Root finding methods

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1 Bisection method

The bisection is a slow but very robust method of finding the root of a function $f(x)$. The disadvantage is that it requires two start points, x_{left} and x_{right}, where the function has opposite sign, $f(x_{\text{left}}) \cdot f(x_{\text{right}}) < 0$. These two points are the boundaries of an interval that contains the root that we are looking for.

At each iteration we compute the midpoint of the interval $x_2 = \dfrac{x_{\text{left}} + x_{\text{right}}}{2}$ and evaluate the function at this point $f_2 = f(x_2)$. Finally we update the interval bounds:

- If f_2 has the same sign as $f(x_{\text{left}})$ we update $x_{\text{left}} \leftarrow x_2$.
- If f_2 has the same sign as $f(x_{\text{right}})$ we update $x_{\text{right}} \leftarrow x_2$.

We iterate these steps until we reach the desired precision in the value of the function, $|f_2| < \epsilon$, or in the value of the root, $x_{\text{right}} - x_{\text{left}} < \epsilon$.

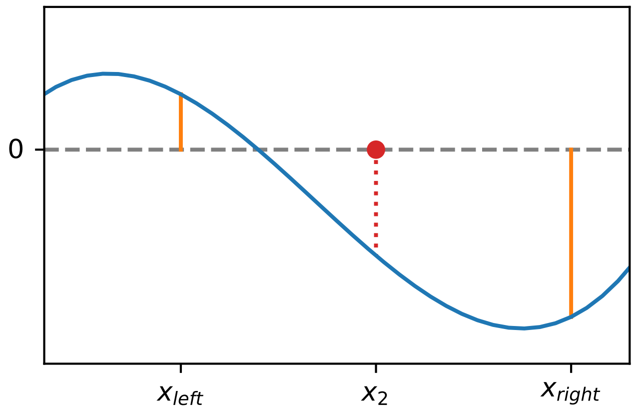

Figure 1: Schematic representation of an iteration of the bisection method. In this case we would update $x_{\text{right}} \leftarrow x_2$.

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2 Secant method

The secant method uses the last two points to compute the slope of the function $f(x)$ and approximate the function to a straight line. It is quicker than the bisection method but less stable. It requires two start points, x_0 and x_1, with arbitrary values $f_0 = f(x_0)$ and $f_1 = f(x_1)$.

At each iteration we approximate the function to a straight line $y(x)$, i.e. $\frac{f_1 - f_0}{x_1 - x_0} = \frac{y - f_0}{x - x_0}$, and compute its root $x_2 = x_0 - f_0 \frac{x_1 - x_0}{f_1 - f_0}$. Finally we update the values $x_0 \leftarrow x_1$ and $x_1 \leftarrow x_2$.

We iterate these steps until we reach the desired precision in the value of the function, $|f_2| < \epsilon$, or in the value of the root, $x_{\text{right}} - x_{\text{left}} < \epsilon$.

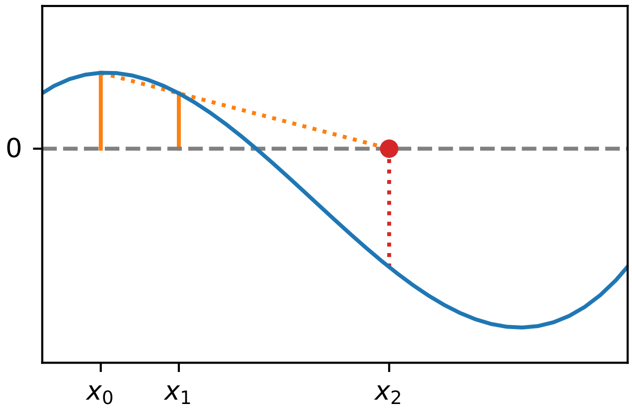

Figure 2: Schematic representation of an iteration of the secant method.

3 Hybrid secant-bisection method

The secant method fails if the slope becomes very small, causing the new root x_2 to diverge. Additionally, in the later stage the differences between f_0 and f_1 become very small which can cause problems with the evaluation of the fraction $\frac{x_1 - x_0}{f_1 - f_0}$.

A possibility to control these deviations is to use the bisection method when the secant method fails. This hybrid method requires two start points, x_{left} and x_{right}, where the function has opposite sign, $f(x_{\text{left}}) \cdot f(x_{\text{right}}) < 0$. We use these same two points as the starting points of the secant method, $x_0 = x_{\text{left}}$ and $x_1 = x_{\text{right}}$ (or vice versa, the order is irrelevant).

At each iteration we compute the secant root $x_2 = x_0 - f_0 \frac{x_1 - x_0}{f_1 - f_0}$.

- If x_2 falls inside of the bisection interval, $x_2 \in [x_{\text{left}}, x_{\text{right}}]$, we accept it.
- Else we reject the secant value and compute a new x_2 with the bisection method, $x_2 = \frac{x_{\text{left}} + x_{\text{right}}}{2}$.

Finally we update the secant values, $x_0 \leftarrow x_1$ and $x_1 \leftarrow x_2$, and also the bisection interval bounds:

- If $f(x_2)$ has the same sign as $f(x_{\text{left}})$ we update $x_{\text{left}} \leftarrow x_2$.
- If $f(x_2)$ has the same sign as $f(x_{\text{right}})$ we update $x_{\text{right}} \leftarrow x_2$.

We iterate these steps until we reach the desired precision in the value of the function, $|f_2| < \epsilon$, or in the value of the root, $x_{\text{right}} - x_{\text{left}} < \epsilon$.

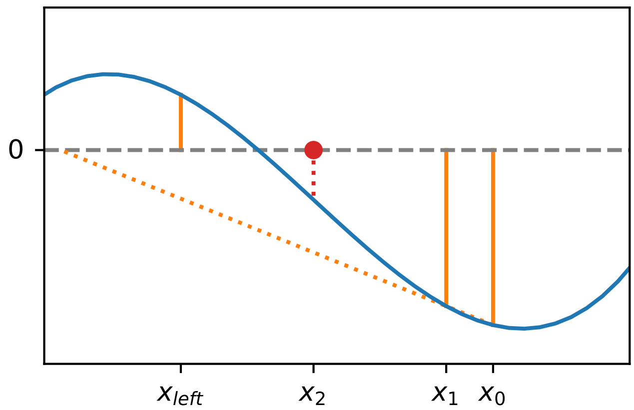

Figure 3: Schematic representation of an iteration of the hybrid secant-bisection method. In this case we reject the secant value and compute x_2 with the bisection method. Notice that at the beginning of the iteration we had $x_{\text{right}} = x_1$ and at the end we will have updated it to $x_{\text{right}} = x_2$.