Chapter 3. Games of Incomplete Information

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Games of Incomplete Information

- In this chapter, we will study games of incomplete information, also called Bayesian games.
- In such games, at least one player is uncertain about another player's payoff function.

3.A. Static Games of Incomplete Information

- In this section, we will study simultaneous-move game of incomplete information, also called static Bayesian game.
- In Section 3.A.1, we will study incomplete information Cournot duopoly model.
- In Section 3.A.2, we will develop normal-form representation of general static Bayesian game and corresponding solution concept Bayesian Nash Equilibrium.
- In Sections 3.A.3 and 3.A.4, we will study two applications on auctions.

3.A.1. Cournot Competition under Asymmetric Information

- Quantities (of a homogeneous product) produced by firms 1 and 2: q_1 and q_2
- Market-clearing price when aggregate quantity is $Q = q_1 + q_2$: $P(Q) = a - Q$.
- Firms choose quantities simultaneously.

Asymmetric Cournot

- Firm 1's cost function is $C_1(q_1) = cq_1$.
- Firm 2's cost function is
  - $C_2(q_2) = c_H q_2$ with probability θ, and
  - $C_2(q_2) = c_L q_2$ with probability $1 - \theta$,

  where $c_L < c_H$.

Asymmetric Cournot

Information is asymmetric:

- Firm 2 knows
  - its own cost function (realization of c_H, c_L) and
  - Firm 1's cost function
- Firm 1 knows
  - its own cost function and
  - only that Firm 2's marginal cost is c_H with probability θ and c_L with probability $1 - \theta$.

Asymmetric Cournot: Analysis

- Naturally, Firm 2 may choose different quantities depending on whether its marginal cost is high or low.
- Moreover, Firm 1 should anticipate this.
- Let
  - $q_2^*(c_H)$ and $q_2^*(c_L)$ denote Firm 2's equilibrium quantity choice;
  - q_1^* denote Firm 1's equilibrium quantity choice.

Asymmetric Cournot: Analysis

Then,

- $q_2^*(c_H)$ solves $\max_{q_2} \Big[(a - q_1^* - q_2) - c_H\Big]q_2$.
- $q_2^*(c_L)$ solves $\max_{q_2} \Big[(a - q_1^* - q_2) - c_L\Big]q_2$.
- q_1^* solves

$$\max_{q_1} \theta\Big[(a - q_1 - q_2^*(c_H)) - c\Big]q_1 +$$

$$(1-\theta)\Big[(a - q_1 - q_2^*(c_L)) - c\Big]q_1$$

$$\implies \max_{q_1}\Big[a - q_1 - \Big(\theta q_2^*(c_H) + (1-\theta)q_2^*(c_L)\Big) - c\Big]q_1$$

Asymmetric Cournot: Analysis

The solution is (assuming that solutions are all positive)

$$
\begin{aligned}
q_2^*(c_H) &= \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L);\\
q_2^*(c_L) &= \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L);\\
q_1^* &= \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}.
\end{aligned}
$$

Remark. $q_2^*(c_H) > \frac{a-2c_H+c}{3}$ and $q_2^*(c_L) < \frac{a-2c_L+c}{3}$: Firm 2 not only tailors its quantity to its cost but also responds to the fact that Firm 1 cannot do so.

3.A.2. Static Bayesian Games and Bayesian Nash Equilibrium

To characterize static Bayesian games, we need to capture the idea that

1. each player knows his/her own payoff function;
2. each player may be uncertain about the other players' payoff functions.

Harsanyi (1967) introduced type spaces to model players' information on payoff-relevant parameters.

Type and Belief.

Player i's payoff functions is represented by

$$u_i(a_1, ..., a_n; t_i),$$

where t_i is called Player i's type.

- $t_i \in T_i$
- T_i is the set of possible types, or type space.

Type and Belief.

For the Cournot competition model in Section 3.A.1,

- Firm 2 has two types and its type space is $T_2 = \{c_L, c_H\}$;
- Firm 1 has only one type and its type space is $T_1 = \{c\}$.

Type and Belief.

Given this definition of types,

1. "Player i knows his/her own payoff function" is equivalent to "Player i knows his/her own type".
2. "Player i may be uncertain about the other players' payoff functions" is equivalent to "Player i maybe uncertain about the types of other players, $t_{-i} = \{t_1, ..., t_{i-1}, t_{i+1}, ..., t_n\}$".

Type and Belief.

- We use T_{-i} to denote the set of possible types of other players.
- We use $p_i(t_{-i}|t_i)$ to denote Player i's belief about t_{-i} when his/her own type is t_i.
- Belief is computed by Bayes' rule from prior probability distribution $p(t)$:

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} = \frac{p(t_{-i}, t_i)}{\sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)}.$$

Bayes' Rule

Example 3.A.1. Consider a two player game with the following prior distribution of types:

		Player 2	
		Type C	Type D
Player 1	Type A	30%	40%
	Type B	10%	20%

Question. What is the posterior probability $p_1(C|A)$?

Example 3.A.2. A certain disease affects about 1 out of 10,000 people. There is a screening test to check whether a person has the disease. The test is quite accurate.

- When a person has the disease, it gives a positive result 99% of the time.
- When a person does not have the disease, it gives a negative result 98% of the time.

Question 3.1. A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?

Normal-form Representation

The normal-form representation of n-player static Bayesian game specifies

- players' action spaces $A_1, ..., A_n$;
- their type spaces $T_1, ..., T_n$;
- their beliefs $p_1(t_{-1}|t_1), ..., p_n(t_{-n}|t_n)$;
- their payoff functions $u_i(a_1, ..., a_n; t_i)$ for all i.[1]

[1] More generally, a player's payoff function could also depend on the other players' types. In this case, we write $u_i(a_1, ..., a_n; t_1, ..., t_n)$.

Normal-form Representation

Following Harsanyi (1967), timing of a static Bayesian game is as follows:

1. nature draws a type vector $t = (t_1, ..., t_n)$ where t_i is drawn from set of possible types T_i;
2. nature reveals t_i to Player i but not to any other player;
3. players simultaneously choose actions, Player i choosing $a_i \in A_i$; and then
4. payoffs $u_i(a_1, ..., a_n; t_i)$ are received.

Normal-form Representation

Remark 3.1. Note that by introducing the fictional moves by nature, incomplete information game is transformed to imperfect information game.

- Here, Player i does not know complete history of game when actions are chosen in Step 3.
- In particular, Player i does not know what nature has revealed to other players.

Definition 3.A.1 (Strategy)**.** In static Bayesian game, a strategy for Player i is a function $s_i(t_i)$ that specifies action $a_i \in A_i$ when type $t_i \in T_i$ is drawn by nature.

Bayesian Nash Equilibrium

For Cournot competition model in Section 3.A.1,

- Firm 2's strategy is $(q_2^*(c_H), q_2^*(c_L))$;
- Firm 1's strategy is q_1^*.

Bayesian Nash Equilibrium

- Next, we define the solution concept in a static Bayesian game, called Bayesian Nash Equilibrium.
- The central idea is the same: each player's strategy must be a best response to the other players' strategies.

Bayesian Nash Equilibrium

Definition 3.A.2 (Bayesian Nash Equilibrium)**.** In the static Bayesian game, the strategies $s^* = (s_1^*, ..., s_n^*)$ are a (pure strategy) Bayesian Nash Equilibrium (BNE) if for each player i and for each of i's type $t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), ...s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), ..., s_n^*(t_n); t) p_i(t_{-i}|t_i).$$

3.A.3. First-Price Sealed-Bid Auction

- We have learned second-price auction in Chapter 1.
- Recall that second-price auction is dominant strategy solvable: bidding one's own valuation is a weakly dominant strategy.
- Now, we will study first-price auction.
- Note that first-price and second-price auctions only differ in winner's payments.
- We will consider a simple version of first-price auction with only two bidders.

First-Price Sealed-Bid Auction

- There is one indivisible good for sale.
- Valuations of two potential buyers are independently drawn from a uniform distribution with support $[0, 1]$.
- Denote Buyer i's valuation by v_i.

First-Price Sealed-Bid Auction

Auction rule is as follows:

- Buyers bid simultaneously and each submits a bid $b_i \in [0, +\infty)$.
- Bidder with the highest bid wins the auction and pays his/her own bid.
- If the two buyers submit the same highest bid, then each of the buyers has 1/2 chance of winning the good. The payment is the highest bid (since there is a tie).

Normal-form Representation

- Buyer i's action: submit a bid $b_i \in A_i \in [0, \infty)$.
- Buyer i's type: her valuation $v_i \in T_i = [0, 1]$.
- Buyer i's belief about Buyer j's type: v_j is uniformly distributed on $[0, 1]$, given any v_i

  (valuations are independent)
- Buyer i's payoff when submitting bid b_i is

$$u_i = \begin{cases} 0 & \text{if } b_i < b_j \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j \\ v_i - b_i & \text{if } b_i > b_j \end{cases}$$

Bayesian Nash Equilibrium

- A strategy for Buyer i is a function $b_i(v_i)$.
- In a Bayesian Nash Equilibrium, Buyer 1's strategy $b_1(v_1)$ is a best response to Buyer 2's strategy $b_2(v_2)$, and vice versa.
- Thus, $b_i(v_i)$ solves

$$\max_{b_i}(v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\}.$$

Bayesian Nash Equilibrium

We focus on symmetric Bayesian Nash equilibrium where the two players adopt the same strictly increasing, continuous and differentiable bidding strategy $b(\cdot)$.

Bayesian Nash Equilibrium

Suppose Buyer j adopts $b(\cdot)$. Then

- $\text{Prob}\{b_i = b(v_j)\} = 0$ since v_j is uniformly distributed and $b(\cdot)$ is strictly increasing.
- $\text{Prob}\{b_i > b(v_j)\} = \text{Prob}\{b^{-1}(b_i) > v_j\} = b^{-1}(b_i)$ since v_j is uniformly distributed on $[0, 1]$ and $b(\cdot)$ is strictly increasing, continuous and differentiable.

So Buyer i solves

$$\max_{b_i} (v_i - b_i) b^{-1}(b_i).$$

Bayesian Nash Equilibrium

- Use FOC.
- Recognizing solution is $b_i = b(v_i)$ (symmetric Bayesian Nash equilibrium)
- Solve and get $b(v_i)v_i = \frac{1}{2}v_i^2 + c$.
- Use boundary condition $b(0) = 0 \implies c = 0$.
- Thus, equilibrium bidding strategy is

$$b(v_i) = \frac{1}{2}v_i.$$

3.A.4. Common Value Auction

In a common value auction, the value of good for sale is the same for all bidders.

- "Oil well" is an often cited example of common value auctions.

Jar of coins

Let us play the following auction game.

- There is a jar with some coins.
- Every bidder bids for the coins in the jar.
- Rules of the auction are as follows:
  - Do not open the jar.
  - The winner is the bidder with the highest bid.
  - The winner pays his/her own bid and gets the coins in the jar.

This is a common value auction: the amount of money in the jar is certain.

Jar of coins

Question 3.2. What is your bidding strategy?

Should you bid less or more than your estimate?

Winner's curse

In common value auctions, winning bid tends to be higher than true value of the good. Such a phenomenon is called winner's curse.

Winner's curse

Question 3.3. Why winner's curse exists?

Winner's curse

- Let v be the common value, and b_i be Bidder i's bid.
- Then Bidder i's payoff is

$$\begin{cases} v - b_i & \text{if } b_i \text{ is the highest bid;} \\ 0 & \text{otherwise.} \end{cases}$$

Winner's curse

Bidders only have estimates of value of the good.

- Let y_i be Player i's estimate: $y_i = v + \tilde{\varepsilon}_i$, where $\tilde{\varepsilon}_i$ is Bidder i's estimation error.
- y_i is also Bidder i's type.
- Suppose that on average bidders estimate correctly.
- If bidders bid roughly the same as their estimate, winner would be bidder with largest $\tilde{\varepsilon}_i$.
- Then, winning bid would be higher (actually much higher) than true value.

Winner's curse

Question 3.4. After learning winner's curse, how should you bid?

Winner's curse

1. If everyone bids roughly their own estimates, then when you (Player i) win, you know that $y_j < y_i$ for all j.
2. You only care how many coins are in the jar if you win.

So, you should bid based

- not only on your initial estimate y_i;
- but also on the fact that $y_i > y_j$ for all j.

Put differently, you should bid as if you know you win.

3.B. Dynamic Games of Incomplete Information

We will study three specific models of dynamic games of incomplete information:

- asymmetric information Cournot model with verifiable information in Section 3.B.1,
- job market signaling model in Section 3.B.2 and
- a screening model in Section 3.B.3.

Perfect Bayesian Equilibrium

Solution concept associated with dynamic games of incomplete information is Perfect Bayesian Equilibrium (PBE).

- PBE was invented in order to refine BNE in a similar way that SPE refined NE.
- We will not study PBE in detail in this course.
- Definition of PBE is in lecture note for your reference.
- In essence, PBE requires
  - strategies to be best responses given the belief system and
  - beliefs to be consistent with strategy profile.

3.B.1. Asymmetric Cournot with Verifiable Information

- Quantities (of a homogeneous product) produced by firms 1 and 2: q_1 and q_2
- Market-clearing price when aggregate quantity is $Q = q_1 + q_2$: $P(Q) = a - Q$.
- Firms choose quantities simultaneously. (Cournot model)

Asymmetric Cournot with Verifiable Information

Firm 1's cost function is

$$C_1(q_1) = c_M q_1.$$

Firm 2's cost function is

$$C_2(q_2) = \begin{cases} c_H q_2 = (c_M + x) q_2 & \text{with probability } 1/3 \\ c_M q_2 & \text{with probability } 1/3 \\ c_L q_2 = (c_M - x) q_2 & \text{with probability } 1/3 \end{cases}$$

Asymmetric Cournot with Verifiable Information

Information is asymmetric:

- Firm 2 knows
  - its own cost function (realization of c_H, c_M, c_L) and
  - Firm 1's cost function
- Firm 1 knows
  - its own cost function and
  - only that Firm 2's marginal cost is c_H, c_M or c_L, each with 1/3 probability.

Asymmetric Cournot with Verifiable Information

Before the firms choose quantities, Firm 2 can costlessly and verifiably reveal its cost information to Firm 1.

Asymmetric Cournot with Verifiable Information

Question 3.5. Should Firm 2 reveal its cost information?

Asymmetric Cournot with Verifiable Information

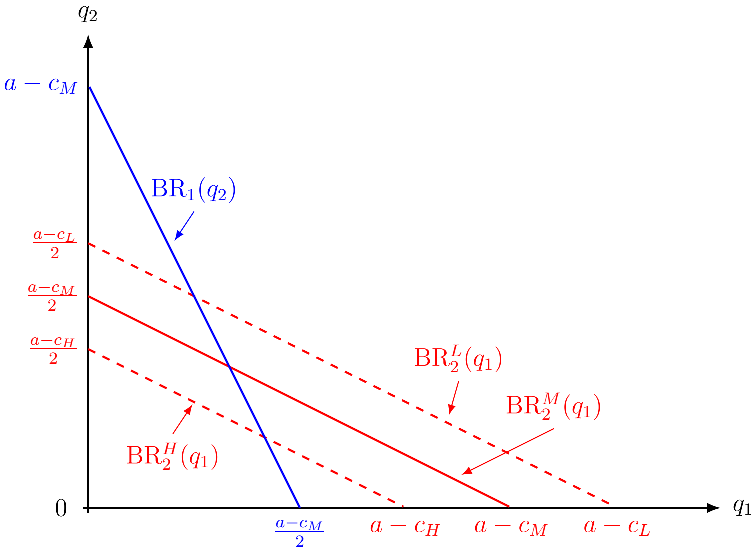
Perhaps it is easier to first consider the following question:

Question 3.6. Would Firm 2 want Firm 1 to know if it has high, middle, or low cost?

Asymmetric Cournot with Verifiable Information

- In Cournot model, one firm's profit would be higher if the other firm produces less.
- Result is: compared to not knowing Firm 2's cost, Firm 1 produces less (more) if it knows that Firm 2 has low (high) cost.
- Firm 2 would want Firm 1 to know if it has low cost.
- Firm 2 with low cost would reveal its cost information.

Asymmetric Cournot with Verifiable Information



Asymmetric Cournot with Verifiable Information

- The argument is not over.
- Let us now consider whether Firm 2 should reveal its cost information when it has middle cost.
- If Firm 2 doesn't reveal that it has middle cost, then Firm 1 knows that cost is not low.
  - Firm 2 would reveal its cost information if it has low cost, as is argued previously.
- Put it differently, Firm 1 knows that the cost is either middle or high.

Asymmetric Cournot with Verifiable Information

- As a result, Firm 2 with middle cost would want Firm 1 to know it so that Firm 1 would produce less.
- Firm 2 with middle cost would also reveal its cost information.

Asymmetric Cournot with Verifiable Information

For Firm 2 with high cost, it really doesn't matter whether it reveals or not.

- Even if it does not reveal, since Firm 2 with middle or low costs would reveal, the fact of no revealing reveals that Firm 2 has high cost.

Remark 3.2. The same argument goes through if Firm 2 has more types.

This idea is called information unraveling.

3.B.2. Job-Market Signaling

- Suppose that there are two types of workers, high-ability and low-ability.
- They differ in productivity: high-ability worker has productivity of 100 whereas low-ability worker has productivity of 60.
- In the population, 20% of workers are high-ability and 80% are low-ability.

	Productivity	Proportion
High-ability Worker	100	20%
Low-ability Worker	60	80%

Job-Market Signaling

- Suppose that firms are competitive.
- Firms would offer 100 to a high-ability worker and 60 to a low-ability worker if they could identify worker's types.
- If firms cannot identify worker's types, they would offer $100 * 20\% + 60 * 80\% = 68$.

Question 3.7. Suppose that you are a high-ability worker, how can you make the firms know it?

In particular, would it work if you simply tell the firms "I am a high-ability worker"?

Job-Market Signaling

- Spence (1973) brings up the idea that "education" could be used as a costly signal to differentiate high-ability workers from low-ability ones.
- The crucial assumption in Spence's model is that low-ability workers find education more costly than high-ability workers.
- Assume

	Cost
High-ability Worker	9
Low-ability Worker	21

Job-Market Signaling

We argue that there exists an equilibrium where

- High-ability workers take three-year graduate education but low-ability workers do not.
- Employers identify those workers with graduate degrees as high-ability workers and those without degrees as low-ability workers.
  - Employers offer 100 to a worker with degree and 60 to a worker without degree.

Job-Market Signaling

Recall that in essence, PBE requires

- strategies to be best responses given belief system and
- beliefs to be consistent with strategy profile.

For this particular game, we need to check

1. Both types of workers would not deviate in their respective education choices.
2. Employers' beliefs are consistent with the equilibrium behavior.

The second point is obvious.

Job-Market Signaling

For the first point,

- A high-ability worker obtains $100 - 9 * 3 = 73$ if he/she takes education and 60 if not.
- A low-ability worker obtains $100 - 21 * 3 = 37$ if he/she takes education and 60 if not.

Thus, a high-ability worker would not deviate to not taking education and a low-ability worker would not deviate to taking education.

Remark 3.3. This is called a separating equilibrium because in equilibrium the types separate and get identified.

Question 3.8. What is the education program only takes two years? How about one year?

Remark 3.4. For separation to work, there must be enough differences in costs for two types of workers.

Remark 3.5. If standard of obtaining education becomes lower, then probably we will see qualification inflation.

Remark 3.6. Education increases inequality: Compared to the no education outcome, a three-year education program makes high-ability workers better-off ($73 > 68$) and low-ability workers worse-off ($60 < 68$).

Remark 3.7. It is possible that high-ability workers are also worse-off.

- To see this, consider a four-year education program.
- In separating equilibrium, no education is interpreted as evidence of low ability.

3.B.3. Screening

- In the last section, we have seen a signaling model in which informed parties (i.e., workers) move first.
- Signaling models are closely related to screening models, in which uninformed parties take the lead.
- Classic references of screening models concern insurance markets.
- But in this course, we still take job market as example.

Job Market Example

Now consider the following timing, which corresponds to a screening setting:

1. Two firms simultaneously announce a menu of contracts specifying required years of education and wage offer (e, w).
2. Given these contracts, workers choose which contract to accept, if any.

Job Market Example

Question 3.9. Is it an equilibrium that both firms offer the same two contracts

$$(e_H = 3, w_H = 100) \text{ and } (e_L = 0, w_L = 60)?$$

Job Market Example

- For workers, similar arguments as in signaling model apply.
- Both types of workers would self-select the contracts designed for them.
  - A high-ability worker obtains $100 - 9 * 3 = 73$ if he/she takes contract $(e_H = 3, w_H = 100)$ and 60 if takes $(e_L = 0, w_L = 60)$.
  - A low-ability worker obtains $100 - 21 * 3 = 37$ if he/she takes contract $(e_H = 3, w_H = 100)$ and 60 if takes $(e_L = 0, w_L = 60)$.

Job Market Example

- In the proposed equilibrium, each firm obtains 0.
- A firm could be better-off by offering $(e'_H = 2, w'_H = 95)$ and $(e_L = 0, w_L = 60)$.
  - High-ability workers prefer $(e'_H = 2, w'_H = 95)$ to $(e_H = 3, w_H = 100)$: they obtain $95 - 9 * 2 = 77$ (> 73) if taking $(e'_H = 2, w'_H = 95)$.
  - Low-ability workers would not take $(e'_H = 2, w'_H = 95)$: they obtain $95 - 21 * 2 = 53$ (< 60) if taking $(e'_H = 2, w'_H = 95)$.

The firm obtains $(100 - 95) * 20\% = 1 > 0$.

Job Market Example

Question 3.10. How about both firms offer the same two contracts $(e_H = 2, w_H = 100)$ and $(e_L = 0, w_L = 60)$?

Remark 3.8. Separating equilibria do not always exist. For example, if we change the proportion of high-ability workers to 80%, then there will be no separating equilibria.