1. $V = E^Q[h(S_T)]$

$= E^Q[I(S_T < S_0 e^{-b})]$

$= Q(S_T < S_0 e^{-b})$

$= Q(uT + \sigma W_T < -b)$

$= Q\left(W_T/\sqrt{T} < \frac{-b-uT}{\sigma\sqrt{T}}\right)$

$= \Phi\left(\frac{-b-uT}{\sigma\sqrt{T}}\right) = \Phi(-2)$.

$= 0.0228$.

4. $\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} I(x_i < -2)$

$x_i \sim N(0,1)$ under Q

$E^Q[\hat{\alpha}] = \frac{1}{n}\sum_{i=1}^{n} E^Q[I(x_i < -2)]$

$= \frac{1}{n}\sum_{i=1}^{n} Q(x_i < -2)$

$= \frac{1}{n}\sum_{i=1}^{n} Q(x_i < -2)$

$= \frac{1}{n}\cdot n \cdot \Phi(-2)$

$= \Phi(-2)$

$MSE(\hat{\alpha}) = \cancel{4}\, var(\hat{\alpha})$

$= \frac{1}{n} var(I(x < -2))$

$= \frac{1}{n}[\Phi(-2) - \Phi(-2)^2]$

$= 0.02223/n$.

$\lim_{n\to\infty} MSE(\hat{\alpha}) = 0 \implies$ is optimal

For IS(2)

$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n}[I(x_i < -2)\exp(\frac{1}{2}\cdot 4 + 2x_i)]$

$x_i \sim N(-2,1)$ under Q,

$E[\hat{\alpha}] = \Phi(-2)$

$MSE(\hat{\alpha}) = var(\hat{\alpha}) = E[I^2(x < -2)\exp(4+4x)] - E[I(x < -2)\exp(2+2x)]^2$.

$E^Q[I^2(x < -2)\exp(4+4x)] = E^Q\left[I^2(x < -2)\frac{\exp(4+4x)}{E^Q[\exp(4+4x)]}\right]E^Q[\exp(4+4x)]$.

since $4+4x \sim N(-4, 16)$ $= \cancel{E^Q[I^2(x<-2)]}\ E^{Q^T}[I^2(x < -2)e^4]$

$= Q^T(x < -2)e^4$

$\frac{dQ^T}{dQ} = \frac{\exp(4+4x)}{E^Q[\exp(4+4x)]} = e^{-4+4+\int_0^1 4dW^Q \cancel{+\int_0^1 \ldots} - \int_0^1 8dt}$.

$\implies Q = 4$.

$dW^{Q^T} = dW^Q - 4dt$

$\implies x = \int_0^1 (dW^{Q^T} + 4dt) - \int_0^1 2dt = \int_0^1 1dW^{Q^T} + \int_0^1 2dt$

$\sim N(2, 1)$.

~~MSE = dW^Q - 4dt~~

$\implies MSE(\hat{\alpha}) = \frac{1}{n}(Q^T(x < -4)e^4 - (\Phi(-2))^2)$

$= \frac{1}{n}[\Phi(-4)^{e4} - (\Phi(-2))^2]$

$= 0.00121/n$.

$\lim_{n\to\infty} MSE(\hat{\alpha}) = 0$

$\implies$ IS(2) is optional.

5. $MSE_{\text{ORC}} = \frac{0.02223}{n}$

$= MSE_{IS(2)} = 0.00121/n$.

$\implies$ More ~~diff~~ efficient ORC.