

# FIBER: A Language with Functions, Integers, Booleans, Eagerness, and Recursion

## 1 INTRODUCTION

FIBER is a toy language for the KAIST CS320 course. FIBER stands for a language with **f**unctions, **i**ntegers, **b**ooleans, **e**agerness<sup>1</sup>, and **r**ecursion. As the name implies, it features integers, booleans, first-class functions, and recursive functions. In addition, it is an eager language. More precisely, FIBER supports the following features:

- integers and booleans
- basic arithmetic operators, including negation, addition, subtraction, multiplication, division, and modulo
- basic relational operators, including equal-to, not-equal-to, less-than, less-than-or-equal-to, greater-than, and greater-than-or-equal-to.
- basic boolean operators, including negation, conjunction, and disjunction
- conditional expressions (if-else expressions)
- tuples of arbitrary lengths greater than one
- projections for tuples
- lists, which are cons or nil
- primitives for lists: isEmpty, nonEmpty, head, and tail
- immutable local variables
- immutable local variable binding via pattern matching on tuples
- first-class functions and function application
- anonymous functions
- mutually recursive functions
- dynamic type tests

This document is the specification of FIBER. First, it gives the syntax of FIBER: Section 2 describes the concrete syntax; Section 3 formalizes the desugaring rules; Section 4 shows the abstract syntax. Second, it defines the semantics of FIBER in a natural language in Section 5. The formal big-step operational semantics of FIBER can be found in A.

## 2 CONCRETE SYNTAX

The concrete syntax of FIBER is written in the **extended Backus–Naur form**. To improve the readability, we use different colors for different kinds of objects. Syntactic elements of the extended Backus–Naur form, rather than FIBER, are written in **purple**. For example, we use **=**, **|**, and **;**. Note that **{ }** denotes a repetition of zero or more times, and **[ ]** denotes an optional existence. Nonterminals are written in **blue**. For example, **expr** is a nonterminal denoting expressions. Any other objects written in black are terminals. For instance, "true" and "false" are terminals representing boolean literals.

The following is the concrete syntax of FIBER:

```
ltr = "A" | "B" | "C" | "D" | "E" | "F" | "G" | "H" | "I" | "J" | "K" | "L"
    | "M" | "N" | "O" | "P" | "Q" | "R" | "S" | "T" | "U" | "V" | "W" | "X"
    | "Y" | "Z" | "a" | "b" | "c" | "d" | "e" | "f" | "g" | "h" | "i" | "j"
```

<sup>1</sup> *Eagerness* denotes the most usual function application semantics, that the arguments of a function application are evaluated before the function body is evaluated.

```

    | "k" | "l" | "m" | "n" | "o" | "p" | "q" | "r" | "s" | "t" | "u" | "v"
    | "w" | "x" | "y" | "z" ;
pdgt = "1" | "2" | "3" | "4" | "5" | "6" | "7" | "8" | "9" ;
dgt  = "0" | pdgt ;

sch  = ltr | "_" ;
ch   = sch | dgt ;
id   = sch {ch} ;

idx  = pdgt {dgt} ;
num  = ["-"] dgt {dgt} ;

expr = id | num | "true" | "false" | "-" expr | "!" expr
      | expr "+" expr | expr "-" expr | expr "*" expr | expr "/" expr
      | expr "%" expr | expr "==" expr | expr "!=" expr | expr "<" expr
      | expr "<=" expr | expr ">" expr | expr ">=" expr | expr "&&" expr
      | expr "||" expr | "if" "(" expr ")" expr "else" expr
      | "(" expr "," expr {"", " id"} ")" | expr "." "-" idx
      | "Nil" | expr "::" expr | expr "." "isEmpty"
      | expr "." "nonEmpty" | expr "." "head" | expr "." "tail"
      | "val" id "=" expr ";" expr
      | "val" "(" id "," id {"", " id"} ")" "=" expr ";" expr
      | "(" ")" "=>" expr | id "=>" expr | "(" id {"", " id"} ")" "=>" expr
      | fdef {fdef} expr
      | expr "(" ")" | expr "(" expr {"", " expr"} ")"
      | expr "." "isInstanceOf" "[" type "]"
      | "(" expr ")" | "{" expr "}" ;

fdef = "def" id "(" ")" "=" expr ";"
      | "def" id "(" id {"", " id"} ")" "=" expr ";" ;

type = "Int" | "Boolean" | "Tuple" | "List" | "Function" ;

```

Note that whitespaces, such as ' ', '\t', and '\n', are omitted from the above specification. You can insert any kinds of whitespaces between any two terminals to make a valid program. For example, since we have `expr = num | "-" expr`, if one parses `-1` and `- 1`, then both will succeed, and the results will be the same. On the other hand, because you cannot insert whitespaces at the middle of terminals, `tr ue` cannot be parsed while `true` can be parsed correctly.

The concrete syntax of FIBER is *ambiguous*. It means that a single string can be parsed in multiple ways. For example, `1 + 2 * 3` can result in both Tree a1 and Tree a2 in Figure 1.

To resolve the ambiguity of the concrete syntax, we define *precedence* between binary operators. If  $op_1$  precedes  $op_2$ , then  $e_1 op_1 e_2 op_2 e_3$  can result in only Tree b1. On the other hand, if  $op_2$  precedes  $op_1$ , Tree b2 is the only possible result.

Figure 2 shows precedence. One appearing earlier in the table precedes one appearing later. For example, since `*` precedes `+`, `1 + 2 * 3` is parsed to only Tree a2. Operators in the same box of the table have the same precedence. If they appear in a single expression, then one appearing first in the expression has the higher precedence in the expression. For instance, `1 - 2 + 3` results in Tree c because `-` and `+` have the same precedence, but `-` appears first in the expression.

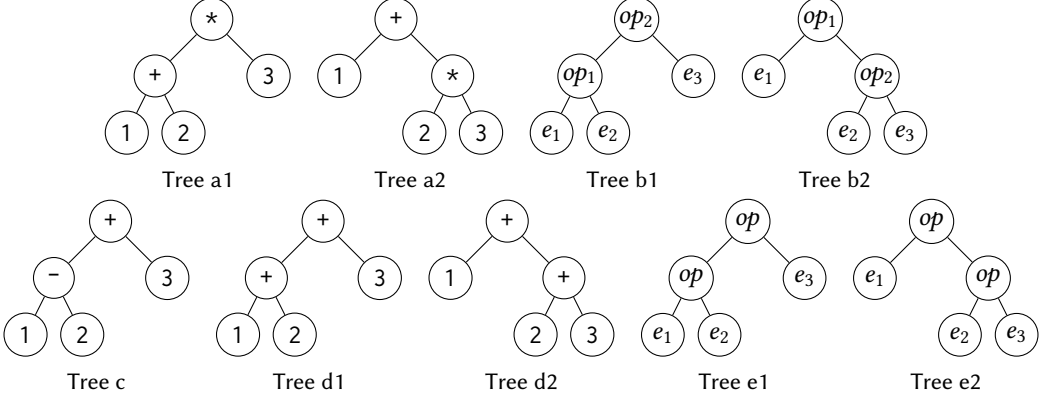


Fig. 1. Parse Trees

		==			
*		!=			
/	+	<	&&		::
%	-	<=			
		>			
		>=			
← higher			lower		

Fig. 2. Operator Precedence

Alas, precedence is not enough to resolve the ambiguity. We have problems when an operator appears more than once in an expression. For example,  $1 + 2 + 3$  can result in both Tree d1 and Tree d2.

We introduce *associativity* of binary operators to solve the problem. A binary operator can be either left-associative or right-associative. If  $op_1$  is left-associative, then  $e_1 op e_2 op e_3$  can result in only Tree e1. On the other hand, if  $op$  is right-associative, Tree e2 is the only possible result. In FIBER, all the binary operators except  $::$  are left-associative. Only  $::$  is right-associative. Thus,  $1 + 2 + 3$  is parsed to only Tree d1.

### 3 DESUGARING

To simplify the implementation of the interpreting phase, the parsing phase of the interpreter desugars a given expression. Desugaring rewrites some subexpressions with other expressions. Due to desugaring, the abstract syntax of FIBER consists of less sorts of expressions than the concrete syntax.

Figure 3 defines desugaring of FIBER expressions. Let  $e$  and  $x$  respectively denote an expression and an identifier. An expression  $e$  is desugared to  $\llbracket e \rrbracket$ . For example, after desugaring,  $-(1 + 2)$  becomes  $(1 + 2) * -1$ . Note that lines under identifiers imply that the identifiers must be fresh, i.e. have different names from existing ones. For instance, if the entire program is  $1 <= 2$ , then below is a valid desugaring result.

```
val x = 1;
val y = 2;
```

$\llbracket -e \rrbracket = \llbracket e \rrbracket * -1$	$\llbracket e_1 \ \&\& \ e_2 \rrbracket = \text{if } (\llbracket e_1 \rrbracket) \ \llbracket e_2 \rrbracket \ \text{else } \text{false}$
$\llbracket !e \rrbracket = \text{if } (\llbracket e \rrbracket) \ \text{false} \ \text{else } \text{true}$	$\llbracket e_1 \    \ e_2 \rrbracket = \text{if } (\llbracket e_1 \rrbracket) \ \text{true} \ \text{else } \llbracket e_2 \rrbracket$
$\llbracket e_1 - e_2 \rrbracket = \llbracket e_1 \rrbracket + \llbracket -e_2 \rrbracket$	$\llbracket e.\text{nonEmpty} \rrbracket = \llbracket !(\llbracket e \rrbracket.\text{isEmpty}) \rrbracket$
$\llbracket e_1 != e_2 \rrbracket = \llbracket !(\llbracket e_1 \rrbracket == \llbracket e_2 \rrbracket) \rrbracket$	$\llbracket \text{val } (x_1, \dots, x_i) = e_1; e_2 \rrbracket = \text{val } \underline{x} = \llbracket e_1 \rrbracket;$ $\text{val } x_1 = x._1;$ $\dots$ $\text{val } x_i = x._i;$ $\llbracket e_2 \rrbracket$
$\llbracket e_1 <= e_2 \rrbracket = \text{val } \underline{x_1} = \llbracket e_1 \rrbracket;$ $\text{val } \underline{x_2} = \llbracket e_2 \rrbracket;$ $\llbracket x_1 == x_2 \    \ x_1 < x_2 \rrbracket$	
$\llbracket e_1 > e_2 \rrbracket = \llbracket !(\llbracket e_1 \rrbracket <= \llbracket e_2 \rrbracket) \rrbracket$	$\llbracket (e) \rrbracket = \llbracket e \rrbracket$
$\llbracket e_1 >= e_2 \rrbracket = \llbracket !(\llbracket e_1 \rrbracket < \llbracket e_2 \rrbracket) \rrbracket$	$\llbracket \{e\} \rrbracket = \llbracket e \rrbracket$

Any other cases recursively desugar their subexpressions.

Fig. 3. Desugaring Rules

Identifier	$x$	$\in Id$	Expression	$e ::= x$	(variable)
				$  n$	(integer)
Index	$i$	$\in \mathbb{Z}^+$		$  b$	(boolean)
				$  e + e$	(addition)
Number	$n$	$\in \mathbb{Z}$		$  e \times e$	(multiplication)
				$  e \div e$	(division)
Boolean	$b$	$::= \text{true}$ $  \text{false}$		$  e \bmod e$	(modulo)
				$  e = e$	(equal-to)
Function	$d$	$::= \text{def } x(x, \dots, x) = e$		$  e < e$	(less-than)
				$  \text{if } e \ e \ e$	(conditional)
Types	$\tau$	$::= \text{Int}$ $  \text{Boolean}$ $  \text{Tuple}$ $  \text{List}$ $  \text{Function}$		$  (e, \dots, e)$	(tuple; length > 1)
				$  e.i$	(projection)
				$  \text{Nil}$	(nil)
				$  e :: e$	(cons)
				$  e.\text{isEmpty}$	(is-empty)
				$  e.\text{head}$	(head)
				$  e.\text{tail}$	(tail)
				$  \text{val } x=e \text{ in } e$	(local variable)
				$  \lambda x \dots x. e$	(anonymous function)
				$  d \dots d \ e$	(recursive function)
				$  e(e, \dots, e)$	(function application)
				$  e \text{ is } \tau$	(type test)

Fig. 4. Abstract Syntax

$x == y \ || \ x < y$

Since the desugaring rules are defined recursively, they can handle complex programs correctly. For example,  $--(1 + 2)$  becomes  $(1 + 2) * -1 * -1$  instead of  $-(1 + 2) * -1$  after desugaring.

#### 4 ABSTRACT SYNTAX

Figure 4 describes the abstract syntax of FIBER. Metavariable  $x$  ranges over identifiers;  $i$  ranges over indices of tuples, which are positive integers;  $n$  ranges over integers;  $b$  ranges over boolean literals, which are either true or false;  $e$  ranges over expressions;  $d$  ranges over recursive function definitions;  $\tau$  ranges over types, which are either Int, Boolean, Tuple, List, or Function.

The following briefly describes expressions:

- $e_1 + e_2$ ,  $e_1 \times e_2$ ,  $e_1 \div e_2$ ,  $e_1 \bmod e_2$ ,  $e_1 = e_2$ , and  $e_1 < e_2$  are binary operations on integers.
- if  $e_1 \ e_2 \ e_3$  is a conditional expression.
- $(e_1, \dots, e_i)$  creates a tuple of length  $i$ . Length  $i$  must be greater than one.
- $e.i$  is a projection from a tuple. The beginning index is one.
- Nil creates the empty list.
- $e_1 :: e_2$  creates a nonempty list.
- $e.\text{isEmpty}$ ,  $e.\text{head}$ , and  $e.\text{tail}$  are unary operations on a list.
- $\text{val } x = e_1 \text{ in } e_2$  defines a local variable whose name is  $x$  and scope is  $e_2$ .
- $\lambda x_1 \dots x_i. e$  defines an anonymous function whose parameters are  $x_1, \dots, x_i$  and body is  $e$ . The names of the parameters must be distinct from each other.
- $\text{def } x(x_1, \dots, x_i) = e$  defines a (possibly recursive) function whose name is  $x$ , parameters are  $x_1, \dots, x_i$ , and body is  $e$ . The names of the parameters must be distinct from each other.
- $d_1 \dots d_i \ e$  defines functions from  $d_1$  to  $d_i$ . The names of the functions must be distinct from each other. They can be mutually recursive and used in  $e$ .
- $e(e_1, \dots, e_i)$  is a function application.  $e$  is a function;  $e_1, \dots, e_i$  are arguments.
- $e \text{ is } \tau$  tests the type of a given value.

#### 5 SEMANTICS

This section explains the semantics of FIBER in a natural language. See Appendix A to find the formal big-step operational semantics.

To explain the semantics, we need the definition of a value. A value is one of the following:

- an integer
- a boolean
- a tuple whose length is greater than one and elements are values
- the empty list
- a nonempty list, which consists of a value and a (empty or nonempty) list
- a closure, which is a function with an environment

In this section, we use the following metavariables and terminologies:

- Metavariable  $v$  ranges over values.
- Metavariable  $\sigma$  ranges over environments, which are maps from identifiers to values.
- If we say “the result is  $v$ ” while explaining evaluation of  $e$ , then  $v$  is the result of  $e$ .
- We use the word “must” to represent requirements. If a requirement is violated, then a run-time error occurs. Any occurrence of a run-time error immediately terminates the execution.

The following explains how each expression is evaluated.

**Case  $x$ :**

- (1) Let  $\sigma$  be the current environment.
- (2)  $x$  must be in the domain of  $\sigma$ .
- (3) The result is  $\sigma(x)$ .

**Case  $n$ :**

(1) The result is  $n$ .

**Case  $b$ :**

- (1) The result is  $b$ .

**Case  $e_1 \oplus e_2$ :**

- (\*) Suppose that  $\oplus \in \{+, \times, \div, \bmod, =, <\}$ .

- (1) Evaluate  $e_1$ .
- (2) Let  $v_1$  be the result of  $e_1$ .
- (3) Evaluate  $e_2$ .
- (4) Let  $v_2$  be the result of  $e_2$ .
- (5)  $(v_1, v_2)$  must be in the domain of  $\oplus$ . Note that  $+, \times \in (\mathbb{Z}, \mathbb{Z}) \rightarrow \mathbb{Z}$ ,  $\div, \text{mod} \in (\mathbb{Z}, \mathbb{Z} \setminus \{0\}) \rightarrow \mathbb{Z}$ , and  $=, < \in (\mathbb{Z}, \mathbb{Z}) \rightarrow \{\text{true}, \text{false}\}$ .
- (6) The result is  $v_1 \oplus v_2$ .

**Case** if  $e_1 \ e_2 \ e_3$ :

- (1) Evaluate  $e_1$ .
- (2) Let  $v_1$  be the result of  $e_1$ .
- (3)  $v_1$  must be a boolean.
- (4) If  $v_1$  is true, then
  - (a) Evaluate  $e_2$ .
  - (b) Let  $v_2$  be the result of  $e_2$ .
  - (c) The result is  $v_2$ .
- (5) If  $v_1$  is false, then
  - (a) Evaluate  $e_3$ .
  - (b) Let  $v_3$  be the result of  $e_3$ .
  - (c) The result is  $v_3$ .

**Case**  $(e_1, \dots, e_i)$ :

- (1) Evaluate  $e_1$ .
- (2) Let  $v_1$  be the result of  $e_1$ .
- (3) Evaluate  $e_{k+1}$  in the same manner after evaluating  $e_k$ .
- (4) Repeat (3) until  $e_i$  is evaluated.
- (5) The result is a tuple consisting of the values from  $v_1$  to  $v_i$ .

**Case**  $e.i$ :

- (1) Evaluate  $e$ .
- (2) Let  $v$  be the result of  $e$ .
- (3)  $v$  must be a tuple whose length is greater than or equal to  $i$ .
- (4) The result is the  $i$ th element of  $v$ . Note the the beginning index is one.

**Case** Nil:

- (1) The result is the empty list.

**Case**  $e_1 :: e_2$ :

- (1) Evaluate  $e_1$ .
- (2) Let  $v_1$  be the result of  $e_1$ .
- (3) Evaluate  $e_2$ .
- (4) Let  $v_2$  be the result of  $e_2$ .
- (5)  $v_2$  must be either the empty list or a nonempty list.

- (6) The result is a nonempty list whose head is  $v_1$  and tail is  $v_2$ .

**Case**  $e.\text{isEmpty}$ :

- (1) Evaluate  $e$ .
- (2) Let  $v$  be the result of  $e$ .
- (3)  $v$  must be either the empty list or a nonempty list.
- (4) If  $v$  is the empty list, then
  - (a) The result is true.
- (5) Else if  $v$  is a nonempty list, then
  - (a) The result is false.

**Case**  $e.\text{head}$ :

- (1) Evaluate  $e$ .
- (2) Let  $v$  be the result of  $e$ .
- (3)  $v$  must be a nonempty list.
- (4) The result is the head of  $v$ .

**Case**  $e.\text{tail}$ :

- (1) Evaluate  $e$ .
- (2) Let  $v$  be the result of  $e$ .
- (3)  $v$  must be a nonempty list.
- (4) The result is the tail of  $v$ .

**Case**  $\text{val } x = e_1 \text{ in } e_2$ :

- (1) Evaluate  $e_1$ .
- (2) Let  $v_1$  be the result of  $e_1$ .
- (3) Add a mapping from  $x$  to  $v_1$  to the current environment.
- (4) Let  $\sigma_{\text{new}}$  be the new environment.
- (5) Evaluate  $e_2$  under  $\sigma_{\text{new}}$ .
- (6) Let  $v_2$  be the result of  $e_2$ .
- (7) The result is  $v_2$ .

**Case**  $\lambda x_1 \dots x_i. e$ :

- (1) Let  $\sigma$  be the current environment.
- (2) The result is a closure whose parameters are from  $x_1$  to  $x_i$ , body is  $e$ , and environment is  $\sigma$ .

**Case**  $d_1 \dots d_i \ e$ :

- (1) Let  $x_1, \dots, x_i$  be the names of  $d_1, \dots, d_i$ .
- (2) Let  $v_1, \dots, v_i$  be the closures of  $d_1, \dots, d_i$ . If  $d_j$  equals  $\text{def } x_j(x_{j1}, \dots, x_{jk}) = e_j$ , then  $v_j$  consists of the parameters  $x_{j1}, \dots, x_{jk}$  and the body  $e_j$ .
- (3) Add a mapping from  $x$ 's to  $v$ 's to the current environment.
- (4) Let  $\sigma_{\text{new}}$  be the new environment.

- (5) The environment of every  $v_k$  needs to be  $\sigma_{new}$ .
- (6) Evaluate  $e$  under  $\sigma_{new}$ .
- (7) The result is  $v$ .
- (12) Add a mapping from  $x$ 's to  $v$ 's to  $\sigma_c$ .
- (13) Let  $\sigma_{new}$  be the new environment.
- (14) Evaluate  $e_c$  under  $\sigma_{new}$ .
- (15) Let  $v_c$  be the result of  $e_c$ .
- (16) The result is  $v_c$ .

**Case**  $e(e_1, \dots, e_i)$ :

- (1) Evaluate  $e$ .
- (2) Let  $v$  be the result of  $e$ .
- (3)  $v$  must be a closure.
- (4) Evaluate  $e_1$ .
- (5) Let  $v_1$  be the result of  $e_1$ .
- (6) Evaluate  $e_{k+1}$  in the same manner after evaluating  $e_k$ .
- (7) Repeat (6) until  $e_i$  is evaluated.
- (8) The number of parameters must equal the number of arguments.
- (9) Let  $x_1, \dots, x_i$  be the names of the parameters of  $v$ .
- (10) Let  $e_c$  be the body of  $v$ .
- (11) Let  $\sigma_c$  be the environment of  $v$ .

**Case**  $e$  is  $\tau$ :

- (\*) The type of a value is as the following:
  - The type of an integer is Int.
  - The type of a boolean is Boolean.
  - The type of a tuple is Tuple.
  - The type of the empty list is List.
  - The type of a nonempty list is List.
  - The type of a closure is Function.
- (1) Evaluate  $e$ .
- (2) Let  $v$  be the result of  $e$ .
- (3) If the type of  $v$  is  $\tau$ , then
  - (a) The result is true.
- (4) If the type of  $v$  is not  $\tau$ , then
  - (a) The result is false.

The division and modulo operations are defined as the following, which is the same as the semantics of many real-world languages:

- If  $n_1 \geq 0$  and  $n_2 > 0$ , then  $n_1 \div n_2$  is the quotient when  $n_1$  is divided by  $n_2$ .
- If  $n_1 \geq 0$  and  $n_2 < 0$ , then the  $n_1 \div n_2$  is the negation of the quotient when  $n_1$  is divided by  $-n_2$ .
- If  $n_1 < 0$  and  $n_2 > 0$ , then the  $n_1 \div n_2$  is the negation of the quotient when  $-n_1$  is divided by  $n_2$ .
- If  $n_1 < 0$  and  $n_2 < 0$ , then the  $n_1 \div n_2$  is the quotient when  $-n_1$  is divided by  $-n_2$ .
- If  $n_1 \geq 0$ , then the  $n_1 \bmod n_2$  is the remainder when  $n_1$  is divided by  $|n_2|$ .
- If  $n_1 < 0$ , then the  $n_1 \bmod n_2$  is the negation of the remainder when  $-n_1$  is divided by  $|n_2|$ .

### 5.1 Interpreter Specification

An interpreter of FIBER must satisfy the following conditions:

- It provides a function named `interp` that takes a FIBER expression as an argument and returns a FIBER value.
- If  $v$  is the result of  $e$  under the empty environment, then `interp( $e$ )` equals  $v$ .
- If the evaluation of  $e$  under the empty environment terminates due to a run-time error, then `interp( $e$ )` throws an exception. Error messages are implementation-dependent.
- If the evaluation of  $e$  runs forever under the empty environment, then `interp( $e$ )` runs forever or terminates due to stack overflow.

A reference interpreter of FIBER is available on <https://plrg.kaist.ac.kr/fiber>.

## A BIG-STEP OPERATIONAL SEMANTICS

Value	$v$	$\in \mathbb{V}$
	$v$	$::= n \mid b \mid (v, \dots, v) \mid \text{Nil} \mid v :: v \mid \langle \lambda x \dots x.e, \sigma \rangle$
Environment	$\sigma$	$\in Id \xrightarrow{\text{fin}} \mathbb{V}$

Fig. 5. Values and Environments

$\boxed{\text{type}(v) = \tau}$	$\text{type}(n) = \text{Int}$	$\text{type}(b) = \text{Boolean}$
	$\text{type}((v_1, \dots, v_i)) = \text{Tuple}$	$\text{type}(\text{Nil}) = \text{List}$
	$\text{type}(v_1 :: v_2) = \text{List}$	$\text{type}(\langle \lambda x_1 \dots x_i.e, \sigma \rangle) = \text{Function}$

Fig. 6. Types of Values

$\boxed{\sigma \vdash e \Rightarrow v}$	$\frac{x \in \text{Domain}(\sigma)}{\sigma \vdash x \Rightarrow \sigma(x)}$	$\sigma \vdash n \Rightarrow n$	$\sigma \vdash b \Rightarrow b$
$\frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 + e_2 \Rightarrow n_1 + n_2}$	$\frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 \times e_2 \Rightarrow n_1 \times n_2}$		
$\frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2 \quad n_2 \neq 0}{\sigma \vdash e_1 \div e_2 \Rightarrow n_1 \div n_2}$	$\frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2 \quad n_2 \neq 0}{\sigma \vdash e_1 \bmod e_2 \Rightarrow n_1 \bmod n_2}$		
$\frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 = e_2 \Rightarrow n_1 = n_2}$	$\frac{\sigma \vdash e_1 \Rightarrow n_1 \quad \sigma \vdash e_2 \Rightarrow n_2}{\sigma \vdash e_1 < e_2 \Rightarrow n_1 < n_2}$		
$\frac{\sigma \vdash e_1 \Rightarrow \text{true} \quad \sigma \vdash e_2 \Rightarrow v}{\sigma \vdash \text{if } e_1 \text{ } e_2 \text{ } e_3 \Rightarrow v}$	$\frac{\sigma \vdash e_1 \Rightarrow \text{false} \quad \sigma \vdash e_3 \Rightarrow v}{\sigma \vdash \text{if } e_1 \text{ } e_2 \text{ } e_3 \Rightarrow v}$		
$\frac{\sigma \vdash e_1 \Rightarrow v_1 \quad \dots \quad \sigma \vdash e_i \Rightarrow v_i}{\sigma \vdash (e_1, \dots, e_i) \Rightarrow (v_1, \dots, v_i)}$	$\frac{\sigma \vdash e \Rightarrow (v_1, \dots, v_i, \dots, v_{i'})}{\sigma \vdash e.i \Rightarrow v_i}$		
$\sigma \vdash \text{Nil} \Rightarrow \text{Nil}$	$\frac{\sigma \vdash e_1 \Rightarrow v_1 \quad \sigma \vdash e_2 \Rightarrow v_2 \quad \text{type}(v_2) = \text{List}}{\sigma \vdash e_1 :: e_2 \Rightarrow v_1 :: v_2}$		

Fig. 7. Evaluation of Expressions (1/2)



$$\begin{array}{c}
\frac{\sigma \vdash e \Rightarrow \text{Nil}}{\sigma \vdash e.\text{isEmpty} \Rightarrow \text{true}} \qquad \frac{\sigma \vdash e \Rightarrow v_1 :: v_2}{\sigma \vdash e.\text{isEmpty} \Rightarrow \text{false}} \\
\\
\frac{\sigma \vdash e \Rightarrow v_1 :: v_2}{\sigma \vdash e.\text{head} \Rightarrow v_1} \qquad \frac{\sigma \vdash e \Rightarrow v_1 :: v_2}{\sigma \vdash e.\text{tail} \Rightarrow v_2} \\
\\
\frac{\sigma \vdash e_1 \Rightarrow v_1 \quad \sigma[x \mapsto v_1] \vdash e_2 \Rightarrow v_2}{\sigma \vdash \text{val } x = e_1 \text{ in } e_2 \Rightarrow v_2} \qquad \sigma \vdash \lambda x_1 \cdots x_i. e \Rightarrow \langle \lambda x_1 \cdots x_i. e, \sigma \rangle \\
\\
\frac{
\begin{array}{c}
d_1 = \text{def } x_1(x_{11}, \dots, x_{1i_1}) = e_1 \quad \cdots \quad d_i = \text{def } x_i(x_{i1}, \dots, x_{ii_i}) = e_i \\
v_1 = \langle \lambda x_{11} \cdots x_{1i_1}. e_1, \sigma' \rangle \quad \cdots \quad v_i = \langle \lambda x_{i1} \cdots x_{ii_i}. e_i, \sigma' \rangle \\
\sigma' = \sigma[x_1 \mapsto v_1, \dots, x_i \mapsto v_i] \quad \sigma' \vdash e \Rightarrow v
\end{array}
}{\sigma \vdash d_1 \cdots d_i e \Rightarrow v} \\
\\
\frac{
\begin{array}{c}
\sigma \vdash e \Rightarrow \langle \lambda x_1 \cdots x_i. e', \sigma' \rangle \quad \sigma \vdash e_1 \Rightarrow v_1 \quad \cdots \quad \sigma \vdash e_i \Rightarrow v_i \\
\sigma'[x_1 \mapsto v_1, \dots, x_i \mapsto v_i] \vdash e' \Rightarrow v
\end{array}
}{\sigma \vdash e(e_1, \dots, e_i) \Rightarrow v} \\
\\
\frac{\sigma \vdash e \Rightarrow v}{\sigma \vdash e \text{ is } \tau \Rightarrow \text{type}(v) = \tau}
\end{array}$$

Fig. 8. Evaluation of Expressions (2/2)