

Sampling power law degree distributions

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1 Unclustered scale-free networks

The degree distribution of scale-free networks is given by a power law with exponent $\gamma > 2$,

$$p(x) = (\gamma - 1)x_{\min}^{\gamma-1}x^{-\gamma}, \quad (1)$$

for $x \geq x_{\min}$. In finite systems, the maximum degree is limited by the fact that a node cannot have more neighbors than available nodes. This natural cut-off is given by

$$\int_{x_{\max}}^{\infty} p(x)dx < \frac{1}{N}, \quad (2)$$

with N the system size. In short, Eq. (2) states that the number of nodes with degree x_{\max} or higher must be smaller than N . Solving Eq. (2) yields

$$x_{\max} > x_{\min}N^{\frac{1}{\gamma-1}}. \quad (3)$$

In practice, we sample the degree sequence from a discrete probability function

$$P(k) = Ak^{-\gamma}, \quad (4)$$

for $k_{\min} \leq k \leq k_{\max}$. In order to guarantee a connected network we choose $k_{\min} = 2$, and from Eq. (3) we find

$$k_{\max} = \left\lfloor k_{\min}N^{\frac{1}{\gamma-1}} \right\rfloor. \quad (5)$$

Finally, the normalization constant yields $A = \zeta(\gamma, k_{\min}) - \zeta(\gamma, k_{\max} + 1)$, with $\zeta(x, \ell)$ the Hurwitz zeta function

$$\zeta(x, \ell) = \sum_{k=0}^{\infty} (k + \ell)^{-x} = \sum_{k=\ell}^{\infty} k^{-x}. \quad (6)$$

2 Rejection method

It is very simple to sample the continuous distribution given in (1). Sampling a uniform random number u_1 we solve for the complementary cumulative function

$$u_1 = \int_x^\infty p(x)dx \rightarrow x = x_{\min} u_1^{\frac{1}{1-\gamma}}. \quad (7)$$

The discrete distribution is a bit trickier, but feasible using an adequate rejection method. First we make a continuous version of the discrete distribution $P(k)$ such that $\hat{P}(x) = P(k) \forall x \in [k, k+1)$. Next we realize that $p(x)$ is generally larger than $\hat{P}(x)$ except for an area around $x = x_0 + 1$ (see Figure 1).¹ If we construct the function $q(x) = Cp(x)$, with $C = P(k_0)/p(x_0 + 1)$, we see that $q(x)$ is always larger than $\hat{P}(x)$.

So we follow this algorithm:

1. Sample $u_1 \in U(0, 1)$ and sample x .
2. Evaluate the integer part $k = \lfloor x \rfloor$.
3. If $k > k_{\max}$, reject the candidate k .
4. Else, sample $u_2 \in U(0, 1)$ and evaluate $y = u_2 Cp(x)$.
 - If $y \leq P(k)$, accept the sampled degree k .
 - Else, reject the candidate k .

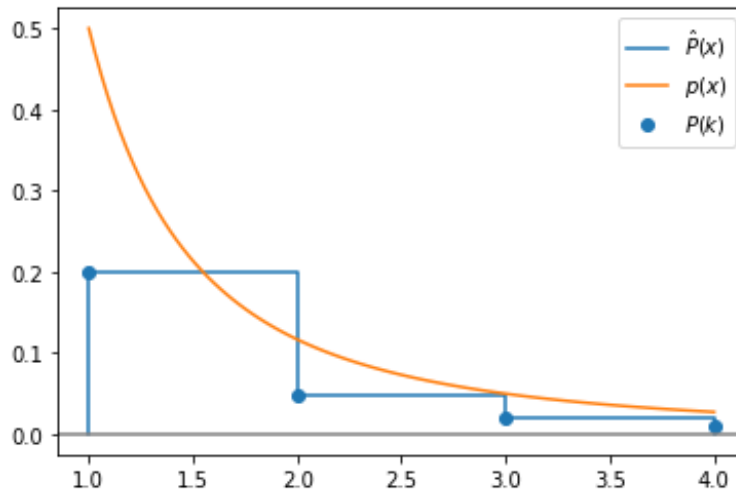


Figure 1: Schematic representation of $p(x)$, $P(k)$ and $\hat{P}(x)$ with $k_0 = x_0 + 1$.

¹This is technically not true. Other points can also show this behavior, depending on the value of γ . However, the largest difference occurs always around $x = x_0 + 1$, so the factor C that we have computed guarantees that all the differences are solved.