

# Chapter 9

---

## Naïve Bayes

David J. Hand

### Contents

9.1	Introduction .....	163
9.2	Algorithm Description .....	164
9.3	Power Despite Independence .....	167
9.4	Extensions of the Model .....	169
9.5	Software Implementations .....	171
9.6	Examples .....	171
9.6.1	Example 1 .....	171
9.6.2	Example 2 .....	173
9.7	Advanced Topics .....	174
9.8	Exercises .....	175
	References .....	176

---

### 9.1 Introduction

Given a set of objects, each of which belongs to a known class, and each of which has a known vector of variables, our aim is to construct a rule which will allow us to assign future objects to a class, given only the vectors of variables describing the future objects. Problems of this kind, called problems of *supervised classification*, are ubiquitous, and many methods for constructing such rules have been developed. One very important method is the *naïve Bayes* method—also called *idiot’s Bayes*, *simple Bayes*, and *independence Bayes*. This method is important for several reasons, including the following. It is very easy to construct, not needing any complicated iterative parameter estimation schemes. This means it may be readily applied to huge data sets. It is easy to interpret, so users unskilled in classifier technology can understand why it is making the classification it makes. And, particularly important, it often does surprisingly well: It may not be the best possible classifier in any given application, but it can usually be relied on to be robust and to do quite well. For example, in an early classic study comparing supervised classification methods, Titterton et al. (1981) found that the independence model yielded the best overall result, while Mani et al. (1997) found that the model was most effective in predicting

breast cancer recurrence. Many further examples showing the surprising effectiveness of the naïve Bayes method are listed in Hand and Yu (2001) and further empirical comparisons, with the same result, are given in Domingos and Pazzani (1997). Of course, there are also some other studies which show poorer relative performance from this method: For a comparative assessment of such studies, see Jamain and Hand (2008).

For convenience, most of this chapter will describe the case in which there are just two classes. This is, in fact, the most important special case as many situations naturally form two classes (right/wrong, yes/no, good/bad, present/absent, and so on). However, the simplicity of the naïve Bayes method is such that it permits ready generalization to more than two classes.

Labeling the classes by  $i = 0, 1$ , our aim is to use the initial set of objects which have known class memberships (known as the training set) to construct a score such that larger scores are associated with class 1 objects (say) and smaller scores with class 0 objects. New objects are then classified by comparing their score with a “classification threshold.” New objects with a score larger than the threshold will be classified into class 1, and new objects with a score less than the threshold will be classified into class 0.

There are two broad perspectives on supervised classification, termed the *diagnostic paradigm* and the *sampling paradigm*. The diagnostic paradigm focuses attention on the differences between the classes—on *discriminating* between the classes—while the sampling paradigm focuses attention on the individual distributions of the classes, comparing these to indirectly produce a comparison between the classes. As we show below, the naïve Bayes method can be viewed from either perspective.

## 9.2 Algorithm Description

Beginning with the sampling paradigm, define  $P(i|x)$  to be the probability that an object with measurement vector  $x = (x_1, \dots, x_p)$  belongs to class  $i$ ,  $f(x|i)$  to be the conditional distribution of  $x$  for class  $i$  objects,  $P(i)$  to be the probability that an object will belong to class  $i$  if we know nothing further about it (the “prior” probability of class  $i$ ), and  $f(x)$  to be the overall mixture distribution of the two classes:

$$f(x) = f(x|0)P(0) + f(x|1)P(1)$$

Clearly, an estimate of  $P(i|x)$  itself would form a suitable score for use in a classification rule. We would need to choose some suitable threshold probability to act as the classification threshold to yield a classification. For example, it is very common to use a threshold of  $1/2$ , so that each new object is assigned to the class it is estimated as most likely to have come from. More sophisticated approaches take into account the relative severities of different kinds of misclassifications when choosing the threshold.

A simple application of Bayes theorem yields  $P(i|x) = f(x|i)P(i)/f(x)$ , and to obtain an estimate of  $P(i|x)$  from this, we need to estimate each of the  $P(i)$  and each of the  $f(x|i)$ .

If the training set is a simple random sample drawn from the overall population distribution  $f(x)$ , the  $P(i)$  can be estimated directly from the proportion of class  $i$  objects in the training set. Sometimes, however, the training set is obtained by more complicated means. For example, in many problems the classes are *unbalanced*, with one being much larger than the other (e.g., in credit card fraud detection, where only 1 in 1,000 transactions may be fraudulent; in rare disease detection, where the ratio may be even more extreme; and so on). In such cases, the larger of the two classes is often subsampled. For example, perhaps only 1 in 10 or 1 in 100 of the larger class will be used in the training set. If this is the case, then it is necessary to reweight the simple observed proportion in the training set to yield an estimate of  $P(i)$ . In general, if the observations are not drawn as a simple random sample from the training set, some thought will need to go into how best to estimate the  $P(i)$ .

The core of the naïve Bayes method lies in the method for estimating the  $f(x|i)$ . The naïve Bayes method assumes that the components of  $x$  are independent within each class, so that  $f(x|i) = \prod_{j=1}^p f(x_j|i)$ —hence the alternative name of “independence Bayes.” Each of the univariate marginal distributions,  $f(x_j|i)$ ,  $j = 1, \dots, p$ ;  $i = 0, 1$ , is then estimated separately. By this means, the  $p$  dimensional multivariate problem is reduced to  $p$  univariate estimation problem. Univariate estimation is familiar and simple, and requires smaller training set sizes to obtain accurate estimates than does the estimation of multivariate distributions.

If the marginal distributions  $f(x_j|i)$  are discrete, with  $x_j$  taking only a few values, one can estimate each of the  $f(x_j|i)$  by simple multinomial histogram-type estimators. Because this is so straightforward, this is a very common approach to the naïve Bayes estimator, and many implementations adopt this approach. Indeed, it is so straightforward that many implementations partition any continuous variables (age, weight, income, and so on) into cells so that a multinomial histogram-type estimator can be constructed for all of the variables. At first glance, this strategy might seem to be a weak one. After all, it means that any notion of continuity between neighboring cells of the histogram has been sacrificed. It also requires the cells to be wide enough to contain sufficient data points that accurate probability estimates can be obtained. On the other hand, it can be regarded as providing a very general nonparametric estimate of the univariate distribution, so avoiding any distributional assumptions. In particular, it is a nonlinear transformation, so that, for example, the relationship between estimates of  $f(x_j|i)$  does not need to be monotonic in  $x_j$ .

At a cost of more computational expense (in particular, at the cost of losing the simple counting procedure which underlies histogram-type estimates), one can fit more elaborate models to the univariate marginals. For example, one can assume particular parametric forms for the distributions (e.g., normal, lognormal, and so on) and estimate their parameters by standard and very familiar estimators, or one can adopt more sophisticated nonparametric estimators, such as kernel density estimation. While these do sacrifice the speed of the histogram approach, this is less important in the modern world in which all the calculations will be done by machine. Having

said that, there is another reason why one might prefer to use the histogram approach based on forcing all the variables to be discrete—that of interpreting the results. We discuss this below.

The assumption of independence at the core of the naïve Bayes method is clearly a strong one. It is unlikely to be true for most real problems. (How often does a diagonal covariance matrix arise from real data in practice?) A priori, then, one might expect the method to perform poorly precisely because of this improbable assumption lying at its core. However, the fact is that it often does surprisingly well in real practical applications. Reasons for this counterintuitive result are discussed below.

So far we have approached the naïve Bayes method from the sampling paradigm, describing it as being based on estimating the separate class conditional distributions using the simplifying assumption that the variables in each of these distributions were independent. However, the elegance of the naïve Bayes method only really becomes apparent when we note that we can obtain classifications equivalent to the above if we use any strictly monotonic transformation of  $P(i|x)$ , transforming the classification threshold in a similar way. To see this, note that if  $T$  is a strictly monotonic increasing transformation then

$$P(i|x) > P(i|y) \Leftrightarrow T(P(i|x)) > T(P(i|y))$$

and, in particular,  $P(i|x) > t \Leftrightarrow T(P(i|x)) > T(t)$ . This means that if  $t$  is the classification threshold with which  $P(i|x)$  is compared, then comparing  $T(P(i|x))$  with  $T(t)$  will yield the same classification results. (We will assume only monotonic increasing transformations, though the extension to monotonic decreasing transformations is trivial.)

One such monotonic transformation is the ratio

$$P(1|x)/(1 - P(1|x)) = P(1|x)/P(0|x) \quad (9.1)$$

Using the naïve Bayes assumption that the variables within each class are independent, so that the distribution for class  $i$  has the form  $f(x|i) = \prod_{j=1}^p f(x_j|i)$ , the ratio  $P(1|x)/(1 - P(1|x))$  can be rewritten:

$$\frac{P(1|x)}{1 - P(1|x)} = \frac{P(1) \prod_{j=1}^p f(x_j|1)}{P(0) \prod_{j=1}^p f(x_j|0)} = \frac{P(1)}{P(0)} \prod_{j=1}^p \frac{f(x_j|1)}{f(x_j|0)} \quad (9.2)$$

The log transformation is also monotonic (and combination of monotonic functions yields monotonic functions) so that another alternative score is given by

$$\ln \frac{P(1|x)}{1 - P(1|x)} = \ln \frac{P(1)}{P(0)} + \sum_{j=1}^p \ln \frac{f(x_j|1)}{f(x_j|0)} \quad (9.3)$$

If we define  $w_j(x_j) = \ln(f(x_j|1)/f(x_j|0))$  and  $k = \ln\{P(1)/(P(0))\}$  we see that Equation (9.3) takes the form of a simple sum

$$\ln \frac{P(1|x)}{1 - P(1|x)} = k + \sum_{j=1}^p w_j(x_j) \quad (9.4)$$

of contributions from the separate variables. Since the score  $S = k + \sum_{j=1}^p w_j(x_j)$  is a direct estimate of (a monotonic transformation of)  $P(1|x)$ , it is based on the diagnostic paradigm. The ease of interpretation now becomes apparent: The naïve Bayes model is simply a sum of transformed values of the raw  $x_j$  values.

In cases when each variable is discrete, or is made to be discrete by partitioning it into cells, Equation (9.4) takes a particularly simple form. Suppose that variable  $x_j$  takes a value in the  $k_j$ th cell of the variable, denoted  $x_j^{(k_j)}$ . Then  $w_j(x_j^{(k_j)})$  is simply a logarithm of a ratio of proportions: the proportion of class 1 points which fall into the  $k_j$ th cell of variable  $x_j$  divided by the proportion of class 0 points which fall into the  $k_j$ th cell of variable  $x_j$ . These  $w_j(x_j^{(k_j)})$  are called *weights of evidence* in some applications:  $w_j(x_j^{(k_j)})$  shows the contribution the  $j$ th variable makes toward the total score, or the evidence in favor of the object belonging to class 1 that is provided by the  $j$ th variable. Such weights of evidence are useful in identifying which variables are important in assigning any particular object to a class. (This is vital in some applications, such as credit scoring in personal banking, where the law requires that reasons must be given if an application for a loan is declined.)

---

### 9.3 Power Despite Independence

The assumption of independence of the  $x_j$  within each class implicit in the naïve Bayes model might seem unduly restrictive. After all, as noted above, variables are rarely independent in real problems. In fact, however, various factors may come into play which means that the assumption is not as detrimental as it might seem (Hand and Yu, 2001).

Firstly, the complexity of  $p$ -univariate marginal distributions is far lower than that of a single  $p$ -variate multivariate distribution. This means that far fewer data points are needed to obtain a given accuracy under the independence model than are needed without this assumption. Put another way, the available sample will lead to an estimator with smaller variance if one is prepared to restrict the model form by assuming independence of the variables within classes. Of course, if the assumption is not true, then there is a risk of bias. This is a manifestation of the classic bias/variance trade-off, which applies to all data analysis modeling, and is not specific to the naïve Bayes model.

To decrease the risk of bias arising from the assumption of independence, a simple modification of the basic naïve Bayes model has been proposed. To understand the reasoning behind this modification, consider the special case in which the marginal distributions of all the variables are the same, and the extreme in which the variables are perfectly correlated. This means that, for any given class, the probability that the  $x_j$ th variable takes a value  $r$  is the same for all variables. In this perfectly correlated

case, the naïve Bayes estimator is

$$\frac{P(1|x)}{P(0|x)} = \frac{P(1)}{P(0)} \left[ \frac{f(x_k|1)}{f(x_k|0)} \right]^p$$

while the true odds ratio is

$$\frac{P(1|x)}{P(0|x)} = \frac{P(1)}{P(0)} \frac{f(x_k|1)}{f(x_k|0)}$$

for any  $k \in \{1, \dots, p\}$ . We can see from this that if  $f(x_k|1)/f(x_k|0)$  is larger than 1, the presence of correlation will mean that the naïve Bayes estimator tends to overestimate  $P(1|x)/P(0|x)$ , and if  $f(x_k|1)/f(x_k|0)$  is less than 1, the presence of correlation will mean that the naïve Bayes estimator tends to underestimate  $P(1|x)/P(0|x)$ . This phenomenon immediately suggests modifying the naïve Bayes estimator by raising the  $f(x_k|1)/f(x_k|0)$  ratios by some power less than 1, to shrink the overall estimator toward the true odds. In general, this yields the improved naïve Bayes estimator

$$\frac{P(1|x)}{P(0|x)} = \frac{f(x|1)P(1)}{f(x|0)P(0)} = \frac{P(1)}{P(0)} \prod_{j=1}^p \left[ \frac{f(x_j|1)}{f(x_j|0)} \right]^\beta$$

with  $\beta < 1$ .  $\beta$  is typically chosen by searching over possible values and choosing that which gives best predictive results by means of a method such as cross-validation. We can also see that this leads to a shrinkage factor appearing as a coefficient of the  $w_j(x_j)$  terms in Equation (9.4).

A second reason why the assumption of independence is not as unreasonable as might at first seem is that often data might have undergone an initial variable selection procedure in which highly correlated variables have been eliminated on the grounds that they are likely to contribute in a similar way to the separation between classes. Think of variable selection methods in linear regression, for example. This means that the relationships between the remaining variables might well be approximated by independence.

A third reason why the independence assumption may not be too detrimental is that only the decision surface matters. While the assumption might lead to poor estimates of probability or of the ratio  $P(1|x)/P(0|x)$ , this does not necessarily imply that the decision surface is far from (or even different from) the true decision surface. Consider, for example, a situation in which the two classes have multivariate normal distributions with the same (nondiagonal) covariance matrix, and with the vector of differences between the means lying parallel to the first principal axis of the covariance matrix. The optimal decision surface is linear and is the same with the true covariance matrices and under the independence assumption.

Finally, of course, the decision surface produced by the naïve Bayes model can in fact have a complicated nonlinear shape: The surface is linear in the  $w_j(x_j)$  but highly nonlinear in the original variables  $x_j$ , so that it can fit quite elaborate surfaces.

## 9.4 Extensions of the Model

We have seen that the naïve Bayes model is often surprisingly effective. It also has the singular merit of being very easy to compute, especially if the discrete variable version is used. Coupled with the ease of understanding and interpretation of the model, perhaps especially in terms of the simple points-scoring perspective in Equation (9.4), these factors explain why it is so widely used. However, its very simplicity, along with the fact that its core assumption often appears unrealistic, has led many researchers to propose extensions of it in an attempt to improve its predictive accuracy.

We have already seen one of these above, to ease the independence assumption by shrinking the probability estimates. Shrinking has also been proposed to improve the simplistic multinomial estimate of the proportions of objects falling into each category in the case of discrete predictor variables. So, if the  $j$ th discrete predictor variable,  $x_j$ , has  $c_r$  categories, and if  $n_{jr}$  of the total of  $n$  objects fall into the  $r$ th category of this variable, the usual multinomial estimator of the probability that a future object will fall into this category,  $n_{jr}/n$ , is replaced by  $(n_{jr} + c_r^{-1})/(n + 1)$ . This shrinkage, which is also sometimes called the *Laplacian correction*, also has a direct Bayesian interpretation. It can be useful if the sample size and cell widths are such that there may not be very many objects in a cell.

Perhaps the most obvious way of easing the independence assumption is by introducing extra terms in the models of the distributions of  $x$  in each class, to allow for interactions. This has been attempted in a large number of ways, but all of them necessarily introduce complications, and so sacrifice the basic simplicity and elegance of the naïve Bayes model. In particular, if an interaction between two of the variables in  $x$  is to be included in the model, then the estimate cannot be based merely on the univariate marginals.

Within the  $i$ th class, the joint distribution of  $x$  is

$$f(x|i) = f(x_1|i)f(x_2|x_1, i)f(x_3|x_1, x_2, i) \dots f(x_p|x_1, x_2, \dots, x_{p-1}, i) \quad (9.5)$$

and this can be approximated by simplifying the conditional probabilities. The extreme arises with  $f(x_j|x_1, \dots, x_{j-1}, i) = f(x_j|i)$  for all  $j$ , and this is the naïve Bayes method. Obviously, however, models between these two extremes can be used. If the variables are discrete, one can estimate appropriate models, with arbitrary degrees of interaction included, by using log-linear models. For continuous variables, graphical models and the literature on conditional independence graphs are appropriate. An example which is appropriate in some circumstances is the Markov model

$$f(x|i) = f(x_1|i)f(x_2|x_1, i)f(x_3|x_2, i), \dots, f(x_p|x_{p-1}, i) \quad (9.6)$$

This is equivalent to using a subset of two-way marginal distributions instead of merely the univariate marginal distributions in the naïve Bayes model.

Yet other extensions combine naïve Bayes models with tree methods (e.g., Langley, 1993), for example splitting the overall population into subsets on the basis of the values the objects take on some of the variables and then fitting naïve Bayes models

to each subset. Such models are popular in some applications, where they are known as *segmented scorecards*. The segmentation is a way to allow for interactions which would cause difficulties if a single overall independence model was fitted.

Another way of embedding naïve Bayes models in higher-level approaches is by means of multiple classifier systems, for example, in a random forest or via boosting.

There is a very close relationship between the naïve Bayes model and another very important model for supervised classification: the logistic regression model. This was originally developed within the statistical community, and is very widely used in medicine, banking, marketing, and other areas. It is more powerful than the naïve Bayes model, but this extra power comes at the cost of necessarily requiring a more complicated estimation scheme. In particular, as we will see, although it has the same attractively simply basic form as the naïve Bayes model, the parameters (e.g., the  $w_j(x_j^{(k_j)})$ ) cannot be estimated simply by determining proportions, but require an iterative algorithm.

In examining the naïve Bayes model above, we obtained the decomposition Equation (9.2) by adopting the independence assumption. However, exactly the same structure for the ratio results if we model  $f(x|1)$  by  $g(x) \prod_{j=1}^p h_1(x_j)$  and  $f(x|0)$  by  $g(x) \prod_{j=1}^p h_0(x_j)$ , where the function  $g(x)$  is the same in each model. If  $g(x)$  does not factorize into a product of components, one for each of the raw  $x_j$ , we are not assuming independence of the  $x_j$ . The dependence structure implicit in  $g(x)$  can be as complicated as we like—the only restriction being that it is the same in the two classes; that is, that  $g(x)$  is common in the factorizations of  $f(x|1)$  and  $f(x|0)$ . With these factorizations of the  $f(x|i)$ , we obtain

$$\frac{P(1|x)}{1 - P(1|x)} = \frac{P(1)g(x) \prod_{j=1}^p h_1(x_j)}{P(0)g(x) \prod_{j=1}^p h_0(x_j)} = \frac{P(1)}{P(0)} \cdot \frac{\prod_{j=1}^p h_1(x_j)}{\prod_{j=1}^p h_0(x_j)} \quad (9.7)$$

Since the  $g(x)$  terms cancel, we are left with a structure identical to Equation (9.2), although the  $h_i(x_j)$  are not the same as the  $f(x_j|i)$  (unless  $g(x) \equiv 1$ ). Note that in this factorization it is not even necessary that the  $h_i(x_j)$  be probability density functions. All that is needed is that the overall products  $g(x) \prod_{j=1}^p h_i(x_j)$  are densities.

The model in Equation (9.7) is just as simple as the naïve Bayes model, and takes exactly the same form. In particular, by taking logs we end up with a points-scoring model as in Equation (9.4). But the model in Equation (9.7) is more flexible than the naïve Bayes model because it does not assume independence of the  $x_j$  in each class. Of course, this considerable extra flexibility of the logistic regression model is not obtained without cost. Although the resulting model form is identical to the naïve Bayes model form (with different parameter values, of course), it cannot be estimated by looking at the univariate marginals separately: An iterative procedure must be used. Standard statistical texts (e.g., Collett, 1991) give algorithms for estimating the parameters of logistic regression models. Often an iterative proportional weighted least squares method is used to find the parameters which maximize the likelihood.

The version of the naïve Bayes model based on the discretization transformation of the raw  $x_j$  can be generalized to yield other extensions. In particular, the more general



class of *generalized additive models* (Hastie and Tibshirani, 1990) take exactly the form of additive combinations of transformations of the  $x_j$ .

The naïve Bayes model is tremendously appealing because of its simplicity, elegance, robustness, as well as the speed with which such a model can be constructed, and the speed with which it can be applied to produce a classification. It is one of the oldest formal classification algorithms, and yet even in its simplest form it is often surprisingly effective. A large number of modifications have been introduced, by the statistical, data mining, machine learning, and pattern recognition communities, in an attempt to make it more flexible, but one has to recognize that such modifications are necessarily complications, which detract from its basic simplicity.

---

## 9.5 Software Implementations

The simplicity of the naïve Bayes algorithm means that, in its basic form, it has been very widely implemented, and many free versions are available on the Web. The open-source Weka implementation (<http://www.cs.waikato.ac.nz/ml/weka/>) allows the individual variables to be modeled by normal distributions, by kernel estimates, or by splitting them into discrete categories.

Perhaps it is worthwhile making a cautionary comment. The term *Bayesian* has several different interpretations, and its now common use in the phrase “naïve Bayes classifier” can mislead the unwary. In particular, “Bayesian networks” are more general classes of models, which include the naïve Bayes model as a special case, but which generally also allow various interactions to be included in the model. An example of the sorts of confusion this can lead to is described in Jamain and Hand (2005).

---

## 9.6 Examples

### 9.6.1 Example 1

To illustrate the principles of the naïve Bayes method, consider the artificial data set shown in Table 9.1. The aim is to use these data as a training set to construct a rule which will allow prediction of variable  $D$  for future customers, where  $D$  is default on a bank loan (the last column, labeled 1 for default and 0 for nondefault). The variables which will be used for the prediction are columns 1 to 3: time with current employer,  $T$ , in years; size of loan requested,  $S$ , in dollars; and  $H$ , whether the applicant is a homeowner (1), rental tenant (2), or “other” (3). In fact, the naïve Bayes method is a common approach to credit default problems of this kind, although typically in such applications the training set will contain hundreds of thousands of accounts and will use many more variables, and the naïve Bayes method will be used as leaves in a segmented scorecard of the kind described above.

**TABLE 9.1** Data for Example 1

Time with Emp, $T$	Size of Loan, $S$	Homeowner, $H$	Default, $D$
5	10,000	1	0
20	10,000	1	0
1	25,000	1	0
1	15,000	3	0
15	2,000	2	0
6	12,000	1	0
1	5,000	2	1
12	8,000	2	1
3	10,000	1	1
1	5,000	3	1

Time with employer is a continuous variable. For each of the two classes separately, we could estimate the distribution  $f(T|i)$ ,  $i = 0, 1$  using a kernel method or some assumed parametric form (lognormal would probably be a sensible choice for such a variable), or we could use the naïve Bayes approach in which the variable is split into cells, estimating the probability of falling in each cell by the proportion of cases from class  $i$  which fall in that cell. We shall take this third approach and, to keep things as simple as possible, will split  $T$  into only two cells, whether or not the customer has been with the employer for 10 or more years. This yields probability estimates

$$\begin{aligned}\hat{f}(T < 10|D = 0) &= 4/6, & \hat{f}(T \geq 10|D = 0) &= 2/6 \\ \hat{f}(T < 10|D = 1) &= 3/4, & \hat{f}(T \geq 10|D = 1) &= 1/4\end{aligned}$$

Similarly, we shall do the same sort of thing with size of loan, splitting it into just two cells (purely for convenience of explanation) according to the intervals  $\leq 10,000$  and  $> 10,000$ . This yields probability estimates

$$\begin{aligned}\hat{f}(S \leq 10000|D = 0) &= 3/6, & \hat{f}(S > 10000|D = 0) &= 3/6 \\ \hat{f}(S \leq 10000|D = 1) &= 3/4, & \hat{f}(S > 10000|D = 1) &= 1/4\end{aligned}$$

For the nondefaulter class, the homeowner column yields three estimated probabilities:

$$\hat{f}(H = 1|D = 0) = 4/6, \quad \hat{f}(H = 2|D = 0) = 1/6, \quad \hat{f}(H = 3|D = 0) = 1/6$$

For the defaulter class, the respective probabilities are

$$\hat{f}(H = 1|D = 1) = 1/4, \quad \hat{f}(H = 2|D = 1) = 2/4, \quad \hat{f}(H = 3|D = 1) = 1/4$$

Suppose now that a new application form is received, from an applicant who has been with his or her (this phrasing is chosen deliberately: It is illegal to use sex as a predictor for making loan decisions such as this.) employer for less than 10 years

( $T < 10$ ), is seeking a loan of \$10,000 ( $S \leq 10000$ ), and is a homeowner ( $H = 1$ ). This leads to an estimated value of the ratio  $\hat{P}(1|x)/\hat{P}(0|x)$  of

$$\begin{aligned}\frac{P(1|x)}{P(0|x)} &= \frac{P(1)}{P(0)} \prod_{j=1}^p \frac{\hat{f}(x_j|1)}{\hat{f}(x_j|0)} = \frac{P(1)}{P(0)} \times \frac{\hat{f}(T|1)\hat{f}(S|1)\hat{f}(H|1)}{\hat{f}(T|0)\hat{f}(S|0)\hat{f}(H|0)} \\ &= \frac{4/10}{6/10} \times \frac{3/4 \times 3/4 \times 1/4}{4/6 \times 4/6 \times 3/6 \times 4/6} = 0.422\end{aligned}$$

Since  $P(1|x) = 1 - P(0|x)$ , this is equivalent to  $P(1|x) = 0.296$  and  $P(0|x) = 0.703$ . If the classification threshold is 0.5 [i.e., if we decide to classify a customer with vector  $x$  to class 1 if  $P(1|x) > 0.5$  and to class 0 otherwise], then this customer will be classified as likely to belong to class 0—the nondefaulter class. This customer would be a good bet for making a loan to.

### 9.6.2 Example 2

An important and relatively new application domain for the naïve Bayes method is spam filtering. *Spams* are unsolicited and typically unwanted emails, often direct marketing of some kind and frequently offering dubious financial or other opportunities. Some of them are so-called *phishing* exercises. The principle behind them is that even a low response rate is profitable if (a) the cost of mailing the emails is negligible and (b) enough are sent. Since they are sent out automatically to millions of email addresses, one may receive many hundreds of these daily. With this number, even to move the cursor and physically hit the delete button would consume significant amounts of time. For this reason researchers have developed classification rules called *spam filters*, which examine incoming emails, and assign them to spam or not-spam classes. Those assigned to the spam class can be deleted automatically, or sent to a holding file for later examination, or treated in any other way deemed appropriate.

Naïve Bayes models are very popular for use as spam filters, going back to the early seminal work by Sahami et al. (1998). In their simplest form, the variables in the model are binary variables corresponding to the presence or absence, in the email, of each word. However, the naïve Bayes model also permits the ready addition of other binary variables corresponding to the presence or absence of other syntactic features such as punctuation marks, currency units (\$, £, €, and so on), combinations of words, whether the source of the email was an individual or a list, and so on. In addition, other nonbinary variables are useful as further predictors, for example, the type of domain of the source, the percentage of nonalphanumeric characters in the subject heading, and so on. It will be clear from the above that the potential number of variables is very large. Because of this, a feature selection step is typically undertaken (recall the discussion of why the naïve Bayes model may do well, despite its underlying independence assumption).

One important aspect of spam filtering is the imbalance in the severity of the misclassification costs. Misclassifying a legitimate email as spam is much more serious than the reverse. Both this and the relative size of the two classes play roles in

determining the classification threshold. In their experiments, Sahami et al. (1998) chose a threshold of 0.999 with which to compare  $P(\text{spam} | x)$ .

One strength of the naïve Bayes model is that it can just as easily be applied to count variables as to binary variables. The multivariate binary spam filter described above is easy to extend to more elaborate models for the distributions of the values of the variables. We have already referred to the use of multinomial models earlier, in which continuous variables are partitioned into more than two cells (and the homeowner variable in the artificial data of Example 1 was a case of a trinomial variable). Experiments suggest that, at least for spam filtering, the multinomial approach using frequencies of word appearances in the emails is superior to using mere presence/absence variables. Metsis et al. (2006) carried out a comparative analysis of different versions of the naïve Bayes model, in which the marginal variables are treated in different ways, applying the methods to some real email data sets.

---

## 9.7 Advanced Topics

The chief attraction of the naïve Bayes model is its extreme simplicity, permitting easy (univariate) estimation and straightforward interpretation via the weights of evidence. The first of these properties is also associated with robustness, provided the estimates of the marginal distributions are robust. In particular, if the marginal distributions are categorical, then each cell needs to contain sufficient data points to yield accurate estimates. With this in mind, researchers have explored optimal partitioning of each variable. The approach, most in tune with the straightforward naïve Bayes estimator, is to examine each variable separately—perhaps splitting into equal quantiles (this is generally superior to splitting into equal length cells). A more sophisticated approach will choose the cells based on the relative number from each class in each cell. This can also be done by considering each variable separately. Finally, one can partition each cell taking into account the overall fit to the distribution in each (or both) classes, but this moves away from the simple marginal approach. Investigations of some of these issues are described in Hand and Adams (2000).

Missing data are a potential problem in all data analysis. Classification methods which cannot handle incomplete data are at a disadvantage. When the data are missing completely at random, then the naïve Bayes model copes without any difficulty: Valid estimates are obtained by simply estimating the marginal distributions from the observed data. If the data are informatively missing, however, then more complex procedures are needed. This is an area meriting further research.

More and more problems involve dynamic data, and data sets which sequentially accrue. The naïve Bayes method can be adapted very readily to such problems, by virtue of its straightforward estimation.

So-called “small  $n$ , large  $p$ ” problems have become important in certain areas, such as bioinformatics, genomics, and proteomics, especially in the analysis of microarray data. These are problems characterized by the fact that the number of variables is

much larger than the sample size. Such problems pose difficulties; for example, the covariance matrix will be singular, leading to overfitting. To tackle such problems, it is necessary to make some kinds of assumptions or (equivalently) to shrink the estimators in some way. One approach to such problems in the context of supervised classification is to use the naïve Bayes method. This has its in-built assumption of independence, which acts to protect against overfitting. More elaborate versions of this idea combine naïve Bayes models with more sophisticated classifiers, trying to strike the best balance.

## 9.8 Exercises

1. Using a package such as the open-source package R, generate samples of size 100 from each of the two classes. Class 1 is bivariate normal, with zero means and identity covariance matrix. Class 2 is bivariate normal, with mean vector  $(0, 2)$  and diagonal covariance matrix with leading diagonal  $(1, 2)$ . Fit a naïve Bayes model to these data, based on assuming (correctly) that the marginal distributions are normal. Plot the decision surface to see that it is not linear.
2. The tables below show the bivariate distributions from samples for two classes, where the variables each have three categories. Show that the two variables are independent in each of the two classes. Taking the classification threshold as  $1/2$ , calculate the decision surface for a naïve Bayes classifier and show that it is nonlinear.

144	144	144
144	144	144
144	144	144

9	90	9
90	900	90
9	90	9

3. For the data from Exercise 2, calculate the weights of evidence for the categories of each variable, so that the naïve Bayes classifier can be expressed as a weighted sum.
4. The tables below show the bivariate distributions from samples for two classes, where the variables each have three categories. Show that the two variables are *not* independent in each of the two classes. Taking the classification threshold as  $1/2$ , fit a naïve Bayes classifier to these data and show that nevertheless its decision surface is optimal.

27	30	27
30	2700	30
27	30	27

432	48	432
48	432	48
432	48	432

5. Using data simulated from multivariate normal distributions, compare the relative performance of a naïve Bayes classifier and a simple linear discriminant classification rule as the (assumed common) correlation between the variables increases.
6. Using a suitable data set from the UCI Machine Learning Repository, with continuous variables which are partitioned into discrete cells, investigate the effect of changing the number and width of the cells in each variable.
7. Using the same data set as in Exercise 6, compare the models produced by the naïve Bayes classifier and logistic regression.
8. A common way to extend the naïve Bayes classifier in some applications is to partition the data into segments, with separate naïve Bayes classifiers constructed for each segment. Clearly such partitioning will be most effective if its splits allow for interactions which the naïve Bayes classifier would not pick up. Develop guidelines to assist people in making such splits.
9. The idea of modeling the distribution of each class by assuming independence extends immediately to more than two classes. For more than two classes write down appropriate classification models in the weights of evidence format.
10. One of the particular attractions of the naïve Bayes classifier is that it permits very simple estimation. Develop updating rules which allow the classifier to be sequentially updated as new data arrive.

---

## References

- Collett D. (1991) *Modelling Binary Data*. London: Chapman and Hall.
- Domingos P. and Pazzani M. (1997) On the optimality of the simple Bayesian classifier under zero-one loss. *Machine Learning*, **29**, 103–130.
- Hand D.J. and Adams N.M. (2000) Defining attributes for scorecard construction. *Journal of Applied Statistics*, **27**, 527–540.

- Hand D.J. and Yu K. (2001) Idiot's Bayes—not so stupid after all? *International Statistical Review*, **69**, 385–398.
- Hastie T.J. and Tibshirani R.J. (1990) *Generalized Additive Models*. London: Chapman and Hall.
- Jamain A. and Hand D.J. (2005) The naïve Bayes mystery: A statistical detective story. *Pattern Recognition Letters*, **26**, 1752–1760.
- Jamain A. and Hand D.J. (2008) Mining supervised classification performance studies: A meta-analytic investigation. *Journal of Classification*, **25**, 87–112.
- Langley P. (1993) Induction of recursive Bayesian classifiers. *Proceedings of the Eighth European Conference on Machine Learning*, Vienna, Austria: Springer-Verlag, 153–164.
- Mani S., Pazzani M.J., and West J. (1997) Knowledge discovery from a breast cancer database. *Lecture Notes in Artificial Intelligence*, **1211**, 130–133.
- Metsis V., Androutsopoulos I., and Paliouras G. (2006) Spam filtering with naïve Bayes—which naïve Bayes? *CEAS 2006—Third Conference on Email and Anti-Spam*, Mountain View, California.
- Sahami M., Dumains S., Heckerman D., and Horvitz E. (1998) A Bayesian approach to filtering junk e-mail. In *Learning for Text Categorization—Papers from the AAAI Workshop*, Madison, Wisconsin, pp. 55–62.
- Titterton D.M., Murray G.D., Murray L.S., Spiegelhalter D.J., Skene A.M., Habbema J.D.F., and Gelpke G.J. (1981) Comparison of discrimination techniques applied to a complex data set of head injured patients. *Journal of the Royal Statistical Society, Series A*, **144**, 145–175.