Operations Research, Spring 2024 (112-2)

Homeworrk 1

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1. During the next n months, the IEDO company has demand D_t for air conditioners in month t, $t = 1, ..., n$. Air conditioners can be produced in m sites. It takes L_i hours of skilled labor to produce an air conditioner in site i, $i = 1, ..., m$. It costs C_i^P to produce an air conditioner in site i, $i = 1, ..., m$. During each month, each city has K hours of skilled labor available. It costs C^H to hold an air conditioner in inventory for a month. At the beginning of month 1, IEDO has I air conditioners in stock.

   (a) Suppose that IEDO must meet all the demands on time. Formulate a linear program whose solution will tell IEDO how to minimize the cost of meeting air conditioner demands for the next six months.

   solution:

   x_{it} is the amount of air conditioners produced in site i in month t. $\forall t = 1, ..., 6$
   y_t is the amount of air conditioners in stock in month t. $\forall t = 1, ..., 6$

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{6} (\sum_{i=1}^{m} x_{it} C_i^P + y_t C^H) \\
\text{s.t.} \quad & y_0 = I \\
& y_{t-1} + \sum_{i=1}^{m} x_{it} - D_t = y_t \quad \forall t = 1, ..., 6 \\
& x_{it} \leqslant \frac{K}{L_i} \quad \forall t = 1, ..., 6, i = 1, ..., m \\
& x_{it} \geqslant 0 \quad \forall t = 1, ..., 6, i = 1, ..., m \\
& y_t \geqslant 0 \quad \forall t = 1, ..., 6
\end{aligned}
$$

   (b) Suppose that those demands are the maximum number that IEDO may sell, and IEDO can decide the sales quantity in each month. Each air conditioner can be sold at \$600. Formulate an LP whose solution maximizes the profit of selling air conditioners for the next six months. Your formulation must be a compact one.

   solution:

   x_{it} is the amount of air conditioners produced in site i in month t. $\forall t = 1, ..., 6$
   y_t is the amount of air conditioners in stock in month t. $\forall t = 1, ..., 6$
   z_t is the amount of air conditioners sold in month t. $\forall t = 1, ..., 6$

$$
\begin{aligned}
\max \quad & \sum_{t=1}^{6}(600 z_t - \sum_{i=1}^{m} x_{it} C_i^P - y_t C^H) \\
\text{s.t.} \quad & y_0 = I \\
& y_{t-1} + \sum_{i=1}^{m} x_{it} - z_t = y_t \quad \forall t = 1, ..., 6 \\
& z_t \leqslant D_t \\
& x_{it} \leqslant \frac{K}{L_i} \quad \forall t = 1, ..., 6, i = 1, ..., m \\
& x_{it} \geqslant 0 \quad \forall t = 1, ..., 6, i = 1, ..., m \\
& y_t \geqslant 0 \quad \forall t = 1, ..., 6 \\
& z_t \geqslant 0 \quad \forall t = 1, ..., 6
\end{aligned}
$$

(c) Continue from Part (a) and solve the linear program to obtain an optimal solution according to the following information. During the next six months, the IEDO company has following demands for air conditioners: month 1, 2500; month 2, 4000; month 3, 4500; month 4, 4200; month 5, 3800; month 6, 4400. Air conditioners can be produced in either Hsinchu or Taoyuan. It takes 2 hours of skilled labor to produce an air conditioner in Hsinchu, and 2.5 hours in Taoyuan. It costs \$400 to produce an air conditioner in Hsinchu, and \$350 in Taoyuan. During each month, each city has 4000 hours of skilled labor available. It costs \$80 to hold an air conditioner in inventory for a month. At the beginning of month 1, IEDO has 2000 air conditioners in stock.

solution:

Optimal solution:
$x_{11} = 1800,\ x_{21} = 1600,\ x_{12} = 2000,\ x_{22} = 1600,\ x_{13} = 2000,\ x_{23} = 1600$
$x_{14} = 2000,\ x_{24} = 1600,\ x_{15} = 2000,\ x_{25} = 1600,\ x_{16} = 2000,\ x_{26} = 1600$

Objective value: 8784000

2. A city is divided into n districts. The time (in minutes) it takes an ambulance to travel from district i to j is D_{ij}. The population of district i (in thousands) is H_i. The city has p ambulances and wants to locate them. Formulate an integer program that can achieve each of the following goals.

(a) To maximize the number of people who live within (no greater than) B minutes of at least one ambulance.

solution:

w_{ij} is 1 if the time travel from district i to district j is within B minutes; otherwise, is 0. $\forall i = 1, ..., n, j = 1, ..., n$.
x_i is 1 if there is an ambulance in district i; otherwise, is 0. $\forall i = 1, ..., n$.
y_i is 1 if there is at least one ambulance to district i; otherwise, is 0. $\forall i = 1, ..., n$.

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} H_i y_i \\
\text{s.t.} \quad & D_{ij} w_{ij} \leqslant B \quad \forall i = 1, ..., n, j = 1, ..., n \\
& \sum_{i=1}^{n} x_i \leqslant p \\
& y_j \leqslant \sum_{i=1}^{n} w_{ij} \quad \forall j = 1, ..., n \\
& w_{ij} \in \{0, 1\} \quad \forall i = 1, ..., n, j = 1, ..., n \\
& x_i \in \{0, 1\} \quad \forall i = 1, ..., n \\
& y_i \in \{0, 1\} \quad \forall i = 1, ..., n
\end{aligned}$$

(b) To maximize the number of people who live within (no greater than) B minutes of at least two ambulances.

solution:

w_{ij} is 1 if the time travel from district i to district j is within B minutes; otherwise, is 0. $\forall i = 1, ..., n, j = 1, ..., n$.
x_i is the amount of ambulances in district i. $\forall i = 1, ..., n$.
y_i is 1 if there are at least two ambulances to district i; otherwise, is 0. $\forall i = 1, ..., n$.
Let $z_{ij} = x_i w_{ij} \quad \forall i = 1, ..., n, j = 1, ..., n$.

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} H_i y_i \\
\text{s.t.} \quad & D_{ij} w_{ij} \leqslant B \quad \forall i = 1, ..., n, j = 1, ..., n \\
& \sum_{i=1}^{n} x_i \leqslant p \\
& z_{ij} \leq x_i \quad \forall i = 1, ..., n, j = 1, ..., n \\
& z_{ij} \leq p w_{ij} \quad \forall i = 1, ..., n, j = 1, ..., n \\
& 2y_j \leqslant \sum_{i=1}^{n} z_{ij} \quad \forall j = 1, ..., n \\
& w_{ij} \in \{0, 1\} \quad \forall i = 1, ..., n, j = 1, ..., n \\
& x_i \geq 0 \quad \forall i = 1, ..., n \\
& y_i \in \{0, 1\} \quad \forall i = 1, ..., n
\end{aligned}$$

(c) To maximize the minimum of the following two quantities: (1) the number of people who live within (no greater than) B_1 minutes of at least two ambulances, and (2) the number of people who live within (no greater than) B_2 minutes of at least one ambulance or within (no greater than) B_3 minutes of at least three ambulances.

solution:

w_{kij} is 1 if the time travel from district i to district j is within B_k minutes; otherwise, is 0. $\forall i = 1, ..., n, j = 1, ..., n, k = 1, 2, 3$.
x_i is the amount of ambulances in district i. $\forall i = 1, ..., n$.
y_{1i} is 1 if there are at least 2 ambulances to district i; otherwise, is 0. $\forall i = 1, ..., n$
y_{2i} is 1 if there are at least 1 ambulance to district i; otherwise, is 0. $\forall i = 1, ..., n$
y_{3i} is 1 if there are at least 3 ambulances to district i; otherwise, is 0. $\forall i = 1, ..., n$

v_i is 1 if there is at least 1 ambulance within B_2 minutes of or at least 3 ambulances within B_3 minutes; otherwise, is 0. $\forall i = 1, ..., n$.
Let $z_{kij} = x_i w_{kij} \quad \forall i = 1, ..., n, j = 1, ..., n$.

$$
\begin{aligned}
\max \quad & M \\
\text{s.t.} \quad & D_{ij} w_{kij} \leqslant B_k \quad \forall i = 1, ..., n, j = 1, ..., n, k = 1, 2, 3 \\
& \sum_{i=1}^{n} x_i \leqslant p \\
& z_{kij} \leq x_i \quad \forall i = 1, ..., n, j = 1, ..., n, k = 1, 2, 3 \\
& z_{kij} \leq p w_{kij} \quad \forall i = 1, ..., n, j = 1, ..., n, k = 1, 2, 3 \\
& 2y_j \leqslant \sum_{i=1}^{n} z_{1ij} \quad \forall j = 1, ..., n \\
& y_{1j} \leqslant \sum_{i=1}^{n} z_{2ij} \quad \forall j = 1, ..., n \\
& 3y_j \leqslant \sum_{i=1}^{n} z_{3ij} \quad \forall j = 1, ..., n \\
& v_i \leq y_{2i} + y_{3i} \quad \forall i = 1, ..., n \\
& M \leq \sum_{i=1}^{n} H_i y_{1i} \\
& M \leq \sum_{i=1}^{n} H_i v_i \\
& w_{ij} \in \{0, 1\} \quad \forall i = 1, ..., n, j = 1, ..., n \\
& x_i \geq 0 \quad \forall i = 1, ..., n \\
& y_i \in \{0, 1\} \quad \forall i = 1, ..., n \\
& v_i \in \{0, 1\} \quad \forall i = 1, ..., n
\end{aligned}
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3. Michelle is traveling from Germany to Taiwan. She bought two bags, each can carry up to K kg of items. There are several items in a set $I = \{1, 2, ..., n\}$ that she considers to carry. The weights and values of item i is W_i and $V_i, i \in I$. Michelle wants to maximize the total values of the items she carry while satisfying the capacity constraint, i.e., each bag cannot carry more than K kg. Do each of the following problems independently.

   (a) Formulate an integer program that solves Michelle's problem.

   solution:

   x_{ij} is 1 if item i is in bag j; otherwise, is 0. $\forall i \in I, j = 1, 2$.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{2} V_i x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{n} W_i x_{ij} \leqslant K \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} \leqslant 1 \quad \forall i \in I \\
& x_{ij} \in \{0, 1\} \quad \forall i \in I, j = 1, 2
\end{aligned}
$$

   (b) Suppose that items 2 and 3 cannot be put in the same bag, items 4, 5, and 6 cannot be put in the same bag, at least two of items 8 to 12 must be carried, and at least one of items 1 and 2 must be carried if item 3 is not carried. Formulate an integer program that solves Michelle's problem.

solution:

x_{ij} is 1 if item i is in bag j; otherwise, is 0. $\forall i \in I, j = 1, 2$.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{2} V_i x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{n} W_i x_{ij} \leqslant K \quad \forall j = 1, 2 \\
& x_{ij} \in \{0, 1\} \quad \forall i \in I, j = 1, 2 \\
& x_{2j} + x_{3j} \leqslant 1 \quad \forall j = 1, 2 \\
& x_{4j} + x_{5j} + x_{6j} \leqslant 2 \quad \forall j = 1, 2 \\
& \sum_{i=8}^{12}\sum_{j=1}^{2} x_{ij} \geqslant 2 \\
& \sum_{i=1}^{3}\sum_{j=1}^{2} x_{ij} \geqslant 1 \\
& \sum_{j=1}^{2} x_{ij} \leqslant 1 \quad \forall i \in I
\end{aligned}
$$

(c) Suppose that if items i and $i+1$ are both brought in the trip, $i = 1, ..., n-1$, an additional value A_i will be created for Michelle. Moreover, suppose that if items i and $i+1$ are both put in the same bag, an additional value B_i will be created for Michelle. For example, if Michelle put items 1 and 2 in bag 1 and items 3 and 4 in bag 2, her total value is $V_1 + V_2 + V_3 + V_4 + A_1 + A_2 + A_3 + B_1 + B_3$. Formulate an integer program that solves Michelle's problem.

solution:

x_{ij} is 1 if item i is in bag j; otherwise, is 0. $\forall i \in I, j = 1, 2$.
a_i is 1 if item i and $i+1$ are both brought; otherwise, is 0. $\forall i = 1, ..., n-1$.
b_i is 1 if item i and i+1 are both put in the same bag; otherwise, is 0. $\forall i = 1, ..., n-1$.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{2} V_i x_{ij} + \sum_{i=1}^{n-1} a_i A_i + \sum_{i=1}^{n-1} b_i B_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} W_i x_{ij} \leqslant K \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} + \sum_{j=1}^{2} x_{i+1j} \geqslant 2a_i \quad \forall i = 1, ..., n-1 \\
& x_{ij} + x_{i+1j} \geqslant 2b_i \quad \forall i = 1, ..., n-1, j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} \leqslant 1 \quad \forall i \in I \\
& x_{ij} \in \{0, 1\} \quad \forall i \in I, j = 1, 2 \\
& a_i \in \{0, 1\} \quad \forall i = 1, ..., n-1 \\
& b_i \in \{0, 1\} \quad \forall i = 1, ..., n-1
\end{aligned}
$$

4. Consider the following n-item ordering problem. You purchase n items from a supplier to sell to a market. The demands for item i is D_i per day (i.e., for all items the demand rates are constants), the ordering cost is K per order, and the holding cost of item i is h_i per unit per day. You are allowed to choose an ordering cycle and n order quantities for the n products, but the supplier forces you to order all items with the same cycle. For example, suppose that $n = 2$, you are allowed order 5 and 10 units of items 1 and 2 every six days, but you are not allowed to order 5 units of item 1 every six days and 5 units of item 2 every seven days. Formulate a nonlinear program that minimizes the average daily total cost, which includes ordering and holding costs.

solution:

T is the cycle time.
q_i is the order quantity of product i. $\forall i = 1, ..., n$

$$\begin{aligned} \min \quad & \frac{K}{T} + \frac{1}{2}\sum_{i=1}^{n} q_i h_i \\ \text{s.t.} \quad & q_i = D_i T \quad \forall i = 1, ..., n \\ & T \geq 1 \end{aligned}$$