Statistics and Prediction

Assignment 2 (Solution)

Exercise 7.1 (p.99) Basic Model of Competitive Obfuscation

Let $n = 2$ and $c = 0$. Modify the model by allowing prices to get values in $\{0, 1\}$ only. In addition, assume that for every firm $k = 1, 2$, the consumer observes $m_k = k$ independent sample points from the firm's price distribution and chooses the firm with the lowest average price in the sample. In case of a tie, he chooses firm 2. If the average price of both firms in the consumer's sample is 1, he opts out and chooses none of the firms. Formulate the interaction between the firms as a strategic game and find its Nash equilibrium.

Solution.

Consumer observes 1 sample from Firm 1, and 2 samples from Firm 2. The table below summarizes possibilities of the consumer's sample and which firm the consumer would choose after observing his sample.

Firm 1	0	0	0	0	1	1	1	1
Firm 2	00	01	10	11	00	01	10	11
Winning Firm	2	1	1	1	2	2	2	no

Suppose Firm 1 chooses $p = 0$ with probability α and $p = 1$ with probability $1 - \alpha$. Similarly, suppose Firm 2 chooses $p = 0$ with probability β and $p = 1$ with probability $1 - \beta$.

Firm 1 wins with probability $\alpha(1 - \beta^2)$;

Firm 2 wins with probability $(1 - \alpha)[1 - (1 - \beta)^2] + \alpha\beta^2$.

Firm 1' problem:

$$\max_{\alpha}(1 - \alpha) \cdot [\alpha(1 - \beta^2)]$$

The optimal choise is $\alpha^* = \frac{1}{2}$, independent of β.

Firm 2' problem:

$$\max_{\beta}(1 - \beta) \cdot \{(1 - \alpha)[1 - (1 - \beta)^2] + \alpha\beta^2\}$$

Since Firm 2 knows that Firm 1 would choose $\alpha^* = \frac{1}{2}$ independent of his choice, it is without loss to plug $\alpha = \frac{1}{2}$ into Firm 2's problem. Therefore, Firm 2' problem is reduced to

$$\max_{\beta}(1-\beta)\cdot\{\frac{1}{2}[1-(1-\beta)^2]+\frac{1}{2}\beta^2\} \iff \max_{\beta}(1-\beta)\cdot\beta$$

The optimal choise is $\beta^* = \frac{1}{2}$

Exercise 7.2 (p.99) Basic Model of Competitive Obfuscation

Let $n = 2$ and $c = 0$. Modify the model by assuming that consumers draw a sequence of two sample points (p_i^1, p_i^2) from each firm i's *cdf* G_i. If $p_i^k < p_j^k$ for both $k = 1, 2$ (i.e., if firm i dominates firm j in the consumer's sample) the consumer chooses firm i. If no firm dominates another in the sample, the consumer chooses each firm with probability $\frac{1}{2}$. Formulate the interaction as a strategic game and show that there is a Nash equilibrium in which $G_1 = G_2 \equiv U[0, 1]$.

Solution. Given Firm 2's strategy $G_2 \equiv U[0, 1]$, we show that Firm 1's strategy being $G_1 \equiv U[0, 1]$ is a best response.

For Firm 1, the probability of winning in the first sample when Firm 1's realization is p is $H_1(p|G_2) = (1 - G_2(p)) = 1 - p$.

The expected probability of winning in the first sample is $\mathbb{E}_{G_i}[H_1(p|G_2)] = 1 - \mu_{G_i}$, where μ_{G_i} is the expected price according to G_i.

The draws are independent, so the probability of winning in the second sample is also $1 - \mu_{G_i}$.

Therefore, the probability of being chosen is

$$(1-\mu_{G_i})^2 + \frac{1}{2}\mu_{G_i}(1-\mu_{G_i}) + \frac{1}{2}(1-\mu_{G_i})\mu_{G_i} = 1-\mu_{G_i}.$$

The expected profit is

$$\pi_1 = \underbrace{\mu_{G_i}}_{\text{expected markup}} \cdot \underbrace{(1-\mu_{G_i})}_{\uparrow \atop \text{size of clientele}}.$$

$\mu_{G_1} = \frac{1}{2}$ maxiimzes π_1. Therefore, any $G_1(p)$ with μ_{G_1} is a best response. And $U[0, 1]$ is such a distribution.

Thus, we have shown that the given Firm 2's strategy $G_2 \equiv U[0,1]$, Firm 1's strategy being $G_1 \equiv U[0,1]$ is a best response. By symmetry, given Firm 1's strategy $G_1 \equiv U[0,1]$, Firm 2's strategy being $G_2 \equiv U[0,1]$ is a best response. Therefore, $G_1 = G_2 \equiv U[0,1]$ is a Nash Equilibrium.

Exercise 7.3 (p.106) "Simple" Options

Suppose that there is only one firm in the market. The firm chooses a *cdf* over $[c, 1]$. In addition to this firm, the consumer has access to a simple option at a fixed price p_0. For simplicity, assume that the consumer breaks ties in favor of the firm. Show how the firm's optimal *cdf* varies with the firm's marginal cost c.

Solution.

If $p_0 < c$, the firm's product would never be chosen, independent of the firm's price distribution. If $p_1 \geq 1$, the firm's product would always be chosen, independent of the firm's price distribution, and thus the firm will always choose $p = 1$.

We next consider the more interesting case when $p_0 \in [c, 1)$.

Because the probability of being chosen is 0 for all $p > p_0$, the firm has no incentive to assign any weight to prices in $(p_0, 1)$. Any price $p \in (p_0, 1)$ is dominated by $p = 1$.

On the other hand, since there is no other competitors except the simple option, and the firm wins over the simple option as long as $p \leq p_0$. Any price $p < p_0$ is dominated by $p = p_0$. Therefore, the firm would at most choose two prices $p = p_0$ and $p = 1$.

Assume that the firm chooses $p = p_0$ with probability α and $p = 1$ with probability $1 - \alpha$. The firm's profit is

$$\max_{\alpha} \underbrace{[\alpha(p_0 - c) + (1-\alpha)(1-c)]}_{\text{expected markup}} \cdot \underbrace{\alpha}_{\text{size of clientele}}.$$

The optimal α is

$$\alpha^* = \frac{(1-c)}{2(1-p_0)}.$$

For α^* to be meaningful, we require $\alpha^* \in [0, 1]$.

$\alpha^* \geq 0$ is satisfied since $c < 1$ and $p_0 < 1$. $\alpha^* \leq 1$ requires $p_0 \leq \frac{1+c}{2}$.

When $p_0 > \frac{1+c}{2}$, under the constraint $\alpha \in [0, 1]$, we will obtain a corner solution and it is optimal to choose $\alpha = 1$, i.e., to assign probability 1 to $p = p_0$.

To conclude,

(i) If $p_0 \in [c, \frac{1+c}{2}]$, the firm chooses the following *cdf*:

$$\begin{cases} p_0 & \text{with probability } \frac{(1-c)}{2(1-p_0)} \\ 1 & \text{with the remaining probability} \end{cases}$$

(ii) If $p_0 \in (\frac{1+c}{2}, 1)$, the firm assigns probability 1 to $p = p_0$.

(iii) If $p_0 \geq 1$, the firm assigns probability 1 to $p = 1$.

(iv) If $p_0 < c$, the firm's pricing strategy is irrelevant since no matter what *cdf* the firm chooses, its product would never be chosen by the consumer.

Exercise 8.1 (p.115) DeBruijn Sequence

Construct a DeBruijn sequence of order 4.

Solution.

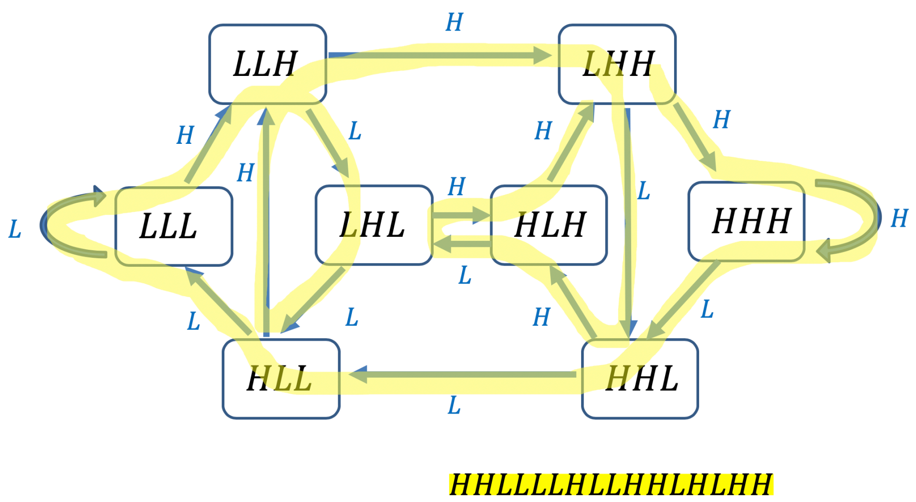

Figure 1: DeBruijn sequence of order 4

Exercise 8.2 (p.118) Complex Price Patterns

Assume $T_h = 2$. Construct a cyclic price sequence, such that after every price history, type h can only predict that the probability of L is $\frac{2}{3}$. Find a value T_l^* such that whenever type l has $T_l \geq T_l^*$, he can make a perfect prediction of the price in each period.

Solution.

To construct a price sequence, similar to the method introduced in the main text, we create two "copies" of L, labeled L_1 and L_2. The copies are relevant only for the monopolist's construction of the price sequence; consumers who observe the price sequence cannot distinguish between L_1 and L_2.

Next, we construct a complete directed graph in which the set of nodes is $V = \{L_1, L_2, H\}^2$. Every node has two elements, so knowledge of the $T_h = 2$ most recent price realizations conveys no information about the next price, because it is equivalent to knowing only that the next link in the cycle will originate from a particular node. Besides, with two copies of L, type h can only predict after every price history that the probability of L in the next period is $\frac{2}{3}$.

One such graph is illustrated in Figure 2.

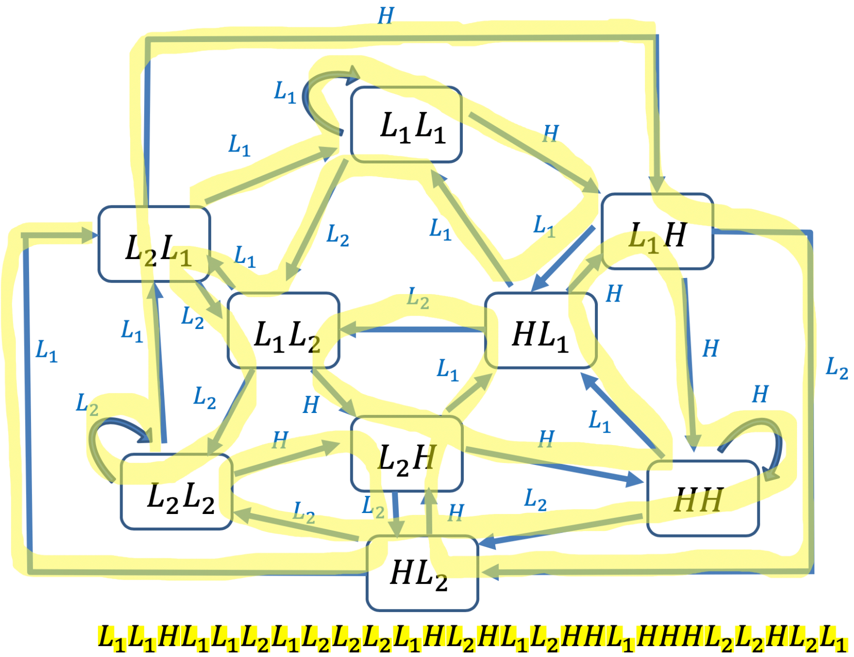

Figure 2: Construction of the Price Sequence

(i) From Figure 2, a possible price sequence is $LLHLLLLLLLHLHLLHHLHHHLLHLL$.

(ii) According to the constructed sequence, as long as $T_l \geq T_l^* = 8$, type l can make a perfect prediction of the price in each period.

Exercise 8.3 (p.122) "Partially" Coarse Buyers

Let $b = 0$ and $\Pi = \{[0, d), [d, 1]\}$. Show that if $d \in (0, \frac{1}{6})$, the buyer's optimal action given this partition is $p^* = \frac{1}{4}(1 + d)$, such that trade occurs if and only if $\omega \leq \frac{1}{4}(1 + d)$.

Solution.

The same method used for the K partition case discussed in the main text applies for this problem.

Let $\pi_1 = [0, d)$ and $\pi_2 = [d, 1]$.

The expected values conditional on the sate being π_1 and π_2 are

$$\mathbb{E}(v|\pi_1) = \frac{d}{2} \quad \text{and} \quad \mathbb{E}(v|\pi_2) = \frac{d+1}{2}.$$

The probabilities of trade conditional on the state being π_1 and π_2 are

$$\Pi[f(\omega \leq p)|\pi_1] = \begin{cases} 1 & \text{if } p \in \pi_2 \\ \frac{p}{d} & \text{if } p \in \pi_1 \end{cases}$$

and

$$\Pi[f(\omega \leq p)|\pi_2] = \begin{cases} \frac{p-d}{1-d} & \text{if } p \in \pi_2 \\ 0 & \text{if } p \in \pi_1 \end{cases}$$

We need to consider two cases, $p \in \pi_1$ and $p \in \pi_2$.

Case I: $p \in \pi_1$

The buyer chooses p to maximize

$$U_1 = d \cdot \underbrace{\frac{p}{d}}_{\Pi[f(\omega \leq p)|\pi_1]} \cdot \Big(\underbrace{\frac{d}{2}}_{\uparrow \atop \mathbb{E}(v|\pi_1)} - p \Big) + (1-d) \cdot \underbrace{0}_{\Pi[f(\omega \leq p)|\pi_2]} \cdot \Big(\underbrace{\frac{d+1}{2}}_{\uparrow \atop \mathbb{E}(v|\pi_2)} - p \Big).$$

The optimal p is $p^* = \frac{d}{4}$. And the maximized payoff is $U_1^* = \frac{1}{16} d^2$.

Case II: $p \in \pi_2$

The buyer chooses p to maximize

$$U_2 = d \cdot \underbrace{1}_{\Pi[f(\omega \leq p)|\pi_1]} \cdot \Big(\underbrace{\frac{d}{2}}_{\substack{\uparrow \\ \mathbb{E}(v|\pi_1)}} - p \Big) + (1-d) \cdot \underbrace{\frac{p-d}{1-d}}_{\Pi[f(\omega \leq p)|\pi_2]} \cdot \Big(\underbrace{\frac{d+1}{2}}_{\substack{\uparrow \\ \mathbb{E}(v|\pi_2)}} - p \Big).$$

The optimal p is $p^* = \frac{d+1}{4}$. And the maximized payoff is $U_2^* = \frac{1}{16}d^2 - 6d + 1$.

Compare the result in Case I and Case II, $U_2^* > U_1^* \iff d < \frac{1}{6}$.

Therefore, if $d \in (0, \frac{1}{6})$, the buyer's optimal action given this partition is $p^* = \frac{1}{4}(1+d)$, and trade occurs if and only if $\omega \leq \frac{1}{4}(1+d)$.