Solution Manual

If you find mistakes in the solutions or you find better ways to solve the problems, please contact me directly via Github Issue or email sherryecon@qq.com or qq.

Contents

Chapter 2. Lagrange's Method

Exercise 2.1: Cobb-Douglas Utility Function Consider a consumer choosing between two goods x and y, with prices p and q respectively. His income is I, so the budget constraint is

$$px + qy = I.$$

Suppose the utility function is

$$\widetilde{U}(x, y) = x^{\alpha} y^{\beta}.$$

What is the consumer's optimal bundle (x, y)? Compare your answer to the answers in Example 2.1.

Solution. Steps to solve the problem:

1. State the problem:

$$\max_{x,\, y} \widetilde{U}(x, y) \equiv \max_{x,\, y} \alpha \ln(x) + \beta \ln(y)$$
$$\text{s.t. } px + qy = I.$$

2. Set up the Lagrangian:

$$\mathcal{L}(x, y, \lambda) = x^{\alpha} y^{\beta} + \lambda \left(I - px - qy\right).$$

3. Write out the first-order necessary conditions:

$$\partial \mathcal{L} / \partial x = \alpha x^{\alpha - 1} y^{\beta} - \lambda p = 0,$$
$$\partial \mathcal{L} / \partial y = \beta x^{\alpha} y^{\beta - 1} - \lambda q = 0,$$
$$\partial \mathcal{L} / \partial \lambda = I - px - qy = 0.$$

4. Solve the above equation system. The method introduced in Example 2.1 is applicable. The solution is

$$x = \frac{\alpha I}{(\alpha + \beta) p}, \qquad y = \frac{\beta I}{(\alpha + \beta) q}, \qquad \lambda = \frac{(\alpha + \beta)}{I}.$$

Comparison with Example 2.1. The problem in Example 2.1 is

$$\max_{x,y} U(x,y) \equiv \max_{x,y} \alpha \ln(x) + \beta \ln(y)$$
$$\text{s.t. } px + qy = I.$$

The solutions to these two problems are exactly the same.

To see why this is the case, note that the two utility functions are linked

$$U(x,y) = \ln\left[\widetilde{U}(x,y)\right], \quad \text{or} \quad \widetilde{U}(x,y) = \exp\left[U(x,y)\right].$$

Both ln and exp are strictly increasing functions.

The strictly increasing transformations are called *monotonic transformations* and are *rank-preserving* transformations. *Rank-preserving* means that if (x,y) is ranked higher than (x',y') in the first function, for example, $U(x,y) > U(x',y')$ in our case, than (x,y) is also ranked higher than (x',y') in the transformed function, $\widetilde{U}(x,y) > \widetilde{U}(x',y')$ in our case. Since the ranks remain the same after the transformation, the solution to the maximization problem (x^*,y^*) would not change.

Remark. You could make use of the *monotonic transformations* when you face other maximization problems. Sometimes, after such transformations, the caluclations could be much simpler.

Exercise 2.2: The Linear Expenditure System

Consider a consumer choosing between two goods x and y, with prices p and q respectively. His income is I, so the budget constraint is

$$px + qy = I.$$

Suppose the utility function is

$$\widehat{U}(x,y) = \alpha \ln(x - x_0) + \beta \ln(y - y_0).$$

What is the consumer's optimal bundle (x,y)?

Solution. We could make use of the solutions of the previous exercise.

This problem is equivalent to the following

$$\max_{x,y} \widehat{U}(x,y) \equiv \max_{x,y} \alpha \ln(x - x_0) + \beta \ln(y - y_0)$$
$$\text{s.t. } p(x - x_0) + q(y - y_0) = I - px_0 - qy_0.$$

Let $x' = x - x_0$, $y' = y - y_0$ and $I' = I - px_0 - qy_0$, this problem becomes

$$\max_{x',y'} \alpha \ln x' + \beta \ln y'$$
$$\text{s.t. } px' + qy' = I'.$$

From the previous exercise, we know that the solution is

$$x' = \frac{\alpha\,(I')}{(\alpha+\beta)p}, \qquad y' = \frac{\beta\,(I')}{(\alpha+\beta)q}$$
$$\implies x - x_0 = \frac{\alpha\,(I - px_0 - qy_0)}{(\alpha+\beta)p}, \qquad y - y_0 = \frac{\beta\,(I - px_0 - qy_0)}{(\alpha+\beta)q}$$
$$\implies x = \frac{\alpha I + \beta p x_0 - \alpha q y_0}{(\alpha+\beta)p}, \qquad y = \frac{\beta I - \beta p x_0 + \alpha q y_0}{(\alpha+\beta)q}.$$

Of course, you could also solve the problem directly following the procedures in Exercise 2.1.

Exercise 2.3: Production and Cost-Minimization

Part I Consider a producer who rents machines K at r per year and hires labor L at wage w per year to produce output Q, where

$$Q = \sqrt{K} + \sqrt{L}.$$

Suppose he wishes to produce a fixed quantity Q at minimum cost.

Find his factor demand function, that is, the optimal amount of K and L.

Interpret the Lagrange multiplier.

Solution. State the problem:

$$\min_{K,L} Kr + Lw$$
$$\text{s.t. } \sqrt{K} + \sqrt{L} = Q.$$

We could transform the problem into a maximization problem and solve the problem in the usual way. (Details skipped.)

The solution is

$$K = \left(\frac{wQ}{r+w}\right)^2, \qquad L = \left(\frac{rQ}{r+w}\right)^2, \qquad \lambda = \frac{2wrQ}{w+r}.$$

λ is the marginal cost of production.

Part II Now let p denote the price of output. Suppose the producer can vary the quantity of output, and seeks to maximize profit. Factor prices and the production function are the same as in Part I.

Find the optimal output.

Relate this to your interpretation of the Lagrange multiplier.

Solution. From Part I, we get the cost function:

$$C(r, w, Q) = \left(\frac{wQ}{r+w}\right)^2 r + \left(\frac{rQ}{r+w}\right)^2 w.$$

The profit maximization problem is

$$\max_{Q} pQ - C(r, w, Q) \equiv \max_{Q} pQ - \left[\left(\frac{wQ}{r+w}\right)^2 r + \left(\frac{rQ}{r+w}\right)^2 w\right]$$

(Details skipped.) Solving the problem, we get

$$Q^* = \frac{p(w+r)}{2wr}.$$

The optimal quantity Q^* is exactly the quantity solved from $p = \lambda$. Therefore, it is optimal for the producer to produce at at level where price is equal to the marginal cost.

Chapter 3. Extensions and Generalizations

Exercise 3.1: Rationing

Consider a consumer choosing between three goods x_1, x_2 and x_3, with prices p_1, p_2 and p_3 respectively. Suppose his utility function is

$$U(x_1, x_2, x_3) = \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3),$$

where $\alpha_1 + \alpha_2 + \alpha_3 = 1$.

His income is I, so the budget constraint is

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I.$$

In addition, the consumer faces a rationing constraint: he is not allowed to buy more than k units of good 1.

Question 1: Find the consumer's optimal bundle (x_1, x_2, x_3).

Solution. The consumer's maximization problem is

$$\begin{aligned}
&\max_{x_1,x_2,x_3} U(x_1, x_2, x_3) \equiv \max_{x_1,x_2,x_3} \alpha_1 \ln(x_1) + \alpha_2 \ln(x_2) + \alpha_3 \ln(x_3) \\
\text{s.t. } & p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I, \\
& x_1 \leq k, \\
& x_1, x_2, x_3 \geq 0.
\end{aligned}$$

(Details skipped.) The solution is

1. When $k \geq \frac{\alpha_1 I}{p_1}$,

$$x_1 = \frac{\alpha_1 I}{p_1}, \qquad x_2 = \frac{\alpha_2 I}{p_2}, \qquad x_3 = \frac{\alpha_3 I}{p_3}.$$

2. When $k < \frac{\alpha_1 I}{p_1}$,

$$x_1 = k, \qquad x_2 = \frac{\alpha_2 (I - p_1 k)}{(\alpha_2 + \alpha_3) p_2}, \qquad x_3 = \frac{\alpha_3 (I - p_1 k)}{(\alpha_2 + \alpha_3) p_3}.$$

Question 2: Show that when the rationing cosntraint binds, the consumer splits his income between goods 2 and 3 in the proportions $\alpha_2 : \alpha_3$. Would you expect rationing of bread purchases (good 1) to affect demands for butter (good 2) and rice (good 3) in this way? Why and why not?

Solution. The consumer spends $p_2 x_2 = \frac{\alpha_2(I-p_1k)}{(\alpha_2+\alpha_3)}$ on good 2 and $p_3 x_3 = \frac{\alpha_3(I-p_1k)}{(\alpha_2+\alpha_3)}$ on good 3. The proportions are $p_2x_2 : p_3x_3 = \alpha 2 : \alpha 3$.
For bread, butter and rice, the effect would be different. The reason is that bread and butter are complements whereas bread and rice are substitutes. If the rationing constraint on bread is binding, then the butter consumption are expected to go down and the rice consumption are expected to go up. Therefore, the proportions of spendings on good 2 and good 3 are expected to change.

Exercise 3.2: Distribution Between Envious Consumers

There is a fixed total Y of goods at the disposal of society. There are two consumers who envy each other. if consumer 1 gets Y_1 and consumer 2 gets Y_2, their utilties are

$$U_1 = Y_1 - kY_2^2, \quad U_2 = Y_2 - kY_1^2,$$

where k is a positive constant. The allocation must satisfy $Y_1 + Y_2 \leq Y$, and maximize $U_1 + U_2$.
Show that if $Y > 1/k$, the resource constraint will be slack at the optimum.

Solution. State the problem:

$$\begin{aligned} &\max_{Y_1,Y_2} U_1 + U_2 \equiv \max_{Y_1,Y_2} (Y_1 - kY_2^2) + (Y_2 - kY_1^2) \\ \text{s.t. } & Y_1 + Y_2 \leq Y, \\ & Y_1, Y_2 \geq 0. \end{aligned}$$

Write our the Lagrangian and the first-order necessary conditions. (Details skipped.) First, we could conclude that $Y_1 > 0$ and $Y_2 > 0$ must be true. Then, from the slack resource constraint, we could get the condition $Y > 1/k$.

Exercise 3.3: Investment Allocation

A capital sum C is available for allocation among n investment projects. If the non-negative amount x_j is allocated to project j for $j = 1, 2, ..., n$, the expected return from this portfolio of projects is

$$R(x) = \sum_{j=1}^{n} \left[\alpha_j x_j - \frac{1}{2}\beta_j x_j^2\right].$$

The objective is to maximize $R(x)$.

Question 1: Fine the first-order necessary conditions.

Solution.

1. State the problem:

$$\max_{x_1,...,x_n} R(x) \equiv \max_{x_1,...,x_n} \sum_{j=1}^{n} \left[\alpha_j x_j - \frac{1}{2}\beta_j x_j^2\right]$$

$$\text{s.t. } \sum_{j=1}^{n} x_j \leq C$$

$$x_j \geq 0 \text{ for all } j = 1, ..., n$$

2. Set up the Lagrangian:

$$\mathcal{L}(x_1, ..., x_n, \lambda) = \sum_{j=1}^{n} \left[\alpha_j x_j - \frac{1}{2}\beta_j x_j^2\right] + \lambda \left(C - \sum_{j=1}^{n} x_j\right)$$

3. We could write out the first-order conditions by Kuhn-Tucker Theorem:

$$\partial\mathcal{L}/\partial x_j = \alpha_j - \beta_j x_j - \lambda \leq 0 \text{ and } x_j \geq 0, \text{ with complementary slackness for all } j = 1, ..., n.$$

$$\partial\mathcal{L}/\partial\lambda = C - \sum_{j=1}^{n} x_j \geq 0 \text{ and } \lambda \geq 0, \text{ with complementary slackness.}$$

Question 2: Define

$$H = \sum_{j=1}^{n}(\alpha_j/\beta_j), \quad K = \sum_{j=1}^{n}(1/\beta_j).$$

Show that

(i) If $C > H$, then a part of the total sum available is left unused.

(ii) If $\alpha_j > (H - C)/K$ for all j, then every project will receive some funding.

(iii) If any project receives zero funding, then it must have a lower α than any project that gets some funding.

Solution.

(i) A part of the total sum available is left unused means

$$C - \sum_{j=1}^{n} x_j > 0. \tag{1}$$

By complementary slackness, we have $\lambda = 0$. Then the first-order conditions for x_j becomes

$$\alpha_j - \beta_j x_j \leq 0 \text{ and } x_j \geq 0, \text{ with complementary slackness for all } j = 1, ..., n..$$

We only need the first part $\alpha_j - \beta_j x_j \leq 0$ for all $j = 1, ..., n$. Adding up x_j's, we have

$$\sum_{j=1}^{n} x_j \geq \sum_{j=1}^{n} x_j \frac{\alpha_j}{\beta_j} = H.$$

Together with (1), we have $C > H$.

Therefore, this case is internally consistent if $C > H$.

(ii) Every project receives some funding means

$$x_j > 0 \text{ for } j = 1, ..., n. \tag{2}$$

By complementary slackness, we have

$$\alpha_j - \beta_j x_j - \lambda = 0 \equiv x_j = \frac{\alpha_j - \lambda}{\beta_j}. \tag{3}$$

Adding up x_j's, we have

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} \frac{\alpha_j - \lambda}{\beta_j} = \sum_{j=1}^{n} \frac{\alpha_j}{\beta_j} - \lambda \sum_{j=1}^{n} \frac{1}{\beta_j} = H - \lambda K. \tag{4}$$

By resource constraint,

$$\sum_{j=1}^{n} x_j \leq C. \tag{5}$$

(4) and (5) implies

$$\lambda \geq \frac{H - C}{K}. \tag{6}$$

(2), (3) and (6) give

$$\alpha_j > \frac{H - C}{K}.$$

Therefore, this case is internally consistent if $\alpha_j > \frac{H-C}{K}$ for all $j = 1, ..., n$.

(iii) The first-order condition with respect to α_j could be equivalently written as follows:

$$x_j = \max\left\{0, \frac{\alpha_j - \lambda}{\beta_j}\right\}, \tag{7}$$

which could also be equivalently written as

$$x_j = \begin{cases} 0 & \text{if } \frac{\alpha_j - \lambda}{\beta_j} \leq 0 \\ \frac{\alpha_j - \lambda}{\beta_j} > 0 & \text{if } \frac{\alpha_j - \lambda}{\beta_j} > 0 \end{cases} \iff x_j = \begin{cases} 0 & \text{if } \alpha_j \leq \lambda \\ \frac{\alpha_j - \lambda}{\beta_j} > 0 & \text{if } \alpha_j > \lambda \end{cases}$$

Therefore, if a project i receives zero funding, i.e., if $x_i = 0$, then it must be the case that $\alpha_i \leq \lambda$; if a project k gets some funding, i.e., if $x_k > 0$, then we must have $\alpha_k > \lambda$. Thus, $\alpha_k > \lambda \geq \alpha_i$, i.e., $\alpha_k > \alpha_i$. The claim is proved.

Chapter 4. Shadow Prices

Exercise 4.1: The invisible hand - Production Continue with the notation of Example 4.1, but now allow production of the goods. Let there be F factor inputs, available in fixed quantities Z_f for $f = 1, 2, ..., F$. If z_{fg} of factor f is used in the production of good g, the output X_g is given by the production function

$$X_g = \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}). \tag{8}$$

Add these constraints to the earlier problem.

Question 1: Verify that the first-order conditions of optimum distribution are the same as before, but new conditions for optimum factor allocation are added.

Solution. The new problem is as follows:

$$\begin{aligned}
&\max_{x_{cg}, z_{fg}} W\left(U^1(x_{11}, ..., x_{1G}), U^2(x_{21}, ..., x_{2G}), ..., U^C(x_{C1}, ..., x_{CG})\right) \\
\text{s.t. } & x_{1g} + x_{2g} + ... + x_{Cg} = \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}), \text{ for } g = 1, 2, ..., G, \\
& z_{f1} + z_{f2} + ... + z_{fG} = Z_f, \text{ for } f = 1, 2, ..., F.
\end{aligned}$$

Form Lagrangian:

$$\begin{aligned}
\mathcal{L}(x, z, \pi, \mu) = & W\left(U^1(x_{11}, ..., x_{1G}), U^2(x_{21}, ..., x_{2G}), ..., U^C(x_{C1}, ..., x_{CG})\right) \\
& + \sum_{g=1}^{G} \pi_g \left[\Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) - \sum_{c=1}^{C} x_{cg}\right] + \sum_{f=1}^{F} \mu_f \left[Z_f - \sum_{g=1}^{G} z_{fg}\right].
\end{aligned}$$

It is easy to check that the first-order conditions of optimum distribution are the same as before. (Details skipped.)

The first-order conditions with respect to z_{fg} gives rise to the additional conditions for optimum factor allocation:

$$\frac{\partial \mathcal{L}}{\partial z_{fg}} = \pi_g \frac{\partial \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg})}{\partial z_{fg}} - \mu_f = 0, \text{ for } f = 1, 2, ..., F \text{ and } g = 1, 2, ..., G. \tag{9}$$

Question 2: Interpret the Lagrange multipliers.

Solution. The new Lagrange multipliers μ are the shadow prices for factors.

Question 3: Can the production be decentralized, with one firm producing one good?

Solution. Let μ be the factor prices. Consider the following decentralized problem where the firm minimizes the costs of producing the amount X_g:

$$\min_z \sum_{f=1}^{F} \mu_f z_{fg}$$

$$\text{s.t. } \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) = X_g.$$

Form the Lagrangian:

$$\mathcal{L}(z, \zeta_g) = -\sum_{f=1}^{F} \mu_f z_{fg} + \zeta_g \left[\Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) - X_g\right]$$

ζ_g is the Lagrange multiplier. First-order conditions give

$$-\mu_f + \zeta_g \frac{\partial \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg})}{\partial z_{fg}} = 0 \text{ for } f = 1, 2, ..., F. \tag{10}$$

The conditions hold for all g.

(9) and (10) coincide when $\pi_g = \zeta_g$. This is true since π_g and ζ_g are both shadow prices for output g.

Question 4: Show that the sum of income I_c handed out to the consumers equals the value of aggregate output.

Solution. The value of aggregate output is

$$\sum_{g=1}^{G} \pi_g \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) = \sum_{g=1}^{G} \pi_g X_g = \sum_{g=1}^{G} \pi_g \sum_{c=1}^{C} x_{cg} = \sum_{g=1}^{G} \sum_{c=1}^{C} (\pi_g x_{cg}) \tag{11}$$

On the other hand, the sum of income I_c is

$$\sum_{c=1}^{C} I_c = \sum_{c=1}^{C} \left[\sum_{g=1}^{G} (\pi_g x_{cg})\right] = \sum_{g=1}^{G} \sum_{c=1}^{C} (\pi_g x_{cg}). \tag{12}$$

By (11) and (12), we have

$$\sum_{g=1}^{G} \pi_g \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) = \sum_{c=1}^{C} I_c,$$

that is, the sum of income I_c handed out to the consumers equals the value of aggregate output.

Exercise 4.2: The Invisible Hand - Factor Supplies Now let even the factor supplies Z_f be a part of the optimization. Suppose each consumer c supplies z_{cf} of factor f. These amounts affect his utility adversely.

Question 1: Find the first-order conditions. Interpret Lagrange multipliers and discuss the implementation of the optimum in a market framework.

Solution. The new problem is as follows:

$$\begin{aligned}
&\max_{x_{cg}, z_{fg}, z_{cf}} W\left(U^1(x_{11}, ..., x_{1G}, z_{1f}, ..., z_{1F}), ..., U^C(x_{C1}, ..., x_{CG}, z_{Cf}, ..., z_{CF})\right)\\
&\text{s.t. } x_{1g} + x_{2g} + ... + x_{Cg} = \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}), \text{ for } g = 1, 2, ..., G,\\
&\qquad z_{f1} + z_{f2} + ... + z_{fG} = z_{1f} + z_{2f} + ... + z_{Cf}, \text{ for } f = 1, 2, ..., F.
\end{aligned}$$

Form Lagrangian:

$$\begin{aligned}
\mathcal{L}(x, z, \pi, \mu) =& W\left(U^1(x_{11}, ..., x_{1G}, z_{1f}, ..., z_{1F}), ..., U^C(x_{C1}, ..., x_{CG}, z_{Cf}, ..., z_{CF})\right)\\
&+ \sum_{g=1}^{G} \pi_g \left[\Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) - \sum_{c=1}^{C} x_{cg}\right] + \sum_{f=1}^{F} \mu_f \left[\sum_{c=1}^{C} z_{cf} - \sum_{g=1}^{G} z_{fg}\right].
\end{aligned}$$

First-order conditions for x_{cg}, z_{fg} and z_{cf} are respectively:

$$\frac{\partial \mathcal{L}}{\partial x_{cg}} = \left(\frac{\partial W}{\partial u_c}\right)\left(\frac{\partial U^c}{\partial x_{cg}}\right) - \pi_g = 0, \text{ for } c = 1, 2, ..., C \text{ and } g = 1, 2, ..., G; \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial z_{fg}} = \pi_g \frac{\partial \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg})}{\partial z_{fg}} - \mu_f = 0, \text{ for } f = 1, 2, ..., F \text{ and } g = 1, 2, ..., G; \tag{14}$$

$$\frac{\partial \mathcal{L}}{\partial z_{cf}} = \left(\frac{\partial W}{\partial u_c}\right)\left(\frac{\partial U^c}{\partial z_{cf}}\right) + \mu_f = 0, \text{ for } c = 1, 2, ..., C \text{ and } f = 1, 2, ..., F. \tag{15}$$

Interpretation of the Lagrange multipliers:

(i) π_g are the shadow prices for the outputs;

(ii) μ_f are the shadow prices for factors.

In the market equilibrium, the firm's problem is the same as in Exercise 4.1 with one firm producing one good.

The consumer's problem is as follows:

$$\max_{x_{cg}, z_{cg}} U^c(x_{c1}, ..., x_{cG}, z_{cf}, ..., z_{cF})$$
$$\text{s.t. } \pi_1 x_{c1} + \pi_2 x_{c2} + ... + \pi_G x_{cG} = \mu_1 z_{c1} + \mu_2 z_{c2} + ... + \mu_F z_{cF} + I_c.$$

The Lagrangian is

$$\mathcal{L}(x_{cg}, z_{cg}, \iota_c) = U^c(x_{c1}, ..., x_{cG}, z_{cf}, ..., z_{cF}) + \iota_c \left[\sum_{f=1}^{F} (\mu_f z_{cf}) + I_c - \sum_{g=1}^{G} \pi_g x_{cg} \right]$$

First-order conditions for x_{cg} and z_{cg} are respectively

$$\frac{\partial \mathcal{L}}{\partial x_{cg}} = \frac{\partial U^c}{\partial x_{cg}} - \iota_c \pi_g = 0 \tag{16}$$
$$\frac{\partial \mathcal{L}}{\partial z_{cg}} = \frac{\partial U^c}{\partial z_{cg}} + \iota_c \mu_f = 0 \tag{17}$$

Comparing (13) and (15) to (16) and (17), we have

$$\frac{\partial W}{\partial u_c} \iota_c = 1.$$

ι_c is the marginal utility of income, the same as λ_c in Example 4.1.

Question 2: Now you must distinguish two sources of income for the consumers: their earnings from the factor services, and the lump sum I_c they get from the government. Show that the total of the lump sums $\sum_{c=1}^{C} I_c$ handed out to consumers equal the total profit in production, that is, the value of output minus the payments to the factors.

Solution. The value of aggregate output is

$$\sum_{g=1}^{G} \pi_g \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) = \sum_{g=1}^{G} \pi_g X_g = \sum_{g=1}^{G} \pi_g \sum_{c=1}^{C} x_{cg} = \sum_{g=1}^{G} \sum_{c=1}^{C} (\pi_g x_{cg}) \tag{18}$$

The sum of income I_c is

$$\begin{aligned} \sum_{c=1}^{C} I_c &= \sum_{c=1}^{C} \left[\sum_{g=1}^{G} (\pi_g x_{cg}) - \sum_{f=1}^{F} (\mu_f z_{cf}) \right] = \sum_{g=1}^{G} \sum_{c=1}^{C} (\pi_g x_{cg}) - \sum_{c=1}^{C} \sum_{f=1}^{F} (\mu_f z_{cf}) \\ &= \sum_{g=1}^{G} \sum_{c=1}^{C} (\pi_g x_{cg}) - \sum_{f=1}^{F} \left[\mu_f \sum_{c=1}^{C} (z_{cf}) \right] = \sum_{g=1}^{G} \sum_{c=1}^{C} (\pi_g x_{cg}) - \sum_{f=1}^{F} (\mu_f Z_f) . \end{aligned} \tag{19}$$

By (18) and (19), we have

$$\sum_{c=1}^{C} I_c = \sum_{g=1}^{G} \pi_g \Phi^g(z_{1g}, z_{2g}, ..., z_{Fg}) - \sum_{f=1}^{F} (\mu_f Z_f).$$

$\sum_{f=1}^{F} (\mu_f Z_f)$ is the payments to the factors. Therefore, we have the result: the total of the lump sums $\sum_{c=1}^{C} I_c$ handed out to consumers equal the total profit in production, that is, the value of output minus the payments to the factors.

Exercise 4.3: Borrowing and Lending Consider a consumer planning his consumption over two years. He will have income I_1 during the first year and I_2 during the second. In each year, there are two goods to consume. In year 1, the prices are p_1 and q_1, and the corresponding quantities x_1 and y_1. In year 2, we similarly have p_2, q_2 and x_2, y_2. The utility function is

$$u_1 = \alpha_1 \ln(x_1) + \beta_1 \ln(y_1) + \alpha_2 \ln(x_2) + \beta_2 \ln(y_2).$$

This is to be maximized subject to two budget constraints, one for each year.

Question 1: Solve the problem, and find the multipliers λ_1 and λ_2 for the two constraints.

Solution. The consumer's problem is

$$\begin{aligned} &\max_{x_1,y_1,x_2,y_2} \alpha_1 \ln(x_1) + \beta_1 \ln(y_1) + \alpha_2 \ln(x_2) + \beta_2 \ln(y_2) \\ \text{s.t. } &p_1 x_1 + q_1 y_1 = I_1 \\ &p_2 x_2 + q_2 y_2 = I_2 \end{aligned}$$

The problem is solvable using the Lagrange's method. (Details skipped.)

The solution is

$$x_1 = \frac{\alpha_1 I_1}{(\alpha_1 + \beta_1) p_1},\ y_1 = \frac{\beta_1 I_1}{(\alpha_1 + \beta_1) q_1},\ x_2 = \frac{\alpha_2 I_2}{(\alpha_2 + \beta_2) p_2},\ y_2 = \frac{\beta_2 I_2}{(\alpha_2 + \beta_2) q_2}$$
$$\lambda_1 = \frac{\alpha_1 + \beta_1}{I_1},\ \lambda_2 = \frac{\alpha_2 + \beta_2}{I_2}$$

Question 2: How much more of year-2 income will the consumer require if he is to give up $\mathrm{d}I_1$ of year-1 income? In other words, what is the rate of return needed to induce him to save a little? You would expect borrowing and lending institutions arise in an economy populated by such consumers. What governs who will borrow and who will lend?

Solution. This problem is about the economic interpretation of λ_1 and λ_2.

(a) (i) Giving up $\mathrm{d}I_1$ of year-1 income will lead to $\lambda_1 \mathrm{d}I_1$ decrease of utility.

(ii) $\mathrm{d}I_2$ of year-2 income will give the consumer $\lambda_2 \mathrm{d}I_2$ of utility.

Therefore, to make the consumer willing to give up $\mathrm{d}I_1$ of year-1 income, we need

$$\lambda_2 \mathrm{d}I_2 \geq \lambda_1 \mathrm{d}I_1 \implies \mathrm{d}I_2 \geq \frac{\lambda_1}{\lambda_2}\mathrm{d}I_1 = \frac{(\alpha_1 + \beta_1)I_2}{(\alpha_2 + \beta_2)I_1}\mathrm{d}I_1.$$

That is, the consumer would require at least $\frac{(\alpha_1+\beta_1)I_2}{(\alpha_2+\beta_2)I_1}\mathrm{d}I_1$ more of year-2 income.

(b) The require rate of return to induce the consumer to save is

$$r = \frac{\mathrm{d}I_2 - \mathrm{d}I_1}{\mathrm{d}I_1} \geq \frac{\lambda_1}{\lambda_2} - 1 = \frac{(\alpha_1 + \beta_1)I_2}{(\alpha_2 + \beta_2)I_1} - 1.$$

(c) Let the market rate be r.

(i) Those consumers with $\frac{(\alpha_1+\beta_1)I_2}{(\alpha_2+\beta_2)I_1} - 1 < r$ is willing to lend;

(ii) Those with $\frac{(\alpha_1+\beta_1)I_2}{(\alpha_2+\beta_2)I_1} - 1 > r$ is willing to borrow;

(iii) Those with $\frac{(\alpha_1+\beta_1)I_2}{(\alpha_2+\beta_2)I_1} - 1 = r$ is indifferent between borrowing and lending.

Chapter 5. Maximum Value Functions

Exercise 5.1: The Cobb-Douglas Cost Function. Consider a production function

$$y = A\prod_{j=1}^{n} x_j^{\alpha_j},$$

where y is output, x_j's are inputs, A and α_j's are positive constants. Let $w = (w_j)$ be the vector of input prices. Suppose the producer wishes to produce a fixed quantity y at minimum cost.

Question 1: Write out the cost minimization problem and solve for the minimum cost function $C(w, y)$. Hint: the minimum cost function should be

$$C(w, y) = \beta(y/A)^{1/\beta}\prod_{j=1}^{n}(w_j/\alpha_j)^{\alpha_j/\beta},$$

where $\beta = \sum_{j=1}^{n}\alpha_j$.

Solution. The cost minimization problem is

$$\begin{aligned}
&\min_{x_1,\ldots,x_n}\sum_{j=1}^{n} w_j x_j \\
\text{s.t. } &A\prod_{j=1}^{n} x_j^{\alpha_j} = y, \\
&x_j \geq 0 \text{ for all } j = 1, ..., n.
\end{aligned}$$

$x_j > 0$ must hold since otherwise the constraint would not hold.

The problem is solvable using the Lagrange's method. (Details skipped.) You will get the result

$$\begin{aligned}
x_j^* &= \frac{\alpha_j}{w_j}(y/A)^{1/\beta}\prod_{j=1}^{n}(w_j/\alpha_j)^{\alpha_j/\beta} \\
\lambda &= (y/A)^{1/\beta}\prod_{j=1}^{n}(w_j/\alpha_j)^{\alpha_j/\beta}
\end{aligned}$$

And the minimum cost function follows differectly from $C(w, y) = \sum_{j=1}^{n} w_j x_j^*$.

You may find it simpler to transform the constraint:

$$\ln A + \sum_{j=1}^{n}\alpha_j \ln x_j = \ln y.$$

Question 2: If $\beta < 1$, calculate the corresponding maximum profit function $\pi(p, w)$, where p is the output price. What goes wrong if $\beta \geq 1$?

Solution. The profit maximization problem is

$$\pi(p, w) = \max_y py - C(w, y) \equiv \max_y py - \beta(y/A)^{1/\beta} \prod_{j=1}^{n} (w_j/\alpha_j)^{\alpha_j/\beta}$$

$$\text{s.t. } y \geq 0.$$

The problem is solvable using the Lagrange's method. (Details skipped.)

We could get

$$y^* = A^{\frac{1}{1-\beta}} p^{\frac{\beta}{1-\beta}} \prod_{j=1}^{n} (w_j/\alpha_j)^{-\frac{\alpha_j}{1-\beta}},$$

$$\text{and } \pi(p, w) = (1-\beta)(pA)^{\frac{1}{1-\beta}} \prod_{j=1}^{n} (w_j/\alpha_j)^{-\frac{\alpha_j}{1-\beta}}.$$

When $\beta < 1$, we are in the situation shown in Figure 1. And the solution is the maximum.

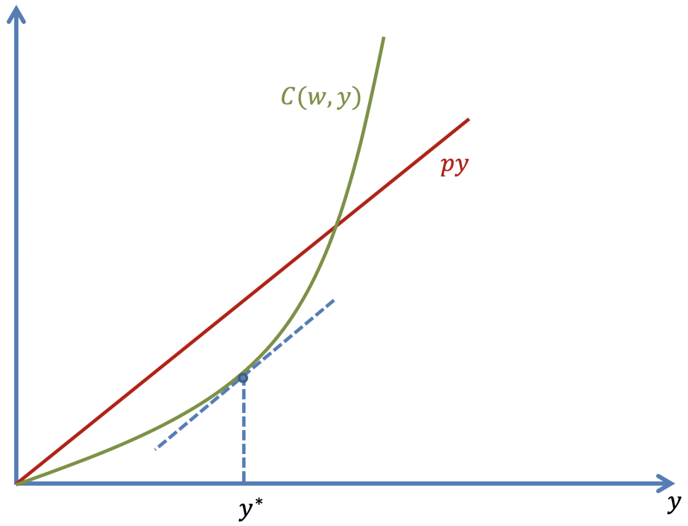


Figure 1: $\beta < 1$

When $\beta = 1$, the maximization problem becomes

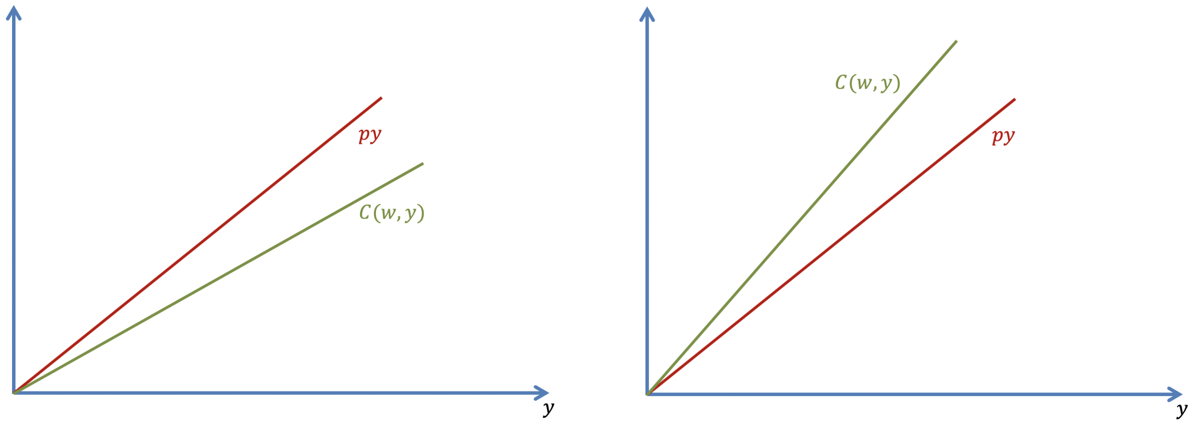
$$\pi(p, w) = \max_y py - (y/A) \prod_{j=1}^{n} (w_j/\alpha_j)^{\alpha_j} = \max_y \left[p - \frac{\prod_{j=1}^{n} (w_j/\alpha_j)^{\alpha_j}}{A} \right] y$$

$$\text{s.t. } y \geq 0.$$

Then,

$$\pi(p,w)=\begin{cases}\infty & \text{if } p>\frac{\prod_{j=1}^{n}(w_j/\alpha_j)^{\alpha_j}}{A}\\ 0 & \text{if } p\leq\frac{\prod_{j=1}^{n}(w_j/\alpha_j)^{\alpha_j}}{A}\end{cases}$$

Graphically,



(a) $p > C_y(w,y)$ (b) $p \leq C_y(w,y)$

Figure 2: $\beta = 1$

(i) Figure 2a corresponds to the first case where $\pi(p,w)\to\infty$ when $y\to\infty$. In this case, p is larger than the constant marginal cost $C_y(w,y)=\frac{\prod_{j=1}^{n}(w_j/\alpha_j)^{\alpha_j}}{A}$.

(ii) Figure 2b corresponds to the second case. In this case, p is less than or equal to the constant marginal cost $C_y(w,y)$ so it is better not to produce. (When $p=C_y(w,y)$, producing or not is indifferent.)

When $\beta > 1$, we are in the situation shown in Figure 3. And the solution is the minimum.

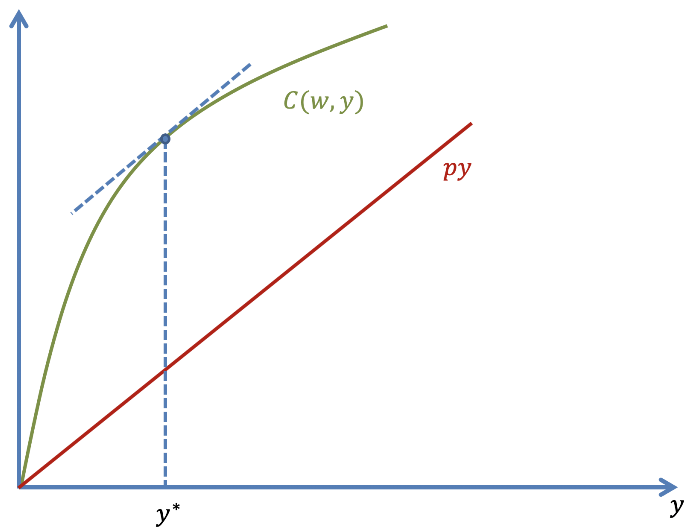


Figure 3: $\beta > 1$

Exercise 5.2: The CES Expenditure Function. Suppose the direct utility function is

$$U(x,y) = \left[\alpha x^{\rho} + \beta y^{\rho}\right]^{1/\rho},$$

where x and y are the quantities of the two goods, and $\alpha > 0$, $\beta > 0$, $\rho < 1$ are given constants. The prices of good x and y are p and q respectively.

Question 1: Show that the expenditure function is of the form

$$E(p,q,u) = \left[ap^{r} + bq^{r}\right]^{1/r} u,$$

where u is the target utility level, and a, b, and r are constants that can be expressed in terms of α, β and ρ.

Solution. The expenditure minimization problem is

$$\begin{aligned} E(p,q,u) = \min_{x,y}\, & px + qy \\ \text{s.t. } & \left[\alpha x^{\rho} + \beta y^{\rho}\right]^{1/\rho} \geq u \end{aligned}$$

Since $U_x(x,y) > 0, U_y(x,y) > 0$, the constraint must be binding. Otherwise, we could decrease x or y without violating the constraint and the expenditure is lower.

Besides, you may find it simpler to transform the constraint:

$$\alpha x^{\rho} + \beta y^{\rho} = u^{\rho}.$$

The problem is solvable using the Lagrange's method. (Details skipped.)

You will find the parameters a, b and r equal to the following:

$$a = \alpha^{\frac{1}{1-\rho}},\ b = \beta^{\frac{1}{1-\rho}},\ r = \frac{\rho}{\rho - 1}.$$

And the optimal solutions are

$$\begin{aligned} x^* &= ap^{r-1}\left[ap^{r} + bq^{r}\right]^{\frac{1}{r}-1} u; \\ y^* &= bq^{r-1}\left[ap^{r} + bq^{r}\right]^{\frac{1}{r}-1} u. \end{aligned}$$

Question 2: Show that the ratio of the cost-minimizing quantities is

$$x/y = (a/b)(q/p)^{1-r}.$$

The elasticity of (x/y) with respect to (q/p):

$$\frac{\mathrm{d}\ln(x/y)}{\mathrm{d}\ln(q/p)}.$$

is called the elasticity of substitution in production. Show that in this example, it is constant and equal to $(1-r)$. What condition must be imposed on ρ to ensure a non-negative elasticity of substitution, that is, $r < 1$?

Solution. x/y could be calculated directly from the solutions we get in Question 1. From the equation of x/y:

$$x/y = (a/b)(q/p)^{1-r} \implies \ln(x/y) = \ln(a/b) + (1-r)\ln(q/p)$$

Therefore,

$$\frac{\mathrm{d}\ln(x/y)}{\mathrm{d}\ln(q/p)} = 1 - r.$$

The condition for ρ is

$$r < 1 \iff \frac{\rho}{\rho - 1} < 1 \iff \rho < 1.$$

Chapter 6. Convex Sets and Their Separations

Exercise 6.1: Commodities that Cause Disutility. How is Figure 6.1 altered when

(a) one of the choice variable is labor, which gives disutility to consumers and is an input to production,

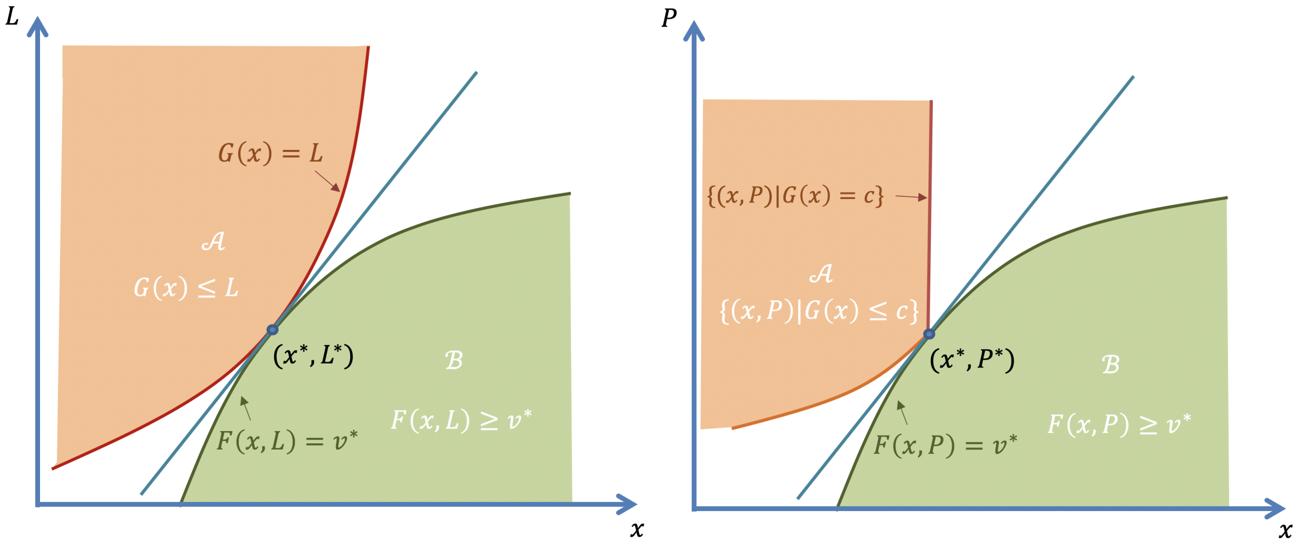
(b) when one of the goods is pollution, which gives disutility to consumers and is the by-product of an economically desirable good which is the other choice variable?

Interpret the associated prices in each of these contexts.

Solution.

(a) Figure 4a shows Labor L and Consumption Good x. The price of labor is the wage rate; the price of the consumption good is the price in the usual sense. Wage is income to the consumer whereas the consumption good is related to the spendings, so they are of opposite signs.

(b) Figure 4b shows Pollution P and Consumption Good x. The price of pollution is cost of pollution; the price of the consumption good is the price in the usual sense. Pollution is a "bad", so the associated price is negative.



(a) Labor L and Consumption Good x (b) Pollution P and Consumption Good x

Figure 4: Separation

Exercise 6.2: Convexity of a Firm's Profit Function. A firm chooses vectors x of inputs and y of outputs subject to a production possibility constraint $G(x, y) \leq 0$, to maximize profit $qy - px$, where q denotes the row vector of output prices and p that of input prices. Let $\Pi(q, p)$ be the maximized profit expressed as a function of the prices. Prove that Π is a convex function of (q, p).

Solution. Let $(x(q, p), y(q, p))$ be the optimal choice. Then, for any (q_1, p_1), (q_2, p_2) and any $\alpha \in [0, 1]$, let $q' = \alpha q_1 + (1 - \alpha)q_2$ and $p' = \alpha p_1 + (1 - \alpha)p_2$,

$$\Pi(q', p') \underbrace{=}_{\text{definition of } \Pi \text{ and } x,\, y} q'y(q', p') - p'x(q', p') \underbrace{=}_{\text{definition of } q',\, p'} (\alpha q_1 + (1 - \alpha)q_2)y(q', p') - (\alpha p_1 + (1 - \alpha)p_2)x(q', p')$$

$$= \alpha \left[q_1 y(q', p') - p_1 x(q', p')\right] + (1 - \alpha) \left[q_2 y(q', p') - p_2 x(q', p')\right]$$

$$\underbrace{\leq}_{\text{definition of } \Pi \text{ and the constraint independent of } p,\, q} \alpha\Pi(q_1, p_1) + (1 - \alpha)\Pi(q_2, p_2).$$

The constraint is independent of q, p ensures that $(x(q', p'), y(q', p'))$ is feasible when the parameters are (q_1, p_1) or (q_2, p_2). From the calculations, we have the result $\Pi(q', p') \leq \Pi(q_1, p_1) + (1 - \alpha)\Pi(q_2, p_2)$ for any (q_1, p_1), (q_2, p_2) and any $\alpha \in [0, 1]$. Thus, Π is a convex function of (q, p).

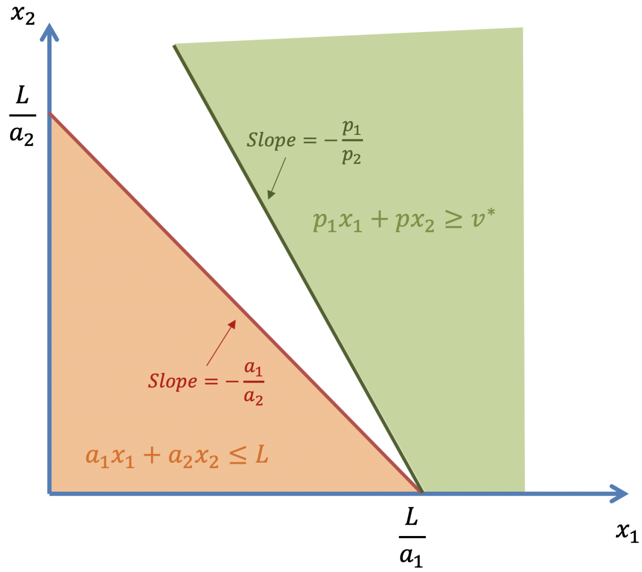
Exercise 6.3: Corner Solutions. Consider an economy with labor endowment L. It can produce two goods x_1 and x_2. A unit of good j needs a fixed amount of a_j units of labor, so the production possibility constraint is

$$a_1 x_1 + a_2 x_2 \leq L.$$

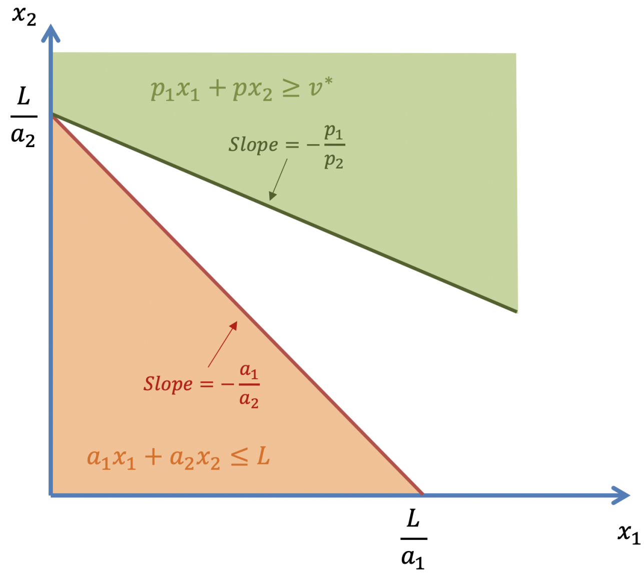
The world prices of the two goods are p_1 and p_2, independent of the levels of production chosen by this country. The aim is to maximize the value of national product, $(p_1 x_1 + p_2 x_2)$.

Question 1: Draw a figure and solve the problem by separation of two convex sets. You will need to consider two cases separately, depending on which of p_1/p_2 and a_1/a_2 is larger.

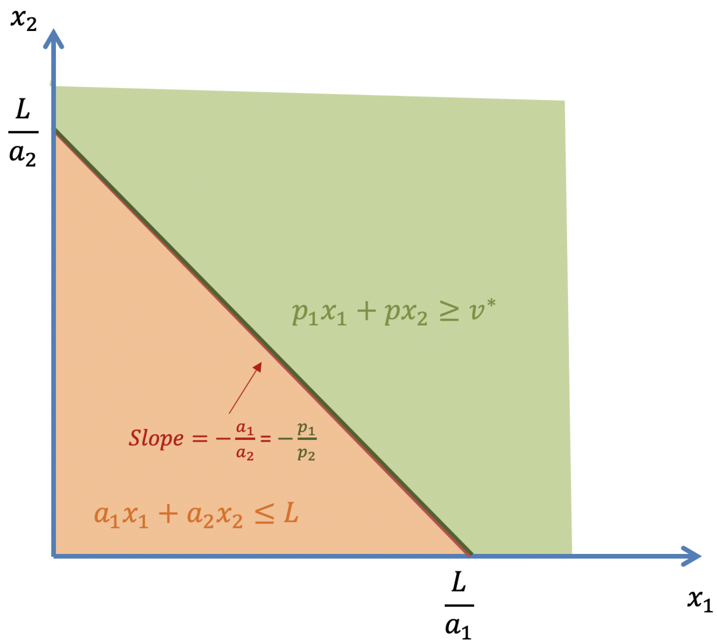
Solution. There are three cases, depending on the relative magnitude of p_1/p_2 and a_1/a_2. See Figure 5 below.



(a) $\frac{a_1}{a_2} < \frac{p_1}{p_2}$



(b) $\frac{a_1}{a_2} > \frac{p_1}{p_2}$



(c) $\frac{a_1}{a_2} = \frac{p_1}{p_2}$

Figure 5: Three Cases

1. When $\frac{a_1}{a_2} < \frac{p_1}{p_2}$ (Figure 5a), the solution is $(x_1^*, x_2^*) = (\frac{L}{a_1}, 0)$.
2. When $\frac{a_1}{a_2} > \frac{p_1}{p_2}$ (Figure 5b), the solution is $(x_1^*, x_2^*) = (0, \frac{L}{a_2})$.
3. When $\frac{a_1}{a_2} = \frac{p_1}{p_2}$ (Figure 5c), the solution is $(x_1^*, x_2^*) = (x_1, -\frac{a_1}{a_2}x_1 + \frac{L}{a_2})$ for $x_1 \in [0, \frac{L}{a_1}]$.

Question 2: Having the solutions in the figure, verify Lagrange's conditions. Find and interpret the Lagrange multiplier. Show that the maximized national product, expressed as a function of the prices (the revenue function or the GNP function) is

$$R(p_1, p_2) = \max\left\{\frac{p_1}{a_1}, \frac{p_2}{a_2}\right\} L.$$

Solution. The maximization problem is

$$\begin{aligned} &\max_{x_1, x_2} p_1 x_1 + p_2 x_2 \\ \text{s.t. } &a_1 x_1 + a_2 x_2 \leq L, \\ &x_1, x_2 \geq 0. \end{aligned}$$

The Lagrangian is $\mathcal{L}(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 + \lambda(L - a_1 x_1 - a_2 x_2)$.
The first-order necessary conditions are

$$\begin{aligned} &\partial\mathcal{L}/\partial x_1 = p_1 - \lambda a_1 \leq 0 \text{ and } x_1 \geq 0, \text{ with complementary slackness;} \\ &\partial\mathcal{L}/\partial x_2 = p_2 - \lambda a_2 \leq 0 \text{ and } x_2 \geq 0, \text{ with complementary slackness;} \\ &\partial\mathcal{L}/\partial \lambda = L - a_1 x_1 - a_2 x_2 \geq 0 \text{ and } \lambda \geq 0, \text{ with complementary slackness.} \end{aligned}$$

Given the first-order conditions, it should not be hard to verify the Lagrange's conditions for the three cases. (Details skipped.)
The Lagrange multiplier is

$$\lambda = \max\left\{\frac{p_1}{a_1}, \frac{p_2}{a_2}\right\},$$

And it measures the marginal product of labor.
For the last part, we already have the optimum choices (x_1^*, x_2^*), and the function $R(p_1, p_2)$ could be easily calculated. (Details skipped.)

Chapter 7. Concave Programming

Exercise 7.1: Minimization. Develop the theory of minimization of a convex function along lines parallel to those used in this chapter for maximization of a concave function.

Solution. You should be able to show this. If you encounter any difficulties, please let me know.

Exercise 7.2: Convexity of Maximum Value Function. Let θ be a vector of parameters and consider the problem of choosing x to maximize $F(x,\theta)$ subject to $G(x) \leq c$. Let $V(\theta)$ denote the maximum value as a function of the parameters.

Question 1: Prove that if F is convex as a function of θ for each fixed x, then V is convex.

Solution. Let $X(\theta)$ be the optimal choice. Then, for any θ_1, θ_2 and any $\alpha \in [0,1]$,

$$\begin{aligned}
V(\alpha\theta_1 + (1-\alpha)\theta_2) &\underbrace{=}_{\text{definition of } V \text{ and } X} F(X(\alpha\theta_1 + (1-\alpha)\theta_2), \alpha\theta_1 + (1-\alpha)\theta_2) \\
&\underbrace{\leq}_{F \text{ is convex as a function of } \theta.} \alpha F(X(\alpha\theta_1 + (1-\alpha)\theta_2), \theta_1) + (1-\alpha) F(X(\alpha\theta_1 + (1-\alpha)\theta_2), \theta_2) \\
&\underbrace{\leq}_{\text{definition of } V \text{ and the constraint independent of } \theta} \alpha V(\theta_1) + (1-\alpha) V(\theta_2).
\end{aligned}$$

The constraint is independent of θ ensures that $X(\alpha\theta_1 + (1-\alpha)\theta_2)$ is feasible when the parameters are θ_1 or θ_2.

Question 2: In Chapter 5, we saw geomatrically that the minimum cost of producing a given quantity of output, regarded as a function of input prices, is concave. Derive that formally as a corollary of the above general result.

Solution. The cost minimization problem is as follows:

$$\begin{aligned}
C(w,q) = &\min_x wx \\
&\text{s.t. } f(x) = q,
\end{aligned}$$

where w is a set of input prices, x is the input quantity, f is the production function and q is the output quantity. $C(w, q)$ is the minimized cost.

Equivalently, we could rewrite the problem into a maximization problem:

$$\begin{aligned} -C(w, q) = &\max_x -wx \\ &\text{s.t. } f(x) = q. \end{aligned}$$

Mapping into Question 1, $F(x, w) = -wx$, $V(w) = -C(w, q)$.

We will first show that F is a convex function of w. For any w_1, w_2 and $\alpha \in [0, 1]$,

$$\begin{aligned} F(x, \alpha w_1 + (1-\alpha)w_2) &= -(\alpha w_1 + (1-\alpha)w_2)x = -\alpha w_1 x - (1-\alpha)w_2 x \\ &= -\alpha F(x, w_1) - (1-\alpha)F(x, w_2). \end{aligned}$$

Thus, F is convex in w.

Then, by the previous result V is convex in w, that is $-C(w, q)$ is convex in w.

$$\begin{aligned} &-C(\alpha w_1 + (1-\alpha)w_2, q) \leq -\alpha C(w_1, q) - (1-\alpha)C(w_2, q) \\ \Longrightarrow\ &C(\alpha w_1 + (1-\alpha)w_2, q) \geq \alpha C(w_1, q) + (1-\alpha)C(w_2, q) \end{aligned}$$

Therefore, $C(w, q)$ is concave in w.

Exercise 7.3: More on Linear Programming.

Question 1: Show that the optimal solution x^* of the linear-programming problem of Example 7.1, and the corresponding vector of multipliers λ^* are such that

$$\mathcal{L}(x, \lambda^*) \leq \mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x^*, \lambda)$$

for all non-negative x and λ. In other words, x^* maximizes the Lagrangian when $\lambda = \lambda^*$, and λ^* minimizes the Lagrangian when $x = x^*$. In other words, the graph of the Lagrangian in (x, λ) space is shaped like a saddle. Therefore, (x^*, λ^*) is said to be a *saddle-point* of the Lagrangian.

Solution. Linear programming belongs to concave programming.

Since x^* maximizes the constrained problem,

$$\max_x ax$$
$$\text{s.t. } Bx \le c,\ x \ge 0.$$

then x^* maximizes the unconstrained problem given by the Lagrangian $\mathcal{L}(x, \lambda)$.

Similarly, by duality, λ^* maximizes the constrained problem

$$\max_\lambda -\lambda c$$
$$\text{s.t. } -\lambda B \le a,\ \lambda \ge 0.$$

Then λ^* maximizes the unconstrained problem given by

$$M(\lambda, x) = -\lambda c + [-a + \lambda B]\, x = -\mathcal{L}(x, \lambda).$$

The result follows from the above two facts.

Question 2: Let $V(a, c)$ denote the maximum value function of the linear-programming problem. Show that V is convex in a for each fixed c, and concave in c for each fixed a.

Solution. The first part "V is convex in a":

1. ax is a convex function is a;

2. Follow the steps in Exercise 7.2 Question 1.

The second part "V is concave in c" could be shown similarly by considering the dual problem. (Details skipped.)

Chapter 8. Second-Order Conditions

Exercise 8.1: Production Theory

In Exercise 6.2, we examined a firm's profit function

$$\Pi(q,p) = \max_{x,y}\{qy - px \mid G(x,y) \leq 0\},$$

where q and p are respectively the vectors of prices of outputs and inputs, y and x the corresponding quantity vectors, and the constraint reflects technological feasibility. We have proved that Π is a convex function of (q,p).

Question 1: Show that the optimum choices of y and x are given in terms of the partial derivatives of Π by

$$y = \Pi_q(q,p), \quad x = -\Pi_p(q,p).$$

Solution. The Lagrangian of the problem is

$$\mathcal{L}(x,y,\lambda,q,p) = qy - px + \lambda G(x,y).$$

By Envelope Theorem

$$\Pi_p(p,q) = \mathcal{L}_p(x,y,\lambda,q,p) = -x^*; \tag{20}$$

$$\Pi_q(p,q) = \mathcal{L}_p(x,y,\lambda,q,p) = y^*. \tag{21}$$

Question 2: Show that output supply curves are upward-sloping and input demand curves are downward-sloping:

$$\frac{\partial y_j}{\partial q_j} \geq 0, \text{ and } \frac{\partial x_k}{\partial p_k} \leq 0,$$

for all j, k.

Solution.

1. We first prove $\frac{\partial x_k}{\partial p_k} \leq 0$. Differentiate (20) with respect to p, we have

$$\Pi_{pp}(p,q) = -x_p^*.$$

Since $\Pi(p,q)$ is convex, $\Pi_{pp}(p,q)$ is positive semi-definite, i.e., $y^T\Pi_{pp}(p,q)y \geq 0$ for all y. Therefore, $y^T x_p^* y \leq 0$ for all y. In particular, pick $y = e^k$, e^k is a vector where e^k is a vector with its k^{th} component equal to 1 and all other components 0. Then, we have the result: $\frac{\partial x_k}{\partial p_k} \leq 0$.

2. $\frac{\partial y_j}{\partial q_j} \geq 0$ could be similarly proved. (Details skipped.)

Exercise 8.3: Minimization

Question 1: Develope second-order sufficient conditions for the unconstrained miminization problem.

You may use the definitions and the determinantal tests of *positive (semi-)definite* matrix in the Lecture Notes directly. If you want, you could also develope the determinantal test from the tests for *negative (semi-)definite* matrix. In your derivation, you will find the following result useful: $\det(-A) = (-1)^n \det(A)$ for an $n \times n$ matrix A.

Question 2: Use Theorem 8.4 in the Lecture Notes to develope the second-order sufficient condition for the constrained minimization problem. (Hint: $\det(-A) = (-1)^n \det(A)$ for an $n \times n$ matrix A.)

Solution. This is a problem in Assignment 2. Solutions will be updated after the assignment is handed in.