ECE1254 Assignment 6

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1 Part a

The spatial second derivative can be discretized

$$\frac{d^2T}{dx^2} = \frac{\frac{T(x+\Delta x)-T(x)}{\Delta x} - \frac{T(x)-T(x-\Delta x)}{\Delta x}}{\Delta x} = \frac{T(x+\Delta x)+T(x-\Delta x)-2T(x)}{\Delta x^2}$$

Move all temperature dependent terms to the right hand side

$$(2+\alpha\Delta x^2)T(x) - T(x+\Delta x) - T(x-\Delta x) + \Delta x^2\frac{dT}{dt} = \Delta x^2 h(x)u(t)$$

Rewriting in vector form

$$\begin{bmatrix} -1 & 2+\alpha\Delta x^2 & -1 \end{bmatrix} \begin{bmatrix} T(x-\Delta x) \\ T(x) \\ T(x+\Delta x) \end{bmatrix} + \begin{bmatrix} 0 & \Delta x^2 & 0 \end{bmatrix} \begin{bmatrix} \frac{dT(x-\Delta x)}{dt} \\ \frac{dT(x)}{dt} \\ \frac{dT(x+\Delta x)}{dt} \end{bmatrix} = \begin{bmatrix} \Delta x^2 h(x) \end{bmatrix} u(t)$$

This is in the form

$$Gx + C\dot{x} = Bu(t)$$

The node voltages would be the temperature at each discretization point.

G is a $N \times N$ tridiagonal matrix. C is a $N \times N$ diagonal matrix,and B is a $N \times 1$ vector. N is the number of discretizations for the rod.

The only special case are at the boundaries. The boundary conditions for all time t

$$\frac{dT}{dx}(x=0,t) = \frac{dT}{dx}(x=1,t) = 0$$

requires there to be no heat flow out of node 1 and node N into the surroundings. This means

$$\frac{T(N+\Delta x)-T(N)}{\Delta x} = 0$$

$$\frac{T(0)-T(0-\Delta x)}{\Delta x} = 0$$

From the circuit perspective, it means there are no pathway from node 1 and node N to the external environment. Therefore

$$G(1,1) = G(N,N) = 1+\alpha\Delta x^2$$

The equivalent circuit is a cascade of parallel R-C circuits stretching from node 1 to N. However, there are also grounding resistors, which contribute to the $\Delta x^2 \alpha$ term that appears in the diagonals of G matrix.

2 Part b

2.1 case 1

Input only affect V_1 at node 1. h(x) is only non zero at the first node. The B matrix, which is a $N \times 1$ vector, will have the first element as Δx^2, and every other element is 0.

For the output the L^T matrix will pick out T_N, the right most node. L will be a vector of size $N \times 1$ with all elements 0 except the last, which will have a value of 1.

2.2 case 2

B will be the same as case 1. There is only one output, which is

$$T_{ave} = \frac{T_1 + T_2 + ...T_N}{N}$$

L matrix would be a vector of size $N \times 1$ with each element being T_i/N

2.3 case 3

Output L matrix will be the same as case 1. Input is uniform, meaning h(x) is actually invariant across the bar. B is a $N \times 1$ vector with all elements having the value Δx

2.4 case 4

The input B will be the same as in case 1. The output L will be the same as in case 2.

3 Part c

The frequency response is the ratio of input over output in frequency space

$$H(s)u(s) = y(s)$$

Since the matrices themselves have no time-variance

$$y(s) = L^T x(s)$$

and

$$(G + sC)x(s) = Bu(s)$$
$$u(s) = B^{-1}(G + sC)x(s)$$

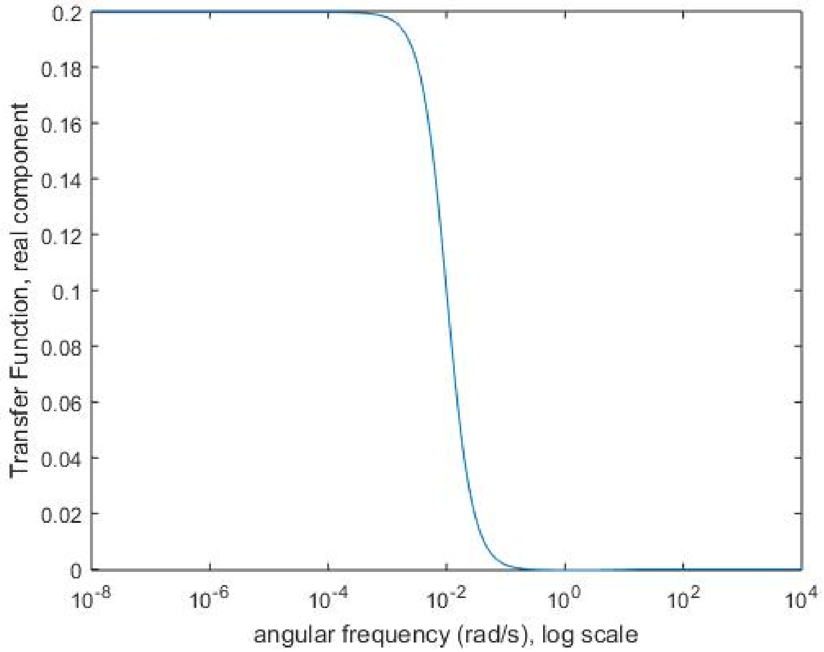

Figure 1: The real component frequency response of the circuit equivalent. The behaviour is that of a low-pass filter.

Combining the two to solve for H(s)

$$H(s)[B^{-1}(G + sC)]x(s) = L^T x(s)$$
$$H(s) = L^T [G + sC]^{-1} B$$

4 part c

The transfer function is plotted as a function of frequency. The magnitude behaviour is that of a low-pass circuit, which makes sense given the circuit equivalent of the scenario is mostly a cascade of parallel R-C circuits. The grounding resistors have resistances ($\alpha \Delta x^2$)that are very small compared to the resistance of 1 in the series resistors. Since the Transfer Function of the equivalent circuit here is

$$H(s) = \frac{V_{out}}{I_{in}}$$

It is actually a measure of the (heat) transmission impedance.

5 part d

As can be seen, the frequency response of all truncated systems approximate the actual behaviour closely. It would seem any choice of q greater than 4 would yield highly accurate results.

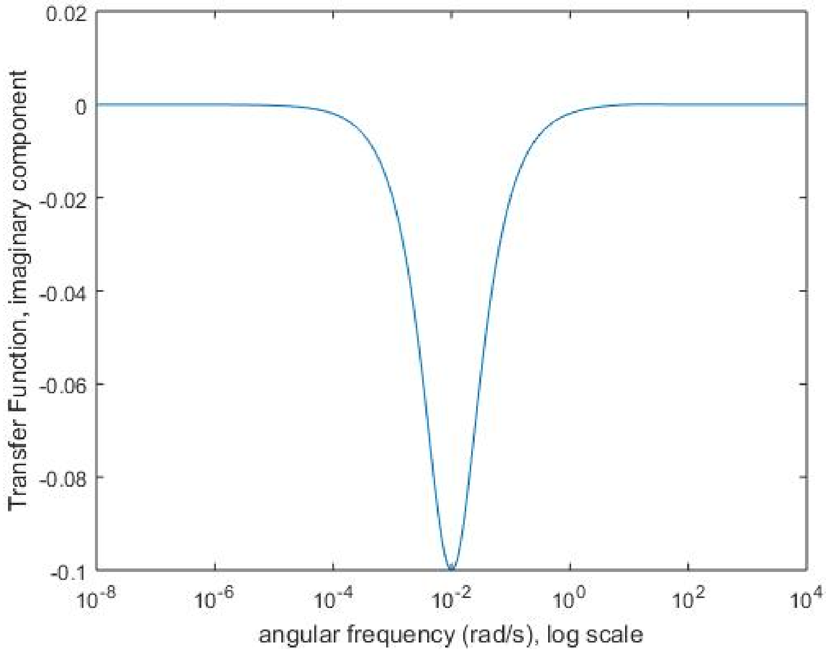

Figure 2: The imaginary component frequency response of the circuit equivalent. The reactance seem to be concentrated around $\omega = 10^{-2}$

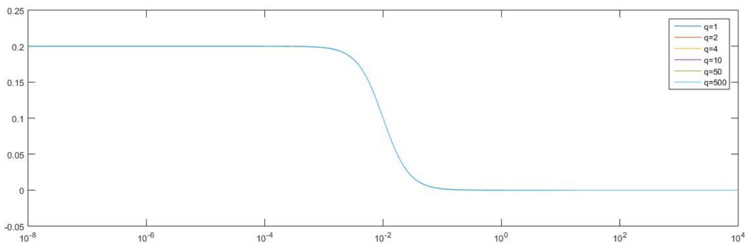

Figure 3: The frequency response (real component) for a set of results with different amount of modal truncation. The matching is excellent

The eigenvalues with the smallest magnitudes are: -0.01, -9.88,-39.49, -88.83, -157.82,-246.73. By the fifth term, the eigenvalues are already more than 5 orders of magnitudes larger than the first, so they have become negligible. The first four eigenvalues are taken. Graphically it also appears that the difference has become quite small after four eigenvalues.

6 Part e

The values are verified by performing trapezoidal rule calculations on two separate cases.

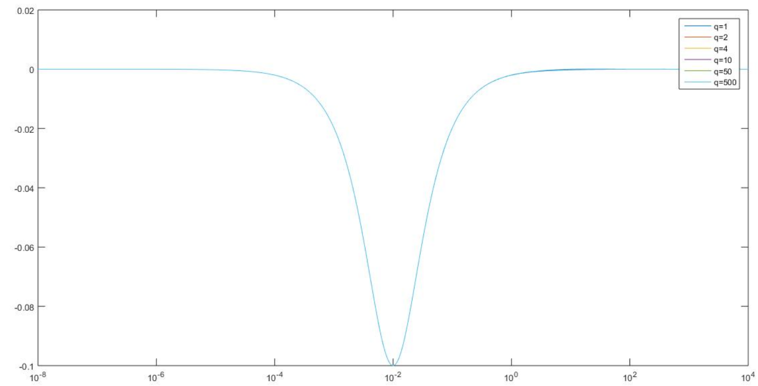

Figure 4: The frequency response (imaginary component) for a set of results with different amount of modal truncation. The matching is excellent

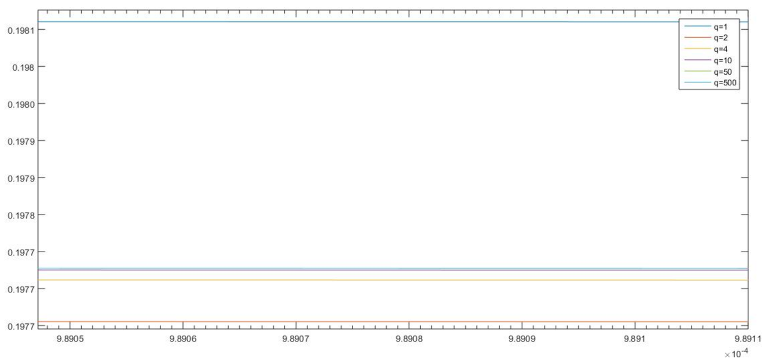

Figure 5: The frequency response (real component) for a set of results with different amount of modal truncation. The results is zoomed in to see there is a maximum of 0.0003 in the frequency response when using modal truncation.

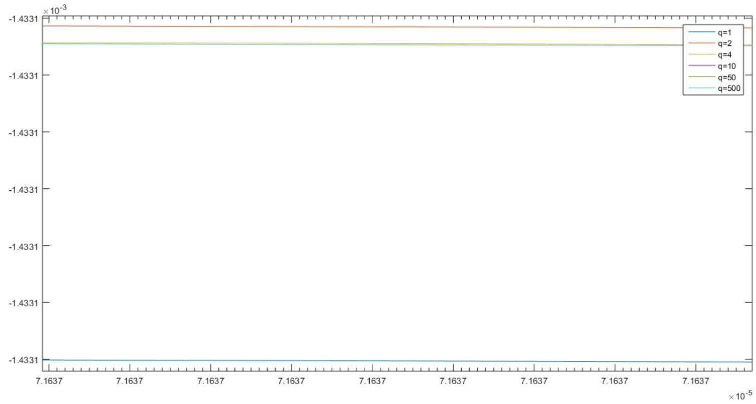

Figure 6: The frequency response (imaginary component) for a set of results with different amount of modal truncation. The results is zoomed in to see that the difference due to truncation is negligible.

u(t)	Regular (s)	Modal Truncation (s)	PRIMA (s)
step function	5.198	0.041	0.036
sine	51.031	0.117	0.105

Table 1: Comparison of Modal Truncation and PRIMA model reduction methods computation time.

In the first case, a step function input is used for a time range from 0 to 1000s, with a time step of 1s. In the second case, a time range from 0 to 10000s is used, with the same time step of 1s. The results are very close between the truncated and stand-alone results. This time value is chosen because a steady state has been clearly reached. The time step is chosen to be 1s because lower time steps results in a matrix that is too large for MATLAB to handle, and an error will be thrown.

7 part f-h

PRIMA algorithm is tested using the exact same parameters used to test Modal Truncation. The results are graphed.

The timing comparison includes both the time-stepping and the model reduction sections. PRIMA reduction took around 0.016s to run while Modal Truncation reduction required 0.032s, about double that of PRIMA.For larger data sets, the difference in running the reduction algorithm seems to be less significant. The bulk of computation time seems to be spent on time-stepping.

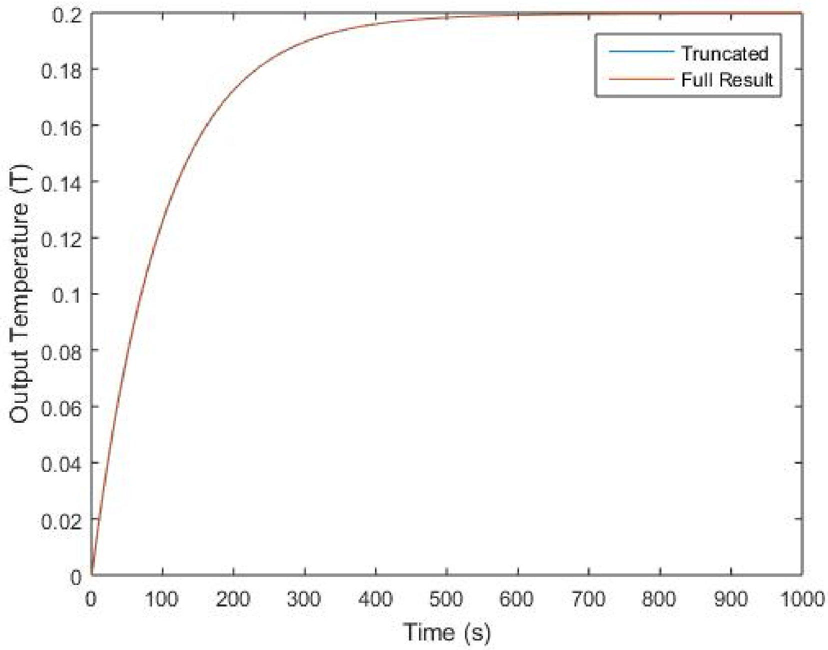

Figure 7: The Trapezoidal Rule time-stepping solution of the step function input. The truncated solution is almost identical to the full solution.

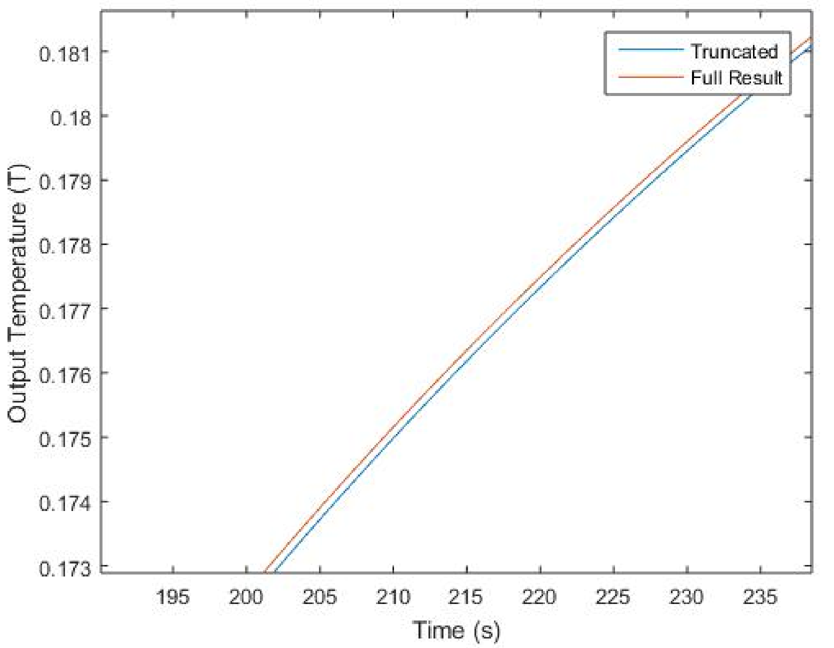

Figure 8: Zoomed in view of one segment in figure 7. The difference is very small

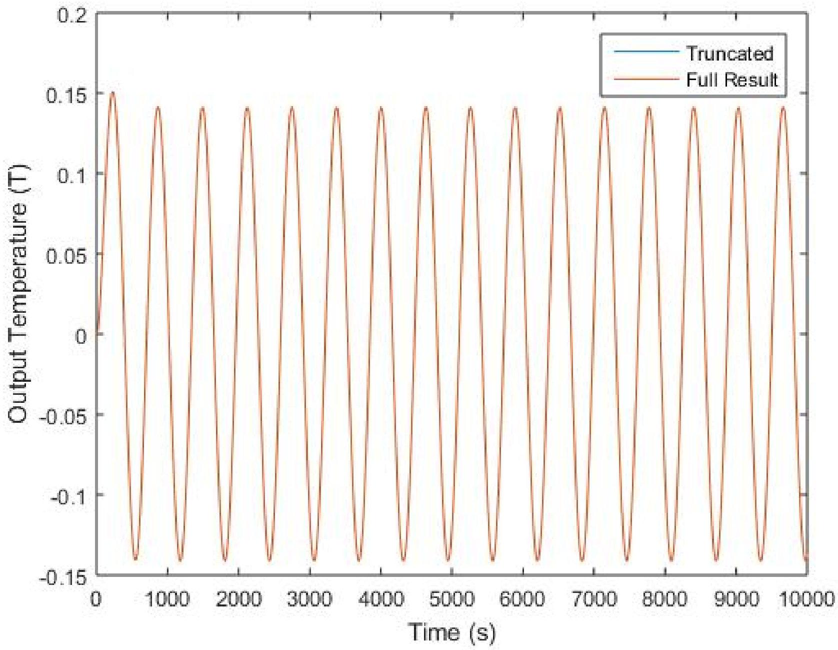

Figure 9: The Trapezoidal Rule time-stepping solution of the sine function input. The truncated solution is almost identical to the full solution.

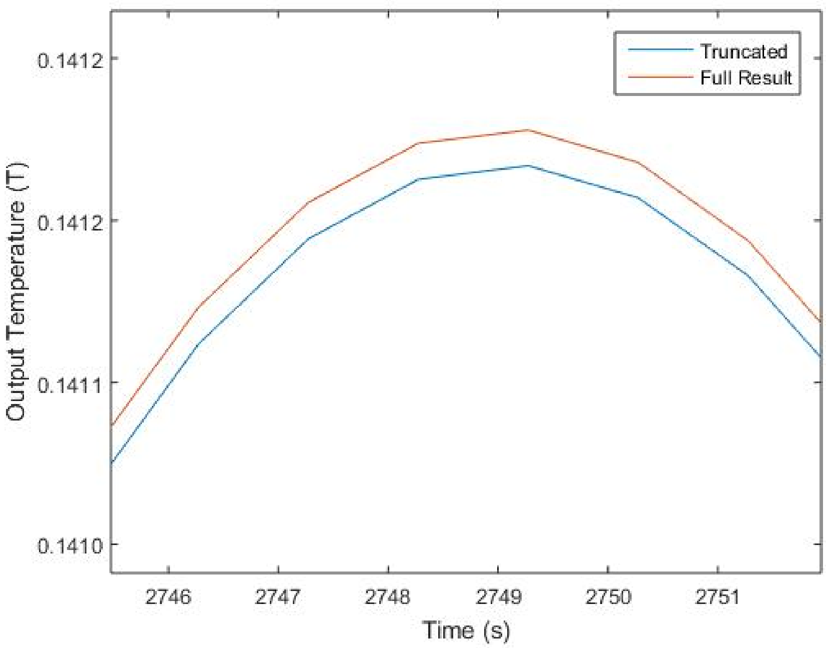

Figure 10: Zoomed in view of one segment in figure 9. The difference is very small

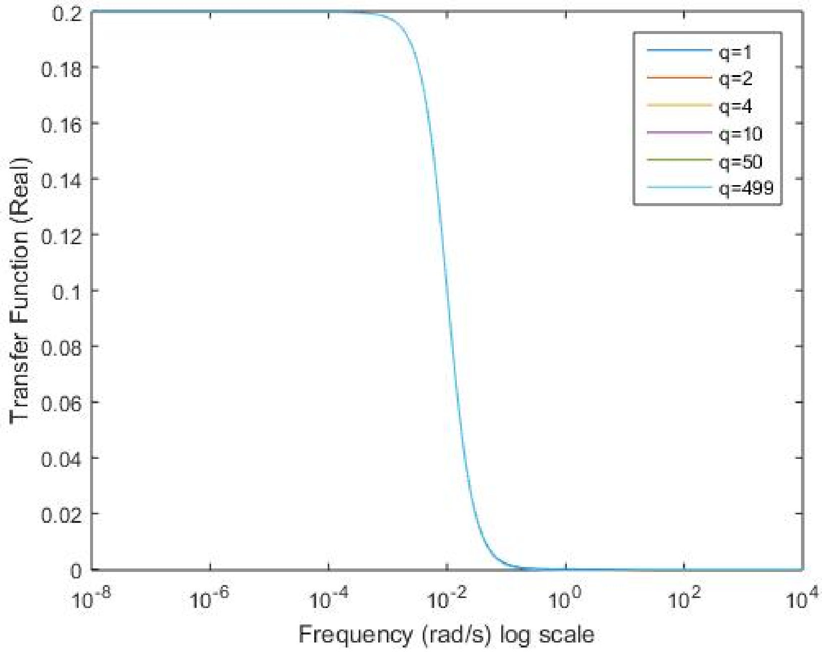

Figure 11: Frequency response obtained from the reduced system using PRIMA.

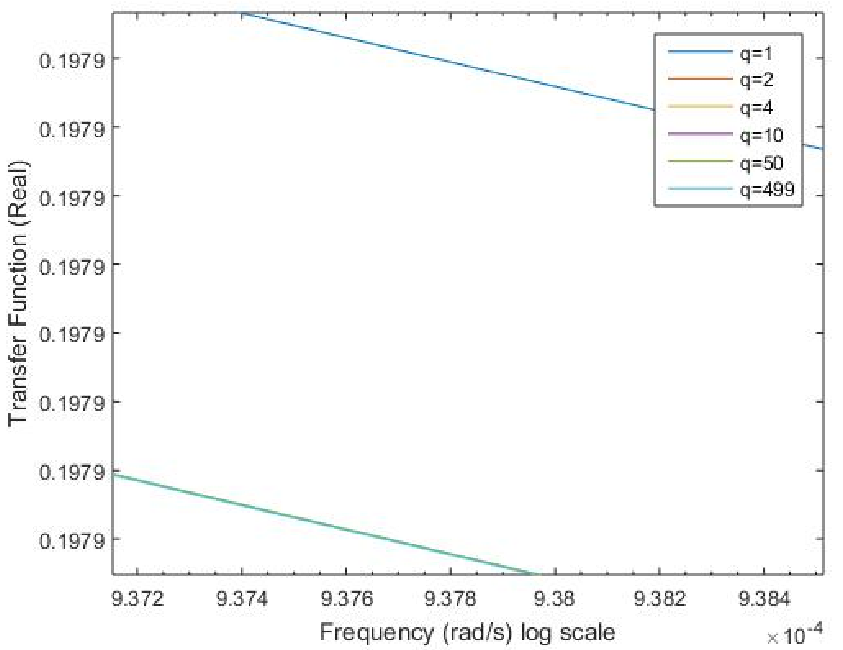

Figure 12: Zoomed in view of one segment in figure 11. The difference is very small

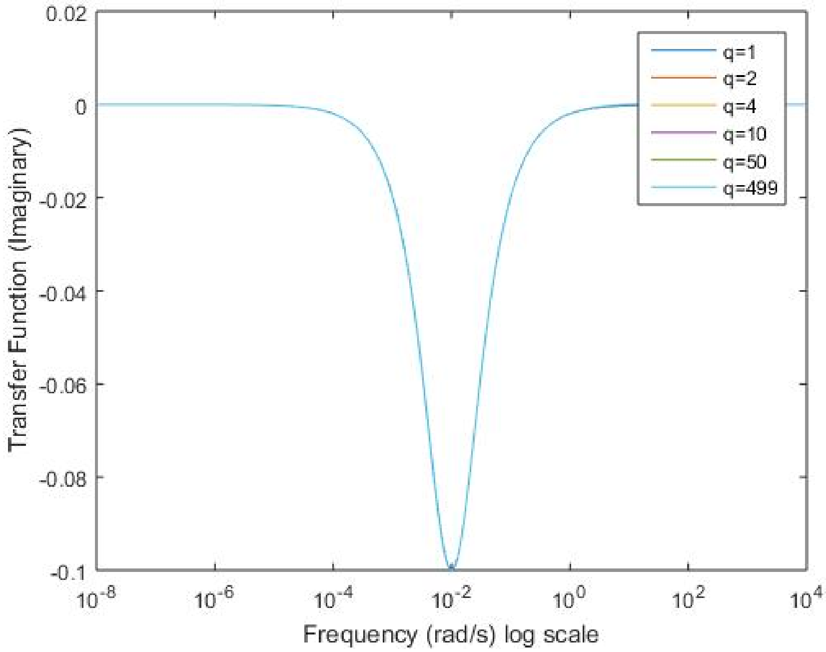

Figure 13: Frequency response obtained from the reduced system using PRIMA.

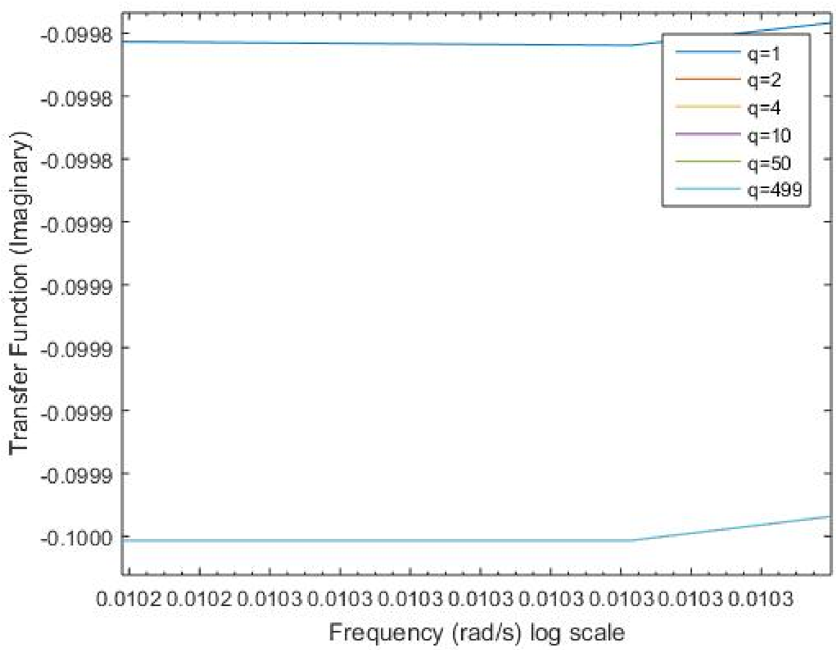

Figure 14: Zoomed in view of one segment in figure 13. The difference is very small

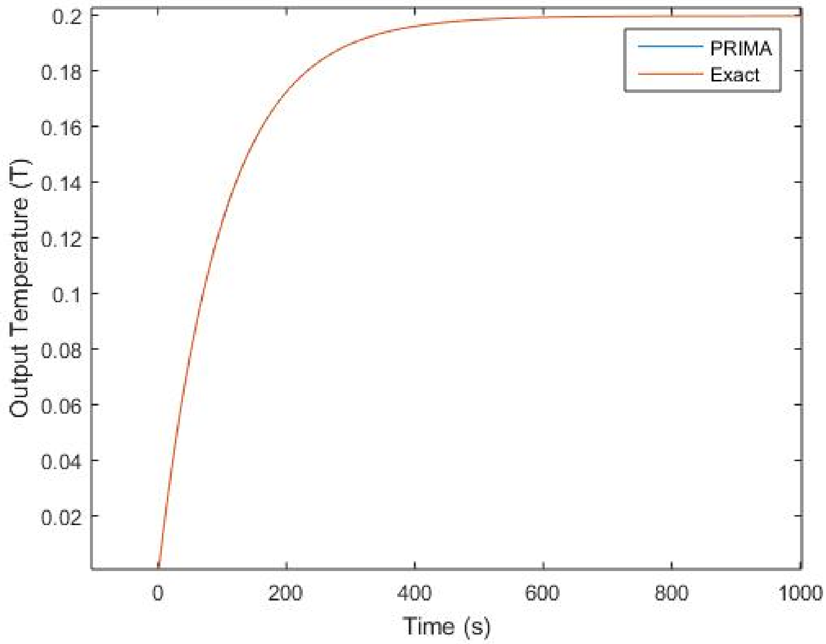

Figure 15: Trapezoidal Rule solution of the full and PRIMA-reduced systems excited by a step function. The agreement is high

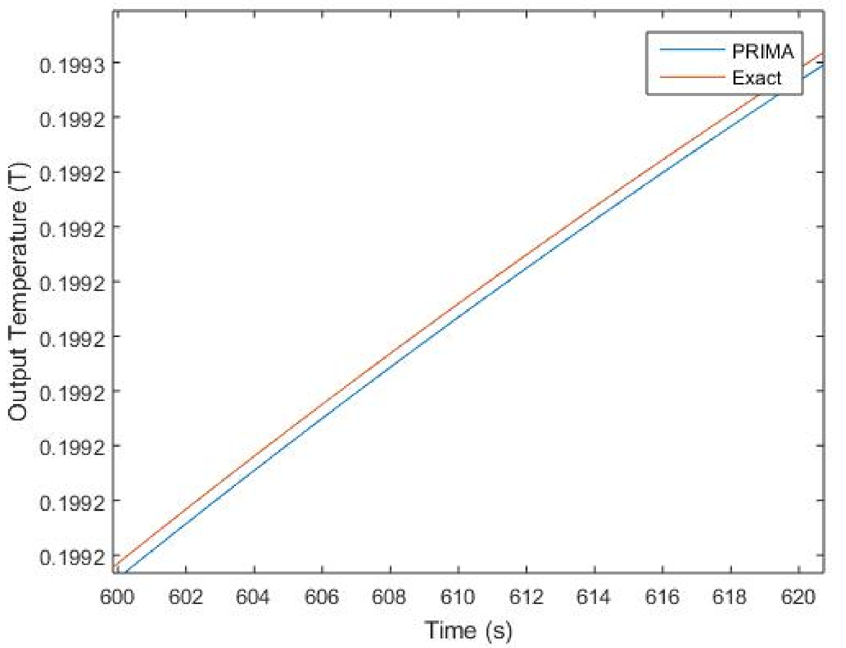

Figure 16: Zoomed in view of one segment in figure 15. The difference is very small

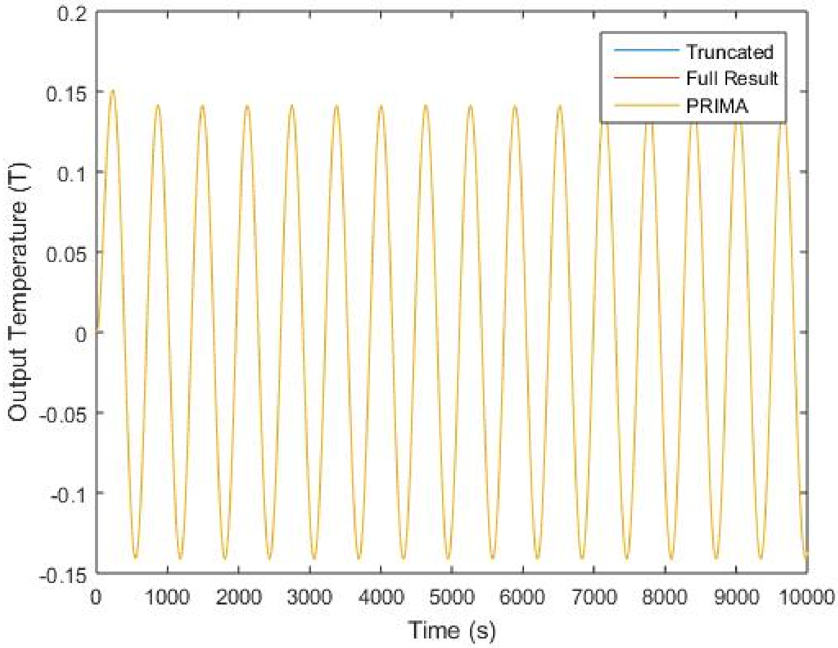

Figure 17: Trapezoidal Rule solution of the full and PRIMA-reduced systems excited by a sine function. The agreement is high

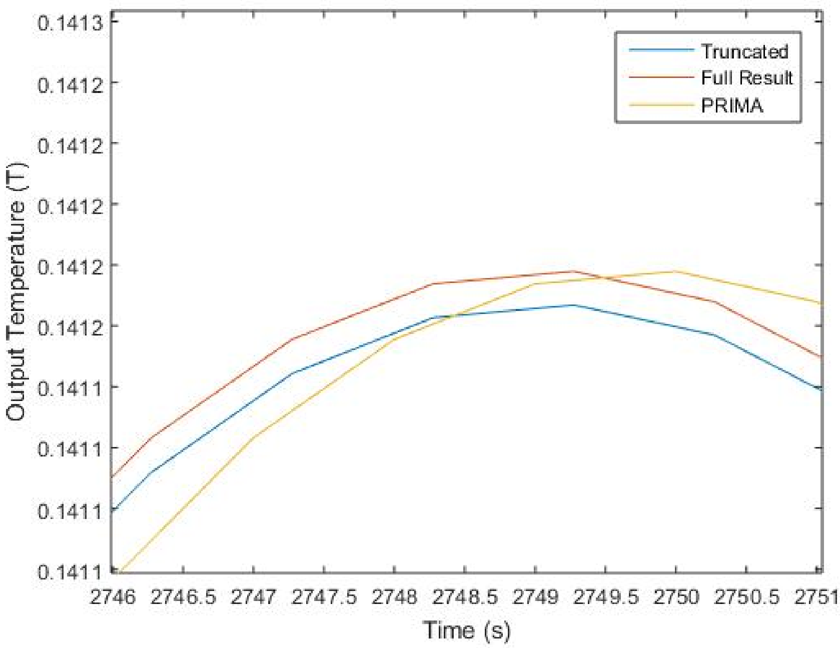

Figure 18: Zoomed in view of one segment in figure 17. The difference is very small