Chapter 5. Maximum Value Functions

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Introduction

- In Chapter 4, we learned that

$$\mathrm{d}v = \lambda\, \mathrm{d}c.$$

- Apart from the parameter c, there are other parameters in the objective funtion $F(x)$ as well as the constraint functions $G(x)$.

- In this chapter, we will learn how these parameters would affect the maximum attainable value in general.

5.A. Parameters in the Objective Function

- We will begin with equality constraints.
- And consider first the case where the parameters affect the maximand alone.
- We have already seen such an example in Exercise 2.3 *Production and Cost Minimization*.

Parameters in the Objective Function

Exercise 2.3: Consider a producer who rents machines K at r per year and hires labor L at wage w per year to produce output Q, where $Q = \sqrt{K} + \sqrt{L}$.

Suppose he wishes to produce a fixed quantity Q at minimum cost.

Find his factor demand function, that is, the optimal amount of K and L.

Parameters in the Objective Function

The maximization problem for this example is:

$$\max_{K,L} -Kr - Lw \tag{MP1}$$
$$\text{s.t. } \sqrt{K} + \sqrt{L} = Q.$$

The parameters r and w only appear in the objective function, i.e., the maximand.

Parameters in the Objective Function

- General representation for such maximization problem is

$$v = \max_x F(x, \theta)$$
$$\text{s.t. } G(x) = c,$$

where θ is a vector of parameters.

- The Lagrangian is

$$\mathcal{L}(x, \lambda, \theta) = F(x, \theta) + \lambda \left[c - G(x) \right].$$

Parameters in the Objective Function

- The first-order necessary conditions are

$$\mathcal{L}_x(x^*, \lambda, \theta) = F_x(x^*, \theta) - \lambda G_x(x^*) = 0,$$

$$\text{and } \mathcal{L}_\lambda(x^*, \lambda, \theta) = c - G(x^*) = 0.$$

- Now suppose that θ changes to $\theta + \mathrm{d}\theta$.

- Correspondingly, the optimum x^* changes to $x^* + \mathrm{d}x^*$, and the maximum value v changes to $v + \mathrm{d}v$.

Parameters in the Objective Function

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*, \theta + \mathrm{d}\theta) - F(x^*, \theta)$$

$$\underbrace{=}_{\text{Taylor approximation}} F_x(x^*, \theta)\mathrm{d}x^* + F_\theta(x^*, \theta)\mathrm{d}\theta \underbrace{=}_{\text{First-order condition}} \lambda G_x(x^*)\mathrm{d}x^* + F_\theta(x^*, \theta)\mathrm{d}\theta$$

$$\underbrace{=}_{\text{Taylor approximation}} \lambda \left[G(x^* + \mathrm{d}x^*) - G(x^*)\right] + F_\theta(x^*, \theta)\mathrm{d}\theta \underbrace{=}_{G(x^*+\mathrm{d}x^*)=G(x^*)=c} F_\theta(x^*, \theta)\mathrm{d}\theta.$$

Therefore, we get

$$\mathrm{d}v = F_\theta(x^*, \theta)\mathrm{d}\theta. \tag{5.1}$$

Parameters in the Objective Function

- The result tells us that to find the first-order change in the maximum value of the objective function in response to changes in parameters that do not affect the constraints, we need not worry about the simultaneous change in the optimum choice x^* itself.

- All we have to do is to calculate the partial effect of the parameter change, and evaluate the expression at the initial optimum choice.

Parameters in the Objective Function

- An equivalent way of writing

$$\mathrm{d}v = F_\theta(x^*, \theta)\mathrm{d}\theta \tag{5.1}$$

  is $\mathrm{d}v/\mathrm{d}\theta = F_\theta(x^*, \theta)$.

- We could check $\mathrm{d}v/\mathrm{d}\theta = F_\theta(x^*, \theta)$ in our cost minimization example.

Parameters in the Objective Function

- Solving the problem

$$\max_{K,L} -Kr - Lw \tag{MP1}$$
$$\text{s.t. } \sqrt{K} + \sqrt{L} = Q,$$

we obtain[1] $K^* = \left(\frac{wQ}{r+w}\right)^2$ and $L^* = \left(\frac{rQ}{r+w}\right)^2$.

- In the problem, $F(K, L, r, w) = -Kr - Lw$ and $G(K, L) = \sqrt{K} + \sqrt{L}$.

[1] Please solve the problem by yourself. If you cannot solve the problem, please review Chapter 2.

Parameters in the Objective Function

- Plugging the optimal K^* and L^* into the objective function, we obtain

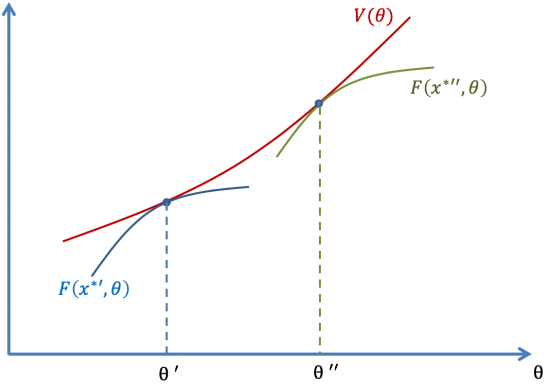
$$v(w, r, Q) = F(K^*, L^*) = -\frac{wrQ^2}{r + w}.$$

- Here, we only check $\mathrm{d}v(w, r, Q)/\mathrm{d}w$ and $F_w(K^*, L^*, r, w)$:

$$\mathrm{d}v(w, r, Q)/\mathrm{d}w = -\left(\frac{rQ}{r + w}\right)^2 = F_w(K^*, L^*, r, w).$$

5.B. The Envelope Theorem

The algebra of the previous section is illustrated geometrically:



The Envelope Theorem

- For particular value of θ, say θ', the optimal choice is $x^{*\prime}$.

- That is , $x^{*\prime}$ solves the below maximization problem:

$$\max_x F(x, \theta')$$

$$\text{s.t. } G(x) = c.$$

The Envelope Theorem

- The curve $F(x^{*\prime}, \theta)$ represents the value of the objective function with respect to θ, where x is held fixed at $x^{*\prime}$.

- The curve $V(\theta)$ represents the optimum value function with respect to θ, where x is allowed to vary optimally as θ varies.

The Envelope Theorem

Formally, the function $V(\theta)$ is defined by

$$V(\theta) = \max_x \{F(x, \theta) | G(x) = c\}, \tag{5.2}$$

which is read as "$V(\theta)$ is the maximum over x of $F(x, \theta)$ subject to $G(x) = c$."

The Envelope Theorem

- Next, write the optimum choice x^* as a function of θ: $x^* = X(\theta)$.

- Pratically, this could be done by solving the maximization problem for fixed θ.

- For instance, we have $x^{*\prime} = X(\theta')$.

- Then, we could rewrite $V(\theta)$ as follows:

$$V(\theta) = F(x^*, \theta) = F(X(\theta), \theta).$$

The Envelope Theorem

- The two functions $V(\theta)$ and $F(x^{*\prime}, \theta)$ coincide at θ', because $x^{*\prime}$ happens to be the optimal choice there.

- Algebraically, $F(x^{*\prime}, \theta') = F(X(\theta'), \theta') = V(\theta')$.

- For the other values of θ, unless $x^{*\prime}$ remains the optimal choice, the curve $V(\theta)$, which is the optimum value, is higher than that of $F(x^{*\prime}, \theta)$:

$$F(x^{*\prime}, \theta) = F(X(\theta'), \theta) \underbrace{\leq}_{X(\theta) \text{ is the optimal choice given } \theta} F(X(\theta), \theta) = V(\theta) \text{ for } \theta \neq \theta'.$$

- Therefore, the two curves should be **mutually tangential** at θ'.

- This is what

$$\mathrm{d}v = F_\theta(x^*, \theta)\mathrm{d}\theta. \tag{5.1}$$

(5.1) expresses.

The Envelope Theorem

- Similarly, we could draw the graph of $F(x^{*\prime\prime}, \theta)$, where $x^{*\prime\prime}$ is the optimal choice at θ''.

- The curve $F(x^{*\prime\prime}, \theta)$ would touch the curve of the optimal value function $V(\theta)$ at θ''.

- We could draw a whole family of curves of $F(x, \theta)$ for a whole range of fixed values x, each x being the optimal for some θ.

The Envelope Theorem

- No member of this family of curves could ever cross above the graph of $V(\theta)$, and each would be tangential to the optimal value function at that value of θ where its x happens to be the optimal choice.

- In other words, the optimal value function is the upper envelope of the family of the value functions, in each of which the choice variables are held fixed.

- (5.1) is often referred to as the *Envelope Theorem*.

The Envelope Theorem: Cost Minimization

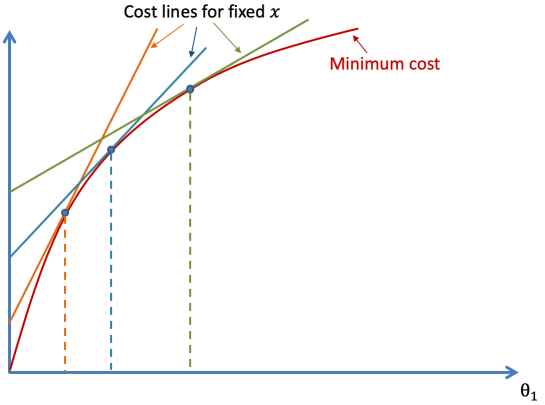
We apply *Envelope Theorem* to cost minimization problem:

$$\max_{x}(-\theta x)$$
$$\text{s.t. } G(x) = c$$

where θ is a vector of input prices.

The Envelope Theorem: Cost Minimization

Figure below shows the the minimum cost curve and the cost lines for fixed x when the first input price θ_1 varies.

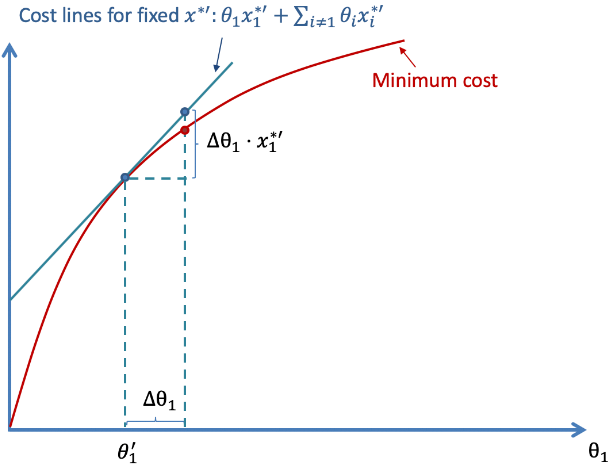


The Envelope Theorem: Cost Minimization

- The cost lines are linear since when x is held fixed, the cost θx is a linear function of θ_1.

- The minimized cost as a function of θ is the lower envelope (not the upper envelope, since this is a minimization problem) of all these straight lines.

The Envelope Theorem: Cost Minimization (Intuition)



The Envelope Theorem: Cost Minimization (Intuition)

- If θ_1' increases by $\Delta\theta_1$, assuming that $x^{*\prime}$ is held fixed, the cost increases with θ_1 by the amount $\Delta\theta_1 \cdot x_1^{*\prime}$ (cost lines for fixed x are linear). However, the producer can lower the cost by adjusting x optimally.

- If θ_1' decreases by $\Delta\theta_1$, assuming that $x^{*\prime}$ is held fixed, the cost decreases with θ_1 by the amount $\Delta\theta_1 \cdot x_1^{*\prime}$. However, the producer can lower the cost by adjusting x optimally (minimum cost curve is the lower envelope of the cost lines).

The Envelope Theorem: Cost Minimization

- The figures also suggest a curvature property.
- The first figure shows each $F(x^*, \theta)$ as a concave curve and $V(\theta)$ as a convex curve.
- The second figure shows a linear cost function for any fixed input choice but the lower envelope is concave.
- In general, the upper envelope must be more convex than any member of the family of which it is the envelope.
- This property will be studied in detail in Chapter 8.

5.C. Parameters Affecting All Functions

- Now suppose G as well as F involves θ.

- General representation for such maximization problem is

$$v = \max_x F(x, \theta)$$

$$\text{s.t. } G(x, \theta) = c,$$

where θ is a vector of parameters.

Parameters Affecting All Functions

- The Lagrangian is

$$\mathcal{L}(x, \lambda, \theta) = F(x, \theta) + \lambda \left[c - G(x, \theta) \right].$$

- The first-order necessary conditions are

$$\mathcal{L}_x(x^*, \lambda, \theta) = F_x(x^*, \theta) - \lambda G_x(x^*, \theta) = 0,$$

$$\text{and } \mathcal{L}_\lambda(x^*, \lambda, \theta) = c - G(x^*, \theta) = 0.$$

Parameters Affecting All Functions

The calculation for a change in θ to $\theta + \mathrm{d}\theta$ proceeds as in Section 5.A, except that the partial derivative of G with respect to x is no longer zero:

$$\begin{aligned} & G(x^* + \mathrm{d}x^*, \theta + \mathrm{d}\theta) - G(x^*, \theta) \\ & \underbrace{=}_{\text{Taylor approximation}} G_x(x^*, \theta)\mathrm{d}x^* + G_\theta(x^*, \theta)\mathrm{d}\theta \underbrace{=}_{G(x^*+\mathrm{d}x^*,\theta+\mathrm{d}\theta)=G(x^*,\theta)=c} 0 \\ \Longrightarrow\ & G_x(x^*, \theta)\mathrm{d}x^* = -G_\theta(x^*, \theta)\mathrm{d}\theta. \end{aligned} \tag{5.3}$$

Parameters Affecting All Functions

Using this, and the previous analysis, we have

$$\mathrm{d}v \underbrace{=}_{\text{by definition}} (v + \mathrm{d}v) - v \underbrace{=}_{\text{by definition}} F(x^* + \mathrm{d}x^*, \theta + \mathrm{d}\theta) - F(x^*, \theta)$$

$$\underbrace{=}_{\text{Taylor approximation}} F_x(x^*, \theta)\mathrm{d}x^* + F_\theta(x^*, \theta)\mathrm{d}\theta$$

$$\underbrace{=}_{\text{First-order condition}} \lambda G_x(x^*, \theta)\mathrm{d}x^* + F_\theta(x^*, \theta)\mathrm{d}\theta$$

$$\underbrace{=}_{\text{Equation (5.3)}} -\lambda G_\theta(x^*, \theta)\mathrm{d}\theta + F_\theta(x^*, \theta)\mathrm{d}\theta = \mathcal{L}_\theta(x^*, \lambda, \theta)\mathrm{d}\theta.$$

Parameters Affecting All Functions

Therefore, we get

$$\mathrm{d}v = \mathcal{L}_\theta(x^*, \lambda, \theta)\mathrm{d}\theta. \tag{5.4}$$

Parameters Affecting All Functions

The difference between (5.4) and (5.1):

$$\mathrm{d}v = \mathcal{L}_\theta(x^*, \lambda, \theta)\mathrm{d}\theta = F_\theta(x^*, \theta)\mathrm{d}\theta - \lambda G_\theta(x^*, \theta)\mathrm{d}\theta; \qquad (5.4)$$

$$\mathrm{d}v = F_\theta(x^*, \theta)\mathrm{d}\theta. \qquad (5.1)$$

Parameters Affecting All Functions

Intuitive explanation of the difference between (5.4) and (5.1):

- When θ affects the constraints, a change $\mathrm{d}\theta$ has the direct effect of increasing the value of G by $G_\theta(x^*, \theta)\mathrm{d}\theta$.
- This acts exactly like an equal reduction in c.
- The interpretation of the Lagrange multiplier tells us that the equivalent reduction in c reduces v by $\lambda G_\theta(x^*, \theta)\mathrm{d}\theta$.
- This is just the additional term in (5.4) when compared to (5.1).

Parameters Affecting All Functions

- In Chapter 4, we have learned a similar comparative static analysis with respect to changes in the parameters c.

- The more general formulation in this chapter can subsume the earlier case.

Parameters Affecting All Functions

- To see this explicitly, define a larger vector of parameters $\widehat{\theta}$, which includes θ and c as subvectors, and write the constraint as $\widehat{G}(x, \widehat{\theta}) = G(x, \theta) - c = 0$.

- The maximization problem is now

$$v = \max_x F(x, \theta)$$

$$\text{s.t. } \widehat{G}(x, \widehat{\theta}) = 0,$$

where $\widehat{\theta}$ is a vector of parameters.

Parameters Affecting All Functions

- The Lagrangian is

$$\widehat{\mathcal{L}}(x, \lambda, \widehat{\theta}) = F(x, \theta) - \lambda \widehat{G}(x, \widehat{\theta}).$$

- (5.4) becomes

$$\begin{aligned}
\mathrm{d}v &= \widehat{\mathcal{L}}_{\widehat{\theta}}(x^*, \lambda, \widehat{\theta})\mathrm{d}\widehat{\theta} = F_{\widehat{\theta}}(x^*, \theta)\mathrm{d}\widehat{\theta} - \lambda \widehat{G}_{\widehat{\theta}}(x, \widehat{\theta})\mathrm{d}\widehat{\theta} \\
&= F_\theta(x^*, \theta)\mathrm{d}\theta - \lambda \left[G_\theta(x, \theta)\mathrm{d}\theta - I_m \mathrm{d}c\right] \\
&= F_\theta(x^*, \theta)\mathrm{d}\theta - \lambda G_\theta(x, \theta)\mathrm{d}\theta + \lambda \mathrm{d}c \\
&= \mathcal{L}_\theta(x^*, \lambda, \theta)\mathrm{d}\theta + \lambda \, \mathrm{d}c.
\end{aligned}$$

Parameters Affecting All Functions

The result

$$\mathrm{d}v = \mathcal{L}_\theta(x^*, \lambda, \theta)\mathrm{d}\theta + \lambda\,\mathrm{d}c$$

includes the previous cases

$$\mathrm{d}v = \mathcal{L}_\theta(x^*, \lambda, \theta)\mathrm{d}\theta, \tag{5.4}$$

$$\text{and } \mathrm{d}v = \lambda\,\mathrm{d}c. \tag{4.3}$$

5.D. Some Choice Variables Fixed

- In this section, we examine the effect of a change in parameters to the optimum value function when some components of x are kept fixed.

- Our main focus is to compare such effect with the case where all components of x could be freely adjusted.

- An economic application is the comparison between the short-run and the long-run outcomes.

Some Choice Variables Fixed

- To tackle this problem, we partition the vector x into two subvectors y and z.

- In the long-run, both y and z are choice variables and could be adjusted freely, while in the short-run, z is held fixed and only y is allowed to vary.

Some Choice Variables Fixed

- Subsuming c into θ, the long-run problem is

$$\max_{y,z} F(y,z,\theta) \tag{MP_LR}$$

$$\text{s.t. } G(y,z,\theta) = 0.$$

- The short-run problem is[2]

$$\max_{y} F(y,z,\theta) \tag{MP_SR}$$

$$\text{s.t. } G(y,z,\theta) = 0.$$

[2] For the short-run problem to be meaningful, the number of constraints must be less than the dimension of y.

Some Choice Variables Fixed

- Write the long-run optimal choices and the resulting value as functions of θ:

$$y = Y(\theta), \quad z = Z(\theta), \quad v = V(\theta). \tag{5.5}$$

- In the short-run, z should be treated as just another parameter along with θ, and the optimal choice y and the resulting value v are functions of (z, θ):

$$y = Y(z, \theta), \quad v = V(z, \theta). \tag{5.6}$$

Some Choice Variables Fixed

- We are now ready to compare the long-run and short-run optimum values.

- The long-run problem (MP_LR) has more choice variables compared to the short-run problem (MP_SR).

- Therefore,

$$V(\theta) \geq V(z, \theta) \text{ for all } (z, \theta).$$

Some Choice Variables Fixed

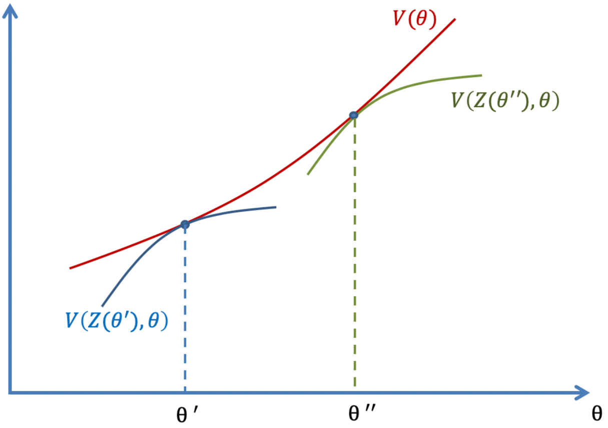
- And the two values $V(\theta)$ and $V(z, \theta)$ coincide when z is just at the long-run optimal level $Z(\theta)$.

- Because when z is at the optimal level $Z(\theta)$, being able to adjust it (the long-run case) or not (the short-run case) will not make a difference.

- Therefore, $V(\theta)$ is the *upper envelope* of the family of value functions $V(z, \theta)$, in each of which z is held fixed.

Some Choice Variables Fixed

We could draw a graph to show the intuition.



Some Choice Variables Fixed

If the functions are differentiable, we would have

$$V'(\theta) = V_\theta(Z(\theta), \theta), \tag{5.7}$$

where the right-hand side is the partial derivative of the short-run optimum value function $V(Z(\theta), \theta)$ taken holding the first argument z fixed, but evaluated at the point $z = Z(\theta)$.

Some Choice Variables Fixed

- Please keep in mind that V functions may not be differentiable even when F and G are.

- The problem may arise when we have inequality and nonnegativity constraints on choice variables.

- At some point, there may be a regime change, one constraint from binding to slack or vice versa, and the graph of maximum value function may have a kink.

- Figure 4.2 provides such an example.

5.E. Examples

Example 5.1: Short-Run and Long-Run Costs

This example is used to illustrate Envelope Theorem. Consider a producer who rents machines K at r per year and hires labor L at wage w per year to produce output Q, where[3]

$$Q = (KL)^{1/\alpha}.$$

Suppose he wishes to produce a fixed quantity Q at minimum cost.

[3] Returns to scale are constant if $\alpha = 2$, increasing if $\alpha < 2$, and decreasing if $\alpha > 2$.

Example 5.1: Short-Run and Long-Run Costs (continued)

Assume that K is **fixed** in the short run; whereas L could be freely adjusted.

Question 1: Calculate the long-run and short-run cost functions.

Question 2: Show that Equation (5.7) holds.

Example 5.1: Solution

See Lecture Notes.

Example 5.2: Consumer Demand.

Part I: Indirect Utility Function

Consider the consumer choice problem:

$$\max_x U(x)$$

$$\text{s.t. } px = I.$$

The resulting maximum utility is a function $V(p, I)$, called the *indirect utility function*,[4] and the utility-maximizing quantities x comprise the demand function $D(p, I)$.

[4] $U(x)$ is called the *direct utility function*.

Part I: Indirect Utility Function (continued)

Show that

$$D(p, I) = -V_p(p, I)/V_I(p, I). \tag{5.8}$$

Example 5.2 (Part I): Solution

See Lecture Notes.

Part II: Expenditure Function

Consider the expenditure minimization problem:

$$\min_x px$$
$$\text{s.t. } U(x) \geq u,$$

where u is the target utility level. The resulting minimized expenditure is a function $E(p, u)$, called the *expenditure function.* Cost-minimizing commodity choices for a given utility level are called *Hicksian compensated demand function* $C(p, u)$.

Part II: Expenditure Function (continued)

Show that

$$C(p,u) = E_p(p,u). \tag{5.9}$$

Example 5.2 (Part II): Solution

See Lecture Notes.