hw2

Hanbei Xiong

Problem 1

An electronic device has lifetime denoted by T. The device has value $V = 5$ if it fails before time $t = 3$; otherwise, it has value $V = 2T$. Find the cdf of V, if T has pdf

$$f_T(t) = \frac{1}{1.5}e^{-t/(1.5)}, \quad t > 0.$$

Answer:

$$\begin{cases} V = 5 & \text{if } 0 < t < 3 \\ V = 2T & \text{if } t \geq 3 \end{cases}$$

So the support of cdf of V with its support is:

$$F_V(v) = \begin{cases} 0 & \text{if } v < 5 \\ \int_0^3 \frac{1}{1.5}e^{-t/(1.5)}dt & \text{if } 5 \leq v < 6 \\ \int_0^3 \frac{1}{1.5}e^{-t/(1.5)}dt + \int_3^{v/2} \frac{1}{1.5}e^{-t/(1.5)}dt & \text{if } v \geq 6 \end{cases}$$

Problem 2

Let λ be a fixed positive constant, and define the function $f(x)$ by

$$f(x) = \left\{ \begin{array}{ll} \frac{1}{2}\lambda e^{-\lambda x} & x \geq 0 \\ & \\ \frac{1}{2}\lambda e^{\lambda x} & x < 0 \end{array} \right\}.$$

a. Verify that $f(x)$ is a pdf.

b. If X is a random variable with pdf given by $f(x)$, find $P(X < t)$ for all t.

c. Find $P(|X| < t)$ for all t.

(a) Answer:

Since λ is positive, $f(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$ is positive for all x.

$\int_{-\infty}^{\infty} f(t) = \int_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda t} + \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda t} = -\frac{1}{2}e^{-\lambda t}\Big|_{0}^{\infty} + \frac{1}{2}e^{\lambda t}\Big|_{-\infty}^{0} = \frac{1}{2} + \frac{1}{2} = 1$

Both properties of pdf have satisfied. We can conclude that $f(x)$ is a pdf.

(b) Answer:

$P(X < t) = F_X(t)$

$$F_X(t) = \begin{cases} \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x} dx & \text{if } t < 0 \\ \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x} dx & \text{if } t \geq 0 \end{cases}$$

$$F_X(t) = \begin{cases} \frac{1}{2}e^{\lambda t} & \text{if } t < 0 \\ \frac{1}{2} - \frac{1}{2}e^{-\lambda t} + \frac{1}{2} = 1 - \frac{1}{2}e^{-\lambda t} & \text{if } t \geq 0 \end{cases}$$

(c) Answer:

By break out the absolute value of X, we have range of X: $X > t$ or $X < -t$

$$F_X(t) = \begin{cases} \int_{t}^{-t} \frac{1}{2}\lambda e^{\lambda x} dx & \text{if } t < 0 \\ \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x} dx & \text{if } t \geq 0 \end{cases}$$

Problem 3

If the random variable X has pdf

$$f(x) = \left\{ \begin{array}{cc} \frac{x-1}{2} & 1 < x < 3 \\ \\ 0 & \text{otherwise} \end{array} \right\},$$

find a monotone function $u(x)$ such that the random variable $Y = u(X)$ has a uniform$(0, 1)$ distribution.

Answer:

We can simply find the cdf of X:

$$u(x) = F_X(t) = \begin{cases} 0 & \text{if } t \leq 1 \\ \int_1^t \frac{x-1}{2} dx = \frac{t^2}{4} - \frac{1}{2}t + \frac{1}{4} & \text{if } 1 < t < 3 \\ \int_1^3 \frac{x-1}{2} dx = 1 & \text{if } t \geq 3 \end{cases}$$

Problem 4

If X is uniformly distributed over $(-1, 1)$, find the pdf of the random variable $|X|$.

Answer:

Since X is uniform distribution, the pdf of X is:

$$f_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2} & \text{if } -1 \leq x \leq 1 \\ 0 & \text{if } x > 1 \end{cases}$$

Then, the cdf of X is:

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{2}x & \text{if } -1 \leq x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$

We can think |X| as a random variable Y, which means Y=|X|

When $-1 \leq x < 0$, Y=-X, so the cdf of Y is:$-\frac{1}{2}y$

When $0 \leq x \leq 1$, Y=X, so the cdf of Y is:$\frac{1}{2}y$

Therefore, the cdf of Y is:

$$F_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ -\frac{1}{2}y & \text{if } -1 \leq y \leq 0 \\ \frac{1}{2} + \frac{1}{2}y & \text{if } 0 \leq y \leq 1 \\ 1 & \text{if } y > 1 \end{cases}$$

Problem 5

Explain in detail how you would generate a sample of 10 observations from $f(x) = 3x^2$ for $0 < x < 1$ and 0 otherwise.

Problem 6

Suppose we select a point at random in the interior of a circle of radius 1. Let X be the distance of the selected point from the center of the circle. For $0 < x < 1$, let the event $\{X \leq x\}$ be equivalent to the point lying in a circle of radius x. Assume that the probability of the selected point lying in a circle of radius x is equal to the ratio of the area of this circle to the area of the full circle. Following this line of thought, first find the cdf and pdf of X. Then find the cdf and pdf of $Y = X^3$.

Answer:

Based on the assumption, we have

$$F_X(x) = P(X < x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{\pi x^2}{\pi} = x^2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

Then, the pdf of X is:

$$f(x) = \begin{cases} 2x & \text{if } 0 < x < 1 \\ 0 & \text{else} \end{cases}$$

Since $Y = X^3$, the cdf of Y is:

$$F_Y(y) = P(Y < y) = \begin{cases} 0 & \text{if } y \leq 0 \\ y^{\frac{1}{3}} & \text{if } 0 < y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

Then, the pdf of Y is:

$$f(y) = \begin{cases} \frac{1}{3}y^{-2/3} & \text{if } 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

Problem 7

Let X be a continuous random variable with pdf given by

$$f(x) = |x|/4, \quad 0 \leq |x| \leq 2.$$

Find $P(1 \leq |X| \leq 2)$.

Answer:

Since pdf of X is $f(x) = \frac{|x|}{4}$, $0 \leq |x| \leq 2$, we can convert the support of X by breaking the absolute value of X based on sign of x

Then, the pdf of X is:

$$f(x) = \begin{cases} 0 & \text{if } x < -2 \\ -\frac{x}{4} & \text{if } -2 \leq x \leq 0 \\ \frac{x}{4} & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$$

$1 \leq |X| \leq 2$ can also be break parts:

If X is positive, we have $1 \leq x \leq 2$

If X is negative, we have $-2 \leq x \leq -1$

Then,

$P(1 \leq |X| \leq 2) = P(1 \leq x \leq 2) + P(-2 \leq x \leq -1) = \int_1^2 \frac{x}{4} dx - \int_{-2}^{-1} \frac{x}{4} dx = \frac{3}{4}$