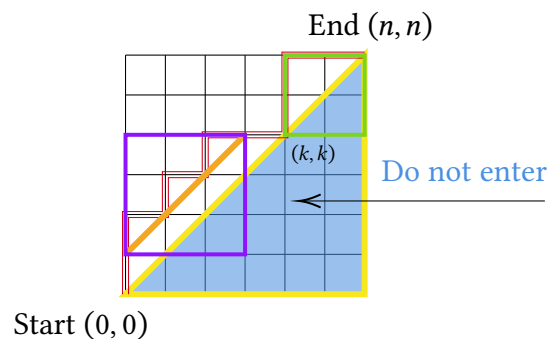
Example 41. C_n is also the number of anagrams of $\underbrace{NN \dots NN}_n \dots \underbrace{EE \dots EE}_n$ such that every initial segment has more N than E .

Conjecture: $C_n = \frac{1}{n+1} \binom{2n}{n}$

E2 First attempt



Question: when is the **first time** we return to the diagonal? That is, what is the smallest $k \in \{1, 2, \dots, n\}$ such that we get (k, k) ?



Steps:

- | | | |
|---|----------------|------------------------|
| 1. $(0, 0) \rightarrow (0, 1)$ | 1 way | |
| 2. $(0, 1) \rightarrow (k-1, k)$ w/o crossing orange line | C_{k-1} ways | ← in the purple square |
| 3. $(k-1, k) \rightarrow (k, k)$ | 1 way | |
| 4. $(0, 1) \rightarrow (k-1, k)$ w/o crossing yellow line | C_{n-k} ways | ← in the green square |

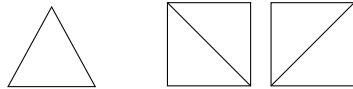
Hence

$$C_n = \sum_{k=1}^n C_{k-1} C_{n-k}$$

← the Catalan recurrence

And so $C_n = C_0 C_{n-1} + \dots + C_{n-1} C_0$.

Example 42. How many ways to triangulate a convex 47-sided polygon by drawing diagonals? ANS: D_{47} !



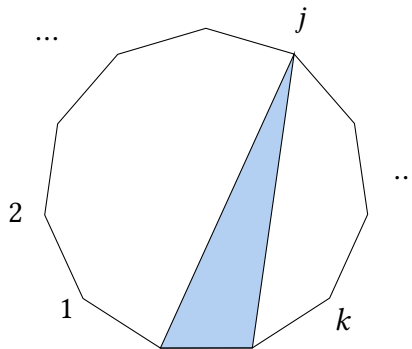
Examples: $D_3 = 1$ $D_4 = 2$...

Conjecture: $D_n = C_{n-2}$

Induction proof. Base case is done as shown above.

Inductive step: suppose it works for $n = 2, 3, \dots, k+1$. Now we want to show $D_{k+2} = C_k$.

Case j : we pick a side that has a triangle that goes to the vertex j .



On the left, we have a $j+1$ sided polygon and so the number of ways of triangulation is $D_{j+1} = C_{j-1}$.

On the right, we have a $k-j+2$ sided polygon and so the number of ways of triangulation is $D_{k-j+2} = C_{k-j}$.

Hence, the number of ways in total is $C_{j-1} \cdot C_{k-j}$. We conclude that

$$D_{k+2} = \sum_{j=1}^k C_{j-1} C_{k-j} = C_k$$

by the Catalan recurrence. □

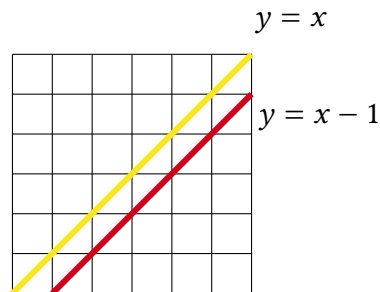
E3 The Catalan bijection

Suppose a path $(0, 0) \rightarrow (n, n)$ is **good** if it does not cross the diagonal $y = x$. We claim that the count of good paths is $\frac{1}{n+1} \binom{2n}{n}$.

Alternatively, we could show that the number of **bad** paths is # all paths $-\frac{1}{n+1} \binom{2n}{n}$, which means we want to show:

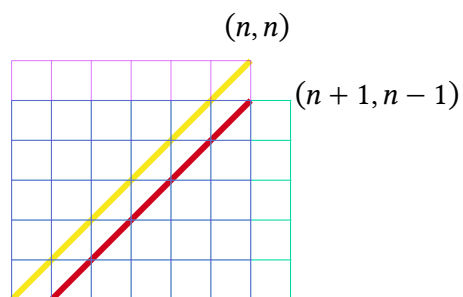
$$\begin{aligned} \# \text{ bad paths} &= \binom{2n}{n} - \frac{1}{n+1} \binom{2n}{n} \\ &= \frac{n}{n+1} \binom{2n}{n} \\ &= \frac{n}{n+1} \frac{(2n)!}{n!n!} \\ &= \frac{(n-1)!(n+1)!}{(2n)!} \\ &= \binom{2n}{n+1} \end{aligned}$$

A **bad** path is one that crosses the diagonal $y = x$, but that is equivalent to touching the subdiagonal $y = x - 1$.



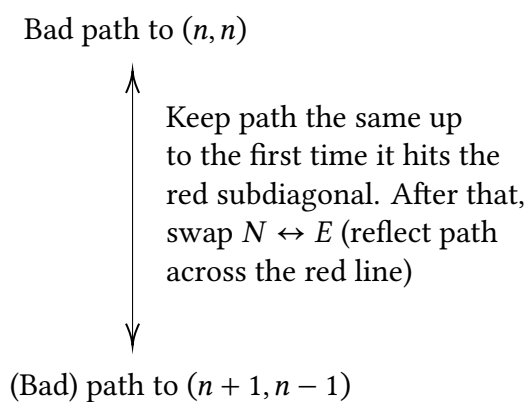
Claim: the number of paths $(0, 0) \rightarrow (n, n)$ that touch the red line is the same as

the number of paths $(0, 0) \rightarrow (n+1, n-1)$.



The latter quantity is just $\binom{2n}{n+1}$.

We use the following bijection:



← all paths are bad to $(n+1, n-1)$!

F Stirling numbers

F1 Table entries 11, 9, 7

Table 3.1 Balls and boxes counting problems.

Number of Ways to Put Balls into Boxes				
$S = S_1 = \{1, 2, \dots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls				
$T = T_1 = \{U_1, U_2, \dots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes				
Box U_i contains u_i balls				
Conditions on S and $T \rightarrow$ on $u_i \downarrow$	$T = T_1$ distinct $S = S_1$ distinct	$T = T_1$ distinct $S = S_2$ identical	$T = T_2$ identical $S = S_1$ distinct	$T = T_2$ identical $S = S_2$ identical
$0 \leq u_i \leq 1$ Assume $t \geq s$	1	2	3	4
$u_i \geq 0$	5	6	7	8
$u_i \geq 1$	9	10	11	12
for $i = 1, \dots, t$, $0 \leq u_i \leq n_i$, $n_i \in \mathbb{Z}^{\geq 0}$	13	14		
$u_i \in N_i \subset \mathbb{Z}^{\geq 0}$ for $i = 1, \dots, t$	15	16		

(11) : How many ways to place s distinct objects into t **unmarked** boxes such that none of the boxes are empty?
 \Leftrightarrow organizing s distinct people into t nonempty teams.
We don't know the (closed-form) answer yet, so we temporarily call it $\left\{s\atop t\right\}$.

\leftarrow Distinct objects,
identical boxes,
 $u_i \geq 1$

Example 43. We have 4 people: A,B,C and D, and we group them into 3 teams. This is the situation $s = 4, t = 3$. We group:

- AB|C|D
- AC|B|D
- AD|B|C
- BC|A|D
- BD|A|C
- CD|A|B

Therefore, $\left\{ \begin{smallmatrix} 4 \\ 3 \end{smallmatrix} \right\} = 6$.

⑨: How many ways to place s distinct objects into t **named** boxes such that none of the boxes are empty?

⇔ organizing s distinct people into t nonempty teams with different team names.

Compared with ⑪, this gives $t!$ more ways due to the labelling. Therefore, the answer is $t! \left\{ \begin{smallmatrix} s \\ t \end{smallmatrix} \right\}$.

← Distinct objects, distinct boxes, $u_i \geq 1$

Remark. This is the same as counting the number of *surjective* functions

$$\{1, 2, \dots, s\} \rightarrow \{1, 2, \dots, t\}$$

we expect some relationship with the quantity t^s , which counts all functions.

⑦: How many ways to place s distinct objects into t **unmarked** boxes?

Answer: $\sum_{k=0}^t \left\{ \begin{smallmatrix} s \\ k \end{smallmatrix} \right\}$ where k is the # of nonempty boxes.

← Distinct objects, identical boxes, $u_i \geq 0$. i.e. ⑪ but boxes can be empty

Remark. Observe:

$u_i \geq 0$	⑤ t^s	⑦ $\sum_{k=0}^t \left\{ \begin{smallmatrix} s \\ k \end{smallmatrix} \right\}$
$u_i \geq 1$	⑨ $t! \left\{ \begin{smallmatrix} s \\ t \end{smallmatrix} \right\}$	⑪ $\left\{ \begin{smallmatrix} s \\ t \end{smallmatrix} \right\}$

We can get ⑨ by multiplying ⑪ by $t!$, but we cannot get ⑤ from ⑦ in the same way due to the empty boxes!

The ways to label boxes in ⑤ is only $(t)_k$ ways for each # of nonempty boxes k .

Conclusion:

$$t^s = \sum_{k=0}^t (t)_k \left\{ \begin{smallmatrix} s \\ k \end{smallmatrix} \right\}$$

It follows that the quantities $0^s, 1^s, \dots, s^s$ are *linearly* related to $\left\{ \begin{smallmatrix} s \\ 0 \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} s \\ 1 \end{smallmatrix} \right\}, \dots, \left\{ \begin{smallmatrix} s \\ s \end{smallmatrix} \right\}$.

Example 44. Let $s = 3$.

$$\begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{bmatrix} \cdot \begin{bmatrix} \begin{Bmatrix} 3 \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} 3 \\ 1 \end{Bmatrix} \\ \begin{Bmatrix} 3 \\ 2 \end{Bmatrix} \\ \begin{Bmatrix} 3 \\ 3 \end{Bmatrix} \end{bmatrix}$$

where every entry (i, j) of the matrix is $(i)_j$.

We can invert the matrix to find what $\begin{Bmatrix} s \\ t \end{Bmatrix}$ is, but it's computationally quite terrible!

F2 Stirling numbers of the second kind

Definition 15 (Stirling numbers of the second kind).

$$\begin{Bmatrix} s \\ t \end{Bmatrix} = \# \text{ of ways to place } s \text{ distinct objects into } t \text{ identical empty boxes}$$

Proposition 23.

1. $\begin{Bmatrix} s \\ t \end{Bmatrix} = 0$ if $t > s$ too many boxes
2. $\begin{Bmatrix} s \\ s \end{Bmatrix} = 1$ must have object in its own box
3. $\begin{Bmatrix} s \\ 1 \end{Bmatrix} = 1$ for $s \geq 1$ everything in 1 box
4. $\begin{Bmatrix} s \\ s-1 \end{Bmatrix} = \binom{s}{2}$ which 2 ppl in the team of 2, rest solo
5. $\begin{Bmatrix} s \\ 2 \end{Bmatrix} = 2^{s-1} - 1$

Ways:

- (a) Ask who else among the $s - 1$ people are my team, count the possibilities. Subtract the case when everyone is in my team for the other team to be nonempty.
- (b) Categorize s people into two *distinct* boxes, which gives us 2^s ways. Then, subtract the 2 ways where everyone is in the same box. Finally, since the boxes aren't distinct after all, we divide the result by $2!$.

$$6. \begin{Bmatrix} s \\ 0 \end{Bmatrix} = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases}$$

$s \backslash t$	0	1	2	3	4	5	6	Row sum
0	1	0	0	0	0	0	0	1
1	0	1	0	0	0	0	0	1
2	0	1	1	0	0	0	0	2
3	0	1	3	1	0	0	0	5
4	0	1	7	6	1	0	0	15
5	0	1	15	25	10	1	0	522
6	0	1	31	90	65	15	1	203

← The row sums are Bell numbers!

Theorem 24.

$$\begin{Bmatrix} s \\ t \end{Bmatrix} = t \begin{Bmatrix} s-1 \\ t \end{Bmatrix} + \begin{Bmatrix} s-1 \\ t-1 \end{Bmatrix}$$

Understand this as placing people $1, 2, \dots, s$ into teams T_1, \dots, T_t .

From perspective of person s :

- Case 1: Person s is alone, so $\begin{Bmatrix} s-1 \\ t-1 \end{Bmatrix}$ ways to organize the rest.
- Case 2: Person s is not alone.
 - Organize the rest into t teams, which is $\begin{Bmatrix} s-1 \\ t \end{Bmatrix}$ ways.
 - Join one of the teams, which is t ways.

Add the two cases.

F3 Stirling numbers of the first kind

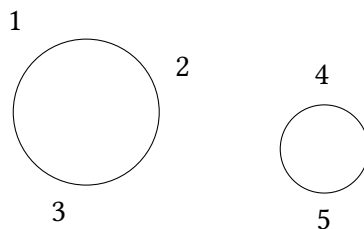
Definition 16 (Stirling numbers of the first kind).

$$\left[\begin{matrix} n \\ k \end{matrix} \right] = \# \text{ of ways of seating } n \text{ people at } k \text{ circular tables (no empty table)}$$

In general, $\left[\begin{matrix} n \\ k \end{matrix} \right] \geq \begin{Bmatrix} n \\ k \end{Bmatrix}$ (usually a lot bigger) because of the extra choices needed to seat the k teams at tables.

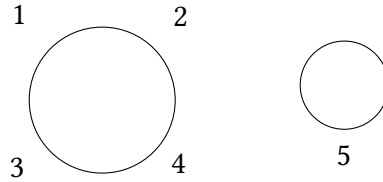
Example 45. We let $n = 5, k = 2$.

Case 1: split into 3+2



- Choose two people to sit at the table for two: $\binom{5}{2}=10$
- Place around the circular tables: $(3-1)! \times (2-1)! = 2$

Case 2: split into 4+1



- Choose 1 person to sit alone: 5
- Place around circular table: $(4-1)! = 6$

Hence, the total ways is $\left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \right] = 20 + 30 = 50$.

Remark. The first task of each case gives us $\left\{ \begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \right\} = 10 + 5 = 15$. We can get crude bounds:

$$2 \left\{ \begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \right\} < \left[\begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \right] < 6 \left\{ \begin{smallmatrix} 5 \\ 2 \end{smallmatrix} \right\}$$

Proposition 25.

1. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = 0$ if $k > n$: too many tables
2. $\left[\begin{smallmatrix} n \\ 0 \end{smallmatrix} \right] = 0$ if $n \geq 1$: not enough tables
3. $\left[\begin{smallmatrix} n \\ n \end{smallmatrix} \right] = 1$: one person per table
4. $\left[\begin{smallmatrix} n \\ 1 \end{smallmatrix} \right] = (n-1)!$ for $n \geq 1$: simple circular permutation
5. $\left[\begin{smallmatrix} n \\ n-1 \end{smallmatrix} \right] = \binom{n}{2}$: which 2 sit at the table for 2

Table of values of $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$

$s \backslash t$	0	1	2	3	4	5	6	Row sum	Alternating sum
0	1	0	0	0	0	0	0	1	1
1	0	1	0	0	0	0	0	1	-1
2	0	1	1	0	0	0	0	2	0
3	0	2	3	1	0	0	0	6	0
4	0	6	11	6	1	0	0	24	0
5	0	24	50	35	10	1	0	120	0
6	0	120	274	225	85	15	1	720	0

Theorem 26.

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right]$$

Proof. Consider from the perspective of yourself. We seat n people at k table.

- You sit alone:
 - You seat the rest of the people at $k - 1$ tables: $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ ways
 - Claim the last table by yourself
- You don't sit alone:
 - Seat the rest of the people at k tables: $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ ways
 - You sit to the left of somebody: $n - 1$ ways

□

Theorem 27.

$$\sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} = n!$$

Proof outline. We claim:

LHS = # ways to seat n people at circular tables

RHS = # permutations of $[n] = \{1, 2, \dots, n\}$

and there is a bijection between ways to seat n people at circular tables and permutations of $[n]$. □

← Consider the elements of S_n .

Theorem 28.

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Sketch. We are effectively counting the permutations with \pm sign according to whether the number of cycles is even or odd.

It turns out:

$$(-1)^{\# \text{ cycles}} = (-1)^n \cdot (\text{sign of permutation})$$

and follows that

$$LHS = (-1)^n \cdot \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

Since $\det[\] = 1$, $\det[1] = 1$ and $\det = 0$ otherwise, we are done. □

Theorem 29.

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} = \frac{\begin{bmatrix} n+1 \\ 2 \end{bmatrix}}{n!}$$

Proof by induction. True for:

$$\begin{aligned} 1 &= \frac{1}{1} \\ 1 + \frac{1}{2} &= \frac{3}{2!} \\ 1 + \frac{1}{2} + \frac{1}{3} &= \frac{11}{3!} \end{aligned}$$

Then, suppose true for $n = k - 1$. Then

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{k-1} + \frac{1}{n} &= \frac{\begin{bmatrix} k \\ 2 \end{bmatrix}}{(k-1)!} + \frac{1}{k} \\ &= \frac{k \begin{bmatrix} k \\ 2 \end{bmatrix} + (k-1)!}{k!} \\ &= \frac{k \begin{bmatrix} k \\ 2 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}}{k!} \\ &= \frac{\begin{bmatrix} k+1 \\ 2 \end{bmatrix}}{k!} \end{aligned}$$

□