Chapter 3. Extensions and Generalizations (Exercises)

Exercise 3.1: Rationing

Consider a consumer choosing between three goods x_1, x_2 and x_3, with prices p_1, p_2 and p_3 respectively. Suppose his utility function is

$$U(x_1, x_2, x_3) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3),$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$. His income is I, so the budget constraint is

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I.$$

In addition, the consumer faces a rationing constraint: he is not allowed to buy more than k units of good 1.

Question 1: Find the consumer's optimal bundle (x_1, x_2, x_3).

Question 2: Show that when the rationing cosntraint binds, the consumer splits his income between goods 2 and 3 in the proportions $\alpha_2 : \alpha_3$. Would you expect rationing of bread purchases (good 1) to affect demands for butter (good 2) and rice (good 3) in this way? Why and why not?

Exercise 3.2: Distribution Between Envious Consumers

There is a fixed total Y of goods at the disposal of society. There are two consumers who envy each other. if consumer 1 gets Y_1 and consumer 2 gets Y_2, their utilties are

$$U_1 = Y_1 - kY_2^2, \quad U_2 = Y_2 - kY_1^2,$$

where k is a positive constant. The allocation must satisfy $Y_1 + Y_2 \leq Y$, and maximize $U_1 + U_2$.

Show that if $Y > 1/k$, the resource constraint will be slack at the optimum.

Exercise 3.3: Investment Allocation

A capital sum C is available for allocation among n investment projects. If the non-negative amount x_j is allocated to project j for $j = 1, 2, ..., n$, the expected return from this portfolio of projects is

$$R(x) = \sum_{j=1}^{n} \left[\alpha_j x_j - \frac{1}{2} \beta_j x_j^2 \right].$$

The objective is to maximize $R(x)$.

Question 1: Fine the first-order necessary conditions.

Question 2: Define

$$H = \sum_{j=1}^{n} (\alpha_j / \beta_j), \quad K = \sum_{j=1}^{n} (1/\beta_j).$$

Show that

(i) If $C > H$, then a part of the total sum available is left unused.

(ii) If $\alpha_j > (H - C)/K$ for all j, then every project will receive some funding.

(iii) If any project receives zero funding, then it must have a lower α than any project that gets some funding.