## **Biostat 202B Homework 4**

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**AUTHOR** 

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## **Problem 1**

Suppose  $X_1, \ldots, X_n$  are iid with pdf  $f(x; \theta) = e^{-x/\theta}/\theta$  for  $0 < x < \infty$ , zero elsewhere. Find the mle of  $P(X \le 2)$ .

#### **Answer:**

$$L( heta) = \prod_{i=1}^n f(x_i; heta) = \prod_{i=1}^n rac{e^{-x_i/ heta}}{ heta} = rac{e^{-\sum x_i/ heta}}{ heta^n}$$

$$l( heta) = \log L( heta) = -\sum x_i/ heta - n\log heta$$

$$rac{dl( heta)}{d heta} = \sum x_i/ heta^2 - n/ heta$$

By setting 
$$rac{dl( heta)}{d heta}=0$$
 , we have  $\hat{ heta}_{MLE}=rac{\sum x_i}{n}=ar{x}$ 

$$F_X(x) = \int_0^x rac{e^{-z/ heta}}{ heta} dz = -e^{-z/ heta}igg|_0^x = 1 - e^{-x/ heta}$$
 for  $0 < x \le heta$ 

$$P(X \le 2) = 1 - e^{-2/\theta} = 1 - e^{-2/\bar{x}}$$

# **Problem 2**

Suppose  $X_1, \ldots, X_n$  are iid with pdf  $f(x; \theta) = 2x/\theta^2$ ,  $0 < x \le \theta$ , zero elsewhere.

- **a.** Find the MLE  $\hat{\theta}$  for  $\theta$ .
- **b.** Prove or disprove that  $\hat{\theta}_{\text{MLE}}$  is unbiased.
- c. Find the MLE for the median of the distribution.

## (a) Answer:

$$L( heta) = \prod_{i=1}^n f(x_i; heta) = rac{2^n \prod_{i=1}^n x_i}{ heta^{2n}} ext{ for } 0 < x_i \leq heta$$

$$L( heta) = rac{2^n \prod_{i=1}^n x_i}{ heta^{2n}}$$
 for  $0 < x_{(1)} < x_{(2)} < \ldots < x_{(n)} \leq heta$ 

$$L( heta) = rac{2^n \prod_{i=1}^n x_i}{ heta^{2n}} I(0 < x_{(1)}) \ldots I(x_{(n)} \leq heta)$$

So,  $\hat{ heta}_{MLE} = x_{(n)}$  is the smallest value that maximizes the likelihood

### (b) Answer:

$$F_X(x) = \int_0^x rac{2z}{ heta^2} dz = rac{z^2}{ heta^2}igg|_0^x = rac{x^2}{ heta^2} ext{ for } 0 < x \leq heta$$

$$F_{X_{(n)}}(x) = P(X_{(n)} < x) = P(X_1 < x, X_2 < x, \dots, X_n < x) = P(X_1 < x)^n = F_X^n(x)$$

$$E(x_{(n)}) = \int_0^{ heta} x n f_X(x) F_X^{n-1}(x) dx = \int_0^{ heta} x n rac{2x}{ heta^2} (rac{x^2}{ heta^2})^{n-1} dx = rac{2n}{ heta^{2n}} \int_0^{ heta} x^{2n} dx = rac{2n heta}{2n+1} 
eq heta$$

Hence,  $\hat{ heta}_{MLE}$  is not unbiased

### (c) Answer:

The median of distribution means  $F_X(m)=0.5$ 

$$\frac{m^2}{ heta^2}=0.5$$
  $\Rightarrow m=rac{ heta}{\sqrt{2}}$ 

$$\hat{m}_{MLE} = rac{\hat{ heta}_{MLE}}{\sqrt{2}} = rac{x_{(n)}}{\sqrt{2}}$$

## **Problem 3**

Suppose that  $X_1, X_2, \ldots, X_{2n}$  are independently distributed as  $X_i \sim N(0, \sigma^2)$  for  $i = 1, \ldots, n$  and  $X_i \sim N(0, 2\sigma^2)$  for  $i = n + 1, \ldots, 2n$ . Find  $\hat{\sigma}^2$ , the maximum likelihood estimate of  $\sigma^2$  based on the entire sample  $X_1, \ldots, X_{2n}$ . What is the **finite-sample** distribution of the appropriately normalized  $\hat{\sigma}^2$ ? Justify your reasoning.

#### **Answer:**

$$L(\sigma) = \prod_{i=1}^{n} rac{1}{\sqrt{2\pi}\sigma} e^{-rac{x_{i}^{2}}{2\sigma^{2}}} \prod_{j=n+1}^{2n} rac{1}{2\sqrt{\pi}\sigma} e^{-rac{x_{j}^{2}}{4\sigma^{2}}}$$

If we take log of likelihood, take derivative and set it to 0,

We get 
$$\hat{\sigma}_{MLE}^2 = rac{2\sum_{i=1}^n x_i^2 + \sum_{j=i+1}^{2n} x_j^2}{4n} = rac{1}{2n} (\sum_{i=1}^n x_i^2 + rac{1}{2} \sum_{j=i+1}^{2n} x_j^2)$$

Since  $rac{\sum_{i=1}^n x_i^2}{\sigma^2}\sim \chi^2(n)$ ,  $rac{\sum_{j=n+1}^{2n} x_i^2}{2\sigma^2}\sim \chi^2(n)$  and  $X_i$  are independent r.v.s, we have

$$rac{\sum_{i=1}^{n}x_{i}^{2}}{\sigma^{2}}+rac{\sum_{j=n+1}^{2n}x_{i}^{2}}{2\sigma^{2}}\sim\chi^{2}(2n)$$

$$rac{2n\hat{\sigma}_{MLE}^2}{\sigma^2} = rac{\sum_{i=1}^n x_i^2}{\sigma^2} + rac{\sum_{j=n+1}^{2n} x_i^2}{2\sigma^2} \sim \chi^2(2n)$$

## **Problem 4**

Let  $X_1, \ldots, X_n$  be iid Bernoulli(p). The object is to estimate  $\theta = 1/p$ .

- **a.** Find the maximum likelihood estimator of  $\theta$ .
- **b.** What is the asymptotic distribution of the appropriately normalized MLE as  $n \to \infty$ ?

### (a) Answer:

$$L(p) = \prod_{i=1}^n p(x_i;p) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\sum_{i=1}^n x_i} (1-p)^{n-\sum_{i=1}^n x_i}$$

$$l(p) = \log L(\theta) = (\sum_{i=1}^{n} x_i) \log(p) + (n - \sum_{i=1}^{n} x_i) \log(1-p)$$

$$\frac{dl(p)}{dn} = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n - \sum_{i=1}^{n} x_i}{1 - \theta} = 0$$

$$p=rac{\sum_{i=1}^n x_i}{n}$$

Since 
$$heta=rac{1}{p},\hat{ heta}_{MLE}=rac{n}{\sum_{i=1}^{n}x_{i}}=rac{n}{nar{X}}=rac{1}{ar{X}}$$

#### (b) Answer:

By CLT, 
$$\sqrt{n}(ar{X}-p)\stackrel{d}{
ightarrow} N(0,p(1-p))$$

By Delta method,

$$\sqrt{n}(rac{1}{X}-rac{1}{p})\stackrel{d}{
ightarrow} N(0,rac{1}{p^4}p(1-p))=N(0,rac{1-p}{p^3})$$

## **Problem 5**

Let  $X_1, \ldots, X_n$  be a random sample from a gamma distribution with  $\alpha = 4$  and  $\beta = \theta > 0$ . (Note that the pdf of  $X \sim \Gamma(\alpha, \beta)$  is  $f(x) = x^{\alpha-1}e^{-x/\beta}/(\Gamma(\alpha)\beta^{\alpha})$  for  $0 < x < \infty$ . In addition  $EX = \alpha\beta$  and  $VarX = \alpha\beta^2$ .)

- **a.** Find the mle of  $\theta$ .
- **b.** Find the limiting distribution of  $\sqrt{n}(\hat{\theta} \theta)$ , where  $\hat{\theta}$  is the mle of  $\theta$ .
- c. Derive an asymptotic 95% CI for  $\theta$ . Show all your work and justify each step in the derivation.

#### (a) Answer:

$$L( heta) = \prod_{i=1}^n p(x_i; heta) = \prod x_i^3 e^{-x_i/ heta}/(6 heta^4)$$

$$l( heta) = \log L( heta) = \sum_{i=1}^n 3\log x_i - x_i/ heta - \log 6 - 4\log heta$$

$$rac{dl( heta)}{d heta} = \sum_{i=1}^n x_i/ heta^2 - 4/ heta = 0$$

$$\hat{ heta}_{MLE}=rac{\sum_{i=1}^{n}x_{i}}{4n}=rac{ar{x}}{4}$$

### (b) Answer:

$$\sqrt{n}(ar{X}-E(X))=\sqrt{n}(ar{X}-4 heta)\stackrel{d}{
ightarrow} N(0,4 heta^2)$$

$$\sqrt{n}(rac{ar{X}}{4}- heta)\stackrel{d}{
ightarrow} N(0,rac{ heta^2}{4})$$

$$\sqrt{n}(\hat{ heta}- heta)\stackrel{d}{
ightarrow} N(0,rac{ heta^2}{4})$$

### (c) Answer:

$$SE(\hat{\theta}) = \sqrt{rac{Var(\hat{\theta})}{n}} = = \sqrt{rac{\hat{ heta^2}}{4n}}$$

$$CI = (\hat{ heta} - 1.96SE(\hat{ heta}), \hat{ heta} + 1.96SE(\hat{ heta})) = (\hat{ heta} - 1.96\sqrt{rac{\hat{ heta}^2}{4n}}, \hat{ heta} + 1.96\sqrt{rac{\hat{ heta}^2}{4n}}) = (rac{ar{x}}{4} - 1.96\sqrt{rac{ar{x}^2}{64n}}, \hat{ heta} + 1.96\sqrt{rac{ar{x}^2}{64n}})$$

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