Chapter 3. Games of Incomplete Information

In this chapter, we will study games of *incomplete information*, also called *Bayesian games*. In such games, at least one player is uncertain about another player's payoff function.

3.A. Static Games of Incomplete Information

In this section, we will study the simultaneous-move game of incomplete information, also called the *static Bayesian game*. We will first look at the incomplete information Cournot duopoly model in Section 3.A.1. Then we develop the normal-form representation of the general static Bayesian game and the corresponding solution concept *Bayesian Nash Equilibrium* in Section 3.A.2. Lastly, we will study several applications in Sections 3.A.3 to 3.A.6.

3.A.1. Cournot Competition under Asymmetric Information

Game Setup. Consider the Cournot duopoly model where the two firms choose their quantities simultaneously. Let q_1 and q_2 denote the quantities (of a homogeneous product) produced by firms 1 and 2, respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. Firm 1's cost function is $C_1(q_1) = cq_1$. Firm 2's cost function is

- $C_2(q_2) = c_H q_2$ with probability θ, and
- $C_2(q_2) = c_L q_2$ with probability $1 - \theta$,

where $c_L < c_H$. Furthermore, information is asymmetric:

- Firm 2 knows both its own cost function (i.e, the realization of the marginal cost c_H, c_L) and Firm 1's cost function, but
- Firm 1 knows its own cost function and only that Firm 2's marginal cost is c_H with probability θ and c_L with probability $1 - \theta$.

Analysis. Naturally, Firm 2 may choose different quantities depending on whether its marginal cost is high or low. Moreover, Firm 1 should anticipate this. Let

- $q_2^*(c_H)$ and $q_2^*(c_L)$ denote Firm 2's equilibrium quantity choice;
- q_1^* denote Firm 1's equilibrium quantity choice.

Then, $q_2^*(c_H)$ solves

$$\max_{q_2} \Big[(a - q_1^* - q_2) - c_H\Big] q_2.$$

Similarly, $q_2^*(c_L)$ solves

$$\max_{q_2} \Big[(a - q_1^* - q_2) - c_L\Big] q_2.$$

Finally, q_1^* solves

$$\begin{aligned} &\max_{q_1} \theta\Big[(a - q_1 - q_2^*(c_H)) - c\Big] q_1 + (1-\theta)\Big[(a - q_1 - q_2^*(c_L)) - c\Big] q_1 \\ \Longrightarrow\ &\max_{q_1}\Big[a - q_1 - \Big(\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)\Big) - c\Big] q_1 \end{aligned}$$

FOCs of the three optimization problem give

$$\begin{aligned} q_2^*(c_H) &= \frac{a - q_1^* - c_H}{2} \\ q_2^*(c_L) &= \frac{a - q_1^* - c_L}{2} \\ q_1^* &= \frac{a - \Big(\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)\Big) - c}{2} \end{aligned}$$

The solution is (assuming that the solutions are all positive)

$$\begin{aligned} q_2^*(c_H) &= \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L); \\ q_2^*(c_L) &= \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L); \\ q_1^* &= \frac{a - 2c + \theta c_H + (1-\theta) c_L}{3}. \end{aligned}$$

Note that $q_2^*(c_H) > \frac{a-2c_H+c}{3}$ and $q_2^*(c_L) < \frac{a-2c_L+c}{3}$: Firm 2 not only tailors its quantity to its cost but also responds to the fact that Firm 1 cannot do so. For example, consider the case where Firm 2 has high marginal cost. If Firm 1 knows that and could adjust its quantity accordingly, Firm 1 would respond by choosing the quantity $\frac{a-2c+c_H}{3}$, which is higher than q_1^*. Therefore, when Firm 1 cannot tailor its quantity (and chooses the smaller quantity q_1^*), Firm 2 would produce more, i.e., $q_2^*(c_H) > \frac{a-2c_H+c}{3}$.

3.A.2. Static Bayesian Games and Bayesian Nash Equilibrium

To characterize the static Bayesian games, we want to capture the idea that

1. each player knows his/her own payoff function;
2. each player may be uncertain about the other players' payoff functions.

Harsanyi (1967) introduced *type spaces* to model the players' information on payoff-relevant parameters.

Type and Belief. Player i's payoff functions is represented by

$$u_i(a_1, ..., a_n; t_i),$$

where t_i is called Player i's *type* and belongs to T_i, the set of possible types, or *type space*. For the Cournot competition model in Section 3.A.1,

- Firm 2 has two types and its type space is $T_2 = \{c_L, c_H\}$;
- Firm 1 has only one type and its type space is $T_1 = \{c\}$.

Given this definition of *types*,

1. "Player i knows his/her own payoff function" is equivalent to "Player i knows his/her own type".
2. "Player i may be uncertain about the other players' payoff functions" is equivalent to "Player i maybe uncertain about the types of other players, $t_{-i} = \{t_1, ..., t_{i-1}, t_{i+1}, ..., t_n\}$".

We use T_{-i} to denote the set of possible types of other players. And we use $p_i(t_{-i} \mid t_i)$ to denote Player i's *belief* about t_{-i} when his/her own type is t_i. The belief is computed by *Bayes' rule* from the prior probability distribution $p(t)$:

$$p_i(t_{-i} \mid t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} = \frac{p(t_{-i}, t_i)}{\sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)}.$$

For concrete examples of applications of Bayes' rule, see the following examples.

Example 3.A.1. Consider a two player game where both players have two types. Player 1's types are denoted as A and B; Player 2's types C and D. The prior distribution of types are given below.

		Player 2	
		Type C	Type D
Player 1	Type A	30%	40%
	Type B	10%	20%

Question 3.1. What is the posterior probability $p_1(C \mid A)$?

We use Bayes' rule to calculate the probability:

$$p_1(C \mid A) = \frac{p(C, A)}{p(A)} = \frac{p(C, A)}{p(C, A) + p(D, A)} = \frac{30\%}{30\% + 40\%} = \frac{3}{7}.$$

Example 3.A.2. A certain disease affects about 1 out of 10,000 people. There is a screening test to check whether a person has the disease. The test is quite accurate. In particular, we know that

- When a person has the disease, it gives a positive result 99% of the time.
- When a person does not have the disease, it gives a negative result 98% of the time.

Question 3.2. A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?

Let D be the event that the person has the disease and T be the event that the test is positive. Accordingly, D^c is the event that the person does not have the disease and T^c be the event that the test is negative.
We know that

$$\begin{aligned} p(D) &= \frac{1}{10,000}; \\ p(T \mid D) &= \frac{99}{100} \quad \& \quad p(T^c \mid D) = \frac{1}{100}; \\ p(T \mid D^c) &= \frac{2}{100} \quad \& \quad p(T^c \mid D^c) = \frac{98}{100}. \end{aligned}$$

And we want to calculate $p(D \mid T)$. By Bayes' rule:

$$p(D \mid T) = \frac{p(D,T)}{p(T)} = \frac{p(D,T)}{p(D,T) + p(D^c,T)} = \frac{p(T \mid D)p(D)}{p(T \mid D)p(D) + p(T \mid D^c)p(D^c)} = 0.0049$$

Normal-form Representation. The normal-form representation of n-player static Bayesian game specifies

- players' action spaces $A_1, ..., A_n$;
- their type spaces $T_1, ..., T_n$;
- their beliefs $p_1(t_{-1} \mid t_1), ..., p_n(t_{-n} \mid t_n)$;
- their payoff functions $u_i(a_1, ..., a_n; t_i)$ for all i.[1]

Following Harsanyi (1967), the timing of a static Bayesian game is as follows:

1. nature draws a type vector $t = (t_1, ..., t_n)$ where t_i is drawn from the set of possible types T_i;
2. nature reveals t_i to Player i but not to any other player;
3. players simultaneously choose actions, Player i choosing $a_i \in A_i$; and then

[1]More generally, a player's payoff function could also depend on the other players' types. In this case, we write $u_i(a_1, ..., a_n; t_1, ..., t_n)$.

4. payoffs $u_i(a_1, ..., a_n; t_i)$ are received.

Remark 3.1. Note that by introducing the fictional moves by nature, the *incomplete* information game is transformed to the *imperfect* information game.

Here, Player i does not know the complete history of the game when actions are chosen in Step 3. In particular, Player i does not know what nature has revealed to the other players.

Bayesian Nash Equilibrium.

Definition 3.A.1 (Strategy)**.** In the static Bayesian game, a **strategy** for Player i is a function $s_i(t_i)$ that specifies the action $a_i \in A_i$ when the type $t_i \in T_i$ is drawn by nature.

For the Cournot competition model in Section 3.A.1,

- Firm 2's strategy is $(q_2^*(c_H), q_2^*(c_L))$;
- Firm 1's strategy is q_1^*.

Next, we define the solution concept in a static Bayesian game, called *Bayesian Nash Equilibrium*. The central idea is the same: each player's strategy must be a *best response* to the other players' strategies.

Definition 3.A.2 (Bayesian Nash Equilibrium)**.** In the static Bayesian game, the strategies $s^* = (s_1^*, ..., s_n^*)$ are a (pure strategy) **Bayesian Nash Equilibrium (BNE)** if for each player i and for each of i's type $t_i \in T_i$, $s_i^*(t_i)$ solves

$$\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), ...s_{i-1}^*(t_{i-1}), a_i, s_{i+1}^*(t_{i+1}), ..., s_n^*(t_n); t) p_i(t_{-i} \mid t_i).$$

3.A.3. Mixed Strategies Revisited

A mixed-strategy Nash equilibrium in a game of complete information can (almost always) be interpreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information.

Example 3.A.3. Consider the following battle of the sexes game:

		Bob: Opera	Bob: Movie
Alice	Opera	$(2+t_a,1)$	$(0,0)$
	Movie	$(0,0)$	$(1,2+t_b)$

Figure 3.1: The Battle of the Sexes

where t_a is privately known by Alice; t_b is privately known by Bob; and t_a and t_b are independently drawn from a uniform distribution on $[0,x]$.

In terms of static Bayesian game,

- action spaces: $A_a = A_b = \{\text{Opera, Movie}\}$;
- type spaces: $T_a = T_b = [0,x]$;
- beliefs: $p_a(t_b \mid t_a) = 1/x$ for all $t_a \in [0,x]$ and $t_b \in [0,x]$, $p_b(t_a \mid t_b) = 1/x$ for all $t_a \in [0,x]$ and $t_b \in [0,x]$;
- payoff functions: see the payoff matrix Figure 3.1.

We construct a pure-strategy Bayesian Nash equilibrium in which

- Alice plays "Opera" if t_a exceeds a critical value a, and plays "Movie" otherwise.
- Bob plays "Movie" if t_b exceeds a critical value b, and plays "Opera" otherwise.

In such an equilibrium, Alice plays "Opera" with probability $\frac{x-a}{x}$ and "Movie" with probability $\frac{a}{x}$; Bob plays "Opera" with probability $\frac{b}{x}$ and "Movie" with probability $\frac{x-b}{x}$. Given Bob's strategy, Alice's expected payoffs from playing "Opera" and "Movie" are respectively

$$\begin{aligned} \mathbb{E}u_a(\text{Opera}, (\frac{b}{x}, \frac{x-b}{x})) &= \frac{b}{x}(2+t_a); \\ \text{and } \mathbb{E}u_a(\text{Movie}, (\frac{b}{x}, \frac{x-b}{x})) &= \frac{x-b}{x}. \end{aligned}$$

Thus, playing "Opera" is optimal if and only if

$$\frac{b}{x}(2+t_a) \geq \frac{x-b}{x} \implies t_a \geq \frac{x}{b} - 3 = a. \tag{3.A.1}$$

Similarly, given Alice's strategy, Bob's expected payoffs from playing "Opera" and "Movie" are respectively

$$\mathbb{E}u_b(\text{Opera}, (\frac{x-a}{x}, \frac{a}{x})) = \frac{x-a}{x};$$
$$\text{and } \mathbb{E}u_b(\text{Movie}, (\frac{x-a}{x}, \frac{a}{x})) = \frac{a}{x}(2+t_b).$$

Thus, playing "Movie" is optimal if and only if

$$\frac{a}{x}(2+t_b) \geq \frac{x-a}{x} \implies t_b \geq \frac{x}{a} - 3 = b. \tag{3.A.2}$$

Equations (3.A.1) and (3.A.2) imply

$$a = b = \frac{-3+\sqrt{9+4x}}{2}.$$

The probability that Alice plays "Opera" (i.e., $\frac{x-a}{x}$) and Bob plays "Movie" (i.e., $\frac{x-b}{x}$) both equal

$$1 - \frac{-3+\sqrt{9+4x}}{2x} = 1 - \frac{2}{3+\sqrt{9+4x}},$$

which approaches 2/3 as x approaches 0. Recall that the mixed strategy Nash equilibrium of the complete information battle of the sexes game (i.e., $t_a = t_b = 0$) is $\left((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\right)$. Thus, as the incomplete information disappears, the players' behavior in this pure-strategy Bayesian Nash equilibrium of the incomplete-information game approaches their behavior in the mixed-strategy Nash equilibrium in the original game of complete information.

3.A.4. First-Price Sealed-Bid Auction

We have learned second-price auction in Chapter 1. Recall that second-price auction is dominant strategy solvable: bidding one's own valuation is a weakly dominant strategy. Now, we will study first-price auction. Note that first-price and second-price auctions only differ in winner's payments. We will consider a simple version of first-price auction with only two bidders.

Game Setup. There is one indivisible good for sale. The valuations of two potential buyers are independently drawn from a uniform distribution with support $[0, 1]$. Denote Buyer i's valuation by v_i. The auction rule is as follows:

- Buyers bid simultaneously and each submits a bid $b_i \in [0, +\infty)$.

- The bidder with the highest bid wins the auction and pays his/her own bid.
- If the two buyers submit the same highest bid, then each of the buyers has 1/2 chance of winning the good. The payment is the highest bid (since there is a tie).

Normal-form Representation.

- Buyer i's *action* is to submit a bid b_i. The action space is $A_i \in [0, \infty)$.
- Buyer i's *type* is her valuation v_i. The type space is $T_i = [0, 1]$.
- Since the valuations are independent, Buyer i's *belief* about Buyer j's type v_j is that v_j is uniformly distributed on $[0, 1]$, given any v_i.
- Buyer i's *payoff* when submitting the bid b_i is

$$u_i = \begin{cases} 0 & \text{if } b_i < b_j \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j \\ v_i - b_i & \text{if } b_i > b_j \end{cases}$$

Bayesian Nash Equilibrium. A strategy for Buyer i is a function $b_i(v_i)$. In a Bayesian Nash equilibrium, Buyer 1's strategy $b_1(v_1)$ is a best response to Buyer 2's strategy $b_2(v_2)$, and vice versa. Thus, $b_i(v_i)$ solves

$$\max_{b_i} (v_i - b_i)\text{Prob}\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)\text{Prob}\{b_i = b_j(v_j)\}.$$

We focus on symmetric Bayesian Nash equilibrium where the two players adopt the same strictly increasing, continuous and differentiable bidding strategy $b(\cdot)$.

Suppose Buyer j adopts $b(\cdot)$. Then

- $\text{Prob}\{b_i = b(v_j)\} = 0$ since v_j is uniformly distributed and $b(\cdot)$ is strictly increasing.
- $\text{Prob}\{b_i > b(v_j)\} = \text{Prob}\{b^{-1}(b_i) > v_j\} = b^{-1}(b_i)$ since v_j is uniformly distributed on $[0, 1]$ and $b(\cdot)$ is strictly increasing, continuous and differentiable.

So Buyer i solves

$$\max_{b_i} (v_i - b_i) b^{-1}(b_i).$$

FOC gives

$$-b^{-1}(b_i) + (v_i - b_i)\frac{db^{-1}(b_i)}{db_i} = 0.$$

For $b(\cdot)$ to be a symmetric Bayesian Nash equilibrium, the solution to the above FOC is

$b_i = b(v_i)$, which gives

$$\begin{aligned} & -v_i + (v_i - b(v_i))\frac{1}{b'(v_i)} = 0 \\ \Longrightarrow\ & b'(v_i)v_i + b(v_i) = v_i \implies \frac{d(b(v_i)v_i)}{dv_i} = v_i \\ \Longrightarrow\ & b(v_i)v_i = \frac{1}{2}v_i^2 + c, \text{ where } c \text{ is a constant.} \end{aligned}$$

Using the boundary condition $b(0) = 0$,[2] we obtain $c = 0$. Thus, the equilibrium bidding strategy is

$$b(v_i) = \frac{1}{2}v_i.$$

3.A.5. Common Value Auction

In a common value auction, the value of good for sale is the same for all bidders. "Oil well" is an often cited example of common value auctions.

Jar of coins. Let us play the following auction game. There is a jar with some coins. Every bidder bids for the coins in the jar. Rules of the auction are as follows:

- Do not open the jar.
- The winner is the bidder with the highest bid.
- The winner pays his/her own bid and gets the coins in the jar.

This is a common value auction: the amount of money in the jar is certain.

Question 3.3. What is your bidding strategy? Should you bid less or more than your estimate?

Winner's curse. In the common value auctions, the winning bid tends to be higher than the true value of the good. Such a phenomenon is called *winner's curse.*

Question 3.4. Why winner's curse exists?

Let v be the common value, and b_i be Bidder i's bid. Then Bidder i's payoff is

$$\begin{cases} v - b_i & \text{if } b_i \text{ is the highest bid;} \\ 0 & \text{otherwise.} \end{cases}$$

[2] No player should bid more than his/her valuation, i.e., $b(v_i) \leq v_i$ for every v_i. In particular, $b(0) \leq 0$. Besides, bids are non-negative. So $b(0) = 0$.

Bidders only have estimates of the value of the good. Let y_i be Player i's estimate:

$$y_i = v + \tilde{\varepsilon}_i,$$

where $\tilde{\varepsilon}_i$ is Bidder i's estimation error. y_i is also Bidder i's type. Suppose that on average bidders estimate correctly. Then around half of the bidders overestimate. If the bidders bid roughly the same as their estimate, the winner would be the bidder with the largest $\tilde{\varepsilon}_i$. Then, the winning bid would be higher (actually much higher) than the true value.

Question 3.5. After learning winner's curse, how should you bid?

1. If everyone bids roughly their own estimates, then when you (Player i) win, you know that $y_j < y_i$ for all j.
2. You only care how many coins are in the jar if you win.

So, you should bid based not only on your initial estimate y_i but also on the fact that $y_i > y_j$ for all j. Put differently, you should bid as if you know you win.

3.A.6. Double Auction

Game Setup. There is one good for sell. The buyer's valuation for the good is v_b, and the seller's is v_s. These valuations are private information and are drawn from independent uniform distributions on $[0, 1]$. To trade, the seller names an asking price, p_s, and the buyer simultaneously names an offer price, p_b.

- If $p_b \geq p_s$, then trade occurs at price $p = (p_b + p_s)/2$;
- if $p_b < p_s$, then no trade occurs.

The players' payoffs are as follows:

- If the buyer gets the good for price p, then the buyer's utility is $v_b - p$; if there is no trade, then the buyer's utility is 0.
- If the seller sells the good for price p, then the seller's utility is $p - v_s$; if there is no trade, then the seller's utility is 0.

Analysis. Buyer's strategy is a function $p_b(v_b)$ specifying the price Buyer will offer for each of Buyer's possible valuations. Likewise, Seller's strategy is a function $p_s(v_s)$ specifying the price Seller will demand for each of Seller's valuations.

In a Bayesian Nash equilibrium, Buyer's strategy $p_b(v_b)$ is a best response to Seller's strategy $p_s(v_s)$, and vice versa. Thus, for each $v_b \in [0,1]$, $p_b(v_b)$ solves

$$\max_{p_b} \left[v_b - \frac{p_b + \mathbb{E}[p_s(v_s) \mid p_b \geq p_s(v_s)]}{2} \right] \text{Prob}\{p_b \geq p_s(v_s)\}; \tag{3.A.3}$$

for each $v_s \in [0,1]$, $p_s(v_s)$ solves

$$\max_{p_s} \left[\frac{p_s + \mathbb{E}[p_b(v_b) \mid p_b(v_b) \geq p_s]}{2} - v_s \right] \text{Prob}\{p_b(v_b) \geq p_s\}. \tag{3.A.4}$$

There are many Bayesian Nash equilibria of this game.

One-price equilibrium Consider the one-price equilibrium in which trade occurs at a single price $x \in [0,1]$ if it occurs at all.

- Buyer's strategy: offer x if $v_b \geq x$ and offer 0 otherwise;
- Seller's strategy: demand x if $v_s \leq x$ and demand 1 otherwise.

Question 3.6. Can you check that the above strategy profile constitutes a Bayesian Nash equilibrium?

In this equilibrium, trade occurs for the (v_s, v_b) pairs indicated in Figure 3.2; trade would be efficient for all (v_s, v_b) pairs such that $v_b > v_s$, but does not occur in the two shaded regions of the figure.

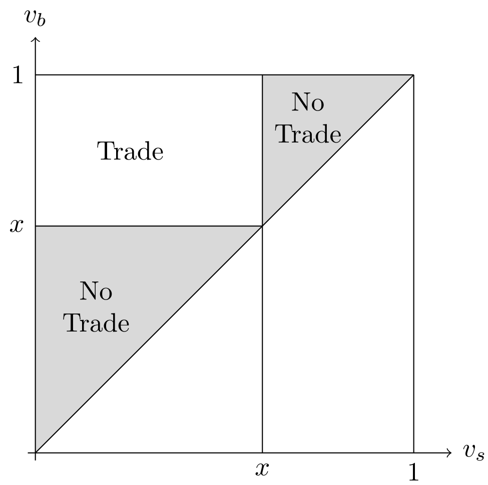


Figure 3.2: One-price Equilibrium

Linear equilibrium Suppose Seller's and Buyer's strategies are

$$p_s(v_s) = a_s + c_s v_s; \tag{3.A.5}$$

$$\text{and } p_b(v_b) = a_b + c_b v_b. \tag{3.A.6}$$

Then p_s is uniformly distributed on $[a_s, a_s + c_s]$; p_b is uniformly distributed on $[a_b, a_b + c_b]$. Equations (3.A.3) and (3.A.4) become:

$$\max_{p_b} \left[v_b - \frac{p_b + \frac{a_s + p_b}{2}}{2} \right] \left(\frac{p_b - a_s}{c_s} \right);$$

$$\text{and } \max_{p_s} \left[\frac{p_s + \frac{a_b + c_b + p_s}{2}}{2} - v_s \right] \left(\frac{a_b + c_b - p_s}{c_b} \right).$$

First-order conditions imply

$$p_b = \frac{2}{3} v_b + \frac{1}{3} a_s; \tag{3.A.7}$$

$$\text{and } p_s = \frac{2}{3} v_s + \frac{1}{3}(a_b + c_b). \tag{3.A.8}$$

Solving from Equations (3.A.5) to (3.A.8), we have

$$p_s(v_s) = \frac{2}{3} v_s + \frac{1}{4};$$

$$\text{and } p_b(v_b) = \frac{2}{3} v_b + \frac{1}{12}.$$

Trade occurs when

$$p_b(v_b) \geq p_s(v_s) \implies v_b \geq v_s + \frac{1}{4}.$$

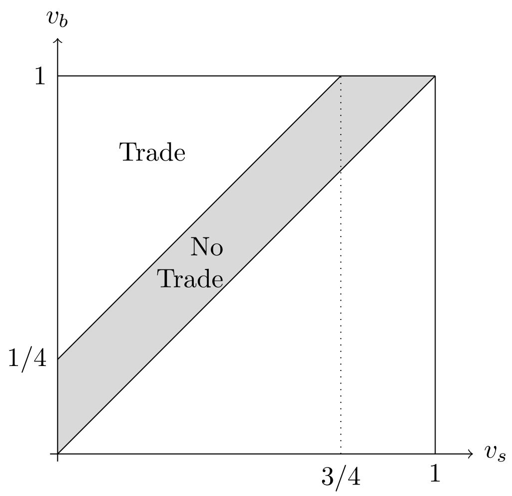


Figure 3.3: Linear-price Equilibrium

3.B. Dynamic Games of Incomplete Information

In this section, we will first study three specific models of dynamic games of incomplete information: asymmetric information Cournot model with verifiable information in Section 3.B.1, job market signaling model in Section 3.B.2 and a screening model in Section 3.B.3. Then, we will study the theory and formally define the solution concept *Perfect Bayesian Equilibrium* in Section 3.B.4. Finally, we will discuss refinements of Perfect Bayesian Equilibrium in Section 3.B.5.

3.B.1. Asymmetric Cournot with Verifiable Information

Game Setup. Consider the Cournot duopoly model where the two firms choose their quantities simultaneously. Let q_1 and q_2 denote the quantities (of a homogeneous product) produced by firms 1 and 2, respectively. Let $P(Q) = a - Q$ be the market-clearing price when the aggregate quantity on the market is $Q = q_1 + q_2$. Firm 1's cost function is

$$C_1(q_1) = c_M q_1.$$

Firm 2's cost function is

$$C_2(q_2) = \begin{cases} c_H q_2 = (c_M + x)q_2 & \text{with probability } 1/3 \\ c_M q_2 & \text{with probability } 1/3 \\ c_L q_2 = (c_M - x)q_2 & \text{with probability } 1/3 \end{cases}$$

Furthermore, information is asymmetric:

- Firm 2 knows both its own cost function (i.e, the realization of the marginal cost) and Firm 1's cost function, but
- Firm 1 knows its own cost function and only that Firm 2's marginal cost is c_H, c_M or c_L, each with 1/3 probability.

Before the firms choose quantities, Firm 2 can costlessly and verifiably reveal its cost information to Firm 1.

Question 3.7. Should Firm 2 reveal its cost information?

Perhaps it is easier to first consider the following question:

Question 3.8. Would Firm 2 want Firm 1 to know if it has high, middle, or low cost?

First, notice that in the Cournot model, one firm's profit would be higher if the other firm produces less. Then the question becomes "would Firm 1 produce less if it knows that Firm 2 has high, middle or low cost?" To answer this question, we could solve for Firm 1's equilibrium quantities under different scenarios. It is also clear from the reaction curves in Figure 3.4. The result is: compared to not knowing Firm 2's cost, Firm 1 produces

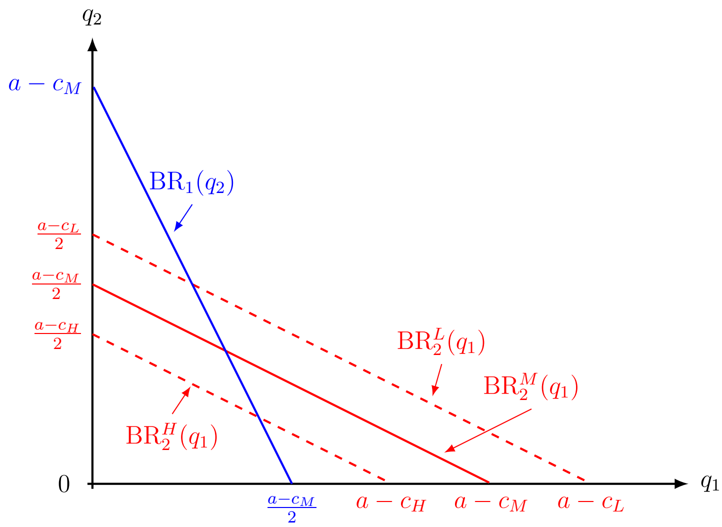


Figure 3.4: Cournot Duopoly

less (more) if it knows that Firm 2 has low (high) cost. Thus, Firm 2 would want Firm 1 to know if it has low cost. That is, **Firm 2 with low cost would reveal its cost information.**

But the above argument is not over. Let us now consider whether Firm 2 should reveal its cost information when it has middle cost. You might think that Firm 2's decision depends on what Firm 1 thought. However, if Firm 2 doesn't reveal that it has middle cost, then Firm 1 knows that the cost is not low. This is because Firm 2 would reveal its cost information if it has low cost, as is argued in the previous paragraph. Put it differently, Firm 1 knows that the cost is either middle or high. As a result, Firm 2 with middle cost would want Firm 1 to know it so that Firm 1 would produce less. That is, **Firm 2 with middle cost would also reveal its cost information.**

Then, **for Firm 2 with high cost, it really doesn't matter whether it reveals or not.** Because even if it does not reveal, since Firm 2 with middle or low costs would reveal, the fact of no revealing reveals that Firm 2 has high cost.

Remark 3.2. The same argument goes through if Firm 2 has more types.

This idea is called *information unraveling.*

3.B.2. Job-Market Signaling

Suppose that there are two types of workers, high-ability and low-ability. They differ in productivity: high-ability worker has productivity of 100 whereas low-ability worker has productivity of 60. In the population, 20% of workers are high-ability and 80% are low-ability.

	Productivity	Proportion
High-ability Worker	100	20%
Low-ability Worker	60	80%

Suppose that firms are competitive. Thus, firms would offer 100 to a high-ability worker and 60 to a low-ability worker if they could identify the worker's types. If firms cannot identify the worker's types, they would offer $100 * 20\% + 60 * 80\% = 68$.

Question 3.9. Suppose that you are a high-ability worker, how can you make the firms know it? In particular, would it work if you simply tell the firms "I am a high-ability worker"?

Spence (1973) brings up the idea that "education" could be used as a costly signal to differentiate high-ability workers from low-ability ones. The crucial assumption in Spence's model is that low-ability workers find education more costly than high-ability workers. Suppose it costs high-ability workers 9 for a year of education and low-ability workers 21.

	Cost
High-ability Worker	9
Low-ability Worker	21

When three-year graduate education is available, we argue that there exists an equilibrium where

- High-ability workers take the education but low-ability workers do not.
- Employers identify those workers with graduate degrees as high-ability workers and those without degrees as low-ability workers. Employers offer 100 to a worker with degree and 60 to a worker without degree.

We have not gone through the solution concept Perfect Bayesian Equilibrium (PBE).[3] In essence, PBE requires

[3]See Section 3.B.4 for a detailed discussion of PBE.

1. the strategies to be best responses given the belief system, and
2. the beliefs to be consistent with the strategy profile.

For this particular game, we need to check

1. Both types of workers would not deviate in their respective education choices.
2. Employers' beliefs are consistent with the equilibrium behavior.

The second point is obvious. For the first point,

- A high-ability worker obtains $100 - 9 * 3 = 73$ if he/she takes the education and 60 if not.
- A low-ability worker obtains $100 - 21 * 3 = 37$ if he/she takes the education and 60 if not.

Thus, a high-ability worker would not deviate to not taking the education and a low-ability worker would not deviate to taking the education.

Remark 3.3. This is called a *separating equilibrium* because in equilibrium the types separate and get identified.

Question 3.10. What if the education program only takes two years? How about one year?

Remark 3.4. For the separation to work, there must be enough differences in costs for the two types of workers.

Remark 3.5. If the standard of obtaining education becomes lower, then probably we will see qualification inflation.

Remark 3.6. Education increases inequality: Compared to the no education outcome, a three-year education program makes high-ability workers better-off ($73 > 68$) and low-ability workers worse-off ($60 < 68$).

Remark 3.7. It is possible that high-ability workers are also worse-off. To see this, consider a four-year education program. The separating equilibrium exists in this case. Recall that in the separating equilibrium, no education is interpreted as evidence of low ability. So, high-ability workers still have the incentive to get educated, obtaining a payoff of 64 (< 68).

3.B.3. Screening

In the last section, we have seen a *signaling* model in which the informed parties (i.e., the workers) move first. Signaling models are closely related to *screening* models, in which the uninformed parties take the lead. Classic references of screening models concern the context of insurance markets. In this course, we still take the job market as an example.

Job market example. In the previous signaling model, the workers choose education first. And after observing the education choices, the firms offer wages accordingly. Now consider the following timing, which corresponds to a screening setting:

1. Two firms simultaneously announce a menu of contracts specifying the required years of education and the wage offer (e, w).
2. Given these contracts, the workers choose which contract to accept, if any.

The productivity and proportion of workers, as well as the cost of a year of education to each types of workers are all the same as in the signaling case.

Question 3.11. Is it an equilibrium that both firms offer the same two contracts $(e_H = 3, w_H = 100)$ and $(e_L = 0, w_L = 60)$? The outcome is the same as a three-year education program in the signaling model.

For the workers, similar arguments as in the signaling model apply. Both types of workers would self-select the contracts designed for them.

- A high-ability worker obtains $100 - 9 * 3 = 73$ if he/she takes the contract $(e_H = 3, w_H = 100)$ and 60 if takes $(e_L = 0, w_L = 60)$.
- A low-ability worker obtains $100 - 21 * 3 = 37$ if he/she takes the contract $(e_H = 3, w_H = 100)$ and 60 if takes $(e_L = 0, w_L = 60)$.

But for the firms, if the contracts $(e_H = 3, w_H = 100)$ and $(e_L = 0, w_L = 60)$ are offered and accepted by the respective types of workers, each firm obtains a payoff of 0 since the wages and the productivities are the same. A firm could be better-off by offering $(e'_H = 2, w'_H = 95)$ and $(e_L = 0, w_L = 60)$.

- The high-ability workers prefer $(e'_H = 2, w'_H = 95)$ to $(e_H = 3, w_H = 100)$: they obtain $95 - 9 * 2 = 77$ (> 73) if taking $(e'_H = 2, w'_H = 95)$.
- The low-ability workers would not take $(e'_H = 2, w'_H = 95)$: they obtain $95 - 21 * 2 = 53$ (< 60) if taking $(e'_H = 2, w'_H = 95)$.

Thus, by deviating to offering $(e'_H = 2, w'_H = 95)$ and $(e_L = 0, w_L = 60)$, the firm obtains $(100 - 95) * 20\% = 1 > 0$.

Question 3.12. How about both firms offer the same two contracts $(e_H = 2, w_H = 100)$ and $(e_L = 0, w_L = 60)$?

Similar to the previous argument, if such contracts are offered, workers would self-select the contract designed for them.

For the firms, a firm could deviate to offering contracts that each attracts only one types of workers or offering a contract that attracts both types of workers. Moreover, when a contract attracts only one types of workers, offering $w_L > 60$ or $w_H > 100$ results in a loss. On the other hand, for a contract with a lower wage to be accepted by the workers, the education requirement must be lower. Therefore, a firm could not benefit from offering a contract that attracts the low-ability workers alone. And we only need to consider the following two types of deviations:

1. A firm could deviate to an offer $(e'_H = 1, w'_H < 100)$ to attract the high-ability workers. The high-ability workers prefer $(e'_H = 1, w'_H)$ to $(e_H = 2, w_H = 100)$ if $w'_H - 9 \geq 100 - 9 * 2 \implies w'_H \geq 91$. However, when this is the case, the low-ability workers also prefer $(e'_H = 1, w'_H)$ since $w'_H - 21 \geq 70$ (> 60). Then the firm obtains $(100 - w'_H) * 20\% + (60 - w'_H) * 80\% = 68 - w'_H < 0$. And thus this is not a profitable deviation.
2. Another possible deviation is to offer $(e = 0, w)$ and attract both types of workers. The low-ability workers would accept this contract as long as $w \geq 60$ and the high-ability workers would accept this contract is $w \geq 100 - 9 * 2 = 82$. Thus, the lowest wage is 82 and the firm obtains $68 - 82 < 0$.

Therefore, it is an equilibrium that both firms offer the two contracts $(e_H = 2, w_H = 100)$ and $(e_L = 0, w_L = 60)$.

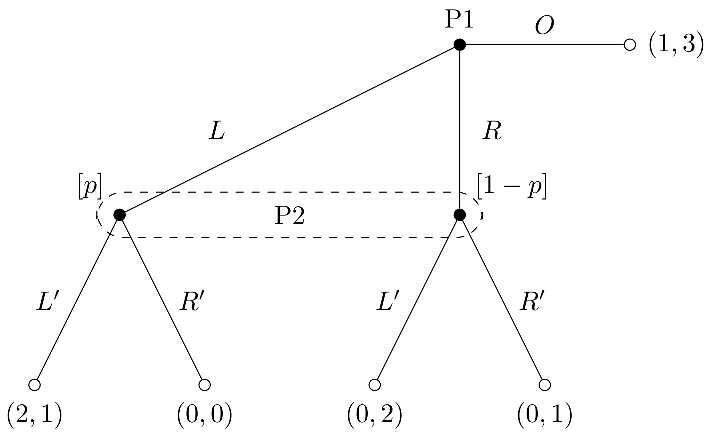
Remark 3.8. Separating equilibria do not always exist. For example, if we change the proportion of high-ability workers to 80%, then there will be no separating equilibria.

3.B.4. Perfect Bayesian Equilibrium

The solution concept associated with dynamic games of incomplete information is the *Perfect Bayesian Equilibrium (PBE).* Perfect Bayesian equilibrium was invented in order to refine Bayesian Nash equilibrium in a similar way that subgame perfect equilibrium refined Nash equilibrium. BNE also fails to capture the idea that threats and promises should be credible.

A complementary perspective is that perfect Bayesian equilibrium strengthens the requirements of subgame perfect equilibrium by explicitly analyzing the players' beliefs, as in Bayesian Nash equilibrium. We will introduce the features of perfect Bayesian equilibrium from this complementary perspective.

Example 3.B.1. Consider the following game:



Question 3.13. What are the pure-strategy Nash equilibria of this game?

Answer: To solve for Nash equilibrium, we transform this extensive-form representation into the normal-form representation:

		Player 2 L'	Player 2 R'
Player 1	L	$(2,1)$	$(0,0)$
Player 1	R	$(0,2)$	$(0,1)$
Player 1	O	$(1,3)$	$(1,3)$

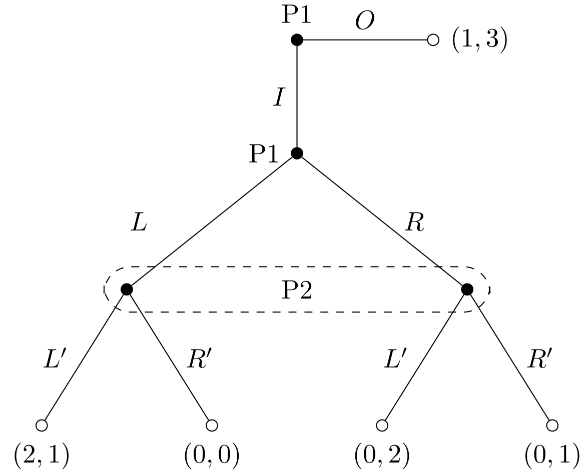
There are two pure strategy Nash equilibria: (L, L') and (O, R').

Question 3.14. Are these Nash equilibria subgame perfect?

Answer: Yes. There are no proper subgames of this game.[4] Hence, every NE is a SPE.

[4] Any subgame other than the entire game itself is called a proper subgame.

Example 3.B.2. Consider the following modified game:



Question 3.15. What are the pure strategy Nash equilibria of this game? Are these Nash equilibria subgame perfect?

Answer: Again, to solve for Nash equilibrium, we use the normal-form representation:

Player 2

Player 1		L'	R'
	[I, L]	$(2,1)$	$(0,0)$
	[I, R]	$(0,2)$	$(0,1)$
	[O, L]	$(1,3)$	$(1,3)$
	[O, R]	$(1,3)$	$(1,3)$

There are three Nash equilibria: $([I,L],L')$, $([O,L],R')$ and $([O,R],R')$.

There is one proper subgame. So, to solve for subgame perfect Nash equilibrium, we solve for the Nash equilibrium in this subgame first:

Player 2

Player 1		L'	R'
	L	$(2,1)$	$(0,0)$
	R	$(0,2)$	$(0,1)$

The only Nash equilibrium is (L,L') and the players' payoffs are $(2,1)$. Then in player 1's first decision node, he/she would choose I. Therefore, there is only one subgame perfect equilibrium: $([I,L],L')$.

Question 3.16. Are the games in Examples 3.B.1 and 3.B.2 really different?

Question 3.17. Is the equilibrium (O, R') in Example 3.B.1 reasonable?

Answer: No. It depends on a non-credible threat: when player 2 gets the move, then playing L' dominates playing R'. So, player 1 should not be induced to play R by player 2's threat to play R' if given the move.
To rule out the unreasonable prediction (O, R'), we impose the *sequential rationality* requirement:

Requirement 1 (Sequential rationality). At each information set, the action taken by the player with the move (and the player's subsequent strategy) must be optimal given the player's belief at the information set and other players' subsequent strategies.

Let us apply the sequential rationality requirement to the (O, R') equilibrium in Example 3.B.1. Let p denote player 2's belief that L has been chosen when the game reaches the information set. Given this belief, the expected payoff from choosing R' is $p \cdot 0 + (1-p) \cdot 1 = 1-p$, while the expected payoff from choosing L' is $p \cdot 1 + (1-p) \cdot 2 = 2-p$. Since $2 - p > 1 - p$ for any value of p, it is never sequentially rational for player 2 to choose R'.
We have only claimed that player 2 should have beliefs and act optimally given the belief. However, we have not yet discussed what beliefs are reasonable. In order to impose such requirements, we first need to distinguish information sets on the equilibrium path and off the equilibrium path.

Definition 3.B.1. For a given equilibrium in a given extensive-form game, an information set is *on the equilibrium path* if it will be reached with positive probability if the game is played according to the equilibrium strategies, and is *off the equilibrium path* if it is certain not to be reached if the game is played according to the equilibrium strategies.

Then we impose the following two Belief consistency requirements:

Requirement 2 (Belief consistency (on the equilibrium path)). At information sets on the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies.

Requirement 3 (**Belief consistency (off the equilibrium path)**). At information sets off the equilibrium path, beliefs are determined by Bayes' rule and the players' equilibrium strategies where possible.

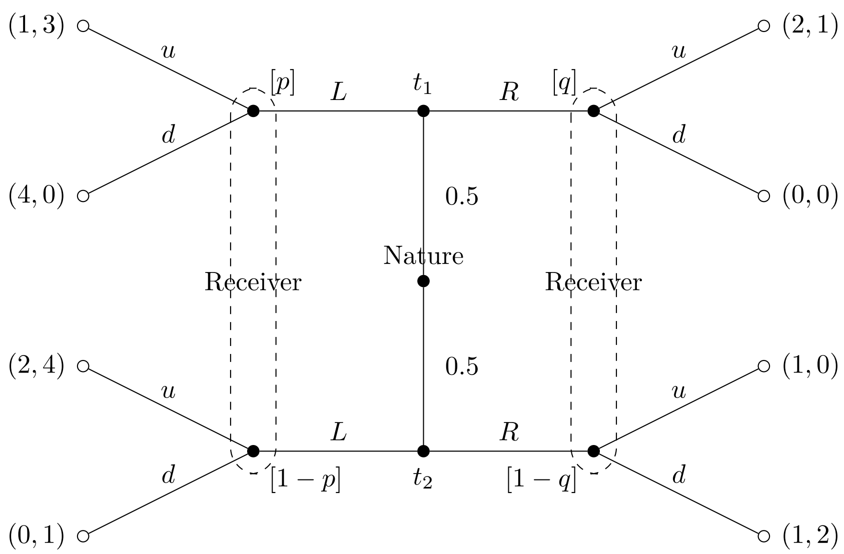
Let us apply the belief consistency requirements to the (L, L') equilibrium in Example 3.B.1. Player 2's belief must be $p = 1$: given player 1's equilibrium strategy (namely, L), player 2 knows which node in the information set has been reached.
As an illustration, suppose that in Example 3.B.1 there were a mixed-strategy equilibrium in which player 1 plays L with probability q_1, R with probability q_2, and O with probability $1 - q_1 - q_2$. Then belief consistency requires player 2's belief to be $p = \frac{q_1}{q_1+q_2}$.

Next, we formally define the solution concept *Perfect Bayesian Equilibrium*.

Definition 3.B.2 (Perfect Bayesian Equilibrium). A **Perfect Bayesian Equilibrium (PBE)** is a strategy profile σ and a belief system $\mu = (\mu_1, \mu_2, ..., \mu_n)$ where μ_i specifies Player i's belief at each of his/her information sets, satisfying Requirements 1 to 3.[5]

Note that because perfect Bayesian equilibrium makes the players' beliefs explicit, such an equilibrium often cannot be constructed by working backwards through the game tree, as we did to construct a subgame perfect equilibrium.

Example 3.B.3 (Signaling game). Consider the following signaling game:



[5]In simple economic applications, such as Example 3.B.3, Requirements 1 and 2 is enough. In richer economic applications, however, more requirements need to be imposed to eliminate implausible equilibria. Different authors have used different definitions of perfect Bayesian equilibrium. All definitions include Requirements 1 and 2; most also include Requirement 3; some impose further requirements. See Section 3.B.5 for details.

- Sender's type space: $T = \{t_1, t_2\}$;
- Sender's action space: $M = \{L, R\}$ for both t_1 and t_2;
- Receiver's action space: $A = \{u, d\}$, independent of the sender's message.

Recall that (in any game) a player's strategy is a complete plan of action–a strategy specifies a feasible action in every contingency in which the player might be called upon to act. In a signaling game, therefore, a **pure strategy for the Sender** is a function $m(t_i)$ specifying which message will be chosen for each type that nature might draw, and a **pure strategy for the Receiver** is a function $a(m_j)$ specifying which action will be chosen for each message that the Sender might send.

In this game, there are four possible pure-strategy perfect Bayesian equilibria based on Sender's pure strategy:

1. pooling on L;
2. pooling on R;
3. separation with t_1 playing L and t_2 playing R; and
4. separation with t_1 playing R and t_2 playing L.

We analyze these possibilities in turn.

Pooling on L Suppose there is an equilibrium in which Sender's strategy is (L, L). Then the Receiver's information set corresponding to L is on the equilibrium path and $p = 0.5$. Given this belief, Receiver's best response to L is u. To determine whether both Sender types are willing to choose L, we need to specify how Receiver would react to R.

- If Receiver's response to R is u, then t_1's payoff from playing R is 2, which exceeds t_1's payoff of 1 from playing L.
- If Receiver's response to R is d, then t_1 and t_2 earn payoffs of 0 and 1 from playing R, whereas they earn 1 and 2 from playing L.

Thus, if there is an equilibrium in which Sender's strategy is (L, L),[6] then Receiver's response to R must be d. So Receiver's strategy is (u, d).[7] It remains to consider Receiver's belief at the information set corresponding to R. For playing d to be optimal, we require $q \cdot 0 + (1-q) \cdot 2 \geq q \cdot 1 + (1-q) \cdot 0 \implies q \leq 2/3$. Thus, $((L, L), (u, d), p = 0.5, q)$ **is a pooling perfect Bayesian equilibrium for any** $q \leq 2/3$.

[6] The first entry denotes type t_1's action and the second denotes t_2's action.

[7] The first entry denotes Receiver's action following L and the second denotes Receiver's action following R.

Pooling on R Suppose Sender's strategy is (R, R). Then $q = 0.5$. Receiver's best response to R is d. Again we need to specify Receiver's reaction to L. However, since t_1 can at least obtain 1 from playing L, which is higher than t_1's payoff of 0 from playing R, there is no equilibrium in which Sender plays (R, R).

Separation, with t_1 **playing** L **and** t_2 **playing** R If the Sender plays the separating strategy (L, R), then both of Receiver's information sets are on the equilibrium path, so $p = 1$ and $q = 0$. Receiver's best response is (u, d). It remains to check whether Sender's strategy is optimal given Receiver's strategy (u, d). It is not: if t_2 deviates by playing L rather than R, then Receiver responds with u, earning t_2 a payoff of 2, which exceeds t_2's payoff of 1 from playing R.

Separation, with t_1 **playing** R **and** t_2 **playing** L If the Sender plays the separating strategy (R, L), then $p = 0$ and $q = 1$. Receiver's best response is (u, u). It remains to check whether Sender's strategy is optimal given Receiver's strategy (u, u).

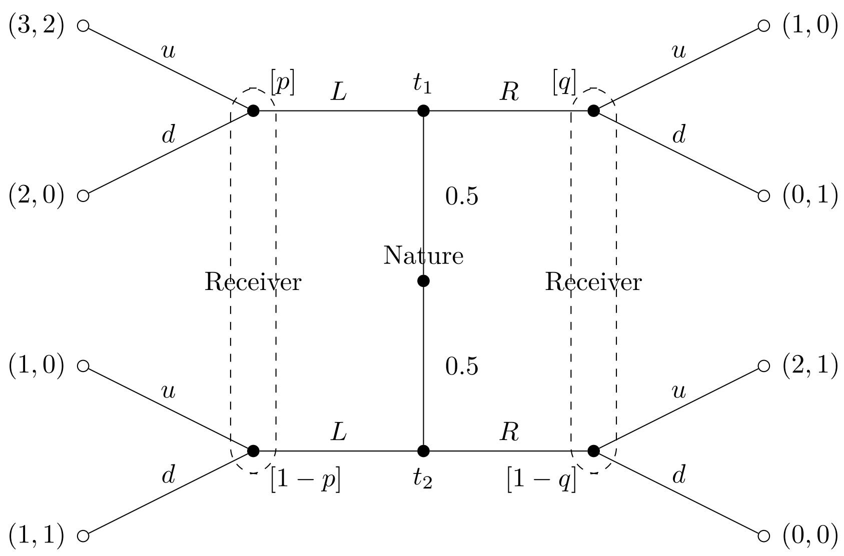
- If t_1 deviates by playing L rather than R, then Receiver responds with u, earning t_1 a payoff of 1, which is lower than t_1's payoff of 2 from playing R.
- If t_2 deviates by playing R rather than L, then Receiver responds with u, earning t_2 a payoff of 1, which is lower than t_2's payoff of 2 from playing L.

Thus, $((R, L), (u, u), p = 0, q = 1)$ **is a separating perfect Bayesian equilibrium.**

3.B.5. Refinement of Perfect Bayesian Equilibrium

We briefly discuss the refinement of perfect Bayesian equilibrium (on beliefs off the equilibrium path) in signaling games through two examples.

Example 3.B.4. Consider the following signalling game:



Question 3.18. Can you find the pure-strategy perfect Bayesian equilibria of this game?

Answer: There are two pure-strategy perfect Bayesian equilibria:

- one pooling equilibrium: $((L, L), (u, d), p = 0.5, q)$ for any $q \geq 1/2$;
- one separating equilibrium: $((L, R), (u, u), p = 1, q = 0)$.

Question 3.19. Is the pooling equilibrium reasonable? In particular, is the off-the-equilibrium-path belief $q \geq 1/2$ reasonable?

Answer: No. Note that it makes no sense for t_1 to play R: if t_1 plays L, the lowest payoff is 2; whereas if t_1 plays R, the highest payoff is 1. However, the belief $q \geq 1/2$ means Receiver believes that a deviation to R is very likely from t_1.

To eliminate such unreasonable predictions, we impose the following requirement:

Requirement 4 (Signalling)**.** If the information set following m_j is off the equilibrium path and m_j is dominated for type t_i, then (if possible) the Receiver's belief $\mu(t_i \mid m_j)$ should place zero probability on type t_i.

The definition of a message being dominated for a type in the requirement is as follows:

Definition 3.B.3. In a signaling game, the message m_j from M is **dominated for type** t_i from T if there exists another message $m_{j'}$ from M such that t_i's lowest possible payoff from $m_{j'}$ is greater than t_i's highest possible payoff from m_j:

$$\min_{a_k \in A} U_S(t_i, m_{j'}, a_k) > \max_{a_k \in A} U_S(t_i, m_j, a_k).$$

Applying Requirement 4 to the pooling equilibrium in Example 3.B.4, we require $q = 0$. Since the pooling equilibrium is a perfect Bayesian equilibrium only if $q \geq 1/2$, this equilibrium cannot satisfy Requirement 4. On the other hand, the separating equilibrium satisfies Requirement 4 trivially.

Example 3.B.5 (Beer and Quiche)**.** In the "Beer and Quiche" signaling game, Sender is one of two **types**:

- t_1 – "wimpy" (with probability 0.1) and
- t_2 – "surly" (with probability 0.9).

Sender's message is the choice of whether to have beer or quiche for breakfast;
Receiver's action is the choice of whether or not to duel with Sender.
The qualitative features of the payoffs are

- wimpy type would prefer to have quiche for breakfast,
- surly type would prefer to have beer,
- both types would prefer not to duel with Receiver (and care about this more than about which breakfast they have), and
- Receiver would prefer to duel with wimpy type but not to duel with surly type.

The game tree is as follows:

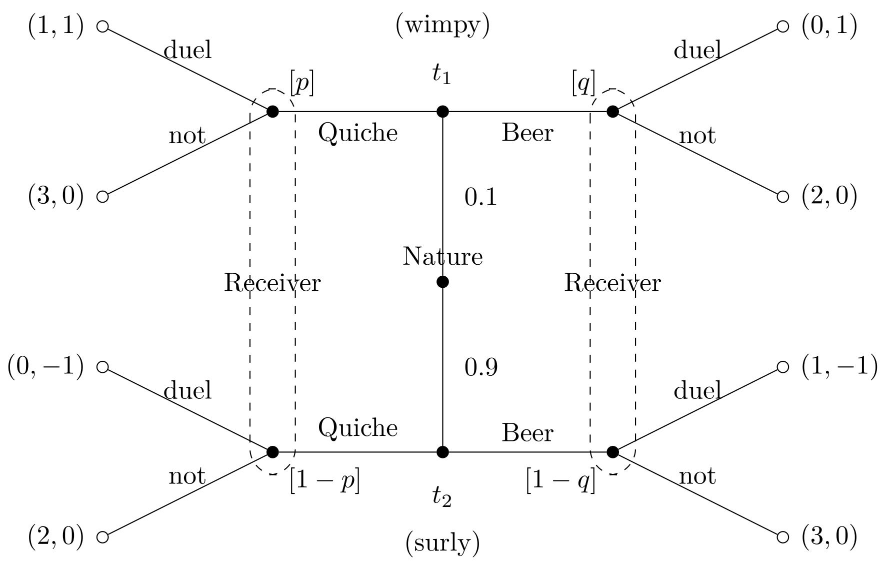


Figure 3.5: Beer and Quiche

Question 3.20. Can you find the pure-strategy perfect Bayesian equilibria of this game?

Answer: There are two pure-strategy perfect Bayesian equilibria, both are pooling:

- $((\text{Quiche}, \text{Quiche}), (\text{not}, \text{duel}), p = 0.1, q)$ for any $q \geq 1/2$;
- $((\text{Beer}, \text{Beer}), (\text{duel}, \text{not}), p, q = 0.1)$ for any $p \geq 1/2$.

Question 3.21. Do these equilibria satisfy Requirement 4?

Answer: Yes. Because both Beer and Quiche are not dominated for either Sender type.

Question 3.22. The first pooling equilibrium requires Receiver to believe that Sender is very likely to be of surly type ($q \geq 1/2$) if the off-the-equilibrium-path message Beer is observed. Is it reasonable?

Answer: No. Because:

1. wimpy type cannot possibly improve on the equilibrium payoff of 3 by having Beer rather than Quiche, while
2. surly type could improve on the equilibrium payoff of 2 as surly type would receive the payoff of 3 if Receiver held a belief $q < 1/2$.

To eliminate such unreasonable predictions, we impose the following requirement:

Requirement 5 ("The Intuitive Criterion", Cho and Kreps (1987))**.** If the information set following m_j is off the equilibrium path and m_j is equilibrium-dominated for type t_i, then (if possible) the Receiver's belief $\mu(t_i \mid m_j)$ should place zero probability on type t_i.

The definition of a message being equilibrium-dominated for a type in the requirement is as follows:

Definition 3.B.4. Given a perfect Bayesian equilibrium in a signaling game, the message m_j from M is **equilibrium-dominated for type** t_i from T if t_i's equilibrium payoff, denoted by $U^*(t_i)$, is greater than t_i's highest possible payoff from m_j:

$$U^*(t_i) > \max_{a_k \in A} U_S(t_i, m_j, a_k).$$

Applying Requirement 5 to the first pooling equilibrium in Example 3.B.5, we require $q = 0$. Since the pooling equilibrium is a perfect Bayesian equilibrium only if $q \geq 1/2$, this equilibrium cannot satisfy Requirement 5. On the other hand, the second pooling equilibrium satisfies Requirement 5.

Remark 3.9. Arguments in the spirit of Requirement 5 are sometimes said to use *forward induction*, because in interpreting a deviation—that is, in forming the belief $\mu(t_i \mid m_j)$—Receiver asks whether Sender's past behavior could have been rational.