Chapter 3. Classical Demand Theory

3.D. Utility Maximization Problem (UMP) (Continued)

We return to Chapter 3, specifically, p.53 of Section 3.D.

The utility maximization problem:

$$\begin{aligned} &\max_{x\in\mathbb{R}^L} u(x) \\ \text{s.t. } &\sum_{l=1}^{L} p_l \cdot x_l = p\cdot x \le w, \\ &x_l \ge 0 \text{ for all } l = 1, ..., L. \end{aligned}$$

Lagrange Function:

$$\mathcal{L}_{x\in\mathbb{R}^L_+,\lambda}(x,\lambda) = u(x) - \lambda(p\cdot x - w).$$

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial x_l} = \frac{\partial u(x^*)}{\partial x_l} - \lambda p_l \le 0, \text{ with equality if } x_l^* > 0, \tag{3.D.1}$$

$$\begin{aligned} &\sum_{l=1}^{L} p_l \cdot x_l = p\cdot x \le w, \\ &x_l \ge 0 \text{ for all } l = 1, ..., L, \\ &\lambda \ge 0, \\ &\lambda(p\cdot x - w) = 0, \text{ i.e., } \lambda = 0 \text{ if } p\cdot x < w. \end{aligned}$$

(3.D.1) can be rewritten as

$$\nabla u(x^*) \le \lambda p \tag{3.D.2}$$

and

$$x^*\cdot[\nabla u(x^*) - \lambda p] = 0. \tag{3.D.3}$$

The constraint $x^* \ge 0$. If we have an **interior solution** (i.e., if $x^* \gg 0$), we must have

$$\nabla u(x^*) = \lambda p. \tag{3.D.4}$$

Condition (3.D.4) shows that at an interior optimum, $\nabla u(x^*)$ must be proportional to p. See Figure 1 below.

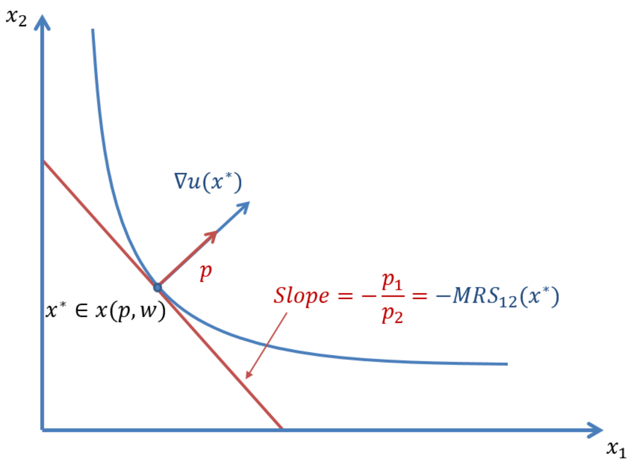

Figure 1: Interior Solution when $L = 2$

Therefore, for any two goods l and k, we have

$$\frac{\partial u(x^*)/\partial x_l}{\partial u(x^*)/\partial x_k} = \frac{p_l}{p_k}. \tag{3.D.5}$$

$\frac{\partial u(x^*)/\partial x_l}{\partial u(x^*)/\partial x_k}$ is the *marginal rate of substitution of good l for good k at x^*, $MRS_{lk}(x^*)$*. It indicates the amount of good k that the consumer needs to get to compensate for 1 unit reduction of good l.

Condition (3.D.5) tells us that at the interior optimum, the consumer's marginal rate of substitution between any two goods must be equal to their price ratio. To see this, suppose on the contrary, $\frac{\partial u(x^*)/\partial x_l}{p_l} > \frac{\partial u(x^*)/\partial x_k}{p_k}$. Then, the consumer can increase her utility by spending ε less on product of k, and ε more on good l. She'll lose ε/p_k units of product k and gain ε/p_l units of product l. This translates into a utility change of $= \varepsilon\left[\frac{\partial u(x^*)/\partial x_l}{p_l} - \frac{\partial u(x^*)/\partial x_k}{p_k}\right] > 0$.

If we have a **boundary solution**, then F.O.C. tells us that $\partial u(x^*)/\partial x_l \leq \lambda p_l$ for those l with $x_l^* = 0$ and $\partial u(x^*)/\partial x_l = \lambda p_l$ for those l with $x_l^* > 0$. See Figure 2 below.

In Figure 2, $MRS_{12}(x^*) > \frac{p_1}{p_2}$. Now, the consumer would want to spend ε less on good 2 and ε more on good 1. But because the consumer's consumption of good 2 is already 0 and thus she is unable to reduce her consumption of good 2 any further.

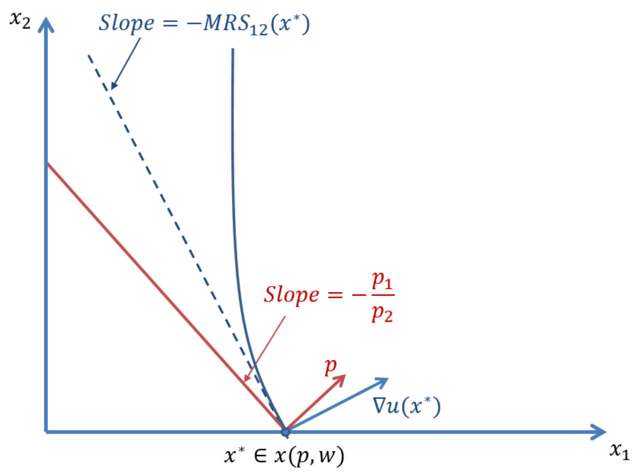

Figure 2: Boundary Solution when $L = 2$

The constraint $p \cdot x \leq w$. If $p \cdot x = w$, then λ measures the marginal, or shadow, value of relaxing the constraint $p \cdot x = w$, or the consumer's *marginal utility of wealth.*

To see this, consider the simple case where $x(p, w)$ is differentiable and $x(p, w) \gg 0$. Then (3.D.1) becomes

$$\nabla u(x^*) = \lambda p.$$

By chain rule, the change in utility from a marginal increase in w gives

$$\nabla u(x(p, w)) \cdot D_w x(p, w) = \lambda p \cdot D_w x(p, w) = \lambda.$$

If $p \cdot x < w$, then the budget constraint is not binding. In this case, relaxing the budget doesn't increase utility, so $\lambda = 0$.

Example 3.D.1. Derive Walrasian Demand Function for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Solution. The problem is:

$$\max_{x \in \mathbb{R}^2} x_1^{\alpha} x_2^{1-\alpha}$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 \leq w$$

$$x_1 \geq 0, x_2 \geq 0.$$

Lagrange Function:

$$\mathcal{L} = x_1^{\alpha} x_2^{1-\alpha} - \lambda(p_1 x_1 + p_2 x_2 - w).$$

Kuhn-Tucker Conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{1-\alpha} - \lambda p_1 \leq 0, \text{ with equality if } x_1 > 0, \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = (1-\alpha) x_1^{\alpha} x_2^{-\alpha} - \lambda p_2 \leq 0, \text{ with equality if } x_2 > 0, \tag{2}$$

$$p_1 x_1 + p_2 x_2 \leq w,$$

$$x_1 \geq 0, x_2 \geq 0,$$

$$\lambda \geq 0,$$

$$\lambda(p_1 x_1 + p_2 x_2 - w) = 0. \tag{3}$$

If $x_1 = 0$ or $x_2 = 0$, we will have $u(x_1, x_2) = 0$. Therefore, utility maximization requires $(x_1, x_2) \gg 0$. Then, (1) and (2) hold with equality. From (1) and (2) with equality,

$$\frac{p_1 x_1}{p_2 x_2} = \frac{\alpha}{1-\alpha}, \tag{4}$$

$$\lambda > 0. \tag{5}$$

(5) and (3) imply

$$p_1 x_1 + p_2 x_2 = w. \tag{6}$$

Therefore, from (4) and (6),

$$x_1 = \frac{\alpha w}{p_1} \text{ and } x_2 = \frac{(1-\alpha) w}{p_2}.$$

The Indirect Utility Function For each $(p, w) \gg 0$, the utility value of UMP (i.e., $u(x^*)$) is denoted $v(p, w) \in \mathbb{R}$. $v(p, w)$ is called the *indirect utility function*.

Example 3.D.2. Derive the indirect utility function for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Solution.

$$v(p, w) = \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha) w}{p_2}\right)^{1-\alpha} = \alpha^{\alpha} (1-\alpha)^{1-\alpha} \frac{w}{p_1^{\alpha} p_2^{1-\alpha}}.$$

Proposition 3.D.3 identifies basic properties of Indirect Utility Function $v(p, w)$.

Proposition 3.D.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The indirect utility function $v(p, w)$ is*

(i) Homogeneous of degree zero.

(ii) Strictly increasing in w and non-increasing in p_l for any l.

(iii) Quasiconvex; that is, the set $\{(p, w) : v(p, w) \le \bar{v}\}$ is convex for any $\bar{v}$.

(iv) Continuous in $p \gg 0$ and w.

Proof.

(i) Recall Proposition 3.D.2, $x(p, w)$ is H.D.$\emptyset$.

$$x(p, w) = x(\alpha p, \alpha w) \iff u(x(p, w)) = u(x(\alpha p, \alpha w)) \iff v(p, w) = v(\alpha p, \alpha w)$$

(ii) $v(p, w)$ **is strictly increasing in** w :

The proof below does not assume that we already have "Proposition 3.D.2, $x(p, w)$ satisfies Walras' law". Suppose $w' > w$. Then $p \cdot x(p, w) \le w < w'$. By continuity, $\exists \varepsilon > 0$ such that for all $\|x' - x(p, w)\| < \varepsilon$, $p \cdot x' < w'$. Let this ball be $b_\varepsilon(x(p, w))$. Since $\succsim$ is locally nonsatiated, $\exists \tilde{x} \in b_\varepsilon(x(p, w))$ such that $u(\tilde{x}) > u(x(p, w))$.

Since $\tilde{x} \in b_\varepsilon(x(p, w))$, we have $p \cdot \tilde{x} < w'$. That is, $\tilde{x} \in B_{p,w'}$. Therefore, $u(x(p, w')) \ge u(\tilde{x})$ since $x(p, w')$ is the solution to the utility maximization problem given (p, w').

Thus, we have $w' > w$ and $u(x(p, w')) \ge u(\tilde{x}) > u(x(p, w))$.

Suppose now that we already know "$x(p, w)$ satisfies Walras' law". Consider $w' > w$. Since $p \cdot x(p, w) = w < w'$, by Walras' law, $x(p, w)$ must not be optimal under the budget $B_{p,w'}$. Therefore, $u(x(p, w')) > u(x(p, w))$.

$v(p, w)$ **is non-increasing in** p :

Suppose $p' \ge p$, then $B_{p',w} \subseteq B_{p,w}$. Therefore, $v(p, w) \ge v(p', w)$.

(iii) Consider (p, w) and (p', w') such that $v(p, w) \le \bar{v}$ and $v(p', w') \le \bar{v}$. The corresponding budgets are denoted by $B_{p,w}$ and $B_{p',w'}$. If for some consumption bundle x, $x \in B_{p,w}$ or $x \in B_{p',w'}$, then we must have $u(x) \le \bar{v}$.

The new budget set is

$$\{x \in \mathbb{R}^L_+ : [\alpha p + (1-\alpha)p'] \cdot x \le \alpha w + (1-\alpha)w'\}$$
$$\iff \{x \in \mathbb{R}^L_+ : \alpha p \cdot x + (1-\alpha)p' \cdot x \le \alpha w + (1-\alpha)w'\}.$$

This implies $p \cdot x \le w$ or $p' \cdot x \le w'$ or both $\implies x \in B_{p,w} \cup B_{p',w'} \implies u(x) \le \bar{v}$.

Figure 3 shows the quasiconvexity of $v(p,w)$ for $L = 2$. $B_{p,w}$ and $B_{p',w'}$ generate the same maximized utility $\bar{u}$. $B_{p'',w''}$, corresponding to $(p'', w'') = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w')$, must generates utility no greater than $\bar{u}$.[1]

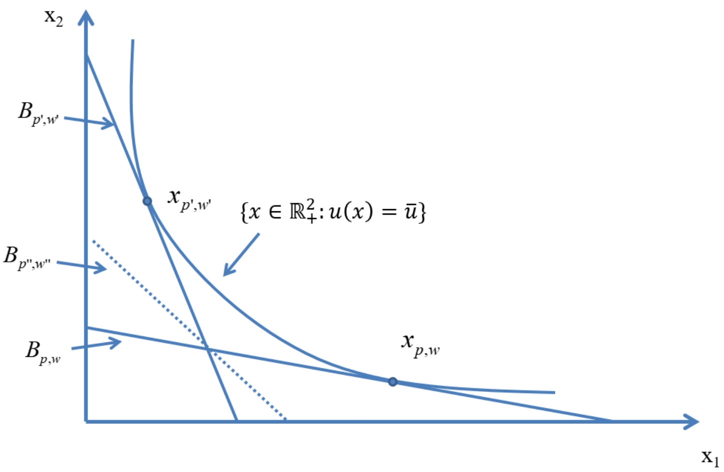

Figure 3: Quasiconvexity of $v(p,w)$

(iv) Consider a sequence $\{(p^n, w^n)\}_{n=1}^{\infty}$, where $\lim_{n\to\infty}(p^n, w^n) = (p,w)$. Let $x(\cdot,\cdot)$ be the solution to the utility maximization problem.

Consider a sufficiently small open ball around (p,w): $b_\varepsilon(p,w) = \{y \in \mathbb{R}^{L+1}_{++} : \|y - (p,w)\| < \varepsilon\}$. For all $(p', w') \in b_\varepsilon(p,w)$, $w' \le w + \varepsilon$ and $p'_l > p_l - \varepsilon$. Therefore, $x_l(p', w') \le \frac{w+\varepsilon}{p_l - \varepsilon}$. There exists N such that for all $n > N$, $(p^n, w^n) \in b_\varepsilon(p,w)$ and $x(p^n, w^n) \le \left(\frac{w+\varepsilon}{p_1-\varepsilon}, \frac{w+\varepsilon}{p_2-\varepsilon}, ..., \frac{w+\varepsilon}{p_L-\varepsilon}\right)$. Since for $n = 1, ..., N$, $x(p^n, w^n)$ are also bounded, we can conclude that for all $n = 1, 2, ..., \infty$, $x(p^n, w^n) \in [0, z]^L$ for some $z > 0$ (sufficiently large). Therefore, by Bolzano-Weierstrass Theorem[2], the infinite sequence $\{\{x(p^n, w^n)\}\}_{n=1}^{\infty}$ must have a convergent subsequence $\{x(p^{m(n)}, w^{m(n)})\}$.

[1]Note that the intersection the budget lines $B_{p,w}$ and $B_{p',w'}$ also satisfies the budget line of $B_{p'',w''}$. In other words, $p \cdot x = w$ and $p' \cdot x = w'$ imply $[\alpha p + (1-\alpha)p'] \cdot x = \alpha w + (1-\alpha)w'$.

[2]Bolzano-Weierstrass Theorem: Each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

We want to show that $\lim_{n\to\infty} x(p^{m(n)}, w^{m(n)}) = x(p, w)$. Suppose to the contrary that $\lim_{n\to\infty} x(p^{m(n)}, w^{m(n)}) = \tilde{x} \neq x(p, w)$. Then since $p^{m(n)} \cdot x(p^{m(n)}, w^{m(n)}) \leq w^{m(n)}$, taking the limit on both sides gives $p \cdot \tilde{x} \leq w$. But since $\tilde{x} \neq x(p, w)$, we must have $u(\tilde{x}) < u(x(p, w))$. Hence, by the continuity of $u(\cdot)$, there exists $\delta \in (0, 1)$ such that $u(\tilde{x}) < u((1-\delta)x(p, w))$. Since $\lim_{n\to\infty} x(p^{m(n)}, w^{m(n)}) = \tilde{x}$, $\exists N_1 \in \mathbb{N}$ such that $\forall n > N_1, u(x(p^{m(n)}, w^{m(n)})) < u((1-\delta)x(p, w))$.

From $p \cdot x(p, w) \leq w$, we have $p \cdot (1-\delta)x(p, w) < w$. Since $\lim_{n\to\infty} p^{m(n)} = p$ and $\lim_{n\to\infty} w^{m(n)} = w$, $\exists N_2 \in \mathbb{N}$ such that $\forall n > N_2, p^{m(n)} \cdot (1-\delta)x(p, w) < w^{m(n)}$.

Thus, $\forall n > \max\{N_1, N_2\}$, we have $u((1-\delta)x(p, w)) > u(x(p^{m(n)}, w^{m(n)}))$ and $p^{m(n)} \cdot (1-\delta)x(p, w) < w^{m(n)}$, which then contradicts the optimality of $x(p^{m(n)}, w^{m(n)})$.

Therefore, every convergent subsequences $x(p^{m(n)}, w^{m(n)})$ converges to $x(p, w)$. That is, $\lim_{n\to\infty} x(p^{m(n)}, w^{m(n)}) = x(p, w)$, and thus $\lim_{n\to\infty} x(p^n, w^n) = x(p, w)$.[3] Therefore, $\lim_{n\to\infty} v(p^n, w^n) = \lim_{n\to\infty} u(x(p^n, w^n)) = u(x(p, w)) = v(p, w)$. □

Exercise 3.D.5

Consider again CES utility function of Exercise 3.C.6, and assume that $\alpha_1 = \alpha_2 = 1$.

(a) Compute Walrasian demand and indirect utility functions.

(b) Verify that the functions satisfy all properties of Propositions 3.D.2 and 3.D.3.

(c) Derive Walrasian demand correspondence and indirect utility function for linear utility and Leontief utility.[a] Show that CES Walrasian demand and indirect utility functions approach these as $\rho \to 1$ and $\rho \to -\infty$, respectively.

(d) The *elasticity of substitution between goods* 1 and 2 is defined as

$$\xi_{12}(p, w) = -\frac{\partial[x_1(p, w)/x_2(p, w)]}{\partial[p_1/p_2]} \frac{p_1/p_2}{x_1(p, w)/x_2(p, w)}.$$

Show that for CES utility function, $\xi_{12}(p, w) = \frac{1}{1-\rho}$, thus justifying the name. What is $\xi_{12}(p, w)$ for linear, Leontief, and Cobb-Douglas utility functions?

[a] See Exercise 3.C.6

[3] If every convergent subsequence converges to a, then so does the original *bounded* sequence.

3.E. The Expenditure Minimization Problem (EMP)

The expenditure minimization problem:

$$\min_{x \in \mathbb{R}^L} p \cdot x$$

$$\text{s.t. } u(x) \geq u$$

$$x \geq 0.$$

The problem is equivalent to

$$\max_{x \in \mathbb{R}^L} -p \cdot x$$

$$\text{s.t. } -u(x) \leq -u$$

$$x \geq 0.$$

Lagrange Function:

$$\underset{x \in \mathbb{R}^L_+, \lambda}{\mathcal{L}(x, \lambda)} = -p \cdot x - \lambda(-u(x) + u)$$

Kuhn-Tucker conditions:

$$\frac{\partial \mathcal{L}}{\partial x_l} = -p_l + \lambda \frac{\partial u(x^*)}{\partial x_l} \leq 0, \text{ with equality if } x_l^* > 0, \tag{7}$$

$$u(x) \geq u,$$

$$x_l \geq 0 \text{ for all } l = 1, ..., L,$$

$$\lambda \geq 0,$$

$$\lambda(u - u(x)) = 0, \text{ i.e., } \lambda = 0 \text{ if } u(x) > u.$$

Equation (7) can be rewritten as

$$p \geq \lambda \nabla u(x^*) \tag{3.E.2}$$

and

$$x^* \cdot [p - \lambda \nabla u(x^*)] = 0. \tag{3.E.3}$$

UMP computes the maximal level of utility that can be obtained given wealth w; EMP computes the minimal level of wealth required to reach utility level u. The two problems are “dual” problems: they capture the same aim of efficient use of consumer’s purchasing power.

To see this, consider the following thought process.

Step 1:

$$\max_{x \geq 0} u(x) \tag{UMP1}$$
$$\text{s.t. } p \cdot x \leq w.$$

Suppose x^* solves (UMP1), and the highest utility is $u(x^*)$.

Step 2:

$$\min_{x \geq 0} p \cdot x \tag{EMP1}$$
$$\text{s.t. } u(x) \geq u(x^*)$$

Claim. x^* solves (EMP1).

Similarly,

Step 1:

$$\min_{x \geq 0} p \cdot x \tag{EMP2}$$
$$\text{s.t. } u(x) \geq u$$

Suppose x^* solves (EMP2), and the lowest expenditure is $p \cdot x^*$.

Step 2:

$$\max_{x \geq 0} u(x) \tag{UMP2}$$
$$\text{s.t. } p \cdot x \leq p \cdot x^*.$$

Claim. x^* solves (UMP2).

Formally, the above claims are stated in Proposition 3.E.1 below.

Proposition 3.E.1. *Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p \gg 0$. We have*

(i) If x^* is optimal in the UMP when wealth is $w > 0$, i.e., $x^* = x(p, w)$, then x^* is optimal in the EMP when the required utility is $u(x^*)$. Moreover, the minimized expenditure in the EMP is w.

*(ii) If x^* is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility in the UMP is u. (*No excess utility)*

Proof.

(i) Suppose x^* is not optimal in the EMP. Then, $\exists x'$ such that $p \cdot x' < p \cdot x^* \leq w$ and $u(x') \geq u(x^*)$. There exists a sufficiently small open ball around x', denoted $b_\varepsilon(x')$, such that for every $\widetilde{x} \in b_\varepsilon(x')$, $p \cdot \widetilde{x} < p \cdot x^* \leq w$ (continuity of $p \cdot x$). Local nonsatiation of u implies that $\exists x'' \in b_\varepsilon(x')$ such that $u(x'') > u(x') \geq u(x^*)$. Since $p \cdot \widetilde{x} < w$ for all $\widetilde{x} \in b_\varepsilon(x')$ and $x'' \in b_\varepsilon(x')$, we have $p \cdot x'' < w$. This implies $x'' \in B_{p,w}$ and $u(x'') > u(x^*)$ and thus contradicts the assumption that x^* is optimal in the UMP. Therefore, x^* must solve the EMP. The minimized expenditure is $p \cdot x^*$. Recall Proposition 3.D.2: Walras' Law is satisfied in the UMP, i.e., $p \cdot x^* = w$.

(ii) Prove $u(x^*) = u$ in EMP first (i.e. no excess utility). Suppose to the contrary, $u(x^*) > u$, then $u(\alpha x^*) > u$ for some $\alpha < 1$ (continuity of $u(\cdot)$). Since αx^* attains the required utility u and is cheaper (since $p \cdot \alpha x^* < p \cdot x^*$), then we reached a contradiction that x^* minimizes expenditure.

Suppose x^* is not optimal in the UMP. Then $\exists x'$ such that $u(x') > u(x^*) \geq u$ and $p \cdot x' \leq p \cdot x^*$. This implies that $\exists \alpha < 1$ such that $u(\alpha x') > u$ and $p \cdot \alpha x' < p \cdot x^*$. This contradicts that x^* solves the EMP.

Therefore, x^* must solve the UMP. And the maximized utility is $u(x^*) = u$. □

The Expenditure Function Let x^* be the/a solution to the EMP. Then $p \cdot x^*$ is the minimized expenditure. Let this be called the *Expenditure Function* and denoted by $e(p, u)$. Proposition 3.E.2 describes the basic properties of $e(p, u)$.

Proposition 3.E.2. *Suppose that $u(\cdot)$ is a continuous utility representing a locally non-satiated preference relation $\succsim$ defined on the consumption set $X = \mathbb{R}^L_+$. The expenditure function $e(p, u)$ is*

(i) Homogeneous of degree one in p.

(ii) *Strictly increasing in u and nondecreasing in p_l for all l.*

(iii) *Concave in p, i.e., $\alpha e(p,u) + (1-\alpha)e(p',u) \leq e(ap + (1-\alpha)p', u)$.*

(iv) *Continuous in $p \gg 0$ and u.*

Proof.

(i) The constraint set $u(x) \geq u$ is unaffected by the change in p.

The solution to

$$\min_{x\geq 0} \alpha p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

and

$$\min_{x\geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u$$

are identical. Therefore, $e(\alpha p, u) = \alpha p \cdot x^* = \alpha e(p,u)$.

(ii) $e(p,u)$ **is strictly increasing in** u**.**

Suppose $e(p,u)$ is NOT strictly increasing in u.

Consider a change from u' to u'' with $u'' > u'$. Let the price be p, and x'' and x' be the optimal bundles for required utility level u'' and u' respectively. Since $e(p,u)$ is not strictly increasing, then we have,

$$p \cdot x'' = e(p,u'') \leq e(p,u') = p \cdot x'.$$

By continuity of $u(\cdot)$ and $u(x'') \geq u'' > u'$, we can find a bundle $\alpha x''$ with $\alpha \in (0,1)$ such that

$$u(\alpha x'') > u'$$
$$\text{and } p \cdot \alpha x'' < p \cdot x'' \leq p \cdot x' = e(p,u').$$

This contradicts that x' minimizes expenditure subject to constraint $u(x) \geq u'$.

Remark. Similar to the Proof of Proposition 3.D.3 (ii), the proof here can be substantially shortened if we already know "Proposition 3.E.3 (ii) No excess utility".

Consider $u'' > u'$. Let the price be p, and x'' and x' be the optimal bundles for required utility level u'' and u' respectively. Since $u(x'') = u'' > u'$, x'' must not be optimal under (p,u'). So, $e(p,u'') > e(p,u')$.

$e(p, u)$ **is nondecreasing in** p_l **for all** l.

Let $e_l = (0, ..., 0, \underbrace{1}_{l^{th} \text{ element}}, 0, ..., 0)$. Consider a price change from p' to $p'' = p' + \alpha e_l$. Let the required utility be u, and x'' and x' be the optimal bundles given prices p'' and p' respectively.

Since $p' \leq p''$,

$$e(p'', u) = p'' \cdot x'' \geq p' \cdot x'' \geq e(p', u)$$

The last inequality follows from the definition of $e(p', u)$: $e(p', u)$ is the minimized expenditure given p' and u, and x'' is a bundle satisfying the constraint $u(x'') \geq u$.

(iii) Let $p'' = ap + (1 - \alpha)p'$ for $\alpha \in [0, 1]$. Let the required utility be u, and x, x' and x'' be the optimal bundles given prices p, p' and p'' respectively. Then,

$$e(p'', u) = p'' \cdot x'' = \alpha p \cdot x'' + (1 - \alpha)p' \cdot x'' \geq \alpha e(p, u) + (1 - \alpha)e(p', u).$$

Intuition of Concavity of $e(p, u)$.

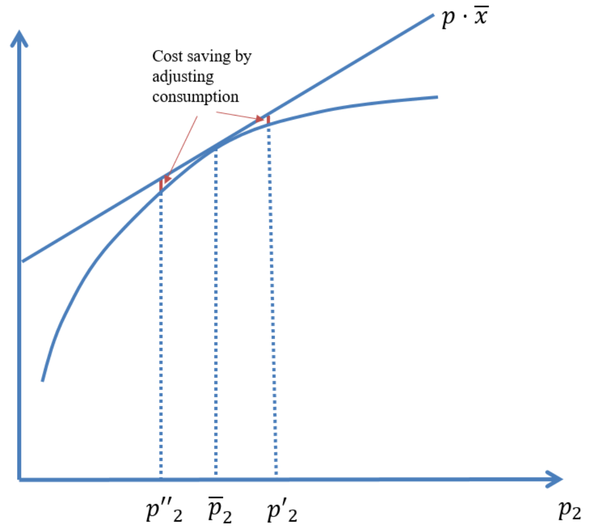

Figure 4: Concavity of $e(p, u)$

If p_2 increases, assuming that x stays at $x = \bar{x}$, the expenditure e increases with p_2 by the amount x_2. However, the consumer can lower the expenditure e by adjusting x to more cost effectively achieve u.

Similarly, if p_2 decreases, assuming that x stays at $x = \bar{x}$, the expenditure e decreases with p_2 by the amount x_2. However, the consumer can further lower the expenditure e by adjusting x to more cost effectively achieve u.

(iv) Suppose the sequence $\{(p^n, u^n)\}_{n=1}^{\infty}$ converges to (p, u). Let $h(\cdot, \cdot)$ be the solution to the expenditure minimization problem. As $h(p^n, u^n)$ is bounded for all n[4], there exists a convergent subsequence $h(p^{m(n)}, u^{m(n)})$.

We want to show that $\lim_{n\to\infty} h(p^{m(n)}, u^{m(n)}) = h(p, u)$. Suppose to the contrary that $\lim_{n\to\infty} h(p^{m(n)}, u^{m(n)}) = \tilde{h} \neq h(p, u)$. Then since $u(h(p^{m(n)}, u^{m(n)})) \geq u^{m(n)}$, taking the limit on both sides gives $u(\tilde{h}) \geq u$. But since $\tilde{h} \neq h(p, u)$, we have $p \cdot \tilde{h} > p \cdot h(p, u)$. There exists a small open ball around $h(p, u)$, denoted $b_\varepsilon(h(p, u))$, such that for every $h' \in b_\varepsilon(h(p, u))$, $p \cdot \tilde{h} > p \cdot h'$. Moreover, by local nonsatiation of u, there exists $\hat{h} \in b_\varepsilon(h(p, u))$, such that $u(\hat{h}) > u(h(p, u)) = u$. Since $p \cdot \tilde{h} > p \cdot h'$ for all $h' \in b_\varepsilon(h(p, u))$ and $\hat{h} \in b_\varepsilon(h(p, u))$, we have $p \cdot \tilde{h} > p \cdot \hat{h}$.

- Since $\lim_{n\to\infty} u^{m(n)} = u$ and $u(\hat{h}) > u$, there exists $N_1 \in \mathbb{N}$ such that $\forall n > N_1, u(\hat{h}) > u^{m(n)}$.
- Since $\lim_{n\to\infty} p^{m(n)} = p$, $\lim_{n\to\infty} h(p^{m(n)}, u^{m(n)}) = \tilde{h}$ and $p \cdot \tilde{h} > p \cdot \hat{h}$, there exists $N_2 \in \mathbb{N}$ such that $\forall n > N_2, p^{m(n)} \cdot h(p^{m(n)}, u^{m(n)}) > p^{m(n)} \cdot \hat{h}$.

Thus, $\forall n > \max\{N_1, N_2\}, p^{m(n)} \cdot \hat{h} < p^{m(n)} \cdot h(p^{m(n)}, u^{m(n)})$ and $u(\hat{h}) > u^{m(n)}$, which then contradicts the optimality of $h(p^{m(n)}, u^{m(n)})$.

Hence, we must have $\lim_{n\to\infty} h(p^{m(n)}, u^{m(n)}) = h(p, u)$, and thus $\lim_{n\to\infty} h(p^n, u^n) = h(p, u)$. Therefore, $\lim_{n\to\infty} e(p^n, u^n) = \lim_{n\to\infty} p^n \cdot h(p^n, u^n) = p \cdot h(p, u) = e(p, u)$. $\square$

Using Proposition 3.E.1, we can connect the expenditure function $e(p, u)$ and the indirect utility function $v(p, w)$:

$$\boldsymbol{e(p, v(p, w)) = w} \quad \text{and} \quad \boldsymbol{v(p, e(p, u)) = u}. \tag{3.E.1}$$

Hicksian (or Compensated) Demand Function The optimal bundle in EMP is denoted as $h(p, u) \subset \mathbb{R}_+^L$ and is called the *Hicksian (or Compensated) demand function/ correspondence.*

[4] It is without loss of generality to modify the expenditure minimization problem to the following:

$$\min_{x \geq 0} p \cdot x$$
$$\text{s.t. } u(x) \geq u \text{ and } p \cdot x \leq M (\text{for some sufficiently large } M).$$

As prices vary, $h(p, u)$ gives the level of demand that would arise if the consumer's wealth were simultaneously adjusted to keep her utility level at u. This type of wealth compensation is called *Hicksian wealth compensation*. From initial price-wealth pair (p, w) and prices change to p', the Hicksian wealth compensation is $\Delta w_{\text{Hicks}} = e(p', u) - w$.

Figure 5 compares Hicksian compensation with Slutsky compensation.

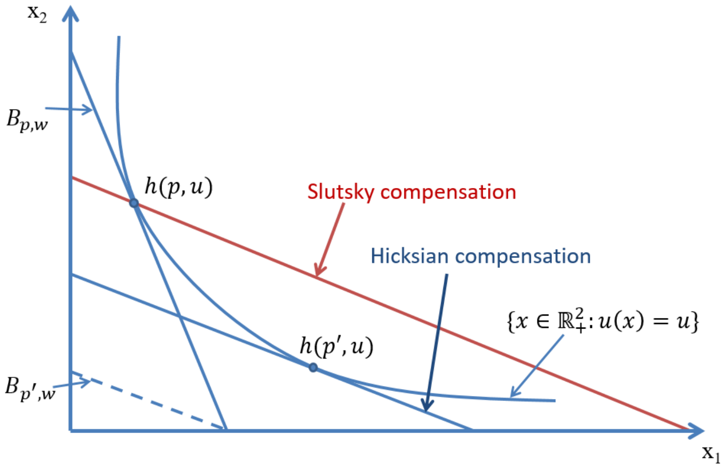

Figure 5: Hicksian compensation and Slutsky compensation

Proposition 3.E.3 describes the basic properties of $h(p, u)$.

Proposition 3.E.3. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. Then for any $p \gg 0$, the Hicksian demand correspondence $h(p, u)$ (i.e., expenditure minimizing demand) possesses the following properties:*

(i) Homogeneity of degree zero in p: $h(\alpha p, u) = h(p, u)$ for all p, u and $\alpha > 0$.

(ii) No excess utility: For any $x \in h(p, u)$, $u(x) = u$.

(iii) Convexity/uniqueness: If $\succsim$ is convex, then $h(p, u)$ is a convex set; and if $\succsim$ is strictly convex, then there is a unique element in $h(p, u)$.

Proof.

(i) The solution to

$$\arg\min_{x \geq 0} \alpha p \cdot x$$

$$\text{s.t. } u(x) \geq u$$

and

$$\arg\min_{x\geq 0} p\cdot x$$
$$\text{s.t. } u(x)\geq u$$

are identical. Therefore, $h(\alpha p,u)=h(p,u)$.

(ii) Suppose $u(x')>u$ for some $x'\in h(p,u)$. By continuity of $u(\cdot)$, $\exists\alpha<1$ such that $u(\alpha x')>u$. However, $p\cdot\alpha x'<p\cdot x'$. This contradicts $x'\in h(p,u)$.

(iii) Suppose $x,x'\in h(p,u)$, then $p\cdot x=p\cdot x'\equiv e(p,u)$. By (ii), $u(x)=u(x')=u$. Let $x''=\alpha x+(1-\alpha)x'$ for some $\alpha\in(0,1)$. $p\cdot x''=\alpha p\cdot x+(1-\alpha)p\cdot x'=e(p,u)$. Convexity of $\succsim$ implies $x''\succsim x$, and $x''\succsim x'$. So $u(x'')\geq u$ and thus $x''\in h(p,u)$.

Suppose $x\neq x'$ and $x,x'\in h(p,u)$. Strict convexity implies $x''\succ x$ and $x''\succ x'$, or $u(x'')>u$. So there is excess utility. Applying the logic in (ii), $\exists\alpha<1$ s.t. $u(\alpha x'')>u$ but $p\cdot\alpha x''<e(p,u)$, constituting a contradiction. □

Using Proposition 3.E.1, we can relate the Hicksian and Walrasian demand correspondences as follows: (assuming single-value demand)

$$\boldsymbol{h(p,u)=x(p,e(p,u))} \quad \text{and} \quad \boldsymbol{x(p,w)=h(p,v(p,w))}. \tag{3.E.4}$$

Exercise 3.E.6

Consider the constant elasticity of substitution utility function studied in Exercises 3.C.6 and 3.D.5 with $\alpha_1=\alpha_2=1$. Derive its Hicksian demand function and expenditure function. Verify the properties of Propositions 3.E.2 and 3.E.3.

Exercise 3.E.9

Use the relations in (3.E.1) to show that the properties of the indirect utility function identified in Proposition 3.D.3 imply Proposition 3.E.2. Likewise, use the relations in (3.E.1) to prove that Proposition 3.E.2 implies Proposition 3.D.3.

Hicksian Demand and the Compensated Law of Demand

Proposition 3.E.4. *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succsim$ and that $h(p,u)$ consists of a single element for* *all $p \gg 0$. Then the Hicksian demand function $h(p,u)$ satisfies the compensated law of demand: for all p' and p'',*

$$(p'' - p') \cdot [h(p'', u) - h(p'.u)] \leq 0. \tag{3.E.5}$$

Proof. $h(p,u)$ is optimal in EMP, so

$$\begin{aligned} & p'' \cdot h(p'', u) \leq p'' \cdot h(p', u) \\ \text{and } & p' \cdot h(p'', u) \geq p' \cdot h(p', u) \\ \Longrightarrow & (p'' - p') \cdot h(p'', u) \leq (p'' - p') \cdot h(p', u) \\ \Longrightarrow & (p'' - p') \cdot [h(p'', u) - h(p'.u)] \leq 0. \quad \square \end{aligned}$$

Example 3.E.1. Suppose $p \gg 0$ and $u > 0$. Derive the Hicksian Demand and Expenditure Functions for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Solution. The problem is:

$$\begin{aligned} & \min_{x \in \mathbb{R}^L} p_1 x_1 + p_2 x_2 \\ \text{s.t. } & x_1^{\alpha} x_2^{1-\alpha} \geq u \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Lagrange Function:

$$\mathcal{L} = -(p_1 x_1 + p_2 x_2) - \lambda(-x_1^{\alpha} x_2^{1-\alpha} + u).$$

Kuhn-Tucker Conditions:

$$\frac{\partial \mathcal{L}}{\partial x_1} = -p_1 + \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha} \leq 0, \text{ with equality if } x_1 > 0, \tag{8}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -p_2 + \lambda(1-\alpha) x_1^{\alpha} x_2^{-\alpha} \leq 0, \text{ with equality if } x_2 > 0, \tag{9}$$

$$x_1^{\alpha} x_2^{1-\alpha} \geq u,$$

$$x_1 \geq 0, x_2 \geq 0,$$

$$\lambda \geq 0,$$

$$\lambda(u - x_1^{\alpha} x_2^{1-\alpha}) = 0. \tag{10}$$

If $x_1 = 0$ or $x_2 = 0$, we will have $x_1^{\alpha} x_2^{1-\alpha} = 0 < u$. Therefore, it requires $(x_1, x_2) \gg 0$.

Then, (8) and (9) hold with equality. From (8) and (9) with equality,

$$\frac{p_1x_1}{p_2x_2} = \frac{\alpha}{1-\alpha}, \tag{11}$$

$$\lambda > 0. \tag{12}$$

(12) and (10) imply

$$x_1^\alpha x_2^{1-\alpha} = u. \tag{13}$$

Therefore, from (11) and (13),

$$\begin{aligned}
h_1(p,u) &= x_1 = u\left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} \\
h_2(p,u) &= x_2 = u\left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} \\
e(p,u) &= p_1h_1(p,u) + p_2h_2(p,u) = \frac{up_1^\alpha p_2^{1-\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}}.
\end{aligned}$$

We will skip Section 3.F Duality.

3.G. Relationships between Demand, Indirect Utility, and Expenditure Functions

This section concerns three relationships:

- Hicksian Demand Function & Expenditure Function;
- Hicksian & Walrasian Demand Functions;
- Walrasian Demand Function & Indirect Utility Function.

Hicksian demand and the expenditure function Recall $e(p,u) = p \cdot h(p,u)$. Now we show $h(p,u) = \nabla_p e(p,u)$

Proposition 3.G.1. *Suppose that $u(\cdot)$ is continuous, representing locally nonsatiated and strictly convex preference relation $\succsim$ defined on $X = \mathbb{R}^L_+$. For all p and u,*

$$h(p,u) = \nabla_p e(p,u).$$

Before proving the result, we will introduce a useful mathematical result called *the Envelope Theorem*.

Consider the following maximization problem:

$$\max_x f(x,\theta)$$
$$\text{s.t. } g_1(x,\theta) = \bar{b}_1$$
$$\vdots$$
$$g_M(x,\theta) = \bar{b}_M$$

where $\theta = (\theta_1, ...\theta_S)$ are parameters affecting f and/or g_i's.

We are now interested in knowing how a marginal change of θ_s would affect the optimal value of f.

The Lagrangian function is:

$$\mathcal{L}(x,\lambda,\theta) = f(x,\theta) - \sum_{m=1}^{M} \lambda_m (g_m(x,\theta) - \bar{b}_m)$$

The first order conditions with respect to x_j's are given by

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial f(x^*(\theta),\theta)}{\partial x_j} - \sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x^*(\theta),\theta)}{\partial x_j} = 0.$$

The effect of a marginal change of θ_s on the optimal value $f(x^*)$ is:

$$\begin{aligned}
\frac{df(x^*(\theta),\theta)}{d\theta_s} &= \frac{\partial f(x^*(\theta),\theta)}{\partial \theta_s} + \sum_{j=1}^{N} \frac{\partial f(x^*(\theta),\theta)}{\partial x_j} \frac{\partial x_j^*(\theta)}{\partial \theta_s} \\
&\underbrace{=}_{\text{F.O.C}} \frac{\partial f(x^*(\theta),\theta)}{\partial \theta_s} + \sum_{j=1}^{N} \left\{ \left[\sum_{m=1}^{M} \lambda_m \frac{\partial g_m(x^*(\theta),\theta)}{\partial x_j} \right] \frac{\partial x_j^*(\theta)}{\partial \theta_s} \right\} \\
&= \frac{\partial f(x^*(\theta),\theta)}{\partial \theta_s} + \sum_{m=1}^{M} \left[\lambda_m \sum_{j=1}^{N} \left\{ \frac{\partial g_m(x^*(\theta),\theta)}{\partial x_j} \frac{\partial x_j^*(\theta)}{\partial \theta_s} \right\} \right] \qquad (14)
\end{aligned}$$

Differentiating the constraints $g_m(x^*(\theta),\theta) = \bar{b}_m$ with respect to θ_s gives

$$\frac{\partial g_m(x^*(\theta),\theta)}{\partial \theta_s} + \sum_{j=1}^{N} \left\{ \frac{\partial g_m(x^*(\theta),\theta)}{\partial x_j} \frac{\partial x_j^*(\theta)}{\partial \theta_s} \right\} = 0 \qquad (15)$$

Substituting (15) into 14) and rearranging yields the **Envelope Theorem**:

$$\frac{df(x^*(\theta),\theta)}{d\theta_s} = \frac{\partial f(x^*(\theta),\theta)}{\partial \theta_s} - \sum_{m=1}^{M} \left[\lambda_m \frac{\partial g_m(x^*(\theta),\theta)}{\partial \theta_s} \right] = \frac{\partial \mathcal{L}(x^*(\theta),\lambda,\theta)}{\partial \theta_s}.$$

In words, the **Envelope Theorem** tells us that the change in the optimal value $f(x^*)$ with respect to a marginal change of the parameter θ_s is given by the partial derivative of the Lagrangian with respect to θ_s.

Remark. The interpretation of Lagrange multiplier we discussed earlier is a special case of the Envelope Theorem. (Please check by yourself.)

We will then use the Envelope Theorem to prove Proposition 3.G.1 above.

Proof. We focus on the case where $h(p,u) \gg 0$ and $h(p,u)$ is differentiable at (p,u). The expenditure minimization problem could be written as

$$\max_{x \in \mathbb{R}^L_+} -p \cdot x$$

$$\text{s.t. } u(x) = u$$

The minimized expenditure is $e(p,u) = p \cdot x^*$, where x^* is the solution to the problem.

Lagrange Function:

$$\underset{x \in \mathbb{R}^L_+, \lambda}{\mathcal{L}(x, \lambda)} = -p \cdot x - \lambda(-u(x) + u)$$

By Envelope Theorem,

$$\frac{\partial(-e(p,u))}{\partial p_l} = \frac{\partial \mathcal{L}(x^*, \lambda^*, p)}{\partial p_l} = -x_l^* = -h_l(p,u).$$

That is, $\frac{\partial(e(p,u))}{\partial p_l} = h_l(p,u)$. In matrix notation, $h(p,u) = \nabla_p e(p,u)$. □

Example. Verify $h(p,u) = \nabla_p e(p,u)$ for Cobb-Douglas Utility Function: $u(x_1,x_2) = x_1^\alpha x_2^{1-\alpha}$.

Solution. $h_1(p,u)$, $h_2(p,u)$ and $e(p,u)$ are solved in Example 3.E.1.

$$\frac{\partial e(p,u)}{\partial p_1} = \frac{u\alpha p_1^{\alpha-1} p_2^{1-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} = u\left[\frac{\alpha p_2}{(1-\alpha)p_1}\right]^{1-\alpha} = h_1(p,u);$$
$$\frac{\partial e(p,u)}{\partial p_2} = \frac{u p_1^\alpha (1-\alpha) p_2^{-\alpha}}{\alpha^\alpha (1-\alpha)^{1-\alpha}} = u\left[\frac{(1-\alpha)p_1}{\alpha p_2}\right]^{\alpha} = h_2(p,u).$$

Proposition 3.G.2 summarizes the properties of $D_p h(p,u)$.

Proposition 3.G.2. *Suppose $u(\cdot)$ is continuous utility function representing a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Suppose $h(p,u)$ is continuously differentiable at (p,u), and denote the $L \times L$ derivative matrix by $D_p h(p,u)$. Then*

(i) $D_p h(p,u) = D_p^2 e(p,u)$.

(ii) $D_p h(p,u)$ *is negative semidefinite.*

(iii) $D_p h(p,u)$ *is symmetric.*

(iv) $D_p h(p,u)p = 0$.

Proof.

(i) Property (i) follows immediately from Proposition 3.G.1 by differentiation.

(ii) Recall that $e(p,u)$ is concave. Below we show that concavity of a function $f(x)$ implies $D^2 f(x)$ is negative semidefinite. Once this is established, then it implies $D_p^2 e(p,u) = D_p h(p,u)$ is negative semidefinite.

By Taylor expansion, $f(x+\alpha z) = f(x) + \nabla f(x)\cdot(\alpha z) + \frac{1}{2}(\alpha z)\cdot D^2 f(x+\beta z)(\alpha z)$ for some $\beta \in (0,\alpha)$. Then, $\frac{\alpha^2}{2} z \cdot D^2 f(x+\beta z) z = f(x+\alpha z) - f(x) - \nabla f(x)\cdot(\alpha z) \le 0$. The inequality follow from the concavity of $f(x)$. This holds for α, β arbitrarily small. Therefore, $z \cdot D^2 f(x) z \le 0$ must hold. To see this, suppose otherwise $z \cdot D^2 f(x) z > 0$. Then for β sufficiently small (as $\alpha \to 0$), $z \cdot D^2 f(x+\beta z) z > 0$, which constitutes a contradiction.

(iii) Symmetry of $D_p^2 e(p,u)$ is due to Schwarz' theorem (or Clairant's theorem) and that $e(p,u)$ is C^2. Therefore, $D_p h(p,u) = D_p^2 e(p,u)$ is symmetric.

(iv) Note that since $h(p,u)$ is H.D.∅ in p, then

$$h(\alpha p, u) = h(p,u)$$

Differentiating both sides of the equation by α gives

$$D_{\alpha p} h(\alpha p, u) \cdot p = 0.$$

This holds for $\alpha = 1$. So, $D_p h(p,u) \cdot p = 0$. □

Remark. Negative semidefiniteness of $D_p h(p,u)$ is the differential analog of compensated law of demand (3.E.5). Condition (3.E.5) implies $dp \cdot dh(p,u) \le 0$. Substituting $dh(p,u) = D_p h(p,u) dp$ gives $dp \cdot D_p h(p,u) dp \le 0\ \forall dp$. Note also that semidefiniteness of $D_p h(p,u)$ implies that $\frac{\partial h_l(p,u)}{\partial p_l} \le 0\ \forall l$; that is, compensated own-price effects are non-positive.

Remark. Symmetry of $D_p h(p,u)$ is not obvious at all ex ante. It's only obvious after we know that $h(p,u) = D_p e(p,u)$.

Remark. Two goods l and k are called *substitutes* at (p,u) if $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$; and *complements* at (p,u) if $\frac{\partial h_l(p,u)}{\partial p_k} \leq 0$.[5] Since $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$, by (iv), there must exist a good k such that $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$; that is, every good has at least one substitute.

The Hicksian and Walrasian Demand Functions Proposition 3.G.3 shows that $D_p h(p,u)$ can be computed from the observable Walrasian demand function $x(p,w)$.[6]

Proposition 3.G.3 (The Slutsky Equation)**.** *Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Then for all (p,w), and $u = v(p,w)$, we have*

For all l, k,

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

or

$$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T$$

Proof. Recall (3.E.4), $h(p,u) = x(p, e(p,u))$. If follows that for any u,

$$\begin{aligned}\frac{\partial h_l(p,u)}{\partial p_k} &= \frac{\partial x_l(p,e(p,u))}{\partial p_k} + \frac{\partial x_l(p,e(p,u))}{\partial e(p,u)} \frac{\partial e(p,u)}{\partial p_k} \\ &= \frac{\partial x_l(p,e(p,u))}{\partial p_k} + \frac{\partial x_l(p,e(p,u))}{\partial e(p,u)} h_k(p,u). \qquad (16)\end{aligned}$$

Since it is assumed that $u = v(p,w)$, we have

$$h(p,u) = h(p, v(p,w)) = x(p,w) \;\; (3.E.4) \quad \text{and} \quad e(p,u) = e(p,v(p,w)) = w \;\; (3.E.1).$$

So, we can write (16) as

$$\left.\frac{\partial h_l(p,u)}{\partial p_k}\right|_{u=v(p,w)} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w). \quad \square$$

Remark. In Chapter 2, we derived the same result, except that it was based on a different compensation (Slutsky compensation). Recall,

- Slutsky compensation: $\Delta w_{\text{Slutsky}} = p' \cdot x(\bar{p}, \bar{w}) - \bar{w}$;
- Hicksian Compensation: $\Delta w_{\text{Hicksian}} = e(p', \bar{u}) - \bar{w}$.

[5]For Walrasian demand, two goods l and k are called *gross substitutes* if $\frac{\partial x_l(p,w)}{\partial p_k} \geq 0$; and *gross complements* if $\frac{\partial x_l(p,w)}{\partial p_k} \leq 0$.

[6]Hicksian demand function is not directly observable. It has consumer's utility level as an argument.

In general, $\Delta w_{\text{Hicksian}} \leq \Delta w_{\text{Slutsky}}$ (see Figure 5). We have just shown that for a differential change in price, Slutsky and Hicksian compensations are identical. This observation is useful because the RHS terms are directly observable.

Example. Verify the Slutsky equation for Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$.

Solution. $h_1(p, u)$ and $h_2(p, u)$ are solved in Example 3.E.1. $x_1(p, w)$ and $x_2(p, w)$ are solved in Example 3.D.1.

Calculation of LHS:

$$D_p h(p,u) = \begin{bmatrix} -\alpha u \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{p_2}{p_1^2} & \alpha u \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{1}{p_1} \\ \alpha u \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{1}{p_1} & -\alpha u \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{1}{p_2} \end{bmatrix}.$$

Substituting $u = \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha}$ into the expression for $D_p h(p, u)$ yields

$$\begin{aligned} D_p h(p,u) &= \begin{bmatrix} -\alpha \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{p_2}{p_1^2} & \alpha \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{1}{p_1} \\ \alpha \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{1}{p_1} & -\alpha \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} \left(\frac{1-\alpha}{\alpha}\frac{p_1}{p_2}\right)^{\alpha} \frac{1}{p_2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\alpha(1-\alpha)w}{p_1^2} & \frac{\alpha(1-\alpha)w}{p_1 p_2} \\ \frac{\alpha(1-\alpha)w}{p_1 p_2} & -\frac{\alpha(1-\alpha)w}{p_2^2}. \end{bmatrix} \end{aligned}$$

Calculation of RHS:

$$\begin{aligned} D_p x(p,w) + D_w x(p,w) x(p,w)^T &= \begin{bmatrix} -\frac{\alpha w}{p_1^2} & 0 \\ 0 & -\frac{(1-\alpha)w}{p_2^2} \end{bmatrix} + \begin{bmatrix} \frac{\alpha}{p_1} \\ \frac{1-\alpha}{p_2} \end{bmatrix} \begin{bmatrix} \frac{\alpha w}{p_1} & \frac{(1-\alpha)w}{p_2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\alpha(1-\alpha)w}{p_1^2} & \frac{\alpha(1-\alpha)w}{p_1 p_2} \\ \frac{\alpha(1-\alpha)w}{p_1 p_2} & -\frac{\alpha(1-\alpha)w}{p_2^2}. \end{bmatrix} \end{aligned}$$

Therefore, $D_p h(p, u) = D_p x(p, w) + D_w x(p, w)x(p, w)^T$.

Walrasian Demand and Indirect Utility Function For EMP, we have $h(p, u) = \nabla_p e(p, u)$ (Proposition 3.G.1). Proposition 3.G.4 below shows the analog statement for UMP.

Proposition 3.G.4 (Roy's Identity)**.** *Suppose that $u(\cdot)$ is A continuous utility function representing a locally nonsatiated and strictly convex $\succsim$ on $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\bar{p}, \bar{w}) \gg 0$.*

Then

$$x(\bar{p},\bar{w}) = -\frac{1}{\nabla_w v(\bar{p},\bar{w})}\nabla_p v(\bar{p},\bar{w})$$

i.e., for every $l = 1, ..., L$:

$$x_l(\bar{p},\bar{w}) = \frac{-\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w}.$$

Proof. We focus on the case where $x(p,w) \gg 0$ and $x(p,w)$ is differentiable at (p,w).

The utility maximization problem could be written as

$$\max_{x\in\mathbb{R}^L_+} u(x)$$

$$\text{s.t. } p \cdot x = w$$

The maximized utility is $v(p,w) = u(x^*)$, where x^* is the solution to the maximization problem.

Lagrange Function:

$$\underset{x\in\mathbb{R}^L_+,\lambda}{\mathcal{L}(x,\lambda)} = u(x) - \lambda(p\cdot x - w)$$

By Envelope Theorem,

$$\begin{aligned}
\frac{\partial(v(p,w))}{\partial p_l} &= \frac{\partial\mathcal{L}(x^*,\lambda^*,\bar{p},\bar{w})}{\partial p_l} = -\lambda x_l^*. \\
\frac{\partial(v(p,w))}{\partial w} &= \frac{\partial\mathcal{L}(x^*,\lambda^*,\bar{p},\bar{w})}{\partial w} = \lambda. \\
\Longrightarrow \frac{\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w} &= -x_l^* = -x_l(\bar{p},\bar{w})
\end{aligned}$$

That is, $x_l(\bar{p},\bar{w}) = \frac{-\partial v(\bar{p},\bar{w})/\partial p_l}{\partial v(\bar{p},\bar{w})/\partial w}$. □

Example. Verify Roy's identity for Cobb-Douglas Utility Function: $u(x_1,x_2) = x_1^\alpha x_2^{1-\alpha}$.

Solution. Direct computation of $-\frac{1}{\nabla_w v(p,w)}\nabla_p v(p,w)$ gives:

$$\begin{aligned}
&-\frac{1}{\nabla_w v(p,w)}\nabla_p v(p,w) \\
&= -\left(\frac{\alpha}{p_1}\right)^{-\alpha}\left(\frac{1-\alpha}{p_2}\right)^{\alpha-1}\left(-\left(\frac{\alpha}{p_1}\right)^{\alpha+1}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} w, -\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{2-\alpha} w\right) \\
&= \left(\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}\right) = x(p,w).
\end{aligned}$$

Hence, Roy's identity holds, i.e., $x(p,w) = -\frac{1}{\nabla_w v(p,w)}\nabla_p v(p,w)$.

Summary Figure 6 below summarizes the relationships between UMP and EMP.

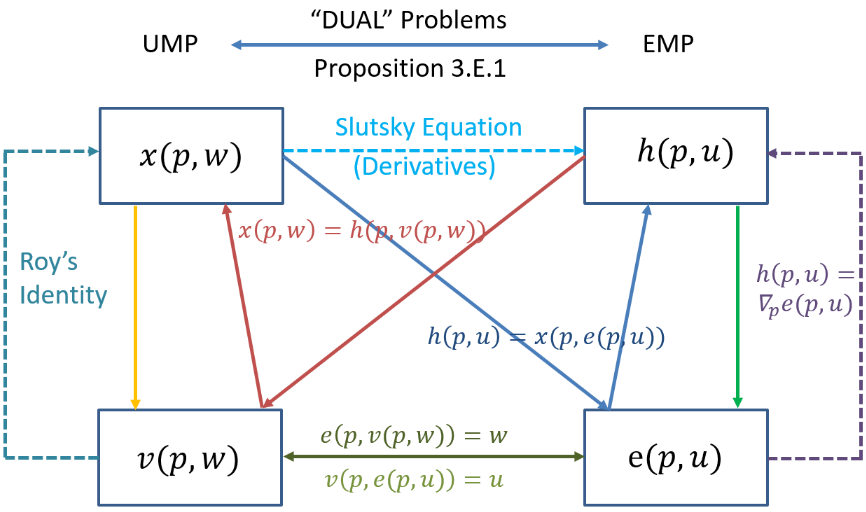

Figure 6: Relationships between UMP and EMP

Exercise 3.G.1

Prove that Proposition 3.G.1 is implied by Roy's identity (Proposition 3.G.4).

Exercise 3.G.8

The indirect utility function $v(p,w)$ is logarithmically homogeneous if $v(p, \alpha w) = v(p,w) + \ln \alpha$ for $\alpha > 0$ [in other words, $v(p,w) = \ln(v^*(p,w))$, where $v^*(p,w)$ is homogeneous of degree one in w]. Show that if $v(\cdot,\cdot)$ is logarithmically homogeneous, then $x(p,1) = -\nabla_p v(p,1)$.

Exercise 3.G.15

Consider the utility function $u = 2x_1^{1/2} + 4x_2^{1/2}$.

(a) Find demand functions for goods 1 and 2 as they depend on prices and wealth.

(b) Find compensated demand function $h(\cdot)$.

(c) Find the expenditure function, and verify that $h(p,u) = \nabla_p e(p,u)$.

(d) Find the indirect utility function, and verify Roy's identity.