Chapter 9. Uncertainty

Xiaoxiao Hu

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Introduction

- This chapter concerns choice under uncertainty.
- It is an important topic in economics, and is of great practical interest.
- In real-life, almost every decision needs to be made under uncertainty.
- For example, when you make up your decision to learn the current course, you will only have some estimates about its usefulness. In the end, it may be more or less useful than you thought.

Introduction

- In this chapter, we will sketch a systematic way of making such decisions.
- In terms of mathematics, there will be nothing new.
- You will see more economic concepts and intuitions.

Introduction

- Let's get familarized with the problems of choice under uncertainty.
- As illustrated in the course-choosing example, uncertainty means that you do not anticipate a sure outcome.
- To make our discussion more concrete, we will need to introduce some concepts.

Introduction

Consider the following simple example:

Example 9.1. *Suppose that you have access to the following lottery: the lottery pays* \$100 *with probability* 1/4, *and pays nothing with the remaining probability. The question is, do you wish to pay* \$25 *for such a lottery?*

Introduction

- This is a problem of choice under uncertainty.
- The outcome is uncertain: you will either get \$100 or \$0.
- There are two important elements in the problem:

  (i) *The outcomes.* In the example, the outcomes refer to the state *paying \$100*, and the state *paying \$0*.

  (ii) *The probabilities associated with the outcomes.* In the example, 1/4 is the probability associated with the state *paying \$100*, and 3/4 is the probability associated with the state *paying \$0*.

Introduction

- Note that the probabilities are *objective* here, but they could be referred to as *subjective probabilities* in certain applications.
- The probabilities in a well-defined problem should be non-negative, and add up to 1.
- In this example, we could write the consumer's utility from the lottery could be written as follows: $U(\$100, \$0; 1/4, 3/4)$.

Introduction

- More generally, denote the possible outcomes by $Y_1, Y_2, ..., Y_m$.
- The probability associated with the outcomes by $p_1, p_2, .., p_m$.
- The utility could be written as

$$U(Y_1, Y_2, ..., Y_m; p_1, p_2, .., p_m).$$

- Next, we will introduce a widely-used method to express the utility in a way that more analysis could be performed.

9.A. Expected Utility

- Since probabilities are involved, it is somewhat natural to make use of mathematical expectation, or probability weighted average.

- For instance, in Example 9.1, we could express the utility as follows:

$$U(\$100, \$0; 1/4, 3/4) = 1/4U(\$100) + 3/4U(\$0).$$

- This is called the *von Neumann-Morgenstern utility function*, and is of *expected utility* form.

Expected Utility

- For a general utility function, the expected utility form is expressed as follows:

$$U(Y_1, Y_2, ..., Y_m; p_1, p_2, .., p_m)$$
$$= p_1 U(Y_1) + p_2 U(Y_2) + ... + p_m U(Y_m) = \sum_{i=1}^{m} p_i U(Y_i). \quad (9.1)$$

- This formulation is useful in its simplicity and its ability to capture economically interesting aspects of behavior.
- We will discuss some implications of this representation.[1]

[1] The applicability of expected utility is out of scope of this course, and thus will not be discussed.

Money Lotteries and Risk-aversion

- Now consider Y_i's as money amounts.
- U is an increasing function.
- The definition of *Risk-aversion* is intuitive.
- In Example 9.1, the lottery gives in expectation

$$\$100 \times 1/4 + \$0 \times 3/4 = \$25.$$

- A risk-averse individual dislikes risk, and thus prefers sure outcome of \$25 to the lottery that gives on average \$25.

Risk-aversion

- In general, for two distinct outcomes Y_1 and Y_2 with (any) positive probability p and $(1-p)$ respectively, a decision maker is risk-averse if

$$U(pY_1 + (1-p)Y_2) > pU(Y_1) + (1-p)U(Y_2).$$

- This is, U is (strictly) concave.

Risk-aversion

- More generally, we could include more than 2 states:
- A decision maker is risk-averse if

$$U(\sum_{i=1}^{m} p_i Y_i) > \sum_{i=1}^{m} p_i U(Y_i). \tag{9.2}$$

- If U is twice differentiable, $U'' < 0$ corresponds to risk-aversion.

Insurance

- Let's now bring back the decision variable x, which affect some or all of the outcomes and probabilities.
- Suppose $Y_1 < Y_2$, which means the first state entails some loss relative to the second.
- Y_1 occurs with probability p; Y_2 with probability $(1 - p)$.
- A risk-averse decision maker would want to purchase insurance.

Insurance

- Consider an insurance policy that requires an advance payment of x (paid independent of the state realization), and gives X if state 1 is realized.
- Suppose the insurance policy is actuarially fair: $pX = x$.
- Actuarially fairness is an outcome of a perfectly competitive insurance industry.
  - Insurance company is risk-free.
  - Zero-profit condition implies $pX = x$.

Insurance

- The decision maker's objective function is

$$\max_{x \geq 0} pU(Y_1 - x + X) + (1-p)U(Y_2 - x)$$

$$\iff \max_{x \geq 0} pU(Y_1 - x + x/p) + (1-p)U(Y_2 - x)$$

- The first-order condition for x gives:

$$pU'(Y_1 - x + x/p)(1/p - 1) - (1-p)U'(Y_2 - x) \leq 0$$

$$\text{and } x \geq 0, \text{ with complementary slackness.} \tag{9.3}$$

Insurance

- When $x = 0$,

$$
\begin{aligned}
&pU'(Y_1)(1/p - 1) - (1-p)U'(Y_2) \\
=&(1-p)[U'(Y_1) - U'(Y_2)] \underbrace{>}_{U''<0} 0,
\end{aligned}
$$

  contradicting with (9.3).

- Therefore, we must have $x > 0$ at the optimum.

Insurance

- Since $x > 0$, by (9.3),

$$pU'(Y_1 - x + x/p)(1/p - 1) - (1-p)U'(Y_2 - x) = 0$$

$$\Longrightarrow U'(Y_1 - x + x/p) = U'(Y_2 - x) \tag{9.4}$$

- When $U'' < 0$, the objective function is concave in x and the first-order condition is also sufficient.
- The first-order condition (9.4) implies

$$Y_1 - x + x/p = Y_2 - x.$$

Insurance

$Y_1 - x + x/p = Y_2 - x$ is the **full-insurance** result:

a risk-averse decision maker would buy the actuarially fair insurance to the point where the outcomes in different states are equal.

Care

- Consider again the previous problem faced by the decision maker, but leave aside insurance for the moment.
- Now suppose that the probability of the bad outcome (state 1) can be reduced by incurring an expense z in advance.
- Specifically, you could think of it as exercising more care by yourself to reduce the probability of being ill.
- In terms of modelling, we make the probability p a function of z, and the function is decreasing.

Care

- The objective function is

$$\max_{z \geq 0} \phi(z) \equiv \max_{z} p(z)U(Y_1 - z) + (1 - p(z))U(Y_2 - z)$$

- Then, derivative of $\phi(z)$ gives

$$\phi'(z) = \underbrace{\underbrace{-p'(z)}_{\text{reduction of prob.}} \underbrace{[U(Y_2 - z) - U(Y_1 - z)]}_{\text{utility diff.}}}_{\text{marginal benefit}}$$

$$- \underbrace{\{p(z)U'(Y_1 - z) + (1 - p(z))U'(Y_2 - z)\}}_{\text{marginal cost}}$$

Care

The optimal solution is defined by the first-order condition

$$\phi'(z^*) = 0.$$

Moral Hazard

- Suppose both insurance and care variables are available.
- The interaction between the insurance company and the decision maker could be formulated as the game in the next page.

Moral Hazard

1. Insurance company sells the acturially fair insurance at constant rate $p(\bar{z})$ per \$1 coverage. That is, if the individual purchases x shares of the insurance, the insurance company would pay the individual $X = \frac{x}{p(\bar{z})}$ when the bad outcome (state 1) occurs.
2. The decision maker chooses how much to purchase x.
3. The decision maker chooses the care parameter z.
4. Outcome realized and the decision maker gets paid from the insurance company if the realized state is 1.

Moral Hazard

- In equilibrium, insurance company holds correct belief on the optimal level of care, that is, $\bar{z} = z^*$ where z^* is the decision maker's actual choice.
- The objective function for the decision maker is

$$\max_{x \geq 0, z \geq 0} \phi(x, z) \equiv \max_{x \geq 0, z \geq 0} p(z)U(Y_1 - z - x + x/p(z^*)) + (1 - p(z))U(Y_2 - z - x)$$

Moral Hazard

- Partial derivative of $\phi(x,z)$ with respect to x gives

$$\begin{aligned}\phi_x(x,z) =& p(z)U'(Y_1 - z - x + x/p(z^*))(1/p(z^*) - 1) \\ & - (1-p(z))U'(Y_2 - z - x)\end{aligned}$$

- Next, we show that the optimal $x^* > 0$ must hold.

$$\begin{aligned}\phi_x(0,z^*) =& p(z^*)U'(Y_1 - z^*)(1/p(z^*) - 1) \\ & - (1-p(z^*))U'(Y_2 - z^*) \\ =& (1-p(z^*))[U'(Y_1 - z^*) - U'(Y_2 - z^*)] \underbrace{>}_{U''<0} 0.\end{aligned}$$

Moral Hazard

- First-order condition on x gives

$$\begin{aligned} &\phi_x(x^*, z^*) = 0 \\ \Longrightarrow\ & U'(Y_1 - z^* - x^* + x^*/p(z^*)) = U'(Y_2 - z^* - x^*) \\ \Longrightarrow\ & Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* \qquad (9.5) \end{aligned}$$

- The optimal choices of x^* and z^* must satisfy (9.5) above.
- Let $Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* = Y_0$.

Moral Hazard

- Partial derivatvie of $\phi_z(x, z)$ with respect to z gives

$$\phi_z(x,z) = \underbrace{-p'(z)\left[U(Y_2 - z - x) - U(Y_1 - z - x + x/p(z^*))\right]}_{\text{marginal benefit}}$$

$$- \underbrace{\left[p(z)U'(Y_1 - z - x + x/p(z^*)) + (1 - p(z))U'(Y_2 - z - x)\right]}_{\text{marginal cost}}$$

- Evaluated at the optimal level (x^*, z^*), we have

$$\phi_z(x^*, z^*) = -p'(z^*) \cdot 0 - U'(Y_0) = -U'(Y_0) < 0.$$

  - Recall $Y_1 - z^* - x^* + x^*/p(z^*) = Y_2 - z^* - x^* = Y_0$

Moral Hazard

- Optimum of care occurs at the corner $z^* = 0$.
- This is known as *"moral hazard"*: the availability of full insurance destroys the incentive to exercise costly care.

9.B. One Safe and One Risky Asset

- Next, we will study portfolio choice.
- Since we will be working with multiple states, for convenience, we introduce a continuous representation.
- The index i is replaced by a continuous random variable r with support $[\underline{r}, \overline{r}]$.

One Safe and One Risky Asset

- The expected utility form in (9.1) is modified by replacing probabilities with densities, and sums with integrals:

$$\mathbb{E}[U(Y)] = \int_{\underline{r}}^{\overline{r}} U(Y(r))f(r)\mathrm{d}r.$$

- The interpretation of risk-aversion parallels (9.2): A decision-maker is risk averse if

$$\begin{aligned} & U(\mathbb{E}(Y)) > \mathbb{E}[U(Y)] \\ \Longleftrightarrow\ & U(\int_{\underline{r}}^{\overline{r}} Y(r)f(r)\mathrm{d}r) > \int_{\underline{r}}^{\overline{r}} U(Y(r))f(r)\mathrm{d}r. \end{aligned}$$

One Safe and One Risky Asset

A *risk-averse* investor has initial wealth W_0, and has the following two investment options:

(i) A risky asset: investing x gives $x(1+r)$, where r is a random variable with density $f(r)$ and support $[\underline{r}, \overline{r}]$.

- Assume $\mathbb{E}[r] > 0$ and $\underline{r} < 0$, so that the risky asset does not always generate a positive return, but on average the return is positive.

(ii) A safe asset: investing x gives x.

One Safe and One Risky Asset

- Investing $x \in [0, W_0]$ in the risky asset and the rest in the safe asset generates final wealth

$$W = x(1+r) + (W_0 - x) = W_0 + xr.$$

- The investor's objective is to maximize the expected final wealth:

$$\max_{x \in [0, W_0]} \mathbb{E}[(U(W))] \equiv \max_{x} \int_{\underline{r}}^{\overline{r}} U(W_0 + xr) f(r) \mathrm{d}r.$$

- Let $\phi(x) = \mathbb{E}[(U(W))]$.

One Safe and One Risky Asset

- Derivative of $\phi(x)$ gives

$$\phi'(x) = \int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r.$$

- Note when $x = 0$:

$$\phi'(0) = \int_{\underline{r}}^{\overline{r}} rU'(W_0)f(r)\mathrm{d}r = U'(W_0)\int_{\underline{r}}^{\overline{r}} rf(r)\mathrm{d}r$$

$$= U'(W_0)\mathbb{E}[r] > 0.$$

- So, $x = 0$ is not optimal.

- Therefore, the risk-averse investor will buy at least some of the actuarially good investment.

One Safe and One Risky Asset

- Typically, the investor will hold some of each asset.
- The first-order condition is

$$\phi'(x) = \int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r = 0. \tag{9.6}$$

- If there is an $x < W_0$ satisfying this, then strict concavity of U guarantees that it is the global maximum:

$$\phi''(x) = \int_{\underline{r}}^{\overline{r}} r^2 \underbrace{U''(W_0 + xr)}_{U''(W)<0 \text{ for all } W} f(r)\mathrm{d}r < 0. \tag{9.7}$$

One Safe and One Risky Asset: Comparative Statics

- Next, assuming an interior maximum, we consider the comparative statics of x with respect to W_0.
- That is, whether the investor would invest more or less in the risky asset when he becomes wealthier.
- Now, we recognize W_0 as a parameter in ϕ, i.e., $\phi(x, W_0)$.

One Safe and One Risky Asset: Comparative Statics

- First-order condition for an interior solution is

$$\phi_x(x, W_0) = 0.$$

- Total differentiation gives

$$\phi_{xx}(x, W_0)\mathrm{d}x + \phi_{xw}(x, W_0)\mathrm{d}W_0 = 0$$

$$\Longrightarrow \mathrm{d}x/\mathrm{d}W_0 = -\phi_{xw}(x, W_0)/\phi_{xx}(x, W_0).$$

- By second-order sufficient condition, $\phi_{xx}(x, W_0) < 0$.
- Then, the sign of $\mathrm{d}x/\mathrm{d}W_0$ is the same as:

$$\phi_{xw}(x, W_0) = \int_{\underline{r}}^{\overline{r}} rU''(W_0 + xr)f(r)\mathrm{d}r.$$

One Safe and One Risky Asset: Comparative Statics

- To gain more insight, we introduce a measure of risk-aversion, called *absolute risk-aversion*, and denoted by $A(W)$:
$$A(W) = -U''(W)/U'(W). \tag{9.8}$$
- Experimental and empirical evidence is consistent with $A(W)$ being decreasing in W.[2]
- If $A(W)$ is decreasing in W, then we would be able to show $\phi_{xw}(x, W_0) \int_{\underline{r}}^{\overline{r}} rU''(W_0 + xr)f(r)\mathrm{d}r > 0$.

[2] Friend, I., & Blume, M. E. (1975). The Demand for Risky Assets. *American Economic Review*, 65(5), 900-922.

One Safe and One Risky Asset: Comparative Statics

(i) For $r < 0$,

$$- U''(W_0 + xr)/U'(W_0 + xr) > -U''(W_0)/U'(W_0) = A(W_0)$$

$$\Longrightarrow rU''(W_0 + xr) > -rA(W_0)U'(W_0 + xr).$$

(ii) For $r > 0$,

$$- U''(W_0 + xr)/U'(W_0 + xr) < -U''(W_0)/U'(W_0) = A(W_0)$$

$$\Longrightarrow rU''(W_0 + xr) > -rA(W_0)U'(W_0 + xr).$$

- So, $rU''(W_0 + xr) > -rA(W_0)U'(W_0 + xr)$ for all $r \neq 0$.

One Safe and One Risky Asset: Comparative Statics

- Recall, $rU''(W_0 + xr) > -rA(W_0)U'(W_0 + xr)\ \forall r \neq 0$.

$$
\begin{aligned}
\phi_{xw}(x, W_0) &= \int_{\underline{r}}^{\overline{r}} rU''(W_0 + xr)f(r)\mathrm{d}r \\
&> \int_{\underline{r}}^{\overline{r}} -rA(W_0)U'(W_0 + xr)f(r)\mathrm{d}r \\
&= -A(W_0)\underbrace{\int_{\underline{r}}^{\overline{r}} rU'(W_0 + xr)f(r)\mathrm{d}r}_{=0 \text{ by Equation (9.6)}} = 0.
\end{aligned}
$$

- Thus, $\phi_{xw}(x, W_0) > 0$, which implies $\mathrm{d}x/\mathrm{d}W_0 > 0$.
- The investor would invest more in the risky asset when he becomes wealthier.

9.C. Examples

Example 9.1: Managerial Incentives

- A risk-neutral owner (she) has to hire a risk-neutral manager (he) to run a project.
- If the project succeeds, it will produce value V.
- Success probability depends on manager's effort.
- The project succeeds with

$$\begin{cases} \text{probability } p & \text{if manager exerts effort;} \\ \text{probability } q(< p) & \text{if no effort.} \end{cases}$$

- Effort cost is e.

Example 9.1: Managerial Incentives

- To make it worthwhile to exert effort, suppose that exerting effort generates higher total surplus:

$$pV - e > qV \implies (p - q)V > e. \tag{9.9}$$

- Assume that the manager's outside job pays him w.

Example 9.1: Managerial Incentives

What is optimal compensation scheme when

(i) The owner can observe manager's effort?

(ii) The owner cannot observe manager's effort?

Example 9.1: Solution (Case I: Observable Effort)

- The owner could compensate effort directly.
- Since inducing effort generates higher total surplus, the owner would be willing to do so as long as it is not too expensive to attract the manager.

Example 9.1: Solution (Case I: Observable Effort)

- Let payment to manager be W, paid when effort exerted.
- Manager is willing to work for owner and exert effort if

$$W - e \geq w \implies W \geq e + w.$$

- After paying the least amount W to the manager, the owner gets

$$pV - W = pV - e - w.$$

- The owner thus is willing to hire the manager if

$$w < pV - e. \tag{9.10}$$

Example 9.1: Solution (Case I: Observable Effort)

- (9.10) $w < pV - e$ is an assumption that we would make throughout the analysis, since otherwise, the manager would not be hired.
- Under the assumption, the owner could offer $w + e$ to the manager, and demand effort in return.
- The owner would get $pV - e - w$ and the manager gets $w + e - e = w$, the same as what he would get from the outside job.

Example 9.1: Solution (Case II: Unovservable Effort)

In this case, compensating effort directly would not work.

- Suppose that compensation is still based on (now unoberservable) effort, then manager could lie about effort: manager could promise to exert effort, but shirk instead.
- Because of

  1. unobservability of effort and
  2. probalistic nature of the outcome,

  owner would not catch such a lie.

Example 9.1: Solution (Case II: Unovservable Effort)

- Therefore, the best thing the owner could do is to base his payment scheme on the thing that he could observe, i.e., the outcome.
- Suppose that the owner pays the manager x if the project succeeds, and y if it fails.

Example 9.1: Solution (Case II: Unovservable Effort)

Two constraints need to be satisfied:

(i) Given such a payment scheme, the manager would exert effort if

$$px + (1-p)y - e \geq qx + (1-q)y$$

$$\Longrightarrow (p-q)(x-y) \geq e. \qquad \text{(IC)}$$

This is called *incentive compatibility constraint.*

(ii) The manager will agree to work if

$$px + (1-p)y - e \geq w \implies y + p(x-y) \geq w + e. \qquad \text{(IR)}$$

This is called *participation constraint* or *individual rationality constraint.*

Example 9.1: Solution (Case II: Unovservable Effort)

- Thus, the owner's problem is to maximize her profit subject to constraints (IC) and (IR).

$$\max_{x,y} pV - [px + (1-p)y] \equiv \max_{x,y} pV - y - p(x-y)$$

$$\text{s.t.} \quad (p-q)(x-y) \geq e; \qquad \text{(IC)}$$

$$y + p(x-y) \geq w + e. \qquad \text{(IR)}$$

- (IR) must be binding.

Example 9.1: Solution (Case II: Unovservable Effort)

- Solving the problem, we get

$$y^* \leq w - eq/(p-q) \text{ and } x^* = (w + e + (1-p)y^*)/p$$
$$(\geq w + e(1-q)/(p-q)).$$

- One interpretation is that the manager's compensation consists of the basic salary w, plus a reward for success and minus a penalty for failure.

- The owner's expected profit is

$$\pi = pV - y^* - p(x^* - y^*) = pV - w - e,$$

same as when she could observe manager's effort.

Example 9.1: Solution (Case II: Unovservable Effort)

- However, one potential problem here is that

$$y^*_{\max} = w - eq/(p - q)$$

  is not guaranteed to be positive.

- $y^*_{\max} < 0$ means that the payment scheme would involve a fine under failure, which is always not feasible.

Example 9.1: Solution (Case II: Unovservable Effort)

- Suppose $y^*_{\max} = w - eq/(p-q) < 0$ and $y \geq 0$ is required.
- The solution is to go as far as possible, i.e., $y = 0$.
- (IC) and (IR) becomes

$$(p-q)(x-0) \geq e \implies x \geq \frac{e}{p-q}; \qquad \text{(IC')}$$

$$0 + p(x-0) \geq w + e \implies x \geq \frac{w+e}{p}. \qquad \text{(IR')}$$

- The problem becomes:

$$\max_x pV - 0 - p(x-0) \equiv \max_x pV - px$$

s.t. (IC') & (IR')

- The owner wants x to be as small as possible.

Example 9.1: Solution (Case II: Unovservable Effort)

- From $y^*_{\max} = w - eq/(p-q) < 0$, we have

$$(w+e)/p < e/(p-q). \tag{9.11}$$

- The minimum x is $x^{**} = e/(p-q)$.
- The profit becomes $\pi = pV - px^{**} = pV - pe/(p-q)$.
- By (9.11), this profit level is lower compared to the previous cases, $\pi = pV - w - e$.
- By (9.9), this profit level is still positive.

Example 9.2: Cost-Plus Contracts

- This example is motivated by the cost-plus contract.
- Government expenditures are often made on such a cost-plus basis, that is, the government reimburses the supplier's cost plus a normal profit.
- In this example, we are concerned with the appropriated amount of reimbersement when the government does not observe the supplier's cost.

Example 9.2: Cost-Plus Contracts

- Suppose true average cost of production can take just two values: c_1 and c_2 (normal profit included), with $c_1 < c_2$.
- We call the supplier with cost c_i Type-i supplier.
- Supplier is privately informed of its own type.
- Before contracting, government's estimate of probability of supplier being Type-1 is β_1, Type-2 is $\beta_2 = 1 - \beta_1$.
- The problem here is that the low cost supplier would pretend to be of high cost and get more reimbersement from the government.

Example 9.2: Cost-Plus Contracts

- To mitigate the problem, the government could purchase different amounts and offer distinct payments, depending on the cost declared by the supplier.
- More specifically, the government would offer the following contract, if the supplier claims to have cost c_i for $i = 1, 2$, the government purchases q_i units and pays R_i.
- In game theory, the use of different contracts to separate supplier types is called "screening".

Example 9.2: Cost-Plus Contracts

- The governments gets benefit $B(q)$ from quantity q.
- $B(q)$ is strictly increasing, strictly concave in q, and

$$B'(0) > c_2 \tag{A}$$

so that the government would demand positive quantities from either type if cost can be observed.

> What is the optimal menu of contracts (q_1, R_1) and (q_2, R_2) when cost is unobservable?

Example 9.2: Solution (Observable Cost)

- Before analyzing the problem with unobservable cost, we will first analyze the problem with observable cost.
- When cost is observable, the government could design the contract based on the supplier's true type.
- The government problem facing a supplier with Type-i is:

$$\max_{q_i, R_i} B(q_i) - R_i$$

$$\text{s.t. } R_i - c_i q_i \geq 0. \tag{IR}$$

$$q_i \geq 0, R_i \geq 0.$$

Example 9.2: Solution (Observable Cost)

- Before solving the problem, we make two observations:

  1. (IR) must be binding. Otherwise, the government could demand a higher R_i without violating the constraints, and the objective function becomes larger.

  2. $R_i \geq 0$ is implied from (IR) and $q_i \geq 0$.

- The government's problem is reduced to

$$\max_{q_i} B(q_i) - c_i q_i$$

$$\text{s.t. } q_i \geq 0.$$

Example 9.2: Solution (Observable Cost)

Next, we solve the problem using the Lagrange's theorem.

- Form the Lagrangian:

$$\mathcal{L}(q_i) = B(q_i) - c_i q_i.$$

- The first-order necessary condition is

$$\partial \mathcal{L}/\partial q_i = B'(q_i) - c_i \leq 0 \text{ and } q_i \geq 0$$

$$\text{with complementary slackness.}$$

Example 9.2: Solution (Observable Cost)

- By Assumption (A) and the strict concavity of $B(q)$, we have $q_i > 0$ and

$$B'(q_i) = c_i. \tag{9.12}$$

- The optimal q_i is given by (9.12).
- The optimal R_i is given by binding (IR)

$$R_i - c_i q_i = 0. \tag{IR}$$

Example 9.2: Solution (Unobservable Cost)

- When cost is unobservable, the government cannot offer contract based on supplier's type.
- As indicated in the question, the government would offer two contracts and let the supplier choose.

Example 9.2: Solution (Unobservable Cost)

- To make the suppliers willing to choose the contract designed for them, we must ensure that Type-1 supplier prefers its contract to Type-2's, and similarly, Type-2 supplier prefers its contract to Type-1's:

$$R_1 - c_1 q_1 \geq R_2 - c_1 q_2; \tag{IC_1}$$

$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1. \tag{IC_2}$$

- These are the *incentive compatibility constraints*.

Example 9.2: Solution (Unobservable Cost)

- Moreover, we need to ensure that the suppliers would want to participate:

$$R_1 - c_1 q_1 \geq 0; \qquad (IR_1)$$

$$R_2 - c_2 q_2 \geq 0. \qquad (IR_2)$$

- These are the *participation constraints*.

Example 9.2: Solution (Unobservable Cost)

- We also need to ensure $q_i \geq 0$ and $R_i \geq 0$.
- The government's problem is

$$\max_{q_1,q_2,R_1,R_2} \beta_1 [B(q_1) - R_1] + \beta_2 [B(q_2) - R_2]$$

$$\text{s.t. } (IC_1),\ (IC_2),\ (IR_1),\ (IR_2)$$

$$q_1 \geq 0, q_2 \geq 0, R_1 \geq 0, R_2 \geq 0.$$

Example 9.2: Solution (Unobservable Cost)

- It is a maximization problem with 4 inequality constraints and 4 non-zero variables.
- These inequality pairs permit $2^8 = 256$ patterns of equations.
- Solving the problem directly involves a lot of work.
- We will make some initial analysis to simplify the problem.

Example 9.2: Solution (Unobservable Cost)

Lemma 1. *$R_i \geq 0$ is implied by (IR_1), (IR_2) and $q_i \geq 0$.*

Proof. We take R_1 as an example.

$$R_1 \underbrace{\geq}_{(IR_1)} c_1 q_1 \underbrace{\geq}_{q_1 \geq 0} 0.$$

$R_2 \geq 0$ follows similarly. $\square$

- We could safely ignore non-negativity constraints on R_i.

Example 9.2: Solution (Unobservable Cost)

Lemma 2. *(IR_1) is implied by (IC_1), (IR_2) and $q_2 \geq 0$.*

Proof. We want to show that (IR_1) holds, i.e., $R_1 - c_1 q_1 \geq 0$.

$$R_1 - c_1 q_1 \underbrace{\geq}_{(IC_1):\ R_1 - c_1 q_1 \geq R_2 - c_1 q_2} R_2 - c_1 q_2 \underbrace{\geq}_{(IR_2):\ R_2 - c_2 q_2 \geq 0} c_2 q_2 - c_1 q_2 = (c_2 - c_1) q_2 \underbrace{\geq}_{q_2 \geq 0,\ c_1 < c_2} 0$$

(IR_1) is implied. □

- We could safely ignore (IR_1).

Example 9.2: Solution (Unobservable Cost)

From Lemma 1 and Lemma 2, the government's problem could be simplified as follows:

$$\max_{q_1,q_2,R_1,R_2} \beta_1 [B(q_1) - R_1] + \beta_2 [B(q_2) - R_2]$$

$$\begin{aligned}
\text{s.t. } & R_1 - c_1 q_1 \geq R_2 - c_1 q_2; && (IC_1) \\
& R_2 - c_2 q_2 \geq R_1 - c_2 q_1; && (IC_2) \\
& R_2 - c_2 q_2 \geq 0; && (IR_2) \\
& q_1 \geq 0, q_2 \geq 0.
\end{aligned}$$

Example 9.2: Solution (Unobservable Cost)

Lemma 3. *(IR_2) must be binding in the optimal scheme, i.e., $R_2 - c_2 q_2 = 0$.*

Example 9.2: Solution (Unobservable Cost)

Proof.

- Suppose not, i.e., $R_2 - c_2 q_2 > 0$, then q_1 and q_2 can be slightly raised by the same amount ε so that (IR_2) still holds. After such a change,

  (i) All constraints still holds.

  (ii) The objective function gets larger.

- Scheme with $R_2 - c_2 q_2 > 0$ must not be optimal.
- (IR_2) must be binding in the optimal scheme. $\square$

Example 9.2: Solution (Unobservable Cost)

Lemma 4. *(IC_1) must be binding in the optimal scheme, i.e.,* $R_1 - c_1 q_1 = R_2 - c_1 q_2$.

Example 9.2: Solution (Unobservable Cost)

Proof.

- Suppose not, i.e., $R_1 - c_1 q_1 > R_2 - c_1 q_2$. Then q_1 can be slightly raised by ε so that (IC_1) still holds. After such a change,

  (i) All constraints still hold.

  (ii) The objective function gets larger.

- Scheme with $R_1 - c_1 q_1 > R_2 - c_1 q_2$ must not be optimal.
- (IC_1) must be binding in the optimal scheme. $\square$

Example 9.2: Solution (Unobservable Cost)

- By Lemma 3 and Lemma 4, we have

$$R_2 = c_2 q_2 \tag{R_2}$$

$$R_1 = c_1 q_1 + (c_2 - c_1) q_2 \tag{R_1}$$

- Plugging (R_2) and (R_1) into (IC_2), we have

$$R_2 - c_2 q_2 \geq R_1 - c_2 q_1 \underbrace{\iff}_{(R_2),\ (R_1)} (c_2 - c_1)(q_1 - q_2) \geq 0$$

$$\underbrace{\iff}_{c_1 < c_2} q_1 \geq q_2 \tag{IC_2'}$$

Example 9.2: Solution (Unobservable Cost)

- Plugging (R_2) and (R_1) into the objective function, and recognizing the remaining constraints, the maximization problem is simplied as follows:

$$\max_{q_1,q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

$$\text{s.t. } q_1 \geq q_2 \qquad (IC_2\text{'})$$

$$q_2 \geq 0$$

- We can solve the maximzation problem in the usual way.
- See Appendix A.

Example 9.2: Solution (Unobservable Cost)

- Here, we introduce another way of solving the problem.
- We first solve the relaxed problem with no (IC_2'), and then show that the solution to the relaxed problem is also the solution to the initial problem.
- The relaxed problem is as follows:

$$\max_{q_1,q_2} \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right]$$

s.t. $q_2 \geq 0$

Example 9.2: Solution (Unobservable Cost)

1. Form the Lagrangian:

$$\mathcal{L}(q_1, q_2) = \beta_1 \left[B(q_1) - (c_1 q_1 + (c_2 - c_1) q_2)\right] + \beta_2 \left[B(q_2) - c_2 q_2\right].$$

2. Write out the first-order necessary conditions:

$$\partial\mathcal{L}/\partial q_1 = \beta_1[B'(q_1) - c_1] = 0 \tag{9.13}$$

$$\partial\mathcal{L}/\partial q_2 = -\beta_1(c_2 - c_1) + \beta_2[B'(q_2) - c_2] \leq 0 \text{ and } q_2 \geq 0 \text{ with complemetary slackness.} \tag{9.14}$$

Example 9.2: Solution (Unobservable Cost)

From (9.13), we have

$$B'(q_1) = c_1.$$

- Note that it is the same as the condition for Type-1 when cost is observable, see (9.12).

Example 9.2: Solution (Unobservable Cost)

From (9.14),

(i) **Case I: $\mathbf{q_2 = 0}$.** Then by (9.14), we must have

$$-\beta_1(c_2 - c_1) + \beta_2[B'(0) - c_2] \leq 0 \implies B'(0) \leq c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1).$$

(ii) **Case II: $\mathbf{q_2 > 0}$.** Then by (9.14), we must have

$$\begin{aligned} & -\beta_1(c_2 - c_1) + \beta_2[B'(q_2) - c_2] = 0 \\ \implies & B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \end{aligned} \tag{9.15}$$

For $q_2 > 0$, we need $B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1)$.

Example 9.2: Solution (Unobservable Cost)

Therefore, the solution to the relaxed problem is

$$\begin{cases} B'(q_1) = c_1,\ q_2 = 0 \\ \qquad\qquad \text{if } B'(0) \le c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1); \\ B'(q_1) = c_1,\ B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1) \\ \qquad\qquad \text{if } B'(0) > c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \end{cases} \tag{9.16}$$

Example 9.2: Solution (Unobservable Cost)

- Next, we show that (9.16) also solves the initial problem.
- We need to show that (IC_2'), i.e., $q_1 \geq q_2$ holds in (9.16).

(i) $\mathbf{q_2 = 0.}$ $q_1 > 0$ and $q_2 = 0$, thus $q_1 > q_2$.

(ii) $\mathbf{q_2 > 0.}$ Since $c_1 < c_2 < c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1)$, $B'(q_1) < B'(q_2)$. Strict concavity of $B(q)$ implies $q_1 > q_2$.

- Solution (9.16) is solution to the simplied problem and is thus solution to government's maximization problem.
- The payments R_1 and R_2 are given by (R_1) and (R_2).

Example 9.2: Solution (Unobservable Cost)

- The logic is similar to "(since you are a student in Economic and Management School of Wuhan university), if you are the best student in Wuhan university, then you are the best student in the EMS of Wuhan university."
- If we want to find "the best student in the EMS of Wuhan university", we could relax the problem and search for the best student in the whole university.

Example 9.2: Solution (Unobservable Cost)

- The relaxed problem is easier to solve, since it involves less constraints.
- However, the solution to the relaxed problem may not be the solution to the initial problem.
- You must make sure that the conditions left out are indeed satisfied.

Example 9.2: Solution (Unobservable Cost)

There are two points worth noticing.

1. When $B'(0) \leq c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1)$, the high cost supplier does not produce. This is likely to happen when

   a) β_2 is small: the probability that the supplier is a high cost one is low, and

   b) $B'(0)$ is small: the benefit of having a high cost supplier producing a little is small.

   The government can effectively eliminate the incentive of a low cost supplier to pretend to be a high cost one.

Example 9.2: Solution (Unobservable Cost)

2. When $q_2 > 0$, the optimal q_2, which is given by (9.15):

$$B'(q_2) = c_2 + \frac{\beta_1}{\beta_2}(c_2 - c_1). \tag{9.15}$$

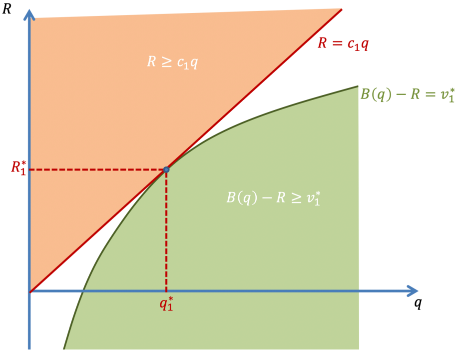
is lower than the optimal q_2 when cost is observable, which is given by (9.12):

$$B'(q_2) = c_2.$$

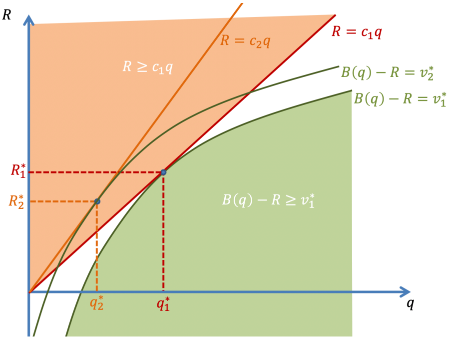
Lowering quantities demanded for high-cost supplier makes it less tempting for low-cost supplier to declare high cost.

Example 9.2: Graphs (Observable Cost)

We could also graphically illustrate the idea.

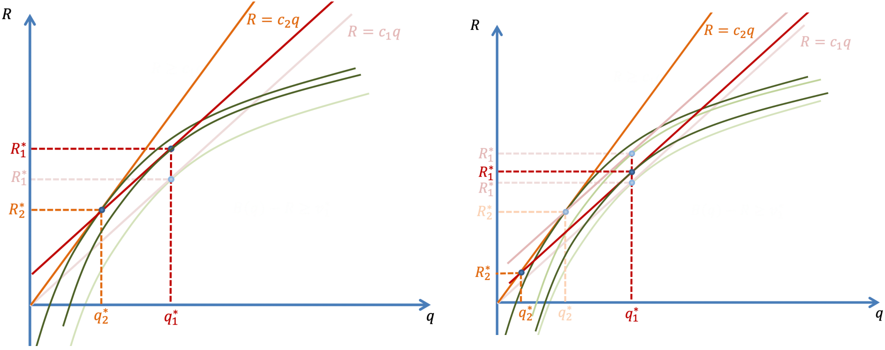


(a)



(b)

Example 9.2: Graphs (Unobservable Cost)



(a) (b)

The *efficiency* concern and the *rent* extraction is the key trade-off faced by the government.