hw4

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Problem 1

A and B agree to meet at a certain place between 1 PM and 2 PM. Suppose they arrive at the meeting place independently and randomly during the hour. Find the distribution of the length of time that A waits for B. (If B arrives before A, then define A's waiting time as 0.)

Answer:

Assume A and B should both have uniform distribution

$A \sim U(1,2)$ and $B \sim U(1,2)$

$f_{A,B}(a,b) = 1$ for $1 \leq a \leq 2$ and $1 \leq b \leq 2$

Let X = B-A

Then:

$$X = \begin{cases} 0 & \text{if } B-A \leq 0 \\ B-A & \text{if } B-A > 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ P(B-A>0)+P(B-A \leq 0) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

If we draw the graph where x-axis represented A and y-axis represented B, the support of X is a square form by $1 \leq a \leq 2$ and $1 \leq b \leq 2$. We can use the 1 minus the probability of landing on area of upper left triangle to represent the distribution of length of time that A waits for B. Here is the distribution:

$$F_X(x) = \begin{cases} 0 \\ 1-\int_1^{2-x}\int_{a+x}^{2} 1dbda = 1-\int_1^{2-x}(2-a-x)da = 1-[(2-x)a-\frac{a^2}{2}]\Big|_1^{2-x} = 1-(2-x)^2+\frac{(2-x)^2}{2}+(2-x)-\frac{1}{2} = -(2-x)^2/2-x+\frac{5}{2} \\ 1 \end{cases}$$

Problem 2

Suppose X and Y have the joint pdf $f_{X,Y}(x,y) = (21x^2y)/4$ for $0 \leq x^2 \leq y \leq 1$, zero elsewhere.

a. Are X and Y independent? Fully justify your answer.

b. Find the marginal pdf of Y.

c. Find $E(Y)$ and $Var(Y)$.

d. Find the marginal cdf of Y.

e. Find $P(X \geq Y)$.

(a) Answer:

X and Y are not independent since their supports are not rectangular support.

(b) Answer:

$f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4}x^2ydx = \frac{7}{4}x^3y\Big|_{-\sqrt{y}}^{\sqrt{y}} = \frac{7}{2}y^{\frac{5}{2}}$ for $0 \leq y \leq 1$

(c) Answer:

$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) dy = \int_0^1 \frac{7}{2} y^{\frac{7}{2}} dy = \frac{7}{9} y^{\frac{9}{2}} \Big|_0^1 = \frac{7}{9}$

$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y) dy = \int_0^1 \frac{7}{2} y^{\frac{9}{2}} dy = \frac{7}{11} y^{\frac{11}{2}} \Big|_0^1 = \frac{7}{11}$

$Var(Y) = E(Y^2) - E(Y)^2 = \frac{7}{11} - \frac{49}{81} \approx 0.0314$

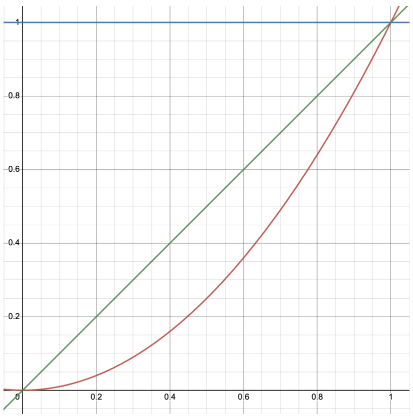
(d) Answer:

$F_Y(y) = \int_0^y \frac{7}{2} z^{\frac{5}{2}} dz = y^{\frac{7}{2}}, 0 \leq y \leq 1$

(e) Answer:

We let y=x and graph it out. By selecting a random point like (1,0), we can tell the point is below the line. We can also let $x^2 = y$ and graph it on the same graph. We can tell the support of the joint pdf is the area between this parabola. Hence, we are looking for the area between these two functions.

The graph looks like this:



$P(X \geq Y) = \int_0^1 \int_{x^2}^{x} \frac{21}{4} x^2 y dy dx = \int_0^1 \frac{21}{8} x^2 y^2 \Big|_{x^2}^{x} dx = \int_0^1 \frac{21}{8} x^2 (x^2 - x^4) dx = \frac{21}{8} \int_0^1 x^4 - x^6 dx = \frac{21}{8} [\frac{1}{5} x^5 - \frac{1}{7} x^7] \Big|_0^1 = \frac{21}{8} [\frac{1}{5} - \frac{1}{7}] = \frac{21}{8} [\frac{2}{35}] = \frac{3}{20}$

Problem 3

Suppose (X_1, X_2) have the joint pdf, $f_{X_1,X_2}(x_1, x_2) = 1$ for $0 < x_1 < 1$, $0 < x_2 < 1$, and zero elsewhere.

a. Find the joint distribution of $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Give a sketch of the support of this distribution.

b. Find the marginal distribution of Y_1.

(a) Answer:

We can find $x_1 = u(y_1, y_2)$ and $x_2 = v(y_1, y_2)$

Since

$Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$

$\Rightarrow X_1 = \frac{Y_1+Y_2}{2}$ and $X_2 = \frac{Y_1-Y_2}{2}$

We can also transform the support of X_1 and X_2 to the support of Y_1 and Y_2.

We have $0 < X_1 < 1$ and $0 < X_2 < 1$

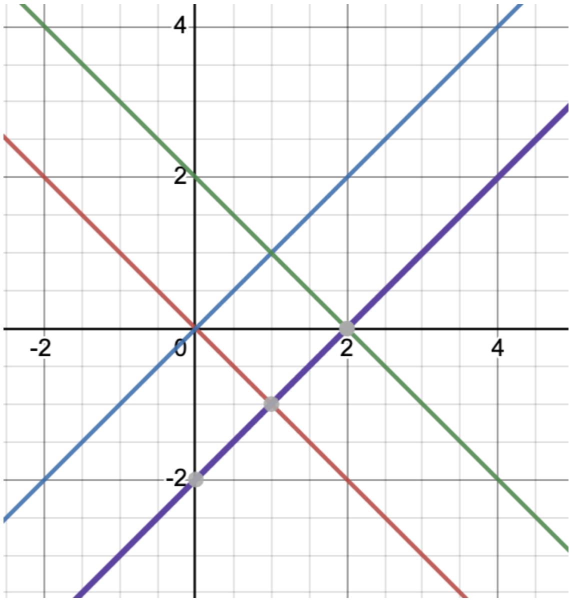
It becomes:

$0 < \frac{Y_1+Y_2}{2} < 1$ and $0 < \frac{Y_1-Y_2}{2} < 1$

The inequality can be saperated into four parts:

$y_2 > -y_1$ and $y_2 < y_1$ and $y_2 < 2 - y_1$ and $y_2 > y_1 - 2$

By setting inequality to equality, we can find the support of Y_1 and Y_2:



The horizontal axis is y_1 and the vertical axis is y_2. By pluging in point as (1,0), we know the support of joint distribution of Y_1 and Y_2 is the area between these lines.

$$|J| = |det \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}| = |det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}| = \frac{1}{2}$$

$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2)|J| = \frac{1}{2}$

(b) Answer:

$$f_Y(y_1) = \begin{cases} \int_{-y_1}^{y_1} \frac{1}{2} dy_2 = y_1 & \text{if } 0 < y_1 < 1 \\ \int_{y_1-2}^{-y_1+2} \frac{1}{2} dy_2 = -y_1 + 2 & \text{if } 1 < y_1 < 2 \end{cases}$$

Problem 4

Let X and Y be discrete random variables and let g and h be functions such that the following identity holds:

$$P(X = x, Y = y) = g(x)h(y).$$

Assume that the support of the joint pmf of X and Y is rectangular, $S_X \times S_Y$, where S_x and S_Y denote the supports of X and Y, respectively.

a. Express $P(X = x)$ and $P(Y = y)$ in terms of g and h.

b. Show that X and Y are independent.

Hint: First show that $(\sum_{x \in S_X} g(x))(\sum_{y \in S_y} h(y)) = 1.$

(a) Answer:

$P(X = x) = f_X(x) = \sum_{y \in S_y} g(x)h(y) = g(x) \sum_{y \in S_y} h(y)$

$P(Y = y) = f_Y(y) = \sum_{x \in S_x} g(x)h(y) = h(y) \sum_{x \in S_x} g(x)$

(b) Answer:

$(\sum_{x\in S_x} g(x))(\sum_{y\in S_y} h(x)) = \sum_{x\in S_x, y\in S_y} g(x)h(x) = \sum_{x\in S_x, y\in S_y} P(X=x, Y=y) = 1$

$P(X=x)P(Y=y) = (g(x)\sum_{y\in S_y} h(y))(h(y)\sum_{x\in S_x} g(x)) = g(x)h(y)(\sum_{y\in S_y} h(y))(\sum_{x\in S_x} g(x)) = g(x)h(y) = P(X=x, Y=y)$

Hence, X and Y are independent.

Problem 5

Let $X = 1$ with probability p and $X = 0$ with probability $1-p$. Let Y be another random variable that can also be either zero or one. Let $Pr(Y = 1|X = 1) = r$ and $Pr(Y = 0|X = 0) = s$. Find $\mathrm{Var}(Y)$.

Answer:

Given:

$Pr(X=1) = p$

$Pr(X=0) = 1-p$

$Pr(Y=1|X=1) = r$

$Pr(Y=0|X=0) = s$

We know:

$Var(Y) = E(Y^2) - E(Y)^2$

$E(Y) = 1 * Pr(Y=1) + 0 * Pr(Y=0) = 1 * (Pr(Y=1|X=1)Pr(X=1) + Pr(Y=1|X=0)Pr(X=0)) = rp + (1-s)(1-p)$

$E(Y^2) = 1^2 * Pr(Y=1) + 0^2 * Pr(Y=0) = 1^2 * (Pr(Y=1|X=1)Pr(X=1) + Pr(Y=1|X=0)Pr(X=0)) = rp + (1-s)(1-p)$

$Var(Y) = rp + (1-s)(1-p) - [rp + (1-s)(1-p)]^2$