1. prove or disprove: Suppose f is convex and X is submartingale, prove that $g(t) = \mathbb{E}[f(X_t)]$ is increasing.

$X_n \le \mathbb{E}[X_{n+1} | \mathcal{F}_n]$

we can use contradiction for disprove this.

assume. $f(x) = \frac{1}{x}$ and $x_2 > x_1$.

since x is submar, then we can know. $\mathbb{E}[X_{t_2}] > \mathbb{E}[X_{t_1}]$.

$\Rightarrow g(t_2) - g(t_1) = \mathbb{E}[f(X_{t_2}) - f(X_{t_1})] < 0$

$\Leftrightarrow$ disprove.

2. ~~we~~ for submartingale. : $n \to X_n$ increasing plus fluctuations.

① for $S_t \geq k$. and $t_2 > t_1$ $t_2, t_1 \in D$.

$\mathbb{E}[e^{-rt_2}(S_{t_2} - k)^+ | \mathcal{F}_{t_1}] = \mathbb{E}[e^{-rt_2}(S_{t_1} - k) | \mathcal{F}_{t_1}]$

$= e^{-rt_1} S_{t_1} - ke^{-rt_1} + k\mathbb{E}[e^{-rt_1} - e^{-rt_2}]$

since $e^{-rt_2} < e^{-rt_1} \Longrightarrow \mathbb{E}[e^{-rt}(S_{t_2} - k) | \mathcal{F}_{t_1}] \geq e^{-rt}(S_{t_1} - k)]$

② for $S_t < k$.

we can get $\mathbb{E}[e^{-rt_1}(S_{t_1} - k)^+] = 0$

$\Rightarrow$ then $C(t) = \mathbb{E}[e^{-rt}(S_t - k)^+]$ is increasing

3. ① $S_t < k$. $\Rightarrow \mathbb{E}[\mathbb{E}[k - S_{t_2} | \mathcal{F}_{t_1}]] \geq \mathbb{E}[(S_{t_1} - k)^-]$ for $t_2 > t_1$. $\forall t_2, t_1 \in D$

② $S_t \geq k$. $P(t_2) - P(t_1) = 0$

then. $\mathbb{E}[k - S_{t_2}] = \mathbb{E}[(S_{t_2} - k)^-] \geq \mathbb{E}[(S_{t_1} - k)^-]$. increasing