Chapter 2. Dynamic Games of Complete Information

In this chapter, we will study the dynamic games. Unlike the static games we have seen before, in dynamic games, players move sequentially. More importantly, the player who moves second knows what the previous player has done before he moves. Moreover, the previous player knows that this is the case. That is, information is more important than timing.

We will focus on games of complete information, that is, games in which the players' payoff functions are common knowledge. In Section 2.A, we first study games of perfect information, that is, games in which the player who is to move knows the full history of the play of the game. In Section 2.B, we study games of imperfect information. In Section 2.C, we study repeated games.

2.A. Dynamic Games of Complete and Perfect Information

2.A.1. Backward Induction

Let us first look at a simple example.

Example 2.A.1. Consider the following 2-player investment game.

- Player 1 chooses to invest 0, 1, or 3.
- After observing Player 1's choice, Player 2 can either match, i.e., add the same amount, or take the cash.
- Payoffs:
  - For Player 1, $\begin{cases} \text{if invests } 0, \text{ gets } 0; \\ \text{if invests } 1, \text{ doubles if matched and loses 1 if not;} \\ \text{if invests } 3, \text{ doubles if matched and loses 3 if not.} \end{cases}$
  - For Player 2, $\begin{cases} \text{if takes, gets Player 1's investment;} \\ \text{if matches 1, gets 2.5 back;} \\ \text{if matches 3, gets 5 back.} \end{cases}$

This is a sequential-move game. Player 2 observes how much Player 1 has invested before making the matching or taking decision. And Player 1 knows that this is the case.

Question 2.1. Suppose you are Player 1, how much would you invest? Suppose you are Player 2, would you match 1? How about 3?

The game could be more clearly organized in the game tree below.

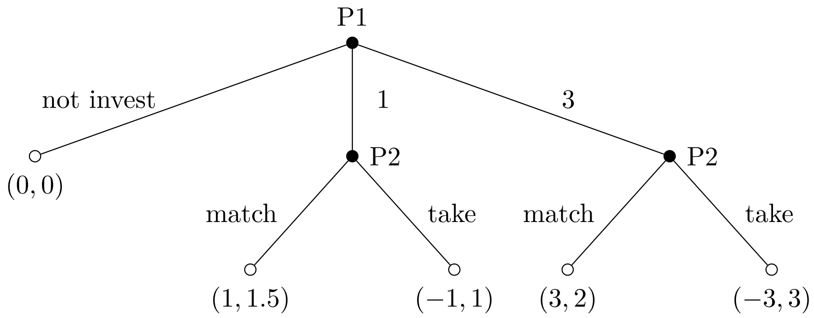

Figure 2.1: The Game Tree (Example 2.A.1)

The solid nodes are called *decision nodes*, besides which we write down the player whose turn it is to move. The hollow nodes are called *end nodes* or *terminal nodes*, besides which we write the players' payoffs.

Let us now work out what the players would do in this game. Player 1 would anticipate what Player 2 would do following each of his own choice, and then work backwards through the tree.

- If Player 1 chooses not to invest, she would get 0.
- If Player 1 chooses to invest 1, she knows Player 2 would match and she will double her money and get 1.
- If Player 1 chooses to invest 3, she knows Player 2 would take the cash and she will lose her investment of 3.

Therefore, Player 1 is essentially choosing among the payoffs of 0 (by not investing), 1 (by investing 1) and -3 (by investing 3). And thus, Player 1 would choose to invest 1. To complete the analysis, we write out Player 2's choice. Seeing that Player 1 chooses 1, Player 2 would indeed match.

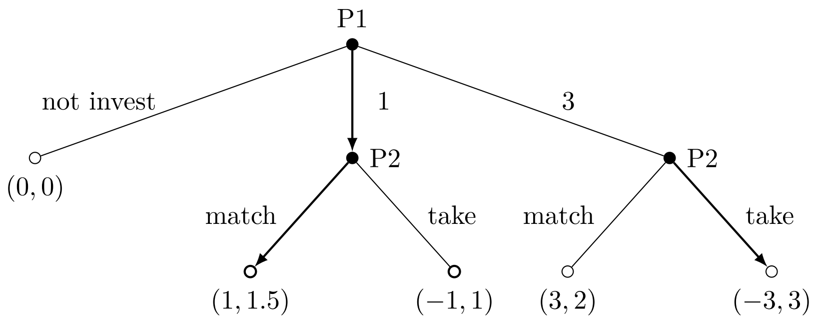

Figure 2.2: Backward Induction

This idea of starting at the player who moves last, solving out what they would do and then work back through the tree is called *Backward Induction*.

Remark 2.1. Note that the mutually beneficial outcome $(3, 2)$ is not played out.

Question 2.2. Can you think of ways to reach the good outcome?

Change the Division. One way to reach the good outcome is to change the division of the gain. Consider the following change the division of the net gain of 5 after the matched investment of 3.

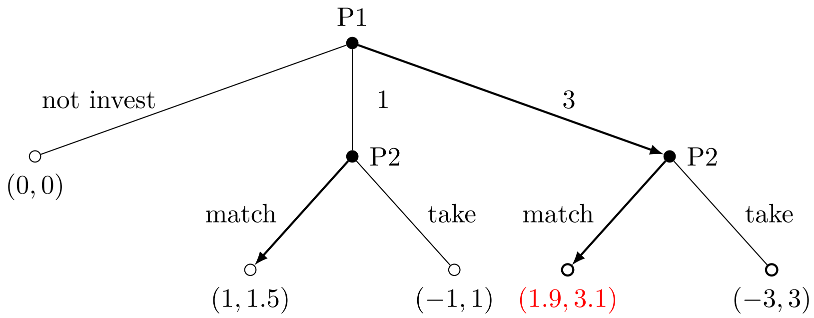

Figure 2.3: Change Division

Remark 2.2. Player 1 also likes such change since now she gets 1.9 instead of 1.

Collateral. Another way to attain the good outcome $(3, 2)$ is to ask Player 2 to make a collateral (and hence change the payoffs of the original game). The reason that the collateral works is not that it gives an extra positive return to Player 1. It works since it imposes an extra negative return to Player 2.

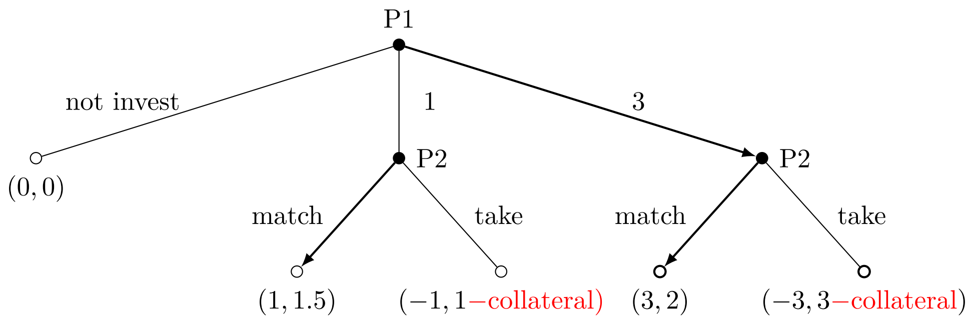

Figure 2.4: Collateral

Remark 2.3. Collateral lowers the payoff to Player 2 at some point of the tree, yet it makes Player 2 better off.

Abstracting from the concrete example, the dynamic games of complete and perfect information takes the following form:

1. Player 1 chooses an action a_1 from the feasible set A_1;
2. Player 2 observes a_1 and then chooses an action a_2 from the feasible set A_2.
3. Payoffs are $u_1(a_1, a_2)$ and $u_2(a_1, a_2)$.

These games could be more clearly represented in game trees as we did in the investment game (Example 2.A.1).

The key features of these games are

1. the moves occur in sequence;
2. all previous moves are observed before the next move is chosen (perfect information);
3. the players' payoffs from each feasible combination of moves are common knowledge (complete information).

This class of games are solved by *Backward Induction*:

1. In the second stage, given the action a_1 previously chosen by Player 1, Player 2 solves
$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$
Denote the solution $R_2(a_1)$. This is Player 2's *reaction* (or best response) to Player 1's action.
2. In the first stage, Player 1 would anticipate Player 2's reaction to each a_1 that Player 1 might take, so Player 1 solves
$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1)).$$

$(a_1^*, R_2(a_1^*))$ is the *backward-induction outcome* of the game.

Remark 2.4. The backward-induction outcome does not involve non-credible threats: Player 1 anticipates that Player 2 would respond optimally to any a_1 that Player 1 might choose, i.e., Player 2 would play $R_2(a_1)$. Player 1 gives no credence to threats that will not be in Player 2's self-interest when the second stage arrives.

For example, in the original game of Example 2.A.1, Player 2's claim that he would always match Player 1's investment would not be believed by Player 1.

Extensive-Form Representation. Our description of the dynamic games of complete and perfect information corresponds to the *Extensive-Form Representation*. Formally, Extensive-Form Representation of a game specifies

1. the players in the game,
2. a) when each player has the move,
   b) what each player can do at each of his/her opportunities to move,
   c) what each player knows at each of his/her opportunities to move,
3. the payoffs received by each player for each combination of moves that could be chosen by the players.

Graphically, the extensive-form representation is shown as the **game tree**.
Recall that we have learned another representation of game, called Normal-Form Representation. Such representation of a game specifies

1. the players in the game,
2. the strategies available to each player, and
3. the payoffs received by each player for each combination of strategies that could be chosen by the players.

Graphically, the normal-form representation is shown as the **payoff matrix**.
It is convenient to represent dynamic games in extensive form and static games in normal form. However, notice that any game could be represented in either normal or extensive form. For the last point, we will discuss in detail later on. See Section 2.A.4 for the case of perfect information and Example 2.B.3 for imperfect information.

After the abstract and formal discussions, let us look at another example to familiarize ourselves with the dynamic games of complete and perfect information, the method of *backward induction* in particular.

Example 2.A.2. Player 1 and Player 2 are two armies in a battle. Each army could choose to "Fight" or to "Run Away". The timing and the payoffs of the game are given in the game tree below.

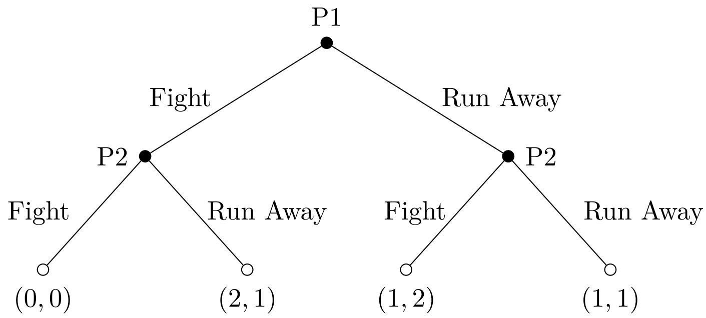

Figure 2.5: Example 2.A.2

Question 2.3. In this game, Player 2 likes the outcome (Run Away, Fight) the most. To make Player 1 run away, Player 2 claims that he would choose to fight no matter Player 1 fights or not. Will such a threat be believed by Player 1?

Answer: The threat of always fighting is non-credible since the best response of Player 2 after observing “Fight” is “Run Away”.

Backward Induction. The game is solved by backward induction.

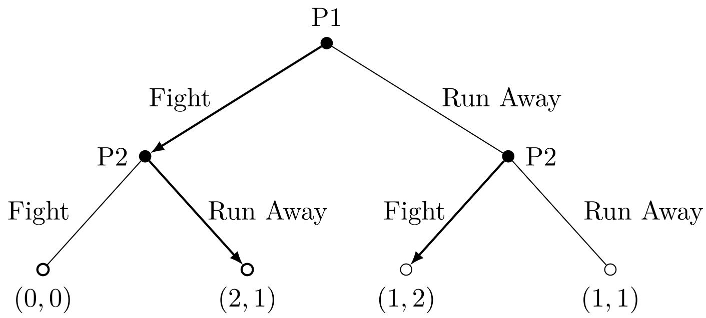

Figure 2.6: Backward Induction: Example 2.A.2

Therefore, Player 1 would “Fight”, and following this action, Player 2 would “Run Away”.

Burning the Ship. Next, consider a modified version of the game. We add the option “Burn the Ship” for Player 2. If Player 2 does not choose to “Burn the Ship”, then the game goes on as in the original example. However, if Player 2 chooses to “Burn the Ship”, then Player 2 has eliminated his own option “Run Away”.

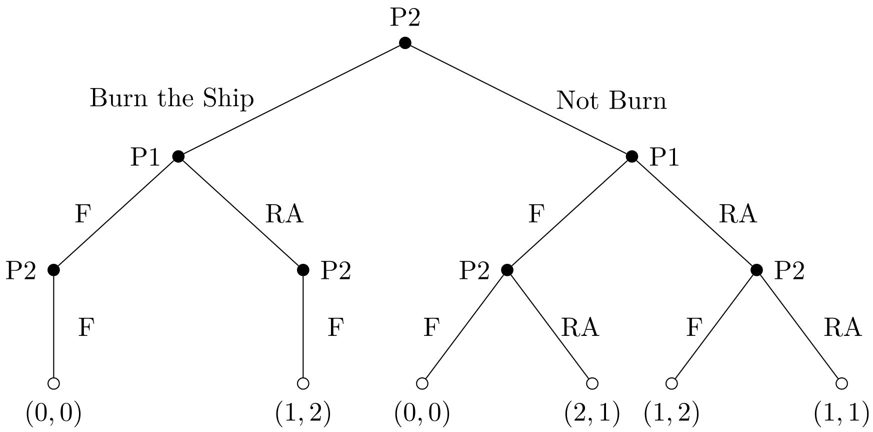

Figure 2.7: Example 2.A.2 Modified

Still, we use backward induction to solve the game.

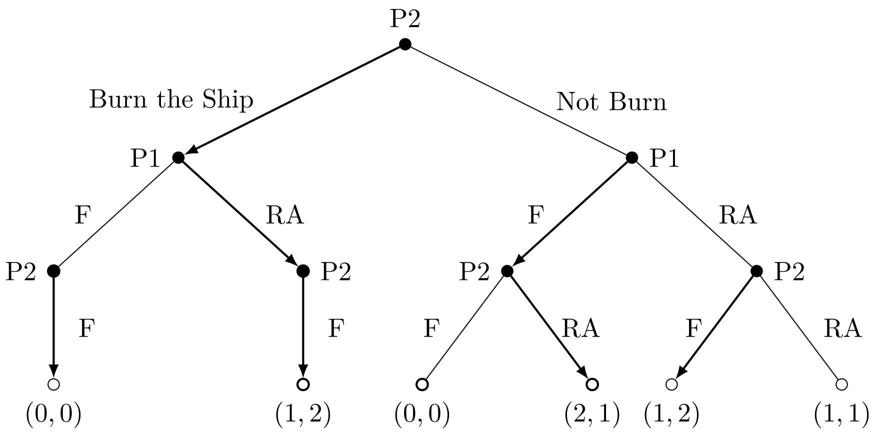

Figure 2.8: Backward Induction: Example 2.A.2 Modified

The backward induction outcome now becomes: Player 2 chooses to "Burn the Ship", followed by Player 1 "Run Away", and at last, Player 2 "Fight". Now, Player 2 would obtain his preferred outcome.

Remark 2.5. Player 2 is better off by getting rid of his own option "Run Away". Essentially, by burning the ship, Player 2 commits not to run away, and thus makes the action "Fight" credible. This commitment strategy is similar to the collateral case in the investment example (Example 2.A.1).

Remark 2.6. Another important aspect is that Player 1 must know that Player 2 has burned the ship. Otherwise, the game tree in Player 1's mind is still the original one and as a result Player 1 would "Fight".

Before moving on to the economic applications, let us play the following game.

Example 2.A.3 (The Lion Game)**.** There are n lions and 1 sheep. The lion society is hierarchical: only the head lion can eat the sheep. However, after the head lion has eaten the sheep, he would fall into postprandial stupor and could be eaten by the second largest lion. After the second largest lion has eaten the head lion, he could be eaten by the third largest lion, so on and so forth.

Question 2.4. Should the head lion eat the sheep?

Next, we will study two economic applications: Stackelberg Model of Duopoly (Subsection 2.A.2) and Sequential Bargaining (Subsection 2.A.3).

2.A.2. Stackelberg Model of Duopoly

Stackelberg (1934) proposed a dynamic model of duopoly in which a dominant (or leader) firm moves first and a subordinate (or follower) firm moves second.
Consider the following Stackelberg game where firms choose quantities. The game is similar to the Cournot game we have learned before except that the firms move sequentially instead of simultaneously. The timing of the game is as follows:

1. Firm 1 chooses a quantity $q_1 \geq 0$;
2. Firm 2 observes q_1 and then chooses a quantity $q_2 \geq 0$.

The payoff to Firm i is given by

$$\pi_i(q_i, q_j) = q_i[P(Q) - c] = q_i[a - q_i - q_j - c].$$

We solve the game by backward induction.

1. At Stage 2, observing q_1, Firm 2 solves

$$\max_{q_2 \geq 0} \pi_2(q_1, q_2) = \max_{q_2 \geq 0} q_2[a - q_1 - q_2 - c].$$

   The solution is

$$q_2 = \frac{a - q_1 - c}{2} = \mathrm{BR}_2(q_1).$$

2. Knowing Firm 2's best response function, at Stage 1, Firm 1 solves

$$\max_{q_1 \geq 0} \pi_1(q_1, \mathrm{BR}_2(q_1)) = \max_{q_1 \geq 0} q_1[a - q_1 - \mathrm{BR}_2(q_1) - c] = \max_{q_1 \geq 0} q_1 \left[\frac{a - q_1 - c}{2}\right].$$

   The solution is

$$q_1^* = \frac{a - c}{2}.$$

Plugging into $\mathrm{BR}_2(q_1)$ gives

$$q_2^* = \frac{a-c}{4}.$$

Comparison with Cournot Outcome.

Question 2.5. Compared to Cournot outcome, is Firm 1 (Firm 2) better-off?

Recall that the Nash equilibrium of the Cournot game is

$$q_1^c = q_2^c = \frac{a-c}{3}.$$

So, in the Stackelberg game, Firm 1 produces more than the Cournot quantity whereas Firm 2 produces less. In terms of aggregate quantity, the firms produces more in the Stackelberg game ($\frac{a-c}{2} + \frac{a-c}{4} > \frac{a-c}{3} + \frac{a-c}{3}$). Since $p(Q) = a - Q$, the price is lower in the Stackelberg game.

To answer Question 2.5, we could directly calculate Firm 1 and Firm 2's profits in the Stackelberg game and the Cournot game and make comparisons. Instead, here we provide an argument without detailed calculations. First, consider the following question.

Question 2.6. In the Stackelberg game, what would Firm 2 do if Firm 1 chooses the Cournot quantity?

Answer: Firm 2 would best respond with the Cournot quantity.

Therefore, Firm 1 could have achieved its Cournot profit level by choosing the Cournot quantity. However, Firm 1 chooses some other quantity $\frac{a-c}{2}$. So, Firm 1's profit in the Stackelberg game must exceed its profit in the Cournot game. For Firm 2, we have already established that in the Stackelberg game, Firm 2's quantity is lower and the price $p(Q)$ is lower. So, Firm 2 is worse-off in the Stackelberg game.

Remark 2.7. Compared to the Cournot game, in the Stackelberg game, Firm 2 has more information (Firm 2 observes q_1 before choosing q_2) and yet it is worse-off.

Remark 2.8. The fact that Firm 1 knows that Firm 2 has more information is important.

Spy. Consider two firms both privately deciding quantities to produce. That is, the two firms play the Cournot game. Now, suppose that Firm 2 sends a spy to Firm 1. Moreover, Firm 1 knows that there is a spy even though they do not know who is the spy.

Question 2.7. What would Firm 1 do?

First-mover Advantage. The Stackelberg game is a game with First-mover Advantage. That is, in this game, you would want to be the first-mover.

Question 2.8. Can you think of games with Second-mover Advantage? And games with neither First-mover nor Second-mover Advantage?

Answer:

1. Rock, Paper, Scissors is a game with Second-mover Advantage.
2. "I split, you choose" is a game with neither First-mover nor Second-mover Advantage.

Example 2.A.4. Consider two players playing with two piles of stones. The players move sequentially. In each turn, the player whose turn it is to move picks one of the two piles and removes some (≥ 1) of the stones. The person who gets the last stone wins.

- If the two piles have equal number of stones, it is a game of second-mover advantage.
- If the two piles have unequal number of stones, it is a game of first-mover advantage.

Zermelo's Theorem. Originally, Zermelo's Theorem concerns the game of chess. Here, we provide a slightly more general version of the theorem.

Theorem (Zermelo's Theorem)**.** Consider a **finite two-person** game with **perfect information**. Assume that there are three outcomes: a win for Player 1, a loss for Player 1, and a tie. Then, either Player 1 can force a win, or Player 1 can at least force a tie, or Player 2 can force a loss on Player 1.

Remark 2.9. Zermelo's Theorem does not tell us the solution of this type of games. It only tells us that a solution exists. You could think of "chess" as an example.

Sketch of Proof. The proof is by induction (on the maximum length of the game).

1. If $N = 1$.
   a) If there is an outcome that Player 1 wins, then Player 1 would choose that branch and force a win.
   b) If there is no outcome that Player 1 wins but an outcome of a tie, then Player 1 would choose that branch and force a tie.

c) If all the outcomes are such that Player 1 loses, then Player 1 loses.

2. Suppose that for all games of length $\leq N$, Zermelo's Theorem is true. We will show that it will be true for games of length $N+1$.
   - Following the first move of Player 1, all the subgames have length $\leq N$.
   - By the induction hypothesis, these subgames have solutions.
   - Then we could translate the original game into a game of length 1. And therefore, the original game has a solution.

2.A.3. Sequential Bargaining

Ultimatum Game. Player 1 and Player 2 are dividing one dollar. Player 1 can make a "take-it-or-leave-it" offer to Player 2: $(s, 1-s)$, i.e., keeping s to herself and giving $(1-s)$ to Player 2. "Take-it-or-leave-it" means:

- If Player 2 accepts the offer, then the payoffs are $(s, 1-s)$.
- If Player 2 rejects the offer, then the payoffs are $(0, 0)$.

By backward induction, Player 2 should accept any amount $(1-s) \geq 0$. Knowing this, Player 1 would keep all 1 to herself. Therefore, the equilibrium division is $(1, 0)$.

Two-stage bargaining. Player 1 and Player 2 are bargaining over one dollar. They alternate in making offers. Both players discount payoffs received in later periods by $\delta < 1$ per period. The bargain will last for two periods at most, with the moves described as follows:

1. Player 1 makes an offer to Player 2: $(s_1, 1-s_1)$, i.e., Player 1 keeps s_1 to herself and gives $(1-s_1)$ to Player 2.
   - If Player 2 accepts the offer, then the game ends and the payoffs are $(s_1, 1-s_1)$.
   - If Player 2 rejects the offer, then the game goes on to stage 2.
2. Player 2 makes an offer to Player 1: $(s_2, 1-s_2)$, i.e., Player 2 keeps $1-s_2$ to himself and gives s_2 to Player 1.
   - If Player 1 accepts the offer, then the game ends and the payoffs are $(s_2, 1-s_2)$.
   - If Player 1 rejects the offer, then the game ends and the payoffs are $(0, 0)$.

Still, we solve this game by backward induction. The Stage 2 game is the same as the Ultimatum game: Player 1 would accept any amount, so Player 2 would offer 0 to Player 1 and keep 1 to himself. Therefore, in Stage 1, Player 2 would accept the offer if and only if $1 - s_1 \geq \delta \cdot 1$, i.e., $s_1 \leq 1 - \delta$. So, Player 1 should keep $1 - \delta$ to herself and leave δ to Player 2.

More Stages. Now consider the sequential bargaining game with more stages:

1. Player 1 makes an offer $(s_1, 1 - s_1)$.
2. If the offer is rejected by Player 2, Player 2 makes an offer $(s_2, 1 - s_2)$.
3. If the offer is rejected by Player 1, Player 1 makes an offer $(s_3, 1 - s_3)$.
4. and so on...
5. The game ends in Period T. If the offer is rejected, then the payoffs are $(0, 0)$.

The game could be solved by backward induction. Consider $T = 3$. The Stage 3 game is the same as the Ultimatum game and the Stage 2 game is the same as the two-stage bargaining. In Stage 2, Player 2 is the offerer and could keep $1 - \delta$ to himself. Therefore, in Stage 1, Player 2 would accept the offer if and only if $1 - s_1 \geq \delta(1 - \delta)$. So, Player 1 should keep $1 - \delta(1 - \delta)$ to herself and leave $\delta(1 - \delta)$ to Player 2.

Figure 2.9 graphically illustrates the outcome of the game with 1, 2, and 3 stages.

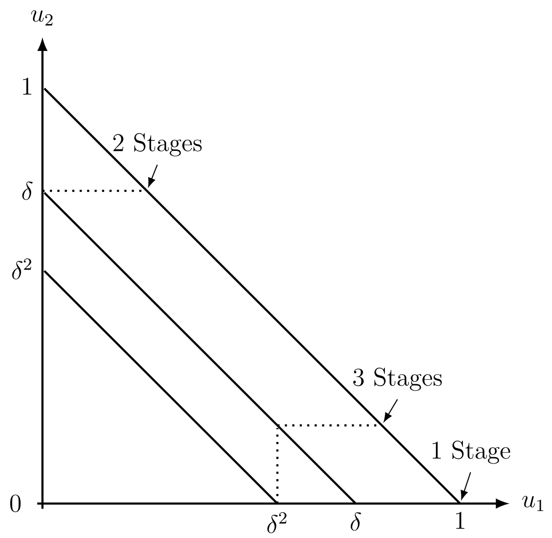

Figure 2.9: Sequential Bargaining

The result could also be summarized in the following table.

	Offerer	Receiver
1 Stage	1	0
2 Stages	$1-\delta$	δ
3 Stages	$1-\delta(1-\delta)=1-\delta+\delta^2$	$\delta(1-\delta)$
4 Stages	$1-\delta[1-\delta(1-\delta)]=1-\delta+\delta^2-\delta^3$	$\delta[1-\delta(1-\delta)]$
$\vdots$	$\vdots$	$\vdots$
2n Stages	$\frac{1-\delta^{2n}}{1+\delta}$	$\frac{\delta+\delta^{2n}}{1+\delta}$
2n + 1 Stages	$\frac{1+\delta^{2n+1}}{1+\delta}$	$\frac{\delta-\delta^{2n+1}}{1+\delta}$

When $n \to \infty$, Player 1 gets $\frac{1}{1+\delta}$ and Player 2 gets $\frac{\delta}{1+\delta}$. Furthermore, when $\delta \to 1$, both players get $\frac{1}{2}$.

Remark 2.10. In the sequential bargaining game, the first offer is accepted.

Remark 2.11. Even split if

(i) $n \to \infty$ (potentially bargaining forever) and

(ii) $\delta \to 1$ (no discounting or rapid offers).

2.A.4. Normal-Form Representations and Credible Threat

Definition 2.A.1 (Pure Strategy)**.** A *Pure Strategy* for Player i in a game of perfect information is a **complete plan** of actions: it specifies which action Player i will take at each of its decision nodes.

Example 2.A.5. Consider the following game.

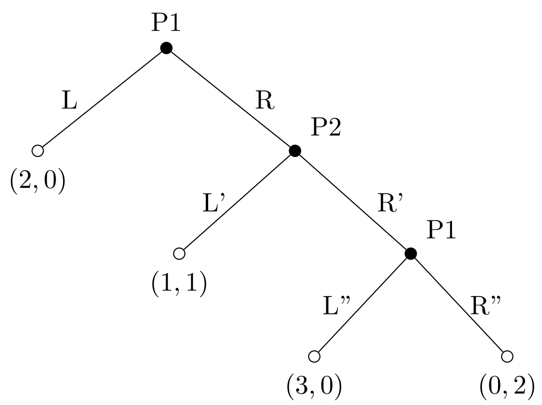

Figure 2.10: Example 2.A.5

According to Definition 2.A.1,

- Player 2's strategies: L' and R'
- Player 1's strategies: [R, L"], [R, R"], [L, L"], [L, R"]

Backward Induction. We could solve the game by backward induction.

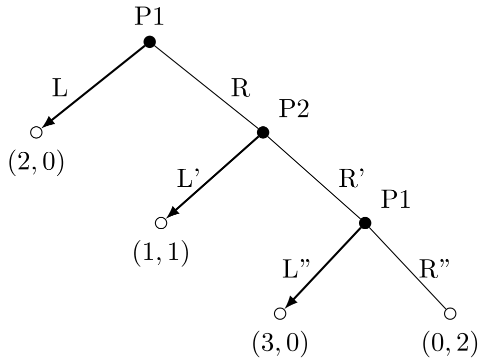

Figure 2.11: Backward Induction (Example 2.A.5)

The backward induction outcome is: Player 1 chooses "L" and the game ends.

Normal-Form Representation. We transform the tree to normal-form representation.

		Player 2	
		L'	R'
Player 1	[R, L"]	$(1, \underline{1})$	$(\underline{3}, 0)$
	[R, R"]	$(1, 1)$	$(0, \underline{2})$
	[L, L"]	$(\underline{2}, \underline{0})$	$(2, \underline{0})$
	[L, R"]	$(\underline{2}, \underline{0})$	$(2, \underline{0})$

Figure 2.12: Normal-Form Representation

There are two Nash equilibria: ([L, L"], L') and ([L, R"], L'). The first one is in accordance with the backward induction outcome whereas the second one is not.
The second solution seems weird since when Player 1's second decision node is reached, it does not make sense for her to choose R". For Player 1, [L, R"] can be an equilibrium strategy because in Player 1's view, it doesn't matter what she chooses in her second decision node because it is never going to be reached as long as Player 2 chooses L'.

Remark 2.12. Nash Equilibrium may not be a good solution concept for dynamic games: it may involve actions that is not rational when some node is actually reached.

Let us look at another example.

Example 2.A.6. Consider the following entry game.

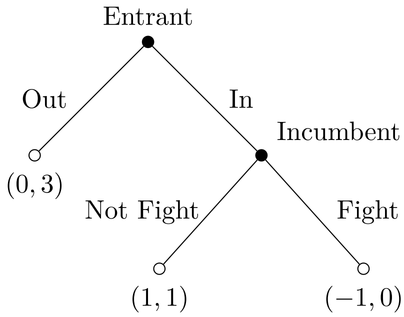

Figure 2.13: Entry Game

Backward Induction. The backward induction outcome of the game is:

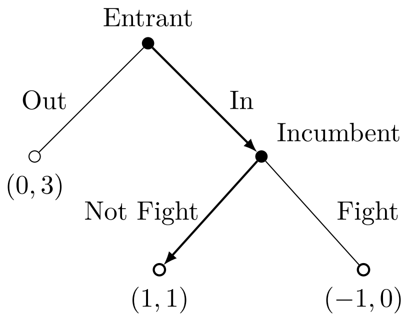

Figure 2.14: Entry Game - Backward Induction

The backward induction outcome is: Entrant chooses "In" and Incumbent "Not Fight".

Nash Equilibrium. We transform the game tree into the normal form representation and find the Nash equilibria of the game.

		Incumbent: Fight	Incumbent: Not Fight
Entrant	In	$(-1, 0)$	$(\underline{1}, \underline{1})$
	Out	$(\underline{0}, \underline{3})$	$(0, \underline{3})$

Figure 2.15: Normal-Form Representation

There are two Nash equilibria: (In, Not Fight) and (Out, Fight). The first one is in accordance with the backward induction outcome, whereas the second one is not. The second equilibrium relies on believing a *non-credible threat*: Entrant needs to believe that once he chooses "In", Incumbent would choose "Fight". However, it is not rational for Incumbent to choose "Fight" when his decision node is actually reached.

2.B. Dynamic Games of Complete but Imperfect Information

Example 2.B.1. Consider the following game:

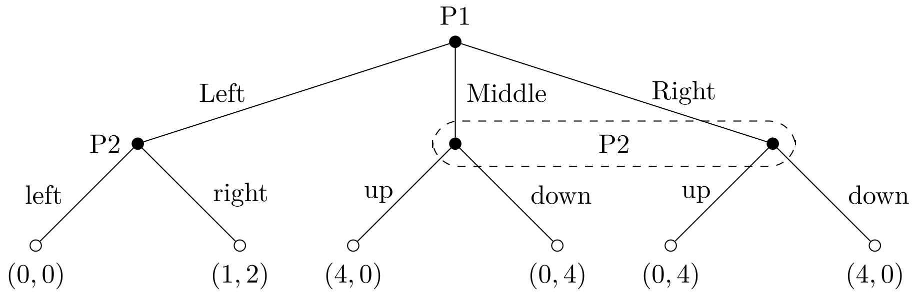

Figure 2.16: Game with Imperfect Information

In this game, the dashed circle is an *information set*. It means that Player 2 cannot distinguish "Middle" and "Right". Player 1 would mix between "Middle" and "Right".

Example 2.B.2. Consider the game in Example 2.B.1 without information set.

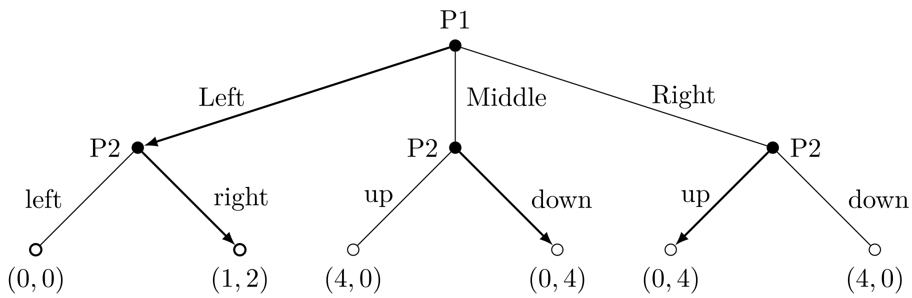

Figure 2.17: Game with Perfect Information

We could solve this game by backward induction. The BI outcome is: P1 chooses "Left" and P2 chooses "right".

Next, we formally define *information set*.

Definition 2.B.1. An *information set* of Player i is a collection of Player i's decision nodes among which Player i cannot distinguish.

Based on Definition 2.B.1, the following scenarios (in Figure 2.18 and 2.19) are **NOT allowed**.

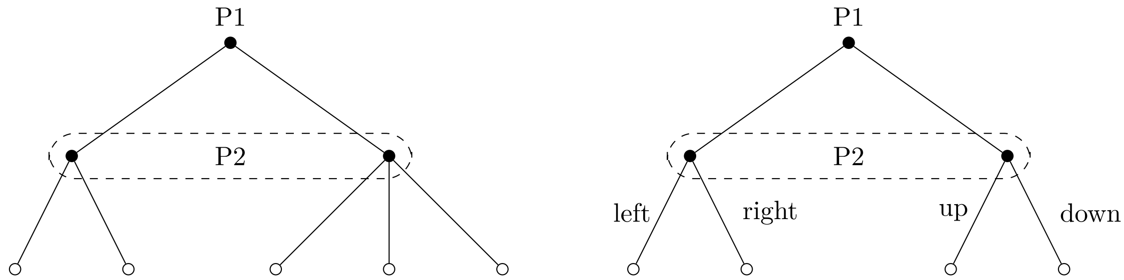

Figure 2.18: Distinct Actions

In the case illustrated by Figure 2.18, Player 2's actions are different following his two decision nodes. Then the two decision nodes cannot form an information set since it is unreasonable that Player 2 cannot distinguish the two nodes: Player 2 should be able to distinguish the two nodes based on his actions when he is called upon to move.

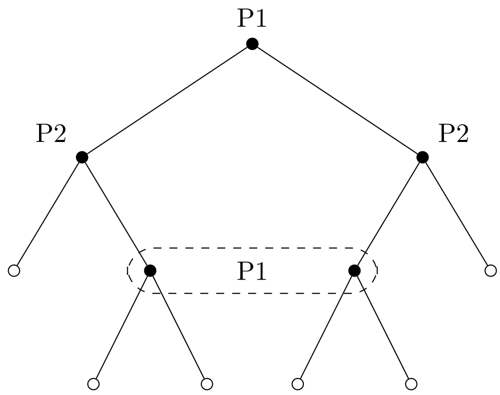

Figure 2.19: Imperfect Recall

In the case illustrated by Figure 2.19, Player 1 moves twice and she cannot distinguish the two decision nodes when she moves the second time. This is unreasonable: she should be able to distinguish the two nodes based on her first move. We call such requirement that the players remember their previous actions the *Perfect Recall*.

With the concept of *information set* in mind, we could formally define games with perfect information and imperfect information.

Definition 2.B.2 (Perfect/Imperfect Information Games)**.** A game is with *perfect information* if all the information sets in the game tree is singleton. A game is with *imperfect information* if it is not a game with perfect information.

Similar to Definition 2.A.1, we could define *Pure Strategy* for games with imperfect information.

Definition 2.B.3 (Pure Strategy)**.** A *Pure Strategy* for Player i is a **complete plan** of actions: it specifies which action Player i will take at each of its information sets.

Next, let us look at some examples.

Example 2.B.3. Consider the following game with imperfect information.

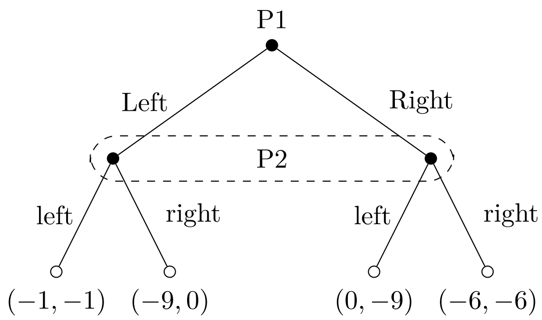

Figure 2.20: Example 2.B.3

In this game, Player 2's two decision nodes are in an information set. Thus, Player 2 cannot distinguish the two nodes and has to choose the same action at the two nodes in the information set. However, a special feature of this game is that Player 2 has the same best response at the two nodes, namely, "right". Therefore, Player 2 would choose "right". Knowing this, Player 1 would choose "Right".

Question 2.9. Do you find this game familiar? What is this game?

Answer: It is the Prisoners' Dilemma game! It is more obvious after we transform the game tree into the normal-form representation.

		P2	
		left	right
P1	Left	$(-1,-1)$	$(-9,0)$
	Right	$(0,-9)$	$(-6,-6)$

Figure 2.21: The Prisoners' Dilemma Game

Remark 2.13. A normal-form representation could have multiple extensive-form representations. For example, the Prisoners' Dilemma game above could be equivalently represented in the following extensive form, with Player 2 moving in the first decision node.

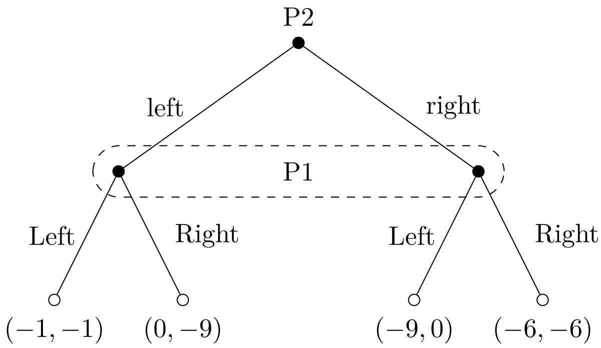

Figure 2.22: Example 2.B.3

2.B.1. Subgame Perfect Equilibrium (SPE)

Next, we will learn how to solve the games with imperfect information.

Example 2.B.4. Consider the following game:

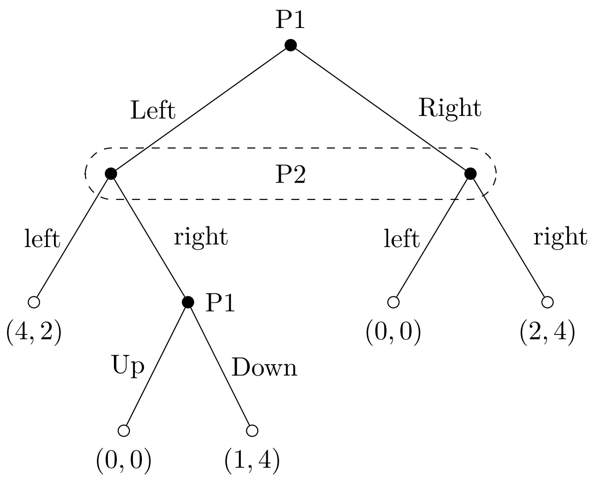

Figure 2.23: Example 2.B.4

Nash Equilibria. First, let us find the Nash equilibria of this game. Similar to what we have done in the perfect information game (See Example 2.A.5 and Example 2.A.6), we represent the game in normal form and look for mutual best responses.

According to Definition 2.B.3,

- Player 1's strategies: [L, U], [L, D], [R, U], [R, D]
- Player 2's strategies: l, r

The normal-form representation of the game is

		Player 2	
		l	r
Player 1	[L, U]	$(\underline{4}, \underline{2})$	$(0, 0)$
	[L, D]	$(\underline{4}, 2)$	$(1, \underline{4})$
	[R, U]	$(0, 0)$	$(\underline{2}, \underline{4})$
	[R, D]	$(0, 0)$	$(\underline{2}, \underline{4})$

Figure 2.24: Normal-Form Representation

There are three Nash equilibria: ([L, U], l), ([R, U], r) and ([R, D], r).

Subgame Perfect Equilibrium. The first two Nash equilibria ([L, U], l) and ([R, U], r) do not seem reasonable: if Player 1 were to move at her second decision node, she would choose D rather than U. To make more reasonable predictions, analogous to what we do in the games of complete and perfect information, we should solve the game backwards. However, we could not proceed backward induction as we did in the games with perfect information since in the games of imperfect information, the information set may involve multiple decision nodes.

Let us try to solve the previous game in Example 2.B.4 and then generalize the idea to other games of imperfect information. Solving the game backwards, at Player 1's second decision node, she would choose D. The players obtain payoffs $(1, 4)$. Substituting the payoffs into the original game, the game becomes

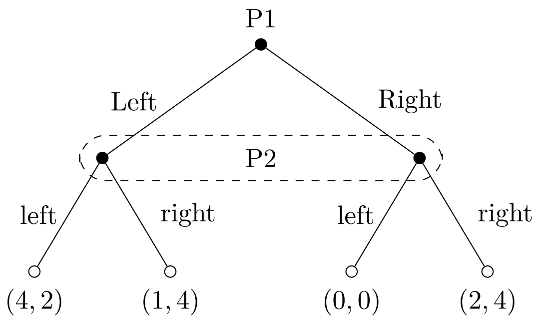

Figure 2.25: Example 2.B.4

This is a game with simultaneous move. It is more conveniently represented in normal form. The game matrix is shown in Figure 2.26 below.

		Player 2: l	Player 2: r
Player 1	L	$(\underline{4}, 2)$	$(1, \underline{4})$
Player 1	R	$(0, 0)$	$(\underline{2}, \underline{4})$

Figure 2.26: Normal-Form Representation

The solution of the game is (R, r). Therefore, when we solve the game backwards, only one equilibrium remains: ([R, D], r).

To summarize what we did to solve the game: we work backwards through the extensive form until we encounter a non-singleton information set. Then, we skip over it and proceed up the tree until a singleton information set is found. What we do is to solve for the *Subgame Perfect Equilibrium*. The definitions of *Subgame* and *Subgame Perfect Equilibrium* are given below.

Definition 2.B.4. A **subgame** in an extensive-form game

1. begins at a decision node n that is a singleton information set,
2. includes all the decision and terminal nodes following n in the game tree, and
3. does not cut any information sets.

Definition 2.B.5. (Selten 1965): A Nash equilibrium is a **Subgame Perfect Equilibrium (SPE)** if the players' strategies constitute a Nash equilibrium in every subgame.

Let us look at more examples.

Example 2.B.5. Consider the following three-player game:

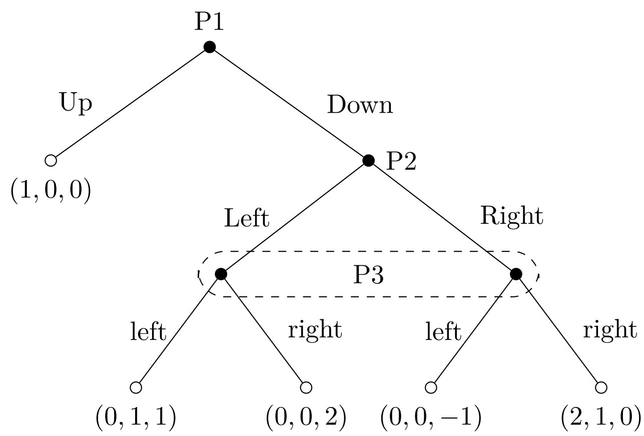

Figure 2.27: Example 2.B.5

There is one subgame involving Player 2 and Player 3. It is a simultaneous move game and we represent the game in normal form:

		Player 3	
		l	r
Player 2	L	$(\underline{1}, 1)$	$(0, \underline{2})$
	R	$(0, -1)$	$(\underline{1}, \underline{0})$

Figure 2.28: Normal-Form Representation of the Subgame

The solution of the subgame is (R, r). In the original game, when (R, r) is played out in the subgame, the three players obtain payoffs $(2, 1, 0)$. Substituting the payoffs into the original game, the original game becomes

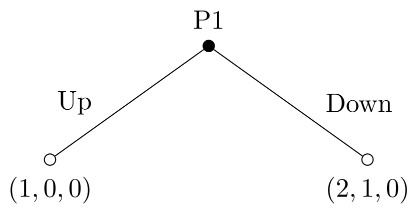

Figure 2.29: Example 2.B.5

Player 1 would choose D. Therefore, the SPE of the game is (D, R, r).

Example 2.B.6. Consider the following matchmaking game. The first player is the matchmaker who could introduce Alice to Bob. If the matchmaker introduces Alice to Bob, the couple would play the battle of the sexes game:

		Bob	
		Opera	Movie
Alice	Opera	$(2, 1)$	$(0, 0)$
	Movie	$(0, 0)$	$(1, 2)$

Figure 2.30: The Battle of the Sexes

As for the matchmaker, if the couple successfully meet, she gets 1; and if the couple fail to meet, she gets -1. All three players get 0 if Alice is not introduced to Bob.

The game could be represented in a game tree. See Figure 2.31.

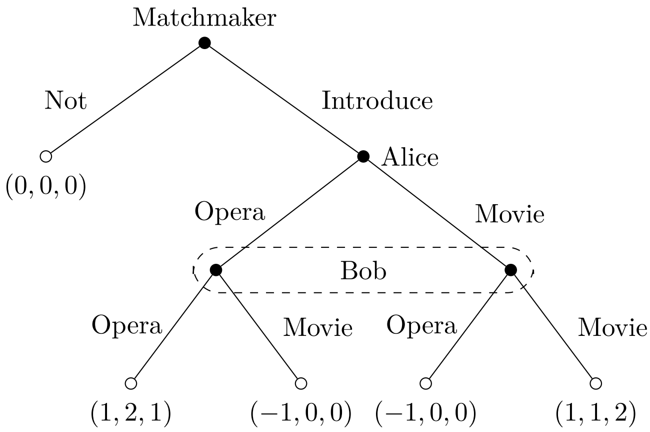

Figure 2.31: The Matchmaking Game

To obtain SPE, we first look at the subgame where Alice and Bob play the Battle of the Sexes game. We already know that there are two pure-strategy Nash equilibria and one mixed-strategy Nash equilibrium in this subgame.

First, consider the two pure-strategy Nash equilibria, namely (Opera, Opera) and (Movie, Movie), both yielding a value of 1 for Matchmaker. In the original game, when (Opera, Opera) is played out in the subgame, the three players obtain payoffs $(1, 2, 1)$; when (Movie, Movie) is played out, they obtain $(1, 1, 2)$. Substituting the payoffs into the original game, the original game becomes

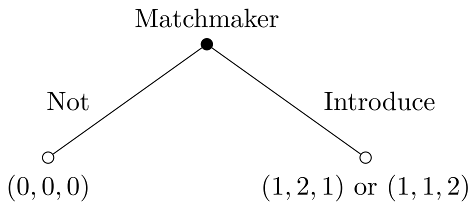

Figure 2.32: The Matchmaking Game

Matchmaker would choose "Introduce". We find two SPEs: (Introduce, Opera, Opera) and (Introduce, Movie, Movie).

Next, consider the mixed strategy Nash equilibrium, namely, $\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right)$. The expected values for the three players are

$$\mathbb{E}_M\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right) = \left[\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3}\right] \cdot 1 + \left[\frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3}\right] \cdot (-1) = -\frac{1}{9};$$
$$\mathbb{E}_A\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right) = \mathbb{E}_A\left(Opera, (\frac{1}{3}, \frac{2}{3})\right) = 2 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{2}{3};$$
$$\mathbb{E}_B\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right) = \frac{2}{3}.$$

Substituting the expected payoffs into the original game, the original game becomes

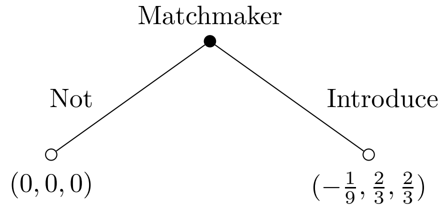

Figure 2.33: The Matchmaking Game

Matchmaker would choose "Not" to introduce Alice to Bob. We find a third SPE: $\big(\text{Not}, (\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\big)$.

Next, we will consider two applications: Bank Runs (Subsection 2.B.2) and War of Attrition (Subsection 2.B.3).

2.B.2. Bank Runs

Game Setup. Two investors have each deposited D with a bank. The bank has invested these deposits in a long-term project:

- If the bank is forced to liquidate its investment before the project matures, a total of $2r$ can be recovered, where $D > r > D/2$.
- If the bank allows the investment to reach maturity, however, the project will pay out a total of $2R$, where $R > D$.

There are two dates at which the investors can make withdrawals: date 1 is before maturity and date 2 is after. Assume that there is no discounting. The investors' decisions and their respective payoffs are as follows.

- At date 1,
  - If both investors make withdrawals, then each receives r and the game ends.
  - If only one investor makes a withdrawal, then that investor receives D, the other investor receives $2r - D$, and the game ends.
  - If neither investor makes a withdrawal, then the project matures and the investors make withdrawal decisions at date 2.
- At date 2 (if the game does not already ends),
  - If both investors make withdrawals, then each receives R and the game ends.
  - If only one investor makes a withdrawal, then that investor receives $2R - D$, the other investor receives D, and the game ends.

- If neither investor makes a withdrawal, then the bank returns R to each investor and the game ends.

Analysis. The game could be represented in a game tree. See Figure 2.34.

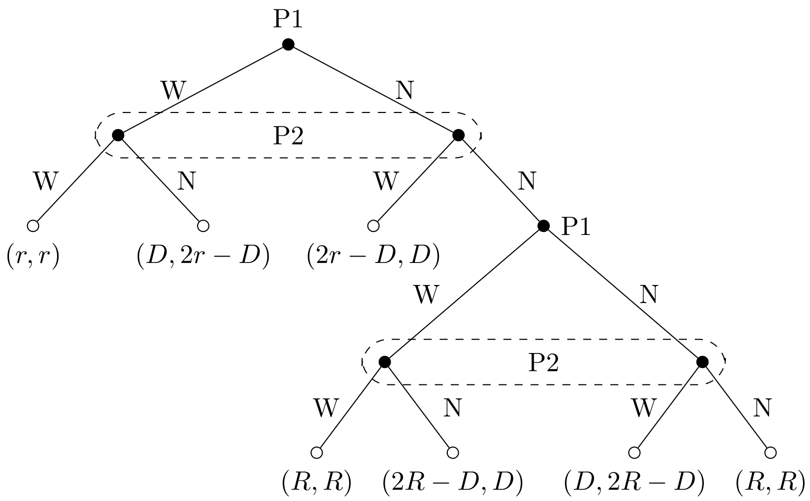

Figure 2.34: Bank Runs

To obtain SPE, we first look at the subgame at date 2. The game could be represented in the matrix form, as follows.

		P2 Withdrawal	P2 Not
P1	Withdrawal	$(\underline{R}, \underline{R})$	$(\underline{2R - D}, D)$
	Not	$(D, \underline{2R - D})$	(R, R)

Figure 2.35: Date 2 Subgame

The Nash equilibrium in this subgame is (Withdrawal, Withdrawal). The payoffs for the players are (R, R). Substituting the payoffs of date 2 subgame into the original game, the original game becomes

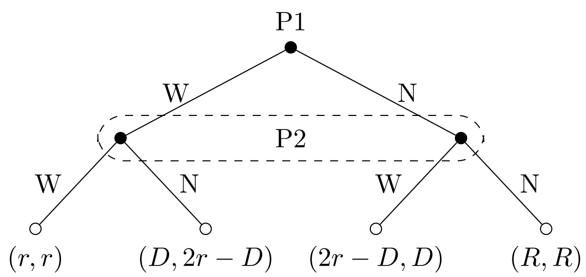

Figure 2.36: Bank Runs

Again, the game is more conveniently represented in the matrix form, as follows.

		P2	
		Withdrawal	Not
P1	Withdrawal	$(\underline{r}, \underline{r})$	$(D, 2r - D)$
	Not	$(2r - D, D)$	$(\underline{R}, \underline{R})$

Figure 2.37: Bank Runs

The game has two pure strategy Nash equilibria:

1. both investors withdraw, leading to payoffs of (r, r);
2. both investors do not withdraw, leading to payoffs of (R, R).

Therefore, the original game has two subgame perfect outcomes:

1. both investors withdraw at date 1, leading to payoffs of (r, r);
2. both investors do not withdraw at date 1 and withdrawal at date 2, leading to payoffs of (R, R).

The first outcome can be interpreted as bank runs. If investor 1 believes that investor 2 will withdraw at date 1, then investor 1's best response is to withdraw as well even though both investors would be better-off if they waited until date 2 to withdraw.

Remark 2.14. There is also a mixed strategy Nash equilibrium for the first-period game in Figure 2.36: $\left((\frac{R-D}{R-r}, \frac{D-r}{R-r}), (\frac{R-D}{R-r}, \frac{D-r}{R-r}) \right)$.
Thus, there is another SPE: $\left(\left[(\frac{R-D}{R-r}, \frac{D-r}{R-r}), R \right], \left[(\frac{R-D}{R-r}, \frac{D-r}{R-r}), R \right] \right)$.

Remark 2.15. Diamond and Dybvig (1983) provide a richer model of bank runs.

2.B.3. Wars of Attrition

Game Setup. Consider the two-period version of Wars of Attrition. Two players choose to "Fight (F)" or "Quit (Q)" in each period. The game ends as soon as at least one of the players chooses Q. Assume that there is no discounting. The payoffs to the players are as follows:

- If one of the players quits first, the player who does not quit win a prize v and the player who quits gets 0;
- If both players quit at once, both get 0;
- At each period in which both players choose F, each player pay a cost $c(< v)$.

Analysis. The game could be represented in a game tree. See Figure 2.38.

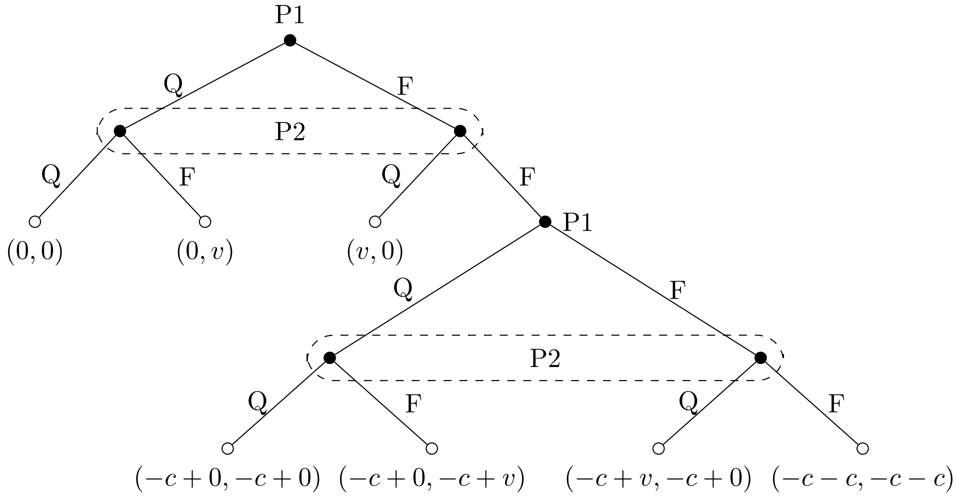

Figure 2.38: Wars of Attrition

To obtain SPE, we first look at the subgame in period 2. The game could be represented in the matrix form, as follows.

		P2	
		Q	F
P1	Q	$(-c+0, -c+0)$	$(\underline{-c+0}, \underline{-c+v})$
	F	$(\underline{-c+v}, \underline{-c+0})$	$(-c-c, -c-c)$

Figure 2.39: Period 2 Subgame

There are **two pure strategy Nash equilibria** in this subgame: (F, Q) and (Q, F). The payoffs for the players are $(-c+v, -c)$ and $(-c, -c+v)$ respectively.

We analyze the two cases separately. Substituting the payoffs from (F, Q) of period 2 subgame into the original game, the original game becomes

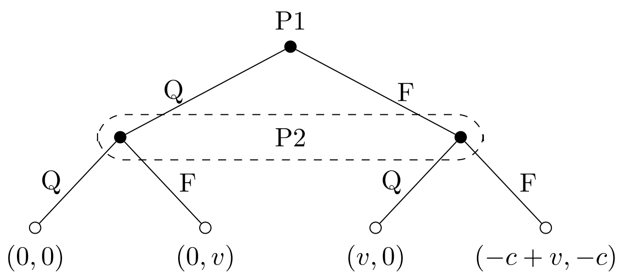

Figure 2.40: Wars of Attrition

Again, the game is more conveniently represented in the matrix form, as follows.

P2

P1		Q	F
	Q	$(0, 0)$	$(0, \underline{v})$
	F	$(\underline{v}, \underline{0})$	$(\underline{-c+v}, -c)$

Figure 2.41: Wars of Attrition

The solution for this first-period game is (F,Q). Thus, we find **a SPE: ([F,F], [Q,Q])**. For the other subgame outcome (Q, F), going through the same analysis, we would obtain **another SPE: ([Q,Q], [F,F])**.

Let us now look for **mixed strategy Nash equilibrium** in the period 2 subgame in Figure 2.39. Note that this subgame is equivalent to the following game plus a sunk cost of $-c$ (since $-c$ is in every entry of the payoff matrix).

P2

P1		Q	F
	Q	$(0, 0)$	$(\underline{0}, \underline{v})$
	F	$(\underline{v}, \underline{0})$	$(-c, -c)$

Figure 2.42: Period 2 Subgame (Net of Sunk Cost $-c$)

Remark 2.16. The cost that has already been incurred and cannot be recovered is sunk cost. Sunk cost is irrelevant when making decisions now. In particular, for our analysis, the equilibrium mix will be the same with and without sunk cost. (You could try to calculate the equilibrium mix with sunk cost and compare with the mix calculated here.)

Let the probability that Player 1 chooses Q be p_Q, then the probability that she chooses F is $p_F = 1 - p_Q$. Recall that we use Player 2's payoffs to solve for Player 1's equilibrium mix. Player 2's expected payoffs from Q and F are respectively:

$$\mathbb{E}U_B(p_Q, Q) = p_Q(0) + (1 - p_Q)(0) = 0$$
$$\mathbb{E}U_B(p_Q, F) = p_Q(v) + (1 - p_Q)(-c) = p_Q v - (1 - p_Q)c$$

In equilibrium,

$$\mathbb{E}U_B(p_Q, Q) = \mathbb{E}U_B(p_Q, F) \implies 0 = p_Q v - (1 - p_Q)c \implies p_Q = \frac{c}{v + c}.$$

So Player 2's equilibrium mix is $(p_Q = \frac{c}{v+c}, p_F = \frac{v}{v+c})$. And Player 2's expected payoff is $\mathbb{E}U_B(p_Q, Q) = 0$.

Since Player 1 and 2 are symmetric in this subgame, Player 1 would also mix with probability $p_Q = \frac{c}{v+c}$ and $p_F = \frac{v}{v+c}$. And Player 1's expected payoff is $\mathbb{E}U_B(p_Q, Q) = 0$.

Adding back the sunk cost $-c$ and substituting the expected payoffs into the original game, the original game becomes

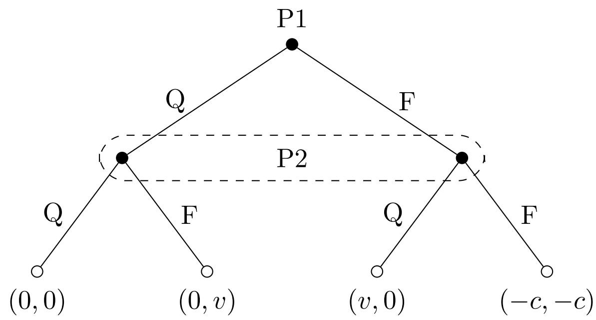

Figure 2.43: Wars of Attrition

This game is exactly the same as the period 2 subgame net of the sunk cost $-c$. So the mixed strategy equilibrium for this game is again $(p_Q = \frac{c}{v+c}, p_F = \frac{v}{v+c})$ for both Player 1 and 2. And each player would obtain an expected payoff of 0. There also exist two pure strategy equilibria for this game: (F, Q) and (Q, F).

Thus, we find **three other SPEs:** $\left(\left[(p_Q^* = \frac{c}{v+c}, p_F^* = \frac{v}{v+c}), (p_Q^*, p_F^*)\right], \left[(p_Q^*, p_F^*), (p_Q^*, p_F^*)\right]\right)$**;** $\left(\left[F, (p_Q^*, p_F^*)\right], \left[Q, (p_Q^*, p_F^*)\right]\right)$**; and** $\left(\left[Q, (p_Q^*, p_F^*)\right], \left[F, (p_Q^*, p_F^*)\right]\right)$**.**

2.C. Repeated Games

Repeated games are games that consist of repetitions of some base game (called a *stage game*). We will first focus on the repeated games with stage games being static games of complete information where the players move simultaneously. We study games where players move sequentially in the stage game in Example 2.C.3 and in Section 2.C.4. When studying repeated games, we are particularly interested in the issue of *cooperation*. As we have learned before, the Prisoners' Dilemma game has a cooperative outcome but this good outcome cannot be sustained when the game is played only once. So we attempt to answer the following question: can repeated interaction induce and sustain cooperative behavior in the Prisoners' Dilemma game and other games in general?

2.C.1. Finitely Repeated Games

We will first learn finitely repeated games, where the stage game is repeated for a fixed number of periods.

Definition 2.C.1. Given a stage game G, let $G(T)$ denote the **finitely repeated game** in which G is played T times with

- the outcomes of all preceding plays observed before the next play begins and
- no discounting: the payoffs of $G(T)$ are simply the sum of the payoffs from the T stage games.

Example 2.C.1. Suppose that the following Prisoner's Dilemma Game is played twice:

		Prisoner 2	
		Cooperate (C)	Defect (D)
Prisoner 1	Cooperate (C)	$(2, 2)$	$(-1, 3)$
	Defect (D)	$(3, -1)$	$(0, 0)$

Figure 2.44: The Prisoners' Dilemma Game

The outcome of the first play is observed before the second play begins. Assume no discounting.

Question 2.10. Could (C, C) be sustained?

Let us find the SPE of the game. In the second stage, what has happened in the first stage is sunk and thus irrelevant in the decision-making process. Therefore, in the second stage, the relevant payoffs are the same as the payoffs in the one-shot game, shown in Figure 2.49. We have already learned that there is a unique Nash equilibrium, namely (D,D), in the Prisoners' Dilemma game. Therefore, SPE requires (D, D) to be played in the second stage.

We then analyze the first-stage game, taking into account the outcome of the second-stage game. In particular, we add to the payoff matrix the payoffs to be obtained in the second stage. See Figure 2.45.

		Prisoner 2: Cooperate (C)	Prisoner 2: Defect (D)
Prisoner 1	Cooperate (C)	$(2+0, 2+0)$	$(-1+0, 3+0)$
	Defect (D)	$(3+0, -1+0)$	$(0+0, 0+0)$

Figure 2.45: The Prisoners' Dilemma Game – First Stage

Notice that the first-stage game is very similar to the original Prisoners' Dilemma game, except that 0 is added to every payoff in the payoff matrix. The Nash equilibrium for the first-stage game is (D, D). This is reasonable: since it is expected that (D, D) would be the outcome of the second stage game independent of the first-stage outcome, the first-stage game is just like a one-shot game.

To sum up, SPE outcome of the two-stage Prisoners' Dilemma game is that both players play D for the two stages.

Question 2.11. What if the Prisoners' Dilemma is played 3 times, 4 times, or more generally, N times?

Remark 2.17. It seems that when the relationship has a known end, cooperation is not sustainable. But this is NOT true.

Example 2.C.2. Suppose that the following game is played twice:

		Player 2: L	Player 2: M	Player 2: R
Player 1	L	$(4, 4)$	$(0, \underline{5})$	$(0, 0)$
	M	$(\underline{5}, 0)$	$(\underline{1}, \underline{1})$	$(0, 0)$
	R	$(0, 0)$	$(0, 0)$	$(\underline{3}, \underline{3})$

Figure 2.46: Cooperation

Again, we assume that the outcome of the first play is observed before the second play begins, and that there is no discounting so that the payoff for the entire game is simply the sum of the payoffs from the two stages.

Question 2.12. Could the good outcome (L, L) be sustained?

It is not hard to find the two pure strategy Nash equilibria of the one-shot game: (M, M) and (R, R). Note that (L, L) is not a Nash equilibrium: the best response to L is M. Since (L, L) is not a Nash equilibrium, we could not sustain (L, L) in the second stage. However, **(L, L) could be sustained in the first stage.** Consider the following strategy:[1]

- In the first stage, play L, and then
- In the second stage,
  - Play R if (L, L) is played in the first stage;
  - Play M otherwise.

Let us check that it is a SPE. In the second stage,

- after (L, L), this strategy induces (R, R). (R, R) is a Nash equilibrium.
- after any other choices, the strategy induces (M, M). (M, M) is a Nash equilibrium.

In the first stage, adding to the original payoffs the payoffs from the second stage, we obtain the following payoff matrix:

		Player 2		
		L	M	R
Player 1	L	$(\underline{4+3}, \underline{4+3})$	$(0+1, 5+1)$	$(0+1, 0+1)$
	M	$(5+1, 0+1)$	$(\underline{1+1}, \underline{1+1})$	$(0+1, 0+1)$
	R	$(0+1, 0+1)$	$(0+1, 0+1)$	$(\underline{3+1}, \underline{3+1})$

Figure 2.47: Cooperation - First Stage

(L, L) is indeed a Nash equilibrium in the first stage.

Remark 2.18. In ongoing relationships, the promise of future rewards and the threat of future punishments may sometimes provide incentives for good behavior today. For this to work, the stage game needs to have more than one Nash equilibrium.

Remark 2.19. The play of different equilibria in the second stage following different first-stage outcomes may seem unreasonable. Here, punishing the deviator involves the punishment of the punisher. There may be a problem of *renegotiation*.

[1] You should check by yourself that it is indeed a strategy.

Example 2.C.3. Let us revisit the entry game in Example 2.A.6. Now suppose that the incumbent is active in N markets. In each market, the incumbent faces a different entrant. That is, the incumbent plays the game in Figure 2.48 with a different entrant each time.

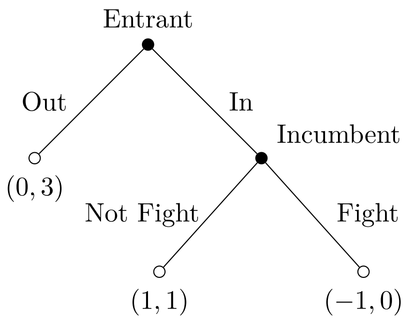

Figure 2.48: Entry Game

The entrants decides whether to enter the market sequentially, and each of the subsequent entrants observes the choices by the previous entrants and by the incumbent.

This game could be solved by backward induction.

1. In the last market, the backward induction outcome is “In” and “Not Fight”.
2. In the second to last market,
   - No matter what happens in this market, the outcome in the last market is “In” and “Not Fight”.
   - Therefore, once the entrant chooses “In”, the incumbent would choose “Not Fight”.
3. The same argument goes through backwards.
4. The BI outcome is: Every entrant chooses “In” and the incumbent “Not Fight”.

Next, suppose that there is 1% chance that the incumbent is crazy and enjoys fighting. For example, the crazy incumbent gets a payoff of 5 when choosing “Fight”.
If there is only one market, then the entrant would still choose “In” and the incumbent would choose “Not Fight” if it is not crazy.
However, the outcome would be different if there are more markets. Suppose that the sane (not crazy) incumbent always chooses “Not Fight”. Then the first entrant, knowing that there is only 1% chance of the incumbent being crazy, enters the market. The entrants in the later market would update their beliefs about the probability of the incumbent being crazy based on their observations in the earlier markets. In this case, if the entrants

observe “Fight” in the first market, then they know that the incumbent is crazy and they would choose to stay out. Then, knowing that the entrants would respond this way, even the sane incumbent would fight in the first market and act as if it is crazy to keep the entrants in the subsequent markets out. Therefore, it is not an equilibrium that the sane incumbent always chooses “Not Fight”.

Remark 2.20. It is also not an equilibrium that the sane incumbent always “Fight”.

Suppose that it is an equilibrium. Then the entrants should not update their beliefs when observing “Fight”. Thus in the last market, the entrant still believes that the incumbent is crazy only with 1% chance. Then the last entrant would choose “In” and the incumbent would choose “Not Fight” if it is not crazy.

2.C.2. Infinitely Repeated Games

Next, we will turn to infinitely repeated games, where the stage game is repeated infinitely. Note that the analysis for the infinitely repeated games would be rather different from that for the finitely repeated games, since there is no last stage in the infinitely repeated games. As it turns out, the result is different too: in the infinitely repeated games, even if the stage game has a unique Nash equilibrium, as in the Prisoners' Dilemma game, there may be subgame-perfect outcomes of the infinitely repeated game in which no stage's outcome is a Nash equilibrium of the stage game.
To see how it works, we will first revisit the Prisoners' Dilemma example.

Example 2.C.4. Suppose that the following Prisoner's Dilemma Game is played infinitely:

		Prisoner 2: Cooperate (C)	Prisoner 2: Defect (D)
Prisoner 1	Cooperate (C)	$(2, 2)$	$(-1, 3)$
	Defect (D)	$(3, -1)$	$(0, 0)$

Figure 2.49: The Prisoners' Dilemma Game

The outcomes of all previous plays are observed before the next play begins. The discount factor is $\delta \in (0, 1)$.

Grim-Trigger Strategy. Consider the following strategy:

- play C in the first period; and
- from the second period onwards
  - play C if no player has played D in the past;
  - play D otherwise.

This strategy is called the *Grim-Trigger Strategy*. The strategy is so called because a player would cooperates until someone fails to cooperates and then a switch to defection forever is triggered.

Question 2.13. Does both players playing *Grim-Trigger Strategy* constitute a subgame perfect equilibrium?

To check whether a strategy profile is a SPE, we need to check whether there exists profitable deviations in every subgame. That is, we fix a subgame and assume that Player 2 has adopted the Grim-Trigger strategy, and then we check whether it is a best response for Player 1 to adopt the Grim-Trigger strategy. Of course, we also need to check the deviation incentive for Player 2, holding fixed Player 1's strategy. But since Player 1 and 2 are symmetric, checking one player is sufficient. We need to do the above analysis for every subgame, and there are infinitely many subgames in the infinitely repeated Prisoners' Dilemma game. It is impossible to check the infinite subgames one by one. The good news is that we could group the subgames. Given that both players follow the Grim-Trigger strategy, all subgames belong to one of the two types:

1. it is the first period or D has never been observed in the past (and the strategy is the same as the initial strategy);
2. D has been observed in the past (and each player plays D in all future periods).

For the second type of subgames, it is not hard to see that both players playing D forever is a SPE. The reason is that given that Player 2 would play D forever, playing D forever gives Player 1 a payoff of 0 in every period and playing C in any period gives Player 1 a payoff of -1 in that period.

For the first type of sugames, the outcome of the subgame would be (C, C) forever, which gives Player 1 a present-discounted payoff of

$$V^C = 2 + \delta \cdot 2 + \delta^2 \cdot 2 + ... = \frac{2}{1-\delta}.$$

If Player 1 deviates to D, then she gains today since now she obtains 3 instead of 2, however, it comes at a cost in the future: the players would both play D forever after the deviation. Such a deviation gives Player 1 a present-discounted payoff of

$$V^D = 3 + \delta \cdot 0 + \delta^2 \cdot 0 + ... = 3.$$

Therefore, for Player 1 not to deviate, we need

$$V^C \geq V^D \implies \delta \geq \frac{1}{3}.$$

Question 2.14. For the first type of subgames, we have only checked a special type of deviation: Player 1 deviates once and then revert to the equilibrium strategy. How about other types of deviations? For example, playing D for one period, followed by C and then playing D forever.

Answer: Other types of deviations is not as profitable as deviating only once.

One-shot Deviation Principle. This result is actually more general. The equivalence of subgame perfection and there being no profitable one-shot deviations is presented in Proposition 2.C.1 below.

Definition 2.C.2. A strategy profile does not have profitable one-shot deviations if no player can increase his payoff in any subgame through a one-shot deviation: a deviation from the strategy profile only in the *first* period of the subgame.

Proposition 2.C.1 (The one-shot deviation principle)**.** A strategy profile is subgame perfect if and only if there are no profitable one-shot deivations.

The proof is not required for this course. If you are interested, please refer to Page 25 of Mailath and Samuelson (2006).[2]

Always Cooperate Strategy. We already know that when $\delta \geq \frac{1}{3}$, both players playing the Grim-Trigger strategy constitutes a SPE, and the equilibrium outcome is that both players play (C, C) forever. Consider the following strategy which also prescribe (C, C) forever as the equilibrium outcome: cooperate in the first period and continue cooperating forever no matter what the other player does. We call this strategy the *Always Cooperate Strategy.*

[2]Mailath, G. J., & Samuelson, L. (2006). *Repeated games and reputations: long-run relationships.* Oxford university press.

Question 2.15. Does both players playing *Alway Cooperate Strategy* constitute a SPE?

We need to check whether there are any profitable one-shot deviations. Given that Player 2 follows the Alway Cooperate strategy,

- If Player 1 also follows the strategy, the outcome would be (C, C) forever. Player 1 receives a present-discounted payoff of

$$V^C = \frac{2}{1-\delta} = 2 + \delta \frac{2}{1-\delta};$$

- If Player 1 deviates to D (one-shot), the outcome would be (D, C) and then (C, C) forever. Player 1 receives a present-discounted payoff of

$$V^D = 3 + \delta \frac{2}{1-\delta}.$$

- Since $V^D > V^C$ for all $\delta \in (0, 1)$, both players playing the Alway Cooperate strategy never constitutes a SPE.

One-period Punishment Strategy. The Grim-Trigger strategy we considered previously may seem too draconian. Consider the more forgiving *One-period Punishment Strategy* as follows:

- play C in the first period; and
- from the second period onwards
  - play C if either (C, C) or (D, D) was played in the last period;
  - play D if either (C, D) or (D, C) was played in the last period.

Question 2.16. Does both players playing *One-period Punishment Strategy* constitute a SPE?

To check SPE, we need to check whether there exists profitable deviations in every subgame. Since the game is symmetric, we only need to consider Player 1. Given that both players follow this strategy, all subgames belong to one of the two types:

1. it is the first period or either (C, C) or (D, D) was played in the last period;
2. either (C, D) or (D, C) was played in the last period.

For the first type of subgames, given that Player 2 follows the strategy,

- if Player 1 also follows the strategy, the outcome would be (C, C) forever.

Player 1 receives a present-discounted payoff of

$$V^C = \frac{2}{1-\delta}.$$

- if Player 1 deviates to D (one-shot), the outcome would be $(D, C), (D, D)$ and then (C, C) forever. Player 1 receives a present-discounted payoff of

$$V^D = 3 + \delta \cdot 0 + \delta^2 \frac{2}{1-\delta}.$$

- For Player 1 not to deviate, we need

$$V^C \geq V^D \implies \delta \geq \frac{1}{2}.$$

For the second type of subgames,

- if Player 1 also follows the strategy, the outcome would be (D, D) and then (C, C) forever. Player 1 receives a present-discounted payoff of

$$V^1 = 0 + \delta \cdot \frac{2}{1-\delta} = 2 + \delta^2 \frac{2}{1-\delta}.$$

- if Player 1 deviates to C (one-shot), the outcome would be $(C, D), (D, D)$ and then (C, C) forever. Player 1 receives a present-discounted payoff of

$$V^2 = (-1) + \delta \cdot 0 + \delta^2 \frac{2}{1-\delta} = -1 + \delta^2 \frac{2}{1-\delta}.$$

- Since $V^1 > V^2$ for all δ, Player 1 has no incentive to deviate.

Therefore, putting together the no-deviation conditions for the two types of subgames, we need $\delta \geq \frac{1}{2}$. Note that the cutoff for the One-period Punishment strategy $(\frac{1}{2})$ is higher than the cutoff for the Grim-Trigger strategy $(\frac{1}{3})$. That is, for a shorter punishment to work, the players need to care more about the future.

Tit-for-Tat Strategy. Tit-for-Tat is also a commonly used strategy when playing the infinitely repeated Prisoner's Dilemma game. In fact, Tit-for-tat is a very effective strategy and it is the winning program of the Alexlrod Tournament.[3]

The *Tit-for-Tat Strategy* is as follows:

- play C in the first period; and

[3] See Axelrod, R. (1980). *More effective choice in the prisoner's dilemma.* Journal of conflict resolution, 24(3), 379-403.

- from the second period onwards
  - play C if the opponent plays C in the last period;
  - play D if the opponent plays D in the last period.

Question 2.17. Does both players playing *Tit-for-Tat Strategy* constitute a SPE?

To check SPE, we need to check whether there exists profitable deviations in every subgame. Since the game is symmetric, we only need to consider Player 1. Given that both players follow the Tit-for-Tat strategy, all subgames belong to one of the four types:

1. it is the first period or (C, C) was played in the last period;
2. (D, D) was played in the last period;
3. (C, D) was played in the last period;
4. (D, C) was played in the last period.

For the first type of subgames, given that Player 2 follows the strategy,

- if Player 1 also follows the strategy, the outcome would be (C, C) forever. Player 1 receives a present-discounted payoff of
$$V^C = \frac{2}{1-\delta}.$$
- if Player 1 deviates to D (one-shot), the outcome would be $(D, C), (C, D)$ repeated forever. Player 1 receives a present-discounted payoff of
$$V^D = 3 + \delta \cdot (-1) + \delta^2 \cdot 3 + \delta^3 \cdot (-1) + ... = \frac{3-\delta}{1-\delta^2}.$$
- For Player 1 not to deviate, we need
$$V^C \geq V^D \implies \delta \geq \frac{1}{3}.$$

For the second type of subgames, given that Player 2 follows the strategy,

- if Player 1 also follows the strategy, the outcome would be (D, D) forever. Player 1 receives a present-discounted payoff of
$$V^1 = 0.$$
- if Player 1 deviates to C (one-shot), the outcome would be $(C, D), (D, C)$ repeated forever. Player 1 receives a present-discounted payoff of
$$V^2 = (-1) + \delta \cdot (3) + \delta^2 \cdot (-1) + \delta^3 \cdot 3 + ... = \frac{-1+3\delta}{1-\delta^2}.$$

- For Player 1 not to deviate, we need

$$V^1 \geq V^2 \implies \delta \leq \frac{1}{3}.$$

For the third type of subgames, given that Player 2 follows the strategy,

- if Player 1 also follows the strategy, the outcome would be $(D, C), (C, D)$ repeated forever.

  Player 1 receives a present-discounted payoff of

$$V^3 = \frac{3 - \delta}{1 - \delta^2}.$$

- if Player 1 deviates to C (one-shot), the outcome would be (C, C) forever.

  Player 1 receives a present-discounted payoff of

$$V^4 = \frac{2}{1 - \delta}.$$

- For Player 1 not to deviate, we need

$$V^3 \geq V^4 \implies \delta \leq \frac{1}{3}.$$

For the fourth type of subgames, given that Player 2 follows the strategy,

- if Player 1 also follows the strategy, the outcome would be $(C, D), (D, C)$ repeated forever. Player 1 receives a present-discounted payoff of

$$V^5 = \frac{-1 + 3\delta}{1 - \delta^2}.$$

- if Player 1 deviates to D (one-shot), the outcome would be (D, D) forever.

  Player 1 receives a present-discounted payoff of

$$V^6 = 0.$$

- For Player 1 not to deviate, we need

$$V^5 \geq V^6 \implies \delta \geq \frac{1}{3}.$$

Therefore, putting together the no-deviation conditions for all four types of subgames, we need $\delta = \frac{1}{3}$.

2.C.3. Collusion between Cournot Duopolists

Let us reconsider the Cournot Duopoly game we learned in Section 1.F.1: Let q_1 and q_2 denote the quantities (of a homogeneous product) produced by firms 1 and 2, respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. Assume that the total cost to a firm with quantity q_i is $C_i(q_i) = cq_i$, where $c < a$. The firms choose their quantities simultaneously.

We have solved this game and find that

- The Nash equilibrium is $(q_1^*, q_2^*) = (\frac{a-c}{3}, \frac{a-c}{3})$.
- Each firm's profit is $\pi_i(q_i^*, q_j^*) = \frac{(a-c)^2}{9}$.

We also know that

- The monopoly quantity is $q^m = \frac{a-c}{2} < q_1^* + q_2^*$.
- The monopoly profit is $\pi(q^m) = \frac{(a-c)^2}{4} > \pi_i(q_i^*, q_j^*) + \pi_j(q_i^*, q_j^*)$.

Now we view the game as the stage game which is played infinitely. The discount factor is δ. Since both firms would be better off if they both produce half the monopoly quantity $\frac{q_m}{2}$ and share the monopoly profit, we are interested in whether the repeated interactions could help the firms achieve this good outcome.

Question 2.18. What is the required value of δ for which it is a SPE for both firms to play the following trigger strategy?

- produce $\frac{q_m}{2}$ in the first period; and
- from the second period onwards,
  - produce $\frac{q_m}{2}$ if both firms have produced $\frac{q_m}{2}$ in each of the previous periods;
  - produce the Cournot quantity, denoted by $q_c = \frac{a-c}{3}$, otherwise.

Given that both firms follow the trigger strategy, all subgames belong to one of the two types:

1. it is the first period or both firms have produced $\frac{q_m}{2}$ in each of the previous period;
2. at least one firm has produced some quantity other than $\frac{q_m}{2}$ in any of the previous period.

Since the two firms are symmetric, it is sufficient to focus on the deviation incentive of Firm 1, holding fixed Firm 2's strategy. For the first type of subgames,

- if Player 1 also follows the strategy, the outcome would be $(\frac{q_m}{2}, \frac{q_m}{2})$ forever.

Player 1 receives a present-discounted payoff of

$$V^C = \frac{1}{1-\delta} \cdot \frac{\pi(q^m)}{2} = \frac{(a-c)^2}{8(1-\delta)}.$$

- if Player 1 wants to make a one-shot deviation, given that Player 2 produces $\frac{q_m}{2}$, the best deviation quantity is solved from the following maximization problem:

$$\max_{q_1}(a - q_1 - \frac{q_m}{2} - c)q_1 \implies q_1 = \frac{3(a-c)}{8}.$$

  This gives a one-shot deviation profit of

$$\pi^d = (a - q_1 - \frac{q_m}{2} - c)q_1 = \frac{9(a-c)^2}{64}.$$

  After the deviation, according to the strategy, the players would revert to setting the Cournot quantity forever and each gets a profit of $\pi^c = \frac{(a-c)^2}{9}$ in every period. In total, Player 1 receives a present-discounted payoff of

$$V^D = \pi^d + \delta\frac{\pi^c}{1-\delta} = \Big[\frac{9}{64} + \frac{\delta}{9(1-\delta)}\Big](a-c)^2.$$

- For Player 1 not to deviate, we need

$$V^C \geq V^D \implies \delta \geq \frac{9}{17}.$$

For the second type of subgames,

- if Player 1 also follows the strategy, the outcome would be (q^c, q^c) forever.
  Player 1 receives a present-discounted payoff of

$$V^1 = \frac{\pi^c}{1-\delta} = \pi^c + \delta\frac{\pi^c}{1-\delta}.$$

- if Player 1 conducts a one-shot deviation to some quantity $\hat{q}$, then compared to choosing the Cournot quantity, she earns less in the current period. Denote the one-shot deviation profit as $\hat{\pi}$ $(< \pi^c)$. Furthermore, according to the strategy, after the one-shot deviation, the players would both set the Cournot quantity forever. Player 1 receives a present-discounted payoff of

$$V^2 = \hat{\pi} + \delta\frac{\pi^c}{1-\delta}.$$

- Since $V^1 > V^2$ for all δ, Player 1 has no incentive to deviate.

Therefore, putting together the no-deviation conditions for the two types of subgames, we need $\delta \geq \frac{9}{17}$.

Remark 2.21. Given a discount factor $\delta < \frac{9}{17}$, the highest quantity that the trigger strategy can support is $q^* = \frac{9-5\delta}{3(9-\delta)}(a-c)$. Notice that

- When $\delta \to \frac{9}{17}$, $q^* \to \frac{q^m}{2}$;
- When $\delta \to 0$, $q^* \to q^c$.

Remark 2.22. Half the monopoly quantity $\frac{q^m}{2}$ can be supported for a wider range of discount factors when both players follow the *Carrot-and-Stick Strategy*:

- produce $\frac{q_m}{2}$ in the first period; and
- from the second period onwards,
  - produce $\frac{q_m}{2}$ if both firms produced $\frac{q_m}{2}$ in the last period or if both firms produced x in the last period;
  - produce x, otherwise.

For example, when $\delta = \frac{1}{2}$, which is lower than $\frac{9}{17}$, both players playing the above carrot-and-stick strategy constitutes a SPE provided that $\frac{3(a-c)}{8} \leq x \leq \frac{(a-c)}{2}$.

2.C.4. Repeated Moral Hazard: Outsource

Stage Game. Suppose that your are running a business and your company is thinking of outsourcing some production to a foreign country with cheaper labor. The normal wage in the foreign country is normalized to be 1. The down side of outsourcing is that it is hard to monitor your worker abroad. If you choose to outsource, you need to make an investment of 1. The return from the investment is 4 if the worker works and 0 if he does not work. You also need to set the wage w. The worker can cheat on you in the following way: he can take your investment of 1 and sell those materials on the market. Then he could go away and just work in his normal job. The game tree is shown in Figure 2.50.

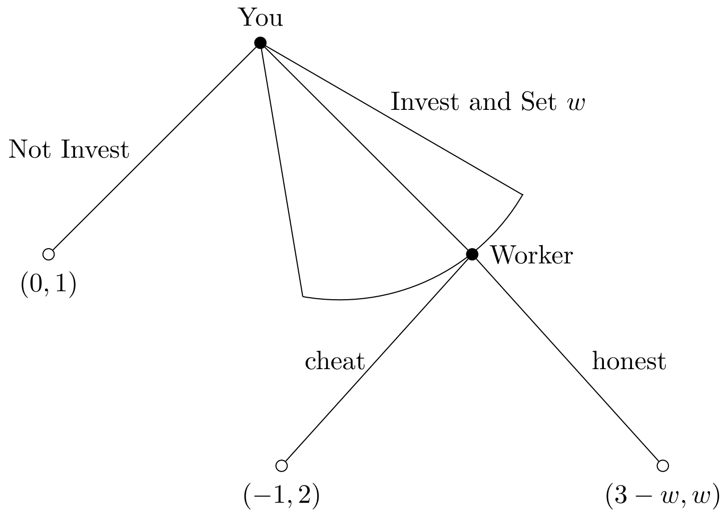

Figure 2.50: Outsourcing

By backward induction, the worker would be honest if and only if $w \geq 2$, and you would invest and set $w = 2$.

Repeated Interaction. Suppose that you will repeatedly invest in the foreign country if the investment works well. The discount factor is $\delta \in (0, 1)$.

Question 2.19. How would you set the wage?

For the worker to work, it requires

$$\frac{w}{1-\delta} \geq 2 + \delta \cdot \frac{1}{1-\delta} \implies w \geq 2 - \delta.$$

Therefore, it is optimal to set $w = 2 - \delta$. Note that this wage level is in-between 1 and 2, i.e., the normal wage in the foreign country and the required wage in the one-shot game.