hw6

AUTHOR
Hanbei Xiong

Question 1

Let X_i, $i = 1, 2, 3$ be independent with $N(i, i^2)$ distributions. For each of the following situations, use the X_i's to construct a statistic with the indicated distributions.

a. χ^2 with 3 degrees of freedom

b. t distribution with 2 degrees of freedom

c. F distribution with 1 and 2 degrees of freedom

(a) Answer:

$Z_i = \frac{X_i - i}{\sqrt{i^2}} = \frac{X_i - i}{i}$ for i = 1, 2, 3

$Z_i \sim N(0, 1)$

$\sum_{i=1}^{3}(\frac{X_i - i}{i})^2 \sim \chi^2(3)$

(b) Answer:

$\sum_{i=1}^{2}(\frac{X_i - i}{i})^2 \sim \chi^2(2)$

By independence of X_1, X_2, and X_3,

$\frac{X_1 - 1}{\sqrt{\sum_{i=2}^{3}(\frac{X_i - i}{i})^2/2}} \sim t_2$

(c) Answer:

$(X_1 - 1)^2 \sim \chi^2(1)$

$\frac{\chi^2(1)/1}{\chi^2(2)/2} \sim F_{1,2}$

By independence of X_1, X_2, and X_3,

$\frac{(X_1 - 1)^2}{\sum_{i=2}^{3}(\frac{X_i - i}{i})^2/2} \sim F_{1,2}$

Question 2

Suppose X and Y are iid normal$(0,1)$. Prove that $W = X^2 + 2Y^2$ does not have a chi-square distribution.

Answer:

By independence of X and Y,

$$M_W(t) = M_{X^2+2Y^2}(t) = E[e^{X^2t+2Y^2t}] = E[e^{X^2t}]E[e^{2Y^2t}] = M_{X^2}(t)M_{Y^2}(2t)$$

Since X and Y are iid normal(0,1)

$$X^2 \sim \chi^2(1)$$

$$Y^2 \sim \chi^2(1)$$

$M_{X^2}(t) = \frac{1}{(1-2t)^{1/2}}$, t<1/2

$M_{Y^2}(2t) = \frac{1}{(1-4t)^{1/2}}$, t<1/4

$M_W(t) = M_{X^2}(t)M_{Y^2}(2t) = \frac{1}{(1-2t)^{1/2}} \frac{1}{(1-4t)^{1/2}}$, t<1/4

Since it is not in the shape of the MGF of chisquare distribution, we can conclude that W does not follow a chisquare distribution.

Question 3

Let X_1 and X_2 be two independent random variables. Let X_1 and $Y = X_1+X_2$ have Poisson distributions with mean μ_1 and $\mu \geq \mu_1$, respectively. Derive the distribution of X_2.

Answer:

$$M_Y(t) = E[e^{X_1t+X^2t}] = M_{X_1}(t)M_{X^2}(t)$$

$$M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)}$$

Since $Y \sim Poisson(\mu)$ and $X_1 \sim Poisson(\mu_1)$,

$$M_Y(t) = e^{\mu(e^t-1)} \; \forall t$$

$$M_{X_1}(t) = e^{\mu_1(e^t-1)} \; \forall t$$

$$M_{X_2}(t) = \frac{M_Y(t)}{M_{X_1}(t)} = e^{\mu(e^t-1)-\mu_1(e^t-1)} = e^{(\mu-\mu_1)(e^t-1)} \; \forall t$$

Therefore, $X_2 \sim Poisson(\mu - \mu_1)$

Question 4

Suppose X is distributed as Geometric(p) with $0 < p < 1$ and $q = 1 - p$. The probability mass function is $f(x) = pq^x$ for $x = 0, 1, 2, \ldots$

a. Find the moment generating function of X.

b. Find the mean and variance μ_x and σ_x^2.

c. Suppose that $X_1, \ldots, X_n$ are iid Geo(p), $0 < p < 1$. Discuss in detail the limiting distribution of $\sqrt{n}(\bar{X} - \mu_x)$.

(a) Answer:

Since $\sum r^x = \frac{1}{1-r}$ for |r| < 1,

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} p(1-p)^x = p\sum_{x=0}^{\infty} e^{tx}(1-p)^x = p\sum_{x=0}^{\infty}(e^t(1-p))^x = \frac{p}{1-e^t(1-p)}$$

To find the bound of t,

$$|e^t(1-p)| < 1 \Rightarrow e^t(1-p) < 1 \Rightarrow t + log(1-p) < 0 \Rightarrow t < -log(1-p)$$

(b) Answer:

Let $\psi_X(t) = logM_X(t) = log(p) - log(1 - e^t(1-p))$

$$\mu_X = \psi'_X(t)\Big|_{t=0} = \frac{e^t(1-p)}{1-e^t(1-p)}\Big|_{t=0} = \frac{1-p}{p}$$

$$\sigma_X^2 = \psi''_X(t)\Big|_{t=0} = \frac{(1-p)e^t\{1-(1-p)e^t\}+(1-p)^2e^{2t}}{\{1-e^t(1-p\}^2}\Big|_{t=0} = \frac{(1-p)p+(1-p)^2}{p^2} = \frac{1-p}{p^2}$$

(c) Answer:

By central limit theorem:

$$\frac{\sqrt{n}(\bar{X}-\mu_x)}{\sigma_x} \sim N(0,1)$$

$$E[\sqrt{n}(\bar{X} - \mu_x)] = 0$$

$$Var[\sqrt{n}(\bar{X} - \mu_x)] = \sigma_x^2$$

$$\sqrt{n}(\bar{X} - \mu_x) \sim N(0, \sigma_x^2)$$

Question 5

Let Y_1, Y_2 be iid $N(0,1)$. Find the moment generating function of $Y = Y_1Y_2$.

Answer:

$E[e^{Y_1 Y_2 t}] = E_{Y_2}\{E[e^{Y_1 Y_2 t}|Y_2]\}$

Let $Y_2 = y_2$

$E[e^{Y_1 Y_2 t}|Y_2 = y_2] = E[e^{Y_1 y_2 t}|Y_2 = y_2] = E[e^{Y_1 y_2 t}] = M_{Y_1}(y_2 t)$

Since $Y_1 \sim N(0,1)$

$M_{Y_1}(y_2 t) = e^{\frac{1}{2} y_2^2 t^2}$ where $-\infty < t < \infty$

Replace y_2 with Y_2

$E_{Y_2}[e^{\frac{1}{2} Y_2^2 t^2}] = M_{Y_2^2}(\frac{1}{2} t^2)$

Since $Y_2 \sim N(0,1)$, we have $Y_2^2 \sim \chi^2(1)$

$M_{Y_2^2}(\frac{1}{2} t^2) = \frac{1}{(1-2\frac{1}{2} t^2)^{1/2}} = \frac{1}{(1-t^2)^{1/2}}$

The support of t can be found below:

$\frac{t^2}{2} < \frac{1}{2} \Rightarrow -1 < t < 1$

$M_Y(t) = \frac{1}{(1-t^2)^{1/2}}$

Question 6

Suppose X is distributed as $n(\mu, \sigma^2)$ and suppose $Y = e^X$. Y is said to have a log-normal distribution.

a. Find the pdf of Y.

b. Find $E(Y^k)$ for any k.

Note that although $E(Y^k)$ exists for every k, Y does not have a mgf.

(a) Answer:

$Y = e^X \Rightarrow log Y = X$

$\infty < X < \infty \Rightarrow 0 < Y < \infty$

$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ with support: $-\infty < x < \infty$

$|\frac{dx}{dy}| = |\frac{dlogy}{dy}| = |\frac{1}{y}| = \frac{1}{y}$

$f_Y(y) = f_X(logy)|\frac{dx}{dy}| = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(log(y)-\mu)^2}{2\sigma^2}}\frac{1}{y}$ with support: $0 < y < \infty$

(b) Answer:

$E[Y^k] = E[e^{Xk}] = M_X(k) = e^{k\mu + \frac{k^2\sigma^2}{2}}$ for all k