example 3. ①

based on proposition 1 that $MSE(\hat{\alpha}) = |Bias(\hat{\alpha})|^2 + Var(\hat{\alpha})$. if $\hat{\alpha}$ is unbiased, then MSE is variance.

then, we can get $\mathbb{E}(\beta_N) \neq Var(\alpha_i)$

$$\mathbb{E}[\beta_N] = \mathbb{E}[\frac{1}{N}\sum(\alpha_i - \hat{\alpha})^2] = \mathbb{E}[\frac{1}{N}\sum((\alpha_i - \mu) - (\hat{\alpha} - \mu))^2]$$

$$\mathbb{E}[\frac{1}{N}\cdot\sum(\alpha_i - \mu)^2 - \frac{1}{N}\cdot 2(\hat{\alpha} - \mu)\sum(\alpha_i - \mu) + \cancel{\tfrac{1}{N}}(\hat{\alpha} - \mu)^2]$$

$$\Rightarrow \hat{\alpha} - \mu = \frac{1}{N}\sum\alpha_i - \frac{1}{N}\sum\mu = \frac{1}{N}\sum\alpha_i - \mu.$$

$$\mathbb{E}(\beta_N) = \sigma^2 - \frac{1}{n}\sigma^2$$

$$= \frac{1}{n}(n-1)\cdot Var(\alpha_i)$$

then we can get β_N is biased.

③ $\mathbb{E}[\beta_N] = Var(\alpha_i)$ (we need to use $\frac{1}{N-1}$ to replace $\frac{1}{N}$) $\longrightarrow \mathbb{E}[\beta_N] = \frac{N-1}{N-1}\sigma^2$

$= Var[\alpha_i]$.

$$\mathbb{E}(\beta_N) = \frac{1}{N-1}\mathbb{E}[\sum(\alpha_i - \mathbb{E}(\alpha_i))^2 - N(\mathbb{E}(\alpha_i) - \hat{\alpha})^2]$$

$$\Rightarrow \frac{1}{N-1}\sum Var(\hat{\alpha}_i) - \frac{N}{N-1}Var(\hat{\alpha}) = Var(\alpha_i)$$

Algorithm 1 MC estimation of π

procedure MCPI(N) ▷ N is total number of samples
  $n \leftarrow 0$ ▷ n is number of hits
  for $i = 1...N$ **do**
    generate two numbers X, Y from $U(-1,1)$
    if $X^2 + Y^2 < 1$ **then** $n \leftarrow n + 1$
  return $\frac{4n}{N}$

Example 4 • *Use β_{100} of Example 3 to estimate $MSE(\hat{\pi}_N)$ by repeating π_N of Example 1. One must write both pseudocode and python code.*

• *Repeat above estimation of $MSE(\hat{\pi}_N)$ for $N = 2^i : i = 5, ...10$ and plot log-log chart.*

Algorithm 4. MC. estimation of $MSE(\hat{\pi}_N)$.

1. procedure MCPI pi (n, N)
    M = [ ]
    m ← a

    for i = 1, 2 ... N. do
        x = MCpi (N)
        M.append (x)
        m = m + x.

  y ← 0.
  for i = 1, 2 ... N. do
      $y = y + (M[i] - \frac{m}{N})^2$
  return $\frac{y}{N}$.

Example 2).

3.2 Evaluation of integral

Back to our Example 1, we write

$$\alpha = \mathbb{E}[X] = \mathbb{E}[h(Y)],$$

where $X = h(Y)$ and $Y \sim U(0,1)$. In other words, although X-sampling is not directly available in python, one can use $U(0,1)$ random generator (see *numpy.random.uniform*) to produce Y_i, then compute $h(Y_i)$ for the sample X_i.

Algorithm 1 Integral by MC - Example 1

```
procedure MCINTEGRAL(N)                          ▷ N is total number of samples
    s ← 0                                        ▷ s is the sum of samples
    for i = 1...N do
        generate two numbers Y from U(0,1)       ▷ use numpy.random.uniform
        s ← s + h(Y)
    return s/N                                   ▷ return the average
```

Example 2 • *Implement Algo 1 for estimator $\hat{\alpha}_N$;*

- *Estimate $MSE(\hat{\alpha}_N)$ for $N = 2^5 \ldots, 2^{10}$ and plot log-log chart.*

Algorithm 1.

1. procedure. MCintegral (N)
2. $S \leftarrow 0$
3. for i = 1. 2.. N do.
4. generate one # Y from U(0.1)
5. $S \leftarrow S + Y$.

   $Y \leq 0.01$.

return $\frac{S}{N}$.