6. UFD on BVP.

Consider ODE $-u''+u=x,\ \forall x\in(0,1),\ u(0)=u(1)=0.$

a) Prove that $u(x)=x-\dfrac{\exp(x+1)-\exp(x-1)}{1-\exp(-2)}$

is unique solution

we need to know $-u''+u=0$.

then we can get the answer $u(x)=C_1e^{x}+C_2e^{-x}$.

$C_1, C_2\in\mathbb{R}$,

$\Rightarrow$ for the ODE of $u(x)=x$'s solution is $-u''+u=x$

$\Rightarrow u(x)=x+C_1e^{x}+C_2e^{-x}$

$$\Rightarrow\begin{cases} C_1+C_2=0 \\ 1+C_1e+C_2e^{-1}=0 \end{cases}$$

as $u(1)=u(0)=0$.

$$\Rightarrow\ C_1=\frac{-e^{-1}}{1-e^{-2}}$$

$$C_2=\frac{e^{-1}}{1-e^{-2}}$$

then the solution is $u(x)=x-\dfrac{\exp(x-1)-\exp(x+1)}{1-\exp(-2)}$.

b) Using the upwind finite difference scheme, find the matrix $L^h$, and vector $R^hf$.

$$L^hu^h=R^hf.$$

$\Rightarrow$ since UFD $u_i\approx\begin{cases}\delta_h u_i=\dfrac{u^h_{i+1}-u^h_i}{h} & b(x)<0\\[2mm] \delta_{-h}u_i=\dfrac{u^h_i-u^h_{i-1}}{h} & b(x)>0.\end{cases}$

$\Rightarrow bu'=(b^+-b^-)u'$

$\approx b^+(\delta_{-h}u)-b^-(\delta_h u)$

$\Rightarrow$ then we can get. $-u^h_{i-1}\frac{1}{h^2}+u^h_i\left(\frac{2}{h^2}+1\right)-u^h_{i+1}\frac{1}{h^2}=\hat{f}_i$.

Assume $x=\frac{1}{h^2}$. $y=\frac{2}{h^2}+1$ $z=\frac{1}{h^2}$.

$$\Rightarrow\begin{cases} u^h_N=0. \\ u^h_0=0 \\ -xu^h_{i-1}+yu^h_i-zu^h_{i+1}=\hat{f}_i \end{cases}$$

$$\Rightarrow\quad L^h=\begin{bmatrix}1&0&0&0&0&0\\-r&s&-r&0&0&0\\0&-r&s&-r&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&-r&s&-r\\0&0&0&0&0&0\end{bmatrix}.$$

is matrix of $(N+1)(N+1)$

and $(N+1)$ vector is $R^h\hat{f}=(0,\frac{1}{N},\frac{2}{N},\dots,\frac{N-1}{N},0)^T$

Then for BVP. $i=1,2,\dots,N-1$.

$$\frac{-u^h_{i+1}-2u^h_i+u^h_{i-1}}{h^2}+b^+\frac{u^h_i-u^h_{i-1}}{h}-b^-\frac{u^h_{i+1}-u^h_i}{h}+Cu^h_i=\hat{f}_i.$$

$$\Rightarrow\begin{cases} u^h_0=0 \\ -ru^h_{i-1}+su^h_i-tu^h_{i+1}=\hat{f}_i. & \text{for } i=1,2,\dots,N-1 \\ u^h_N=0 \end{cases}$$

let $r=\frac{1}{h^2}+\frac{b^+}{h}$

$s=\frac{2}{h^2}+\frac{|b|}{h}+c.$

$t=\frac{1}{h^2}+\frac{b^-}{h}$