1. Pf: $u'(x) = 1 - \frac{1}{\sqrt{\varepsilon}}\left([\exp(\frac{x-1}{\sqrt{\varepsilon}})] + [\exp(-\frac{x+1}{\sqrt{\varepsilon}})]\right) / 1-\exp(-\frac{2}{\sqrt{\varepsilon}})$

$$u''(x) = -\frac{1}{\varepsilon}\cdot\frac{\exp(\frac{x-1}{\sqrt{\varepsilon}}) - \exp(-\frac{x+1}{\sqrt{\varepsilon}})}{1-\exp(-\frac{2}{\sqrt{\varepsilon}})}$$

$-u''(x)\varepsilon + u = x$

$$\Rightarrow \therefore u(x) = x - \frac{\exp(\frac{x-1}{\sqrt{\varepsilon}}) - \exp(-\frac{x+1}{\sqrt{\varepsilon}})}{1-\exp(-\frac{2}{\sqrt{\varepsilon}})} \quad ①$$

then we can use ODE's equation: $r^2 = \frac{1}{\varepsilon}$, $r = \pm\frac{1}{\sqrt{\varepsilon}}$. $-r^2\varepsilon + 1 = 0$

$\Rightarrow u(x) = C_1 e^{\frac{1}{\sqrt{\varepsilon}}x} + C_2 e^{-\frac{1}{\sqrt{\varepsilon}}x}$

$\Rightarrow u(x) = C_1 e^{\frac{1}{\sqrt{\varepsilon}}x} + C_2 e^{-\frac{1}{\sqrt{\varepsilon}}x} + x$ ($u(x) = x$ is a special one)

$$\begin{cases} u(0) = 0 \\ u(1) = C_1 e^{\frac{1}{\sqrt{\varepsilon}}} + C_2 e^{-\frac{1}{\sqrt{\varepsilon}}} + 1 = 0 \end{cases}$$

then we can get C_1 and C_2.

$\Rightarrow$ the answer of ODE is unique. ②

① ②. $\Rightarrow u(x) = x - \frac{\exp(\frac{x-1}{\sqrt{\varepsilon}}) - \exp(-\frac{x+1}{\sqrt{\varepsilon}})}{1-\exp(-\frac{2}{\sqrt{\varepsilon}})}$

2. $-\varepsilon\delta_h\delta_{-h}u_i^h + u_i^h = x_i$. $\quad -\varepsilon\frac{u_i^h - 2u_i^h + u_{i-1}^h}{h^2} + u_i^h = x_i$

$-\frac{\varepsilon}{h^2}u_{i-1}^h + (\frac{2\varepsilon}{h^2} + 1)u_i^h - \frac{\varepsilon}{h^2}u_{i+1}^h = x_i$

$$\Rightarrow \begin{cases} u_0^h = 0 \\ -ru_{i-1}^h + su_i^h - tu_{i+1}^h = f_i \\ u_N^h = 0 \end{cases} \qquad R^h f = \begin{bmatrix} 0 \\ \frac{1}{h} \\ 2/h \\ \vdots \end{bmatrix} \qquad L^h = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -r & s & t & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & -r & s & -t \\ 0 & & & 0 & 1 \end{bmatrix}$$

Constitions

3. ① $(L^hR^hv)_0 = v_0$. $(R^hLv)_0 = v(0)$. $\quad (L^hR^hv)_0 - (R^hLv)_0 = 0$.

② $|(L^hR^hv)_N - (R^hLv)_N| = 0$

③ $(L^hR^hv)_i = -\varepsilon\delta_h\delta_{-h}v_i + v_i$ $\quad (R^hLv)_i = Lv(x_i) = -\varepsilon v''(x_i) + v(x_i)$

$|(L^hR^hv)_i - (R^hLv)_i| = O(h^2)$.

L^h is constistions of $\alpha = 2$.

Stability: $\quad (L^hv)_i = -rv_i - tv_{i+1} + sv_i$

$= r(v_i - v_{i-1}) + t(v_i - v_{i+1}) + (s-r-t)v_i$

$\|L^hv\|_\infty \geq |(L^hv)_i| \geq 2|v_i| \geq \|v\|_\infty$

④ If $-v_i = \|v\|_\infty$ for some $1 \leq i \leq N-1$

$(L^hv)_i = -r(v_{i-1} - v_i) - t(v_{i+1} - v_i) + 2v_i \leq 0$.

$\|L^hv\|_\infty \geq |(L^hv)_i| \geq 2|v_i| = 2\|v\|_\infty \Rightarrow \|v\|_\infty \leq \|L^hv\|_\infty$ $\quad L^2$ is stable

FEM:

4. T^8 is a uniform mesh on $[0,1]$ with 9 intervals.

T^{8+1} is a modified T^8 with one point added at the center of the last interval.

CFD: on uniform on $[0,1]$ with 9 interval

based on the question, we can know CFD is more accurate than FEM.