MATH103 Combinatorics Notes

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Contents

A Recurrence Relations

A1 Intro

Remark. Let there be a set $\{1, 2, \ldots, n\}$. The number of subsets of it is 2^n since for each number, we could say "include" or "exclude".

Example 1. Now consider the number of subsets with no two adjacent elements. Call them *good* subsets, and the count be $f(n)$.

(Scratch work begins)

First consider $n = 0$. Then the only *good* subset is $\emptyset$.

Now consider $n = 1$, both $\emptyset, \{1\}$ are good.

Now consider $n = 2$. We have subsets: $\emptyset, 1, 2, 12$. The set 12 is not good.

← notation simplified for fast typing

Similarly, we have $f(3) = 5, f(5) = 8$.

(Scratch ends here)

We have $f(n) = f(n-1) + f(n-2)$ for all $n \geq 2$. Hence, $f(n)$ is the sequence that satisfies the recurrence relation and the initial conditions $f(0) = 1, f(1) = 2$.

A2 Fibonnacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \ldots$$

Remark. Two notation conventions:

- $F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2$, and ← Textbook
- $f_0 = 0, f_1 = 1, f_n = f_{n-1} + f_{n-2} \quad \forall n \geq 2$. ← Preferred!

Example 2. Prof Rad is climbing 47 steps. Energized by coffee, she sometimes climbds one step per stride, sometimes two steps per stride. In how many ways can she do this?

← It is the same recurrence as A1 but with init conditions shifted: $f(n) = F_{n+1} = f_{n+2}$.

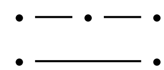
Table 1: Table of the sequence in two notations

n	0	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21	34
f_n	0	1	1	2	3	5	8	13	21

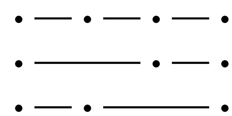
(Scratch work begins) Let $S(n)$ be the number of ways climbing n steps.

- $S(1) = 1$

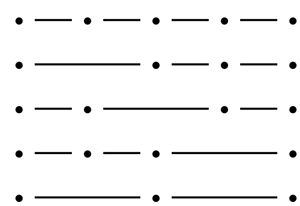
- $S(2) = 2$



- $S(3) = 3$



- $S(4) = 5$



Conjecture: maybe Fibonnacci?

(Scratch ends here)

Proof. Consider the set of ways she can cover n steps. We have two cases:

1. Her first stride is 1 step. Then, the number of ways is the number of ways to cover the remaining $n-1$ steps. Thus, this gives us $S(n-1)$ ways.
2. Her first stride is 2 steps. Then the number of ways is the number of ways to cover the remaining $n-2$ steps. Thus, this gives us $S(n-2)$ ways.

Therefore, we conclude that $S(n) = S(n-1) + S(n-2)$. We account the initial conditions and conclude the closed form:

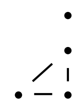
$$S(n) = F_n = f_{n+1}$$

for all n. Since Prof Rad climbs 47 steps, we get $\boxed{S(47) = 4807526976}$. □

A3 Simplex numbers

Definition 1. *Two-dimensional* triangular numbers: $T_2(n) = 1 + 2 + 3 + \cdots + n$

- $T_2(1) = 1$
- $T_2(2) = 1 + 2 = 3$
- …



$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

Theorem 1. $T_2(n) = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

First proof. We prove by induction.

← Not as good of a proof: we must know what we are proving in the first place!

Base case $n = 1$: $T_2(1) = 1$, formula gives $\frac{1(1+1)}{2} = 1$.
Inductive hypothesis: Suppose proved formula for up to $n = k$.
Inductive step: Consider $n = k + 1$.

$$\begin{aligned}
T_2(k+1) &= 1 + \cdots + k + (k+1) \\
&= T_2(k) + k + 1 \\
&= \frac{k(k+1)}{2} + k + 1 \\
&= \frac{k^2 + k + 2(k+1)}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)\left((k+1)+1\right)}{2}
\end{aligned}$$

□

Proof by Gauss. Observe:

← Better proof: concluding the formula without knowing it first!

$$\begin{aligned}
T_2(n) &= 1 + 2 + \cdots + (n-1) + n \\
&= n + (n-1) + \cdots + 2 + 1
\end{aligned}$$

Therefore, we **add** the two rows:

$$
\begin{aligned}
2T_2(n) &= \underbrace{(n+1)+(n+1)+\cdots+(n+1)}_{n} \\
&= n(n+1) \\
\therefore \quad T_2(n) &= \frac{1}{2}n(n+1)
\end{aligned}
$$

□

Definition 2. Tetrahedral numbers: $T_3(n) = T_2(1) + T_2(2) + \cdots + T_2(n)$

- $T_3(5) = 1 + 3 + 6 + 10 + 15 = 35$

Definition 3. Simplex numbers: $T_{k+1}(n) = T_k(1) + \cdots + T_k(n)$

B Ramsey Theory

Invented by Frank Ramsey in 1930. We would need:

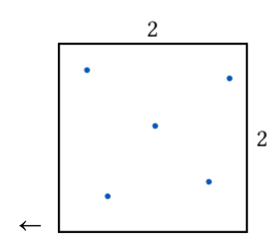
- Graph Theory
- Pigeonhole Principle
- Quantifiers
- Counterexamples

B1 Pigeonhole principle

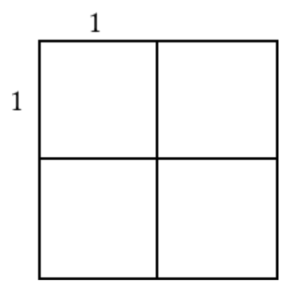
Theorem 2 (Dirichlet's Pigeonhole Principle). If you put $n+1$ pigeons in n pigeonholes, then (at least) one pigeonhole will contain (at least) two pigeons.

Proof omitted. □

Example 3. Given 5 points in a square of side length 2, show that there must exist two points whose mutual distance is $\leq \sqrt{2}$.



Proof. Divide square into 4 smaller squares. We now have 4 pigeonholes and 5 dots:



These two points in the same pigeonhole have distance $\leq \sqrt{1^2+1^2} = \sqrt{2}$. $\square$

Example 4. There exists two people in NYC who have exactly the same number of hairs on their head.

Example 5. There are 30 people at a party talking with each other. Afterwards, there will be two people who talked with the same number of people.

Proof. If we put a person who talked to i people into box i, we get 30 boxes; however, we cannot have someone who talked to 0 people and someone who talked at 29 people at the same time! Hence, we combine the box 0 and box 29, and only one of which could be the case.

Now we have 29 boxes and 30 people. By pigeonhole principle, there must be two people who talked with the same amount of people. $\square$

Theorem 3 (Strong Pigeonhole Principle)**.** Given pigeonholes $1, 2, \ldots, n$ with capacities $c_1, c_2, \ldots, c_n$ where $c_i \geq 0$; if we have at least $c_1 + c_2 + \cdots + c_n + 1$ pigeons in these pigeonholes, then at least one pigeonhole overflows.

Proof. Suppose BWOC that no pigeonhole overflows. Then for all $i = 1, 2, \ldots, n$, we have the number of pigeons in $i \leq c_i$.

We add up and get inequalities:

$$\text{total \# pigeons} \leq c_1 + c_2 + \cdots + c_n$$

Contradiction! $\square$

Example 6. There are five people supporting two teams. Then at least one team is supported by 3 people.

Proof. Assume BWOC that the two teams only have two supporters. Let $c_1 = c_2 = 2$. However, by SPP, $5 \geq 2 + 2 + 1$, hence one pigeonhole overflows. Therefore, one team must have > 2 supporters. $\square$

B2 First Ramsey Theorem

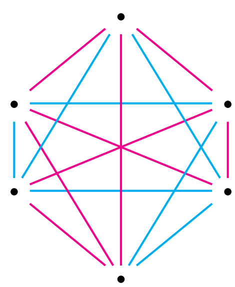
There are 6 people taking a class. Then:

either there exists 3 people such that each pair of them have previously taken a class together,

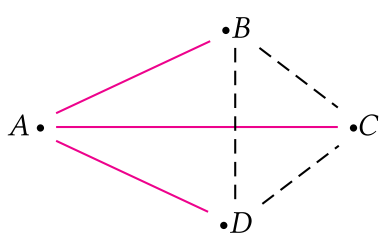
or (inclusive) there exists 3 people such that no two have taken a class together.

Theorem 4. If we have 6 vertices and we draw all edges between them (a K_6 graph), then for every possible way of coloring the edges red and blue, there must exist a **monochromatic** triangle.

← K_6 stands for *complete graph on 6 vertices.* It has 15 edges.



Proof. Pick any vertex and call it A. It has 5 edges colored red and blue. By the SPP, there exists at least 3 edges of the same color. WLOG let these three edges be red and call the other three vertices B, C, D.

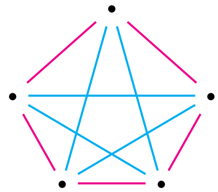


- If BC is red, then ABC is a red triangle.
- If CD is red, then ACD is a red triangle.
- If BD is red, then ABD is a red triangle.
- If none of the above has happened, then BC, CD, BD are all blue, meaning that BCD is a blue triangle!

□

Theorem 5. If there are 5 instead of 6 vertices, then the above coloring prediction cannot be made with certainty.

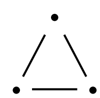
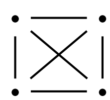
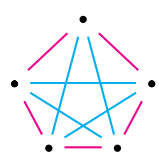
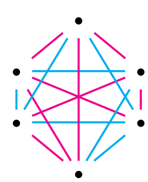
Counterexample.



□

B3 $K_p \to K_q,\ K_r$

In graphy theory, K_n is the **complete** graph on n vertices.

K_1		1 vertex	0 edges
K_2		2 vertices	1 edge
K_3		3 vertices	3 edges
K_4		4 vertices	6 edges
K_5		5 vertices	10 edges
K_6		6 vertices	15 edges

Remark. Note that K_n has $1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$ edges, hence is the $n-1$-th triangular number.

Ramsey Theory uses the following language convention: the expression

$$K_p \to K_q,\ K_r$$

represents a statement with the following meaning:

Definition 4. If the edges of K_p are colored red/blue, then it necessarily follows that either the K_p contains a red K_q, or K_p contains a blue K_r (or possibly both).

We want to know whether this statement is true for a given triple of (p, q, r).

Example 7. We proved in **B2** that $K_6 \to K_3, K_3$.

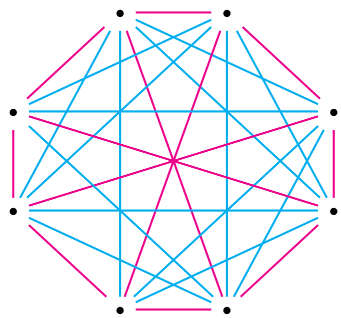
Non-example 8. We also showed that $K_5 \to K_3, K_3$ is false by exhibiting a coloring of K_5 that does not have a red or blue triangle (counterexample).

← write $K_5 \not\to K_3, K_3$

Example 9. It is known that $K_{18} \to K_4, K_4$ and $K_{17} \not\to K_4, K_4$.

Example 10. Also, $K_9 \to K_3, K_4$ and $K_8 \not\to K_3, K_4$.

← Here we have to decide in advance which color goes with the K_3 and which goes with the K_4 due to asymmetry.



This K_8 has no red triangle and no blue K_4.

Theorem 6 (Ramsey)**.** Let q, r be positive integers. Then there always exists a positive integer p such that

$$K_p \to K_q, K_r$$

is true.

We would see the following tabel giving us values of p that work.

Define a function $N(q, r)$ recursively:

← $q, r \in \mathbb{Z}^+$

- Base case: $N(1, r) = N(q, 1) = 1$
- Recurrence: $N(q, r) = N(q-1, r) + N(q, r-1)$ if $q, r \geq 2$.

We compute the value of $N(q, r)$ for:

← They do look like simplex numbers!

$q \backslash r$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	3	4	5	6
3	1	3	6	10	15	21
4	1	4	10	20	35	56
5	1	5	15	35	70	126
6	1	6	21	56	126	252

We would want to prove that $K_{N(q,r)} \to K_q, K_r$ for all $q, r \geq 1$.

Proof. By induction.

Base case: If $q = r = 1$, then $N = 1$, we need to show that

$$K_1 \to K_1, K - r$$
$$\text{and} \quad K_1 \to K_q, K_1$$

for all q, r.
That is, suppose K_1 has its edges colored red/blue, then there exists a red K_1 or a blue K_r, and *vice versa*.
Since there are no edges, this is vacuously true.

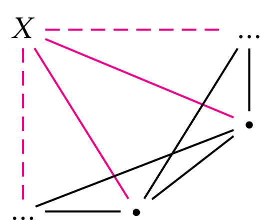
Inductive step: We will show that if we are given that A, B are numbers such that

← This works because as we fill out the table above, each new number we write in will work because it's the sum of the left and above numbers and they both work.

$$K_A \to K_{q-1}, K_r$$
$$\text{and } K_B \to K_q, K_{r-1}$$

are true, then $K_{A+B} \to K_q, K_r$.
Consider K_{A+B} colored red and blue. We will show that it has a red K_q or a blue K_r.
Pick a vertex and call it X. It would have $A + B - 1$ edges in total. We claim that X either has at least A red edges, or at least B blue edges. This is indeed true, because if not, the number of red edges would be $\leq A - 1$ and the number of blue edges would be $\leq B - 1$ and so the total number of edges would be $\leq A + B - 2 < A + B - 1$, which is a contradiction.
Now, if X has a red claw of size A:

← The neighbouring vertices connected by black edges form K_A, yet to be colored.



K_A

From our inductive hypothesis $K_A \to K_{q-1}, K_r$, we must have
either red K_{q-1}, in which case we combine with the vertex X and the red claw to get at least one red K_q.
or blue K_r, in which case we are done.
Similarly, if X has a blue claw of size B, then we make the same argument.

Hence, we know that $K_{A+B} \to K_q, K_r$ is true whenever $K_A \to K_{q-1}, K_r$ and $K_B \to K_q, K_{r-1}$.

□

B4 Ramsey Numbers

Recall: Let m, n be positive integers. We know that there are numbers $p \in \mathbb{N}$ such that $K_p \to K_m, K_n$.

Remark. If p works, then so does any $q \geq p$ as K_q would contain copies of K_p.

So the question becomes, if we have $K_p \to K_m, K_n$, is p the **smallest** such number?

Definition 5. The **Ramsey number** $r(m,n)$ is the smallest such number.

Example 11. We know $K_6 \to K_3, K_3$ but $K_5 \not\to K_3, K_3$, so $r(3,3) = 6$.

Example 12. Mathematicians have proved that

$$K_{48} \to K_5, K_5$$
$$K_{42} \not\to K_5, K_5$$

so we have $43 \leq r(5,5) \leq 48$.

Remark. In general,

$$K_N \to K_m, K_n \iff r(m,n) \leq N$$
$$K_{N-1} \not\to K_m, K_n \iff r(m,n) \geq N$$

Need both to get the precise value of $r(m,n)$.

Proposition 7. Properties of Ramsey numbers:

(a) $r(3,3) = 6$ ← proven in **B2**

(b) $r(m,n) = r(n,m)$ ← symmetry

(c) $r(1,n) = 1$ ← $K_1 \to K_1, K_n$, $K_0 \not\to K_1, K_n$

(d) $r(2,n) = n$

(e) $r(m,n) \leq r(m-1,n) + r(m,n-1)$ for all $m,n \geq 2$

Proof for (d). Claim: $K_2 \to K_2, K_n$, $K_{n-1} \not\to K_2, K_n$.

Color K_n. If all edges are blue then we have a blue K_n. Else we have some red edges, so we have some red K_2.

Now color K_{n-1} all blue: we realize that we don't have any red K_2, but we don't have a blue K_n either! □

Proof for (e). Let $A = r(m-1,n)$, $B = r(m,n-1)$. We have shown that if $K_A \to K_{m-1}, K_n$ and $K_B \to K_m, K_{n-1}$, then $K_{A+B} \to K_m, K_n$. Hence $r(m,n) \leq A + B$. □

Known facts:

$$r(2,2) = 2$$
$$r(3,3) = 6$$
$$r(4,4) = 18$$
$$43 \leq r(5,5) \leq 48$$
$$102 \leq r(6,6) \leq 165$$

B5 A lower bound for $r(m, n)$

Our table of $N(m, n)$ gave us upper bonds for $r(m, n)$. Specifically,

$$r(m, n) \leq N(m, n) = \frac{(n+m-2)!}{(n-1)!(m-1)!} = \binom{m+n-2}{m-1}$$

What about lower bound?

Theorem 8.

$$r(m, n) \geq (m-1)(n-1)+1$$

if and only if $K_{(m-1)(n-1)} \not\to K_m, K_n$

Proof. We prove this by exhibiting a coloring of $K_{(m-1)(n-1)}$ that has no red K_m, no blue K_n.

Place vertices in grid:

$$\begin{array}{cccccc} & \bullet & \bullet & \cdots & \bullet \\ & \bullet & \bullet & \cdots & \bullet \\ m-1 \text{ rows} & \vdots & \vdots & \ddots & \vdots \\ & \bullet & \bullet & \cdots & \bullet \\ & & & n-1 \text{ columns} & \end{array}$$

Coloring rule of edges: If two vertices are in the same row, color the edges blue. If two vertices are in the same column, color the edges red. Every other edge arbitrary.

Claim: there exists no red K_m.

Consider the m vertices of such a K_m. There are $m-1$ rows. Pigeonhole principle ensures that some vertices must be in the same row. But that edge must be blue! So this is not a red K_m. Similarly, there are no blue K_n. $\square$

Thus, we get: $(m-1)(n-1)+1 \leq r(m, n) \leq \dfrac{(n+m-2)!}{(n-1)!(m-1)!} = \dbinom{m+n-2}{m-1}$.

Observe there is still a huge gap between the bounds. Could we get better?

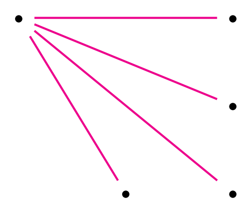
B6 The "parity" improvement

Our methods have shown that $K_{10} \to K_3, K_4$. But it is actually true that $K_9 \to K_3, K_4$. Why?

Proof. Given K_9 colored red or blue. We seek a red K_3 or a blue K_4.

That is to say that if we ever see a red 4-claw, then we are done!

← See this argument.



In addition, if we ever see a blue 6-claw, then we are also done because $K_6 \to K_3, K_3$ and we either have a red K_3 or a blue K_3, which would have to combine with the other vertex to get a blue K_4.

Now suppose we neither have a red 4-claw nor a blue 6-claw. This implies that each vertex has ≤ 3 red edges, and ≤ 5 blue edges. However, in a K_9, each vertex only has 8 edges, so they must exactly each have 3 red edges and 5 blue edges. Does this exist? We realize that to make this happen, we have:

- 9 vertices
- Each vertex has 3 red edges
- Every edge belongs to two vertices

Hence, we need to have exactly $\frac{3 \times 9}{2} = 13.5$ red edges, but this cannot happen because we need a whole number of edges! Thus, it is not possible that we neither have a red 4-claw nor a blue 6-claw. $\square$

Lemma 9 (Ramsey inductive step improved by parity)**.** Suppose

← Also seen here.

$$\begin{aligned} K_A &\to K_{q-1}, K_r \\ \text{and } K_B &\to K_q, K_{r-1} \end{aligned}$$

are true, then $K_{A+B} \to K_q, K_r$.

In addition, if A, B are **both even numbers**, then $K_{A+B-1} \to K_q, K_r$.

B7 Variations

More colors!

For example:

$$K_p \to K_a, K_b, K_c$$

(given K_p colored red, blue, green, it must contain a red K_a, or a blue K_b, or a green K_c.)

Example 13. It is known that $K_{17} \to K_3, K_3, K_3$.

Proof sketch. Pick a vertex which has 16 edges. We observe $16 \div 3 = 5\frac{1}{3} \implies$ at least one color occurs 6 times (i.e. we can see a red/blue/green 6-claws). $\square$

Remark. $r(a, b, c)$ is the smallest number that works for the above.

Other graphs

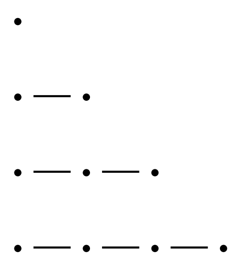
Paths:

P_1

P_2

P_3

P_4



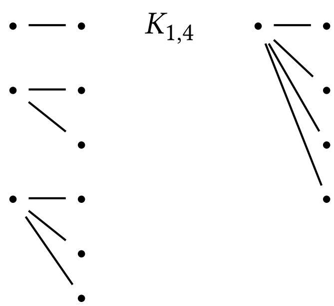
Claws:

$K_{1,1}$

$K_{1,2}$

$K_{1,3}$

$K_{1,4}$



Example 14. Show that $r(K_{1,3}, K_{1,3}) = 6$.

Proof. We know $K_6 \to K_3, K_3$. Pick a vertex that has 5 neighbors. By the strong pigeonhole principle, we must have three edges of the same color $\implies$ either red or blue $K_{1,3}$. $\square$