Chapter 2. Dynamic Games of Complete Information

Xiaoxiao Hu

Dynamic Games of Complete Information

- In dynamic games, players move sequentially.
- Information is more important than timing.
  - Player who moves second knows what the previous player has done before he moves.
  - The previous player knows that this is the case.
- Complete information: players' payoff functions are common knowledge

Dynamic Games of Complete Information

- In Section 2.A, we study games of perfect information: games in which the player who is to move knows the full history of the play of the game.
- In Section 2.B, we study games of imperfect information.
- In Section 2.C, we study repeated games.

2.A. Dynamic Games of Complete and Perfect Information

2.A.1. Backward Induction

Example 2.A.1. Consider the 2-player investment game.

- Player 1 chooses to invest 0, 1, or 3.
- After observing Player 1's choice, Player 2 can either match, i.e., add the same amount, or take the cash.

Investment Game: Payoffs

For Player 1,

- if invests 0, gets 0;
- if invests 1, doubles if matched and loses 1 if not;
- if invests 3, doubles if matched and loses 3 if not.

For Player 2,

- if takes, gets Player 1's investment;
- if matches 1, gets 2.5 back;
- if matches 3, gets 5 back.

Investment Game

This is a sequential-move game.

- Player 2 observes how much Player 1 has invested before making the matching or taking decision.
- Player 1 knows that this is the case.

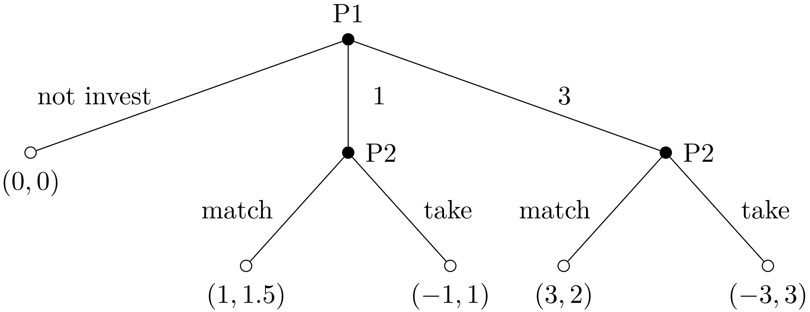
Investment Game

Question.

- Suppose you are Player 1, how much would you invest?
- Suppose you are Player 2, would you match 1? How about 3?

Investment Game

The game could be more clearly organized in the game tree:



Investment Game

- The solid nodes are called decision nodes, besides which we write down the player whose turn it is to move.
- The hollow nodes are called end nodes or terminal nodes, besides which we write the players' payoffs.

Investment Game

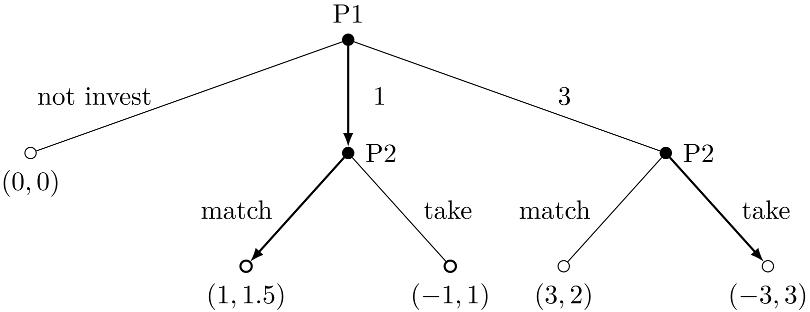
Let us now work out what the players would do in this game.

- P1 would anticipate what P2 would do following each of his own choice, and then work backwards.
  - If P1 chooses not to invest, she would get 0.
  - If P1 chooses to invest 1, she knows P2 would match and she will double her money and get 1.
  - If P1 chooses to invest 3, she knows P2 would take the cash and she will lose her investment of 3.

Investment Game

- P1 is essentially choosing among the payoffs of 0 (not investing), 1 (investing 1) and -3 (investing 3).
- P1 would choose to invest 1.
- To complete the analysis, we write out P2's choice.
- Seeing that P1 chooses 1, P2 would indeed match.

Investment Game: Backward Induction



Backward Induction

This idea of starting at the player who moves last, solving out what they would do and then work back through the tree is called Backward Induction.

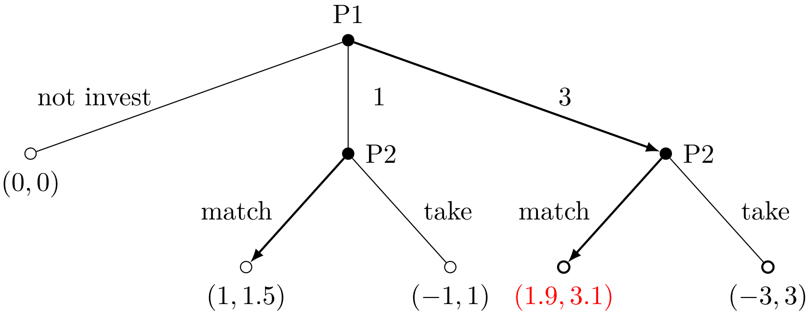
Investment Game

Remark. Note that the mutually beneficial outcome $(3, 2)$ is not played out.

Question. Can you think of ways to reach the good outcome?

Change the Division

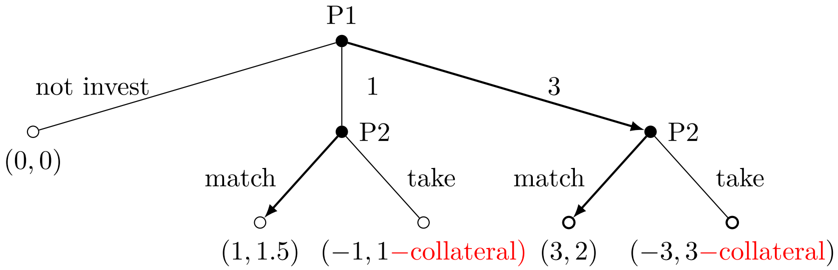
Change the division of the net gain of 5 after the matched investment of 3.



Remark. P1 also likes such change since now she gets 1.9 instead of 1.

Collateral

Ask P2 to make a collateral: imposing an extra negative return to P2



Remark. Collateral lowers the payoff to P2 at some point of the tree, yet it makes P2 better off.

Dynamic Games of Complete and Perfect Information

Abstracting from the concrete example, the dynamic games of complete and perfect information takes the following form:

1. Player 1 chooses an action a_1 from the feasible set A_1;
2. Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2.
3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.

These games could be more clearly represented in game trees as we did in the investment game (Example 2.A.1).

Dynamic Games of Complete and Perfect Information

The key features of these games are

1. the moves occur in sequence;
2. all previous moves are observed before the next move is chosen (perfect information);
3. the players' payoffs from each feasible combination of moves are common knowledge (complete information).

Dynamic Games of Complete and Perfect Information

This class of games are solved by Backward Induction:

1. In the second stage, given a_1 previously chosen by P1, P2 solves
$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$
Denote the solution $R_2(a_1)$.
This is P2's reaction (or best response) to P1's action.
2. In the first stage, P1 would anticipate P2's reaction to each a_1 that P1 might take, so P1 solves
$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1)).$$

$(a_1^*, R_2(a_1^*))$ is the backward-induction outcome.

Backward Induction

Remark. The backward-induction outcome does not involve non-credible threats.

- P1 anticipates that P2 would respond optimally to any a_1 that P1 might choose, i.e., P2 would play $R_2(a_1)$.
- P1 gives no credence to threats that will not be in P2's self-interest when the second stage arrives.

For example, in the original game of Example 2.A.1, P2's claim that he would always match P1's investment would not be believed by P1.

Extensive-Form Representation

Our description of the game corresponds to the Extensive-Form Representation that specifies

1. the players in the game,
2. a) when each player has the move,
   b) what each player can do at each of her opportunities to move,
   c) what each player knows at each of her opportunities to move,
3. the payoffs received by each player for each combination of moves that could be chosen by the players.

Extensive-Form and Normal-Form Representation

Graphically, the extensive-form representation is shown as the game tree.

Recall the normal-form representation of a game specifies

1. the players in the game,
2. the strategies available to each player, and
3. the payoffs received by each player for each combination of strategies that could be chosen by the players.

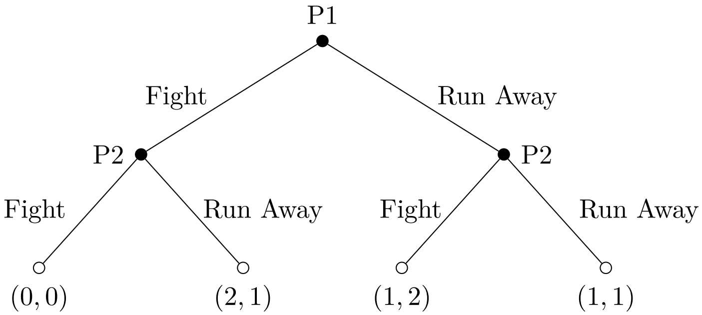
Graphically, the normal-form representation is shown as the payoff matrix.

Extensive-Form and Normal-Form Representation

- It is convenient to represent dynamic games in extensive form and static games in normal form.
- However, notice that any game could be represented in either normal or extensive form.
- For the last point, we will discuss in detail later on.

Backward Induction: Example

Example 2.A.2. P1 and P2 are two armies in a battle.



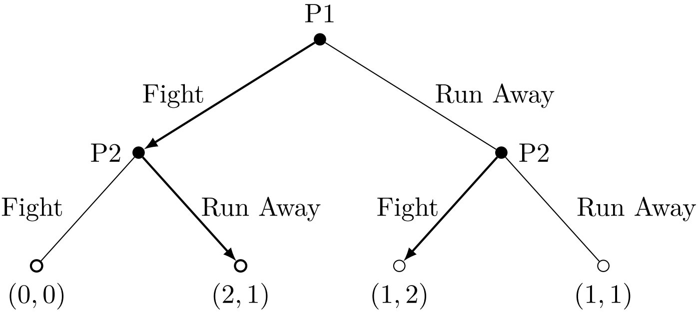
Example 2.A.2

Question. P2 likes outcome (Run Away, Fight) the most. To make P1 run away, P2 claims that he would choose to fight no matter P1 fights or not.

Will such a threat be believed by P1?

Backward Induction

The game is solved by backward induction.



P1 would “Fight”.

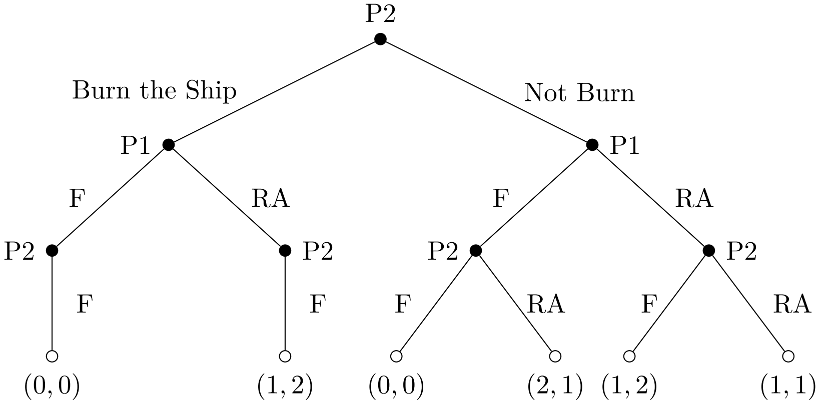
Following this action, P2 would “Run Away”.

Burning the Ship

We add the option "Burn the Ship" for Player 2.

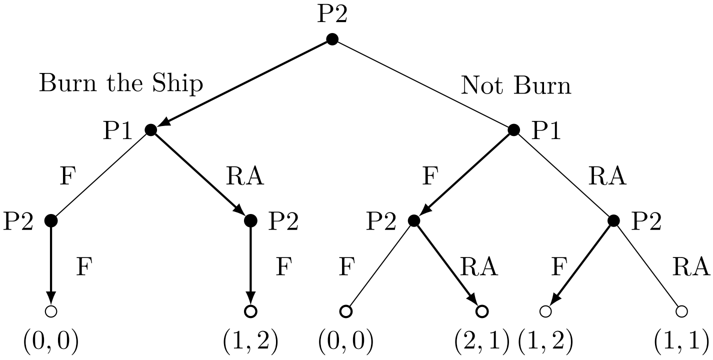
- If P2 does not choose to "Burn the Ship", then the game goes on as in the original example.
- If P2 chooses to "Burn the Ship", then P2 has eliminated his own option "Run Away".

Burning the Ship



Burning the Ship: Backward Induction

Still, we use backward induction to solve the game.



P2 chooses to "Burn the Ship", followed by P1 "Run Away".

At last, P2 "Fight".

Burning the Ship

Remark. P2 is better off by getting rid of his own option "Run Away".

- Essentially, by burning the ship, P2 commits not to run away, and thus makes the action "Fight" credible.
- This commitment strategy is similar to the collateral case in the investment example (Example 2.A.1).

Burning the Ship

Remark. Another important aspect is that P1 must know that P2 has burned the ship. Otherwise, the game tree in P1's mind is still the original one and as a result P1 would "Fight".

Lion Game

Example 2.A.3 (The Lion Game).

- n lions and 1 sheep
- Lion society is hierarchical: only the head lion can eat the sheep.
- However, after the head lion has eaten the sheep, he would fall into postprandial stupor and could be eaten by the second largest lion.
- After the second largest lion has eaten the head lion, he could be eaten by the third largest lion...

Question. Should the head lion eat the sheep?

2.A.2. Stackelberg Model of Duopoly

Stackelberg (1934) proposed a dynamic model of duopoly in which a dominant (or leader) firm moves first and a subordinate (or follower) firm moves second.

Stackelberg Model of Duopoly

The timing of the game is as follows:

1. Firm 1 chooses a quantity $q_1 \geq 0$;
2. Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$.

Payoff to Firm i is given by

$$\pi_i(q_i, q_j) = q_i[P(Q) - c] = q_i[a - q_i - q_j - c].$$

Backward Induction

1. At Stage 2, observing q_1, Firm 2 solves

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2) = \max_{q_2 \geq 0} q_2[a - q_1 - q_2 - c]$$
$$\Longrightarrow q_2 = \frac{a - q_1 - c}{2} = \mathrm{BR}_2(q_1).$$

2. Knowing $\mathrm{BR}_2(q_1)$, at Stage 1, Firm 1 solves

$$\max_{q_1 \geq 0} \pi_1(q_1, \mathrm{BR}_2(q_1)) = \max_{q_1 \geq 0} q_1 \left[\frac{a - q_1 - c}{2}\right]$$
$$\Longrightarrow q_1^* = \frac{a - c}{2}.$$

   Plugging into $\mathrm{BR}_2(q_1)$ gives $q_2^* = \frac{a-c}{4}$.

Comparison with Cournot Outcome

Question. Compared to Cournot outcome, is Firm 1 better-off? How about Firm 2?

Comparison with Cournot Outcome

Recall that the Nash equilibrium of the Cournot game is

$$q_1^c = q_2^c = \frac{a-c}{3}.$$

- In Stackelberg game, Firm 1 produces more than Cournot quantity whereas Firm 2 produces less.
- In terms of aggregate quantity, the firms produces more in Stackelberg game ($\frac{a-c}{2} + \frac{a-c}{4} > \frac{a-c}{3} + \frac{a-c}{3}$).
- Since $p(Q) = a - Q$, price is lower in Stackelberg game.

Comparison with Cournot Outcome

- We could directly calculate Firm 1 and Firm 2's profits in Stackelberg game and Cournot game and make comparisons.
- Instead, here we provide an argument without detailed calculations.

Comparison with Cournot Outcome

Question. In Stackelberg game, what would Firm 2 do if Firm 1 chooses Cournot quantity?

Comparison with Cournot Outcome

- Firm 2 would best respond with Cournot quantity.
- Thus, Firm 1 could have achieved its Cournot profit level by choosing Cournot quantity.
- But Firm 1 chooses some other quantity $\frac{a-c}{2}$.
- Firm 1's profit in Stackelberg game must exceed its profit in Cournot game.
- For Firm 2, we have already established that in Stackelberg game, Firm 2's quantity is lower and the price $p(Q)$ is lower.
- Firm 2 is worse-off in the Stackelberg game.

Comparison with Cournot Outcome

Remark. Compared to Cournot game, in Stackelberg game, Firm 2 has more information (Firm 2 observes q_1 before choosing q_2) and yet it is worse-off.

Remark. The fact that Firm 1 knows that Firm 2 has more information is important.

Spy

- Two firms both privately deciding quantities to produce. (Cournot game)
- Firm 2 sends a spy to Firm 1.
- Moreover, Firm 1 knows that there is a spy even though they do not know who is the spy.

Question. What would Firm 1 do?

First-mover/Second-mover Advantage

The Stackelberg game is a game with First-mover Advantage.

Question. Can you think of games with Second-mover Advantage? And games with neither First-mover nor Second-mover Advantage?

First-mover/Second-mover Advantage

1. Rock, Paper, Scissors is a game with Second-mover Advantage.
2. "I split, you choose" is a game with neither First-mover nor Second-mover Advantage.

First-mover/Second-mover Advantage

Example 2.A.4.

- Two players play with two piles of stones.
- Players move sequentially.
- In each turn, the player whose turn it is to move picks one of the two piles and removes some (≥ 1) of the stones.
- The person who gets the last stone wins.

Question. Is it a game of First-mover or Second-mover advantage?

Zermelo's Theorem

Theorem (Zermelo's Theorem)**.** Consider a finite two-person game with perfect information. Assume that there are three outcomes: a win for Player 1, a loss for Player 1, and a tie. Then, either Player 1 can force a win, or Player 1 can at least force a tie, or Player 2 can force a loss on Player 1.

Originally, Zermelo's Theorem concerns the game of chess.

2.A.3. Sequential Bargaining

Ultimatum Game

- Player 1 and Player 2 are dividing one dollar.
- P1 can make "take-it-or-leave-it" offer to P2: $(s, 1-s)$.
- "Take-it-or-leave-it" means:
  - If P2 accepts the offer, then payoffs are $(s, 1-s)$.
  - If P2 rejects the offer, then payoffs are $(0, 0)$.

Ultimatum Game

- By backward induction, P2 should accept any amount $(1-s) \geq 0$.
- Knowing this, P1 would keep all 1 to herself.
- The equilibrium division is $(1, 0)$.

Two-stage bargaining

- Player 1 and Player 2 are bargaining over one dollar.
- They alternate in making offers.
- Both players discount payoffs received in later periods by $\delta < 1$ per period.
- The bargain will last for two periods at most.

Two-stage bargaining

1. P1 makes an offer to P2: $(s_1, 1 - s_1)$

   - If P2 accepts offer, then game ends and payoffs are $(s_1, 1 - s_1)$.
   - If P2 rejects offer, then game goes on to stage 2.

2. P2 makes an offer to P1: $(s_2, 1 - s_2)$

   - If P1 accepts offer, then game ends and payoffs are $(s_2, 1 - s_2)$.
   - If P1 rejects offer, then game ends and payoffs are $(0, 0)$.

Two-stage bargaining

- Still, we solve this game by backward induction.
- Stage 2 game is the same as Ultimatum game:

  P1 would accept any amount, so P2 would offer 0 to P1 and keep 1 to himself.
- In Stage 1, P2 would accept offer if and only if $1 - s_1 \geq \delta \cdot 1$, i.e., $s_1 \leq 1 - \delta$.
- P1 should keep $1 - \delta$ to herself and leave δ to P2.

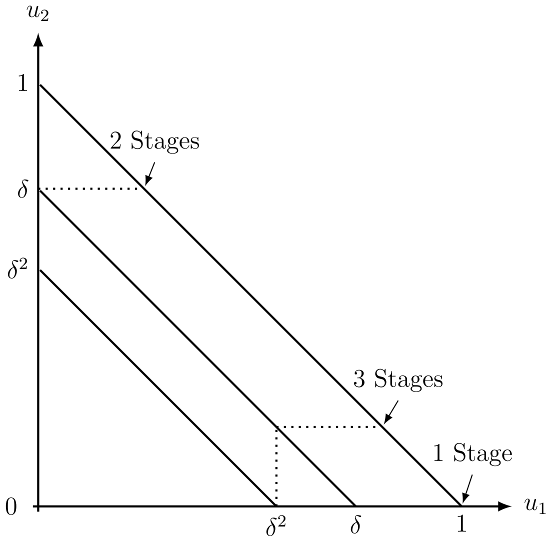
More Stages

Now consider the sequential bargaining game with more stages:

1. P1 makes an offer $(s_1, 1 - s_1)$.
2. If offer is rejected by P2, P2 makes an offer $(s_2, 1 - s_2)$.
3. If offer is rejected by P1, P1 makes an offer $(s_3, 1 - s_3)$.
4. and so on...
5. The game ends in Period T. If offer is rejected, then payoffs are $(0, 0)$.

We could solve the game by backward induction.

More Stages



More Stages

	Offerer	Receiver
1 Stage	1	0
2 Stages	$1-\delta$	δ
3 Stages	$1-\delta(1-\delta)$	$\delta(1-\delta)$
$\vdots$	$\vdots$	$\vdots$
2n Stages	$\frac{1-\delta^{2n}}{1+\delta}$	$\frac{\delta+\delta^{2n}}{1+\delta}$
2n + 1 Stages	$\frac{1+\delta^{2n+1}}{1+\delta}$	$\frac{\delta-\delta^{2n+1}}{1+\delta}$

- When $n \to \infty$, P1 gets $\frac{1}{1+\delta}$ and P2 gets $\frac{\delta}{1+\delta}$.
- When $\delta \to 1$, both players get $\frac{1}{2}$.

Sequential Bargaining

Remark. In the sequential bargaining game, the first offer is accepted.

Remark. Even split if

(i) $n \to \infty$ (potentially bargaining forever) and

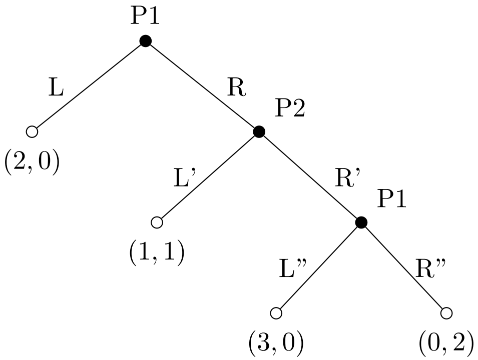
(ii) $\delta \to 1$ (no discounting or rapid offers).

2.A.4. Normal-Form Representations and Credible Threat

Definition 2.A.1 (Pure Strategy)**.** A Pure Strategy for Player i in a game of perfect information is a complete plan of actions: it specifies which action Player i will take at each of its decision nodes.

Strategy

Example 2.A.5.



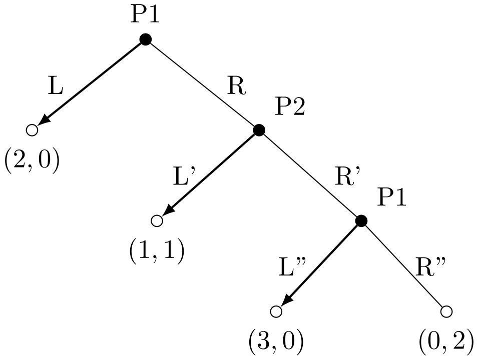
Strategy

According to Definition 2.A.1,

- Player 2's strategies: L' and R'
- Player 1's strategies: [R, L"], [R, R"], [L, L"], [L, R"]

Backward Induction

We could solve the game by backward induction.



Player 1 chooses "L" and the game ends.

Normal-Form Representation

		Player 2	
		L'	R'
Player 1	[R, L"]	$(1, \underline{1})$	$(\underline{3}, 0)$
	[R, R"]	$(1, 1)$	$(0, \underline{2})$
	[L, L"]	$(\underline{2}, \underline{0})$	$(2, \underline{0})$
	[L, R"]	$(\underline{2}, \underline{0})$	$(2, \underline{0})$

Two Nash equilibria: ([L, L"], L') and ([L, R"], L').

Normal-Form Representation

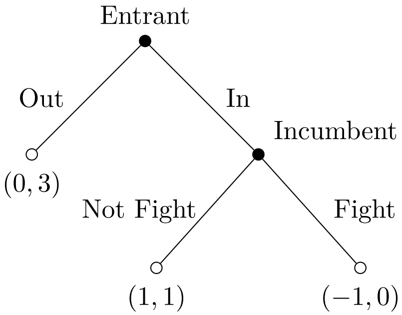
- ([L, L”], L’) is in accordance with the backward induction outcome whereas ([L, R”], L’) is not.
- ([L, R”], L’) seems weird since when Player 1’s second decision node is reached, it does not make sense for her to choose R”.
  - [L, R”] can be an equilibrium strategy because in P1’s view, it doesn’t matter what she chooses in her second decision node because it is never going to be reached as long as P2 chooses L’.

Normal-Form Representation

Remark. Nash Equilibrium may not be a good solution concept for dynamic games: it may involve actions that is not rational when some node is actually reached.

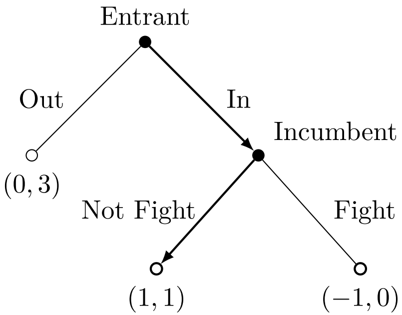
Entry Game

Example 2.A.6. Consider the entry game.



Backward Induction

Backward induction outcome of the game is:



Entrant chooses “In” and Incumbent “Not Fight”.

Normal-Form Representation and Nash Equilibrium

		Incumbent	
		Fight	Not Fight
Entrant	In	$(-1, 0)$	$(\underline{1}, \underline{1})$
	Out	$(\underline{0}, \underline{3})$	$(0, \underline{3})$

Two Nash equilibria: (In, Not Fight) and (Out, Fight).

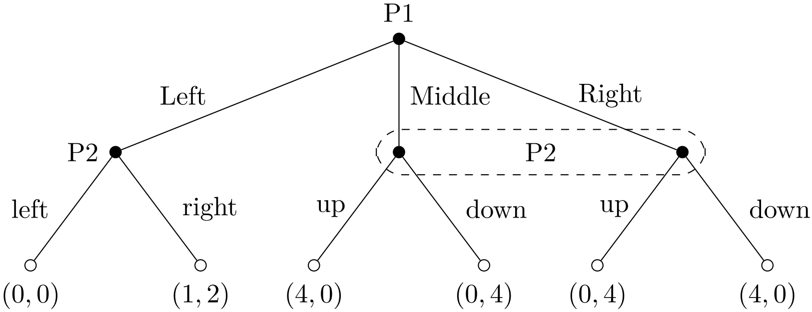
- The first one is in accordance with the backward induction outcome, whereas the second one is not.
- The second equilibrium relies on believing a non-credible threat.

2.B. Dynamic Games of Complete but Imperfect Information

- Previous games are all games of perfect information: the player who is to move knows the full history of the play of the game.
- We now study games of imperfect information.

Dynamic Game of Imperfect Information

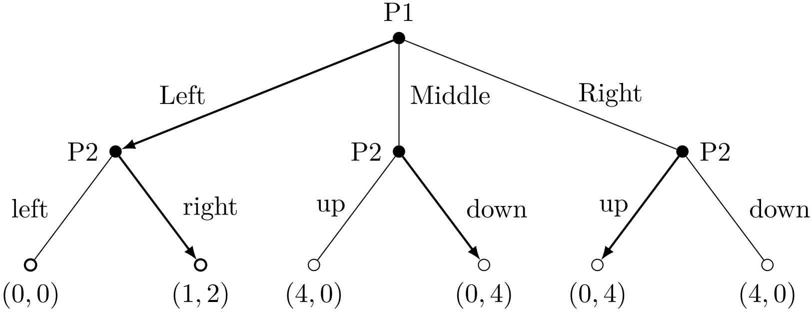
Example 2.B.1.



- The dashed circle is an information set: P2 cannot distinguish "Middle" and "Right".
- Player 1 would mix between "Middle" and "Right".

Dynamic Game of Perfect Information

Example 2.B.2. Consider the game in Example 2.B.1 without information set.



We could solve this game by backward induction.

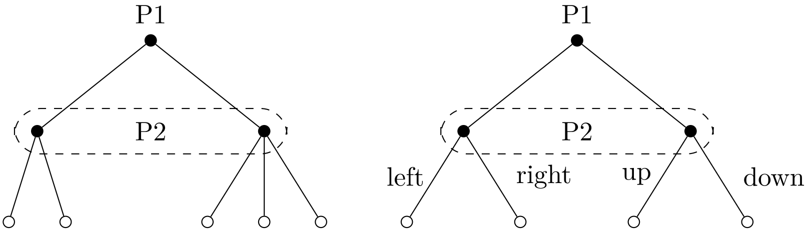
BI outcome is: P1 chooses "Left" and P2 chooses "right".

Information Set

Definition 2.B.1. An information set of Player i is a collection of Player i's decision nodes among which Player i cannot distinguish.

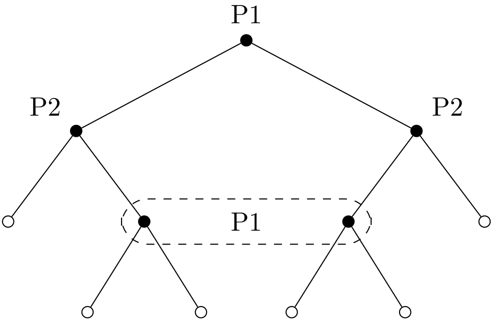
Information Set: Not Allowed

Based on Definition 2.B.1, the following scenarios are NOT allowed.



Distinct Actions

Information Set: Not Allowed



Imperfect Recall

Perfect Recall: players remember their previous actions

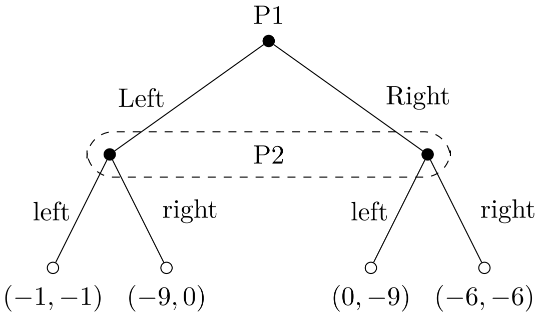
Perfect/Imperfect Information Games

Definition 2.B.2 (Perfect/Imperfect Information Games)**.**
A game is with perfect information if all the information sets in the game tree is singleton.
A game is with imperfect information if it is not a game with perfect information.

Definition 2.B.3. A Pure Strategy for Player i is
a complete plan of actions: it specifies which action Player i will take at each of its information sets.

Game with Imperfect Information: Example

Example 2.B.3.



Example 2.B.3

Question. Do you find this game familiar? What is this game?

Example 2.B.3

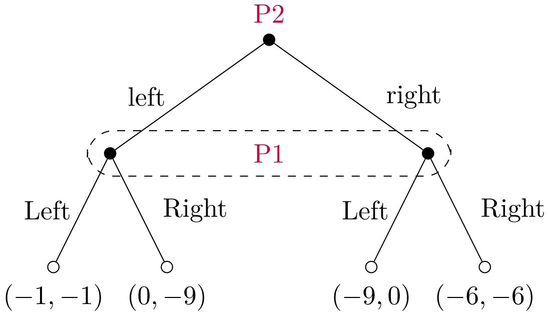
It is the Prisoners' Dilemma game! It is more obvious after we transform the game tree into normal-form representation.

		P2	
		left	right
P1	Left	$(-1, -1)$	$(-9, 0)$
	Right	$(0, -9)$	$(-6, -6)$

Normal-Form Representation

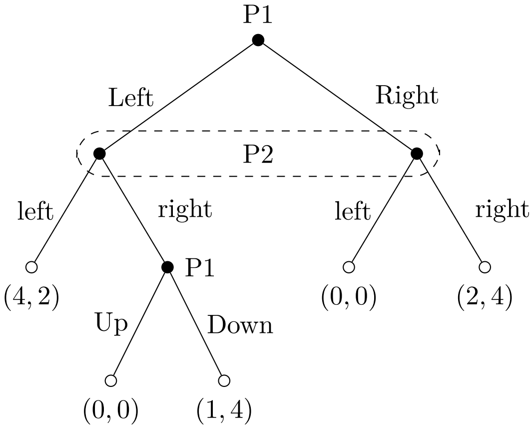
Remark. A normal-form representation could have multiple extensive-form representations.

- For example, PD game above could be equivalently represented in the following extensive form:



2.B.1. Subgame Perfect Equilibrium (SPE)

Example 2.B.4.



Nash Equilibria

- We represent the game in normal form and look for mutual best responses.
- According to Definition 2.B.3,
  - Player 1's strategies: [L, U], [L, D], [R, U], [R, D]
  - Player 2's strategies: l, r

Nash Equilibria

The normal-form representation of the game is

		Player 2	
		l	r
Player 1	[L, U]	$(\underline{4}, \underline{2})$	$(0, 0)$
	[L, D]	$(\underline{4}, 2)$	$(1, \underline{4})$
	[R, U]	$(0, 0)$	$(\underline{2}, \underline{4})$
	[R, D]	$(0, 0)$	$(\underline{2}, \underline{4})$

Three Nash equilibria: ([L, U], l), ([R, U], r) and ([R, D], r).

Subgame Perfect Equilibrium

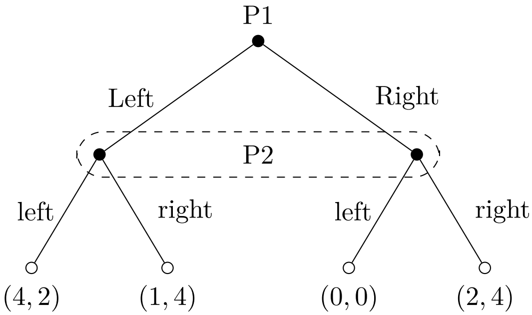
- ([L, U], l) and ([R, U], r) do not seem reasonable: if P1 were to move at her second decision node, she would choose D rather than U.
- To make more reasonable predictions, we should solve the game backwards.
- However, we could not proceed backward induction since in the games of imperfect information, the information set may involve multiple decision nodes.

Subgame Perfect Equilibrium

- Let us try to solve the previous game in Example 2.B.4 and then generalize the idea to other games of imperfect information.
- Solving the game backwards, at P1's second decision node, she would choose D.
- Players obtain payoffs $(1, 4)$.

Subgame Perfect Equilibrium

Substituting the payoffs into the original game, the game becomes



This is a game with simultaneous move.

Subgame Perfect Equilibrium

- The game is more conveniently represented in normal form.

		Player 2	
		l	r
Player 1	L	$(\underline{4}, 2)$	$(1, \underline{4})$
	R	$(0, 0)$	$(\underline{2}, \underline{4})$

- The solution of the game is (R, r).
- When we solve the game backwards, only one equilibrium remains: ([R, D], r).

Subgame Perfect Equilibrium

To summarize what we did to solve the game:

- We work backwards through the extensive form until we encounter a non-singleton information set.
- Then, we skip over it and proceed up the tree until a singleton information set is found.
- What we do is to solve for Subgame Perfect Equilibrium.
- The definitions of Subgame and Subgame Perfect Equilibrium are given below.

Subgame Perfect Equilibrium

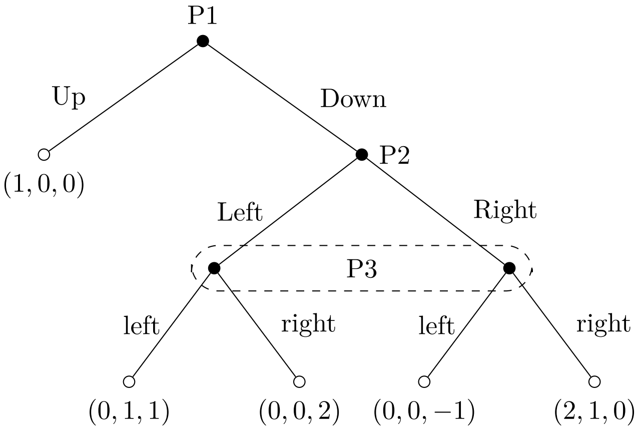
Definition 2.B.4. A subgame in an extensive-form game

1. begins at a decision node n that is a singleton information set,
2. includes all the decision and terminal nodes following n in the game tree, and
3. does not cut any information sets.

Definition 2.B.5. (Selten 1965): A Nash equilibrium is a Subgame Perfect Equilibrium (SPE) if the players' strategies constitute a Nash equilibrium in every subgame.

Subgame Perfect Equilibrium

Example 2.B.5. Consider the following three-player game:



Subgame Perfect Equilibrium

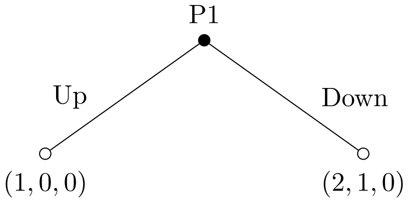
- There is one subgame involving Player 2 and Player 3.
- It is a simultaneous move game and we represent the game in normal form:

		Player 3	
		l	r
Player 2	L	$(\underline{1}, 1)$	$(0, \underline{2})$
	R	$(0, -1)$	$(\underline{1}, \underline{0})$

- The solution of the subgame is (R, r).

Subgame Perfect Equilibrium

- In the original game, when (R, r) is played out in the subgame, the three players obtain payoffs $(2, 1, 0)$.
- Substituting the payoffs into the original game, the original game becomes



- Player 1 would choose D.
- SPE of the game is (D, R, r).

Example 2.B.6. Consider the following matchmaking game.

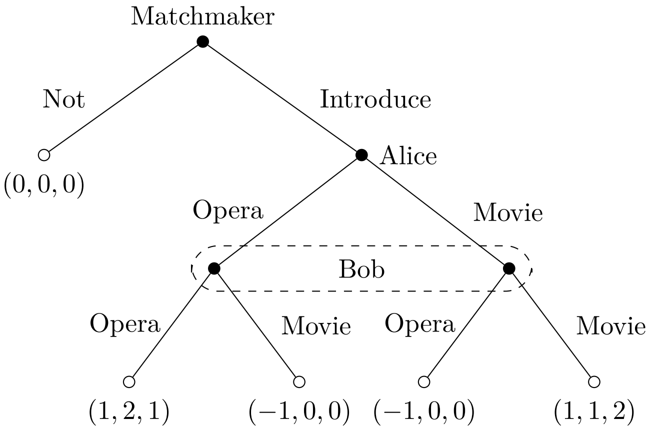
- The first player is the matchmaker who could introduce Alice to Bob.
- If the matchmaker introduces Alice to Bob, the couple would play the battle of the sexes game:

		Bob	
		Opera	Movie
Alice	Opera	$(2, 1)$	$(0, 0)$
	Movie	$(0, 0)$	$(1, 2)$

Example 2.B.6

- As for the matchmaker, if the couple successfully meet, she gets 1; and if the couple fail to meet, she gets -1.
- All three players get 0 if Alice is not introduced to Bob.

Example 2.B.6

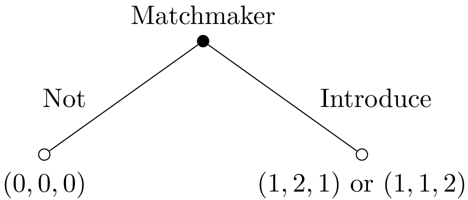

Matchmaker
Not
Introduce
(0, 0, 0)
Alice
Opera
Movie
Bob
Opera
Movie
Opera
Movie
(1, 2, 1)
(−1, 0, 0)
(−1, 0, 0)
(1, 1, 2)

Example 2.B.6

- To obtain SPE, we first look at the subgame where Alice and Bob play the Battle of the Sexes game.
- We already know that there are two pure-strategy Nash equilibria and one mixed-strategy Nash equilibrium in this subgame.

Example 2.B.6: Pure Strategy NE

- Consider pure-strategy NE.
- Substituting the payoffs into the original game,



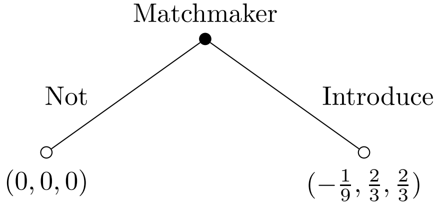
- Matchmaker would choose "Introduce".
- Two SPEs: (Introduce, Opera, Opera) and (Introduce, Movie, Movie).

Example 2.B.6: Mixed Strategy NE

- Consider mixed strategy NE, namely, $\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right)$.
- The expected values for the three players are
  - Matchmaker: $\mathbb{E}_M\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right) = -\frac{1}{9}$
  - Alice: $\mathbb{E}_A\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right) = \frac{2}{3}$
  - Bob: $\mathbb{E}_B\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right) = \frac{2}{3}$

Example 2.B.6: Mixed Strategy NE

- Substituting the expected payoffs into the original game,



- Matchmaker would choose "Not".
- Third SPE: $\Big(\text{Not}, (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\Big)$.

2.B.2. Bank Runs

- Two investors have each deposited D with a bank.
- Bank has invested these deposits in a long-term project:
  - If bank is forced to liquidate its investment before project matures, a total of $2r$ can be recovered, where $D > r > D/2$.
  - If bank allows the investment to reach maturity, project will pay out a total of $2R$, where $R > D$.

Bank Runs

- Two dates at which the investors can make withdrawals: date 1 is before maturity and date 2 is after.
- Assume that there is no discounting.

Bank Runs

At date 1,

- If both investors make withdrawals, game ends.
  - Each investor receives r.
- If only one investor makes a withdrawal, game ends.
  - That investor receives D,
  - the other investor receives $2r - D$.
- If neither investor makes a withdrawal, project matures and investors make withdrawal decisions at date 2.

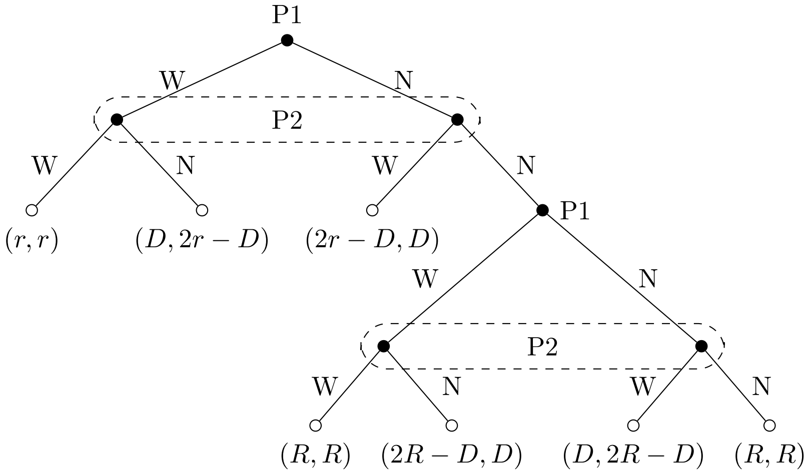
Bank Runs

At date 2 (if the game does not already ends),

- If both investors make withdrawals, game ends.
  - Each investor receives R.
- If only one investor makes a withdrawal, game ends.
  - That investor receives $2R - D$,
  - the other investor receives D.
- If neither investor makes a withdrawal, game ends.
  - Bank returns R to each investor.

Bank Runs: Analysis

The game could be represented in a game tree.



Bank Runs: Analysis

To obtain SPE, we first look at the subgame at date 2.

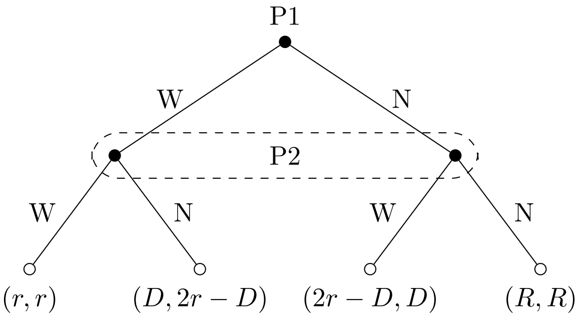
		P2	
		Withdrawal	Not
P1	Withdrawal	$(\underline{R}, \underline{R})$	$(\underline{2R - D}, D)$
	Not	$(D, \underline{2R - D})$	(R, R)

Date 2 Subgame

- NE in this subgame is (Withdrawal, Withdrawal).
- Payoffs for players are (R, R).

Bank Runs: Analysis

Substituting payoffs of date 2 subgame into original game,



Bank Runs: Analysis

The game is more conveniently represented in matrix form,

		P2	
		Withdrawal	Not
P1	Withdrawal	$(\underline{r}, \underline{r})$	$(D, 2r - D)$
	Not	$(2r - D, D)$	$(\underline{R}, \underline{R})$

Two pure strategy Nash equilibria:

1. both investors withdraw, leading to payoffs of (r, r);
2. both investors do not withdraw, leading to payoffs of (R, R).

Bank Runs: Analysis

The original game has two subgame perfect outcomes:

1. both investors withdraw at date 1, leading to payoffs of (r, r);
2. both investors do not withdraw at date 1 and withdrawal at date 2, leading to payoffs of (R, R).

First outcome can be interpreted as bank runs.

Bank Runs: Analysis

Remark. There is also a mixed strategy Nash equilibrium for the first-period game: $\left((\frac{R-D}{R-r}, \frac{D-r}{R-r}), (\frac{R-D}{R-r}, \frac{D-r}{R-r}) \right)$.
Thus, there is another SPE: $\left(\left[(\frac{R-D}{R-r}, \frac{D-r}{R-r}), R \right], \left[(\frac{R-D}{R-r}, \frac{D-r}{R-r}), R \right] \right)$.

Remark. Diamond and Dybvig (1983) provide a richer model of bank runs.

2.B.3. Wars of Attrition

- Consider two-period version of Wars of Attrition.
- Two players choose to "Fight (F)" or "Quit (Q)" in each period.
- Game ends as soon as at least one player chooses Q.
- Assume no discounting.

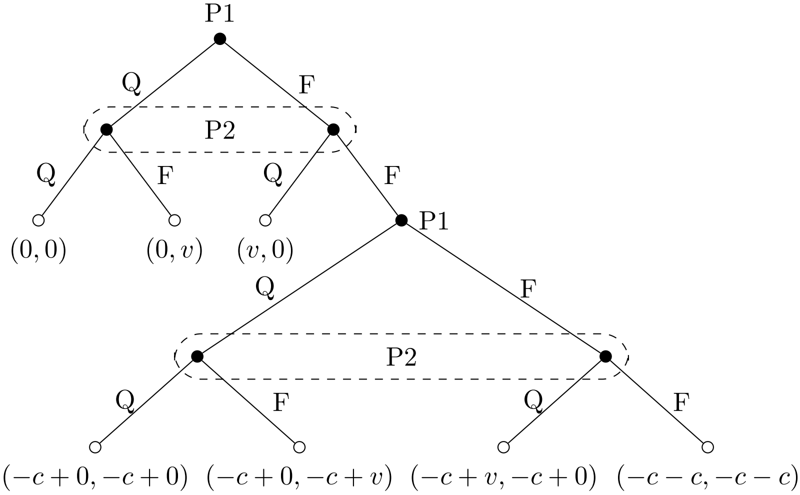
Wars of Attrition

Payoffs to players are as follows:

- If one of the players quits first,
  - the player who does not quit win a prize v and
  - the player who quits gets 0;
- If both players quit at once, both get 0;
- At each period in which both players choose F, each player pay a cost $c(< v)$.

Wars of Attrition: Analysis

The game could be represented in a game tree.



Wars of Attrition: Result

Five SPEs

- Two SPEs with pure strategies in each period: ([F,F], [Q,Q]) and ([Q,Q], [F,F])
- One SPE with mixed strategies in each period:

$$\left(\left[(p_Q^* = \frac{c}{v+c}, p_F^* = \frac{v}{v+c}), (p_Q^*, p_F^*)\right], \left[(p_Q^*, p_F^*), (p_Q^*, p_F^*)\right]\right)$$

- Two other SPEs: $\left(\left[F, (p_Q^*, p_F^*)\right], \left[Q, (p_Q^*, p_F^*)\right]\right)$ and $\left(\left[Q, (p_Q^*, p_F^*)\right], \left[F, (p_Q^*, p_F^*)\right]\right)$.

2.C. Repeated Games

- Repeated games are games that consist of repetitions of some base game (called a stage game).
- We will first focus on repeated games with stage games being static games of complete information where players move simultaneously.
- In Example 2.C.3 and Section 2.C.4, we study games where players move sequentially in stage game.
- When studying repeated games, we are particularly interested in the issue of cooperation.

2.C.1. Finitely Repeated Games

We will first learn finitely repeated games, where the stage game is repeated for a fixed number of periods.

Definition 2.C.1. Given a stage game G, let $G(T)$ denote the finitely repeated game in which G is played T times with

- the outcomes of all preceding plays observed before the next play begins and
- no discounting: the payoffs of $G(T)$ are simply the sum of the payoffs from the T stage games.

Two-stage PD Game

Example 2.C.1. Suppose PD game played twice.

		P2	
		Cooperate (C)	Defect (D)
P1	Cooperate (C)	$(2,2)$	$(-1,3)$
	Defect (D)	$(3,-1)$	$(0,0)$

- The outcome of the first play is observed before the second play begins.
- Assume no discounting.

Two-stage PD game

Question. Could (C, C) be sustained?

Two-stage PD game: SPE

- Second stage: (D, D)
- First stage:

P2

P1	Cooperate (C)	Defect (D)
Cooperate (C)	$(2+0, 2+0)$	$(-1+0, 3+0)$
Defect (D)	$(3+0, -1+0)$	$(0+0, 0+0)$

  Nash equilibrium for the first-stage game is (D, D).

- SPE outcome of the two-stage PD game is that both players play D for the two stages.

More stages

Question. What if the Prisoners' Dilemma is played 3 times, 4 times, or more generally, N times?

More stages

Remark. It seems that when the relationship has a known end, cooperation is not sustainable. But this is NOT true.

An Example

Example 2.C.2. Suppose the following game played twice.

		P2 L	P2 M	P2 R
P1	L	$(4, 4)$	$(0, \underline{5})$	$(0, 0)$
P1	M	$(\underline{5}, 0)$	$(\underline{1}, \underline{1})$	$(0, 0)$
P1	R	$(0, 0)$	$(0, 0)$	$(\underline{3}, \underline{3})$

- The outcome of the first play is observed before the second play begins.
- Assume no discounting

Example 2.C.2

Question. Could the good outcome (L, L) be sustained?

Example 2.C.2

- NE of the one-shot game: (M, M) and (R, R).
- Since (L, L) is not a Nash equilibrium, we could not sustain (L, L) in the second stage.
- However, (L, L) could be sustained in the first stage.
- Consider the following strategy:
  - In the first stage, play L, and then
  - In the second stage,
    * Play R if (L, L) is played in the first stage;
    * Play M otherwise.

Finitely Repeated Games

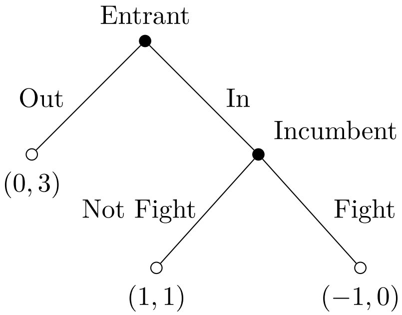
Remark. In ongoing relationships, the promise of future rewards and the threat of future punishments may sometimes provide incentives for good behavior today.

For this to work, the stage game needs to have more than one Nash equilibrium.

Remark. The play of different equilibria in the second stage following different first-stage outcomes may seem unreasonable. Here, punishing the deviator involves the punishment of the punisher. There may be a problem of renegotiation.

Finitely Repeated Games: Entry Game

Example 2.C.3. Let us reconsider the following entry game:



Entry Game

- Now suppose that incumbent is active in N markets.
- In each market, incumbent faces a different entrant.
- Entrants decides whether to enter market sequentially.
- Each of the subsequent entrants observes the choices by the previous entrants and by the incumbent.

Entry Game

- This game could be solved by backward induction.
- BI outcome is:
  - Every entrant chooses "In";
  - Incumbent "Not Fight".

Entry Game

- Next, suppose that there is 1% chance that the incumbent is crazy and enjoys fighting.
- For example, the crazy incumbent gets a payoff of 5 when choosing "Fight".

Entry Game

If there is only one market, then

- entrant would still choose “In” and
- incumbent would choose “Not Fight” if it is not crazy.

However, the outcome would be different if there are more markets.

- It is not an equilibrium that sane incumbent always chooses "Not Fight".
  - Sane incumbent could deter entry by fighting: act as if it is crazy.

Remark 2.1. It is also not an equilibrium that the sane incumbent always "Fight".

2.C.2. Infinitely Repeated Games

- Next, we will turn to infinitely repeated games, where the stage game is repeated infinitely.
- Analysis for the infinitely repeated games would be rather different, since there is no last stage in the infinitely repeated games.
- Result is different too: in the infinitely repeated games, even if the stage game has a unique Nash equilibrium, as in the PD game, there may be subgame-perfect outcomes in which no stage's outcome is a NE of the stage game.

Infinitely Repeated Games

Example 2.C.4. Suppose PD Game played infinitely.

		P2	
		Cooperate (C)	Defect (D)
P1	Cooperate (C)	$(2, 2)$	$(-1, 3)$
	Defect (D)	$(3, -1)$	$(0, 0)$

- The outcomes of all previous plays are observed before the next play begins.
- The discount factor is $\delta \in (0, 1)$.

Grim-Trigger Strategy

- play C in the first period; and
- from the second period onwards
  - play C if no player has played D in the past;
  - play D otherwise.

A player would cooperates until someone fails to cooperates and then a switch to defection forever is triggered.

Question. Does both players playing Grim-Trigger Strategy constitute a SPE?

Grim-Trigger Strategy

- To check whether a strategy profile is a SPE, we need to check whether there exists profitable deviations in every subgame.
- Given that both players follow Grim-Trigger strategy, all subgames belong to one of the two types:
    1. it is the first period or D has never been observed in the past;
    2. D has been observed in the past.

Grim-Trigger Strategy

- Players have no incentive to deviate in second type of subgames.
- For first type of subgames:
  - Follow the strategy:

$$V^C = 2 + \delta \cdot 2 + \delta^2 \cdot 2 + ... = \frac{2}{1-\delta}$$

  - Deviate to D: $V^D = 3 + \delta \cdot 0 + \delta^2 \cdot 0 + ... = 3$.
  - No deviation requires $V^C \geq V^D \implies \delta \geq \frac{1}{3}$.

Grim-Trigger Strategy

For first type of subgames, we have only checked a special type of deviation: Player 1 deviates once and then revert to the equilibrium strategy.

Question. How about other types of deviations? For example, playing D for one period, followed by C and then playing D forever.

One-shot Deviation Principle

Definition. A strategy profile does not have profitable one-shot deviations if no player can increase his payoff in any subgame through a one-shot deviation: a deviation from the strategy profile only in the first period of the subgame.

Proposition (The one-shot deviation principle)**.** *A strategy profile is subgame perfect if and only if there are no profitable one-shot deivations.*

Always Cooperate Strategy

Consider the following strategy which also prescribe (C, C) forever as the equilibrium outcome:

- cooperate in the first period and
- continue cooperating forever no matter what the other player does.

Question. Does both players playing Alway Cooperate Strategy constitute a SPE?

Always Cooperate Strategy

- Follow the strategy, outcome is (C, C) forever.

$$V^C = \frac{2}{1-\delta} = 2 + \delta\frac{2}{1-\delta};$$

- Deviate to D (one-shot), outcome is (D, C) and then (C, C) forever.

$$V^D = 3 + \delta\frac{2}{1-\delta}.$$

- Since $V^D > V^C$ for all $\delta \in (0, 1)$, it is never a SPE.

One-period Punishment Strategy

- play C in the first period; and
- from the second period onwards
  - play C if either (C, C) or (D, D) was played in the last period;
  - play D if either (C, D) or (D, C) was played in the last period.

Question. Does both players playing the One-period Punishment Strategy constitute a SPE?

One-period Punishment Strategy

Given that both players follow the One-period Punishment strategy, all subgames belong to one of the two types:

1. it is the first period or either (C, C) or (D, D) was played in the last period;
2. either (C, D) or (D, C) was played in the last period.

One-period Punishment Strategy

For first type of subgames,

- Follow the strategy, outcome is (C, C) forever.

$$V^C = \frac{2}{1-\delta}.$$

- Deviate to D (one-shot), outcome is $(D, C), (D, D)$ and then (C, C) forever.

$$V^D = 3 + \delta \cdot 0 + \delta^2 \frac{2}{1-\delta}.$$

- No deviation requires $V^C \geq V^D \implies \delta \geq \frac{1}{2}$.

One-period Punishment Strategy

For second type of subgames,

- Follow the strategy, outcome is (D, D) and then (C, C) forever.

$$V^1 = 0 + \delta \cdot \frac{2}{1-\delta} = 2 + \delta^2 \frac{2}{1-\delta}.$$

- Deviate to C (one-shot), outcome is $(C, D), (D, D)$ and then (C, C) forever.

$$V^2 = (-1) + \delta \cdot 0 + \delta^2 \frac{2}{1-\delta} = -1 + \delta^2 \frac{2}{1-\delta}.$$

- Since $V^1 > V^2$ for all δ, no incentive to deviate.

One-period Punishment Strategy

Putting together the no-deviation conditions for the two types of subgames, we need $\delta \geq \frac{1}{2}$.

Note that

- Cutoff for One-period Punishment strategy ($\frac{1}{2}$) is higher than cutoff for Grim-Trigger strategy ($\frac{1}{3}$).
- For a shorter punishment to work, the players need to care more about the future.

Tit-for-Tat Strategy

- play C in the first period; and
- from the second period onwards
  - play C if the opponent plays C in the last period;
  - play D if the opponent plays D in the last period.

Tit-for-tat is a very effective strategy and it is the winning program of the Alexlrod Tournament.

Question. Does both players playing Tit-for-Tat Strategy constitute a SPE?

Tit-for-Tat Strategy

Given that both players follow the Tit-for-Tat strategy, all subgames belong to one of the four types:

1. it is the first period or (C, C) was played in the last period;
2. (D, D) was played in the last period;
3. (C, D) was played in the last period;
4. (D, C) was played in the last period.

Tit-for-Tat Strategy

For first type of subgames,

- Follow the strategy, outcome is (C, C) forever.

$$V^C = \frac{2}{1-\delta}.$$

- Deviate to D (one-shot), outcome is $(D, C), (C, D)$ repeated forever.

$$V^D = 3 + \delta \cdot (-1) + \delta^2 \cdot 3 + \delta^3 \cdot (-1) + ... = \frac{3-\delta}{1-\delta^2}.$$

- No deviation requires $V^C \geq V^D \implies \delta \geq \frac{1}{3}$.

Tit-for-Tat Strategy

For second type of subgames,

- Follow the strategy, outcome is (D, D) forever.

$$V^1 = 0.$$

- Deviate to C (one-shot), outcome is $(C, D), (D, C)$ repeated forever.

$$V^2 = (-1) + \delta \cdot (3) + \delta^2 \cdot (-1) + \delta^3 \cdot 3 + ... = \frac{-1 + 3\delta}{1 - \delta^2}.$$

- No deviation requires $V^1 \geq V^2 \implies \delta \leq \frac{1}{3}$.

Tit-for-Tat Strategy

For third type of subgames,

- Follow the strategy, outcome is $(D,C),(C,D)$ repeated forever.
$$V^3 = \frac{3-\delta}{1-\delta^2}.$$
- Deviates to C (one-shot), outcome is (C,C) forever.
$$V^4 = \frac{2}{1-\delta}.$$
- No deviation requires $V^3 \geq V^4 \implies \delta \leq \frac{1}{3}$.

Tit-for-Tat Strategy

For fourth type of subgames,

- Follow the strategy, outcome is $(C, D), (D, C)$ repeated forever.
$$V^5 = \frac{-1 + 3\delta}{1 - \delta^2}.$$
- Deviate to D (one-shot), outcome is (D, D) forever.
$$V^6 = 0.$$
- No deviation requires $V^5 \geq V^6 \implies \delta \geq \frac{1}{3}$.

Tit-for-Tat Strategy

Putting together the no-deviation conditions for all four types of subgames, we need $\delta = \frac{1}{3}$.

2.C.3. Collusion between Cournot Duopolists

- Quantities (of a homogeneous product) produced by firms 1 and 2: q_1 and q_2
- Market-clearing price when aggregate quantity is $Q = q_1 + q_2$: $P(Q) = a - Q$.
- Total cost to a firm with q_i: $C_i(q_i) = cq_i$, where $c < a$.
- Firms choose quantities simultaneously.

Collusion between Cournot Duopolists

Previous results:

- Nash equilibrium of Cournot game: $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$.
- Each firm's profit: $\pi_i(q_i^*, q_j^*) = \frac{(a-c)^2}{9}$.
- Monopoly quantity: $q^m = \frac{a-c}{2} < q_1^* + q_2^*$.
- Monopoly profit: $\pi(q^m) = \frac{(a-c)^2}{4} > \pi_i(q_i^*, q_j^*) + \pi_j(q_i^*, q_j^*)$.

Collusion between Cournot Duopolists

Now

- We view the game as stage game, played infinitely.
- Discount factor is δ.

We are interested in whether the repeated interactions could help the firms achieve $q_1 = q_2 = \frac{q_m}{2}$.

Collusion between Cournot Duopolists

Question. What is the required value of δ for which it is a SPE for both firms to play the following trigger strategy?

- produce $\frac{q_m}{2}$ in the first period; and
- from the second period onwards,
  - produce $\frac{q_m}{2}$ if both firms have produced $\frac{q_m}{2}$ in each of the previous periods;
  - produce Cournot quantity $(q_c = \frac{a-c}{3})$ otherwise.

Collusion between Cournot Duopolists

Given that both firms follow trigger strategy, all subgames belong to one of the two types:

1. it is the first period or both firms have produced $\frac{q_m}{2}$ in each of the previous period;
2. at least one firm has produced some quantity other than $\frac{q_m}{2}$ in any of the previous period.

Collusion between Cournot Duopolists

For first type of subgames,

- Follows the strategy, outcome is $(\frac{q_m}{2}, \frac{q_m}{2})$ forever.

$$V^C = \frac{1}{1-\delta} \cdot \frac{\pi(q^m)}{2} = \frac{(a-c)^2}{8(1-\delta)}.$$

Collusion between Cournot Duopolists

For first type of subgames (continue),

- Deviate to q_d (best deviation quantity):

$$\max_{q_d}(a - q_d - \frac{q_m}{2} - c)q_d \implies q_d = \frac{3(a-c)}{8}.$$

  – One-shot profit: $\pi^d = (a - q_d - \frac{q_m}{2} - c)q_d = \frac{9(a-c)^2}{64}$.

  – Afterwards, $\pi^c = \frac{(a-c)^2}{9}$ in every period.

  – Present-discounted payoff:

$$V^D = \pi^d + \delta\frac{\pi^c}{1-\delta} = \Big[\frac{9}{64} + \frac{\delta}{9(1-\delta)}\Big](a-c)^2.$$

Collusion between Cournot Duopolists

For first type of subgames (continue),

- No deviation requires

$$V^C \geq V^D \implies \delta \geq \frac{9}{17}.$$

Collusion between Cournot Duopolists

For the second type of subgames,

- Follow the strategy, outcome is (q^c, q^c) forever.

$$V^1 = \frac{\pi^c}{1-\delta} = \pi^c + \delta \frac{\pi^c}{1-\delta}.$$

- Deviation to some quantity $\hat{q}$ (one-shot),
  - Deviation profit: $\hat{\pi} \ < \pi^c$
  - Afterwards, $\pi^c = \frac{(a-c)^2}{9}$ in every period.
  - Present-discounted payoff: $V^2 = \hat{\pi} + \delta \frac{\pi^c}{1-\delta}$.
- Since $V^1 > V^2$ for all δ, no incentive to deviate.

Collusion between Cournot Duopolists

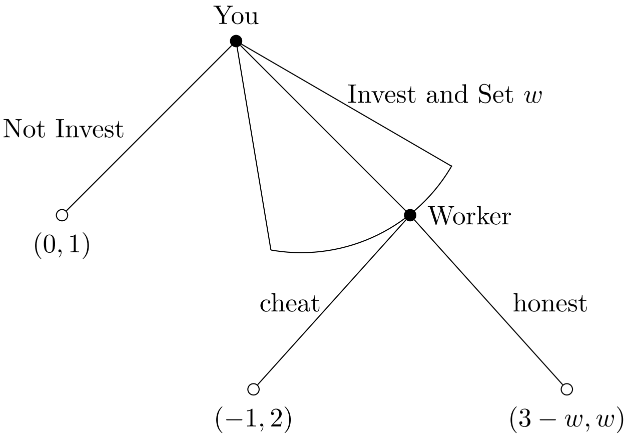
Putting together the no-deviation conditions for the two types of subgames, we need $\delta \geq \frac{9}{17}$.

2.C.4. Repeated Moral Hazard: Outsource

Stage Game

- Outsource to foreign country with cheaper labor
- normal wage in foreign country 1
- If outsource, investment 1; return from investment
  - 4 if worker works
  - 0 otherwise
- need to determine wage w
- Worker may cheat: take investment of 1 and sell on the market, then go away and just work in normal job

Repeated Moral Hazard: Outsource



Repeated Moral Hazard: Outsource

By backward induction, the worker would be honest if and only if $w \geq 2$, and you would invest and set $w = 2$.

Repeated Interaction

Now

- Repeatedly invest in foreign country if investment works well.
- Discount factor is $\delta \in (0, 1)$.

Question. How would you set the wage?

For the worker to work, it requires

$$\frac{w}{1-\delta} \geq 2 + \delta \cdot \frac{1}{1-\delta} \implies w \geq 2 - \delta.$$

Therefore, it is optimal to set $w = 2 - \delta$.

Remark 2.2. Note that this wage level is in-between 1 and 2, i.e., the normal wage in the foreign country and the required wage in the one-shot game.