1. Suppose that we wish to approximate the first derivative $u'(x)$ of a very smooth function with an error of only $O(h)^4$, where h is the step size. Which difference approximation could we use? (Hint: you may consider to use more than two points in the neighborhood)

$$u(x-h) = u(x) - h\cdot u'(x) + h^2\cdot\frac{1}{2}\cdot u''(x) + h^3\cdot\frac{1}{6}u'''(x) + h^4\cdot\frac{1}{24}\cdot u^{(4)}(x) - O(h^5)$$
$$u(x+h) = u(x) + hu'(x) + h^2\frac{1}{2}u''(x) + h^3\frac{1}{6}u'''(x) + h^4\frac{1}{24}u^{(4)}(x) + O(h^5)$$

Substract.

$$u(x+h) - u(x-h) = 2hu'(x) + h^3\cdot\frac{1}{3}u'''(x) \quad ①$$

then similiar step.

$$u(x+2h) = u(x) + 2hu'(x) + 2h^2u''(x) + \frac{4}{3}h^3u'''(x) + \frac{4}{3}h^4u^{(4)}(x) + O(h^5)$$
$$u(x-2h) = u(x) - 2hu'(x) + 2h^2u''(x) - \frac{4}{3}h^3u'''(x) + \frac{4}{3}h^4u^{(4)}(x) - O(h^5)$$

$$\Rightarrow u(x+2h) - u(x-2h) = 4hu'(x) + \frac{8}{3}\cdot h^3u'''(x) \quad ②$$

$\Rightarrow$ multiplying ① by 8, and substract ②
we can get

$$\Rightarrow 8[u(x+h) - u(x-h)] - [u(x+2h) - u(x-2h)] = 16hu'(x) + 8h^3\cdot\frac{1}{3}u'''(x)$$
$$- \left[4hu'(x) + \frac{8}{3}h^3u'''(x)\right]$$

Divided by 12h.

$$= 12hu'(x) + O(h^5)$$

$$\Rightarrow u'(x) + O(h^4) = \frac{-u(x+2h) + u(x-2h) + 8u(x+h) - 8u(x-h)}{12h}$$

2. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a smooth even function satisfying $f(0) = 0$. Our objective is to approximate the second order derivative $f''(0)$.

- Prove that $f'(0) = 0$.

1) $f(x) = f(-x)$ $\forall x$. f is a even function.

Differentiate both side

$f'(x) = -f'(-x)$. chain rule.

or $f'(-x) = -f'(x)$

$\Rightarrow f'(0) = -f'(0) \Rightarrow f'(0) = 0.$

- Prove that $f'(0) = 0$.

2) - Gwan proposes the following estimator for $f''(0)$: for a step size h

$$a_h = \frac{2f(h)}{h^2}.$$

Please justify that Chenyu's estimation has its convergence $O(h^2)$.

2). we can use taylor series.

$\Rightarrow f(h+0) = f(0) + hf''(0) + h^2 \cdot \frac{1}{2} f''(0) + h^3 \frac{1}{6} f'''(0) + O(h^4)$ ①

$\Rightarrow f(h) = h^2 \frac{1}{2} f''(0) + O(h^4)$ $\quad f(0-h) = f(0) - f'(0)h + h^2 \cdot \frac{f''(0)}{2}$

$- h^3 \cdot \frac{1}{6} f'''(0) + O(h^4)$ ②

$f(h) + f(-h) = 2f(h) = h^2 f''(0) + O(h^4)$.

divided h^2. then $\Rightarrow f''(0) = \frac{2f(h)}{2} - O(h^2)$

3). - Is there anyway to improve the above convergence to $O(h^4)$ in the form of

$$b_h = \frac{c_1 f(h) + c_2 f(2h)}{h^2}$$

for some constants c_1 and c_2?

3) $f(0+h) = f(0) + h^1 f'(0) + h^2 \frac{1}{2} f''(0) + h^3 \frac{1}{6} f'''(0) + h^4 \frac{1}{24} f^{(4)}(0)$

$+ h^5 \frac{1}{5!} f^{(5)}(0) + O(h^6)$ ①

$f(0-h) = f(0) - f'(0) \cdot h^1 + h^2 \frac{1}{2} f''(0) - \frac{1}{6} f'''(0) h^3 + h^4 \frac{1}{24} f^{(4)}(0)$

$- h^5 \frac{1}{5!} f^5(0) + O(h^6)$ ②

$\Rightarrow$ add ① and ②

$\Rightarrow \quad 2f(0) + f''(0) h^2 + h^4 \cdot \frac{1}{12} f^{(4)}(0) + O(h^6)$ ③

the similiar.

we can get. $f(0+2h) + f(0-2h) = 2f(0) + 4h^2 f''(0)$

$+ 8h^4 \cdot \frac{1}{12} \cdot f''''(0) + O(h^6)$ ④

$\Rightarrow$ we can get.

$$f''(0) + O(h^4) = \frac{C_1 f(h) + C_2 f(h)}{h^2}$$

$$= \frac{4 f(h) - \frac{1}{2} f(2h)}{h^2}$$

$\Rightarrow C_1 = 4$ and $C_2 = -\frac{1}{2}$.

4) if $f(x)$ is an odd function. $f(0) = 0$.

$f(-x) = -f(x)$

for $x = 0$. $f(0) = -f(-0) = -f(0)$.

$$f''(0) = \lim_{h \to \infty} \frac{f(h) - 2f(0) + f(-h)}{h^2} = 0.$$