hw1

Hanbei Xiong

Problem 1

Let S be a sample space.

a. Show that the collection $B = \{\emptyset, S\}$ is a sigma algebra.

b. Let $B = \{$all subsets of S, including S itself$\}$. Show that B is a sigma algebra.

c. Show that the intersection of two sigma algebras is a sigma algebra.

Note:

$\sigma - algebra$ B is any collection of sets such that

1. $\emptyset \in B$
2. If $A \in B$, then $A^c \in B$
3. If $A_1, A_2, ... \in B$, then $\cup_{i=1}^{\infty} A_i \in B$

(a) Answer:

Condition 1 is satisfied because $\emptyset \in B$

Condition 2 is satisfied because $S \in B$ and $S^c = \emptyset \in B$

Condition 3 is satisfied because $S, \emptyset \in B$ and $S \cup \emptyset = S \in B$

(b) Answer:

Condition 1 is satisfied because $\emptyset \in S \Rightarrow \emptyset \in B$

Condition 2 is satisfied because for all subsets of S denoted A_i, $A_i \subseteq S \iff A_i^c \subseteq S$. Then $A_i \in B \iff A_i^c \in B$

Condition 3 is satisfied because for all subsets of S denoted A_i, $A_i \subseteq S$ and $\cup_{i=1}^{\infty} A_i \subseteq S$. Then $A_i \in S$ and $\cup_{i=1}^{\infty} A_i \in S$.

(c) Answer:

Assume B_1, B_2 are two $\sigma - algebra$, we want to prove $B_1 \cap B_2$ is also a $\sigma - algebra$

Condition 1 is satisfied because $\emptyset \in B_1$ and $\emptyset \in B_2 \Rightarrow \emptyset \in B_1 \cap B_2$

Condition 2 is satisfied because if $A \in B_1 \cap B_2$, then $A \in B_1$ and $A \in B_2 \Rightarrow A^c \in B_1$ and $A^c \in B_2 \Rightarrow A^c \in B_1 \cap B_2$

Condition 3 is satisfied because if $A_1, A_2, ... \in B_1 \cap B_2$, then $A_1, A_2, ... \in B_1$ and $A_1, A_2, ... \in B_2$ $\Rightarrow \cup_{i=1}^{\infty} A_i \in B_1$ and $\cup_{i=1}^{\infty} A_i \in B_2 \Rightarrow \cup_{i=1}^{\infty} A_i \in B_1 \cap B_2$

Problem 2

If n balls are placed at random into n cells, find the probability that exactly one cell remains empty.

Answer:

Pick the cell to remain empty, the number of ways are C_1^n

Pick the cell among n-1 cells to be filled with two balls, the number of ways are C_1^{n-1}

Pick the two balls to be filled in the same cell, the number of ways are C_2^n

Assign n-2 balls to n-2 cells, the number of ways are $(n-2)!$

Total possible ways with no restriction are n^n

So the probability is $\frac{C_1^n \times C_1^{n-1} \times C_2^n \times (n-2)!}{n^n}$

Problem 3

A closet contains n pairs of shoes. If $2r$ shoes are chosen at random ($2r < n$), what is the probability that there will be no matching pair in the sample?

Answer:

Pick 2r shoes from the n unique shoes, the number of ways are C_{2r}^n

Pick a shoe from a pair of shoes, the number of ways are 2 (Left or Right), hence the number of ways to pick 2r pairs of shoes are 2^{2r}

Total possible ways with no restriction are C_{2r}^{2n}

So the probability is $\frac{C_{2r}^n \times 2^{2r}}{C_{2r}^{2n}}$

Problem 4

Suppose there is a music festival consisting of 6 concerts: C1, C2, C3, C4, C5, and C6. There is a separate ticket package available for each possible subset of concerts one could attend. Suppose a ticket package is selected completely at random from among the available packages. What is the probability that it includes concert C1?

Answer:

We can exclude C1 and see how many subsets the remaining concert can form. The number of subsets are 2^5

Adding C1 to any one of the subsets above can form a package, so the number of valuable packages are 2^5

Total possible ways with no restriction are $2^6 - 1$ because we exclude the empty set.

So the probability is $\frac{2^5}{2^6-1}$

Problem 5

Suppose we are playing poker, where each player is dealt five cards from a well shuffled deck. What is the probability of getting exactly 3 of a kind (3 cards of the same rank (ace, 2, 3, . . . , 10, Jack, Queen, King) and the other two cards of two other ranks)?

Answer:

Pick the three suites among four suites, the number of ways are C_3^4

Pick a rank from the 13 rank, the number of ways are C_1^{13}

Pick 2 ranks (denoted a and b) from the 12 rank (exclude the rank being picked above), the number of ways are C_2^{12}

Pick the suite of the rank a being picked above, the number of ways are C_1^4

Pick the suite of the rank b being picked above, the number of ways are C_1^4

Total possible ways with no restriction are C_5^{52}

So the probability is $\frac{C_3^4 \times C_1^{13} \times C_2^{12} \times C_1^4 \times C_1^4}{C_5^{52}}$