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8.9 선형연립미분방정식

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8.6 SVD와 일반화된 역행렬

$$\mathbf{A}v_i = \sigma_i u_i \quad (v_i \perp v_j, u_i \perp u_j) \longrightarrow \mathbf{AV} = \mathbf{U\Sigma}$$

- SVD : Singular Value Decomposition

$$\mathbf{A} = \mathbf{U\Sigma V}^T \qquad \mathbf{A}^T\mathbf{A} = \mathbf{V\Sigma U}^T\mathbf{U\Sigma V}^T = \mathbf{V\Sigma}^2\mathbf{V}^T$$

$$\mathbf{AA}^T = \mathbf{U\Sigma V}^T\mathbf{V\Sigma U}^T = \mathbf{U\Sigma}^2\mathbf{U}^T$$

symmetric matrices

m × n matrix

m ×m orthogonal matrix

m × n diagonal matix

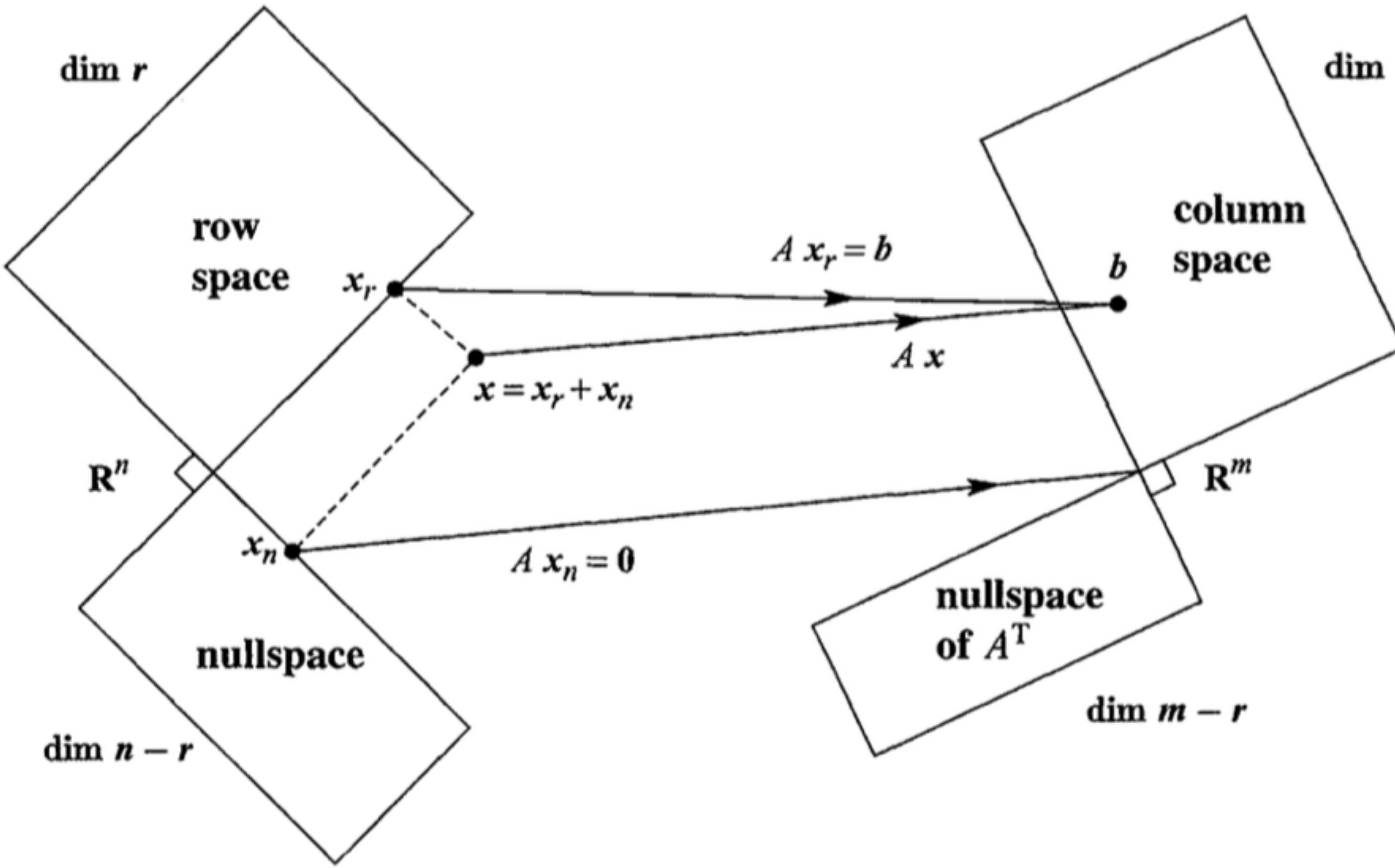
n × n orthogonal matrix

$$\mathbf{A} = [\underbrace{u_1 \ \cdots \ u_r}_{\text{column space of A}} \ \underbrace{u_{r+1} \ \cdots \ u_m}_{\text{null space of } A^T}] \left[\begin{array}{ccc|ccc} \sigma_1 & & \mathbf{0} & & & \\ & \ddots & & & \mathbf{0} & \\ \mathbf{0} & & \sigma_r & & & \\ \hline & & & 0 & & \\ & \mathbf{0} & & & \ddots & \\ & & & & & 0 \end{array}\right] \begin{array}{c} \left[\begin{array}{c} v_1^T \\ \vdots \\ v_r^T \\ v_{r+1}^T \\ \vdots \\ v_n^T \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{row space of A} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{null space of A} \end{array}$$

$\sigma_1 \geq \cdots \geq \sigma_r$

- singular value : $\sigma = \sqrt{\lambda}$

$$\mathbf{A}^T\mathbf{A}$$



The 4 subspaces of a matrix A [3]

Visualization of SVD !! [1]

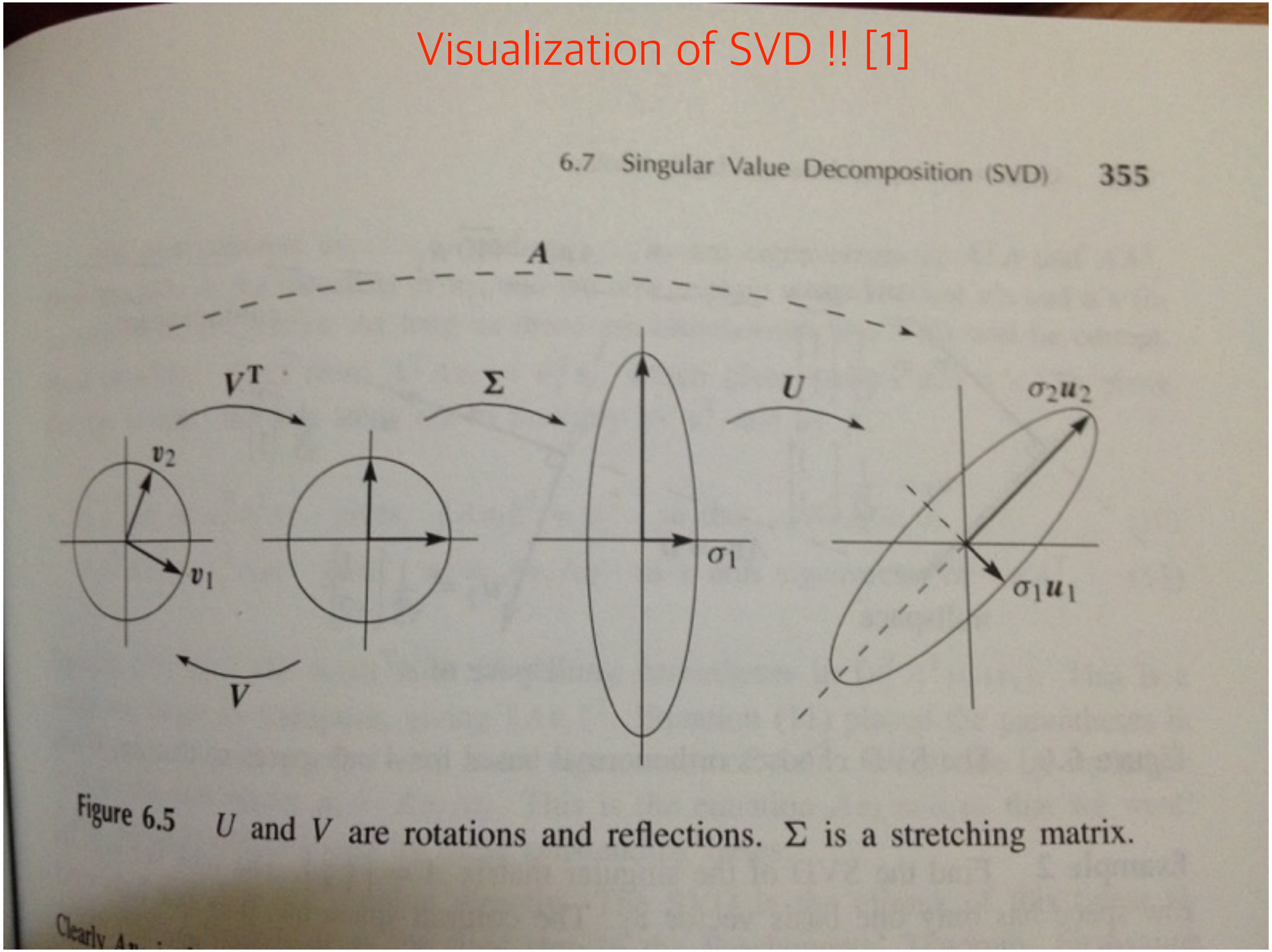

6.7 Singular Value Decomposition (SVD) 355

Figure 6.5 U and V are rotations and reflections. Σ is a stretching matrix.

$$\mathbf{A}_{n\times n} \longrightarrow \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$$

n × m matrix

$\mathbf{A}_{m\times n} \longrightarrow$ left Inverse : full column matrix (r=n) → $\underline{(A^TA)^{-1}A^T}\ A = A^{-1}_{left}\ A = I$

right Inverse : full row matrix (r=m) → $A\ \underline{A^T(AA^T)^{-1}} = A\ A^{-1}_{right} = I$

m × n (r < n, r < m)

- Pseudo-Inverse : $\underset{n\times m}{\mathbf{A}^+} = \mathbf{V}\mathbf{\Sigma}^+\mathbf{U}^T \longleftarrow \underset{m\times n}{\mathbf{A}} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$

$$\mathbf{\Sigma}^+ = \begin{bmatrix} 1/\sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{n\times m} \quad \mathbf{\Sigma}^+\mathbf{\Sigma} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{n\times n} \quad \mathbf{\Sigma}\mathbf{\Sigma}^+ = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{m\times m}$$

- Projection matrix onto the column space of (all) A : $\mathbf{A}\mathbf{A}^+$

- Projection matrix onto the row space of (all) A : $\mathbf{A}^+\mathbf{A}$

- The shortest least squares solution of (all) : $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\mathbf{x}^+ = \mathbf{A}^+\mathbf{b} \longleftarrow \mathbf{A}^+\mathbf{A}\mathbf{x} = \mathbf{A}^+\mathbf{b}$$

8.7 복소고유값과 고유벡터

- Complex n-Space : $\mathbf{C}^n$

- Complex conjugate : $\bar{z} = a - bi \longleftarrow z = a + bi$

- Inner product of complex : $\mathbf{u} \cdot \mathbf{v} = \bar{\mathbf{u}}^T \mathbf{v} = \mathbf{u}^H \mathbf{v}$

Hermitian

- Real matrix : can have complex (conjugate) eigenvalues

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \longrightarrow \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \bar{\mathbf{x}}^{\mathbf{T}}\lambda\mathbf{x}$$

$$\mathbf{A}\bar{\mathbf{x}} = \overline{\mathbf{A}\mathbf{x}} = \overline{\lambda\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \xrightarrow[\text{if } A = A^T]{\text{transpose}} \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A} = \bar{\mathbf{x}}^{\mathbf{T}}\bar{\lambda}$$

- Symmetric matrix : real eigenvalues

$$\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \bar{\mathbf{x}}^{\mathbf{T}}\bar{\lambda}\mathbf{x}$$

$$\lambda = \bar{\lambda}$$

8.8 Hermitian, Unitary, Normal Matrix

- Hermitian matrix : conjugate transpose square matrix

$$\mathbf{A} = \mathbf{A}^H (= \overline{\mathbf{A}}^T)$$

1×1 same conjugate transpose = same conjugate = real number

- Complex vector : $(\mathbf{z}^H \mathbf{A} \mathbf{z})^H = \mathbf{z}^H \mathbf{A}^H (\mathbf{z}^H)^H = \mathbf{z}^H \mathbf{A} \mathbf{z}$

- Real eigenvalues :

$$\mathbf{A}\mathbf{z} = \lambda \mathbf{z}$$

$$\mathbf{z}^H \mathbf{A} \mathbf{z} = \lambda \mathbf{z}^H \mathbf{z}$$

$$\lambda = \frac{\mathbf{z}^H \mathbf{A} \mathbf{z}}{\mathbf{z}^H \mathbf{z}}$$

- eigenvectors are orthogonal : $\mathbf{A}\mathbf{y} = \beta \mathbf{y}$ $\quad \mathbf{y}^H \mathbf{A}^H = \beta \mathbf{y}^H$

$$\mathbf{y}^H \mathbf{A} \mathbf{z} = \lambda \mathbf{y}^H \mathbf{z} \qquad \mathbf{y}^H \mathbf{A}^H \mathbf{z} = \beta \mathbf{y}^H \mathbf{z}$$

$$\mathbf{y}^H \mathbf{z} = 0$$

- Skew-Hermitian :

$$\mathbf{A}^H = -\overline{\mathbf{A}}^T$$

- Unitary : complex square matrix, orthonormal columns

$$\mathbf{U}^H\mathbf{U} = \mathbf{I} \qquad \mathbf{U}_{n\times n}$$

$$\mathbf{U}^H = \mathbf{U}^{-1}$$

- Vector length stays the same : $\|\mathbf{Uz}\| = \|\mathbf{z}\|$

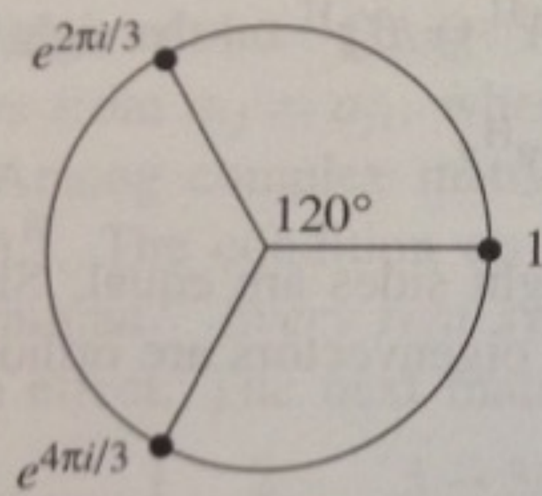
$$\|\mathbf{z}\|^2 = \mathbf{z}^H\mathbf{z}$$

$$(\mathbf{Uz})^H\mathbf{Uz} = \mathbf{z}^H\mathbf{U}^H\mathbf{Uz} = \mathbf{z}^H\mathbf{z}$$

- All eigenvalues : $|\lambda| = 1$

- All eigenvectors are orthogonal

Fourier matrix can be a unitary matrix !! [1]



Fourier matrix $$F = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix}.$$

Figure 10.4 The cube roots of 1 go into the Fourier matrix $F = F_3$.

10G ***The matrix U has orthonormal columns when $U^{\mathrm{H}}U = I$.***
If U is square, it is a ***unitary matrix***. Then $U^{\mathrm{H}} = U^{-1}$.

- Normal matrix : $\mathbf{A}^H\mathbf{A} = \mathbf{A}\mathbf{A}^H$ $\qquad$ $\mathbf{A} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1}$
- complex eigenvalue when : $\mathbf{A} \neq \mathbf{A}^H$
- Its orthonormal eigenvectors are columns of U
- Hermitian, Unitary, Symmetric are Normal matrix : real λ
- Unitary diagonalizable $\Longleftrightarrow$ Normal matrix
- but, SVD is more general !$\cdots$? $\qquad$ $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$

8.9 선형연립미분방정식

- Differential Equations : $\mathbf{x}' = \mathbf{A}\mathbf{x}$

- Diagonal Matrix : decoupling of entries

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{aligned} \mathbf{A} &= \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1} \\ \mathbf{A} &= \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{T} \\ \mathbf{A} &= \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^{-1} \end{aligned} \quad \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$\mathbf{A}^2 = \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1}\mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^{-1} = \mathbf{S}\boldsymbol{\Lambda}^2\mathbf{S}^{-1}$$

$$\mathbf{A}^k = \mathbf{S}\boldsymbol{\Lambda}^k\mathbf{S}^{-1} \qquad e^{\mathbf{A}t} = \mathbf{S}e^{\boldsymbol{\Lambda}t}\mathbf{S}^{-1}$$

- Companion matrix : n-th order differential equations

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- [1] Introduction to Linear Algebra 3rd edition, Gilbert Strang, Wellesley-Cambridge Press, 2003

- [2] 현대 선형대수학 with Sage, 이상구, 경문사, 2012

- [3] The Fundamental Theorem of Linear Algebra, Gilbert Strang, The American Mathematical Monthly, 1993