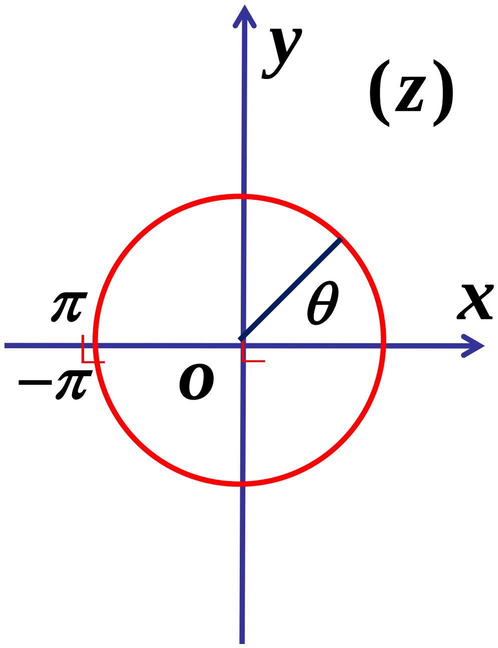
3.6 积分的典型例题解析

例1　设$C: z=e^{i\theta}$，θ从$-\pi$到π的一周，求 $\oint_{|z|=1} x\mathrm{d}z.$

解1 $\oint_{|z|=1} x\mathrm{d}x + ix\mathrm{d}y$　　　$C:\begin{cases} x=\cos\theta, \\ y=\sin\theta, \end{cases} -\pi<\theta\le\pi.$

$$= \int_{-\pi}^{\pi} -\cos\theta\sin\theta\mathrm{d}\theta + i\cos^2\theta\mathrm{d}\theta$$

$$= 4i\int_0^{\frac{1}{2}\pi} \cos^2\theta\mathrm{d}\theta = \pi i$$



解2 $\oint_{|z|=1} \dfrac{z+\bar{z}}{2}\mathrm{d}z$

$$= \oint_{|z|=1} \frac{z}{2}\mathrm{d}z + \oint_{|z|=1} \frac{1}{2z}\mathrm{d}z = \pi i$$

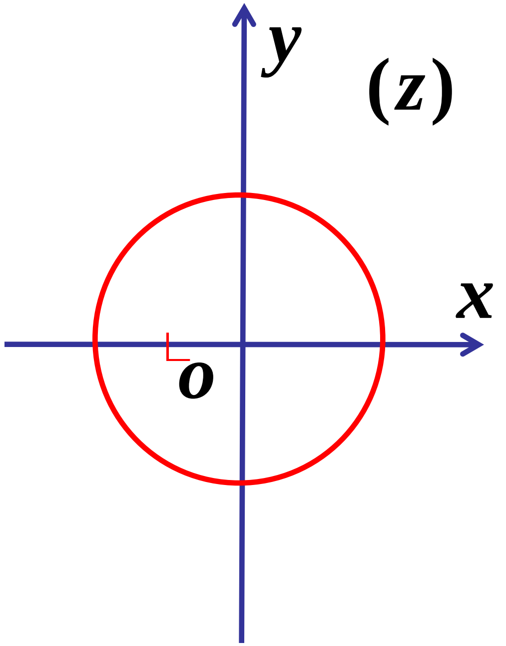
例2　计算 $\oint\limits_{|z|=1} \dfrac{\sin \pi z}{2z+1} dz.$

解　$2z+1=0,\quad z=-\dfrac{1}{2}.$

$$\oint\limits_{|z|=1} \frac{\sin \pi z}{2z+1} dz = \frac{1}{2}\oint\limits_{|z|=1} \frac{\sin \pi z}{z+\frac{1}{2}} dz$$

$$= \pi i \sin \pi z \Big|_{-\frac{1}{2}}$$

$$= -\pi i.$$



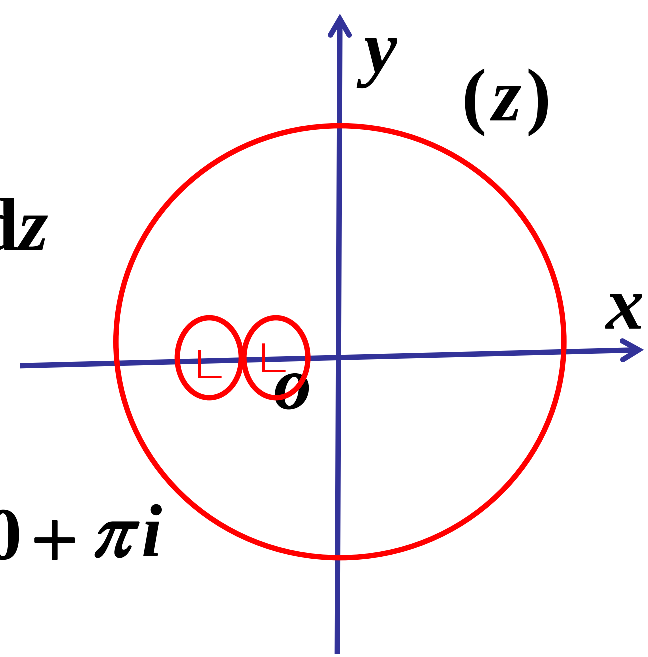
例3　计算积分 $\oint\limits_{|z|=1} \dfrac{\cos 2\pi z}{8z^2+6z+1}\mathrm{d}z.$

解　$8z^2+6z+1=(4z+1)(2z+1)$　　$z_1=-\dfrac{1}{4}, z_2=-\dfrac{1}{2}.$

$$\oint\limits_{|z|=1} = \oint\limits_{|z+\frac{1}{4}|=\frac{1}{8}} + \oint\limits_{|z+\frac{1}{2}|=\frac{1}{8}} \mathrm{d}z$$

$$= \oint\limits_{|z+\frac{1}{4}|=\frac{1}{8}} \frac{\dfrac{\cos 2\pi z}{4(2z+1)}}{z+\dfrac{1}{4}}\mathrm{d}z + \oint\limits_{|z+\frac{1}{2}|=\frac{1}{8}} \frac{\dfrac{\cos 2\pi z}{2(4z+1)}}{z+\dfrac{1}{2}}\mathrm{d}z$$

$$=2\pi i \left.\frac{\cos 2\pi z}{4(2z+1)}\right|_{-\frac{1}{4}} + 2\pi i \left.\frac{\cos 2\pi z}{2(4z+1)}\right|_{-\frac{1}{2}} = 0+\pi i$$

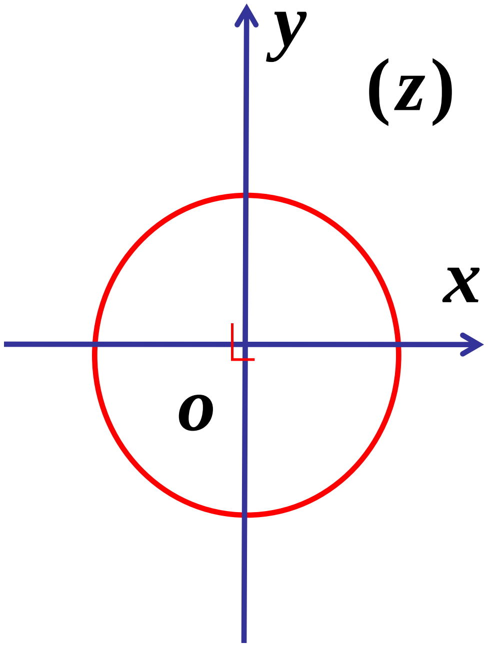


例4　计算下列积分 $\oint\limits_{|z|=1} \frac{e^z}{z^3}\mathrm{d}z.$

解　$\oint\limits_{|z|=1} \frac{f(z)}{(z-0)^3}\mathrm{d}z=\frac{2\pi i}{2!}f''(0)$

$$\oint\limits_{|z|=1} \frac{e^z}{z^3}\mathrm{d}z = \frac{2\pi i}{2!}(e^z)''_{z=0}$$

$$=\pi i.$$

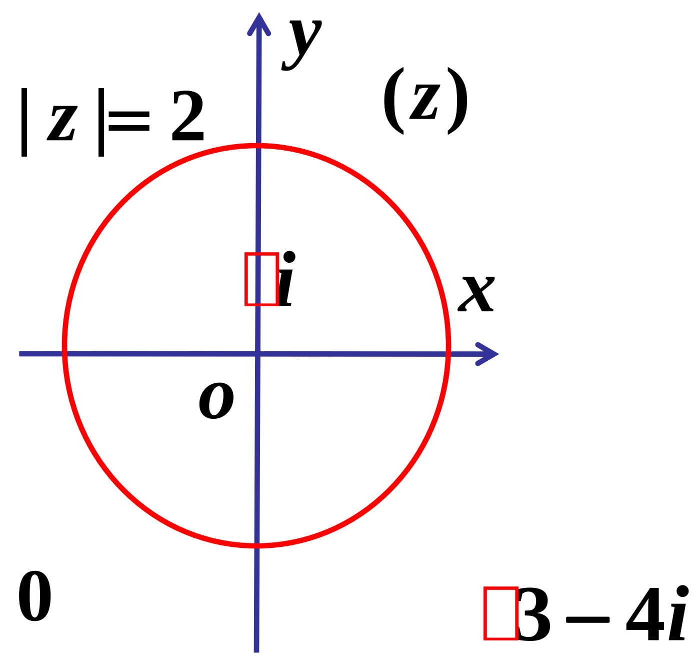


例5 设$f(z)=\oint\limits_{|\xi|=2}\dfrac{e^{\frac{\pi}{4}\xi}}{\xi-z}\mathrm{d}\xi$，求$f(i)$，$f(3-4i)$.

解 $$f(i)=\oint\limits_{|\xi|=2}\frac{e^{\frac{\pi}{4}\xi}}{\xi-i}\mathrm{d}\xi=2\pi ie^{\frac{\pi}{4}i}$$

$$=\pi i(\sqrt{2}+i\sqrt{2})$$

$$=-\pi\sqrt{2}+i\pi\sqrt{2}.$$



$$f(3-4i)=\oint\limits_{|\xi|=2}\frac{e^{\frac{\pi}{4}\xi}}{\xi-(3-4i)}\mathrm{d}\xi=0$$