

Biostat 202B Homework 3

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AUTHOR

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Question 1

Let $W_1 < W_2 < \dots < W_n$ be the order statistics of n independent observations from a $U(0, 1)$ distribution.

- For $1 \leq r \leq n$, derive the pdf of W_r , the r th order statistics.
- Use the pdf you've found in a. to find $E(W_1)$ and $E(W_n)$.
- Let X_1, X_2, X_3 be three independent observations from $U(0, 1)$ distribution. What is $P(X_1 < X_2 < X_3)$?

(a) Answer:

By definition, we know the pdf of i th order statistic is:

$$f_{W_i}(x) = \frac{n!}{(i-1)!(n-i)!} F_X(x)^{i-1} (1 - F_X(x))^{n-i} f_X(x)$$

Since $X_i \sim U(0, 1)$

$$f(x) = 1 \text{ for } 0 \leq x \leq 1$$

$$F(x) = x \text{ for } 0 \leq x \leq 1$$

$$f_{W_r}(x) = \frac{n!}{(i-1)!(n-r)!} x^{r-1} (1-x)^{n-r} \text{ for } 0 \leq x \leq 1$$

(b) Answer:

$$E[W_1] = \int_0^1 w \frac{n!}{(1-1)!(n-1)!} w^{1-1} (1-w)^{n-1} dw = \int_0^1 nw(1-w)^{n-1} dw = \frac{1}{n+1}$$

$$E[W_n] = \int_0^1 w \frac{n!}{(n-1)!(n-n)!} w^{n-1} (1-w)^{n-n} dw = \int_0^1 nw^n dw = \frac{n}{n+1}$$

(c) Answer:

There are only $3!$ combinations of cases to form the order.

Therefore, $P(X_1 < X_2 < X_3) = \frac{1}{6}$

Question 2

Answer:

$$\begin{aligned} f_{Y(i), Y(j)}(y, z) &= \lim_{\Delta y \rightarrow 0, \Delta z \rightarrow 0} \frac{P(i-1 \text{ } Y_t' s < y, \text{ one } Y_t \in (y, y + \Delta y), j-i-1 \text{ } Y_t' s > y + \Delta y \text{ and } < z, \text{ one } Y_t \in (z, z + \Delta z), n-j \text{ } Y_t' s > z + \Delta z)}{\Delta y \Delta z} \\ &= \binom{n}{i-1} \binom{n-i+1}{j-i-1} \binom{n-i+1-j+i+1}{n-j} F_X(y)^{i-1} \{F_X(z) - F_X(y)\}^{j-i-1} \{1 - F_X(z)\}^{n-j} f_X(y) f_X(z) \\ &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} F_X(y)^{i-1} \{F_X(z) - F_X(y)\}^{j-i-1} \{1 - F_X(z)\}^{n-j} f_X(y) f_X(z), \text{ for } -\infty < y < z < \infty \end{aligned}$$

Question 3

Let X_1, \dots, X_n be a random sample from a population with pdf

$$f(x) = \begin{cases} 1/\theta & \text{if } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Let $X_{(1)} < \dots < X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

Answer:

$$\text{Let } V = \frac{X_{(1)}}{X_{(n)}}, R = X_{(n)}$$

$$\text{Then, } X_{(1)} = RV, X_{(n)} = R$$

$$J(V, R) = \begin{vmatrix} \frac{\partial X_{(1)}}{\partial V} & \frac{\partial X_{(1)}}{\partial R} \\ \frac{\partial X_{(n)}}{\partial V} & \frac{\partial X_{(n)}}{\partial R} \end{vmatrix} = \begin{vmatrix} R & V \\ 0 & 1 \end{vmatrix} = R$$

$$\begin{aligned} f_{V,R}(v, r) &= f_{X_{(1)}, X_{(n)}}(rv, r) |J(v, r)| \\ &= f_{X_{(1)}, X_{(n)}}(rv, r) r \\ &= \frac{n!}{(n-2)!} \{F_X(r) - F_X(rv)\}^{n-2} f_X(rv) f_X(r) r \\ &= \frac{n!}{(n-2)!} \left(\frac{r}{\theta} - \frac{rv}{\theta}\right)^{n-2} \frac{r}{\theta^2} \\ &= \frac{n!}{(n-2)!} \frac{r^{n-1}}{\theta^n} (1-v)^{n-2}, \text{ for } 0 < r < \theta, 0 < v < 1 \end{aligned}$$

Since $f_{V,R}(v, r) = f_V(v) f_R(r)$, for $0 < r < \theta, 0 < v < 1$, V and R are independent

Hence, $\frac{X_{(1)}}{X_{(n)}} \perp X_{(n)}$

Question 4

One observation is taken on a discrete random variable X with pmf $f(x|\theta)$, where $\theta \in \{1, 2, 3\}$. Find the MLE of θ .

x	$f(x 1)$	$f(x 2)$	$f(x 3)$
0	1/3	1/4	0
1	1/3	1/4	0
2	0	1/4	1/4
3	1/6	1/4	1/2
4	1/6	0	1/4

Answer:

This is non regular model since $f(x; \theta) \geq 0$

For $X = 0$: The MLE of θ is 1 since $f(0|1) = \frac{1}{3}$ is the largest probability of observing $X = 0$.

For $X = 1$: Similarly, the MLE of θ is 1.

For $X = 2$: The MLE of θ is 2 or 3, as both $f(2|2)$ and $f(2|3)$ are $\frac{1}{4}$.

For $X = 3$: The MLE of θ is 3 since $f(3|3) = \frac{1}{2}$ is the largest.

For $X = 4$: The MLE of θ is 3 as well since $f(4|3) = \frac{1}{4}$ is the largest