Difference Equations

1.A. First order linear constant coefficient difference equations

Homogeneous linear constant coefficient difference equations: $y_{t+1} + ay_t = 0$

- General solution: $y_t = C(-a)^t$
- Particular solution: given y_0, then $y_t = y_0(-a)^t$

Linear constant coefficient difference equations: $y_{t+1} + ay_t = b$

- Equilibrium: $y_{t+1} = y_t = y^* \implies y^* = \frac{b}{1+a}$. Define $x_t = y_t - y^*$ (x_t is the deviation from the equilibrium). Then $x_{t+1} + ax_t = 0$ and thus $x_t = C(-a)^t$.
- General solution: $y_t = C(-a)^t + y^*$ where $y^* = \frac{b}{1+a}$.
- Particular solution: given y_0, then $y_t = (y_0 - y^*)(-a)^t + y^*$

Method of undetermined coefficient: $y_{t+1} + ay_t = b(t)$

- Particular solution: $y^p(t)$
- General solution: $y_t = C(-a)^t + y^p(t)$

Example 1.1. $y_{t+1} - ay_t = 4b^t$. Try $y^p(t) = cb^t$. Then

$$cb^{t+1} - acb^t = 4b^t \implies c = \frac{4}{b-a}.$$

Thus, the general solution is $y_t = C(-a)^t + \frac{4}{b-a}b^t$.

Recursive method: $y_{t+1} = ay_t + b_{t+1}$

$$y_T = a^T y_0 + \sum_{t=1}^{T} a^{T-t} b_t.$$

1.B. Autonomous difference equations: $y_{t+1} = f(y_t)$ and phase diagram

- Equilibrium y^*: $y^* = f(y^*)$.
- Phase diagram

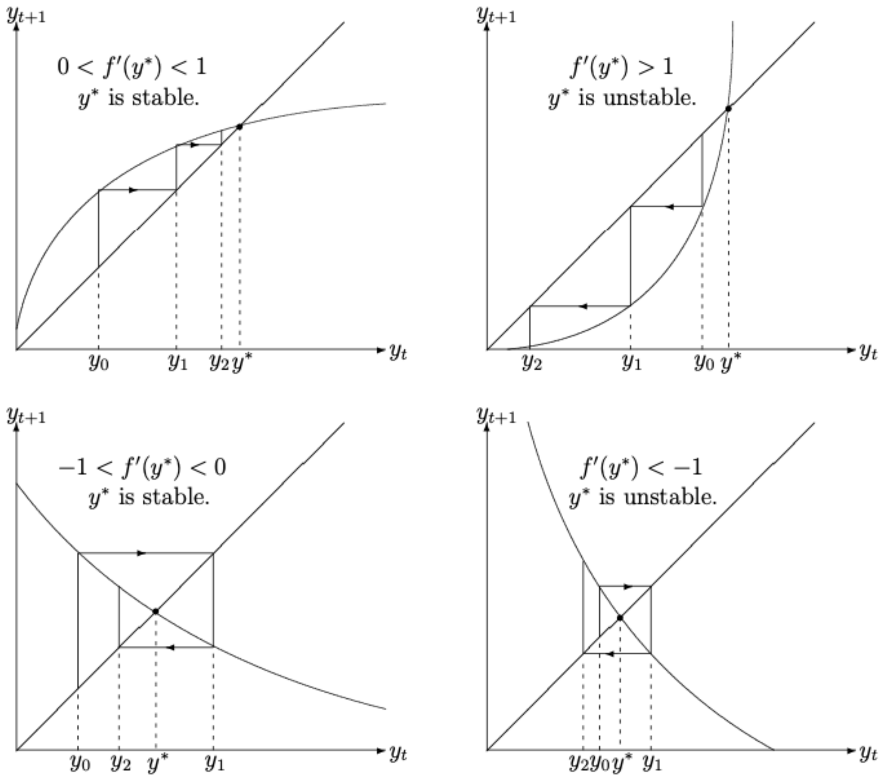

Figure 1.1: Phase diagram

1.C. Second order linear constant coefficient difference equation

Homogeneous linear constant coefficient difference equation $y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$

Suppose that there is a solution $y_t = \lambda^t$. Then $y_{t+1} = \lambda^{t+1} = \lambda y_t$ and $y_{t+2} = \lambda^{t+2} = \lambda^2 y_t$. Then λ satisfies the *characteristic equation*:

$$\phi(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0.$$

- If λ_1 and λ_2 are distinct reals, then the general solution $y_t = C_1\lambda_1^t + C_2\lambda_2^t$.
- If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution $y_t = C_1\lambda_1^t + C_2 t\lambda_2^t$.
- If $\lambda_1, \lambda_2 = \alpha \pm \beta i = \rho e^{\pm i\theta}$ where $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$, then the general solution is $y_t = \rho^t[C_1 \cos\theta t + C_2 \sin\theta t]$.

Linear constant coefficient difference equation $y_{t+2} + a_1 y_{t+1} + a_2 y_t = b$

- Equilibrium: $y_{t+2} = y_{t+1} = y_t = y^* \implies y^* = \frac{b}{1+a_1+a_2}$. Define $x_t = y_t - y^*$ (x_t is the deviation from the equilibrium). Then $x_{t+2} + a_1 x_{t+1} + a_2 x_t = 0$.
- General solution: $y_t = x_t + y^*$.