hw2

Hanbei Xiong

Problem 1

An electronic device has lifetime denoted by T. The device has value $V = 5$ if it fails before time $t = 3$; otherwise, it has value $V = 2T$. Find the cdf of V, if T has pdf

$$f_T(t) = \frac{1}{1.5}e^{-t/(1.5)}, \quad t > 0.$$

Answer:

We first need to find the pdf of V. Since V=2T when $t \geq 3$, we have $g^{-1}(v) = \frac{1}{2}v$. Therefore, we need to find the pdf of T.

$$F_T(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \int_0^\infty \frac{1}{1.5}e^{-t/1.5}dt = -e^{-t/1.5}\Big|_0^\infty = 0 - (-1) = 1 & \text{if } t > 0 \end{cases}$$

Since V=2T, we wish to find the cdf of V. If V<3, $F_V(v) = 3$. If V>3, $F_V(v) = P(V \leq v) = P(2T \leq v) = P(T \leq \frac{1}{2}v) = F_T(\frac{1}{2}v)$. Therefore, the cdf of V is

Problem 2

Let λ be a fixed positive constant, and define the function $f(x)$ by

$$f(x) = \left\{ \begin{array}{cc} \frac{1}{2}\lambda e^{-\lambda x} & x \geq 0 \\ \\ \frac{1}{2}\lambda e^{\lambda x} & x < 0 \end{array} \right\}.$$

a. Verify that $f(x)$ is a pdf.

b. If X is a random variable with pdf given by $f(x)$, find $P(X < t)$ for all t.

c. Find $P(|X| < t)$ for all t.

(a) Answer:

Since λ is positive, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$ is positive for all x.

$\int_{-\infty}^{\infty} f(t) = \int_0^{\infty} \frac{1}{2}\lambda e^{-\lambda t} + \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda t} = -\frac{1}{2}e^{-\lambda t}\Big|_0^{\infty} + \frac{1}{2}e^{\lambda t}\Big|_{-\infty}^{0} = \frac{1}{2} + \frac{1}{2} = 1$

Both properties of pdf have satisfied. We can conclude that $f(x)$ is a pdf.

(b) Answer:

$P(X < t) = F_X(t)$

$$F_X(t) = \begin{cases} -\frac{1}{2}e^{-\lambda t} & \text{if } t \geq 0 \\ \frac{1}{2}e^{\lambda t} & \text{if } t < 0 \end{cases}$$

(c) Answer:

Problem 3

If the random variable X has pdf

$$f(x) = \left\{ \begin{array}{cc} \frac{x-1}{2} & 1 < x < 3 \\ \\ 0 & \text{otherwise} \end{array} \right\},$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a uniform$(0, 1)$ distribution.

Problem 4

If X is uniformly distributed over $(-1, 1)$, find the pdf of the random variable $|X|$.

Problem 5

Explain in detail how you would generate a sample of 10 observations from $f(x) = 3x^2$ for $0 < x < 1$ and 0 otherwise.

Problem 6

Suppose we select a point at random in the interior of a circle of radius 1. Let X be the distance of the selected point from the center of the circle. For $0 < x < 1$, let the event $\{X \leq x\}$ be equivalent to the point lying in a circle of radius x. Assume that the probability of the selected point lying in a circle of radius x is equal to the ratio of the area of this circle to the area of the full circle. Following this line of thought, first find the cdf and pdf of X. Then find the cdf and pdf of $Y = X^3$.

Problem 7

Let X be a continuous random variable with pdf given by

$$f(x) = |x|/4, \quad 0 \leq |x| \leq 2.$$

Find $P(1 \leq |X| \leq 2)$.