Ordinary Differential Equations

1.A. First order linear constant coefficient differential equations

Homogeneous linear constant coefficient differential equations: $\dot{y} + ay = 0$

- General solution: $y(t) = Ce^{-at}$
- Particular solution: given $y(0) = y_0$, then $y(t) = y_0 e^{-at}$

Linear constant coefficient differential equations: $\dot{y} + ay = b$

- Equilibrium: $y^* = \frac{b}{a}$. Define $x(t) = y(t) - y^*$ ($x(t)$ is the deviation from the equilibrium). Then $\dot{x} + ax = 0$ and thus $x(t) = Ce^{-at}$.
- General solution: $y(t) = Ce^{-at} + y^*$ where $y^* = \frac{b}{a}$.
- Particular solution: given y_0, then $y_t = (y_0 - y^*)e^{-at} + y^*$

Method of undetermined coefficient: $\dot{y} + ay = b(t)$

- Particular solution: $y_p(t)$
- General solution: $y(t) = Ce^{-at} + y_p(t)$

Example 1.1 (K-I model when $I = be^{\theta t}$)**.** Solve $\dot{K} + \delta K = be^{\theta t}$.

Try $K_p(t) = ce^{\theta t}$. Then

$$c\theta e^{\theta t} + \delta c e^{\theta t} = be^{\theta t} \implies c = \frac{b}{\theta + \delta}.$$

Thus, the general solution is $K(t) = Ce^{-\delta t} + \frac{b}{\theta+\delta}e^{\theta t}$.

If $K(0) = K_0$, then $K_0 = C + \frac{b}{\theta+\delta} \implies C = K_0 - \frac{b}{\theta+\delta}$. Thus, the particular solution is $K(t) = (K_0 - \frac{b}{\theta+\delta})e^{-\delta t} + \frac{b}{\theta+\delta}e^{\theta t}$.

1.B. General first order linear differential equation: $\dot{y} + a(t)y = b(t)$

$$y(t) = e^{-\int_0^t a(s)\mathrm{d}s}\Big(y_0 + \int_0^t b(s)e^{\int_0^s a(\tau)\mathrm{d}\tau}\mathrm{d}s\Big)$$

This can be shown using the method of integrating factors: $\phi(y,t) = e^{\int_0^t a(s)\mathrm{d}s}$. (See Section 1.C)

1.C. Some nonlinear first order differential equations

Separable DE: $f(y)\dot{y} = b(t)$

$$\int f(y)\mathrm{d}y = \int b(t)\mathrm{d}t$$

Exact DE: $M(y,t)\mathrm{d}y + N(y,t)\mathrm{d}t = \mathrm{d}F(y,t)$

Example 1.2. $\dot{y} = -\frac{2yt+y^2}{t^2+2yt}$. We have $(t^2+2yt)\mathrm{d}y + (2yt+y^2)\mathrm{d}t = 0$. On the other hand, $(t^2+2yt)\mathrm{d}y + (2yt+y^2)\mathrm{d}t = \mathrm{d}(yt^2+y^2t)$. Therefore, $yt^2+y^2t = C$.

Integrating factors: $\phi(y,t)(M(y,t)\mathrm{d}y + N(y,t)\mathrm{d}t) = \mathrm{d}F(y,t)$

1.D. Autonomous differential equations: $\dot{y} = f(y)$ and phase diagram

- Equilibrium y^*: $f(y^*) = 0$.
- Phase diagram

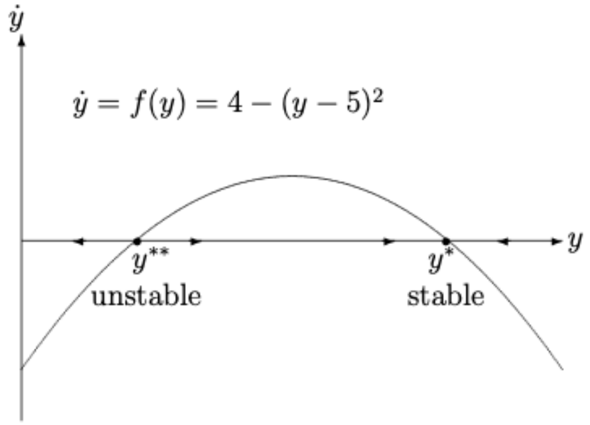

Figure 1.1: Phase diagram

1.E. Second order linear constant coefficient differential equation

Homogeneous linear constant coefficient differential equation $\ddot{y} + a_1\dot{y} + a_2 y = 0$

Suppose that there is a solution $y = e^{\lambda t}$. Then $\dot{y} = \lambda e^{\lambda t} = \lambda y$ and $\ddot{y} = \lambda^2 e^{\lambda t} = \lambda^2 y$. Then λ satisfies the *characteristic equation*:

$$\phi(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0.$$

- If λ_1 and λ_2 are distinct reals, then the general solution $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$.
- If $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$, then the general solution $y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$.
- If $\lambda_1, \lambda_2 = \alpha \pm \beta i$, then the general solution is $y(t) = e^{\alpha t}[C_1 \cos(\beta t) + C_2 \sin(\beta t)]$.

Linear constant coefficient differential equation $\ddot{y} + a_1 \dot{y} + a_2 y = b$

- Equilibrium: $y^* = \frac{b}{a_2}$. Define $x(t) = y(t) - y^*$ (x_t is the deviation from the equilibrium). Then $\ddot{x} + a_1 \dot{x} + a_2 x = 0$.
- General solution: $y(t) = x(t) + y^*$.

Method of undetermined coefficients: $\ddot{y} + a_1 \dot{y} + a_2 y = b(t)$

- Particular solution: $y_p(t)$
- General solution: $y(t) = x(t) + y_p(t)$

Example 1.3. Solve $\ddot{y} - \dot{y} - 2y = 4t^2$.

Try $y_p(t) = c_2 t^2 + c_1 t + c_0$. We have $2c_2 - (2c_2 t + c_1) - 2(c_2 t^2 + c_1 t + c_0) = 4t^2 \implies c_0 = -3, c_1 = 2, c_2 = -2$. Thus $y_p(t) = -2t^2 + 2t - 3$. The general solution is $y = C_1 e^{-t} + C_2 e^{2t} - 2t^2 + 2t - 3$.

Example 1.4. Solve $\ddot{y} - \dot{y} - 2y = e^{3t}$.

Try $y_p(t) = ce^{3t}$. We have $9ce^{3t} - 3ce^{3t} - 2ce^{3t} = e^{3t} \implies c = 0.25$. Thus $y_p(t) = 0.25e^{3t}$. The general solution is $y = C_1 e^{-t} + C_2 e^{2t} + 0.25e^{3t}$.