Dalhousie University
CSCI 3110 — Design and Analysis of Algorithms I
Fall 2014
Final Examination
December 5
3:30pm-6:30pm

Student Name: _____

Student ID Number: _____

FCS Username (CSID): _____

Signature: _____

Instructions (Read Carefully):

1. Aids allowed: one 8.5" by 11" piece of paper with anything written or printed on it (both sides). No textbooks, computers, calculators, or other aids.
2. This exam booklet has 11 pages, including this page. Ensure that you have a complete paper.
3. Understanding the exam questions is part of the exam. Therefore, questions will **not** be interpreted. Proctors will confirm or deny errors or ambiguities only. If you are unsure of your own understanding, clearly state reasonable assumptions that will not trivialize the questions.
4. The blank sheet given with this exam book is a piece of scratch paper. Do not submit.

Question	1	2	3	4	5	6	7	8	9	Total
Marks	15	12	20	13	10	7	8	7	8	100
Scores										
Marker										

1. [15 marks] True-false: 3 marks each. No justification necessary.

   (a) $\lg n = O(n)$.
   True

   (b) Dijkstra's algorithm returns false when the input graph has a negative weight cycle reachable from the source vertex.
   False

   (c) Kruskal's algorithm does not work when some edges of the graph have negative weights.
   False

   (d) If $f(n) = \Theta(g(n))$ then $g(n) = \Theta(f(n))$.
   True

   (e) $P \subseteq NP$.
   True

2. [12 marks] Multiple-choice — **no justification necessary**. Circle the *single* best answer. 4 marks each.

(a) What is the relationship between $f(n) = 2^{(\lg n)/2}$ and $g(n) = n/\lg n$?

ⓐ. $f(n) = O(g(n))$

b. $f(n) = \Theta(g(n))$

c. $f(n) = \omega(g(n))$

d. $f(n) = \Omega(g(n))$

(b) Which of the following four problems is NP-complete?

a. The maximum subarray problem

ⓑ. The subset-sum problem

c. The longest common subsequence problem

d. The single-source shortest path problem

(c) In class, we learned a randomized algorithm for the selection problem, and proved that its expected running time is $O(n)$. What does this result imply?

a. There is no $O(n)$-time deterministic algorithm for this problem

b. The running time of this algorithm is $O(n)$ only when the input distribution is uniform

c. Any deterministic algorithm that solves this problem requires sorting

ⓓ. The running time of this algorithm, when computed as an expectation over the distribution of the random number generator used in this algorithm, is $O(n)$

3. [20 marks] (Order of Growth)

(a) [4 marks] Complete the formal definition of $\Theta(g(n))$ by filling in the blank below:

$$\Theta(g(n)) = \{f(n) : \text{there exist positive constants } c_1,\ c_2 \text{ and } n_0 \text{ such that}$$

$$\underline{0 \le c_1 g(n) \le f(n) \le c_2 g(n) \text{ for all } n \ge n_0\}}$$

(b) [4 marks] Prove that $100 \lg n + 100\sqrt{n} = o(n)$.
This can be proved using either the definition of little-oh, or the limit approach.

$$\begin{aligned}
\lim_{n\to\infty} \frac{100 \lg n + 100\sqrt{n}}{n} &= \lim_{n\to\infty} \frac{100 \lg n}{n} + \lim_{n\to\infty} \frac{100}{\sqrt{n}} \\
&= 100 \lim_{n\to\infty} \frac{\lg n}{n} \\
&= 100 \lim_{n\to\infty} \frac{1/(n \ln 2)}{1} \quad \text{(l'Hôpital)} \\
&= 0.
\end{aligned}$$

(c) [12 marks] For each of the following recurrences, use the "master theorem" and give the solution using big-Θ notation. Simply write down your answer and no justification is necessary.
If the "master theorem" does not apply to a recurrence, indicate this, but you need not show your reasoning or give a solution.

- $T(n) = 2T(n/2) + \Theta(n^3)$

  $\Theta(n^3)$

- $T(n) = 4T(\lceil n/2 \rceil) + n^2$

  $\Theta(n^2 \lg n)$

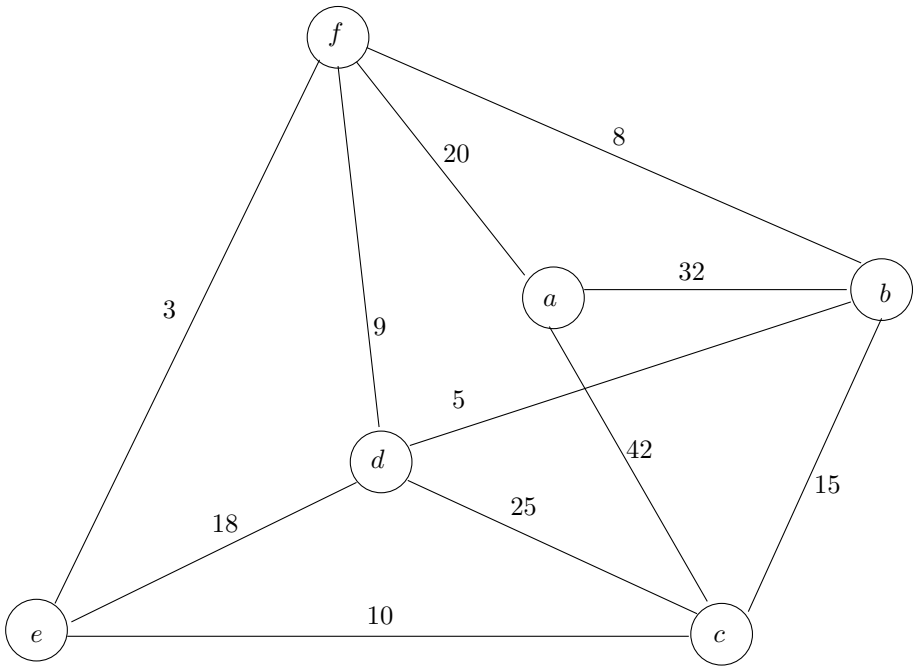
- $T(n) = 9T(n/3) + \Theta(n^2/\lg n)$

  Does not apply

- $T(n) = 2T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + n$

  $\Theta(n^{\lg 3})$

4. [13 marks] Consider the weighted, connected and undirected graph G below:



(a) [3 marks] The number of edges in a minimum spanning tree of G is (just write down the answer, no justification is needed):

5

(b) [5 marks] Write down the edges of a minimum spanning tree of G constructed using Kruskal's algorithm, *in the order that they are selected by Kruskal's algorithm*, in the line below.

You can use (a, b) to denote an edge between vertices a and b.

You need not actually draw the tree or show your steps.

$(e, f), (b, d), (b, f), (c, e), (a, f)$

(c) [5 marks] Choose a as the *starting vertex* for Prim's algorithm, and write down the edges of a minimum spanning tree of G constructed using Prim's algorithm, *in the order that they are selected by Prim's algorithm*, in the line below.

Use (a, b) to denote an edge between vertices a and b.

You need not actually draw the tree or show your steps.

$(a, f), (e, f), (b, f), (b, d), (c, e)$

5. [10 marks] (Computational Problems and Complexity)

(a) [3 marks] What is the main difference between the complexity classes NP-hard and NP-complete?

If a problem is NP-complete, then it must also be in NP. However, a problem that is NP-hard does not have to be in NP.

(b) [3 marks] Consider the following decision problem:
Given three integers u, v and w. Is there a common divisor of u and v that is greater than w?
Is this problem in the complexity class P? Justify your answer.

Yes.
This can be solved in polynomial time by first computing the greatest common divisor, x, of u and v using Euclid's algorithm (or the binary algorithm), and then compare x to w. If x is greater than w, return true. otherwise, return false.

(b) [4 marks] Let A, B and C be three decision problems. Use the definition of polynomial-time reduction to prove that if $A \leq_P B$ and $B \leq_P C$, then $A \leq_P C$.

Since $A \leq_P B$, there is a function f mapping instances of A to instances of B, preserving answers. Since $B \leq_P C$, there is a function g mapping instances of B to instances of C, preserving answers. f and g can be computed in polynomial time. Therefore, $g \circ f$ can also be computed in polynomial time. We further claim that $g \circ f$ maps instances of A to instances of C, preserving answers. To see this, let x be an arbitrary instance of A. If x is a "yes" instance of A, then $f(x)$ is a "yes" instance of B, and $g(f(x))$ is a "yes" instance of C. If x is a "no" instance of A, then $f(x)$ is a "no" instance of B, and $g(f(x))$ is a "no" instance of C.

6. [7 marks] Suppose that Alice has been using a communication channel to transmit messages to Bob. However, for some reason, this channel becomes noisy. As a result, some characters in the message could be lost during the transmission, so that only a selected portion of the original message is successfully transmitted.

   For example, Alice sent the following message to Bob:

   `WHO IS HARRY POTTER'S BEST FRIEND?`

   As some character are lost during transmission, Bob could receive the following message:

   `HO HARY POT'S BST END`

   We can model this phenomenon as a problem for strings. Given a string $X[1..n]$, we say that string $Y[1..m]$ is a *subsequence* of X if there are a set of indices $\{i_1, i_2, \ldots, i_k\}$, such that $y_1 = x_{i_1}, y_2 = x_{i_2}, \ldots, y_k = x_{i_k}$, and $i_j < i_{j+1}$ for $j = 1, 2, \ldots, k-1$. That is, we follow the definition of subsequence given in class. Then, the message that Bob received from Alice is a subsequence of the message that Alice sent.

   In a case of transmission via a noisy channel, it could be useful to know if the received message is indeed a subsequence of the message sent. Therefore, give an $O(m+n)$-time algorithm for determining whether a given string, $Y[1..m]$, is a subsequence of a given string, $X[1..n]$. Describe your algorithm in English. You can provide pseudocode if it helps you describe your algorithm, but it is not required. Analyze the running time of your algorithm.

   You are NOT required to prove the correctness of your algorithm.

   Hint: design a greedy algorithm.

   Solution: Search for the first occurrence of character $Y[1]$ in X. If $Y[1]$ does not appear in X, then return false. Otherwise, let $X[n_1]$ be the first occurrence of $Y[1]$ in X. Then, search for the first occurrence of $Y[2]$ in $X[n_1+1..n]$. If $Y[2]$ does not appear in $X[n_1+1..n]$, then return false. Otherwise, let $X[n_2]$ be the first occurrence of $Y[2]$ in $X[n_1+1..n]$ and repeat this process. If the process terminates without returning false, then return true.

   Since we scan each character of X and Y at most once, the total running time is $O(n+m)$.

7. [8 marks] Assume that you are given an array $A[1..n]$ of distinct elements. A *local minimum* of A is defined as follows:

   - $A[1]$ is a local minimum if it is smaller than $A[2]$;
   - $A[n]$ is a local minimum if it is smaller than $A[n-1]$;
   - An element A[i] with $1 < i < n$ is a local minimum if it is smaller than both $A[i-1]$ and $A[i+1]$.

   Thus an array may have one or more local minimum elements. Your tasks is to design an algorithm that can find one local minimum of A.

   (a) [3 marks] Consider an algorithm that scans the array from left to right, while doing some comparisons until it sees a local minimum. Give an array of n elements that will make this algorithm scan the entire array.

   The content of this array is $\{n, n-1, \ldots, 1\}$.
   (Any array in which all elements are in descending order will do.)

   (b) [2 marks] Describe in concise English how to solve the above problem efficiently using divide-and-conquer in $O(\lg n)$ time. You need not give pseudocode, but you can if it helps with your explanation.

   Let $A[m]$ be the element in the middle of the array, i.e., $m = \lfloor (n+1)/2 \rfloor$. We compare $A[m]$ with $A[m-1]$ and $A[m+1]$. If $A[m]$ is less than both $A[m-1]$ and $A[m+1]$, then we return $A[m]$. Otherwise, if $A[m] > A[m-1]$, then we recursively look for a local minimum element in $A[1..m-1]$ and return it as the result. Otherwise, we have $A[m] > A[m+1]$, and we recursively look for a local minimum element in $A[m+1..n]$ and return it as the result.

   (c) [2 marks] Justify the correctness of your algorithm briefly.

   To make it easier to justify the correctness of the algorithm, we conceptually add two elements to the array: $A[0]$ which stores an arbitrary number greater than $A[1]$ (e.g. $A[1]+1$), and $A[n+1]$ which stores an arbitrary number greater than $A[n]$ (e.g. $A[n]+1$).
   By the algorithm described above, we observe that when we recurse on a subarray $A[p..q]$, we always have $A[p-1] > A[p]$ and $A[q+1] > A[q]$. This guarantees that the local minimum element in $A[p..q]$ is also a local minimum element in $A[1..n]$.

   (d) [1 marks] Analyze the running time of your algorithm.

   The running time satisfies the recurrence $T(n) = T(n/2)+1$. By master's theorem, $T(n) = O(\lg n)$.

8. [7 marks] Suppose that you are hired by a company, which set up its computer network a year ago to link together its n offices spread across the globe. You have reviewed the work done at that time, and noted that they modeled their network as a connected, undirected graph, G, with n vertices, one for each office, and m edges, one for each possible connection. Each edge is assigned a weight, which was equal to the annual rent that it costs to use that edge for communication purposes. Then they computed a minimum spanning tree, T, for G to decide which of the m edges in G to lease.

   Now it is time to renew the leases for connecting the vertices in G. You noticed that the rent for each possible connection is unchanged, with the exception of one connection used in T, which has been increased. That is, in the graph, one edge, e, in T now has a greater weight than before, while the weights of all other edges remain unchanged.

   Your tasks is to design an $O(n+m)$-time algorithm to update T, to find a new minimum spanning tree T' for G given the change in weight for the edge e.

   (a) [2 marks] Describe in concise English how to solve the above problem. You need not give pseudocode, but you can if it helps with your explanation.

   We first remove the edge e from T. This will divide T into two trees, T_1 and T_2. Starting from each endpoint of e, we perform a depth-first traversal in T_1 or T_2, and set a flag for each vertex of G to indicate whether this vertex is in T_1 or T_2. Then we loop thorough all the edges of G, and find the lowest-weight edge, e', among all the edges that have one endpoint in T_1 and one endpoint in T_2. We then use e' to connect T_1 and T_2, and the resulting tree is a minimum spanning tree of the new graph.

   (b) [3 marks] Justify the correctness of your algorithm.

   Let V_1 denote the set of vertices of T_1, and V_2 the set of vertices of T_2. Let G_1 and G_2 denote the subgraphs induced by V_1 and V_2, respectively. Then, any spanning tree consists of three components: a spanning tree for G_1, a spanning tree for G_2, and an edge with one endpoint in V_1 and another endpoint in V_2. Our algorithm minimizes the weight of the edge chosen to connect the two spanning trees of G_1 and G_2. Before we update the weight of e, T_1 is a minimum spanning tree of G_1, for otherwise, we could replace T_1 by a spanning tree with a lower total weight and improve T. Since no edge weight in G_1 has been changed, T_1 remains to be a MST of G_1 after the update. Similarly, T_2 is a MST of G_2 after the update. Therefore, the new tree that we construct minimizes the weights of each of its three components, and is thus a MST.

   (c) [2 marks] Analyze the running time of your algorithm.

   As T has $n-1$ edges, it requires $O(n)$ time to perform a DFS in T_1 and T_2. It requires $O(m)$ time to go through all edges to find e'. Therefore, the total running time is $O(n+m)$.

9. [8 marks] Given an array $A[1..n]$ of integers, the *maximum product sum* is the largest sum that can be formed by multiplying adjacent elements in the array. Each element can be matched with at most one of its neighbours.

   For example, if $A = \{1, 2, 3, 1\}$, then the maximum product sum is $1 + (2 \times 3) + 1 = 8$. If $A = \{2, 2, 1, 3, 2, 1, 2, 2, 1, 2\}$, then the maximum product sum is $(2 \times 2) + 1 + (3 \times 2) + 1 + (2 \times 2) + 1 + 2 = 19$.

   More precisely, for a set of array indices $S = \{i_1, i_2, \ldots, i_k\}$, with $1 \leq i_1 < i_2 < \ldots < i_k < n$, and $i_j - i_{j-1} > 1$ for any $j \in [2, k]$, the *product sum* is defined as $\sum_{i \in S}(A[i]A[i+1]) + \sum_{i \notin S} A[i]$. Then, the maximum product sum is the largest among product sums for all possible S.

   (a) [2 marks] Compute the maximum product sum of $A = \{1, 4, 3, 2, 3, 4, 2\}$.

   $29 = 1 + (4 \times 3) + 2 + (3 \times 4) + 2$

   (b) [6 marks] Describe a dynamic programming algorithm that computes the maximum product sum for any given array A.
   Provide an English description AND pseudocode.
   Analyze the running time of your algorithm.
   No justification of correctness is required.

   Our algorithm computes an array, $P[1..i]$, in which $P[i]$ stores the maximum product sum for $A[1..i]$. We compute entries of $P[i]$ starting from $i = 0$. In each iteration, we compute $P[i]$ using the following recurrence:

$$P[i] = \begin{cases} 0 & \text{if } i = 0; \\ A[i] & \text{if } i = 1; \\ \max(P[i-1] + A[i], P[i-2] + A[i-1] \cdot A[i]) & \text{otherwise.} \end{cases} \quad (1)$$

   The following is the pseudocode:

   `maximum_product_sum`$(A[1..n])$

```
1: P[0] ← 0
2: P[1] ← A[1]
3: for i ← 2 to n do
4:     if P[i − 1] + A[i] ≥ P[i − 2] + A[i − 1] · A[i] then
5:         P[i] ← P[i − 1] + A[i]
6:     else
7:         P[i] ← P[i − 2] + A[i − 1] · A[i]
8: return P[n]
```

   To analyze the running time, we observe that the while loop dominates the running time. As each iteration of the loop uses $O(1)$ time and the loop iterates at most n times, the total running time is $O(n)$.