Chapter 3. Classical Demand Theory

(Part 2)

Xiaoxiao Hu

3.D. Utility Maximization Problem (UMP) (Continued)

We return to Chapter 3, specifically, p.53 of Section 3.D.

The utility maximization problem:

$$\max_{x \in \mathbb{R}^L} u(x)$$

$$\text{s.t. } \sum_{l=1}^{L} p_l \cdot x_l = p \cdot x \leq w,$$

$$x_l \geq 0 \text{ for all } l = 1, ..., L.$$

Utility Maximization Problem (UMP)

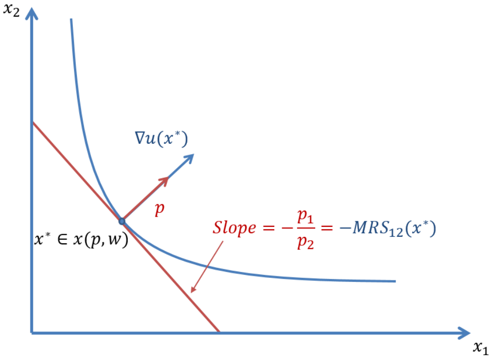
- Lagrange Function:

$$\underset{x \in \mathbb{R}^L_+, \lambda}{\mathcal{L}(x, \lambda)} = u(x) - \lambda(p \cdot x - w).$$

- Kuhn-Tucker conditions

Interior Solution

$$\nabla u(x^*) = \lambda p. \tag{3.D.4}$$



Interior Solution

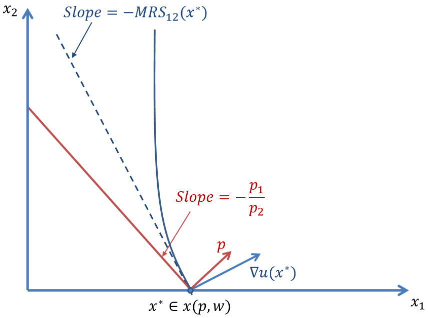
For any two goods l and k, we have

$$\frac{\partial u(x^*)/\partial x_l}{\partial u(x^*)/\partial x_k} = \frac{p_l}{p_k}. \tag{3.D.5}$$

$\frac{\partial u(x^*)/\partial x_l}{\partial u(x^*)/\partial x_k}$ is the *marginal rate of substitution of good* l *for good* k *at* x^*, $MRS_{lk}(x^*)$.

Boundary Solution

- $\partial u(x^*)/\partial x_l \leq \lambda p_l$ for those l with $x_l^* = 0$;
- $\partial u(x^*)/\partial x_l = \lambda p_l$ for those l with $x_l^* > 0$.



The constraint $p \cdot x \leq w$.

- If $p \cdot x = w$, then λ measures the marginal, or shadow, value of relaxing the constraint $p \cdot x = w$, or the consumer's *marginal utility of wealth*.

- If $p \cdot x < w$, then the budget constraint is not binding. In this case, relaxing the budget doesn't increase utility, so $\lambda = 0$.

Utility Maximization Problem

Example 3.D.1. Derive Walrasian Demand Function for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Indirect Utility Function

For each $(p, w) \gg 0$, the utility value of UMP (i.e., $u(x^*)$) is denoted $v(p, w) \in \mathbb{R}$. $v(p, w)$ is called the *indirect utility function*.

Indirect Utility Function

Example 3.D.2. Derive the indirect utility function for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Indirect Utility Function

Proposition 3.D.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. $v(p, w)$ is*

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and nonincreasing in p_l for any l.

(iii) Quansiconvex; that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for any $\bar{v}$.

(iv) Continuous in $p \gg 0$ and w.

3.E. Expenditure Minimization Problem (EMP)

The expenditure minimization problem:

$$\min_{x \in \mathbb{R}^L} p \cdot x$$

$$\text{s.t. } u(x) \geq u \quad \& \quad x \geq 0.$$

The problem is equivalent to

$$\max_{x \in \mathbb{R}^L} \; - p \cdot x$$

$$\text{s.t. } - u(x) \leq -u \quad \& \quad x \geq 0.$$

Expenditure Minimization Problem

- Lagrange Function:

$$\underset{x\in\mathbb{R}^L_+,\lambda}{\mathcal{L}(x,\lambda)} = -p\cdot x - \lambda(-u(x)+u)$$

- Kuhn-Tucker conditions

UMP and EMP

- UMP computes the maximal level of utility that can be obtained given wealth w.

- EMP computes the minimal level of wealth required to reach utility level u.

- The two problems are "dual" problems: they capture the same aim of efficient use of consumer's purchasing power.

UMP and EMP

Proposition 3.E.1. *Suppose* $u(\cdot)$ *is a continuous utility function representing a locally nonsatiated preference relation* $\succsim$ *defined on the consumption set* $X = \mathbb{R}^L_+$ *and that the price vector is* $p \gg 0$. *We have*

(i) If x^* *is optimal in the UMP when wealth is* $w > 0$, *i.e.,* $x^* = x(p, w)$, *then* x^* *is optimal in the EMP when the required utility is* $u(x^*)$. *Moreover, the minimized expenditure in the EMP is* w.

UMP and EMP

Proposition 3.E.1 (continued).

*(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility in the UMP is u. (*No excess utility)*

The Expenditure Function

Let x^* be the/a solution to the EMP. Then $p{\cdot}x^*$ is the minimized expenditure. Let this be called the *Expenditure Function* and denoted by $e(p, u)$.

The Expenditure Function

Proposition 3.E.2. *Suppose that $u(\cdot)$ is a continuous utility representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. $e(p, u)$ is*

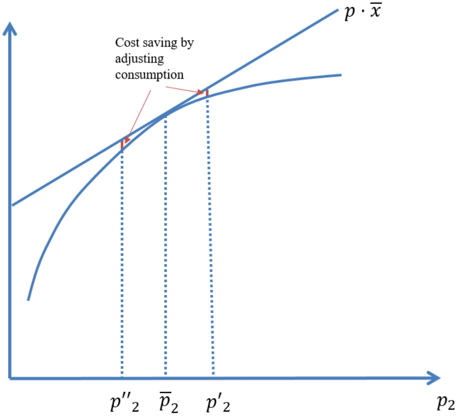
(i) Homogeneous of degree one in p.

(ii) Strictly increasing in u and nondecreasing in p_l for all l.

(iii) Concave in p, i.e., $\alpha e(p, u) + (1 - \alpha)e(p', u) \leq e(ap + (1 - \alpha)p', u)$.

(iv) Continuous in $p \gg 0$ and u.

Intuition of Concavity of $e(p, u)$.



Relationship between $e(p, u)$ and $v(p, w)$

$$e(p, v(p, w)) = w \quad \text{and} \quad v(p, e(p, u)) = u \qquad (3.E.1)$$

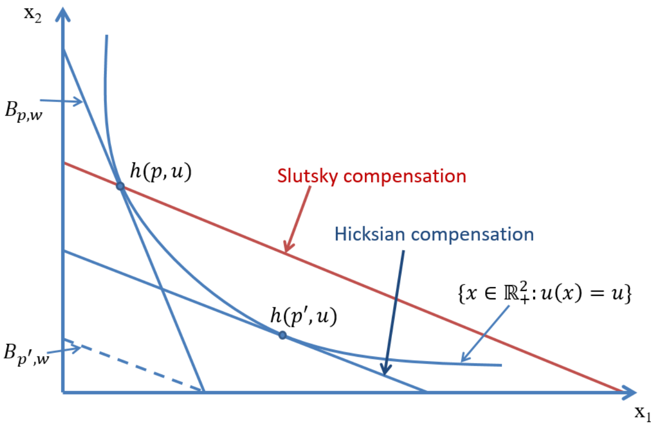
Hicksian (or Compensated) Demand Function

- The optimal bundle in EMP is denoted as $h(p,u) \subset \mathbb{R}^L_+$ and is called the *Hicksian (or Compensated) demand function/ correspondence*.

- As prices vary, $h(p,u)$ gives the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility level at u.

- This type of wealth compensation is called *Hicksian wealth compensation*.

Hicksian (or Compensated) Demand Function



Hicksian (or Compensated) Demand Function

Proposition 3.E.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ (i.e., expenditure minimizing demand) possesses the following properties:*

(i) Homogeneity of degree zero in p: $h(\alpha p, u) = h(p, u)$ for all p, u and $\alpha > 0$.

(ii) No excess utility: For any $x \in h(p, u)$, $u(x) = u$.

Hicksian (or Compensated) Demand Function

Proposition 3.E.3 (continued).

(iii) *Convexity/uniqueness: If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is strictly convex, then there is a unique element in $h(p, u)$.*

Hicksian and Walrasian demand

$$h(p, u) = x(p, e(p, u)) \quad \text{and} \quad x(p, w) = h(p, v(p, w)) \tag{3.E.4}$$

Hicksian Demand and the Compensated Law of Demand

Proposition 3.E.4. *Suppose that* $u(\cdot)$ *is a continuous utility function representing a locally nonsatiated preference relation* $\succsim$ *and that* $h(p,u)$ *consists of a single element for all* $p \gg 0$. *Then the Hicksian demand function* $h(p,u)$ *satisfies the compensated law of demand: for all* p' *and* p'',

$$(p'' - p') \cdot [h(p'', u) - h(p'.u)] \leq 0. \qquad (3.E.5)$$

Hicksian Demand and Expenditure Function

Example 3.E.1. Suppose $p \gg 0$ and $u > 0$. Derive the Hicksian Demand and Expenditure Functions for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

3.G. Relationships between Demand, Indirect Utility, and Expenditure Functions

This section concern three relationships:

- Hicksian Demand Function & Expenditure Function;
- Hicksian & Walrasian Demand Functions;
- Walrasian Demand Function & Indirect Utility Function.

Hicksian Demand and Expenditure Function

Proposition 3.G.1. *Suppose that $u(\cdot)$ is continuous, representing locally nonsatiated and strictly convex preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. For all p and u,*

$$h(p, u) = \nabla_p e(p, u).$$

- We will introduce a useful mathematical result called *the Envelope Theorem*.

Hicksian Demand and Expenditure Function

Example. Verify $h(p, u) = \nabla_p e(p, u)$ for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Hicksian Demand

Proposition 3.G.2. *Suppose* $u(\cdot)$ *is continuous utility function representing a locally nonsatiated and strictly convex* $\succsim$ *on* $X = \mathbb{R}^L_+$. *Suppose* $h(p,u)$ *is continuously differentiable at* (p,u), *and denote the* $L \times L$ *derivative matrix by* $D_p h(p,u)$. *Then*

(i) $D_p h(p,u) = D_p^2 e(p,u)$.

(ii) $D_p h(p,u)$ *is negative semidefinite.*

(iii) $D_p h(p,u)$ *is symmetric.*

(iv) $D_p h(p,u)p = 0$.

Hicksian Demand

Remark 1. Negative semidefiniteness of $D_p h(p,u)$ is the differential analog of compensated law of demand (3.E.5).

Remark 2. Symmetry of $D_p h(p,u)$ is not obvious at all ex ante. It's only obvious after we know that $h(p,u) = \nabla_p e(p,u)$.

Remark 3. Two goods l and k are called *substitutes* at (p,u) if $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$; and *complements* at (p,u) if $\frac{\partial h_l(p,u)}{\partial p_k} \leq 0$. Since $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$, there must exist a good k such that $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$; that is, every good has at least one substitute.

Proposition 3.G.3 (The Slutsky Equation). *Suppose that* $u(\cdot)$ *is a continuous utility function representing a locally nonsatiated and strictly convex* $\succsim$ *on* $X = \mathbb{R}^L_+$. *Then for all* (p, w), *and* $u = v(p, w)$, *we have*

For all l, k,

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

or

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$$

Hicksian and Walrasian Demand Functions

Remark. Recall,

- Slutsky compensation: $\Delta w_{\text{Slutsky}} = p' \cdot x(\bar{p}, \bar{w}) - \bar{w}$;
- Hicksian Compensation: $\Delta w_{\text{Hicksian}} = e(p', \bar{u}) - \bar{w}$.

In general, $\Delta w_{\text{Hicksian}} \leq \Delta w_{\text{Slutsky}}$. We have just shown that for a differential change in price, Slutsky and Hicksian compensations are identical. This observation is useful because the RHS terms are directly observable.

Hicksian and Walrasian Demand Functions

Example. Verify the Slutsky equation for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Walrasian Demand and Indirect Utility Function

Proposition 3.G.4 (Roy's Identity). *Suppose that* $u(\cdot)$ *is A continuous utility function representing a locally nonsatiated and strictly convex* $\succsim$ *on* $X = \mathbb{R}^L_+$. *Suppose also that the indirect utility function is differentiable at* $(\bar{p}, \bar{w}) \gg 0$.

Then

$$x(\bar{p}, \bar{w}) = -\frac{1}{\nabla_w v(\bar{p}, \bar{w})} \nabla_p v(\bar{p}, \bar{w})$$

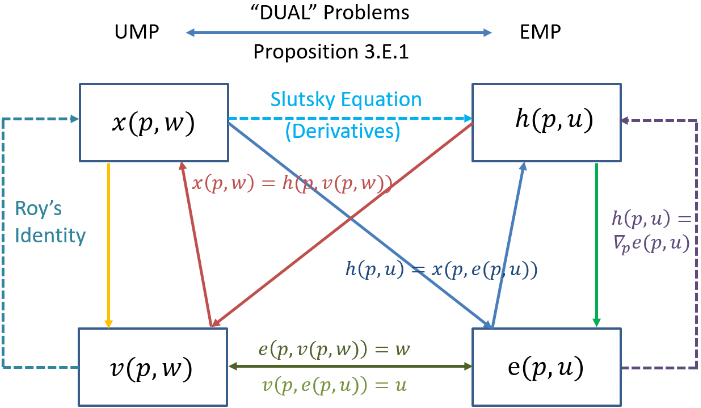
i.e., for every $l = 1, ..., L$:

$$x_l(\bar{p}, \bar{w}) = \frac{-\partial v(\bar{p}, \bar{w}) / \partial p_l}{\partial v(\bar{p}, \bar{w}) / \partial w}.$$

Walrasian Demand and Indirect Utility Function

Example. Verify Roy's identity for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Summary



"DUAL" Problems
UMP
EMP
Proposition 3.E.1
$x(p,w)$
Slutsky Equation
(Derivatives)
$h(p,u)$
$x(p,w) = h(p, v(p,w))$
Roy's Identity
$h(p,u) = x(p, e(p,u))$
$h(p,u) = \nabla_p e(p,u)$
$v(p,w)$
$e(p, v(p,w)) = w$
$v(p, e(p,u)) = u$
$e(p,u)$