# MATH103 Combinatorics Notes

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# Contents

# A Recurrence Relations

## A1 Intro

**Remark.** Let there be a set $\{1, 2, \dots, n\}$. The number of subsets of it is $2^n$ since for each number, we could say "include" or "exclude".

**Example 1.** Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

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*(Scratch work begins)*

First consider $n = 0$. Then the only *good* subset is $\varnothing$.

Now consider $n = 1$, both $\varnothing, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\varnothing, 1, 2, 12$. The set 12 is not good. ← notation simplified for fast typing

Similarly, we have $f(3) = 5, f(5) = 8$.

*(Scratch ends here)*

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We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

## A2 Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$$

**Remark.** Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and ← Textbook
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$. ← Preferred!

**Example 2.** Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbds one step per stride, sometimes two steps per stride. In how many ways can she do this?

← It is the same recurrence as A1 but with init conditions shifted: $f(n) = F_{n+1} = f_{n+2}$.

Table 1: Table of the sequence in two notations

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 |
| $f_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 |

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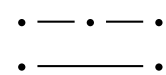
*(Scratch work begins)* Let $S(n)$ be the number of ways climbing $n$ steps.

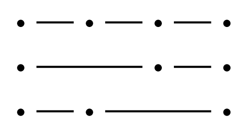
- $S(1) = 1$



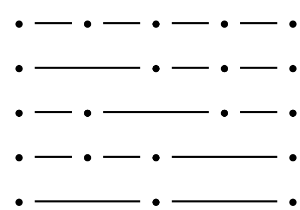
- $S(2) = 2$



- $S(3) = 3$



- $S(4) = 5$



Conjecture: maybe Fibonacci?

*(Scratch ends here)*

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*Proof.* Consider the set of ways she can cover $n$ steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n-1$ steps. Thus, this gives us $S(n-1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n-2$ steps. Thus, this gives us $S(n-2)$ ways.

Therefore, we conclude that $S(n) = S(n-1) + S(n-2)$. We account the initial conditions and conclude the closed form:

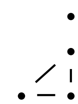
$$S(n) = F_n = f_{n+1}$$

for all $n$. Since Prof Rad climbs 47 steps, we get $\boxed{S(47) = 4807526976}$. □

## A3 Simplex numbers

**Definition 1.** *Two-dimensional* triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- …



$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

**Theorem 1.** $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

*First proof.* We prove by induction.

← Not as good of a proof: we must know what we are proving in the first place!

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.
Inductive hypothesis: Suppose proved formula for up to $n = k$.
Inductive step: Consider $n = k + 1$.

$$\begin{aligned} T_2(k+1) &= 1 + \cdots + k + (k+1) \\ &= T_2(k) + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k^2 + k + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)\left((k+1)+1\right)}{2} \end{aligned}$$

□

*Proof by Gauss.* Observe:

← Better proof: concluding the formula without knowing it first!

$$\begin{aligned} T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\ &= n + (n-1) + \cdots + 2 + 1 \end{aligned}$$

Therefore, we **add** the two rows:

$$\begin{aligned} 2T_2(n) &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n} \\ &= n(n+1) \\ \therefore \quad T_2(n) &= \frac{1}{2} n(n+1) \end{aligned}$$

□

**Definition 2.** Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

- $T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$

**Definition 3.** Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$

Some examples of simplex numbers $T_d(n)$:

| $d \backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| 3 | 1 | 4 | 10 | 20 | 35 | 56 | 84 |
| 4 | 1 | 5 | 15 | 35 | 70 | 126 | 210 |
| 5 | 1 | 6 | 21 | 56 | 126 | 252 | 462 |

# B Ramsey Theory

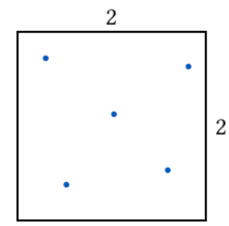
Invented by Frank Ramsey in 1930. We would need:

- Graph Theory
- Pigeonhole Principle
- Quantifiers
- Counterexamples

## B1 Pigeonhole principle

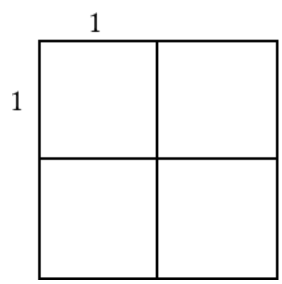
**Theorem 2** (Dirichlet's Pigeonhole Principle)**.** If you put $n+1$ pigeons in $n$ pigeonholes, then (at least) one pigeonhole will contain (at least) two pigeons.

*Proof omitted.* □




**Example 3.** Given 5 points in a square of side length 2, show that there must exist two points whose mutual distance is $\leq \sqrt{2}$.

*Proof.* Divide square into 4 smaller squares. We now have 4 pigeonholes and 5 dots:




These two points in the same pigeonhole have distance $\leq \sqrt{1^2+1^2} = \sqrt{2}$. $\square$

**Example 4.** There exists two people in NYC who have exactly the same number of hairs on their head.

**Example 5.** There are 30 people at a party talking with each other. Afterwards, there will be two people who talked with the same number of people.

*Proof.* If we put a person who talked to $i$ people into box $i$, we get 30 boxes; however, we cannot have someone who talked to 0 people and someone who talked at 29 people at the same time! Hence, we combine the box 0 and box 29, and only one of which could be the case.

Now we have 29 boxes and 30 people. By pigeonhole principle, there must be two people who talked with the same amount of people. $\square$

**Theorem 3** (Strong Pigeonhole Principle)**.** Given pigeonholes $1, 2, \dots, n$ with capacities $c_1, c_2, \dots, c_n$ where $c_i \geq 0$; if we have at least $c_1 + c_2 + \cdots + c_n + 1$ pigeons in these pigeonholes, then at least one pigeonhole overflows.

*Proof.* Suppose BWOC that no pigeonhole overflows. Then for all $i = 1, 2, \dots, n$, we have the number of pigeons in $i \leq c_i$.

We add up and get inequalities:

$$\text{total \# pigeons} \leq c_1 + c_2 + \cdots + c_n$$

Contradiction! $\square$

**Example 6.** There are five people supporting two teams. Then at least one team is supported by 3 people.

*Proof.* Assume BWOC that the two teams only have two supporters. Let $c_1 = c_2 = 2$. However, by SPP, $5 \geq 2 + 2 + 1$, hence one pigeonhole overflows. Therefore, one team must have $> 2$ supporters. $\square$

## B2 First Ramsey Theorem

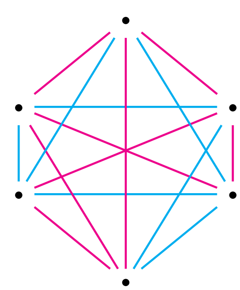
There are 6 people taking a class. Then:

either there exists 3 people such that each pair of them have previously taken a class together,

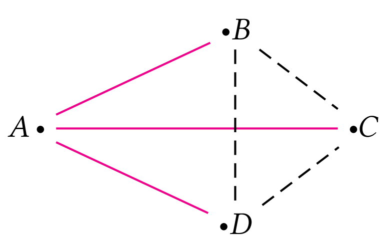
or (inclusive) there exists 3 people such that no two have taken a class together.

**Theorem 4.** If we have 6 vertices and we draw all edges between them (a $K_6$ graph), then for every possible way of coloring the edges red and blue, there must exist a **monochromatic** triangle.

← $K_6$ stands for *complete graph on 6 vertices.* It has 15 edges.



*Proof.* Pick any vertex and call it $A$. It has 5 edges colored red and blue. By the SPP, there exists at least 3 edges of the same color. WLOG let these three edges be red and call the other three vertices $B, C, D$.

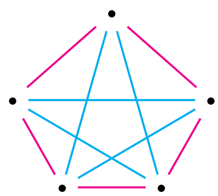



- If $BC$ is red, then $ABC$ is a red triangle.
- If $CD$ is red, then $ACD$ is a red triangle.
- If $BD$ is red, then $ABD$ is a red triangle.
- If none of the above has happened, then $BC, CD, BD$ are all blue, meaning that $BCD$ is a blue triangle!

□

**Theorem 5.** If there are 5 instead of 6 vertices, then the above coloring prediction cannot be made with certainty.

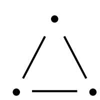
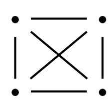
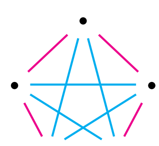
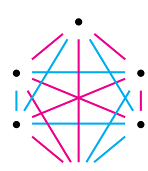
*Counterexample.*



□

## B3 $K_p \to K_q,\ K_r$

In graphy theory, $K_n$ is the **complete** graph on $n$ vertices.

| | | | |
|---|---|---|---|
| $K_1$ | | 1 vertex | 0 edges |
| $K_2$ | | 2 vertices | 1 edge |
| $K_3$ |  | 3 vertices | 3 edges |
| $K_4$ |  | 4 vertices | 6 edges |
| $K_5$ |  | 5 vertices | 10 edges |
| $K_6$ |  | 6 vertices | 15 edges |

**Remark.** Note that $K_n$ has $1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$ edges, hence is the $n-1$-th triangular number.

Ramsey Theory uses the following language convention: the expression

$$K_p \to K_q,\ K_r$$

represents a statement with the following meaning:

**Definition 4.** If the edges of $K_p$ are colored red/blue, then it necessarily follows that either the $K_p$ contains a red $K_q$, or $K_p$ contains a blue $K_r$ (or possibly both).

We want to know whether this statement is true for a given triple of $(p, q, r)$.

**Example 7.** We proved in B2 that $K_6 \to K_3, K_3$.

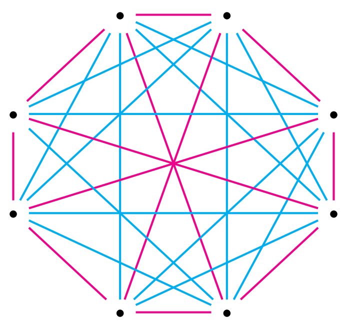
**Non-example 8.** We also showed that $K_5 \to K_3, K_3$ is false by exhibiting a coloring of $K_5$ that does not have a red or blue triangle (counterexample).

← write $K_5 \not\to K_3, K_3$

**Example 9.** It is known that $K_{18} \to K_4, K_4$ and $K_{17} \not\to K_4, K_4$.

**Example 10.** Also, $K_9 \to K_3, K_4$ and $K_8 \not\to K_3, K_4$.

← Here we have to decide in advance which color goes with the $K_3$ and which goes with the $K_4$ due to asymmetry.



This $K_8$ has no red triangle and no blue $K_4$.

**Theorem 6** (Ramsey)**.** Let $q, r$ be positive integers. Then there always exists a positive integer $p$ such that

$$K_p \to K_q, K_r$$

is true.

We would see the following tabel giving us values of $p$ that work.

Define a function $N(q, r)$ recursively:

← $q, r \in \mathbb{Z}^+$

- Base case: $N(1, r) = N(q, 1) = 1$
- Recurrence: $N(q, r) = N(q-1, r) + N(q, r-1)$ if $q, r \geq 2$.

We compute the value of $N(q, r)$ for:

← They do look like simplex numbers!

| $q \backslash r$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 1 | 3 | 6 | 10 | 15 | 21 |
| 4 | 1 | 4 | 10 | 20 | 35 | 56 |
| 5 | 1 | 5 | 15 | 35 | 70 | 126 |
| 6 | 1 | 6 | 21 | 56 | 126 | 252 |

We would want to prove that $K_{N(q,r)} \to K_q, K_r$ for all $q, r \geq 1$.

*Proof.* By induction.

*Base case*: If $q = r = 1$, then $N = 1$, we need to show that

$$\begin{aligned} & K_1 \to K_1, K - r \\ \text{and} \quad & K_1 \to K_q, K_1 \end{aligned}$$

for all $q, r$.
That is, suppose $K_1$ has its edges colored red/blue, then there exists a red $K_1$ or a blue $K_r$, and *vice versa.*
Since there are no edges, this is vacuously true.

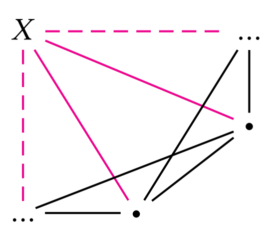
*Inductive step*: We will show that if we are given that $A, B$ are numbers such that

← This works because as we fill out the table above, each new number we write in will work because it's the sum of the left and above numbers and they both work.

$$\begin{aligned} K_A &\to K_{q-1}, K_r \\ \text{and } K_B &\to K_q, K_{r-1} \end{aligned}$$

are true, then $K_{A+B} \to K_q, K_r$.
Consider $K_{A+B}$ colored red and blue. We will show that it has a red $K_q$ or a blue $K_r$.
Pick a vertex and call it $X$. It would have $A + B - 1$ edges in total. We claim that $X$ either has at least $A$ red edges, or at least $B$ blue edges. This is indeed true, because if not, the number of red edges would be $\leq A - 1$ and the number of blue edges would be $\leq B - 1$ and so the total number of edges would be $\leq A + B - 2 < A + B - 1$, which is a contradiction.
Now, if $X$ has a red claw of size $A$:

← The neighbouring vertices connected by black edges form $K_A$, yet to be colored.




$K_A$

From our inductive hypothesis $K_A \to K_{q-1}, K_r$, we must have
**either** red $K_{q-1}$, in which case we combine with the vertex $X$ and the red claw to get at least one red $K_q$.
**or** blue $K_r$, in which case we are done.
Similarly, if $X$ has a blue claw of size $B$, then we make the same argument.

Hence, we know that $K_{A+B} \to K_q, K_r$ is true whenever $K_A \to K_{q-1}, K_r$ and $K_B \to K_q, K_{r-1}$.

□

## B4 Ramsey numbers

Recall: Let $m, n$ be positive integers. We know that there are numbers $p \in \mathbb{N}$ such that $K_p \to K_m, K_n$.

**Remark.** If $p$ works, then so does any $q \geq p$ as $K_q$ would contain copies of $K_p$.

So the question becomes, if we have $K_p \to K_m, K_n$, is $p$ the **smallest** such number?

**Definition 5.** The **Ramsey number** $r(m, n)$ is the smallest such number.

**Example 11.** We know $K_6 \to K_3, K_3$ but $K_5 \not\to K_3, K_3$, so $r(3, 3) = 6$.

**Example 12.** Mathematicians have proved that

$$K_{48} \to K_5, K_5$$
$$K_{42} \not\to K_5, K_5$$

so we have $43 \leq r(5, 5) \leq 48$.

**Remark.** In general,

$$K_N \to K_m, K_n \iff r(m, n) \leq N$$
$$K_{N-1} \not\to K_m, K_n \iff r(m, n) \geq N$$

Need both to get the precise value of $r(m, n)$.

**Proposition 7.** Properties of Ramsey numbers:

(a) $r(3, 3) = 6$ ← proven in B2
(b) $r(m, n) = r(n, m)$ ← symmetry
(c) $r(1, n) = 1$ ← $K_1 \to K_1, K_n$, $K_0 \not\to K_1, K_n$
(d) $r(2, n) = n$
(e) $r(m, n) \leq r(m-1, n) + r(m, n-1)$ for all $m, n \geq 2$

*Proof for (d).* Claim: $K_2 \to K_2, K_n$, $K_{n-1} \not\to K_2, K_n$.

Color $K_n$. If all edges are blue then we have a blue $K_n$. Else we have some red edges, so we have some red $K_2$.

Now color $K_{n-1}$ all blue: we realize that we don't have any red $K_2$, but we don't have a blue $K_n$ either! □

*Proof for (e).* Let $A = r(m-1, n)$, $B = r(m, n-1)$. We have shown that if $K_A \to K_{m-1}, K_n$ and $K_B \to K_m, K_{n-1}$, then $K_{A+B} \to K_m, K_n$. Hence $r(m, n) \leq A + B$. □

Known facts:

$$
\begin{aligned}
r(2,2) &= 2 \\
r(3,3) &= 6 \\
r(4,4) &= 18 \\
43 \leq r(5,5) &\leq 48 \\
102 \leq r(6,6) &\leq 165
\end{aligned}
$$

## B5 A lower bound for $r(m,n)$

Our table of $N(m,n)$ gave us upper bonds for $r(m,n)$. Specifically,

$$r(m,n) \leq N(m,n) = \frac{(n+m-2)!}{(n-1)!(m-1)!} = \binom{m+n-2}{m-1}$$

What about lower bound?

**Theorem 8.**

$$r(m,n) \geq (m-1)(n-1)+1$$

*if and only if* $K_{(m-1)(n-1)} \not\to K_m, K_n$

*Proof.* We prove this by exhibiting a coloring of $K_{(m-1)(n-1)}$ that has no red $K_m$, no blue $K_n$.

Place vertices in grid:

$$
\begin{array}{cc}
 & \begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ \bullet & \bullet & \cdots & \bullet \\ \vdots & \vdots & \ddots & \vdots \\ \bullet & \bullet & \cdots & \bullet \end{array} \\
m\!-\!1 \text{ rows} & \\
 & n\!-\!1 \text{ columns}
\end{array}
$$

Coloring rule of edges: If two vertices are in the same row, color the edges blue. If two vertices are in the same column, color the edges red. Every other edge arbitrary.

Claim: there exists no red $K_m$.

Consider the $m$ vertices of such a $K_m$. There are $m-1$ rows. Pigeonhole principle ensures that some vertices must be in the same row. But that edge must be blue! So this is not a red $K_m$. Similarly, there are no blue $K_n$. □

Thus, we get: $(m-1)(n-1)+1 \leq r(m,n) \leq \dfrac{(n+m-2)!}{(n-1)!(m-1)!} = \dbinom{m+n-2}{m-1}$.

Observe there is still a huge gap between the bounds. Could we get better?

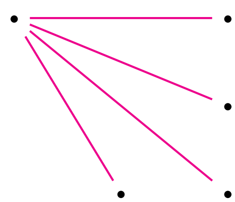
## B6 The "parity" improvement

Our methods have shown that $K_{10} \to K_3, K_4$. But it is actually true that $K_9 \to K_3, K_4$. Why?

*Proof.* Given $K_9$ colored red or blue. We seek a red $K_3$ or a blue $K_4$.

That is to say that if we ever see a red 4-claw, then we are done!

← See this argument.



In addition, if we ever see a blue 6-claw, then we are also done because $K_6 \to K_3, K_3$ and we either have a red $K_3$ or a blue $K_3$, which would have to combine with the other vertex to get a blue $K_4$.

Now suppose we neither have a red 4-claw nor a blue 6-claw. This implies that each vertex has $\leq 3$ red edges, and $\leq 5$ blue edges. However, in a $K_9$, each vertex only has 8 edges, so they must exactly each have 3 red edges and 5 blue edges. Does this exist? We realize that to make this happen, we have:

- 9 vertices
- Each vertex has 3 red edges
- Every edge belongs to two vertices

Hence, we need to have exactly $\frac{3\times 9}{2} = 13.5$ red edges, but this cannot happen because we need a whole number of edges! Thus, it is not possible that we neither have a red 4-claw nor a blue 6-claw. $\square$

**Lemma 9** (Ramsey inductive step improved by parity)**.** Suppose

← Also seen here.

$$\begin{aligned} K_A &\to K_{q-1}, K_r \\ \text{and } K_B &\to K_q, K_{r-1} \end{aligned}$$

are true, then $K_{A+B} \to K_q, K_r$.

In addition, if $A, B$ are **both even numbers**, then $K_{A+B-1} \to K_q, K_r$.

## B7 Variations

### More colors!

For example:

$$K_p \to K_a, K_b, K_c$$

(given $K_p$ colored red, blue, green, it must contain a red $K_a$, or a blue $K_b$, or a green $K_c$.)

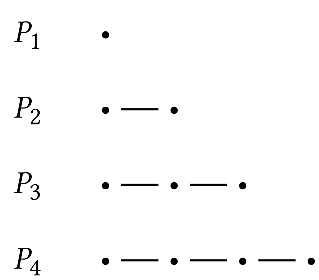
**Example 13.** It is known that $K_{17} \to K_3, K_3, K_3$.

*Proof sketch.* Pick a vertex which has 16 edges. We observe $16 \div 3 = 5\frac{1}{3} \implies$ at least one color occurs 6 times (i.e. we can see a red/blue/green 6-claws). □

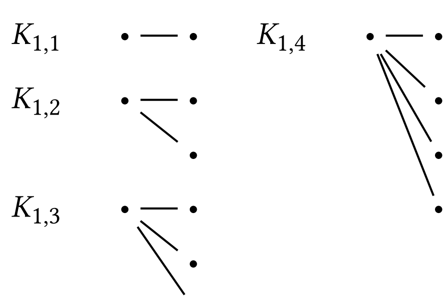
**Remark.** $r(a, b, c)$ is the smallest number that works for the above.

### Other graphs

Paths:




Claws:




**Example 14.** Show that $r(K_{1,3}, K_{1,3}) = 6$.

*Proof.* We know $K_6 \to K_3, K_3$. Pick a vertex that has 5 neighbors. By the strong pigeonhole principle, we must have three edges of the same color $\implies$ either red or blue $K_{1,3}$. $\square$

# C Counting

## C1 Three principles

### Addition principle

← aka. counting by cases

**Definition 6.** If a set $S$ is *partitioned* into subsets $S_1, S_2, \ldots S_n$, then the cardinality of $S$ is

$$|S| = |S_1| + \cdots + |S_n|.$$

← that is, $S = \bigcup S_i$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$

The art lies in:

- making each $S_i$ easy to count, and
- not having too many $S_i$ if there is no formula for $|S_i|$.

**Remark** (Variations)**.** If the $S_i$ cover $S$ but they overlap, then we have the inequality $|S| < \sum_{i=1}^{n} |S_i|$ because the overlap implies that we are *overcounting*.

← The inclusion/exclusion principle handles overlaps precisely

**Example 15.** Let $S$ be the set of *good* subsets of $[5] = \{1, 2, 3, 4, 5\}$. We could first try:

← *good* meaning no adjacent elements

- $S_1$ contains the subsets that contain 5
- $S_2$ contains the subsets that don't contain 5

We have previously shown that $|S_1|$ = number of *good* subsets of [3] and $|S_2|$ = number of *good* subsets of [4].

Alternatively, we could also let $T_i$ be the *good* subsets of [5] with cardinality $i$. Then

$S$ is partitioned into $T_0 \cup T_1 \cup T_2 \cup T_3$. We count:

$$\left.\begin{array}{lccc} T_0 & : & \emptyset & |T_0| = 1 \\ T_1 & : & 1,2,3,4,5 & |T_1| = 5 \\ T_2 & : & 13,14,15,24,25,35 & |T_2| = 6 \\ T_3 & : & 135 & |T_3| = 1 \end{array}\right) |S|=13$$

### Subtraction principle

**Definition 7.** Let $A \subseteq S$ and $A^c$ be its complement in $S$. Then $A, A^c$ partition $S$ and $|S| = |A| + |A^c|$. This means that

$$|A| = |S| - |A^c|.$$

**Example 16.** How many 2-digit numbers have distinct nonzero digits?

Let $S$ be the set of all 2-digit numbers $\{10, 11, \dots, 99\}$ and let $A$ be the subset of those with nonzero distinct digits. We count:

$$\begin{array}{rll} A^c & : 11, 22, \dots, 99 & \text{(distinct fails)} \\ & \;\; 10, 20, \dots, 90 & \text{(nonzero fails)} \end{array}$$

Hence $|A| = |S| - |A^c| = 90 - 18 = 72$.

### Multiplication principle

**Definition 8.** Suppose we have to do two tasks in sequence. We suppose:

← Note: sometimes the 2nd task could depend on the 1st one

- Task 1 has $m$ outcomes
- Task 2 has $n$ outcomes, regardless of how Task 1 was carried out.

Then there are $mn$ ways of carrying out both tasks.

← Similarly for 3 or more tasks in sequence

**Example 17.** How many 2-digit numbers have distinct nonzero digits?

Let $\boxed{a}\,\boxed{b}$ be the two digits in these numbers. Let Task 1 be selecting digit $\boxed{a}$ and Task 2 be selecting digit $\boxed{b}$.

- Task 1: 9 ways (1,2,...,9)
- Task 2: 8 ways (1,2...,9 but not same as a )

Hence there are $9 \times 8 = 72$ such numbers.

**Tricky example 18.** How many **odd** numbers in the range 1000-9999 have distinct digits?

**Attempt 1**: Let a b c d be the 4 digits in these numbers and assign them Tasks 1-4. We have:
  - Task 1: 9 ways (1-9)
  - Task 2: 9 ways (0-9 except a )
  - Task 3: 8 ways (0-9 except a , b )
  - Task 4: Could be 2 or 3 or 4 or 5 (depending on how many odd digits had already been used)

Hence, the best we can say here is that the answer is between $9 \times 9 \times 8 \times 2$ and $9 \times 9 \times 8 \times 5$.

← BAD! This is a large range!

**Attempt 2**: Let a b c d be the 4 digits in these numbers and try the order d a b c for Tasks 1-4. We have:
  - Task 1: 5 ways $(1, 3, 5, 7, 9)$
  - Task 2: 8 ways (1-9 except d )
  - Task 3: 8 ways (0-9 except a , d )
  - Task 4: 7 ways (0-9 except a , d , b )

Therefore, the number of ways is $5 \times 8 \times 8 \times 7 = 2240$.

**Example 19.** How many numebrs $0, 1, \dots 99999$ have exactly one digit 6?

We can assign tasks:

- Task 1: choose a location for 6, giving us 5 ways
- Task 2: assign the remaining digits from left to right, giving us $9^4$ ways

Hence there are $5 \times 9^4 = 32805$ such numbers.

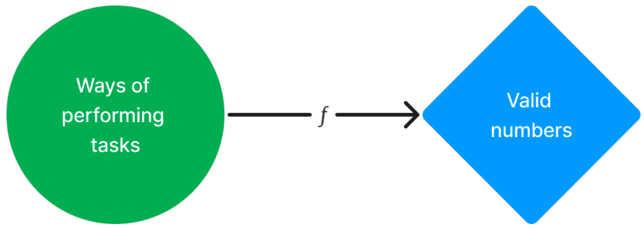
**Non-example 20.** How many integers $0, 1, \dots 99999$ have *at least* one digit 6?

**Attempt:** We can assign tasks:

- Task 1: choose a location for 6, giving us 5 ways
- Task 2: assign the remaining digits from left to right, giving us $10^4$ ways

But the answer of 50000 is **wrong**! But why?

The counting process for this problem is corresponding the ways of performing tasks to valid 5-digit numbers:




We have correctly counting the green set. However, for this to count the blue set, we need $f$ to be **bijective**. That is, every valid number must be obtained in exactly one way. However, in this case, our $f$ is surjective but not injective. For instance, 62516 will be counted *twice*:

- 6____ then 62516; or
- ____6 then 62516.

Hence, 50000 > correct answer!

**Correct way:** Using the subtract principle to deduct numbers that don't have 6: $10^5 - 9^5 = 40951 < 50000$.

## C2 Probability

**Definition 9.**

$$\text{Probability} = \frac{\text{number of favourable cases}}{\text{number of total cases}}$$

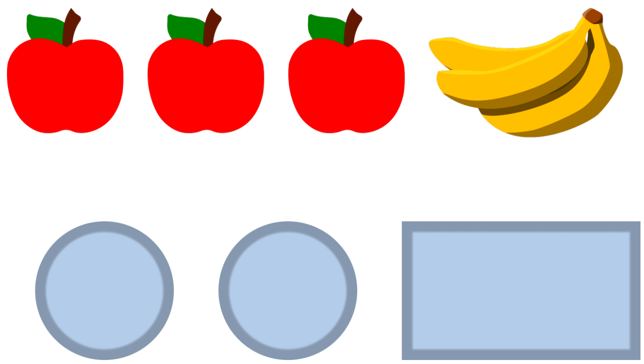
**Example 21.** Probability of 3 dice rolling the same number: $P = \frac{6}{6^3} = \frac{1}{36}$.

## C3 The counting framework

Here is a very general problem:

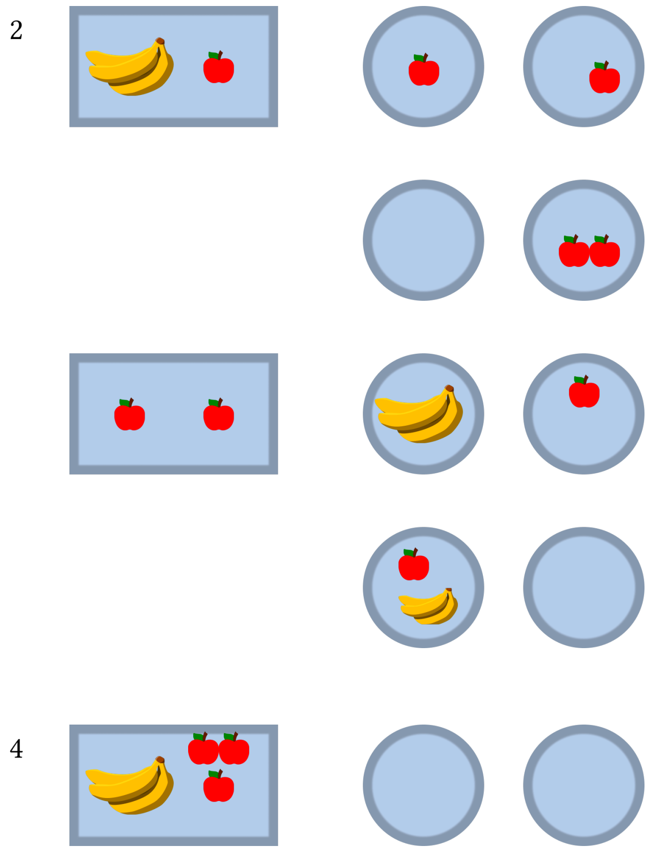
"How many distinguishable ways to map a multiset $S$ to a multiset $T$ satisfying given constraints?"

**Example 22.** There are fruits and plates. Let $S = \{3 \cdot \text{apple}, 1 \cdot \text{banana}\}$ and $T = \{2 \cdot \text{circle}, 1 \cdot \text{rectangle}\}$. How many ways are there to serve fruit on plates such that the rectangular plate has an even number of fruit?



**Ans:** Organize by number of fruit on rectangle.

0




Hence there are 9 ways.

### The general counting problem

Let there be multisets

$$
\begin{aligned}
S &= \{a_1 \cdot 1, a_2 \cdot 2, \dots, a_s \cdot s\} && \text{object types } 1, 2, \dots, s \\
T &= \{b_1 \cdot U_1, b_2 \cdot U_2, \dots, b_t \cdot U_t\} && \text{box types } U_1, \dots, U_t
\end{aligned}
$$

How many distinguishable maps $f : S \to T$ are there, subject to restrictions on the numbers $u_i = \left|f^{-1}(U_i)\right|$?

← The number of items mapped to boxes of type $U_i$

**Remark** (Special cases)**.** To recognize which situation applies to the problem:

- By objects:
  - Distinct: $S = \{1, 2, \dots, s\}$
  - Identical: $S = \{s \cdot 1\}$
- By boxes:

  - Distinct: $T = \{U_1, \dots, U_t\}$
  - Identical: $T = \{t \cdot U_1\}$
- For each case above, we can apply constraints:
  - $0 \le u_i \le 1$
  - $0 \le u_i < \infty$ no constraint
  - $1 \le u_i < \infty$ nonempty
  - $0 \le u_i \le n_i$ for some $n_i \in \mathbb{N}_0$ max. capacity
  - $u_i \in N_i \subseteq \mathbb{N}$

3.3 A Framework for Counting Questions: The Counting Table 95

**Table 3.1 Balls and boxes counting problems.**

Number of Ways to Put Balls into Boxes

$S = S_1 = \{1, 2, \dots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls

$T = T_1 = \{U_1, U_2, \dots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes

Box $U_i$ contains $u_i$ balls

| Conditions on $S$ and $T$ → on $u_i$ ↓ | $T = T_1$ distinct $S = S_1$ distinct | $T = T_1$ distinct $S = S_2$ identical | $T = T_2$ identical $S = S_1$ distinct | $T = T_2$ identical $S = S_2$ identical |
|---|---|---|---|---|
| $0 \le u_i \le 1$ Assume $t \ge s$ | 1 | 2 | 3 | 4 |
| $u_i \ge 0$ | 5 | 6 | 7 | 8 |
| $u_i \ge 1$ | 9 | 10 | 11 | 12 |
| for $i = 1, \dots, t$, $0 \le u_i \le n_i$, $n_i \in \mathbb{Z}^{>0}$ | 13 | 14 | | |
| $u_i \in N_i \subset \mathbb{Z}^{\ge 0}$ for $i = 1, \dots, t$ | 15 | 16 | | |

**Example 23.** We have 10 distinct books to be shared between 2 children. Each child needs 2 books to avoid a crisis. ← Table entry $\boxed{15}$

- Distinct objects $\{1, 2, \dots, 10\}$
- Distinct boxes $\{U_1, U_2\}$
- Constraints: $u_i \in [2, \infty[$

**Example 24** (Attempt 1)**.** 10 distinct books, 5 of them to be arranged on shelf (order matters) ← Table entry $\boxed{15}$

- $S = \{1, 2, \dots, 10\}$
- $T = \{U_1, U_2, \dots, U_5, U_6\}$ (positions on shelf + extra for unshelved books)
- $u_1 = \dots = u_5 = 1, u_6 = 5$

**Example 25** (Attempt 2)**.** 10 distinct books, 5 of them to be arranged on shelf (order matters) ← Table entry $\boxed{1}$, easier!

- $S = \{1, 2, \dots, 5\}$ (numbered stickers to arrange books)
- $T = \{U_1, U_2, \dots, U_{10}\}$ (10 books)
- $0 \le u_1 \le 1$ (each book can get 0 or 1 sticker)

**Example 26.** I have 10 books and will take 5 on holiday.

- Take 1: ← Table entry $\boxed{15}$
  - $S = \{1, 2, \dots, 10\}$
  - $T = \{U_1, U_2\}$
  - $u_1 = u_2 = 5$
- Take 2: ← Table entry $\boxed{2}$
  - $S = \{5 \cdot 1\}$ (identical 'stickers' marking on-holiday)
  - $T = \{U_1, U_2 \ \dots, U_{10}\}$
  - $0 \le u_1 \le 1$ (each book can get 0 or 1 sticker)

## C4 Permutations of a set

**Remark.** Recall $[n] = \{1, 2, \dots, n\}$.

**Definition 10.** Let $0 \le s \le t$.

- An $s$-**permutation** of $[t]$ is an ordered list of $s$ distinct elements of $[t]$.
- A $t$-**permutation** of $[t]$ is just a **permutation** of $[t]$.

**Example 27.** 10 books, arrange 5 on shelf: 5-perm of [10]

**Example 28.** 20 athletes, Gold, Silver and Bronze awarded: 3-perm of [20]

**Theorem 10.** The number of $s$-perms of $[t]$ is

$$\underbrace{t(t-1)\dots(t-s+1)}_{s \text{ terms}}$$

*Proof.* Select elements of the list one-by-one. We have $t$ ways to pick the first, $t-1$ ways of picking the second, etc. □

**Definition 11** ($s$-th falling-factorial function)**.** This inspires the following notation

$$(x)_s = x(x-1)\dots(x-s+1) = \frac{t!}{(t-s)!}$$

← There is also a rising $(x)^s = x(x+1)\dots(x+s-1)$

**Remark.** $(x)_0 = 1, (n)_n = n!$

**Example 29.** Number of ways to shelve 5 books out of 10 is

$$(10)_5 = 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

## C5 Circular permutations

A circular 5-perm of $[n]$ is an arrangement of $s$ distinct elements of $[n]$ around a round table. The difference from a non-circular perm is that the orientation of the table does not matter! How many ways to do so?

If there is a head of the table and positions are marked clockwise, then it would be the same as the $s$-perm $(n)_s$.

However, we consider all ways of marking the 'head' of the tablen to be equivalent, so we divide $s$ upon that. Hence, we get the answer $\frac{1}{s}(n)_s = \frac{n!}{s(n-s)!}$.

## C6 Table entries 3,4,5

- 3 : $s$ distinct objects, $t$ identical boxes, 0 or 1 per box.

$$\#\text{ ways} = \begin{cases} 1 & s \le t \\ 0 & \text{otherwise} \end{cases}$$

- $\boxed{4}$ : $s$ identical objects, $t$ identical boxes, 0 or 1 per box.

$$\# \text{ ways} = \begin{cases} 1 & s \leq t \\ 0 & \text{otherwise} \end{cases}$$

- $\boxed{5}$ : $s$ distinct objects, $t$ distinct boxes, no restrictions.

$$\# \text{ ways} = \underbrace{t \times t \times \cdots \times t}_{s \text{ times}} = t^s$$

**Remark.** $0^0 = 1$ in combinatorics. ← NOT in analysis!!

3.3 A Framework for Counting Questions: The Counting Table 95

**Table 3.1 Balls and boxes counting problems.**

| Number of Ways to Put Balls into Boxes | | | | |
|---|---|---|---|---|
| $S = S_1 = \{1, 2, \ldots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls | | | | |
| $T = T_1 = \{U_1, U_2, \ldots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes | | | | |
| Box $U_i$ contains $u_i$ balls | | | | |
| Conditions on $S$ and $T$ → on $u_i$ ↓ | $T = T_1$ distinct $S = S_1$ distinct | $T = T_1$ distinct $S = S_2$ identical | $T = T_2$ identical $S = S_1$ distinct | $T = T_2$ identical $S = S_2$ identical |
| $0 \leq u_i \leq 1$ Assume $t \geq s$ | $\boxed{1}$ | $\boxed{2}$ | $\boxed{3}$ | $\boxed{4}$ |
| $u_i \geq 0$ | $\boxed{5}$ | $\boxed{6}$ | $\boxed{7}$ | $\boxed{8}$ |
| $u_i \geq 1$ | $\boxed{9}$ | $\boxed{10}$ | $\boxed{11}$ | $\boxed{12}$ |
| for $i = 1, \ldots, t$, $0 \leq u_i \leq n_i$, $n_i \in \mathbb{Z}^{>0}$ | $\boxed{13}$ | $\boxed{14}$ | | |
| $u_i \in N_i \subset \mathbb{Z}^{\geq 0}$ for $i = 1, \ldots, t$ | $\boxed{15}$ | $\boxed{16}$ | | |

## C7 Combinations of sets: table entries 2,6,10

If we let $0 \le s \le t$, then an $s$-**combination of** $[t]$ is a subset of size $s$ of $[t]$.

**Definition 12.** The binomial coefficient $\binom{t}{s}$ is defined (combinatorially) to be the number of $s$-combinations of $[t]$.

**Theorem 11.**

$$\binom{t}{s} = \frac{(t)_s}{s!} = \frac{t!}{(t-s)!s!}$$

*Proof.* Consider the number of $s$-perms of $[t]$.

We can count it directly: $(t)_s$.

Alternatively, make task 1 'selecting a subset of size $s$', which gives us $\binom{t}{s}$ ways. Make task 2 the ways of ordering a subset, which give $s!$ ways. By multiplication principle, the number of $s$-perms of $[t]$ is $\binom{t}{s}s!$.

We have $(t)_s = \binom{t}{s}s!$, giving us the formula above. □

**Example 30.** All of the following are equivalent and satisfy table entry 2 : ← $0 \le u_i \le 1$

- Select $s$ books out of $t$ distinct books;
- Place $s$ stickers on $t$ books, at most one sticker per book;
- Put $s$ identical objects into $t$ distinct boxes, each box can only have at most 1 object.

**Example 31.** These equivalent problems satisfy table entry 6 : ← $0 \le u_i < \infty$

- Solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ where $x_i \ge 0$ are integers.
- Packing a box of 12 bagels of 5 different types of bagels with unlimited supply.
- Number of permutations of 12 objects and 4 'drawer dividers':

$$\bullet\bullet \mid \bullet\bullet\bullet \mid \bullet\bullet \mid \bullet\bullet\bullet\bullet \mid \bullet$$

**Remark.** In general, when we count the number of anagrams $s$ •s and $t-1$ |s, there are $s+t-1$ total symbols. We must select $s$ of the $s+t-1$ positions to be •. Hence, the number of ways would be

← LHS is choose •, RHS is choose |.

$$\binom{s+t-1}{s} = \binom{s+t-1}{t-1}$$

**Example 32.** Entry 10 is the same as 6 except that each box must be nonempty:

- Solutions to $x_1 + \cdots + x_s = t$ where $x_i \geq 1$ integers.
- Select $s$ bagels from $t$ types, with each type chosen at least once.
  This reduces to the prev problem: select one of each type of bagel first; then, we choose $s - t$ bagels of $t$ types.
- Anagrams with $s$ •s and $t-1$ |: must avoid || and | at beginning or end. Then we could think about placing $t-1$ dividers in the $s-1$ spaces between •s, giving us $\binom{s-1}{t-1}$ ways.

## C8 Anagrams

**Example 33.** Find the number of anagrams of the word

$$\text{COMBINATORIALISTICALLY}$$

First, suppose all 22 letters were distinct. We put subscripts:

$$\mathrm{C_1O_1MBI_1NA_1T_1O_2RI_2A_2L_1I_3ST_2I_4C_2A_3L_2L_3Y}$$

Now we have this multiset:

$$\{2\mathrm{C}, 2\mathrm{O}, \mathrm{M}, \mathrm{B}, 4\mathrm{I}, \mathrm{N}, 3\mathrm{A}, 2\mathrm{T}, \mathrm{R}, 3\mathrm{L}, \mathrm{S}, \mathrm{Y}\}$$

We want to find the ways of arranging this!

- Assuming everything is distinct: 22! ways.
- Same assumption but with two tasks in multiplication:
  - Task 1: Arrange without subscripts (what we want)
  - Task 2: Add subscripts: the number of ways to add them is

$$2! \times 2! \times 1! \times 1! \times 4! \times 1! \times 3! \times 2! \times 1! \times 3! \times 1! \times 1!$$

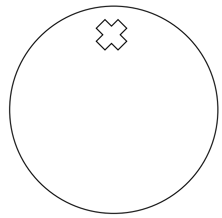
Therefore, our answer would be (in multinomial coefficient) ← DO NOT LEAVE OUT 1s

$$\frac{22!}{2! \times 2! \times 1! \times 1! \times 4! \times 1! \times 3! \times 2! \times 1! \times 3! \times 1! \times 1!} = \binom{22}{2,2,1,1,4,1,3,2,1,3,1,1}$$

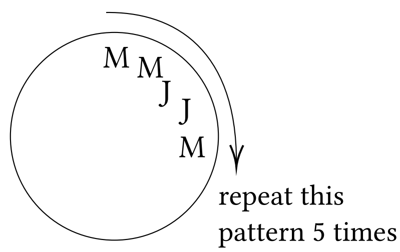
## C9 More circular tables

**Example 34.** An alien conference has 9 Martian hare delegates and 16 Jovian hare delegates, with each type of hare identical. How many distinguishable ways are there to seat them at a circular table?

Method: first consider the ways of arrangement at a marked table.



- We seat 9 Martian hares and fill out the rest with Jovian ones: $\binom{25}{9}$.
- We place all hares (the answer we want) and then mark the table. In the end we should get the same answer of $\binom{25}{9}$.
  - How do we mark the table? There are 25 places that can be marked, but not all of them are distinct! For instance:




then the table only has 5 distinct markings due to rotational symmetry.

**Theorem 12.** Consider an arrangement on a circular table with $n$ spots. Let $R_k$ be the action of rotating this table by $k$ places. Define:

← $F$ is the *stabilizer* of the group action

$$F = \{k \in \mathbb{Z} \mid R_k \text{ leaves the arrangement unchanged}\}$$

Then we have:

1. $F$ is the set of multiples of some $d|n$.
2. A length $d$ pattern would be repeated $\frac{n}{d}$ times (so there are only $\frac{n}{d}$ distinct markings)

*Proof.* Let $d$ be the smallest positive integer such that $R_d \in F$. Suppose BWOC $k \in F$ but $d \nmid k$. Then let $k = md + r$ with $r \leq d$, and rotation by $r$ would be: $R_k R_{-md}$. Since $k \in F$, this also fixes the arrangement. However, $r < d$ contradicts the fact that $d$ is the smallest positive integer such that $R_d \in F$. □

← Review Abstract Alg. 1

Back to Example 34: $F = d\mathbb{Z}$ where $d = 1, 5, 25$.

- If $d = 1$, 25 ways of marking the table.

- $d = 5$, 5 ways of marking.
- $d = 25$, 1 way of marking.

**Definition 13** (Multichoose notation)**.** Define the ways to select a bag of $k$ items from $n$ different types of item to be $\left(\!\binom{n}{k}\!\right) = \binom{k+n-1}{k} = \binom{k+n-1}{n-1}$.

# D Binomial Coefficients

We know that $\binom{n}{k}$ is the number of $k$-subsets of $[n]$, which is $\frac{n!}{k!(n-k)!}$.

## D1 Binomial identities

**Proposition 13.**

$$\binom{n}{k} = \binom{n}{n-k}$$

That is, the number of ways to get $k$-subsets in $[n]$ is the same as that of $n-k$-subsets.

**Proposition 14.**

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

This is because the RHS counts **all** subsets of $[n]$, while each summand on the LHS counts the number of $k$-subsets thereof.

**Proposition 15.** For $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor} = 2^{n-1}$$

This is because the RHS chooses any subset of $[n-1]$, then makes the subset **even** by putting or not putting the $n$ into it (the $n$ doesn't get to choose).

← $\lfloor a \rfloor$ is the largest integer $\leq a$

**Proposition 16** (Binomial recurrence)**.**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is because the LHS chooses $k$-subsets of $[n]$, and the RHS splits the case into 1) the subset contains $n$, which gives us $\binom{n-1}{k-1}$ ways to choose the rest, and 2) the subset doesn't contain $n$, which gives us $\binom{n-1}{k}$.

This allows us to construct the table:

| $n\backslash k$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 |
| 2 | 1 | 2 | 1 | 0 | 0 |
| 3 | 1 | 3 | 3 | 1 | 0 |
| 4 | 1 | 4 | 6 | 4 | 1 |

**Proposition 17.**

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Say there are $2n$ students: $n$ in class G and $n$ in class S. We need to choose $n$ students.

- Method 1: choose $n$ students: $\binom{2n}{n}$
- Method 2: let $k = 0, 1, \dots, n$. Select $k$ S to go and select $k$ G to NOT go. Then we have $\binom{n}{k}$ for each of the process. Hence, the total number of ways is $\sum_{k=0}^{n} \binom{n}{k}^2$.

## D2 Binomial theorem

**Theorem 18.** Let $n \geq 0$ be an integer. Then:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

← This gives a relationship between the Karaji/Pascal triangle and polynomial algebra.

*Proof.*

$$(x+y)^n = (x+y)(x+y)\dots(x+y)$$

A typical term looks like $n$ terms $x, y$ multiplied together, in some amount: $x^{n-k}y^k$.

We get that term by selecting $k$ partentheses to take the $y$ from, and we take $x$ from the rest. This gives $\binom{n}{k}$ ways. □

**Example 35.**

$$
\begin{aligned}
&(x+y)^0 = 1,\\
&(x+y)^1 = x+y,\\
&(x+y)^2 = x^2+2xy+y^2,\\
&(x+y)^3 = x^3+3x^2y+3xy^2+y^3,\\
&(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4,\\
&(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5,\\
&(x+y)^6 = x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6,\\
&(x+y)^7 = x^7+7x^6y+21x^5y^2+35x^4y^3+35x^3y^4+21x^2y^5+7xy^6+y^7,\\
&\qquad \ldots
\end{aligned}
$$

**Example 36.**

$$
\begin{aligned}
1.01^5 &= \sum_{k=0}^{5} \binom{5}{k} 0.01^k\\
&= 1.010510100501
\end{aligned}
$$

Table 2: Jia-Karaji triangle up to $n = 20$.

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | | | | | | | | | | |
| 1 | 2 | 1 | | | | | | | | | | | | | | | | | | |
| 1 | 3 | 3 | 1 | | | | | | | | | | | | | | | | | |
| 1 | 4 | 6 | 4 | 1 | | | | | | | | | | | | | | | | |
| 1 | 5 | 10 | 10 | 5 | 1 | | | | | | | | | | | | | | | |
| 1 | 6 | 15 | 20 | 15 | 6 | 1 | | | | | | | | | | | | | | |
| 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 | | | | | | | | | | | | | |
| 1 | 8 | 28 | 56 | 70 | 56 | 28 | 8 | 1 | | | | | | | | | | | | |
| 1 | 9 | 36 | 84 | 126 | 126 | 84 | 36 | 9 | 1 | | | | | | | | | | | |
| 1 | 10 | 45 | 120 | 210 | 252 | 210 | 120 | 45 | 10 | 1 | | | | | | | | | | |
| 1 | 11 | 55 | 165 | 330 | 462 | 462 | 330 | 165 | 55 | 11 | 1 | | | | | | | | | |
| 1 | 12 | 66 | 220 | 495 | 792 | 924 | 792 | 495 | 220 | 66 | 12 | 1 | | | | | | | | |
| 1 | 13 | 78 | 286 | 715 | 1287 | 1716 | 1716 | 1287 | 715 | 286 | 78 | 13 | 1 | | | | | | | |
| 1 | 14 | 91 | 364 | 1001 | 2002 | 3003 | 3432 | 3003 | 2002 | 1001 | 364 | 91 | 14 | 1 | | | | | | |
| 1 | 15 | 105 | 455 | 1365 | 3003 | 5005 | 6435 | 6435 | 5005 | 3003 | 1365 | 455 | 105 | 15 | 1 | | | | | |
| 1 | 16 | 120 | 560 | 1820 | 4368 | 8008 | 11440 | 12870 | 11440 | 8008 | 4368 | 1820 | 560 | 120 | 16 | 1 | | | | |
| 1 | 17 | 136 | 680 | 2380 | 6188 | 12376 | 19448 | 24310 | 24310 | 19448 | 12376 | 6188 | 2380 | 680 | 136 | 17 | 1 | | | |
| 1 | 18 | 153 | 816 | 3060 | 8568 | 18564 | 31824 | 43758 | 48620 | 43758 | 31824 | 18564 | 8568 | 3060 | 816 | 153 | 18 | 1 | | |
| 1 | 19 | 171 | 969 | 3876 | 11628 | 27132 | 50388 | 75582 | 92378 | 92378 | 75582 | 50388 | 27132 | 11628 | 3876 | 969 | 171 | 19 | 1 | |
| 1 | 20 | 190 | 1140 | 4845 | 15504 | 38760 | 77520 | 125970 | 167960 | 184756 | 167960 | 125970 | 77520 | 38760 | 15504 | 4845 | 1140 | 190 | 20 | 1 |

## D3 Further binomial identities

Immediate consequences from the binomial theorem:

**Corollary 19.** $(1+x)^n = \sum_{k=1}^{n} \binom{n}{k} x^k$

And therefore,

**Corollary 20.** $2^n = \sum_{k=0}^{n} \binom{n}{k}$, $0^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$

← take $x = 1$ or $x = -1$

**Theorem 21.** Observe:

(a)
$$\sum_{k=1}^{n} k\binom{n}{k} = 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}$$

(b)
$$\sum_{k=1}^{n} k\binom{n}{k} = 1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = (n+1)n2^{n-2}$$

---

*(Scratch work begins)*

(a)
$$\begin{array}{llllll}
0 \cdot 1 & & & & & \\
0 \cdot 1 & 1 \cdot 1 & & & & = 1 = 1 \times 1 \\
0 \cdot 1 & 1 \cdot 2 & 2 \cdot 1 & & & = 4 = 2 \times 2 \\
0 \cdot 1 & 1 \cdot 3 & 2 \cdot 3 & 3 \cdot 1 & & = 12 = 3 \times 4 \\
0 \cdot 1 & 1 \cdot 4 & 2 \cdot 6 & 3 \cdot 4 & 4 \cdot 1 & = 32 = 4 \times 8
\end{array}$$

(b)
$$\begin{array}{llllll}
0 \cdot 1 & & & & & \\
0 \cdot 1 & 1 \cdot 1 & & & & = 1 = 1 \times 1 \\
0 \cdot 1 & 1 \cdot 2 & 4 \cdot 1 & & & = 6 = 3 \times 2 \\
0 \cdot 1 & 1 \cdot 3 & 4 \cdot 3 & 9 \cdot 1 & & = 24 = 6 \times 4 \\
0 \cdot 1 & 1 \cdot 4 & 4 \cdot 6 & 9 \cdot 4 & 16 \cdot 1 & = 80 = 10 \times 8
\end{array}$$

*(Scratch ends here)*

---

*Proof (a), method 1.* We know $\sum_{k=0}^{n} \binom{n}{k} x^k \equiv (1+x)^n$. Hence, we can take the derivative of both sides:

$$\sum_{k=1}^{n} k\binom{n}{k}x^{k-1} \equiv n(1+x)^{n-1}$$

Now let $x = 1$ and obtain $\sum_{k=1}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$. □

*Proof (b), method 1.* Similar to proof (a) but we first multiply both sides of the differentiated equation in (a) by $x$: $\sum_{k=1}^{n} k\binom{n}{k}x^k \equiv xn(1+x)^{n-1}$. Then we differentiate it again and plug in $x = 1$. □ ← check this!

*Proof (a), method 2.* We let there be $n$ people, with $k$ of them selected to join the elite squad™. One of the people in elite squad™ is given a secret microfiche. In how many ways can this be done?

Way 1: Choose $k$ be the size of the elite squad™, so $k$ could be anything from 1 to $n$. We need to choose $k$ people among $n$, giving us $\binom{n}{k}$ ways. Then, we choose one person among the $k$ to give a microfiche. The total number of ways is $\sum_{k=1}^{n} \binom{n}{k} k$.

Way 2: Assign a microfiche, which can be done in $n$ ways. Decide who else is in the squad: it is either 'yes' (in the elite squad™) or 'no'. Hence, we make a binary decision for $n-1$ people, which gives us $2^{n-1}$ ways. The total number of ways is $n \cdot 2^{n-1}$.

Both methods give the same answer. □

*Proof (b), method 2.* We let there be $n$ people, with $k$ of them selected to jointhe elite squad™. One of the person in elite squad™ is given a homework problem. One of the person (could be the same) in elite squad™ is given an investigation problem. In how many ways can this be done?

Way 1: Choose $k$ be the size of the elite squad™, so $k$ could be anything from 1 to $n$. We need to choose $k$ people among $n$, giving us $\binom{n}{k}$ ways. Then, we choose one person among the $k$ to give homework, and choose again to give an investigation. The total number of ways is $\sum_{k=1}^{n} \binom{n}{k} k^2$.

Way 2: We first consider the case where the homework and investigation go to the same person. Assign them, which can be done in $n$ ways. Decide who else is in the squad: it is either 'yes' (in the elite squad™) or 'no'. Hence, we make a binary decision for $n-1$ people, which gives us $2^{n-1}$ ways. The total number of ways is $n \cdot 2^{n-1}$.

Next, we consider the case where the homework and investigation go to different people. Assign homework, which can be done in $n$ ways. Then, assign investigation, which can be given to the rest $n-1$ people. We then decide who else is in the squad, giving us $2^{n-2}$ ways.
Hence, we have $n \cdot 2^{n-1} + n(n-1) \cdot 2^{n-2} = n(n+1)2^{n-2}$ ways.

Both methods give the same answer. □

## D4 Newton's Binomial Theorem

**Theorem 22** (Newton)**.** If $x, y \in \mathbb{R}$ and $|\frac{y}{x}| < 1$, let $\alpha > 0$ be real. Then ← not only integers!

$$\begin{aligned}(x+y)^\alpha &= \sum_{k=0}^{\infty} \binom{\alpha}{k} x^{\alpha-k} y^k \\ &= x^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{y}{x}\right)^k\end{aligned}$$

where $\binom{\alpha}{k} = \frac{(\alpha)_k}{k!}$ and $(\alpha)_k = \alpha(\alpha-1)\ldots(\alpha-k+1)$ is the falling factorial.

*Proof.* Let $f(z) = (1+z)^\alpha$. Then $f^{(k)}(z) = (\alpha)_k(1+z)^{\alpha-k}$ and so $f^{(k)}(0) = (\alpha)_k$. By the Taylor series of $f(z)$, we get $f(z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} z^k$ for all $|z| < 1$. □

**Example 37.** What is the $\frac{1}{2}$th row of the Karaji triangle?

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | … |
|---|---|---|---|---|---|---|---|
| $\left(\frac{1}{2}\right)_k$ | 1 | $\frac{1}{2}$ | $\frac{-1}{4}$ | $\frac{3}{8}$ | $\frac{-15}{16}$ | $\frac{105}{32}$ | |
| $\binom{\frac{1}{2}}{k}$ | 1 | $\frac{1}{2}$ | $\frac{-1}{8}$ | $\frac{1}{16}$ | $\frac{-5}{128}$ | $\frac{7}{256}$ | |

**Example 38.**

$$\begin{aligned}\sqrt{103} &= \sqrt{100+3} \\ &= 10 \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \left(\frac{3}{100}\right)^k \\ &= 10 \cdot \left(1 + \frac{15}{1000} - \frac{1125}{10^7} + \ldots\right) \\ &\simeq 10.148875\end{aligned}$$

Actual answer: 10.14889...

### D5 Simplex numbers

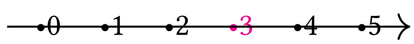
Recall the simplex numbers $T_d(n)$:

| $d\backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| 3 | 1 | 4 | 10 | 20 | 35 | 56 | 84 |
| 4 | 1 | 5 | 15 | 35 | 70 | 126 | 210 |
| 5 | 1 | 6 | 21 | 56 | 126 | 252 | 462 |

Observe that these are the same as binomial coefficients and just organised differently!

Consider teh nonnegative integer lattice in $d+1$ dimensions: $\mathbb{N}^{d+1} \subseteq \mathbb{R}^{d+1}$. What are the points $(a_0, a_1, \dots, a_d)$ such that $a_0 + a_1 + \cdots + a_d = k$ for a given $k \geq 0$? How many are there?

**Example 39.** When $k = 3$ and $d = 0$:




In general, the number of points on the plane satisfying $x_0 + x_1 + \cdots + x_d = k - 1$ is equal to $T_d(k)$. From our multichoose argument, this is $T_d(k) = \binom{d+k-1}{d} = \left(\!\binom{k}{d}\!\right)$.

## E Catalan numbers

### E1 Examples

**Example 40.** How many sequecnes $a_1, a_2, \dots, a_n$ are there, where $n$ of the terms are $+1$ and $n$ are $-1$, such that the partial sums are all nonnegative? That is, $a_1 + a_2 + \cdots + a_k \geq 0$ for all $k$?

The problem concerns a multiset: $n$ objects of type 1 and $n$ of type 2. We want to count the number of permutations such that the number of type 1 occuring in the first $k$ letters is $\geq$ the number of type 2 occuring in the first $k$ letters for all $k = 0, 1, \dots, 2n$.

**Definition 14.** Let $C_n$ be the number of permutations as above. This is the $n$-th **Catalan number**.

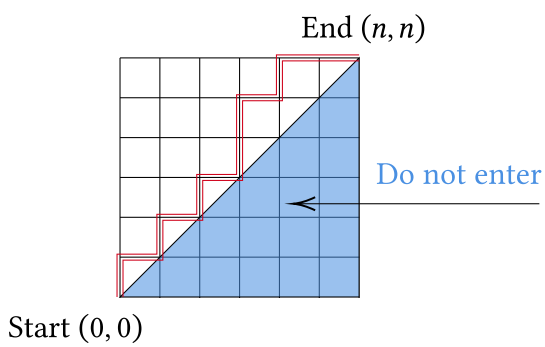
We can calculate a few:

| $n$ | $C_n$ | $\binom{2n}{n}$ | $C_n/\binom{2n}{n}$ |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 1 | 2 | 1/2 |
| 2 | 2 | 6 | 1/3 |
| 3 | 5 | 20 | 1/4 |
| 4 | 14 | 70 | 1/5 |

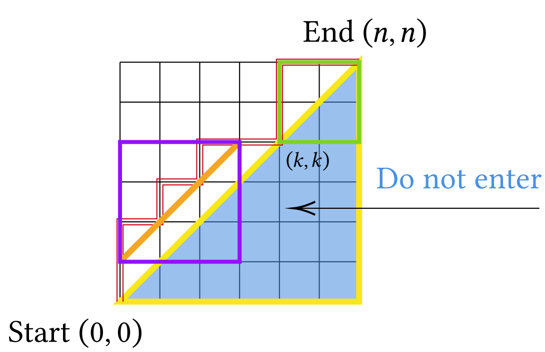
**Example 41.** $C_n$ is also the number of anagrams of $\underbrace{NN\dots N}_{n}\underbrace{EE\dots E}_{n}$ such that every initial segment has more $N$ than $E$.

**Conjecture:** $C_n = \frac{1}{n+1}\binom{2n}{n}$

## E2 First attempt




**Question:** when is the **first time** we return to the diagonal? That is, what is the smallest $k \in \{1, 2, \dots, n\}$ such that we get $(k,k)$?




Steps:

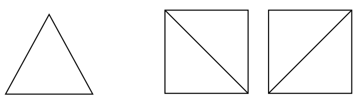
1. $(0,0) \to (0,1)$ — 1 way
2. $(0,1) \to (k-1,k)$ w/o crossing orange line — $C_{k-1}$ ways ← in the purple square
3. $(k-1,k) \to (k,k)$ — 1 way
4. $(0,1) \to (k-1,k)$ w/o crossing yellow line — $C_{n-k}$ ways ← in the green square

Hence ← the Catalan recurrence

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}$$

And so $C_n = C_0C_{n-1} + \cdots + C_{n-1}C_0$.

**Example 42.** How many ways to triangulate a convex 47-sided polygon by drawing diagonals? ANS: $D_{47}$!



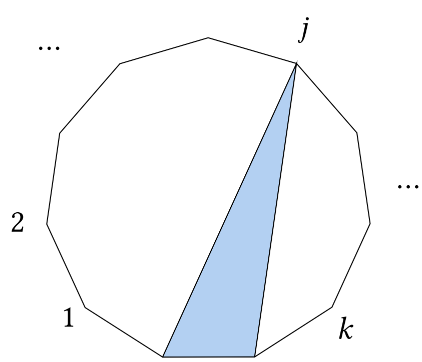
Examples: $D_3 = 1$ $D_4 = 2$ …

**Conjecture:** $D_n = C_{n-2}$

*Induction proof.* Base case is done as shown above.

Inductive step: suppose it works for $n = 2, 3, \dots, k+1$. Now we want to show $D_{k+2} = C_k$.

<u>Case $j$</u>: we pick a side that has a triangle that goes to the vertex $j$.




On the left, we have a $j+1$ sided polygon and so the number of ways of triangulation is $D_{j+1} = C_{j-1}$.

On the right, we have a $k-j+2$ sided polygon and so the number of ways of triangulation is $D_{k-j+2} = C_{k-j}$.

Hence, the number of ways in total is $C_{j-1} \cdot C_{k-j}$. We conclude that

$$D_{k+2} = \sum_{j=1}^{k} C_{j-1}C_{k-j} = C_k$$

by the Catalan recurrence. □

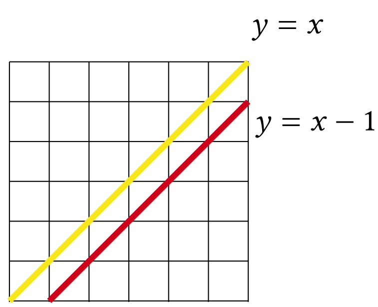
## E3 The Catalan bijection

Suppose a path $(0, 0) \to (n, n)$ is **good** if it does not cross the diagonal $y = x$. We claim that the count of good paths is $\frac{1}{n+1}\binom{2n}{n}$.

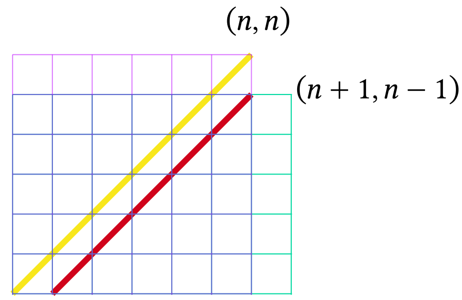
Alternatively, we could show that the number of **bad** paths is # all paths $-\frac{1}{n+1}\binom{2n}{n}$, which means we want to show:

$$\begin{aligned}
\#\ \textbf{bad}\text{ paths} &= \binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n} \\
&= \frac{n}{n+1}\binom{2n}{n} \\
&= \frac{n}{n+1}\frac{(2n)!}{n!n!} \\
&= \frac{(2n)!}{(n-1)!(n+1)!} \\
&= \binom{2n}{n+1}
\end{aligned}$$

A **bad** path is one that crosses the diagonal $y = x$, but that is equivalent to touching the subdiagonal $y = x - 1$.




**Claim**: the number of paths $(0, 0) \to (n, n)$ that touch the red line is the same as the number of paths $(0,0) \to (n+1, n-1)$.




The latter quantity is just $\binom{2n}{n+1}$.

We use the following bijection:

$$\begin{array}{c} \text{Bad path to } (n,n) \\ \Big\updownarrow \; \begin{array}{l}\text{Keep path the same up} \\ \text{to the first time it hits the} \\ \text{red subdiagonal. After that,} \\ \text{swap } N \leftrightarrow E \text{ (reflect path} \\ \text{across the red line)}\end{array} \\ \text{(Bad) path to } (n+1, n-1) \end{array}$$

$\leftarrow$ all paths are bad to $(n+1, n-1)$!

# F Stirling numbers

## F1 Table entries 11, 9, 7



**Table 3.1 Balls and boxes counting problems.**

| Number of Ways to Put Balls into Boxes | | | | |
|---|---|---|---|---|
| $S = S_1 = \{1, 2, \ldots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls<br>$T = T_1 = \{U_1, U_2, \ldots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes<br>Box $U_i$ contains $u_i$ balls | | | | |
| Conditions on $S$ and $T$ → on $u_i$ ↓ | $T = T_1$ distinct $S = S_1$ distinct | $T = T_1$ distinct $S = S_2$ identical | $T = T_2$ identical $S = S_1$ distinct | $T = T_2$ identical $S = S_2$ identical |
| $0 \le u_i \le 1$ Assume $t \ge s$ | $\boxed{1}$ | $\boxed{2}$ | $\boxed{3}$ | $\boxed{4}$ |
| $u_i \ge 0$ | $\boxed{5}$ | $\boxed{6}$ | $\boxed{7}$ | $\boxed{8}$ |
| $u_i \ge 1$ | $\boxed{9}$ | $\boxed{10}$ | $\boxed{11}$ | $\boxed{12}$ |
| for $i = 1, \ldots, t$, $0 \le u_i \le n_i$, $n_i \in \mathbb{Z}^{>0}$ | $\boxed{13}$ | $\boxed{14}$ | | |
| $u_i \in N_i \subset \mathbb{Z}^{\ge 0}$ for $i = 1, \ldots, t$ | $\boxed{15}$ | $\boxed{16}$ | | |

(11) : How many ways to place $s$ distinct objects into $t$ **unmarked** boxes such that none of the boxes are empty?
⇔ organizing $s$ distinct people into $t$ nonempty teams.
We don't know the (closed-form) answer yet, so we temporarily call it $\left\{ s \atop t \right\}$.

← Distinct objects, identical boxes, $u_i \ge 1$

**Example 43.** We have 4 people: A,B,C and D, and we group them into 3 teams. This is the situation $s = 4, t = 3$. We group:
- AB|C|D
- AC|B|D
- AD|B|C
- BC|A|D
- BD|A|C
- CD|A|B

Therefore, $\left\{ {4 \atop 3} \right\} = 6$.

9 : How many ways to place $s$ distinct objects into $t$ **named** boxes such that none of the boxes are empty?
$\Leftrightarrow$ organizing $s$ distinct people into $t$ nonempty teams with different team names.
Compared with 11 , this gives $t!$ more ways due to the labelling. Therefore, the answer is $t! \left\{ {s \atop t} \right\}$.

← Distinct objects, distinct boxes, $u_i \geq 1$

**Remark.** This is the same as counting the number of *surjective* functions

$$\{1, 2, \ldots, s\} \to \{1, 2, \ldots, t\}$$

we expect some relationship with the quantity $t^s$, which counts all functions.

7 : How many ways to place $s$ distinct objects into $t$ **unmarked** boxes?
Answer: $\displaystyle\sum_{k=0}^{t} \left\{ {s \atop k} \right\}$ where $k$ is the # of nonempty boxes.

← Distinct objects, identical boxes, $u_i \geq 0$. i.e. 11 but boxes can be empty

**Remark.** Observe:

| | | |
|---|---|---|
| $u_i \geq 0$ | 5 $t^s$ | 7 $\sum_{k=0}^{t} \left\{ {s \atop k} \right\}$ |
| $u_i \geq 1$ | 9 $t! \left\{ {s \atop t} \right\}$ | 11 $\left\{ {s \atop t} \right\}$ |

We can get 9 by multiplying 11 by $t!$, but we cannot get 5 from 7 in the same way due to the empty boxes!

The ways to label boxes in 5 is only $(t)_k$ ways for each # of nonempty boxes $k$.

**Conclusion:**

$$t^s = \sum_{k=0}^{t} (t)_k \left\{ {s \atop k} \right\}$$

It follows that the quantities $0^s, 1^s, \ldots, s^s$ are *linearly* related to $\left\{ {s \atop 0} \right\}, \left\{ {s \atop 1} \right\}, \ldots, \left\{ {s \atop s} \right\}$.

**Example 44.** Let $s = 3$.

$$\begin{bmatrix} 0 \\ 1 \\ 8 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{bmatrix} \cdot \begin{bmatrix} \left\{ {3 \atop 0} \right\} \\ \left\{ {3 \atop 1} \right\} \\ \left\{ {3 \atop 2} \right\} \\ \left\{ {3 \atop 3} \right\} \end{bmatrix}$$

where every entry $(i, j)$ of the matrix is $(i)_j$.

We can invert the matrix to find what $\left\{ {s \atop t} \right\}$ is, but it's computationally quite terrible!

## F2 Stirling numbers of the second kind

**Definition 15** (Stirling numbers of the second kind)**.**

$$\left\{ {s \atop t} \right\} = \text{\# of ways to place } s \text{ distinct objects into } t \text{ identical empty boxes}$$

**Proposition 23.**

1. $\left\{ {s \atop t} \right\} = 0$ if $t > s$ too many boxes
2. $\left\{ {s \atop s} \right\} = 1$ must have object in its own box
3. $\left\{ {s \atop 1} \right\} = 1$ for $s \geq 1$ everything in 1 box
4. $\left\{ {s \atop s-1} \right\} = \binom{s}{2}$ which 2 ppl in the team of 2, rest solo
5. $\left\{ {s \atop 2} \right\} = 2^{s-1} - 1$

   Ways:
   (a) Ask who else among the $s - 1$ people are my team, count the possibilities. Subtract the case when everyone is in my team for the other team to be nonempty.
   (b) Categorize $s$ people into two *distinct* boxes, which gives us $2^s$ ways. Then, subtract the 2 ways where everyone is in the same box. Finally, since the boxes aren't distinct after all, we divide the result by $2!$.
6. $\left\{ {s \atop 0} \right\} = \begin{cases} 1 & \text{if } s = 0 \\ 0 & \text{if } s \geq 1 \end{cases}$

| $s\backslash t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Row sum |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| 3 | 0 | 1 | 3 | 1 | 0 | 0 | 0 | 5 |
| 4 | 0 | 1 | 7 | 6 | 1 | 0 | 0 | 15 |
| 5 | 0 | 1 | 15 | 25 | 10 | 1 | 0 | 522 |
| 6 | 0 | 1 | 31 | 90 | 65 | 15 | 1 | 203 |

← The row sums are Bell numbers!

**Theorem 24.**

$$\left\{ {s \atop t} \right\} = t \left\{ {s-1 \atop t} \right\} + \left\{ {s-1 \atop t-1} \right\}$$

Understand this as placing people $1, 2, \dots, s$ into teams $T_1, \dots, T_t$.

From perspective of person $s$:

Case 1: Person $s$ is alone, so $\left\{ {s-1 \atop t-1} \right\}$ ways to organize the rest.

Case 2: Person $s$ is not alone.

- Organize the rest into $t$ teams, which is $\left\{ {s-1 \atop t} \right\}$ ways.
- Join one of the teams, which is $t$ ways.

Add the two cases.

## F3 Stirling numbers of the first kind

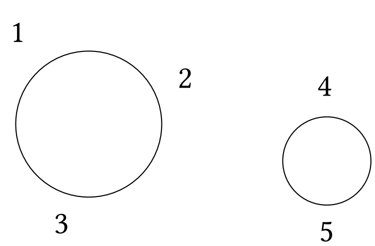
**Definition 16** (Stirling numbers of the first kind)**.**

$\left[ {n \atop k} \right]$ = # of ways of seating $n$ people at $k$ circular tables (no empty table)

In general, $\left[ {n \atop k} \right] \geq \left\{ {n \atop k} \right\}$ (usually a lot bigger) because of the extra choices needed to seat the $k$ teams at tables.

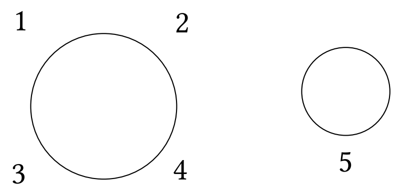
**Example 45.** We let $n = 5, k = 2$.

Case 1: split into 3+2



- Choose two people to sit at the table for two: $\binom{5}{2}$=10
- Place around the circular tables: $(3-1)! \times (2-1)! = 2$

Case 2: split into 4+1




- Choose 1 person to sit alone: 5
- Place around circular table: $(4-1)! = 6$

Hence, the total ways is $\begin{bmatrix}5\\2\end{bmatrix} = 20 + 30 = 50$.

**Remark.** The first task of each case gives us $\begin{Bmatrix}5\\2\end{Bmatrix} = 10 + 5 = 15$. We can get crude bounds:

$$2\begin{Bmatrix}5\\2\end{Bmatrix} < \begin{bmatrix}5\\2\end{bmatrix} < 6\begin{Bmatrix}5\\2\end{Bmatrix}$$

**Proposition 25.**

1. $\begin{bmatrix}n\\k\end{bmatrix} = 0$ if $k > n$: too many tables
2. $\begin{bmatrix}n\\0\end{bmatrix} = 0$ if $n \geq 1$: not enough tables
3. $\begin{bmatrix}n\\n\end{bmatrix} = 1$: one person per table
4. $\begin{bmatrix}n\\1\end{bmatrix} = (n-1)!$ for $n \geq 1$: simple circular permutation
5. $\begin{bmatrix}n\\n-1\end{bmatrix} = \binom{n}{2}$: which 2 sit at the table for 2

### Table of values of $\begin{bmatrix}n\\k\end{bmatrix}$

| $s\backslash t$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | Row sum | Alternating sum |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | -1 |
| 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 |
| 3 | 0 | 2 | 3 | 1 | 0 | 0 | 0 | 6 | 0 |
| 4 | 0 | 6 | 11 | 6 | 1 | 0 | 0 | 24 | 0 |
| 5 | 0 | 24 | 50 | 35 | 10 | 1 | 0 | 120 | 0 |
| 6 | 0 | 120 | 274 | 225 | 85 | 15 | 1 | 720 | 0 |

**Theorem 26.**

$$\begin{bmatrix}n\\k\end{bmatrix} = (n-1)\begin{bmatrix}n-1\\k\end{bmatrix} + \begin{bmatrix}n-1\\k-1\end{bmatrix}$$

*Proof.* Consider from the perspective of yourself. We seat $n$ people at $k$ table.

- You sit alone:
  - You seat the rest of the people at $k-1$ tables: $\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ ways
  - Claim the last table by yourself
- You don't sit alone:
  - Seat the rest of the people at $k$ tables: $\begin{bmatrix} n-1 \\ k \end{bmatrix}$ ways
  - You sit to the left of somebody: $n-1$ ways

□

**Theorem 27.**

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!$$

*Proof outline.* We claim:

$$\begin{aligned} \text{LHS} &= \#\text{ ways to seat } n \text{ people at circular tables} \\ \text{RHS} &= \#\text{ permutations of } [n] = \{1, 2, \dots, n\} \end{aligned}$$

and there is a bijections between ways to seat $n$ people at circular tables and permutations of $[n]$. □

← Consider the elements of $S_n$.

**Theorem 28.**

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

*Sketch.* We are effectively counting the permutations with ± sign according to whether the number of cycles is even or odd.

It turns out:

$$(-1)^{\#\text{ cycles}} = (-1)^n \cdot (\text{sign of permutation})$$

and follows that

$$LHS = (-1)^n \cdot \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

Since $\det[] = 1$, $\det[1] = 1$ and $\det = 0$ otherwise, we are done. □

**Theorem 29.**

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \frac{\begin{bmatrix} n+1 \\ 2 \end{bmatrix}}{n!}$$

*Proof by induction.* True for:

$$\begin{aligned} 1 &= \frac{1}{1} \\ 1 + \frac{1}{2} &= \frac{3}{2!} \\ 1 + \frac{1}{2} + \frac{1}{3} &= \frac{11}{3!} \end{aligned}$$

Then, suppose true for $n = k - 1$. Then

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{k-1} + \frac{1}{n} &= \frac{\begin{bmatrix} k \\ 2 \end{bmatrix}}{(k-1)!} + \frac{1}{k} \\ &= \frac{k\begin{bmatrix} k \\ 2 \end{bmatrix} + (k-1)!}{k!} \\ &= \frac{k\begin{bmatrix} k \\ 2 \end{bmatrix} + \begin{bmatrix} k \\ 1 \end{bmatrix}}{k!} \\ &= \frac{\begin{bmatrix} k+1 \\ 2 \end{bmatrix}}{k!} \end{aligned}$$

□

# G The inclusion-exclusion (IE) principle

*Sometimes AND is easy, but OR is difficult.* This is when IE can help.

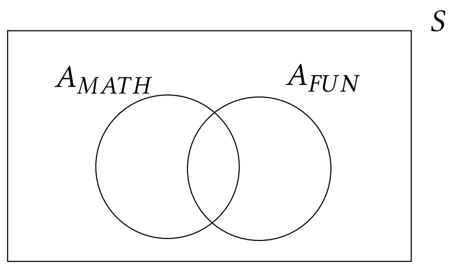
It is really about handling Venn diagrams or indicator functions.

## G1 Introduction

**Example 46.** How many anagrams of MATHFUN have **neither** of the substrings MATH or FUN?

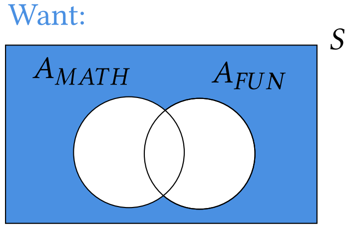
Let $S =$ {anagrams of MATHFUN}, $A_{MATH} =$ {anagrams of MATH} and $A_{FUN} =$

{anagrams of FUN}. We can draw the following Venn diagram:

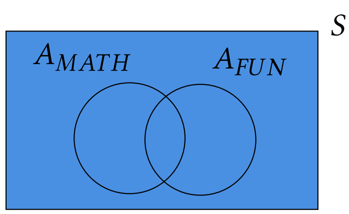



Where:

Want:




$=$

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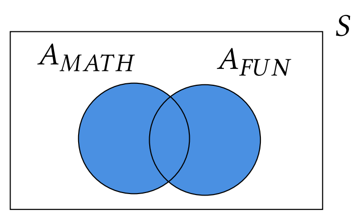
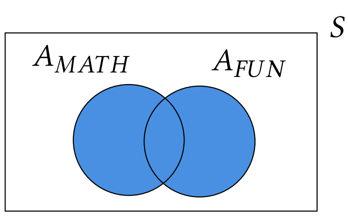
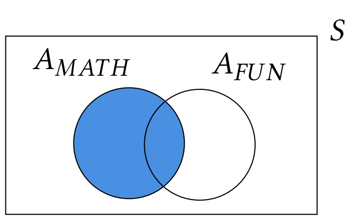



Note that

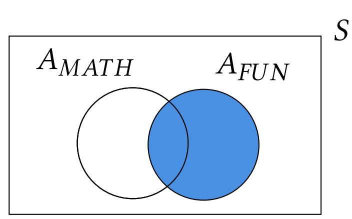



$=$



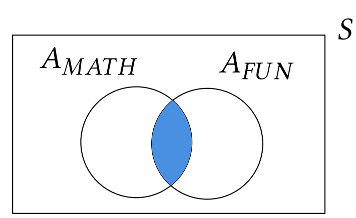

$4!$

$+$




$5!$

$-$




$2!$

And hence, the answer is $7! - (4! + 5! - 2!) = 4898$.

## Indicator functions

These are $\{0, 1\}$-valued functions on $S$.

**Example 47.** Let:

$$\begin{aligned} \mathbb{1}_s(x) &= 1 \quad \text{for all anagrams of } x \\ \mathbb{1}_{MATH}(x) &= \begin{cases} 1 & \text{if } x \text{ contains MATH} \\ 0 & \text{otherwise} \end{cases} \\ \mathbb{1}_{FUN}(x) &= \begin{cases} 1 & \text{if } x \text{ contains FUN} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

**Proposition 30.** Key facts about indicator functions:

AND:

$$\mathbb{1}_{A\cap B} = \mathbb{1}_A \mathbb{1}_B$$

COMPLEMENT:

$$\mathbb{1}_A^c = 1 - \mathbb{1}_A$$

OR:

$$\begin{aligned} \mathbb{1}_{A\cup B} &= 1 - \mathbb{1}_{A^c \cap B^c} \\ &= 1 - \mathbb{1}_{A^c}\mathbb{1}_{B^c} \\ &= 1 - (1 - \mathbb{1}_A)(1 - \mathbb{1}_B) \\ &= \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B} \end{aligned}$$

So we get the same formula for the previous example:

$$\mathbb{1}_{\text{MATH or FUN}} = \mathbb{1}_{MATH} + \mathbb{1}_{FUN} - \mathbb{1}_{\text{MATH and FUN}} \qquad (*)$$

**Definition 17.** For any function $f : S \to \mathbb{R}$, we can define

$$\int f = \sum_{x \in S} f(x)$$

For indicator functions,

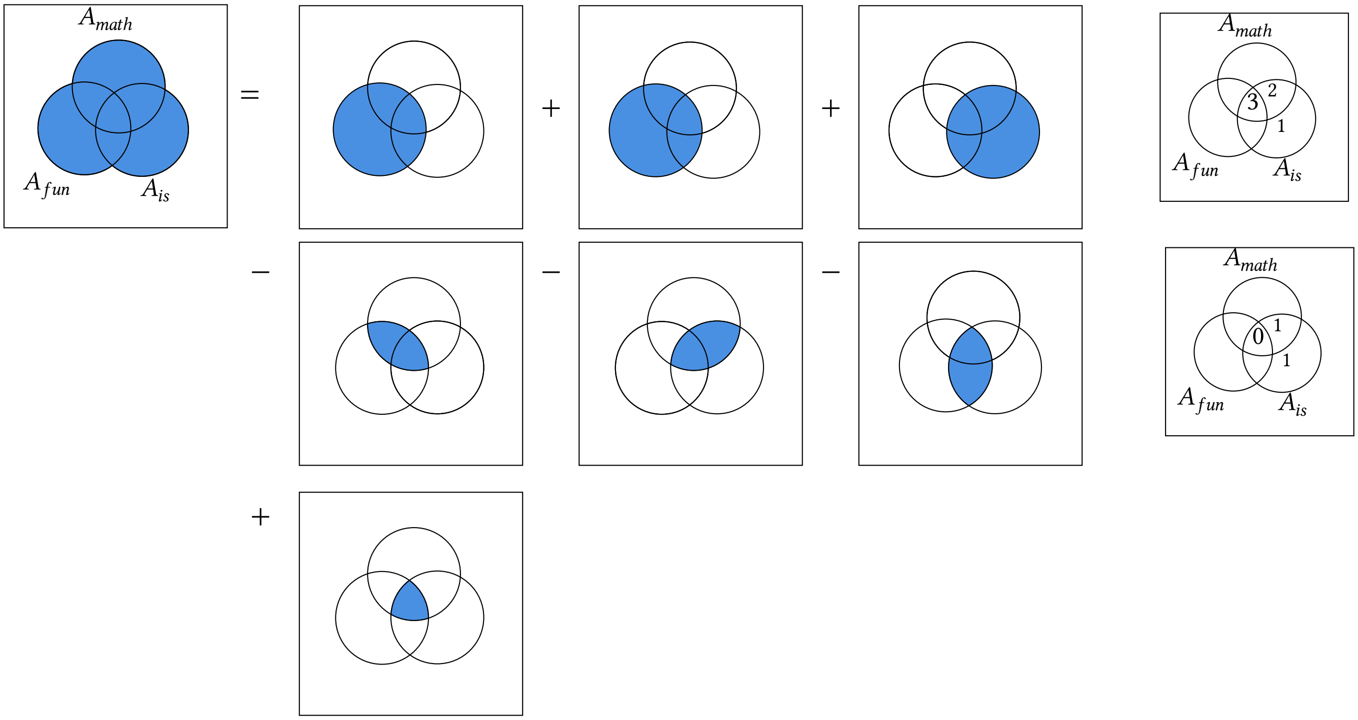
$$\int \mathbb{1}_A = \sum_{x\in S} \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} = \sum_{x \in A} 1 = |A|$$

we get the cardinality of $A$.

Therefore, we integrate both sides of $(*)$ and get the cardinality of $A_{MATH} \cup A_{FUN}$.

**Example 48.** How many anagrams of MATHISFUN do contain at least one of MATH, IS or FUN?

**Method 1:**




Hence:

$$
\begin{aligned}
ANS =& 6! + 8! + 7! \\
&- 5! - 6! - 4! \\
&+ 3! \\
=& 45222
\end{aligned}
$$

**Method 2:** Let $\mathbb{1}_M, \mathbb{1}_I, \mathbb{1}_F$ be the indicator functions. Then:

$$
\begin{aligned}
\mathbb{1}_{M\vee I\vee F} &= 1 - \mathbb{1}_{M^c\wedge I^c\wedge F^c} \\
&= 1 - (1 - \mathbb{1}_M)(1 - \mathbb{1}_I)(1 - \mathbb{1}_F)
\end{aligned}
$$

and the formula is the same as method 1.

## G2 The IE formula

Let $S$ be a set of objects (e.g. anagrams). Let $P_1, \dots, P_n$ be properties these objects could satisfy (e.g. "contains MATH"). Then sets $A_i = \{x \in S \mid P(x) \text{ true}\}$.

**Theorem 31.**

0. $$|S| = |S|$$

1. $$|A_1^c| = |S| - |A_1|$$

2. $$|A_1^c \cap A_2^c| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|$$

3. $$\begin{aligned}|A_1^c \cap A_2^c \cap A_3^c| = &|S| \\ &- |A_1| - |A_2| - |A_3| \\ &+ |A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| \\ &- |A_1 \cap A_2 \cap A_3|\end{aligned}$$

$\vdots$

$n$. $$\begin{aligned}|A_1^c \cap \dots \cap A_n^c| &= \sum_{I \subseteq [n]} (-1)^{|I|} \left|\bigcap_{i \in I} A_i\right| \\ &= |S| - (|A_1| + \dots + |A_n|) + (\dots + |A_i \cap A_j| + \dots) - \dots + (-1)^n |A_1 \cap \dots \cap A_n|\end{aligned}$$

*Proof by indicator function.* We observe:

$$\begin{aligned}\mathbb{1}_{A_1^c \cap \dots \cap A_n^c} &= \mathbb{1}_{A_1^c} \cdot \dots \cdot \mathbb{1}_{A_n^c} \\ &= \sum_{I \subseteq [n]} \prod_{i=1}^{n} \begin{cases} -\mathbb{1}_{A_i} & \text{if } c \in I \\ 1 & \text{if } c \notin I \end{cases} \\ &= \sum_{I \subseteq [n]} \prod_{i \in I} (-\mathbb{1}_{A_i}) \\ &= \sum_{I \subseteq [n]} (-1)^{|I|} \mathbb{1}_{\bigcap_{i \in I} A_i}\end{aligned}$$

Now integrate both sides and get the result. □

---

**Example 49.** How many integers in $S = \{1, 2, \dots, 1000\}$ are not divisible by 5 or 6 or 8?

Let $A_i = \{x \in S \mid x \text{ divisible by } i\}$. We want $|A_2^c \cap A_5^c \cap A_8^c|$.

We know that we need to subtract from 1000:

$$|A_5| = \frac{1000}{5} = 200$$

$$|A_6| = \lfloor \frac{1000}{6} \rfloor = 166$$
$$|A_8| = \frac{1000}{8} = 125$$

The least common multiple (lcm) helps with the rest to add back:

$$|A_5 \cap A_6| = |A_{30}| = \lfloor \frac{1000}{30} \rfloor = 33$$
$$|A_6 \cap A_8| = |A_{24}| = \lfloor \frac{1000}{24} \rfloor = 41$$
$$|A_8 \cap A_5| = |A_{40}| = \frac{1000}{40} = 25$$
$$-|A_5 \cap A_6 \cap A_8| = -|A_{120}| = -\lfloor \frac{1000}{120} \rfloor = -8$$

Therefore, ANS = 600.

**Example 50.** How many anagrams of HAPPYMATH contain neither HAPPY nor MATH?

$$\begin{aligned} S &= \{\text{all anagrams}\} & |S| &= \frac{9!}{2!2!2!} & \\ A_1 &= \{\text{Anagrams with HAPPY}\} & |A_1| &= 5! & \boxed{\text{HAPPY}}\ \text{MATH} \\ A_2 &= \{\text{Anagrams with MATH}\} & |A_2| &= \frac{6!}{2!} & \text{HAPPY}\boxed{\text{MATH}} \\ A_1 \cap A_2 &= \{\text{Anagrams with MATH and HAPPY}\} & & & \end{aligned}$$

*Case* 1: $\boxed{\text{HAPPY}}\boxed{\text{MATH}}$ 2!

*Case* 2: $\boxed{\text{MATHAPPY}}$ H 2!

So $|A_1 \cap A_2| = 4$. Hence, ANS is

$$|S| - |A_1| - |A_2| + |A_1 \cap A_2| = 45360 - 120 - 360 + 4 = 44884$$

## G3 Combinations of a multiset

- How many ways to take 7 scrabble tiles from a bag?
- How many ways to get a bag of 12 bagels from limited supply of different bags?
- How many ways to place $r$ identical pigeons in $k$ pigeonholes without exceeding capacities $n_1, n_2, \dots, n_k$.

In general, let $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ be a multiset. How many $r$-combinations of $X$ are there? (i.e. objects of types $a_1, \dots, a_k$ of maximum quantities $n_1, n_2, \dots, n_k$.)

**Special cases:**

1. If $n_1 = \cdots = n_k = 1$: $\binom{k}{r}$
2. If $n_1 = n_2 = \cdots = n_k = \infty$: $\left(\binom{k}{r}\right) = \binom{r+k-1}{r}$
3. If $n_1, n_2, \dots, n_k \geq r$: $\left(\binom{k}{r}\right) = \binom{r+k-1}{r}$
4. If $r > n_1 + n_2 + \cdots + n_k$: $0$ ← by Strong Pigeonhole

**Example 51.** How many ways to select 10 jewels from a bag of 3 Amethyst, 4 Beryl and 5 Citrine?

**Method 1**: Easy answer by complement: choose which 2 to stay in the bag: AA, AB, AC, BB, BC, CC $\boxed{\text{6 ways}}$

**Method 2**: Integer solutions to $a + b + c = 10$ such that

$$0 \leq a \leq 3$$
$$0 \leq b \leq 4$$
$$0 \leq c \leq 5$$

We first pretend there are infinite supply of jewels and call the set of ways $S$. Let the sets $S_A, S_B, S_C$ be the ways with too many A ($a \geq 4$), B ($b \geq 5$) and C ($c \geq 6$) respectively. Then:

$$\begin{aligned}
|S| &= \left(\binom{3}{10}\right) = \binom{12}{10} = 66 \\
|S_A| &= \left(\binom{3}{6}\right) = \binom{8}{6} = 28 \\
|S_B| &= \left(\binom{3}{5}\right) = \binom{7}{5} = 21 \\
|S_C| &= \left(\binom{3}{4}\right) = \binom{6}{4} = 15 \\
|S_A \cap S_B| &= \left(\binom{3}{1}\right) = \binom{3}{1} = 3 \\
|S_A \cap S_C| &= \left(\binom{3}{0}\right) = 1 \\
|S_B \cap S_C| &= 0 \\
|S_A \cap S_B \cap S_C| &= 0
\end{aligned}$$

Hence, the answer would be, by IE, $66 - (28 + 21 + 15) + (3 + 1 + 0) - 0 = 6$.

## G4 Symmetric IE

**Remark.** The previous example is not symmetric since the three jewels have different constraints.

Now suppose $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_h}|$ depends only on the number of distinct indices chosen and not on which indices are chosen.

Then set

$$\begin{aligned}
\alpha_0 &= |S| \\
\alpha_1 &= |A_1| = \cdots = |A_n| \\
\alpha_2 &= |A_1 \cap A_2| = \cdots = |A_i \cap A_j| = \ldots \\
&\vdots \\
\alpha_n &= |A_1 \cap A_2 \cap \cdots \cap A_n|
\end{aligned}$$

Then the $2^n$ terms of the IE formula collapse to $n + 1$ terms:

$$\begin{aligned}
|A_1^c \cap \cdots \cap A_n^c| &= \alpha_0 - n\alpha_1 + \cdots + (-1)^i \binom{n}{i} \alpha_i + \cdots + (-1)^n \alpha_n \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \alpha_k
\end{aligned}$$

**Example 52.** How many integers $0, 1, \ldots, 99999$ contain each of $2, 5, 8$ in their digits?

We know $(5)_3 \cdot 7^2 < ANS < (5)_3 \cdot 10^2$. Using symmetric IE, let $A_i$ be the set of numbers without digit $i$. Then:

$$\begin{aligned}
\alpha_0 &= |S| = 10^5 \\
\alpha_1 &= |A_2| = |A_5| = |A_8| = 9^5 \\
\alpha_2 &= |A_2 \cap A_5| = \cdots = |A_5 \cap A_8| = 8^5 \\
\alpha_3 &= |A_2 \cap A_5 \cap A_8| = 7^5
\end{aligned}$$

Hence,

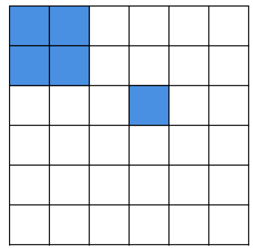
$$ANS = 10^5 - \binom{3}{1} 9^5 + \binom{3}{2} 8^5 - \binom{3}{3} 7^5 = 4350$$

## G5 Rook problems

**Example 53.** How many ways to place 6 identical rooks on $6 \times 6$ board such that

1. no two in the same row or column

2. no rook on blue squares:

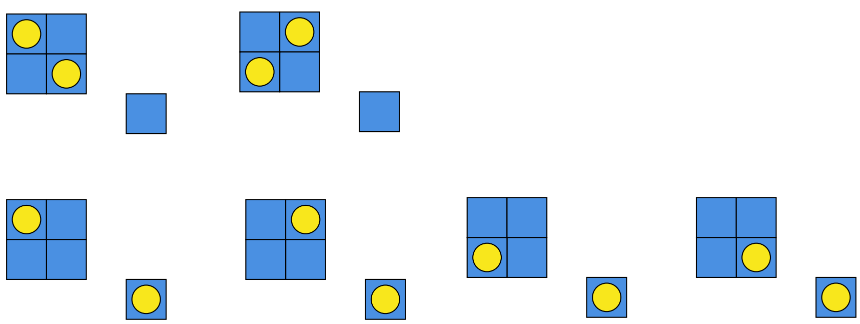


**Solution**: We'll solve the problem for distinct rooks labelled $1, 2, \dots, 6$ and then divide by $6!$. Let $S$ = all arrangements ignoring blue squares and $A_i$ = arrangements where rook $i$ is on a blue square.

We use symmetric IE:

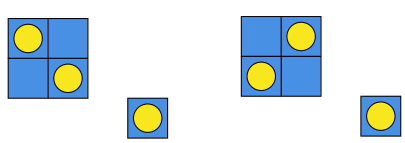
← choose rows and choose columns

$$\begin{aligned} \alpha_0 &= |S| = 6! \times 6! \\ \alpha_1 &= |A_1| = 5 \times (5!)^2 \\ \alpha_2 &= |A_1 \cap A_2| = 6 \cdot 2! \cdot (4!)^2 \end{aligned}$$



(And then label the two rooks)

$$\alpha_3 = |A_1 \cap A_2 \cap A_3| = 2 \cdot 3! \cdot (3!)^2$$



(And then label the 3 rooks)

$$\alpha_4 = \alpha_5 = \alpha_6 = 0$$

Hence, by symmetric IE, the answer for labelled rooks would be:

$$(6!)^2 - \binom{6}{1} \cdot 5 \cdot 1! \cdot (5!)^2 + \binom{6}{2} \cdot 6 \cdot 2! \cdot (4!)^2 - \binom{6}{3} \cdot 2 \cdot 3! \cdot (3!)^2$$

$$=6!6! - \frac{6!}{1!5!} \cdot 5 \cdot 1 \cdot 5!5! + \frac{6!}{2!4!} \cdot 6 \cdot 2! \cdot 4!4! - \frac{6!}{3!3!} \cdot 2 \cdot 3!3!3!$$
$$=6!(6! - 5 \cdot 5! + 6 \cdot 4! - 2 \cdot 3!)$$

Therefore, the **unlabelled** rooks have an answer of $6! - 5 \cdot 5! + 6 \cdot 4! - 2 \cdot 3! = 252$.

#### General rook formula

The number of ways to place $n$ rooks on a $n \times n$ board such that

1. no two in the same row or column
2. no rook on blue squares

is

$$n! - r_1(n-1)! + \cdots + (-1)^n r_n 0!$$

where $r_k$ is the number of ways to place $k$ identical rooks on blue squares satisfying rule 1.

# H Power series

## H1 Geometric series

Let

$$\begin{aligned} S &= 1 + x + x^2 + \cdots + x^n \\ xS &= x + x^2 + \cdots + x^n + x^{n+1} \end{aligned}$$

Hence, $1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$ whenever $x \neq 1$. If $x = 1$, then the sum is simply $n + 1$.

← Works in any division ring!

**Remark.** A variation:

Do notes from 3/25 and 3/27

**Theorem 32.** It is impossible to tile a $10 \times 10$ square using $1 \times 4$ tiles.

## H3 $9899^{-1}$

Mystery:

$$\frac{1}{9899} = 0.00\,01\,01\,02\,03\,05\ldots$$

Explanation: If we have $f_0 = 0, f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$, then the RHS looks like

$$\sum_{k=0}^{\infty} \frac{f_k}{100^{k+1}} = \frac{1}{100} \sum_{k=0}^{\infty} f_k \left(\frac{1}{100}\right)^k$$

Consider $F(x) = \sum_{k=0}^{\infty} f_k x^k$. The series converges for $|x| < \frac{1}{2}$ because we can easily bound it above through $f_k \le 2^k$ by induction. Hence, we observe:

← There exists a better bound, but this is sufficient.

$$\begin{array}{rlll} F & = 0 + x + & x^2 + & 2x^3 + \cdots + f_n x^n + \ldots \\ xF & = & x^2 + & x^3 + \cdots + f_{n-1} x^n + \ldots \\ x^2 F & = & & x^3 + \cdots + f_{n-2} x^n + \ldots \\ \hline F - xF - x^2 F & = 0 + x + & 0x^2 + & \ldots \end{array}$$

Therefore, we have a closed form for $f_n$:

$$\sum f_n x^n = \frac{x}{1 - x - x^2}$$

Therefore, the RHS could be written as:

$$\begin{aligned} 0.0001010203\ldots &= \frac{1}{100} \sum f_n \left(\frac{1}{100}\right)^n \\ &= \frac{1}{100} \cdot \frac{\frac{1}{100}}{1 - \frac{1}{100} - \frac{1}{100^2}} \\ &= \frac{1}{9899} \\ &= LHS \end{aligned}$$

$\square$

**More on the closed form of $f_n$**

Write $1 - x - x^2 = (x - \alpha)(x - \beta)$. Therefore, $\alpha, \beta = -\frac{1 \pm \sqrt{5}}{2}$. We then use partial fractions to get

$$\begin{aligned} \sum f_n x^n &= \frac{x}{1 - x - x^2} \\ &= \frac{-\alpha}{\alpha - \beta} \cdot \frac{1}{x - \alpha} + \frac{\beta}{\alpha - \beta} \cdot \frac{1}{x - \beta} \\ &= \frac{-1}{\alpha - \beta} \left( \frac{1}{\frac{x}{\alpha} - 1} - \frac{1}{\frac{x}{\beta} - 1} \right) \\ &= \frac{1}{\alpha - \beta} \left( \frac{1}{1 - \frac{x}{\alpha}} - \frac{1}{1 - \frac{x}{\beta}} \right) \qquad \text{by geometric series formula,} \end{aligned}$$

$$= \frac{1}{\alpha-\beta}\left(\sum_{i=0}^{\infty}\left(\frac{x}{\alpha}\right)^i - \sum_{i=0}^{\infty}\left(\frac{x}{\beta}\right)^i\right)$$

Hence, we get the constant term of the $n$-th derivative:

$$\begin{aligned}\frac{f^{(n)}(0)}{n!} &= \frac{1}{\alpha-\beta}\cdot\left(n!\frac{1}{\alpha^n} - n!\frac{1}{\beta^n}\right)\cdot\frac{1}{n!}\\ &= \frac{1}{\alpha-\beta}\cdot\left(\frac{1}{\alpha^n}-\frac{1}{\beta^n}\right)\\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\beta^n}-\frac{1}{\alpha^n}\right)\end{aligned}$$

Since $\alpha = \frac{-(1+\sqrt{5})}{2}, \beta = \frac{-(1-\sqrt{5})}{2}$, we observe that $\frac{1}{\alpha} = \frac{1-\sqrt{5}}{2}$ and $\frac{1}{\beta} = \frac{1+\sqrt{5}}{2}$. Thus, we substitute them back in:

$$\begin{aligned}\frac{f^{(n)}(0)}{n!} &= \frac{1}{\sqrt{5}}\left(\frac{1}{\beta^n}-\frac{1}{\alpha^n}\right)\\ &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\end{aligned}$$

which is Binet's formula.

**Theorem 33** (Binet's formula). $f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

**Corollary 34.** $f_n \to \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$ as $n \to \infty$.

Type the $\sigma$ example

## H4 Polynomial power series

We were able to obtain a formula for the Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

by converting the sequence into a function

$$F(x) = \sum_{n=0}^{\infty} f_n x^n$$

We showed that

$$F(x) = \frac{x}{1-x-x^2} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x}$$

and we used the geometric series

$$\frac{1}{1-\alpha x} = \sum_{n=0}^{\infty} (\alpha x)^n$$

to get

$$f_n = A\alpha^n + B\beta^n$$

---

In general, partial fractions will involve terms of the form

$$\frac{1}{1-\alpha x}, \frac{1}{(1-\alpha x)^2}, \frac{1}{(1-\alpha x)^3}, \cdots$$

**Theorem 35.**

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \left(\!\binom{k}{n}\!\right) x^n$$

valid when $|x| < 1$. It follows that: $\leftarrow \left(\!\binom{k}{n}\!\right) = \binom{k+n-1}{n}$

$$\frac{1}{(1-\alpha x)^k} = \sum_{n=0}^{\infty} \left(\!\binom{k}{n}\!\right) \alpha^n x^n$$

when $|x| < \left|\frac{1}{\alpha}\right|$.

*Proof.* We observe that when the geometric sequence converges absolutely:

$$\begin{aligned} LHS &= (1+x+x^2+\dots)^k \\ &= \underbrace{(1+x+x^2+\dots)\cdot(1+x+x^2+\dots)\cdot\dots\cdot(1+x+x^2+\dots)}_{k} \end{aligned}$$

To get the coefficient for $x^n$, we want to select terms from each bracket

$$x^{a_1}, x^{a_2}, \dots, x^{a_k}$$

such that $a_1 + a_2 + \dots + a_k = n$. This is equivalent to the problem of choosing a bag of $n$ bagels of $k$ different types. □

---

Last time, we found a formula

$$\sum_{n=0}^{\infty} n^2 x^n = \frac{x+x^2}{(1-x)^3}$$

using derivatives.

We now have an alternative approach:

$$\left(\!\binom{1}{n}\!\right) = \binom{n+0}{0} = 1$$
$$\left(\!\binom{2}{n}\!\right) = \binom{n+1}{1} = n+1$$
$$\left(\!\binom{3}{n}\!\right) = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$$

Write $n^2$ as a linear combination of them:

$$n^2 = 2 \cdot \frac{(n+2)(n+1)}{2} - 3(n+1) + 1 \cdot 1$$

And so

$$\begin{aligned}\sum_{n=0}^{\infty} n^2 x^n &= \sum_{n=0}^{\infty}\left(2 \cdot \frac{(n+2)(n+1)}{2} - 3(n+1) + 1 \cdot 1\right)x^n \\ &= \frac{2}{(1-x)^3} + \frac{-3}{(1-x)^2} + \frac{1}{1-x} \\ &= \frac{x^2+x}{(1-x)^3}\end{aligned}$$

**Remark.** We have two natural bases for the vector space of polynomials in $n$:

$$\begin{matrix} 1, & n, & n^2, & n^3, \dots \\ \left(\!\binom{1}{n}\!\right), & \left(\!\binom{2}{n}\!\right), & \left(\!\binom{3}{n}\!\right), & \left(\!\binom{4}{n}\!\right), \dots \end{matrix}$$

## H5 Linear recurrence relations

We can apply our analysis of the Fibonacci series $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2}$ to more general recursively-defined sequences.

### Homogeneous order-$k$ linear recurrence

$$g_n = c_1 g_{n-1} + c_2 g_{n-2} + \cdots + c_k g_{n-k}$$

with $n \geq k$ and initial conditions $g_0 = a_0, g_1 = a_1, \dots, g_{k-1} = a_{k-1}$.

**Example 54.** Order 2 with distinct roots:

$$\begin{cases} g_n = 2g_{n-1} + 3g_{n-2} \\ g_0 = 3, g_1 = 5 \end{cases}$$

And so $(g_n) = (3, 5, 19, 53, 163, \dots)$.

Let $G(x) = \sum_{n=0}^{\infty} g_n x^n$.

$$\begin{array}{rlllll}
G & = 3+ & 5x+ & 19x^2 + \dots + g_n x^n + \dots \\
xG & = & 3x+ & 5x^2 + \dots + g_{n-1} x^n + \dots \\
x^2 G & = & & 3x^2 + \dots + g_{n-2} x^n + \dots \\
\hline
G - 2xG - 3x^2 G & = 3- & x+ & 0x^2 + \dots
\end{array}$$

And hence $(1 - 2x - 3x^2)G = 3 - x$. Therefore,

$$\begin{aligned}
G = \frac{3 - x}{1 - 2x - 3x^2} &= \frac{3 - x}{(1 - 3x)(1 + x)} \\
&= \frac{2}{1 - 3x} + \frac{1}{1 + x} \\
&= \sum_{n=0}^{\infty} (2 \cdot 3^n + 1 \cdot (-1)^n) x^n
\end{aligned}$$

And so $g_n = 2 \cdot 3^n + (-1)^n$.

**Example 55.** Order 2 with repeated roots:

$$\begin{cases} g_n = 4g_{n-1} + 4g_{n-2} \\ g_0 = 1, g_1 = 6 \end{cases}$$

Thus, the sequence is $(g_n) = (1, 6, 20, 56, 144 \dots)$.

Let $G(x) = \sum_{n=0}^{\infty} g_n x^n$.

$$\begin{array}{rlllll}
G & = 1+ & 6x+ & 20x^2 + \dots + g_n x^n + \dots \\
xG & = & x+ & 6x^2 + \dots + g_{n-1} x^n + \dots \\
x^2 G & = & & x^2 + \dots + g_{n-2} x^n + \dots \\
\hline
G - 4xG - 4x^2 G & = 1+ & 2x+ & 0x^2 + \dots
\end{array}$$

And hence $(1 - 4x - 4x^2)G = 1 + 2x$. Therefore,

$$\begin{aligned}
G = \frac{1 + 2x}{1 - 4x - 4x^2} &= \frac{1 + 2x}{(1 - 2x)^2} \\
&= \frac{2}{(1 - 2x)^2} + \frac{-1}{1 - 2x} \\
&= \sum_{n=0}^{\infty} (2 \cdot \left(\!\binom{2}{n}\!\right) 2^n + (-1) 2^n) x^n
\end{aligned}$$

We observe $\left(\!\binom{2}{n}\!\right) = n + 1$, so $g_n = 2^n(2n + 1)$.

### Inhomogeneous recurrence

Inhomogeneous: there are some other known sequences added.

**Example 56.** Inhomogeneous, order 1:

$$\begin{cases} g_n = -g_{n-1} + 2^n \\ g_0 = 1 \end{cases}$$

Then $(g_n) = (1, 1, 3, 5, 11, \dots)$.

Let $G = \sum_{n=0}^{\infty} g_n x^n$ and observe that $H = \sum_{n=0}^{\infty} 2^n x^n = \frac{1}{1-2x}$.

$$\begin{array}{rlll} G & = 1+ & x+ & 3x^2 + \dots + g_n x^n + \dots \\ xG & = & x+ & x^2 + \dots + g_{n-1} x^n + \dots \\ H & = 1+ & 2x & 4x^2 + \dots + g_{n-2} x^n + \dots \\ \hline G + xG - H & = 1+ & 2x+ & 4x^2 + 0 + \dots \end{array}$$

Thus, $G = \frac{1}{(1+x)(1-2x)} = \frac{2/3}{1-2x} + \frac{1/3}{1+x} = \sum_{n=0}^{\infty}(\frac{2}{3} \cdot 2^n + \frac{1}{3}(-1)^n)$. Hence, $g_n = \frac{2^{n+1} + (-1)^n}{3}$.

## H6 Nickels and dimes

**Example 57.** How many ways are there to make \$1.03 out of nickels (5¢) and pennies (1¢)?

Answer: that would be the number of solutions to $5a + b = 103$ with $a, b \in \mathbb{N}$. We can have:

$$(a, b) = (0, 103), (1, 98), \dots, (20, 3)$$

More generally, the number of solutions to $5a + b = n$ is $\lfloor \frac{n}{5} \rfloor + 1$ since the number of nickels have to be $a = 0, 1, \dots, \lfloor \frac{n}{5} \rfloor$.

**Example 58.** How many ways are there to make \$1.03 out of dimes (10¢), nickels (5¢) and pennies (1¢)?

We use the previous solution:

- 0 dimes: $\lfloor \frac{103}{5} \rfloor + 1 = 21$
- 1 dimes: $\lfloor \frac{93}{5} \rfloor + 1 = 19$
- …
- 10 dimes: $\lfloor \frac{3}{5} \rfloor + 1 = 1$

Then we sum them up. Since they are consecutive odd numbers,

$$21 + 19 + 17 + \cdots + 1 = 11^2 = 121$$

**Another method**

Revisit Example 57.

Consider the rational function

$$\frac{1}{1-x} \cdot \frac{1}{1-x^5} = (1 + x + x^2 + x^3 + \dots)(1 + x^5 + x^{10} + x^{15} + \dots)$$

we want to know what the coefficient for $x^{103}$ is. Note this is equivalent to the previous example since we are choosing $x^p$ terms from the first part and $x^{5q}$ terms from the second part, and we are looking for ways to get $5q + p = 103$.

To get the coefficient of $x^{103}$, consider

$$1 + x + x^2 + x^3 + x^4 = \frac{1-x^5}{1-x}$$

And hence $\frac{1}{1-x} = \frac{1+x+x^2+x^3+x^4}{1-x^5}$. Furthermore, we see that

$$\frac{1}{(1-x^5)^2} = \sum_{n\geq 0} \left(\!\binom{2}{n}\!\right) (x^5)^n$$

We observe that the LHS becomes:

$$\begin{aligned} \frac{1}{1-x} \cdot \frac{1}{1-x^5} &= \frac{1 + x + x^2 + x^3 + x^4}{1-x^5} \cdot \frac{1}{1-x^5} \\ &= (1 + x + x^2 + x^3 + x^4) \cdot \sum_{n\geq 0} \left(\!\binom{2}{n}\!\right) (x^5)^n \end{aligned}$$

To get $x^{103}$, we want $x^p \cdot x^{5q} = x^{103}$ such that $0 \leq p \leq 4$ and $q \geq 0$. The only solution is $(p, q) = (3, 20)$. The coefficient is therefore

$$1 \cdot \left(\!\binom{2}{20}\!\right) = 21$$

Now revisit Example 58 Similarly, consider

$$\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} = (1+x+x^2+x^3+\dots)(1+x^5+x^{10}+x^{15}+\dots)(1+x^{10}+x^{20}+\dots)$$

We now want to get the coefficient of $x^{103}$, which are from choosing terms $x^a, x^{5b}$, and $x^{10c}$ from the parts above such that $a + 5b + 10c = 103$.

Also, we prepare that

$$\frac{1}{1-x} = \frac{1 + x + x^2 + x^3 + \cdots + x^9}{1 - x^{10}}$$

and

$$\frac{1}{1-x^5} = \frac{1 + x^5}{1 - x^{10}}$$

We observe that the LHS becomes:

$$\begin{aligned}
&\frac{1}{1-x} \cdot \frac{1}{1-x^5} \cdot \frac{1}{1-x^{10}} \\
=&\frac{1 + x + x^2 + x^3 + \cdots + x^9}{1-x^{10}} \cdot \frac{1+x^5}{1-x^{10}} + \frac{1}{1-x^{10}} \\
=&(1 + x + \cdots + x^4 + 2(x^5 + \cdots + x^9) + x^{10} + \cdots + x^{14}) \cdot \sum_{n \geq 0} \left(\!\binom{3}{n}\!\right) x^{10}
\end{aligned}$$

We pick an $x^p$ term from the first part and a $x^{10q}$ term from the 2nd part such that $p + 10q = 103$, $0 \leq p \leq 14$ and $q \geq 0$.

There are two solutions: $(p, q) = (3, 10)$ or $(13, 9)$. The coefficient of $x^{103}$ is

$$1 \cdot \left(\!\binom{3}{10}\!\right) + 1 \cdot \left(\!\binom{3}{9}\!\right) = 66 + 55 = 121$$

## H7 Characteristic ops and egf

Example 57 and Example 58 are equivalent to asking *How many ways to get a bag of 103 bagels*:

1. of 2 types, plain + garlic such that we can get any number of plain but garlic must be a multiple of 5?
2. of 3 types, plain + garlic + onion, where can get any number of plain, but garlic must be a multiple of 5 and onion has to be a multiple of 10?

These are combinations with constraints. That is, we want to put 103 "balls" in buckets

$$U_p, U_g, U_o$$

with capacities

$$U_p \in \{0, 1, 2, \dots\} = N_p$$

$$U_g \in \{0, 5, 10, \dots\} = N_g$$
$$U_o \in \{0, 10, 20, \dots\} = N_0$$

Effectively, we convert each set to a power series. There are two ways to convert a subset $N \subseteq \mathbb{N}$ into a power series.

- Step 1: Convert $N$ into a 0-1 sequence:
$$a = (a_n) \text{ where } a_n = \begin{cases} 1 & \text{if } n \in N \\ 0 & \text{otherwise} \end{cases}$$
- Step 2:
  - Characteristic OPS (ordinary power series) $X_N$ of $N$:
$$(a_n) \mapsto \sum_{n \geq 0} a_n x^n$$
  - Characteristic EGF (exponential generating function) $\mathfrak{X}_n$ of $N$:
$$(a_n) \mapsto \sum_{n \geq 0} \frac{a_n}{n!} x^n$$

**Example 59.** $N = \{0, 1, 2, \dots\}$ (no constraint)

$$\begin{aligned} a^N &= 1, 1, 1, \dots \\ X_N &= 1 + x + x^2 + \cdots = \frac{1}{1-x} \\ \mathfrak{X}_N &= 1 + x + \frac{x^2}{2!} + \cdots = e^x \end{aligned}$$

**Example 60.** $N = \{1, 2, 3, \dots\}$ (at least 1 object)

$$\begin{aligned} a^N &= 0, 1, 1, \dots \\ X_N &= x + x^2 + \cdots = \frac{x}{1-x} \\ \mathfrak{X}_N &= x + \frac{x^2}{2!} + \cdots = e^x - 1 \end{aligned}$$

**Example 61.** $N = \{0, 1\}$ (at most 1 object)

$$\begin{aligned} a^N &= 1, 1, 0, 0, \dots \\ X_N &= 1 + x \\ \mathfrak{X}_N &= 1 + x \end{aligned}$$

**Example 62.** $N = \{0, 2, 4, 6, \dots\}$ (even numbers of object)

$$\begin{aligned} a^N &= 1, 0, 1, 0, \dots \\ X_N &= 1 + x^2 + x^4 + \cdots = \frac{x}{1-x^2} \\ \mathfrak{X}_N &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh x = \frac{e^x + e^{-x}}{2} \end{aligned}$$

**Example 63.** $N = \{1, 3, 5, 7, \dots\}$ (odd numbers of object)

$$\begin{aligned} a^N &= 0, 1, 0, 1, \dots \\ X_N &= x + x^3 + x^5 + \cdots = \frac{x}{1-x^2} \\ \mathfrak{X}_N &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sinh x = \frac{e^x - e^{-x}}{2} \end{aligned}$$

**Example 64.** $N = \{0, 5, 10, 15, \dots\}$ (multiples of 5 numbers of object)

$$\begin{aligned} a^N &= 1, 0, 0, 0, 0, 1, 0, \dots \\ X_N &= 1 + x^5 + x^10 + \cdots = \frac{1}{1-x^5} \\ \mathfrak{X}_N &= 1 + \frac{x^5}{5!} + \frac{x^{10}}{10!} + \cdots = \frac{1}{5}\sum_{k=0}^{4} e^{\omega^k x} \end{aligned}$$

where $\omega = e^{2\pi i/5}$ is a primitive 5th root of unity.

# I Generating functions

## I1 OPS and EGF

Suppose a combiatorial problem has an answer $a_n$ for each $n \geq 0$. We will encode an entire sequence $(a_n)$ as a function and try to find the function.

**Definition 18** (Ordinary power series)**.**

$$g(x) = \sum_n a_n x^n$$

**Definition 19** (Exponential generating function)**.**

$$G(x) = \sum_n \frac{a_n}{n!} x^n$$

**Remark.** Special case mentioned here: if $(a_n)$ is a sequence of 0s and 1s where $a_n = \begin{cases} 1 & n \in N \\ 0 & n \notin N \end{cases}$ for some set $N \subseteq \mathbb{N}$, then $g = X_n$ and $G = \mathfrak{X}_n$.

**Example 65.** Sequence: $1, 1, 1, 1, \dots$

- OPS: $1 + x + x^2 + \cdots = \frac{1}{1-x}$ $\leftarrow |x| < 1$
- EGF: $1 + x + \frac{x^2}{2!} + \cdots = e^x$

**Example 66.** Sequence: $1, \alpha, \alpha^2, \dots$

- OPS: $1 + \alpha x + \alpha x^2 + \cdots = \frac{1}{1-\alpha x}$ ← $|x| < 1/|\alpha|$
- EGF: $1 + \alpha x + \frac{\alpha^2 x^2}{2!} + \cdots = e^{\alpha}$

**Example 67.** Sequence: $a_n = \binom{m}{n}$

$$\binom{m}{0}, \binom{m}{1}, \dots, \binom{m}{m}, 0, 0, 0, \dots$$

- OPS: $\binom{m}{0} + \binom{m}{1}x + \cdots + \binom{m}{m}x^m = (1+x)^m$
- EGF: $\binom{m}{0} + \binom{m}{1}x + \binom{m}{2}\frac{x^2}{2!} + \cdots + \binom{m}{m}\frac{x^m}{m!} = ??$ ← We don't know how to solve this yet!

**Example 68.** Sequence: $a_n = (m)_n$

$$(m)_0, (m)_1, \dots, (m)_m, 0, 0, 0, \dots$$

- OPS: $(m)_0 + (m)_1 x + \cdots + (m)_m x^m = ??$ ← We don't know how to solve this yet!
- EGF: $(m)_0 + (m)_1 x + \frac{(m)_2}{2!}x^2 + \cdots + \frac{(m)_m}{m!}x^m = (1+x)^m$ ← Since $\frac{(m)_k}{k!} = \binom{m}{k}$

**Remark.** Two skills needed:

1. Recognizing OPS/EGF as a function
2. Given OPS/EGF, finding sequence $(a_n)$.

**Definition 20.** Notation: Write $[x^n : g] = x^n$ coefficient of the Taylor series of function $g$.

**Remark.** If $(a_n) \xrightarrow{ops} g(x)$, then $a_n = [x^n : g]$. Similarly, if $(a_n) \xrightarrow{egf} G(x)$, then $a_n = n![x^n : G]$.

## I2 The multiplication rule

**Proposition 36.** Suppose

$$(a_n) \xrightarrow{ops} f$$
$$(b_n) \xrightarrow{ops} g$$

Let $f \circ g \xleftarrow{ops} (c_n)$. Then

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

*Proof.* Make a multiplication table. $\square$

**Proposition 37.** Suppose

$$(a_n) \xrightarrow{egf} F$$
$$(b_n) \xrightarrow{egf} G$$

Let $F \circ G \xleftarrow{egf} (c_n)$. Then

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

*Proof.* We observe

$$(a_n) \xrightarrow{egf} F \xleftarrow{ops} \left(\frac{a_n}{n!}\right)$$
$$(b_n) \xrightarrow{egf} G \xleftarrow{ops} \left(\frac{b_n}{n!}\right)$$
$$(c_n) \xrightarrow{egf} F \circ G \xleftarrow{ops} \left(\frac{c_n}{n!}\right)$$

By Proposition 36, we get

$$\frac{c_n}{n!} = \sum_{k=0}^{n} \left(\frac{a_k}{k!}\right)\left(\frac{b_{n-k}}{(n-k)!}\right)$$

Therefore, $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$. $\square$

**Proposition 38** (Generalization)**.** Sequences

$$\begin{array}{c} (a_n^1), (a_n^2), \dots, (a_n^t) \\ \Big\downarrow {\scriptstyle ops} \\ f_1, f_2, \dots, f_t \end{array}$$

Then $f_1 \circ f_2 \circ \dots \circ f_t =: f \xleftarrow{ops} (c_n)$ where

$$c_n = \sum_{\substack{k_1, k_2, \dots, k_t \geq 0 \\ k_1 + \dots + k_t = n}} a_n^1 a_n^2 \dots a_n^t$$

Similarly, if

$$\begin{array}{c}(a_n^1), (a_n^2), \dots, (a_n^t) \\ \Big\downarrow egf \\ F_1, F_2, \dots, F_t\end{array}$$

Then $F_1 \circ F_2 \circ \dots \circ F_t =: F \xleftarrow{egf} (d_n)$ where

$$d_n = \sum_{\substack{k_1,k_2,\dots,k_t \geq 0 \\ k_1+\cdots+k_t=n}} \binom{n}{k_1, k_2, \dots, k_t} a_n^1 a_n^2 \dots a_n^t$$

**Example 69** (Use of OPS)**.** Let $a_n = \binom{p}{n}, b_n = \binom{q}{n}$. Hence

$$(a_n) \xrightarrow{ops} (1+x)^p = f(x)$$

$$(b_n) \xrightarrow{ops} (1+x)^q = g(x)$$

then

$$f(x) \circ g(x) = (1+x)^p (1+x)^q = (1+x)^{p+q} \xleftarrow{ops} (c_n)$$

where $c_n = \binom{p+q}{n}$. The multiplication rule also gives

$$c_n = \sum_{k=0}^{n} a_n b_n$$

which means

$$\binom{p+q}{n} = \sum_{k=0}^{n} a_n b_n$$

← can also prove by combinatorics: think of Hogwarts!

Similarly, we can use the general version to get

$$\binom{p_1 + \cdots + p_t}{n} = \sum_{\substack{k_i \geq 0 \\ k_1+\cdots+k_t=n}} \binom{p_1}{k_1}\binom{p_2}{k_2} \cdots \binom{p_t}{k_t}$$

**Example 70** (Use of EGF)**.** Say

$$(\alpha^n) \xrightarrow{egf} e^{\alpha x}$$

$$(\beta^n) \xrightarrow{egf} e^{\beta x}$$

Then

$$e^{(\alpha+\beta)x} \xleftarrow{egf} (\alpha+\beta)^n$$

The multiplication rule gives us

$$(\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \alpha^n \beta^n$$

We just reproved the binomial theorem!

# I3 Combinations with constraints

We now tackle $\boxed{16}$ in the table.

3.3 A Framework for Counting Questions: The Counting Table 95

**Table 3.1 Balls and boxes counting problems.**

| Number of Ways to Put Balls into Boxes<br>$S = S_1 = \{1, 2, \ldots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls<br>$T = T_1 = \{U_1, U_2, \ldots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes<br>Box $U_i$ contains $u_i$ balls | | | | |
|---|---|---|---|---|
| Conditions on $S$ and $T$ $\rightarrow$ on $u_i$ $\downarrow$ | $T = T_1$ distinct<br>$S = S_1$ distinct | $T = T_1$ distinct<br>$S = S_2$ identical | $T = T_2$ identical<br>$S = S_1$ distinct | $T = T_2$ identical<br>$S = S_2$ identical |
| $0 \le u_i \le 1$<br>Assume $t \ge s$ | $\boxed{1}$ | $\boxed{2}$ | $\boxed{3}$ | $\boxed{4}$ |
| $u_i \ge 0$ | $\boxed{5}$ | $\boxed{6}$ | $\boxed{7}$ | $\boxed{8}$ |
| $u_i \ge 1$ | $\boxed{9}$ | $\boxed{10}$ | $\boxed{11}$ | $\boxed{12}$ |
| for $i = 1, \ldots, t$,<br>$0 \le u_i \le n_i$,<br>$n_i \in \mathbb{Z}^{>0}$ | $\boxed{13}$ | $\boxed{14}$ | | |
| $u_i \in N_i \subset \mathbb{Z}^{\ge 0}$<br>for $i = 1, \ldots, t$ | $\boxed{15}$ | $\boxed{16}$ | | |

**Theorem 39.** Let $N_1, N_2, \ldots, N_t$ be subsets of $\mathbb{N}$. Let $c_s$ be the # of solutions to

$$x_1 + x_2 + \cdots + x_t = s$$

such that $x_i \in N_i$ for all $i$.

This is equivalent to # of ways to place $s$ identical objects in $t$ distinct boxes s.t. # of objects $u_i$ in box $i$ is in $N_i$ for all $i$.

Also equivalent to # ways to select bag of $s$ bagels from $t$ types, s.t. # of bagels of type $i \in N_i$ for all $i$.

Let $(c_n) \xrightarrow{ops} f$. Then

$$f = \underbrace{X_{N_1} \cdot X_{N_2} \dots X_{N_t}}_{\text{prod. of characteristic ops}}$$

*Proof omitted.* Similar reasoning to Example 58. □

**Example 71.** How many ways to pick 10 marbles from $\{3R, 4G, 5B\}$?

Answer:

$$\begin{aligned}
N_R &= \{0,1,2,3\} && \xrightarrow{ops} \frac{1-x^4}{1-x} = X_R \\
N_G &= \{0,1,2,3,4\} && \xrightarrow{ops} \frac{1-x^5}{1-x} = X_G \\
N_B &= \{0,1,2,3,4,5\} && \xrightarrow{ops} \frac{1-x^6}{1-x} = X_B
\end{aligned}$$

We want to know $c_{10}$. Theorem above tells us $(c_n) \xleftarrow{ops} X_R X_G X_B$ and so we want the $x^1 0$ coeff. of

$$\begin{aligned}
&\frac{1-x^4}{1-x} \cdot \frac{1-x^5}{1-x} \cdot \frac{1-x^6}{1-x} \\
=&(1 - x^4 - x^5 - x^6 + x^9 + x^{10} + x^{11} - x^{15}) \cdot \frac{1}{(1-x)^3} \\
=&(1 - x^4 - x^5 - x^6 + x^9 + x^{10} + x^{11} - x^{15}) \cdot \sum_{n \geq 0} \left(\!\binom{3}{n}\!\right) x^n
\end{aligned}$$

Hence we select $x^a$ and select $x^b$ from the two product factors. We table the possibilities:

| | |
|---|---|
| $(a,b) = (0,10)$ | coefficient: $1 \cdot \left(\!\binom{3}{10}\!\right) = 66$ |
| $(4,6)$ | $-1 \cdot \left(\!\binom{3}{6}\!\right) = -28$ |
| $(5,5)$ | $-1 \cdot \left(\!\binom{3}{5}\!\right) = -21$ |
| $(6,4)$ | $-1 \cdot \left(\!\binom{3}{4}\!\right) = -15$ |
| $(9,1)$ | $1 \cdot \left(\!\binom{3}{1}\!\right) = 3$ |
| $(10,0)$ | $1 \cdot \left(\!\binom{3}{0}\!\right) = 1$ |
| | total $= 6$ |

**Example 72.** How many ways to select a bag of 47 fruits of 4 types, namely:

- # apples is even
- # bananas is a multiple of 5
- # cherries is $\leq 4$
- # dragon fruit is $\leq 1$

**Combinatorial way:** we let $k = \#$ of bananas and cherries.

For each $k = 0, 1, \ldots, 47$:

- Choose $k$ bananas + cherries

$$1 \text{ way} \begin{cases} \#B = \lfloor \frac{k}{5} \rfloor \cdot 5 & \text{largest mult. of 5} \\ \#C = k - \lfloor \frac{k}{5} \rfloor \cdot 5 & \text{remainder} \end{cases}$$

- Choose $47 - k$ apples and dragon fruit

$$1 \text{ way} \begin{cases} \#A = \lfloor \frac{47-k}{2} \rfloor \cdot 2 & \text{largest mult. of 2} \\ \#D = (47 - k) - \lfloor \frac{47-k}{2} \rfloor \cdot 2 & \text{remainder} \end{cases}$$

Hence the only choices made are through choosing $k$, which gives us 48 ways.

**Generating function:** Consider this as a combinations with constraints problem. Let $c_n$ be the number of ways to choose $n$ fruit with the aforementioned conditions. Then

$$f(x) = \sum_n c_n x^n$$

satisfies

$$f = X_A X_B X_c X_D$$

Now since

$$\begin{aligned} N_A &= \{0, 2, 4, \ldots\} \\ N_B &= \{0, 5, 10, \ldots\} \\ N_C &= \{0, 1, 2, 3, 4\} \\ N_D &= \{0, 1\} \end{aligned}$$

We could write their OPS:

$$\begin{aligned} X_A &= 1 + x^2 + x^4 + \ldots = \frac{1}{1 - x^2} \\ X_B &= 1 + x^5 + x^{10} + \ldots = \frac{1}{1 - x^5} \\ X_c &= 1 + x + x^2 + x^3 + x^4 = \frac{1 - x^5}{1 - x} \end{aligned}$$

$$X_D = 1 + x$$

Thus,

$$\begin{aligned} f(x) &= \frac{1}{1-x^2} \cdot \frac{1}{1-x^5} \cdot \frac{1-x^5}{1-x} \cdot (1+x) \\ &= \frac{1+x}{(1+x)(1-x)(1-x)} \\ &= \frac{1}{(1-x)^2} \\ &= \sum_{n=0} \left(\!\binom{2}{n}\!\right) x^n \\ &= \sum_{n\geq 0} (n+1)\, x^n \\ \Rightarrow c_n &= n+1 \end{aligned}$$

Therefore, $c_{47} = 47 + 1 = 48$.

## I4 Permutations with constraints

**Example 73.** We have R, G, B marbles. We want to place 7 of them in a row such that

- #R is odd
- #G is a multiple of 3
- #B is anything

**Combinatorial way:** split into cases by first selecting combinations of (R, G, B):

$$(1,0,6), (1,3,3), (1,6,0), (3,0,4), (3,3,1), (5,0,2), (7,0,0)$$

And then we solve each anagram problem:

$$\binom{7}{1,0,6}, \binom{7}{1,3,3}, \binom{7}{1,6,0}, \binom{7}{3,0,4}, \binom{7}{3,3,1}, \binom{7}{5,0,2}, \binom{7}{7,0,0}$$

Sum them up and we get 351 ways.

**Generating function:** Consider

$$\begin{aligned} \mathfrak{X}_R - &= x + \frac{x^3}{3!} + \cdots = \sinh x = \frac{e^x - e^{-x}}{2} \\ \mathfrak{X}_G &= 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \cdots = \frac{e^x + e^{\zeta_3 x} + e^{\zeta_3^2 x}}{3} \end{aligned}$$

$$\mathfrak{X}_B = 1 + x + \frac{x^2}{2!} + \cdots = e^x$$

Let $F = \mathfrak{X}_R \mathfrak{X}_G \mathfrak{X}_B$. What is $[x^7 : F]$?

← the coeff. of F on $x^7$ term

We need to pick powers $x^r$ in $\mathfrak{X}_R$, $x^g$ in $\mathfrak{X}_G$, $x^b$ in $\mathfrak{X}_B$ such that $r + g + b = 7$. The solutions $(r, g, b)$ are the same ones as before:

$$(1, 0, 6), (1, 3, 3), (1, 6, 0), (3, 0, 4), (3, 3, 1), (5, 0, 2), (7, 0, 0)$$

We want the coefficients for each such case. For instance,

$$(r, g, b) = (1, 3, 3) \qquad \rightarrow \qquad x\frac{x^3}{3!}\frac{x^3}{3!} \qquad \text{coefficient is } \frac{1}{1!3!3!} = \frac{1}{7!}\binom{7}{1, 3, 3}$$

and similar for other cases.

Thus, the answer is $7![x^7 : F]$.

In general, if $p_n$ is the number of ways to line up $n$ marbles with the aforementioned constraints, then $p_n = n![x^n : \mathfrak{X}_R \mathfrak{X}_G \mathfrak{X}_B]$. That is,

$$(p_n) \xrightarrow{egf} \mathfrak{X}_R \mathfrak{X}_G \mathfrak{X}_B$$

General formula for $F = \mathfrak{X}_R \mathfrak{X}_G \mathfrak{X}_B$:

← (Optional content)

$$\begin{aligned} F &= \frac{1}{6}\left(e^x - e^{-x}\right)\left(e^x + e^{\zeta_3 x} + e^{\zeta_3^2 x}\right)e^x \\ &= \frac{1}{6}\left(e^{3x} - e^x + e^{(2+\zeta_3)x} + e^{\zeta_3 x} + e^{(2+\zeta_3^2)x} - e^{\zeta_3^2 x}\right) \end{aligned}$$

Thus

$$ANS = \frac{1}{6}\left(3^n - (-1)^n + (z + \zeta_3)^n - \zeta_3^n + \left(z + \zeta_3^2\right)^n - \zeta_3^{2n}\right)$$

**Theorem 40.** Let constraints $N_1, N_2, \dots, N_t \subseteq \mathbb{N}$. Let $p_s = \#\, s$-permutations of $\{\infty \cdot 1, \infty \cdot 2, \dots, \infty \cdot t\}$ such that # of type $k$ is in the set $N_k$. Let $(p_G) \xrightarrow{egf} F$. Then

$$F = \mathfrak{X}_{N_1} \mathfrak{X}_{N_2} \dots \mathfrak{X}_{N_t}$$

where $\mathfrak{X}_{N_i}$ is the characteristic EGF of $N_i$.

**Example 74.** How many 47-digit numbers have at least 2 digits equal to 1?

← allowing leading 0s

This is a question about permutations with constraints: $\mathfrak{X}_0 = \mathfrak{X}_2 = \cdots = \mathfrak{X}_9 = e^x$, and $\mathfrak{X}_1 = \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x - 1 - x$. Let $a_n$ be the number of the $n$-digit numbers with the aforementioned constraints. Then

$$(a_n) \xrightarrow{egf} \mathfrak{X}_0 \mathfrak{X}_1 \dots \mathfrak{X}_9 = (e^x)^9 (e^x - 1 - x)$$

meaning $F = e^{10x} - e^{9x} - xe^{9x}$. We want to find $47![x^{47} : F]$. We observe that since

$$
\begin{aligned}
e^{10x} &\xleftarrow{egf} (10^n) \\
e^{9x} &\xleftarrow{egf} (9^n) \\
xe^{9x} &\xleftarrow{egf} (n9^{n-1})
\end{aligned}
$$

Therefore, $F \xleftarrow{egf} (10^n - 9^n - n9^{n-1})$. Hence

$$
\begin{aligned}
xe^{9x} &= \sum_{n=0}^{\infty} x \frac{9^n x^n}{n!} \\
&= \sum_{m=1}^{\infty} 9^{m-1} \cdot m \cdot \frac{x^m}{m!} && m = n+1
\end{aligned}
$$

When $n = 47$, we have $10^{47} - 9^{47} - 47 \cdot 9^{46}$ ways.

**Lemma 41.**

$$x^k e^{\alpha x} \xleftarrow{egf} \left((n)_k \alpha^{n0k}\right)$$

*Proof.* For $n < k$, coefficient of LHS is 0 and RHS is also 0.

For $n \geq k$, coefficient of LHS is

$$x^k \frac{(\alpha x)^{n-k}}{(n-k)!} = \frac{\alpha^{n-k} x^n}{(n-k)!} = (n)_k \alpha^{n-k} \frac{x^n}{n!}$$

□

**Example 75.** How many $n$-digit numbers have an odd number of digits of each prime 2,3,5,7? We have

$$
\begin{aligned}
\mathfrak{X}_2 = \mathfrak{X}_3 = \mathfrak{X}_5 = \mathfrak{X}_7 &= \sinh x = \frac{e^x - e^{-x}}{2} \\
\mathfrak{X}_0 = \mathfrak{X}_1 = \mathfrak{X}_4 = \cdots = \mathfrak{X}_9 &= e^x
\end{aligned}
$$

$$
\begin{aligned}
F &= \left(\frac{e^x - e^{-x}}{2}\right)^4 e^{6x} \\
&= \frac{1}{16}\left(e^{10x} - 4e^{8x} + 6e^{6x} - 4e^{4x} + e^{2x}\right)
\end{aligned}
$$

Therefore, the answer is

$$\frac{1}{16}\left(10^n - 4 \cdot 8^n + 6 \cdot 6^n - 4 \cdot 4^n + 2^n\right)$$

# I5 Return of the counting table

Now we re-calculate these entries using generating functions:

| | | | |
|---|---|---|---|
| $u_i \le 1$ | 1 $(t)_s$ | 2 $\binom{t}{s}$ | $N = \{0, 1\}$ |
| $u_i \ge 0$ | 5 $t^s$ | 6 $\left(\binom{t}{s}\right)$ | $N = \{0, 1, 2, \dots\}$ |
| $u_i \ge 1$ | 9 $t!\left\{{s \atop t}\right\}$ | 10 $\left(\binom{t}{s-t}\right) = \binom{s-1}{t-1}$ | $N = \{1, 2, 3, \dots\}$ |

We use OPS for **combinations**.

2 We observe

$$N = \{0, 1\} \xrightarrow[ops]{char} 1 + x$$

since there are $t$ distinct boxes, we need to do this for each box, so we have $(1 + x)^t$. Let

$$F \xleftarrow{ops} (1 + x)^t = \sum_{s=0}^{t} \binom{t}{s} x^s$$

and we have $[x^s : F] = \binom{t}{s}$.

6 We observe

$$N = \{0, 1, 2, \dots\} \xrightarrow[ops]{char} 1 + x + x^2 + \dots \overset{geom}{=} \frac{1}{1 - x}$$

we do that for all $t$ distinct boxes, so we have $\left(\frac{1}{1-x}\right)^t$. Let

$$F \xleftarrow{ops} \left(\frac{1}{1 - x}\right)^t = \sum_{s=0}^{\infty} \left(\binom{t}{s}\right) x^s$$

and we have $[x^s : F] = \left(\binom{t}{s}\right)$.

10 We observe

$$N = \{1, 2, \dots\} \xrightarrow[ops]{char} x + x^2 + \dots \overset{geom}{=} \frac{x}{1 - x}$$

we do that for all $t$ distinct boxes, so we have $\left(\frac{x}{1-x}\right)^t$. Let

$$F \xleftarrow{ops} \left(\frac{x}{1 - x}\right)^t = \sum_{s=0}^{\infty} x^t \left(\binom{t}{s}\right) x^s = \sum_{r=t}^{\infty} \left(\binom{t}{r - t}\right) x^r$$

and we have $[x^s : F] = \left(\binom{t}{s-t}\right) = \binom{s-1}{t-1}$.

Now we have to use EGF for the **permutations**.

1 We observe

$$N = \{0, 1\} \xrightarrow[egf]{char} 1 + x$$

since there are $t$ distinct boxes, we need to do this for each box, so we have $(1+x)^t$. Let

$$F \xleftarrow{egf} (1+x)^t = \sum_{s=0}^{t} \binom{t}{s} x^s$$

and we have $s![x^s : F] = s!\binom{t}{s} = (t)_s$.

5 We observe

$$N = \{0, 1, 2, \dots\} \xrightarrow[egf]{char} 1 + x + \frac{1}{2!}x^2 + \dots \overset{taylor}{=} e^x$$

we do that for all $t$ distinct boxes, so we have $e^{tx}$. Let

$$F \xleftarrow{egf} e^{tx} = \sum_{s=0}^{\infty} \frac{t^s}{s!} x^s$$

and we have $s![x^s : F] = t^s$.

9 We observe

$$N = \{1, 2, \dots\} \xrightarrow[egf]{char} x + \frac{1}{2!}x^2 + \dots \overset{taylor}{=} e^x - 1$$

we do that for all $t$ distinct boxes, so we have $(e^x - 1)^t$. Let

$$\begin{aligned} F \xleftarrow{egf} (e^x - 1)^t &= \sum_{s=0}^{t} \binom{t}{k} (e^x)^{t-k} (-1)^k \qquad \text{binomial thm.} \\ &= \sum_{k=0}^{t} (-1)^k \binom{t}{k} \frac{1}{s!} (t-k)^s x^s \\ &= \sum_{s=0}^{\infty} \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t-k)^s \frac{x^s}{s!} \end{aligned}$$

and we have $s![x^s : F] = \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t-k)^s$.

Surprise: we got a formula for Stirling numbers of the second kind!

**Proposition 42.**

$$t! \left\{ {s \atop t} \right\} = \sum_{k=0}^{t} (-1)^k \binom{t}{k} (t-k)^s$$

*Alternate proof (counting argument).* Use symmetric IE. The LHS means placing $s$ people into $t$ named groups.

Let $S =$ all ways to place $s$ people into $t$ **possibly empty** named groups. We let $A_i$ be the ways where team $i$ is empty. Then $\alpha_k = \left|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right| = (t-k)^s$. We use the formula for symmetric IE. □

← But we want nonempty groups!

# J Partitions

We tackle table entries 8 and 12.

## J1 Integer partitions

12 : How many ways to put $s$ identical objects into $t$ identical boxes with **no empty boxes**?

i.e. ways to write $s = x_1 + \cdots + x_t$ where $x_i$ are positive integer and the *order doesn't matter*. WLOG let $x_1 \geq x_2 \geq \cdots \geq x_t \geq 1$.

Let $p_t(s)$ denote the answer (the number of partitions of the integer $s$ into $t$ parts).

**Example 76.** $s = 8, t = 3$ could be partitioned into:

$$\begin{gathered} 6+1+1 \\ 5+2+1 \\ 4+3+1 \\ 4+2+2 \\ 3+3+2 \end{gathered}$$

Hence, $p_3(8) = 5$.

8 : How many ways to put $s$ identical objects into $t$ identical boxes, possibly with some empty boxes?

**Answer**: $\displaystyle\sum_{k=0}^{t} p_k(s)$ where $k$ is the number of nonempty parts.

Special cases for 8 :

$t \geq s$: define the **partition function**

$$p(s) := p_0(s) + p_1(s) + \cdots + p_s(s)$$

| $s$ | $p(s)$ | |
|---|---|---|
| 0 | 1 | $0 =$ empty sum |
| 1 | 1 | $1$ |
| 2 | 2 | $2 = 1 + 1$ |
| 3 | 3 | $3 = 2 + 1 = 1 + 1 + 1$ |
| 4 | 5 | $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$ |
| 5 | 7 | $5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$ |

**Remark.** *Not Fibonacci*!! But why does it start out like Fibonacci?
**Answer**: Let's consider the generating function

$$P(x) = \sum_{s=0}^{\infty} p(s)x^s$$

As an example, consider the partitioning of 4¢ of money out of coins of **all** integer values: 1¢, 2¢, 3¢, 4¢, ... The generating function for this is

$$P(x) = \underbrace{(1 + x + x^2 + \dots)}_{1¢}\underbrace{(1 + x^2 + x^4 \dots)}_{2¢}\underbrace{(1 + x^3 + x^6 + \dots)}_{3¢} \cdots = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

Euler showed that the denominator is

← Euler's pentagonal number formula

$$\prod_{k=1}^{\infty}(1 - x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \dots$$

Every term either has a 0, 1 or $-1$ coefficient.
We compare with the generating function for Fibonacci numbers:

$$(F_n) = 1, 1, 2, 3, 5, \dots \xrightarrow{ops} \frac{1}{1 - x - x^2}$$

while

$$(p(n)) \xrightarrow{ops} \frac{1}{1 - x - x^2 + x^5 + \dots}$$

should agree up to $n = 4$.

---

*(Scratch work begins)* In 1918, Hardy and Ramanujan showed that

$$p(n) \approx \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \qquad n \to \infty$$

*(Scratch ends here)*

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