ECE1254 Assignment 4

Xiao Han Dong
998407834

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1 1a

Simple Newton converged in 391 iterations to a value of $x = 0.1923$.

2 1b

The equation can be restated as

$$f(x) = x + 10^{-6}e^{80x} - 5\lambda = 0$$

The actual solution x^*should give us the root of $f(x)$ precisely. The error for the approximate solution on the nth iteration x_n is $|x_n - x^*|$. I assumed the value of x in the last iteration would be close enough to the actual value of x^* assuming it converged. During the running of the code, when the value of x* is still undetermined, I checked three criteria: the values of f(x) over each iteration, the improvement between each iteration $|x_{n+1} - x_n|$, and the relative improvement $\frac{|x_{n+1}-x_n|}{|x_n|}$, all of which should eventually decrease down to the convergence criteria.

3 1c

Using increments of $\Delta\lambda = 0.1$, the solution converged quickly from $\lambda = 0$ to $\lambda = 1$. The total number of iterations to convergence is 81, which is much quicker than simple Newton. It converged to a value of 0.1923. The exact break-down is as follows in table 1

4 1d

The improved Newton's method with source stepping did converge a little faster, taking only 68 iterations. The function for which Newton's Method find the roots has changed to

$$F'(x) = \lambda F(x) + (1 - \lambda)x$$

and the error between the approximate and actual solution dropped down more quickly as lambda is stepped up in value. A more detailed breakdown is found in table 2

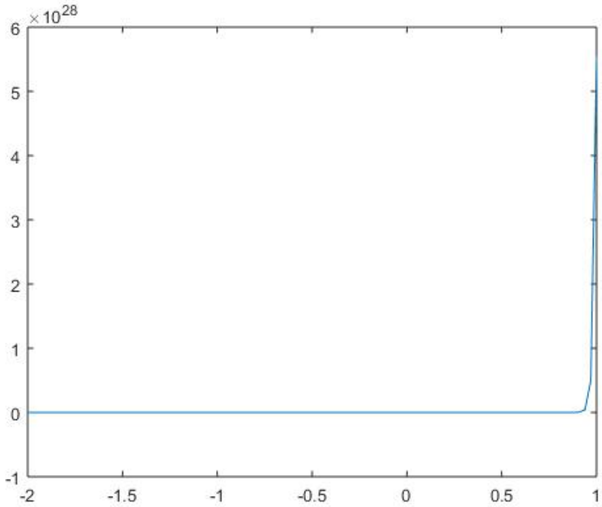

Figure 1: The extreme shape of the diode equation is not ideal for simple Newton's Method.

λ	Iterations	x value
0	1	0
0.1	34	0.1592
0.2	7	0.1704
0.3	6	0.1762
0.4	5	0.1802
0.5	5	0.1832
0.6	5	0.1856
0.7	5	0.1877
0.8	5	0.1894
0.9	5	0.1910
1.0	5	0.1923

Table 1: Convergence behaviour of simple source stepping in Newton's Method.

5 2a

The derivatives of F(x) are

$$\frac{dF}{d\phi_i} = 2 + \Delta x^2 (e^{\phi_i(x)} + e^{-\phi_i(x)}) = 2 + \Delta 2x^2 cosh(\phi_i(x))$$

$$\frac{dF}{d\phi_{i-1}} = \frac{dF}{d\phi_{i+1}} = -1$$

By inspection the Jacobian is tridiagonal. It can be split in to two matrices. The first matrix is a diagonal matrix with the diagonals being $2\Delta x^2(e^{\phi_i(x)} + e^{-\phi_i(x)})$. Since cosh is always greater or equal to 1, the first matrix is positive definite. The second matrix has 2 on the diagonals and -1 on the off diagonals.

λ	Iterations	x value
0	1	0
0.1	31	0.1870
0.2	9	0.1902
0.3	5	0.1911
0.4	5	0.1916
0.5	4	0.1918
0.6	3	0.1920
0.7	3	0.1921
0.8	3	0.1922
0.9	3	0.1923
1.0	3	0.1923

Table 2: Behaviour of stepping using an improved optimization function F'(x)

This is actually the nodal matrix of a resistor grid in series with the same resistor values. We have shown in the previous assignment that this matrix is positive semi-definite. If sum of two positive semi-definite matrices is positive semi-definite, and a positive semi-definite matrix is always invertible, then the Jacobian matrix is non-singular.

Two positive semi-definite matrices A,B, satisfies the property $x^T Ax \geq 0, x^T Bx \geq 0$. To prove that the sum of two positive semi-definite matrices is positive semi-definite, we want:

$$x^T (A + B)x \geq 0$$
$$= (x^T A + X^T B)x \geq 0$$
$$= x^T Ax + x^T Bx \geq 0$$

which is satisfied by the definition of A and B

6 2b

If the jacobian is non-singular, then the norm of its inverse is bounded. Assuming it is Lifschitz continuous, the theorem derived in class ensures that damped Newton's Method will converge quadratically to the solution.

7 2c

Convergence is reached in 3 and 16 iterations for V= 1V and 20V respectively, and the solutions are plotted in figures 2 and 3.If the ratios of the logarithms of the errors between two successive iterations are plotted, such as in figure 4, then the slope is the order of convergence.In this case we found it to be 1.9, close to the expected value of 2. Only the V=20 case is shown, since the V=1 case only have three iteration data points, so fitting would not be meaningful.

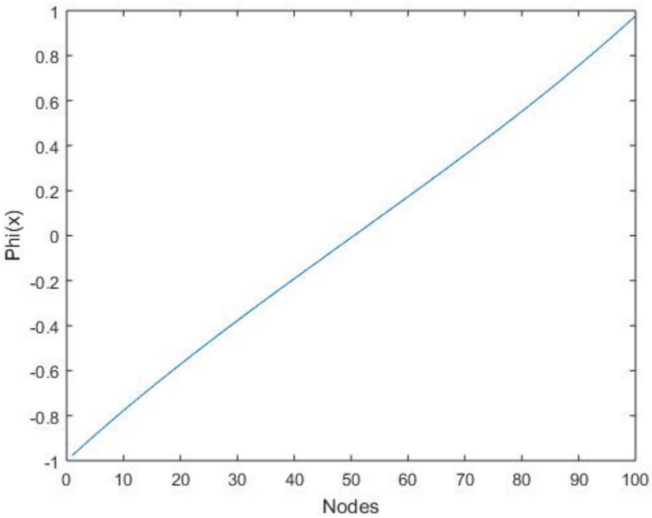

Figure 2: The solution when the boundaries are held at -1V and 1V. Node 0 is near x=0, and node 100 is near x=1.

8 2d

The value of the diagonal elements of the Jacobian is proportional to $cosh(\phi(x))$. Since the initial guess is 0 at all nodes, in the first iterations the diagonal elements are relatively small. However, as Newton's method gradually approximates the actual functional form, the diagonal elements blow up, especially near the ends of the discretization, near the boundary conditions. Here, since cosh(100) is roughly 10^43.The V=1 and V=20 case did not blow up because the hyperbolic cosine of these functions could still be contained in one double value.Since the off-diagonal elements are still -1 and 1, as we saw in class, conditioning number increases dramatically and accuracy drops off. MATLAB displayed a warning about large conditioing in the last 53 iterations. The determinant and rank of the Jacobian is calculated, and is found to quickly converge to infinity and 30 (the matrix is 100 by 100), respectively, which supports my suspicion.

However, the solution still has the same form as the V=1V and 20V cases (figure 5. In 91 iterations it converged.

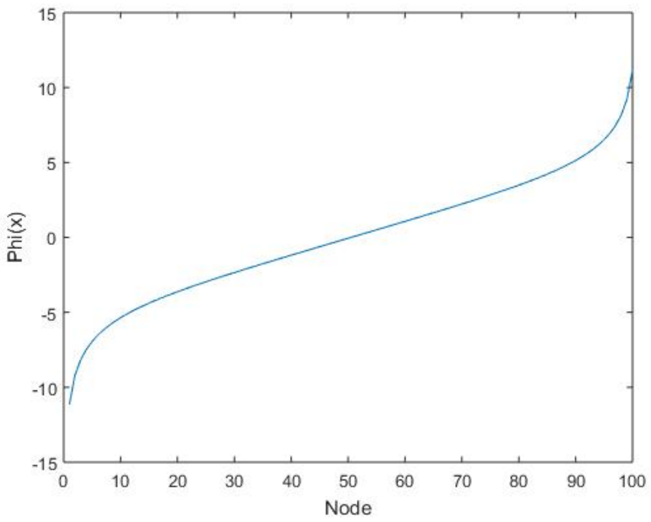

Figure 3: The solution when the boundaries are held at -20V and 20V. Node 0 is near x=0, and node 100 is near x=1.

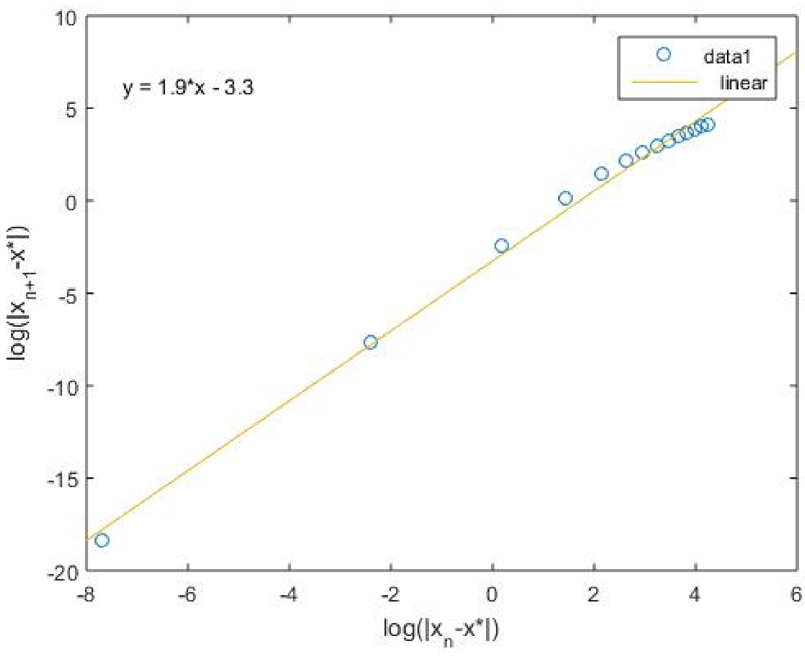

Figure 4: The slope of the linear fit between the logarithms of the error between two successive iterations of the case where V=20 is almost 2.

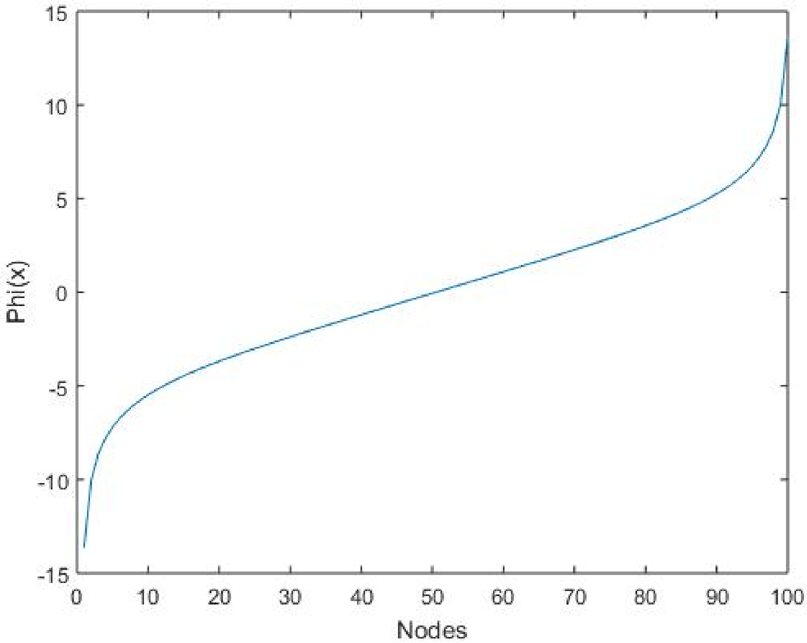

Figure 5: The solution when the boundaries are held at -100V and 100V. Node 0 is near x=0, and node 100 is near x=1.