Operation Research, Spring 2024 (112-2)
Homework 0

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1. Solve the following problems.

   (a) Let $f(x) = ax^2 + 8x + 6$, where $a \in \mathbb{R}$. Find all values of a such that $f(x)$ is maximized at $x = 2$.

   solution:

   If $f(x)$ is maximized at $x = 2$, then $f'(2)$ should be 0 and $a < 0$.

   $f'(x) = 2ax + 8 \;\Rightarrow\; f'(2) = 4a + 8$

   Hence, $a = -2$.

   (b) Let $f(x) = ax^2 + 8x + 6$, where $a \in \mathbb{R}$. Find $F(t) = \int_0^t f(x)dx$ as a function of a and t for all $t > 0$.

   solution:

   $F(t) = \int_0^t f(x)dx = \frac{a}{3}t^3 + 4t^2 + 6t + C$ (C is a constant)

   (c) Find all values of $a \in \mathbb{R}$ such that the inverse of

   $$\begin{bmatrix} 1 & 0 & 1 \\ a & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$

   does not exist. If for all values of a the inverse exists, prove it.

   solution:

   Let $A = \begin{bmatrix} 1 & 0 & 1 \\ a & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$.

   The inverse of A does not exist only when $det(A) = 0$.

   $det(A) = a - 1$

   Hence, for $a = 1$, the inverse of A does not exist.

2. Consider the problem of determining whether a given integer n is a prime number.

   (a) Write down a pseudocode of an algorithm that solves the problem for any given positive integer n.

   solution:

   ```
   Algorithm isPrime(n) :
       if n ⩽ 1 then
           return false // 1 is not a prime number
       i = 2
       while i × i ⩽ n do
   ```

if n mod $i == 0$ then

return false

i++

return true

(b) What is the time complexity of your algorithm? Please use the big-O notation to express your solution.

solution:

Since the while loop stops when $i \times i > n$, it will at most traverse the loop for $\sqrt{n}$ times.
Hence, the time complexity of my algorithm is $O(\sqrt{n})$.

3. For each subproblem, use the graphical approach to solve

$$\begin{aligned} \max \quad & 2x_1 + Ax_2 \\ \text{s.t.} \quad & x_1 + x_2 \leqslant 8 \\ & 2x_1 - x_2 \geqslant 12 \\ & x_2 \leqslant B \\ & x_1 \geqslant 0, x_2 \geqslant 0, \end{aligned}$$

for the given values of A and B. If there are multiple optimal solutions, please list just one of them. As long as there is at least one optimal solution, write one down and also list all constraints binding at that optimal solution. If there is no optimal solution, graphically demonstrate it.

(a) $A = 1,\ B = -4$.

solution:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leqslant 8 \\ & 2x_1 - x_2 \geqslant 12 \\ & x_2 \leqslant -4 \\ & x_1 \geqslant 0, x_2 \geqslant 0, \end{aligned}$$

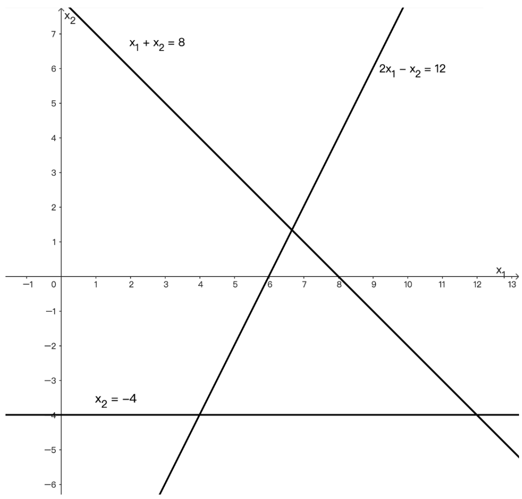

Figure 1: Graphical solution for Problem 3a

There is no optimal solution because from Figure 1, we can see that there is no feasible region satisfying all the constraints.

(b) $A = 5,\ B = 4.$

solution:

$$\begin{array}{rl} \max & 2x_1 + 5x_2 \\ \text{s.t.} & x_1 + x_2 \leqslant 8 \\ & 2x_1 - x_2 \geqslant 12 \\ & x_2 \leqslant 4 \\ & x_1 \geqslant 0, x_2 \geqslant 0, \end{array}$$

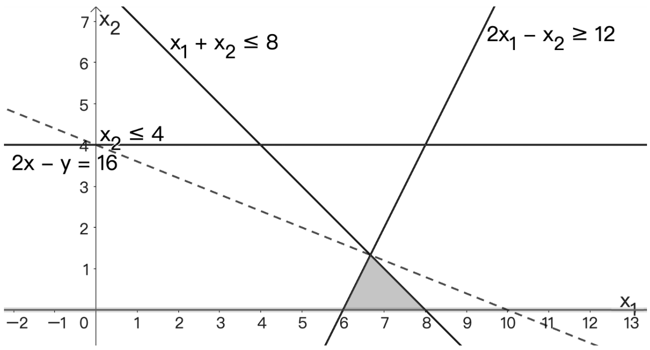

Figure 2: Graphical solution for Problem 3b

The solution is shown in Figure 2, the gray region is the feasible region.The dotted line is an isocost line. According to the graph, $(x_1, x_2) = (\frac{20}{3}, \frac{4}{3})$ is optimal, and the maximum value is 16. Hence, the constraints $x_1+x_2 \leqslant 8$ and $2x_1-x_2 \geqslant 12$ are binding at $(x_1, x_2) = (\frac{20}{3}, \frac{4}{3})$.

(c) $A = -1,\ B = 1.$

solution:

$$\begin{array}{rl} \max & 2x_1 - x_2 \\ \text{s.t.} & x_1 + x_2 \leqslant 8 \\ & 2x_1 - x_2 \geqslant 12 \\ & x_2 \leqslant 1 \\ & x_1 \geqslant 0, x_2 \geqslant 0, \end{array}$$

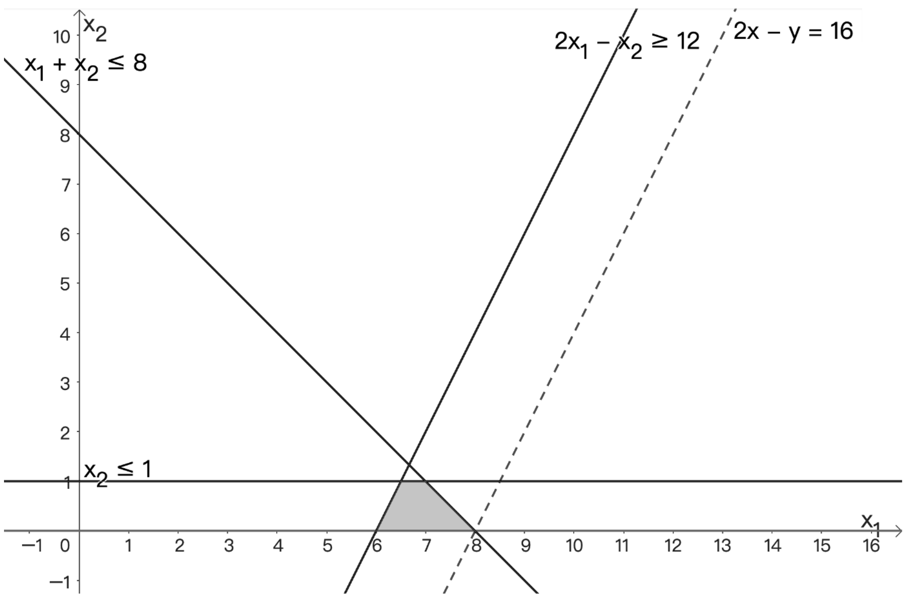

Figure 3: Graphical solution for Problem 3c

The solution is shown in Figure 3, the gray region is the feasible region.The dotted line is an isocost line. According to the graph, $(x_1, x_2) = (8, 0)$ is optimal, and the maximum value is 16. Hence, the constraints $x_1 + x_2 \leqslant 8$ and $x_1 \geqslant 0$ are binding at $(x_1, x_2) = (8, 0)$.

4. For each of the following subproblems, formulate a linear program that maximizes IEDO's profits for the next year in Parts (a) and (b) or for the next T years in Parts (c) and (d).

   (a) IEDO Oil has refineries in Kaohsiung and Taipei. Currently, the Kaohsiung refinery can refine up to K_1 million barrels of oil per year, and the Taipei refinery up to K_2 million. Once refined, oil is shipped to two distribution points: Hsinchu and Taichung. IEDO Oil estimates that each distribution point can sell up to D million barrels per year. Because of differences in shipping and refining costs, the profit earned per million barrels of oil shipped depends on where the oil was refined and on the point of distribution. In particular, the profit per million barrels is P_{11} from Kaohsiung to Hsinchu, P_{12} from Kaohsiung to Taichung, P_{21} from Taipei to Hsinchu, and P_{22} from Taipei to Taichung.

   solution:

$$
\begin{aligned}
\max \quad & x_{11}P_{11} + x_{12}P_{12} + x_{21}P_{21} + x_{22}P_{22} \\
& (x_{ij} \text{ is the number of million barrels of oil from refinery i to distribution point j}) \\
\text{s.t.} \quad & x_{11} + x_{12} \leqslant K_1 \\
& x_{21} + x_{22} \leqslant K_2 \\
& x_{11} + x_{21} \leqslant D, \;\; x_{12} + x_{22} \leqslant D \\
& x_{11} \geqslant 0, \;\; x_{12} \geqslant 0, \;\; x_{21} \geqslant 0, \;\; x_{22} \geqslant 0
\end{aligned}
$$

   (b) IEDO Oil has refineries in n cities. Currently, the refinery in city i can refine up to K_i million barrels of oil per year. Once refined, oil is shipped to m distribution points. IEDO Oil estimates that each distribution point can sell up to D million barrels per year. Because of differences in shipping and refining costs, the profit earned per million barrels of oil shipped depends on where the oil was refined and on the point of distribution. In particular, the profit per million barrels is P_{ij} from refinery in city i to distribution point j.

   solution:

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n}\sum_{j=1}^{m} x_{ij}P_{ij} \\
& (x_{ij} \text{ is the number of million barrels of oil from refinery i to distribution point j}) \\
\text{s.t.} \quad & \sum_{j=1}^{m} x_{ij} \leqslant K_i \\
& \sum_{i=1}^{n} x_{ij} \leqslant D \\
& x_{ij} \geqslant 0 \quad \forall i = 1, ..., n, \;\; j = 1, ..., m
\end{aligned}
$$