Biostat 202B Homework 1

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Question 1

Let $X_1, X_2, \ldots$ be a sequence of random variables that converges in probability to a constant a. Assume that $P(X_i > 0) = 1$ for all i.

(a) Verify that the sequences defined by $Y_i = \sqrt{X_i}$ and $Y_i' = a/X_i$ converge in probability.

(b) Use the results in part (a) to prove the fact used in Example 5.5.18, that σ/S_n converges in probability to 1.

(a) Answer:

Let $g(x) = \sqrt{x}$, g is continuous for $x \geq 0$.

Since $P(X_i > 0) = 1$ for all i, all X_i are greater than 0.

Since $X_n \to a$ in probability,

$Y_n = \sqrt{X_n} = g(X_n) \to \sqrt{a}$ in probability by C.M.T

Let $q(x) = \frac{a}{x}$, q is continuous for $x > 0$.

Since $X_n \to a$ in probability,

$Y_n' = q(X_n) = \frac{a}{X_n} \to \frac{a}{a} = 1$ in probability by C.M.T

(b) Answer:

Since $S_n^2 \to \sigma^2$ in probability

$S_n = \sqrt{S_n^2}$

Hence, by part (a)

$S_n \to \sqrt{\sigma^2} = \sigma$ in probability

$\frac{\sigma}{S_n} \to \frac{\sigma}{\sigma} = 1$ in probability

Question 2

Consider a sequence of random variables X_n for which $E(X_n) \to \mu$ for a fixed value $\mu \in \mathrm{R}$ and $Var(X_n) \to 0$ as $n \to \infty$. Show that this implies $X_n \to \mu$ in probability for $n \to \infty$.

(a) Answer:

Since $Var(X_n) \to 0$ in probability

$(X_n - E(X_n))^2 \to 0$ in probability

$X_n - E(X_n) \to 0$ in probability

Since $E(X_n) \to \mu$ in probability

$X_n \to \mu$ in probability for $n \to \infty$

Question 3

A random sample $X_1, \ldots, X_n$ is drawn from a population with pdf

$$f(x|\theta) = \frac{1}{2}(1 + \theta x), \quad -1 < x < 1, \quad -1 < \theta < 1.$$

Find a consistent estimator of θ, show that it is consistent.

Answer:

$\mu = E(X) = \int_{-1}^{1} x \frac{1}{2}(1 + \theta x) dx = (\frac{x^2}{4} + \frac{\theta x^3}{6})\Big|_{-1}^{1} = \frac{2\theta}{6}$

Then, $\theta = 3\mu$

Since $\bar{X}_n \to \mu$ in probability by WLLN,

$3\bar{X}_n \to 3\mu = \theta$ in probability

Hence, $3\bar{X}_n$ is a consistent estimator of θ

Question 4

Let the random variable Y_n have a distribution that is Binomial(n, p).
(a) Prove that Y_n/n converges in probability p. This result is one form of the weak law of large numbers.
(b) Prove that $1 - Y_n/n$ converges in probability to $1 - p$.
(c) Prove that $(Y_n/n)(1 - Y_n/n)$ converges in probability to $p(1-p)$.

(a) Answer:

$E(Y_n) = np$ and $Var(Y_n) = np(1-p)$

$E(\frac{Y_n}{n}) = \frac{np}{n} = p$ and $Var(\frac{Y_n}{n}) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$

By Chebyshev inequality:

$$\begin{aligned} P(|\frac{Y_n}{n} - p| > \epsilon) &= P(|\frac{Y_n}{n} - E(\frac{Y_n}{n})| > \epsilon) \\ &\leq \frac{Var(\frac{Y_n}{n})}{\epsilon^2} \\ &= \frac{p(1-p)}{n\epsilon^2} \to 0, \text{as n goes large} \end{aligned}$$

Since $P(|\frac{Y_n}{n} - p| > \epsilon) \geq 0$,

$P(|\frac{Y_n}{n} - p| > \epsilon) \to 0$ in probability because it is squeezed on both sides of inequalities.

Hence, $\frac{Y_n}{n} \to p$ in probability.

(b) Answer:

$Var(1 - \frac{Y_n}{n}) = \frac{p(1-p)}{n}$

$$\begin{aligned} P(|1 - \frac{Y_n}{n} - (1-p)| < \epsilon) &= P(|1 - \frac{Y_n}{n} - E(1 - \frac{Y_n}{n})| < \epsilon) \\ &\geq \frac{Var(1 - \frac{Y_n}{n})}{\epsilon^2} \\ &= \frac{p(1-p)}{n\epsilon^2} \to 0 \text{ in probability} \end{aligned}$$

Since $P(|1 - \frac{Y_n}{n} - (1-p)| < \epsilon) \geq 0$,

Then, $1 - \frac{Y_n}{n} \to 1 - p$ in probability.

(c) Answer:

Let $g(x) = x(1-x)$ be continuous for all x

then, $\frac{Y_n}{n}(1 - \frac{Y_n}{n}) = g(\frac{Y_n}{n}) = g(p) \to p(1-p)$ in probability by C.M.T

Question 5

Let W_n denote a random variable with mean μ and variance b/n^p, where $p > 0, \mu$, and b are constants (not functions of n). Prove that W_n converges in probability to μ.
Hint: Use Chebyshev's inequality.

Answer:

Since p>0,

$P(|W_n - \mu| > \epsilon) = P(|W_n - E(W_n)| > \epsilon) \leq \frac{Var(W_n)}{\epsilon^2} = \frac{b}{n^p \epsilon^2} \to 0$ in probability

Since $P(|W_n - \mu| > \epsilon) \geq 0$

$P(|W_n - \mu| > \epsilon) \to 0$ in probability because it is squeezed on both sides

Hence, $W_n \to \mu$ in probability