Chapter 4. Shadow Prices (Exercises)

Exercise 4.1: The invisible hand - Production Continue with the notation of Example 4.1, but now allow production of the goods. Let there be F factor inputs, available in fixed quantities Z_f for $f = 1, 2, ..., F$. If z_{fg} of factor f is used in the production of good g, the output X_g is given by the production function

$$X_g = \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}). \tag{4.1}$$

Add these constraints to the earlier problem.

Question 1: Verify that the first-order conditions of optimum distribution are the same as before, but new conditions for optimum factor allocation are added.

Question 2: Interpret the Lagrange multipliers.

Question 3: Can the production be decentralized, with one firm producing one good?

Question 4: Show that the sum of income I_c handed out to the consumers equals the value of aggregate output.

Exercise 4.2: The Invisible Hand - Factor Supplies Now let even the factor supplies Z_f be a part of the optimization. Suppose each consumer c supplies z_{cf} of factor f. These amounts affect his utility adversely.

Question 1: Find the first-order conditions. And interpret Lagrange multipliers and discuss the implementation of the optimum in a market framework.

Question 2: Now you must distinguish two sources of income for the consumers: their earnings from the factor services, and the lump sum I_c they get from the government. Show that the total of the lump sums $\sum_{c=1}^{C} I_c$ handed out to consumers equal the total profit in production, that is, the value of output minus the payments to the factors.

Exercise 4.3: Borrowing and Lending Consider a consumer planning his consumption over two years. He will have income I_1 during the first year and I_2 during the second. In each year, there are two goods to consume. In year 1, the prices are p_1 and q_1, and the corresponding quantities x_1 and y_1. In year 2, we similarly have p_2, q_2 and x_2, y_2. The utility function is

$$u_1 = \alpha_1 \ln(x_1) + \beta_1 \ln(y_1) + \alpha_2 \ln(x_2) + \beta_2 \ln(y_2).$$

This is to be maximized subject to two budget constraints, one for each year.

Question 1: Solve the problem, and find the multipliers λ_1 and λ_2 for the two constraints.

Question 2: How much more of year-2 income will the consumer require if he is to give up $\mathrm{d}I_1$ of year-1 income? In other words, what is the rate of return needed to induce him to save a little? You would expect borrowing and lending institutions arise in an economy populated by such consumers. What governs who will borrow and who will lend?

The interpretation of Lagrange multipliers in Exercise 2.3

Part I Consider a producer who rents machines K at r per year and hires labor L at wage w per year to produce output Q, where $Q = \sqrt{K} + \sqrt{L}$. Suppose he wishes to produce a fixed quantity Q at minimum cost.
Find his factor demand function, that is, the optimal amount of K and L.
Interpret the Lagrange multiplier.

Part II Now let p denote the price of output. Suppose the producer can vary the quantity of output, and seeks to maximize profit. Factor prices and the production function are the same as in Part I.
Find the optimal output.
Relate this to your interpretation of the Lagrange multiplier.