Chapter 5. Maximum Value Functions (Exercises)

Exercise 5.1: The Cobb-Douglas Cost Function. Consider a production function

$$y = A \prod_{j=1}^{n} x_j^{\alpha_j},$$

where y is output, x_j's are inputs, A and α_j's are positive constants. Let $w = (w_j)$ be the vector of input prices. Suppose the producer wishes to produce a fixed quantity y at minimum cost.

Question 1: Write out the cost minimization problem and solve for the minimum cost function $C(w, y)$. Hint: the minimum cost function should be

$$C(w, y) = \beta (y/A)^{1/\beta} \prod_{j=1}^{n} (w_j/\alpha_j)^{\alpha_j/\beta},$$

where $\beta = \sum_{j=1}^{n} \alpha_j$.

Question 2: If $\beta < 1$, calculate the corresponding maximum profit function $\pi(p, w)$, where p is the output price. What goes wrong if $\beta \geq 1$?

Exercise 5.2: The CES Expenditure Function. Suppose the direct utility function is

$$U(x, y) = \left[\alpha x^{\rho} + \beta y^{\rho}\right]^{1/\rho},$$

where x and y are the quantities of the two goods, and $\alpha > 0$, $\beta > 0$, $\rho < 1$ are given constants. The prices of good x and y are p and q respectively.

Question 1: Show that the expenditure function is of the form

$$E(p, q, u) = \left[a p^{r} + b q^{r}\right]^{1/r} u,$$

where u is the target utility level, and a, b, and r are constants that can be expressed in terms of α, β and ρ.

Question 2: Show that the ratio of the cost-minimizing quantities is

$$x/y = (a/b)(q/p)^{1-r}.$$

The elasticity of (x/y) with respect to (q/p):

$$\frac{\mathrm{d}\ln(x/y)}{\mathrm{d}\ln(q/p)}.$$

is called the elasticity of substitution in production. Show that in this example, it is constant and equal to $(1-r)$. What condition must be imposed on ρ to ensure a non-negative elasticity of substitution, that is, $r < 1$?