MATH103 Combinatorics Notes

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January 22, 2024

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Combinatorics

A1 Intro

Remark. Let there be a set $\{1, 2, \dots, n\}$. The number of subsets of it is 2^n since for each number, we could say "include" or "exclude".

Example 1. Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

(Scratch work begins)

First consider $n = 0$. Then the only *good* subset is $\emptyset$.

Now consider $n = 1$, both $\emptyset, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\emptyset, 1, 2, 12$. The set 12 is not good. ← notation simplified for fast typing

Similarly, we have $f(3) = 5, f(5) = 8$.

(Scratch ends here)

We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

A2 Fibonnacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$$

Remark. Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and ← Textbook
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$. ← Preferred!

Example 2. Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbds one step per stride, sometimes two steps per stride. In how many ways can she do this?

← It is the same recurrence as A1 but with init conditions shifted: $f(n) = F_{n+1} = f_{n+2}$.

Table 1: Table of the sequence in two notations

n	0	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21	34
f_n	0	1	1	2	3	5	8	13	21

(Scratch work begins) Let $S(n)$ be the number of ways climbing n steps.

- $S(1) = 1$

  • — •

- $S(2) = 2$

  • — • — •
  • ——— •

- $S(3) = 3$

  • — • — • — •
  • ——— • — •
  • — • ——— •

- $S(4) = 5$

  • — • — • — • — •
  • ——— • — • — •
  • — • ——— • — •
  • — • — • ——— •
  • ——— • ——— •

Conjecture: maybe Fibonnacci?

(Scratch ends here)

Proof. Consider the set of ways she can cover n steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n-1$ steps. Thus, this gives us $S(n-1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n-2$ steps. Thus, this gives us $S(n-2)$ ways.

Therefore, we conclude that $S(n) = S(n-1) + S(n-2)$. We account the initial conditions and conclude the closed form:

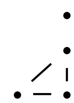
$$S(n) = F_n = f_{n+1}$$

for all n. Since Prof Rad climbs 47 steps, we get $\boxed{S(47) = 4807526976}$. □

A3 Simplex numbers

Definition 1. *Two-dimensional* triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- ...



$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

Theorem 1. $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

First proof. We prove by induction.

← Not as good of a proof: we must know what we are proving in the first place!

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.
Inductive hypothesis: Suppose proved formula for up to $n = k$.
Inductive step: Consider $n = k + 1$.

$$\begin{aligned} T_2(k+1) &= 1 + \cdots + k + (k+1) \\ &= T_2(k) + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k^2 + k + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)\left((k+1)+1\right)}{2} \end{aligned}$$

$\square$

Proof by Gauss. Observe:

← Better proof: concluding the formula without knowing it first!

$$\begin{aligned} T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\ &= n + (n-1) + \cdots + 2 + 1 \end{aligned}$$

Therefore, we **add** the two rows:

$$
\begin{aligned}
2T_2(n) &= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{n} \\
&= n(n+1) \\
\therefore T_2(n) &= \frac{1}{2}n(n+1)
\end{aligned}
$$

□

Definition 2. Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

- $T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$

Definition 3. Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$