MATH103 Combinatorics Notes

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Contents

A Recurrence Relations

A1 Intro

Remark. Let there be a set $\{1, 2, \dots, n\}$. The number of subsets of it is 2^n since for each number, we could say "include" or "exclude".

Example 1. Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

(Scratch work begins)

First consider $n = 0$. Then the only *good* subset is $\emptyset$.

Now consider $n = 1$, both $\emptyset, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\emptyset, 1, 2, 12$. The set 12 is not good.

← notation simplified for fast typing

Similarly, we have $f(3) = 5, f(5) = 8$.

(Scratch ends here)

We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

A2 Fibonnacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \dots$$

Remark. Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and ← Textbook
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$. ← Preferred!

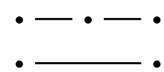
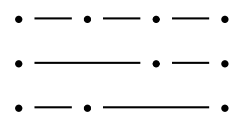
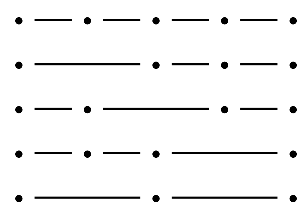
Example 2. Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbds one step per stride, sometimes two steps per stride. In how many ways can she do this?

← It is the same recurrence as A1 but with init conditions shifted: $f(n) = F_{n+1} = f_{n+2}$.

Table 1: Table of the sequence in two notations

n	0	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21	34
f_n	0	1	1	2	3	5	8	13	21

(Scratch work begins) Let $S(n)$ be the number of ways climbing n steps.

- $S(1) = 1$
- $S(2) = 2$ 
- $S(3) = 3$ 
- $S(4) = 5$ 

Conjecture: maybe Fibonnacci?

(Scratch ends here)

Proof. Consider the set of ways she can cover n steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n-1$ steps. Thus, this gives us $S(n-1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n-2$ steps. Thus, this gives us $S(n-2)$ ways.

Therefore, we conclude that $S(n) = S(n-1) + S(n-2)$. We account the initial conditions and conclude the closed form:

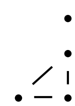
$$S(n) = F_n = f_{n+1}$$

for all n. Since Prof Rad climbs 47 steps, we get $\boxed{S(47) = 4807526976}$. $\square$

A3 Simplex numbers

Definition 1. *Two-dimensional* triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- …



$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

Theorem 1. $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

First proof. We prove by induction.

← Not as good of a proof: we must know what we are proving in the first place!

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.
Inductive hypothesis: Suppose proved formula for up to $n = k$.
Inductive step: Consider $n = k + 1$.

$$\begin{aligned} T_2(k+1) &= 1 + \cdots + k + (k+1) \\ &= T_2(k) + k + 1 \\ &= \frac{k(k+1)}{2} + k + 1 \\ &= \frac{k^2 + k + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \frac{(k+1)\left((k+1)+1\right)}{2} \end{aligned}$$

□

Proof by Gauss. Observe:

← Better proof: concluding the formula without knowing it first!

$$\begin{aligned} T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\ &= n + (n-1) + \cdots + 2 + 1 \end{aligned}$$

Therefore, we **add** the two rows:

$$
\begin{aligned}
2T_2(n) &= \underbrace{(n+1)+(n+1)+\cdots+(n+1)}_{n} \\
&= n(n+1) \\
\therefore \quad T_2(n) &= \frac{1}{2}n(n+1)
\end{aligned}
$$

□

Definition 2. Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

- $T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$

Definition 3. Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$

Some examples of simplex numbers $T_d(n)$:

$d \backslash n$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	1	3	6	10	15	21	28
3	1	4	10	20	35	56	84
4	1	5	15	35	70	126	210
5	1	6	21	56	126	252	462

B Ramsey Theory

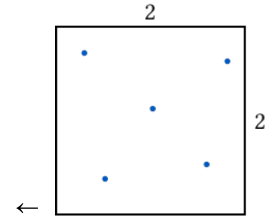
Invented by Frank Ramsey in 1930. We would need:

- Graph Theory
- Pigeonhole Principle
- Quantifiers
- Counterexamples

B1 Pigeonhole principle

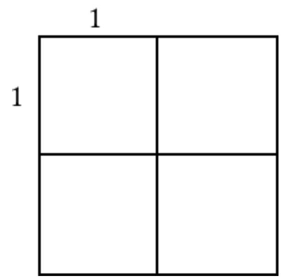
Theorem 2 (Dirichlet's Pigeonhole Principle)**.** If you put $n + 1$ pigeons in n pigeonholes, then (at least) one pigeonhole will contain (at least) two pigeons.

Proof omitted. □



Example 3. Given 5 points in a square of side length 2, show that there must exist two points whose mutual distance is $\leq \sqrt{2}$.

Proof. Divide square into 4 smaller squares. We now have 4 pigeonholes and 5 dots:



These two points in the same pigeonhole have distance $\leq \sqrt{1^2 + 1^2} = \sqrt{2}$. □

Example 4. There exists two people in NYC who have exactly the same number of hairs on their head.

Example 5. There are 30 people at a party talking with each other. Afterwards, there will be two people who talked with the same number of people.

Proof. If we put a person who talked to i people into box i, we get 30 boxes; however, we cannot have someone who talked to 0 people and someone who talked at 29 people at the same time! Hence, we combine the box 0 and box 29, and only one of which could be the case.

Now we have 29 boxes and 30 people. By pigeonhole principle, there must be two people who talked with the same amount of people. □

Theorem 3 (Strong Pigeonhole Principle)**.** Given pigeonholes $1, 2, \ldots, n$ with capacities $c_1, c_2, \ldots, c_n$ where $c_i \geq 0$; if we have at least $c_1 + c_2 + \cdots + c_n + 1$ pigeons in these pigeonholes, then at least one pigeonhole overflows.

Proof. Suppose BWOC that no pigeonhole overflows. Then for all $i = 1, 2, \ldots, n$, we have the number of pigeons in $i \leq c_i$.

We add up and get inequalities:

$$\text{total \# pigeons} \leq c_1 + c_2 + \cdots + c_n$$

Contradiction! □

Example 6. There are five people supporting two teams. Then at least one team is supported by 3 people.

Proof. Assume BWOC that the two teams only have two supporters. Let $c_1 = c_2 = 2$. However, by SPP, $5 \geq 2 + 2 + 1$, hence one pigeonhole overflows. Therefore, one team must have > 2 supporters. □

B2 First Ramsey Theorem

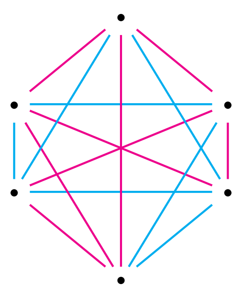
There are 6 people taking a class. Then:

either there exists 3 people such that each pair of them have previously taken a class together,

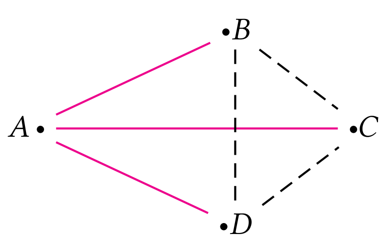
or (inclusive) there exists 3 people such that no two have taken a class together.

Theorem 4. If we have 6 vertices and we draw all edges between them (a K_6 graph), then for every possible way of coloring the edges red and blue, there must exist a **monochromatic** triangle.

← K_6 stands for *complete graph on 6 vertices.* It has 15 edges.



Proof. Pick any vertex and call it A. It has 5 edges colored red and blue. By the SPP, there exists at least 3 edges of the same color. WLOG let these three edges be red and call the other three vertices B, C, D.

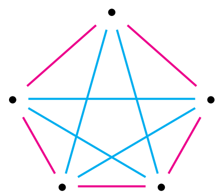


- If BC is red, then ABC is a red triangle.
- If CD is red, then ACD is a red triangle.
- If BD is red, then ABD is a red triangle.
- If none of the above has happened, then BC, CD, BD are all blue, meaning that BCD is a blue triangle!

□

Theorem 5. If there are 5 instead of 6 vertices, then the above coloring prediction cannot be made with certainty.

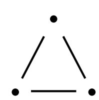
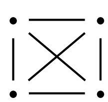
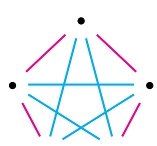
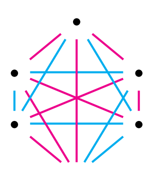
Counterexample.



□

B3 $K_p \to K_q, K_r$

In graphy theory, K_n is the **complete** graph on n vertices.

K_1		1 vertex	0 edges
K_2		2 vertices	1 edge
K_3		3 vertices	3 edges
K_4		4 vertices	6 edges
K_5		5 vertices	10 edges
K_6		6 vertices	15 edges

Remark. Note that K_n has $1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$ edges, hence is the $n-1$-th triangular number.

Ramsey Theory uses the following language convention: the expression

$$K_p \to K_q, K_r$$

represents a statement with the following meaning:

Definition 4. If the edges of K_p are colored red/blue, then it necessarily follows that either the K_p contains a red K_q, or K_p contains a blue K_r (or possibly both).

We want to know whether this statement is true for a given triple of (p, q, r).

Example 7. We proved in B2 that $K_6 \to K_3, K_3$.

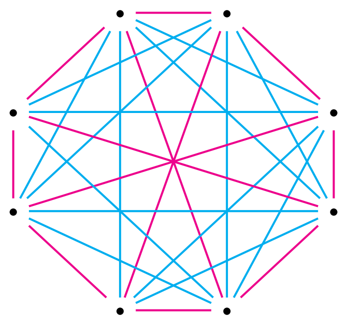
Non-example 8. We also showed that $K_5 \to K_3, K_3$ is false by exhibiting a coloring of K_5 that does not have a red or blue triangle (counterexample).

← write $K_5 \not\to K_3, K_3$

Example 9. It is known that $K_{18} \to K_4, K_4$ and $K_{17} \not\to K_4, K_4$.

Example 10. Also, $K_9 \to K_3, K_4$ and $K_8 \not\to K_3, K_4$.

← Here we have to decide in advance which color goes with the K_3 and which goes with the K_4 due to asymmetry.



This K_8 has no red triangle and no blue K_4.

Theorem 6 (Ramsey). Let q, r be positive integers. Then there always exists a positive integer p such that

$$K_p \to K_q, K_r$$

is true.

We would see the following tabel giving us values of p that work.

Define a function $N(q, r)$ recursively:

← $q, r \in \mathbb{Z}^+$

- Base case: $N(1, r) = N(q, 1) = 1$
- Recurrence: $N(q, r) = N(q - 1, r) + N(q, r - 1)$ if $q, r \geq 2$.

We compute the value of $N(q, r)$ for:

← They do look like simplex numbers!

$q \backslash r$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126
6	1	6	21	56	126	252

We would want to prove that $K_{N(q,r)} \to K_q, K_r$ for all $q, r \geq 1$.

Proof. By induction.

Base case: If $q = r = 1$, then $N = 1$, we need to show that $K_1 \to K_1, K - r$ and $K_1 \to K_q, K_1$ for all q, r.

That is, suppose K_1 has its edges colored red/blue, then there exists a red K_1 or a blue K_r, and *vice versa*.

Since there are no edges, this is vacuously true.

Inductive step: We will show that if we are given that A, B are numbers such that

$$K_A \to K_{q-1}, K_r$$
$$\text{and } K_B \to K_q, K_{r-1}$$

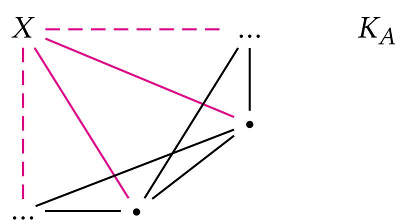
are true, then $K_{A+B} \to K_q, K_r$.

← This works because as we fill out the table above, each new number we write in will work because it's the sum of the left and above numbers and they both work.

Consider K_{A+B} colored red and blue. We will show that it has a red K_q or a blue K_r.

Pick a vertex and call it X. It would have $A + B - 1$ edges in total. We claim that X either has at least A red edges, or at least B blue edges. This is indeed true, because if not, the number of red edges would be $\leq A - 1$ and the number of blue edges would be $\leq B - 1$ and so the total number of edges would be $\leq A + B - 2 < A + B - 1$, which is a contradiction.

Now, if X has a red claw of size A:



← The neighbouring vertices connected by black edges form K_A, yet to be colored.

From our inductive hypothesis $K_A \to K_{q-1}, K_r$, we must have

either red K_{q-1}, in which case we combine with the vertex X and the red claw to get at least one red K_q.

or blue K_r, in which case we are done.

Similarly, if X has a blue claw of size B, then we make the same argument.

Hence, we know that $K_{A+B} \to K_q, K_r$ is true whenever $K_A \to K_{q-1}, K_r$ and $K_B \to K_q, K_{r-1}$.

□

B4 Ramsey Numbers

Recall: Let m, n be positive integers. We know that there are numbers $p \in \mathbb{N}$ such that $K_p \to K_m, K_n$.

Remark. If p works, then so does any $q \geq p$ as K_q would contain copies of K_p.

So the question becomes, if we have $K_p \to K_m, K_n$, is p the **smallest** such number?

Definition 5. The **Ramsey number** $r(m, n)$ is the smallest such number.

Example 11. We know $K_6 \to K_3, K_3$ but $K_5 \not\to K_3, K_3$, so $r(3, 3) = 6$.

Example 12. Mathematicians have proved that

$$K_{48} \to K_5, K_5$$
$$K_{42} \not\to K_5, K_5$$

so we have $43 \leq r(5, 5) \leq 48$.

Remark. In general,

$$K_N \to K_m, K_n \iff r(m, n) \leq N$$
$$K_{N-1} \not\to K_m, K_n \iff r(m, n) \geq N$$

Need both to get the precise value of $r(m, n)$.

Proposition 7. Properties of Ramsey numbers:

(a) $r(3, 3) = 6$ ← proven in B2
(b) $r(m, n) = r(n, m)$ ← symmetry
(c) $r(1, n) = 1$ ← $K_1 \to K_1, K_n$, $K_0 \not\to K_1, K_n$
(d) $r(2, n) = n$
(e) $r(m, n) \leq r(m-1, n) + r(m, n-1)$ for all $m, n \geq 2$

Proof for (d). Claim: $K_2 \to K_2, K_n$, $K_{n-1} \not\to K_2, K_n$.

Color K_n. If all edges are blue then we have a blue K_n. Else we have some red edges, so we have some red K_2.

Now color K_{n-1} all blue: we realize that we don't have any red K_2, but we don't have a blue K_n either! □

Proof for (e). Let $A = r(m-1, n)$, $B = r(m, n-1)$. We have shown that if $K_A \to K_{m-1}, K_n$ and $K_B \to K_m, K_{n-1}$, then $K_{A+B} \to K_m, K_n$. Hence $r(m, n) \leq A + B$. □

Known facts:

$$\begin{aligned} r(2,2) &= 2 \\ r(3,3) &= 6 \\ r(4,4) &= 18 \\ 43 \leq r(5,5) &\leq 48 \\ 102 \leq r(6,6) &\leq 165 \end{aligned}$$

B5 A lower bound for $r(m, n)$

Our table of $N(m, n)$ gave us upper bonds for $r(m, n)$. Specifically,

$$r(m,n) \leq N(m,n) = \frac{(n+m-2)!}{(n-1)!(m-1)!} = \binom{m+n-2}{m-1}$$

What about lower bound?

Theorem 8.

$$r(m,n) \geq (m-1)(n-1)+1$$

if and only if $K_{(m-1)(n-1)} \not\to K_m, K_n$

Proof. We prove this by exhibiting a coloring of $K_{(m-1)(n-1)}$ that has no red K_m, no blue K_n.

Place vertices in grid:

$$\begin{array}{r|cccc} & \bullet & \bullet & \cdots & \bullet \\ & \bullet & \bullet & \cdots & \bullet \\ m-1 \text{ rows} & \vdots & \vdots & \ddots & \vdots \\ & \bullet & \bullet & \cdots & \bullet \\ \hline & \multicolumn{4}{c}{n-1 \text{ columns}} \end{array}$$

Coloring rule of edges: If two vertices are in the same row, color the edges blue. If two vertices are in the same column, color the edges red. Every other edge arbitrary.

Claim: there exists no red K_m.

Consider the m vertices of such a K_m. There are $m-1$ rows. Pigeonhole principle ensures that some vertices must be in the same row. But that edge must be blue! So this is not a red K_m. Similarly, there are no blue K_n. □

Thus, we get: $(m-1)(n-1)+1 \leq r(m,n) \leq \frac{(n+m-2)!}{(n-1)!(m-1)!} = \binom{m+n-2}{m-1}$.

Observe there is still a huge gap between the bounds. Could we get better?

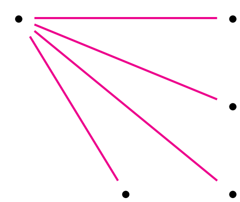
B6 The "parity" improvement

Our methods have shown that $K_{10} \to K_3, K_4$. But it is actually true that $K_9 \to K_3, K_4$. Why?

Proof. Given K_9 colored red or blue. We seek a red K_3 or a blue K_4.

That is to say that if we ever see a red 4-claw, then we are done!

← See this argument.



In addition, if we ever see a blue 6-claw, then we are also done because $K_6 \to K_3, K_3$ and we either have a red K_3 or a blue K_3, which would have to combine with the other vertex to get a blue K_4.

Now suppose we neither have a red 4-claw nor a blue 6-claw. This implies that each vertex has ≤ 3 red edges, and ≤ 5 blue edges. However, in a K_9, each vertex only has 8 edges, so they must exactly each have 3 red edges and 5 blue edges. Does this exist? We realize that to make this happen, we have:

- 9 vertices
- Each vertex has 3 red edges
- Every edge belongs to two vertices

Hence, we need to have exactly $\frac{3\times 9}{2} = 13.5$ red edges, but this cannot happen because we need a whole number of edges! Thus, it is not possible that we neither have a red 4-claw nor a blue 6-claw. $\square$

Lemma 9 (Ramsey inductive step improved by parity)**.** Suppose

← Also seen here.

$$K_A \to K_{q-1}, K_r$$
$$\text{and } K_B \to K_q, K_{r-1}$$

are true, then $K_{A+B} \to K_q, K_r$.

In addition, if A, B are **both even numbers**, then $K_{A+B-1} \to K_q, K_r$.

B7 Variations

More colors!

For example:

$$K_p \to K_a, K_b, K_c$$

(given K_p colored red, blue, green, it must contain a red K_a, or a blue K_b, or a green K_c.)

Example 13. It is known that $K_{17} \to K_3, K_3, K_3$.

Proof sketch. Pick a vertex which has 16 edges. We observe $16 \div 3 = 5\frac{1}{3} \implies$ at least one color occurs 6 times (i.e. we can see a red/blue/green 6-claws). □

Remark. $r(a, b, c)$ is the smallest number that works for the above.

Other graphs

Paths:

P_1 •

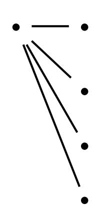
P_2 • — •

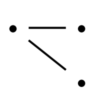
P_3 • — • — •

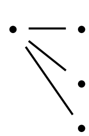
P_4 • — • — • — •

Claws:

$K_{1,1}$

$K_{1,4}$


$K_{1,2}$


$K_{1,3}$


Example 14. Show that $r(K_{1,3}, K_{1,3}) = 6$.

Proof. We know $K_6 \to K_3, K_3$. Pick a vertex that has 5 neighbors. By the strong pigeonhole principle, we must have three edges of the same color $\implies$ either red or blue $K_{1,3}$. $\square$

C Counting

C1 Three principles

Addition principle

← aka. counting by cases

Definition 6. If a set S is *partitioned* into subsets $S_1, S_2, \ldots S_n$, then the cardinality of S is

$$|S| = |S_1| + \cdots + |S_n|.$$

← that is, $S = \bigcup S_i$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$

The art lies in:

- making each S_i easy to count, and
- not having too many S_i if there is no formula for $|S_i|$.

Remark (Variations)**.** If the S_i cover S but they overlap, then we have the inequality $|S| < \sum_{i=1}^{n} |S_i|$ because the overlap implies that we are *overcounting*.

← The inclusion/exclusion principle handles overlaps precisely

Example 15. Let S be the set of *good* subsets of $[5] = \{1, 2, 3, 4, 5\}$. We could first try:

← *good* meaning no adjacent elements

- S_1 contains the subsets that contain 5
- S_2 contains the subsets that don't contain 5

We have previously shown that $|S_1|$ = number of *good* subsets of [3] and $|S_2|$ = number of *good* subsets of [4].

Alternatively, we could also let T_i be the *good* subsets of [5] with cardinality i. Then

S is partitioned into $T_0 \cup T_1 \cup T_2 \cup T_3$. We count:

$$\left.\begin{array}{llcl} T_0 & : & \varnothing & |T_0| = 1 \\ T_1 & : & 1, 2, 3, 4, 5 & |T_1| = 5 \\ T_2 & : & 13, 14, 15, 24, 25, 35 & |T_2| = 6 \\ T_3 & : & 135 & |T_3| = 1 \end{array}\right\} |S|=13$$

Subtraction principle

Definition 7. Let $A \subseteq S$ and A^c be its complement in S. Then A, A^c partition S and $|S| = |A| + |A^c|$. This means that

$$|A| = |S| - |A^c|.$$

Example 16. How many 2-digit numbers have distinct nonzero digits?

Let S be the set of all 2-digit numbers $\{10, 11, \dots, 99\}$ and let A be the subset of those with nonzero distinct digits. We count:

$$\begin{array}{rlcl} A^c & : 11, 22, \dots, 99 & & \text{(distinct fails)} \\ & 10, 20, \dots, 90 & & \text{(nonzero fails)} \end{array}$$

Hence $|A| = |S| - |A^c| = 90 - 18 = 72$.

Multiplication principle

Definition 8. Suppose we have to do two tasks in sequence. We suppose:

← Note: sometimes the 2nd task could depend on the 1st one

- Task 1 has m outcomes
- Task 2 has n outcomes, regardless of how Task 1 was carried out.

Then there are mn ways of carrying out both tasks.

← Similarly for 3 or more tasks in sequence

Example 17. How many 2-digit numbers have distinct nonzero digits?

Let $\boxed{a}\,\boxed{b}$ be the two digits in these numbers. Let Task 1 be selecting digit $\boxed{a}$ and Task 2 be selecting digit $\boxed{b}$.

- Task 1: 9 ways (1,2,...,9)
- Task 2: 8 ways (1,2...,9 but not same as $\boxed{a}$)

Hence there are $9 \times 8 = 72$ such numbers.

Tricky example 18. How many **odd** numbers in the range 1000-9999 have distinct digits?

Attempt 1: Let $\boxed{a}\boxed{b}\boxed{c}\boxed{d}$ be the 4 digits in these numbers and assign them Tasks 1-4. We have:
- Task 1: 9 ways (1-9)
- Task 2: 9 ways (0-9 except $\boxed{a}$)
- Task 3: 8 ways (0-9 except $\boxed{a}$,$\boxed{b}$)
- Task 4: Could be 2 or 3 or 4 or 5 (depending on how many odd digits had already been used)

Hence, the best we can say here is that the answer is between $9 \times 9 \times 8 \times 2$ and $9 \times 9 \times 8 \times 5$.

← BAD! This is a large range!

Attempt 2: Let $\boxed{a}\boxed{b}\boxed{c}\boxed{d}$ be the 4 digits in these numbers and try the order $\boxed{d}\boxed{a}\boxed{b}\boxed{c}$ for Tasks 1-4. We have:
- Task 1: 5 ways $(1, 3, 5, 7, 9)$
- Task 2: 8 ways (1-9 except $\boxed{d}$)
- Task 3: 8 ways (0-9 except $\boxed{a}$, $\boxed{d}$)
- Task 4: 7 ways (0-9 except $\boxed{a}$, $\boxed{d}$, $\boxed{b}$)

Therefore, the number of ways is $5 \times 8 \times 8 \times 7 = 2240$.

Example 19. How many numebrs $0, 1, \ldots 99999$ have exactly one digit 6?

We can assign tasks:

- Task 1: choose a location for 6, giving us 5 ways
- Task 2: assign the remaining digits from left to right, giving us 9^4 ways

Hence there are $5 \times 9^4 = 32805$ such numbers.

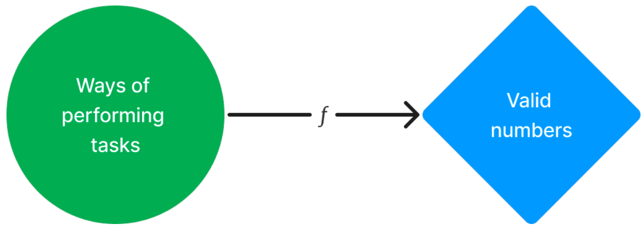
Non-example 20. How many integers $0, 1, \ldots 99999$ have *at least* one digit 6?

Attempt: We can assign tasks:

- Task 1: choose a location for 6, giving us 5 ways
- Task 2: assign the remaining digits from left to right, giving us 10^4 ways

But the answer of 50000 is **wrong**! But why?

The counting process for this problem is corresponding the ways of performing tasks to valid 5-digit numbers:



We have correctly counting the green set. However, for this to count the blue set, we need f to be **bijective**. That is, every valid number must be obtained in exactly one way. However, in this case, our f is surjective but not injective. For instance, 62516 will be counted *twice*:

- 6____ then 62516; or
- ____6 then 62516.

Hence, 50000 > correct answer!

Correct way: Using the subtract principle to deduct numbers that don't have 6: $10^5 - 9^5 = 40951 < 50000$.

C2 Probability

Definition 9.

$$\text{Probability} = \frac{\text{number of favourable cases}}{\text{number of total cases}}$$

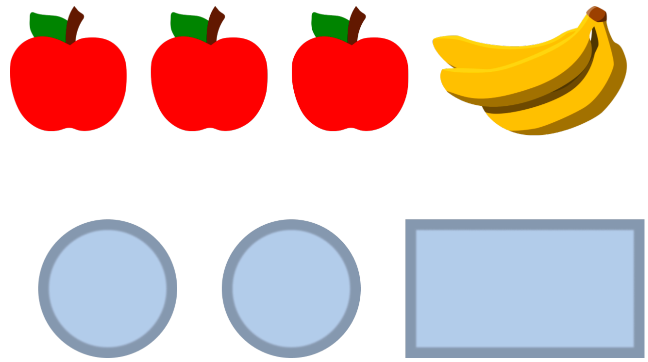
Example 21. Probability of 3 dice rolling the same number: $P = \frac{6}{6^3} = \frac{1}{36}$.

C3 The counting framework

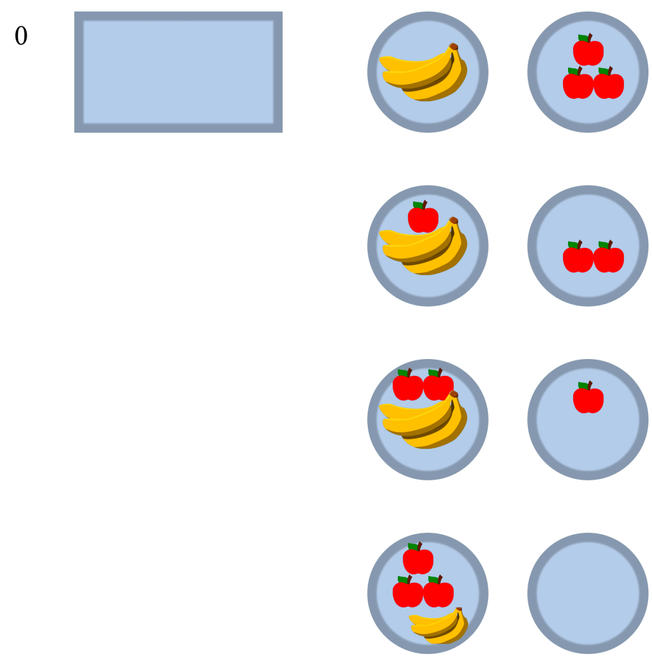
Here is a very general problem:

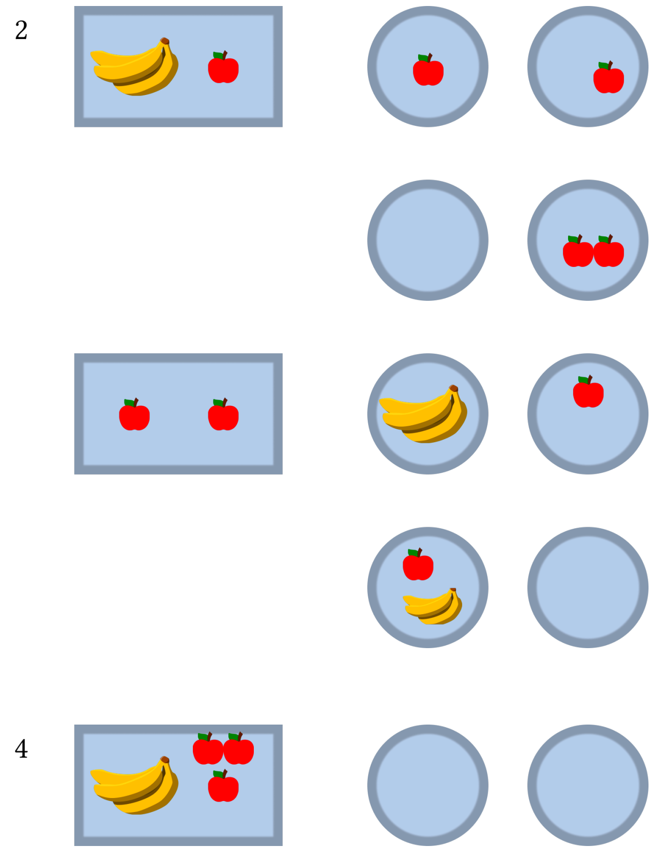
"How many distinguishable ways to map a multiset S to a multiset T satisfying given constraints?"

Example 22. There are fruits and plates. Let $S = \{3 \cdot \text{apple}, 1 \cdot \text{banana}\}$ and $T = \{2 \cdot \text{circle}, 1 \cdot \text{rectangle}\}$. How many ways are there to serve fruit on plates such that the rectangular plate has an even number of fruit?



Ans: Organize by number of fruit on rectangle.





Hence there are 9 ways.

The general counting problem

Let there be multisets

$$S = \{a_1 \cdot 1, a_2 \cdot 2, \dots, a_s \cdot s\} \qquad \text{object types } 1, 2, \dots, s$$
$$T = \{b_1 \cdot U_1, b_2 \cdot U_2, \dots, b_t \cdot U_t\} \qquad \text{box types } U_1, \dots, U_t$$

How many distinguishable maps $f : S \to T$ are there, subject to restrictions on the numbers $u_i = \left|f^{-1}(U_i)\right|$?

← The number of items mapped to boxes of type U_i

Remark (Special cases)**.** To recognize which situation applies to the problem:

- By objects:
  - Distinct: $S = \{1, 2, \dots, s\}$
  - Identical: $S = \{s \cdot 1\}$
- By boxes:

- Distinct: $T = \{U_1, \ldots, U_t\}$
- Identical: $T = \{t \cdot U_1\}$

- For each case above, we can apply constraints:
  - $0 \le u_i \le 1$
  - $0 \le u_i < \infty$ no constraint
  - $1 \le u_i < \infty$ nonempty
  - $0 \le u_i \le n_i$ for some $n_i \in \mathbb{N}_0$ max. capacity
  - $u_i \in N_i \subseteq \mathbb{N}$

3.3 A Framework for Counting Questions: The Counting Table 95

Table 3.1 Balls and boxes counting problems.

Number of Ways to Put Balls into Boxes				
$S = S_1 = \{1, 2, \ldots, s\}$, or $S = S_2 = \{s \cdot 1\}$, a multiset of balls				
$T = T_1 = \{U_1, U_2, \ldots, U_t\}$, or $T = T_2 = \{t \cdot U_1\}$, a multiset of boxes				
Box U_i contains u_i balls				
Conditions on S and T → on u_i ↓	$T = T_1$ distinct $S = S_1$ distinct	$T = T_1$ distinct $S = S_2$ identical	$T = T_2$ identical $S = S_1$ distinct	$T = T_2$ identical $S = S_2$ identical
$0 \le u_i \le 1$ Assume $t \ge s$	1	2	3	4
$u_i \ge 0$	5	6	7	8
$u_i \ge 1$	9	10	11	12
for $i = 1, \ldots, t$, $0 \le u_i \le n_i$, $n_i \in \mathbb{Z}^{>0}$	13	14		
$u_i \in N_i \subset \mathbb{Z}^{\ge 0}$ for $i = 1, \ldots, t$	15	16		

Example 23. We have 10 distinct books to be shared between 2 children. Each child needs 2 books to avoid a crisis.

← Table entry $\boxed{15}$

- Distinct objects $\{1, 2, \dots, 10\}$
- Distinct boxes $\{U_1, U_2\}$
- Constraints: $u_i \in [2, \infty[$

Example 24 (Attempt 1)**.** 10 distinct books, 5 of them to be arranged on shelf (order matters)

← Table entry $\boxed{15}$

- $S = \{1, 2, \dots, 10\}$
- $T = \{U_1, U_2, \dots, U_5, U_6\}$ (positions on shelf + extra for unshelved books)
- $u_1 = \cdots = u_5 = 1, u_6 = 5$

Example 25 (Attempt 2)**.** 10 distinct books, 5 of them to be arranged on shelf (order matters)

← Table entry $\boxed{1}$, easier!

- $S = \{1, 2, \dots, 5\}$ (numbered stickers to arrange books)
- $T = \{U_1, U_2, \dots, U_{10}\}$ (10 books)
- $0 \leq u_1 \leq 1$ (each book can get 0 or 1 sticker)

Example 26. I have 10 books and will take 5 on holiday.

- Take 1: ← Table entry $\boxed{15}$
  - $S = \{1, 2, \dots, 10\}$
  - $T = \{U_1, U_2\}$
  - $u_1 = u_2 = 5$
- Take 2: ← Table entry $\boxed{2}$
  - $S = \{5 \cdot 1\}$ (identical 'stickers' marking on-holiday)
  - $T = \{U_1, U_2 \; \dots, U_{10}\}$
  - $0 \leq u_1 \leq 1$ (each book can get 0 or 1 sticker)

C4 Permutations of a set

Remark. Recall $[n] = \{1, 2, \dots, n\}$.

Definition 10. Let $0 \leq s \leq t$.

- An s-**permutation** of $[t]$ is an ordered list of s distinct elements of $[t]$.
- A t-**permutation** of $[t]$ is just a **permutation** of $[t]$.

Example 27. 10 books, arrange 5 on shelf: 5-perm of [10]

Example 28. 20 athletes, Gold, Silver and Bronze awarded: 3-perm of [20]

Theorem 10. The number of s-perms of $[t]$ is

$$\underbrace{t(t-1)\dots(t-s+1)}_{s \text{ terms}}$$

Proof. Select elements of the list one-by-one. We have t ways to pick the first, $t-1$ ways of picking the second, etc. □

Definition 11 (s-th falling-factorial function)**.** This inspires the following notation

$$(x)_s = x(x-1)\dots(x-s+1) = \frac{t!}{(t-s)!}$$

← There is also a rising $(x)^s = x(x+1)\dots(x+s-1)$

Remark. $(x)_0 = 1, (n)_n = n!$

Example 29. Number of ways to shelve 5 books out of 10 is

$$(10)_5 = 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

C5 Circular permutations

A circular 5-perm of $[n]$ is an arrangement of s distinct elements of $[n]$ around a round table. The difference from a non-circular perm is that the orientation of the table does not matter! How many ways to do so?

If there is a head of the table and positions are marked clockwise, then it would be the same as the s-perm $(n)_s$.

However, we consider all ways of marking the 'head' of the tablen to be equivalent, so we divide s upon that. Hence, we get the answer $\frac{1}{s}(n)_s = \frac{n!}{s(n-s)!}$.

C6 Table entries 3,4,5

- 3 : s distinct objects, t identical boxes, 0 or 1 per box.

$$\# \text{ ways} = \begin{cases} 1 & s \leq t \\ 0 & \text{otherwise} \end{cases}$$

- $\boxed{4}$: s identical objects, t identical boxes, 0 or 1 per box.

$$\#\text{ ways} = \begin{cases} 1 & s \leq t \\ 0 & \text{otherwise} \end{cases}$$

- $\boxed{5}$: s distinct objects, t distinct boxes, no restrictions.

$$\#\text{ ways} = \underbrace{t \times t \times \cdots \times t}_{s\text{ times}} = t^s$$

Remark. $0^0 = 1$ in combinatorics. ← NOT in analysis!!

C7 Combinations of sets: table entries 2,6,10

If we let $0 \leq s \leq t$, then an s-**combination of** $[t]$ is a subset of size s of $[t]$.

Definition 12. The binomial coefficient $\binom{t}{s}$ is defined (combinatorially) to be the number of s-combinations of $[t]$.

Theorem 11.

$$\binom{t}{s} = \frac{(t)_s}{s!} = \frac{t!}{(t-s)!s!}$$

Proof. Consider the number of s-perms of $[t]$.

We can count it directly: $(t)_s$.

Alternatively, make task 1 'selecting a subset of size s', which gives us $\binom{t}{s}$ ways. Make task 2 the ways of ordering a subset, which give $s!$ ways. By multiplication principle, the number of s-perms of $[t]$ is $\binom{t}{s}s!$.

We have $(t)_s = \binom{t}{s}s!$, giving us the formula above. □

Example 30. All of the following are equivalent and satisfy table entry $\boxed{2}$: ← $0 \leq u_i \leq 1$

- Select s books out of t distinct books;
- Place s stickers on t books, at most one sticker per book;
- Put s identical objects into t distinct boxes, each box can only have at most 1 object.

Example 31. These equivalent problems satisfy table entry $\boxed{6}$: ← $0 \leq u_i < \infty$

- Solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 12$ where $x_i \geq 0$ are integers.

- Packing a box of 12 bagels of 5 different types of bagels with unlimited supply.
- Number of permutations of 12 objects and 4 'drawer dividers':

$$\bullet\bullet \mid \bullet\bullet\bullet \mid \bullet\bullet \mid \bullet\bullet\bullet\bullet \mid \bullet$$

Remark. In general, when we count the number of anagrams s $\bullet$s and $t-1$ $|$s, there are $s+t-1$ total symbols. We must select s of the $s+t-1$ positions to be $\bullet$. Hence, the number of ways would be

← LHS is choose $\bullet$, RHS is choose $|$.

$$\binom{s+t-1}{s} = \binom{s+t-1}{t-1}$$

Example 32. Entry $\boxed{10}$ is the same as $\boxed{6}$ except that each box must be nonempty:

- Solutions to $x_1 + \cdots + x_s = t$ where $x_i \geq 1$ integers.
- Select s bagels from t types, with each type chosen at least once.
  This reduces to the prev problem: select one of each type of bagel first; then, we choose $s-t$ bagels of t types.
- Anagrams with s $\bullet$s and $t-1$ $|$: must avoid $||$ and $|$ at beginning or end. Then we could think about placing $t-1$ dividers in the $s-1$ spaces between $\bullet$s, giving us $\binom{s-1}{t-1}$ ways.

C8 Anagrams

Example 33. Find the number of anagrams of the word

COMBINATORIALISTICALLY

First, suppose all 22 letters were distinct. We put subscripts:

$$C_1O_1MBI_1NA_1T_1O_2RI_2A_2L_1I_3ST_2I_4C_2A_3L_2L_3Y$$

Now we have this multiset:

$$\{2C, 2O, M, B, 4I, N, 3A, 2T, R, 3L, S, Y\}$$

We want to find the ways of arranging this!

- Assuming everything is distinct: $22!$ ways.
- Same assumption but with two tasks in multiplication:
  - Task 1: Arrange without subscripts (what we want)

– Task 2: Add subscripts: the number of ways to add them is

$$2! \times 2! \times 1! \times 1! \times 4! \times 1! \times 3! \times 2! \times 1! \times 3! \times 1! \times 1!$$

Therefore, our answer would be (in multinomial coefficient)

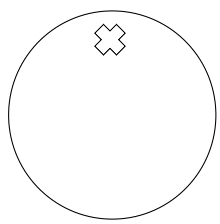
← DO NOT LEAVE OUT 1s

$$\frac{22!}{2! \times 2! \times 1! \times 1! \times 4! \times 1! \times 3! \times 2! \times 1! \times 3! \times 1! \times 1!} = \binom{22}{2,2,1,1,4,1,3,2,1,3,1,1}$$

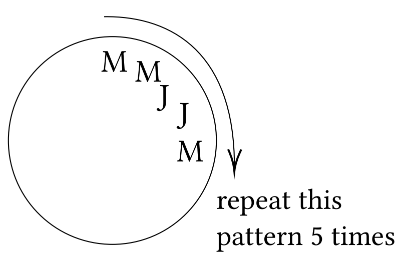
C9 More circular tables

Example 34. An alien conference has 9 Martian hare delegates and 16 Jovian hare delegates, with each type of hare identical. How many distinguishable ways are there to seat them at a circular table?

Method: first consider the ways of arrangement at a marked table.



- We seat 9 Martian hares and fill out the rest with Jovian ones: $\binom{25}{9}$.
- We place all hares (the answer we want) and then mark the table. In the end we should get the same answer of $\binom{25}{9}$.
  - How do we mark the table? There are 25 places that can be marked, but not all of them are distinct! For instance:



then the table only has 5 distinct markings due to rotational symmetry.

Theorem 12. Consider an arrangement on a circular table with n spots. Let R_k be the action of rotating this table by k places. Define:

← F is the *stabilizer* of the group action

$$F = \{k \in \mathbb{Z} \mid R_k \text{ leaves the arrangement unchanged}\}$$

Then we have:

1. F is the set of multiples of some $d|n$.
2. A length d pattern would be repeated $\frac{n}{d}$ times (so there are only $\frac{n}{d}$ distinct markings)

Proof. Let d be the smallest positive integer such that $R_d \in F$. Suppose BWOC $k \in F$ but $d \nmid k$. Then let $k = md + r$ with $r \leq d$, and rotation by r would be: $R_k R_{-md}$. Since $k \in F$, this also fixes the arrangement. However, $r < d$ contradicts the fact that d is the smallest positive integer such that $R_d \in F$. □

← Review Abstract Alg. 1

Back to Example 34: $F = d\mathbb{Z}$ where $d = 1, 5, 25$.

- If $d = 1$, 25 ways of marking the table.
- $d = 5$, 5 ways of marking.
- $d = 25$, 1 way of marking.

Definition 13 (Multichoose notation)**.** Define the ways to select a bag of k items from n different types of item to be $\left(\!\binom{n}{k}\!\right) = \binom{k+n-1}{k} = \binom{k+n-1}{n-1}$.

D Binomial Coefficients

We know that $\binom{n}{k}$ is the number of k-subsets of $[n]$, which is $\frac{n!}{k!(n-k)!}$.

D1 Binomial identities

Proposition 13.

$$\binom{n}{k} = \binom{n}{n-k}$$

That is, the number of ways to get k-subsets in $[n]$ is the same as that of $n-k$-subsets.

Proposition 14.

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

This is because the RHS counts **all** subsets of $[n]$, while each summand on the LHS counts the number of k-subsets thereof.

Proposition 15. For $n \geq 1$,

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor} = 2^{n-1}$$

This is because the RHS chooses any subset of $[n-1]$, then makes the subset **even** by putting or not putting the n into it (the n doesn't get to choose).

← $\lfloor a \rfloor$ is the largest integer $\leq a$

Proposition 16 (Binomial recurrence)**.**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is because the LHS chooses k-subsets of $[n]$, and the RHS splits the case into 1) the subset contains n, which gives us $\binom{n-1}{k-1}$ ways to choose the rest, and 2) the subset doesn't contain n, which gives us $\binom{n-1}{k}$.

This allows us to construct the table:

$n\backslash k$	0	1	2	3	4
0	1	0	0	0	0
1	1	1	0	0	0
2	1	2	1	0	0
3	1	3	3	1	0
4	1	4	6	4	1

Proposition 17.

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

Say there are $2n$ students: n in class G and n in class S. We need to choose n students.

- Method 1: choose n students: $\binom{2n}{n}$
- Method 2: let $k = 0, 1, \ldots, n$. Select k S to go and select k G to NOT go. Then we have $\binom{n}{k}$ for each of the process. Hence, the total number of ways is $\sum_{k=0}^{n} \binom{n}{k}^2$.