Biostat 202B Homework 3

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Question 1

Let $W_1 < W_2 < \ldots < W_n$ be the order statistics of n independent observations from a $U(0,1)$ distribution.

a. For $1 \leq r \leq n$, derive the pdf of W_r, the rth order statistics.

b. Use the pdf you've found in a. to find $E(W_1)$ and $E(W_n)$.

c. Let X_1, X_2, X_3 be three independent observations from $U(0,1)$ distribution. What is $P(X_1 < X_2 < X_3)$?

(a) Answer:

By definition, we know the pdf of ith order statistic is:

$f_{W_i}(x) = \frac{n!}{(i-1)!(n-i)!} F_X(x)^{i-1}(1-F_X(x))^{n-i} f_X(x)$

Since $X_i \sim U(0,1)$

$f(x) = 1$ for $0 \leq x \leq 1$

$F(x) = x$ for $0 \leq x \leq 1$

$f_{W_r}(x) = \frac{n!}{(i-1)!(n-r)!} x^{r-1}(1-x)^{n-r}$ for $0 \leq x \leq 1$

(b) Answer:

$E[W_1] = \int_0^1 w \frac{n!}{(1-1)!(n-1)!} w^{1-1}(1-w)^{n-1} dw = \int_0^1 nw(1-w)^{n-1} dw = \frac{1}{n+1}$

$E[W_n] = \int_0^1 w \frac{n!}{(n-1)!(n-n)!} w^{n-1}(1-w)^{n-n} dw = \int_0^1 nw^n dw = \frac{n}{n+1}$

(c) Answer:

There are only 3! combinations of cases to form the order.

Therefore, $P(X_1 < X_2 < X_3) = \frac{1}{6}$

Question 2

Answer:

$$\begin{aligned}
f_{Y(i),Y(j)}(y,z) &= P(Y_{(i)} = y, Y_{(j)} = z) \\
&= lim_{\Delta y \to \infty, \Delta z \to \infty} \frac{P(\text{i-1 } Y_t's < y,\ \text{one } Y_t \in (y, y+\Delta y),\ \text{j-i-1 } Y_t's > y + \Delta y \text{ and } < z,\ \text{one } Y_t \in (z, z+\Delta z),\ \text{n-j } Y_t's > z + \Delta z)}{\Delta y \Delta z} \\
&= \binom{n}{i-1}\binom{n-i+1}{j-i-1}\binom{n-i+1-j+i+1}{n-j} F_X(y)^{i-1}\{F_X(z) - F_X(y)\}^{j-i-1}\{1-F_X(z)\}^{n-j} f_X(y) f_X(z) \\
&= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} F_X(y)^{i-1}\{F_X(z) - F_X(y)\}^{j-i-1}\{1-F_X(z)\}^{n-j} f_X(y) f_X(z), \text{for} - \infty < y < z < \infty
\end{aligned}$$

Question 3

Let $X_1, \ldots, X_n$ be a random sample from a population with pdf

$$f(x) = \left\{ \begin{array}{cc} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{array} \right\}$$

Let $X_{(1)} < \ldots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

Answer:

Let $V = \frac{X_{(1)}}{X_{(n)}}, R = X_{(n)}$

Then, $X_{(1)} = RV, X_{(n)} = R$

$$J(V,R) = \begin{vmatrix} \frac{\partial X_{(1)}}{\partial V} & \frac{\partial X_{(1)}}{\partial R} \\ \frac{\partial X_{(n)}}{\partial V} & \frac{\partial X_{(n)}}{\partial R} \end{vmatrix} = \begin{vmatrix} R & V \\ 0 & 1 \end{vmatrix} = R$$

$$\begin{aligned} f_{V,R}(v,r) &= f_{X_{(1)},X_{(n)}}(rv,r)|J(v,r)| \\ &= f_{X_{(1)},X_{(n)}}(rv,r)r \\ &= \frac{n!}{(n-2)!}\{F_X(r) - F_X(rv)\}^{n-2} f_X(rv) f_X(r) r \\ &= \frac{n!}{(n-2)!}(\frac{r}{\theta} - \frac{rv}{\theta})^{n-2}\frac{r}{\theta^2} \\ &= \frac{n!}{(n-2)!}\frac{r^{n-1}}{\theta^n}(1-v)^{n-2}, \text{for } 0 < r < \theta, 0 < v < 1 \end{aligned}$$

Therefore, V and R are independent and $\frac{X_{(1)}}{X_{(n)}} \perp X_{(n)}$

Question 4

One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$. Find the MLE of θ.

x	$f(x\|1)$	$f(x\|2)$	$f(x\|3)$
0	1/3	1/4	0
1	1/3	1/4	0
2	0	1/4	1/4
3	1/6	1/4	1/2
4	1/6	0	1/4

Answer:

This is non regular model since $f(x;\theta) \geq 0$

For $X = 0$: The MLE of θ is 1 since $f(0|1) = \frac{1}{3}$ is the largest probability of observing $X = 0$.

For $X = 1$: Similarly, the MLE of θ is 1.

For $X = 2$: The MLE of θ is 2 or 3, as both $f(2|2)$ and $f(2|3)$ are $\frac{1}{4}$.

For $X = 3$: The MLE of θ is 3 since $f(3|3) = \frac{1}{2}$ is the largest.

For $X = 4$: The MLE of θ is 3 as well since $f(4|3) = \frac{1}{4}$ is the largest