Statistics and Prediction

Assignment 1 (Solution)

Exercise 6.1 (p.83) The basic model in Subsection 6.2

Let $n = 2$. Modify the model by letting the success rate of firm 2 be $\alpha_2 > \alpha = \alpha_1 = \alpha_0$.

1. Derive the firms' Nash equilibrium profits. Show that firm 1's profit is independent of α_2. (Hint: When the two firms have different success rates, their equilibrium pricing strategies have the same support, but firm 2's cdf has an atom on $p = 1$.)

2. Suppose that in a stage prior to the price competition game, firms choose their success rates simultaneously and at no cost. Which profiles of success rates are consistent with sub-game perfect equilibrium in this two-stage game?

Solution. **Part 1:**

Claim. There exists a Nash Equilibrium where Firm 1 play the mixed strategy given by the *cdf*:

$$G_1(p) = \frac{1}{\alpha_1} - \frac{1-\alpha_1}{\alpha_1 p}$$

defined over the support $[1-\alpha_1, 1]$; Firm 2 mixes according to the distribution function

$$G_2(p) = \begin{cases} \frac{1}{\alpha_2} - \frac{1-\alpha_1}{\alpha_2 p} & \text{for } p \in [1-\alpha_1, 1) \\ 1 & \text{for } p = 1. \end{cases}$$

That is, $G_2(p)$ places an atom on $p = 1$: $\Pr(p = 1) = 1 - \frac{\alpha_1}{\alpha_2}$.

It is not hard to check that the above construction constitutes a Nash Equilibrium. We will show below that the firms are indifferent among the prices within the support and prices different from those in the support generates strictly lower profit.
For Firm 2, the profit from choosing p is

$$\pi_2 = p \cdot \alpha_2 \cdot (1-\alpha_0) \cdot [1 - \alpha_1 G_1(p)] = \alpha_2(1-\alpha_0)(1-\alpha_1) = \alpha_2(1-\alpha)^2.$$

For Firm 1, the profit from choosing p for some $p \in [1-\alpha_1, 1)$ is

$$\pi_1 = p \cdot \alpha_1 \cdot (1-\alpha_0) \cdot [1-\alpha_2 G_2(p)] = \alpha_1(1-\alpha_0)(1-\alpha_1) = \alpha(1-\alpha)^2.$$

Actually, Firm 1 would not choose $p=1$ since at $p=1$, it needs to share the profit with Firm 2, explicitly,

$$\pi_1(1) = 1 \cdot \alpha_1 \cdot (1-\alpha_0) \cdot \left[(1-\alpha_2) + \alpha_2\left(1-\frac{\alpha_1}{\alpha_2}\right)\frac{1}{2}\right] = \alpha(1-\alpha)(1-\frac{1}{2}\alpha_2 - \frac{1}{2}\alpha) \underbrace{<}_{\alpha_2>\alpha} \pi_1.$$

Deviating to other prices are not profitable since deviating to a lower price $p < 1-\alpha_1$ would induce the same probability of winning as $p = 1-\alpha_1$ does, however, the profit is lower since the price is lower; deviating to a price above 1 would cause no customers purchasing its product.

Therefore, this is a Nash equilibrium and firms' Nash equilibrium profits are $\pi_1 = \alpha(1-\alpha)^2$ (firm 1) and $\pi_2 = \alpha_2(1-\alpha)^2$ (firm 2). It is clear that firm 1's profit is independent of α_2.

Part 2: We assume that after the first stage, the success rates chosen by the two firms are observable by the other firm.

We could find two profiles of success rates that are consistent with sub-game perfect equilibrium in this two-stage game as follows:

1. From Part 1, we have

$$\pi_1 = \alpha_1(1-\alpha_0)(1-\alpha_1) \quad \text{and} \quad \pi_2 = \alpha_2(1-\alpha)^2$$

   Firm 1 setting $\alpha_1 = \frac{1}{2}$ and firm 2 setting $\alpha_2 = 1$ is potentially consistent with the sub-game perfect equilibrium in this two-stage game.

   We claim that this is an equilibrium. On the equilibrium path, in stage 2, the firms play the equilibrium stated in Part 1. This equilibrium is supported by the off-equilibrium path:

   a) if $\alpha_1 \neq \alpha_2$, the firms play the equilibrium stated in Part 1.

b) if $\alpha_1 = \alpha_2 \neq 1$, the firms play the symmetric mixed strategy equilibrium.

c) if $\alpha_1 = \alpha_2 = 1$, the firms play the pure strategy equilibrium $p_1 = p_2 = 0$.

We need to ensure that the off-equilibrium paths themselves constitutes an equilibrium in stage 2. 1a and 1b are already shown to be equilibria. 1c constitutes an equilibrium in stage 2 since now both firms are always successful, so there will be no clientele if a firm deviates to a higher price $p_i > 0$.

It should not be hard to check that for both firm 1 and firm 2, deviating to other success rates would not generate higher profit. Please check by yourself.

2. Symmetrically, firm 1 choosing $\alpha_1 = 1$ and firm 2 choosing $\alpha_2 = \frac{1}{2}$ also constitutes an equilibrium.

Exercise 6.2 (p.86) Spurious Product Differentiation in Subsection 6.3

Construct an asymmetric mixed-strategy Nash equilibrium when $n > m$ and both n and m are even.

Solution.

Since both n and m are even, the simplist way to construct the equilibrium is to divide the firms and actions both into two groups.

One group of firms only choose from one group of actions; the other group of firms only choose from the other group of actions.

Within one group, firms play the symmetric mixed-strategy equilibrium as shown in Proposition 2 (with slight modifications on firm and action numbers).

Exercise 6.3 (p.86) Spurious Product Differentiation in Subsection 6.3

Let $A = \{a_1, a_2\}$. Assume that with probability α (or $(1-\alpha)$) the action a_1 (or a_2) alone satisfies the consumer's need. Assume $\alpha > \frac{1}{2}$. Consider a symmetric Nash equilibrium in which firms randomize over prices and recommendations independently. What is the equilibrium probability that the sub-optimal action a_2 is recommended? What happens to this probability as n tends to infinity?

Solution.

Suppose that each firm chooses a_1 with probability q, and a_2 with probability $1-q$. Suppose also that each firm randomizes independently over prices according to the *cdf* $G(p)$ over the support $[p_L, p_H]$. $G(p)$ must be continuous with $p_H = 1$.

When a_1 is chosen, the firm gets

$$\pi_1(p) = \alpha \left[1 - qG(p)\right]^{n-1} p.$$

The firm is indifferent when it randomizes over p, so

$$\pi_1 = \pi_1(1) = \alpha \left(1-q\right)^{n-1}.$$

Similarly, when a_2 is chosen, the firm gets

$$\pi_2(p) = (1-\alpha)\left[1-(1-q)G(p)\right]^{n-1} p = (1-\alpha)q^{n-1} = \pi_2.$$

Since the firm mixes between a_1 and a_2, it must be indifferent between choosing a_1 and a_2:

$$\pi_1 = \pi_2 \implies q = \frac{1}{1 + \left(\frac{1-\alpha}{\alpha}\right)^{\frac{1}{n-1}}}$$

The equilibrium probability that the sub-optimal action a_2 is recommended is

$$1-q = \frac{1}{1 + \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{n-1}}}.$$

When $n \to \infty$,

$$\lim_{n\to\infty}(1-q) = \lim_{n\to\infty}\left[\frac{1}{1 + \left(\frac{\alpha}{1-\alpha}\right)^{\frac{1}{n-1}}}\right] = \frac{1}{2}.$$