```
clear; close all;
```

Solve Poisson Equation

Consider the 2D Poisson equation with only Dirichlet boundary conditions:

$$u_t - \Delta u = f \quad \text{on } \Omega$$
$$u = g \quad \text{on } \partial\Omega$$
$$u = u_0 \quad t = 0$$

where

$$g = \sin(x)\cos(y)\cos(t).$$

and

$$f = \sin(x)\cos(y)\left(-\sin(t) + 2\cos(t)\right).$$

```
global f g
f = @(x,y,t) sin(x).*cos(y).*(-sin(t)+2*cos(t));
g = @(x,y,t) sin(x).*cos(y).*cos(t);
```

Note that one true solution is $u = g$ if $u_0(x, y) = g(x, y, 0)$.

```
u_true = g;
```

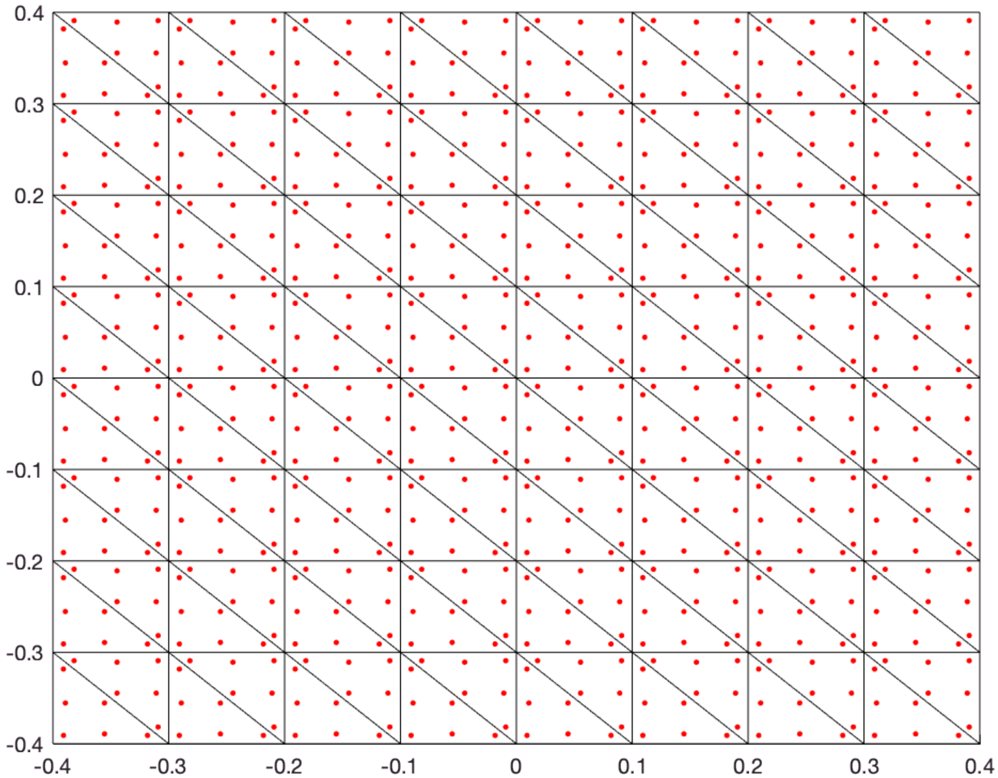
Domain

First we define our domain Ω and generate a function space on it with finite element of choice:

```
addpath('..')
import MatFem.*
domain = [-0.4 0.4 -0.4 0.4]; % define a domain with -0.4 ≤ x ≤ 0.4, -0.4 ≤ y ≤ 0.4
% NOTE: if we choose too large a mesh, each mesh grid will have little communication, which le
% to poor observability.
shape  = [8 8];         % 8 x 8 grids
mesh   = rectMesh(domain, shape);
```

Now let's construct a degree 2 function space with 9 quadrature points in each element:

```
global V
V = MatFem.mesh2spc(mesh, 1, 4); % P1 element with degree 4 quadrature
figure()
V.plot()
```




```
% title('Function space of degree 2 element and degree 4 quadrature points')
```

Variational form

With backward-Euler, the variational form at time t is

$$\int_\Omega (u_t v + \Delta t \nabla u_t \cdot \nabla v) = \int_\Omega (\Delta t\, f v + u_{t-\Delta t} v).$$

The left hand side matrix then can be assembled as

```
import MatFem.assemble
global dt A Auv

dt = 0.001; % Choose a Δt
Auv = assemble(V, [0 0; 0 0]); % the [0 0; 0 0] are derivative orders of
                               % u (first row) and v (second row)
                               % wrt x (first column) and y (second column).
                               % So this term is integrate(u*v)
A = Auv + dt * assemble(V, [1 0; 1 0]); % Δt * integrate(u_x*v_x)
A = A   + dt * assemble(V, [0 1; 0 1]); % Δt * integrate(u_y*v_y)
```

To see what `assemble` does, run

```
% help assemble
```

Boundary Condition

Boundary conditions can be defined as

```
global bc
bc = MatFem.rectBndCond(V, 'd', domain); % 'd' means dirichlet. This creates Dirichlet boundar
[~, A] = bc.applyDir('d', 0, [], A);     % apply Dirichlet boundary conditions to A. The value
```

Solution

Now let us solve the system.

We'd like to write a function F s.t. $u_t = F(u_{t-\Delta t}, t)$.

Definition code is placed at the end of the file.

Now let's examine the solution. Given initial value $u_0 = g(t = 0)$, our solution should be close to g.

```
T = 1;

u = V.project(@(x,y)u_true(x,y,0)); % initial condition
for t = dt:dt:T
    u = F(u, t); % solve for next state
end
MatFem.errorNorm(V, u, @(x,y)u_true(x,y,T), 'L2')
```

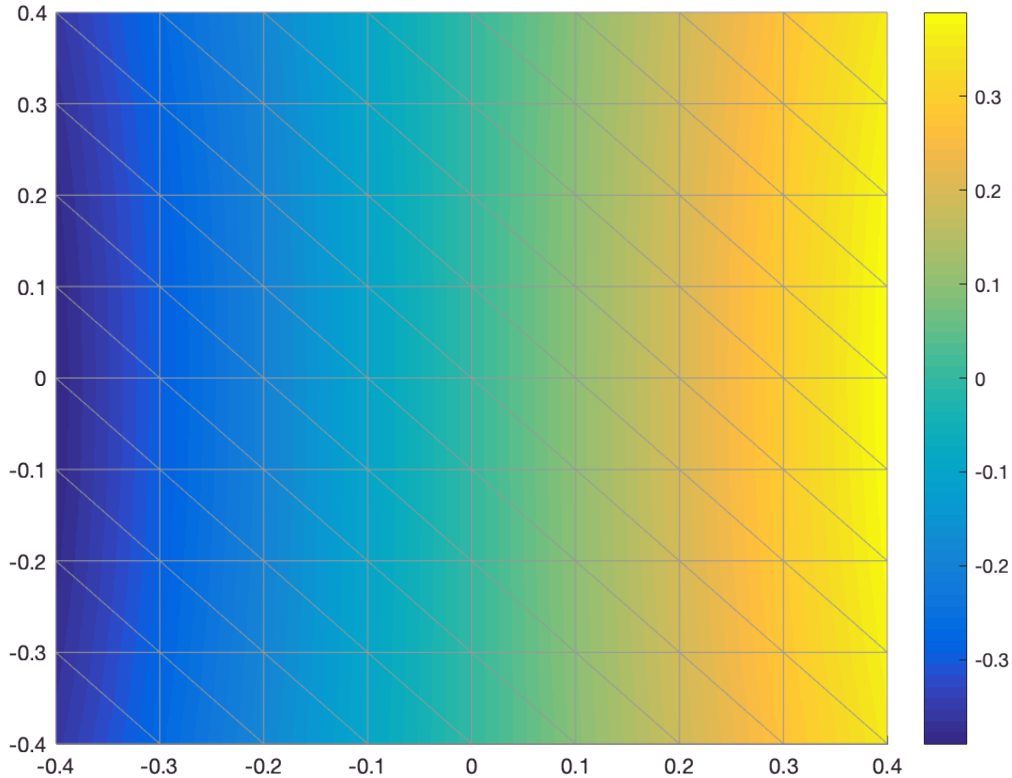
```
ans = 1.9745e-04
```

A more rigorous test would be to examin convergence rate. But it is not our focus here.

Data Assimilation

Suppose we are trying to recover the initial state $u_0 = g(t = 0)$.

```
V.plotu(@(x,y)u_true(x,y,0))
```



Bases

The number of bases in our function space V

```
V.nb
```

```
ans = 81
```

However the Dirichlet boundaries reduces some of them. In order to find all valid bases, we can apply Dirichlet Boundary conditions

```
global bases
bases = ones(V.nb, 1);
bases = bc.applyDir('d', 0, bases);
bases = logical(bases);
```

Now the number of bases is

```
nb = sum(bases)
```

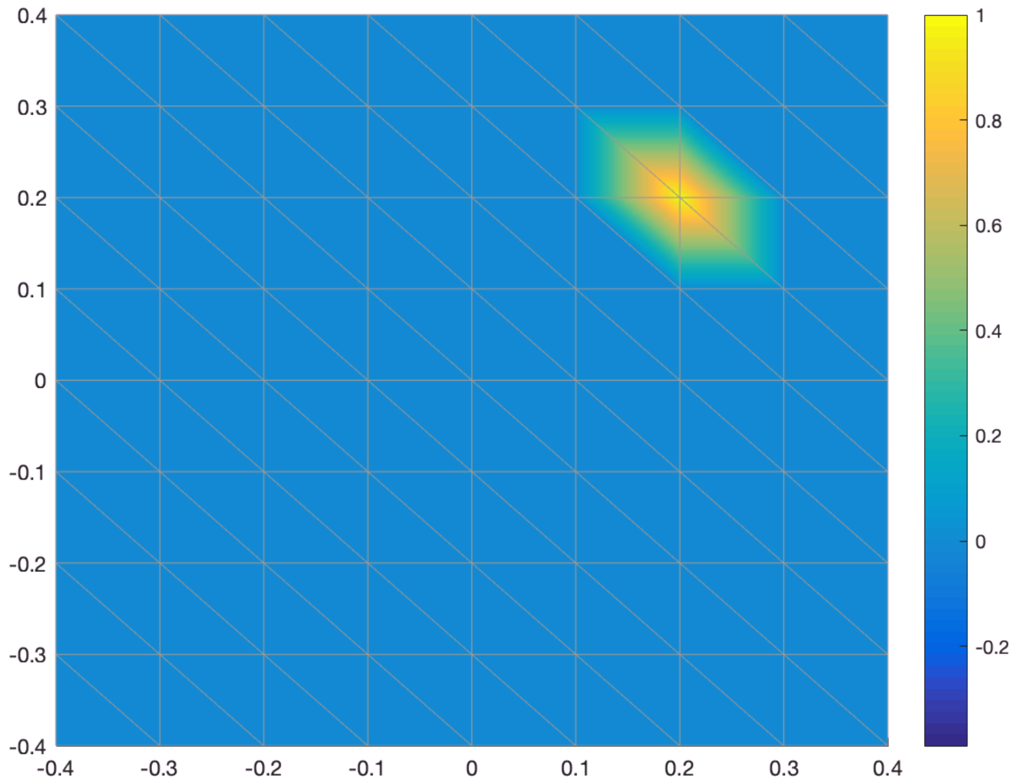
```
nb = 49
```

We can write a function to map a vector $[\alpha_1, \alpha_2, \ldots, \alpha_{nb}]$ to a function in our function space V with value 0 at all Dirichlet boundaries. See the function `alphas2V` at the end of this file. And the function `V2alphas` is vice versa.

Now the i^{th} base function is:

```
i = 41;
basei = zeros(nb,1);
basei(i) = 1;
C1 = alphas2V(basei);

V.plotu(C1)
```



In fact, if we stack all base functions into one matrix, it would be

```
alphas = eye(nb);
C = alphas2V(alphas);
```

Observations

Now let's assume g describes true states, but we don't know it yet.

We made some observations every $2\Delta t$, at positions $(-0.2,-0.2),(-0.2,0),(-0.2,0.2),(0,-0.2),(0,0),(0,0.2),(0.2,-0.2),(0.2,0),(0.2,0.2)$ and we have some error in our observations.

```
[X, Y] = meshgrid([-0.2 0 0.2], [-0.2 0 0.2]);
nobst = 15;         % how many times we make observations
nsensor = numel(X);   % number of sensors
noise = 0;
ndtobs = 1;     % observe every `ndtobs` timestep
```

```
data = zeros(nobst * nsensor, 1);
for i = 1:nobst
    data((nsensor*(i-1)+1):(nsensor*i)) = reshape(g(X,Y,dt*ndtobs*i), nsensor, 1) + randn(nser
end
```

In order to do data assimilation, we also need to define a function for numerical observations. To do that, we need an interpolation function in our finite element package that can evaluate a function given a vector u and location (x, y). This is not implemented in my MatFem. So for now I'll just hard code the observation in `observe` at the end of this file.

Homogenize

Consider the homogenized Poisson system:

$$\begin{aligned} u_t - \Delta u &= 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \\ u &= u_0 \quad t = 0 \end{aligned}$$

Because of the linearity of this system, if u_1, u_2 are two solutions to this system corresponding to initial conditions u_{01}, u_{02}, then for any $\alpha_1, \alpha_2 \in R$, $\alpha_1 u_1 + \alpha_2 u_2$ is a solution corresponding to initial condition $\alpha_1 u_{01} + \alpha_2 u_{02}$.

Moreover, if u^* is a solution to the inhomogeneous system with initial condition u_0^*, then $\sum_i \alpha_i u_i + u^*$ is a solution to the inhomogeneous system with initial condition $\sum_i \alpha_i u_{0i} + u_0^*$.

A solution to the homogenized system is written in function Fh. See the end of this file.

We randomly select an initial condition for inhomogeneous system

```
u0star = zeros(V.nb, 1);
u0star = bc.applyDir('d', @(x,y)g(x,y,0), u0star);
```

Note that a smooth u_0^* is better. Here we just leave it as it is for simplicity.

If we observe this solution, we can get a set of data

```
inhodata = zeros(nobst * nsensor, 1);
ustar = u0star;
for i = 1:(nobst*ndtobs)
    t = i * dt;
    ustar = F(ustar, t); % solve for $\tilde{C_i^0}$ at time $t$.
    if mod(i, ndtobs) == 0
        inhodata((1+nsensor*(i/ndtobs-1)):(nsensor*i/ndtobs)) = observe(ustar);
    end
end
```

Subtracting real data by this one, we get the data that should be observed by homogeneous system.

```
homodata = data - inhodata;
```

Solve for $\widetilde{C}_i$

Now by using the homogeneous solver and the bases we introduced above, we can solve for $\widetilde{C}_i^0$ and $\widetilde{C}_i$.

```
numdata = zeros(nobst*nsensor, nb);

Ct = C;
for i = 1:(nobst*ndtobs)
    Ct = Fh(Ct);
    if mod(i, ndtobs) == 0
        numdata((1+nsensor*(i/ndtobs-1)):(nsensor*i/ndtobs), :) = observe(Ct);
    end
end
```

The i^{th} column of `numdata` corresponds to $\widetilde{C}_i$.

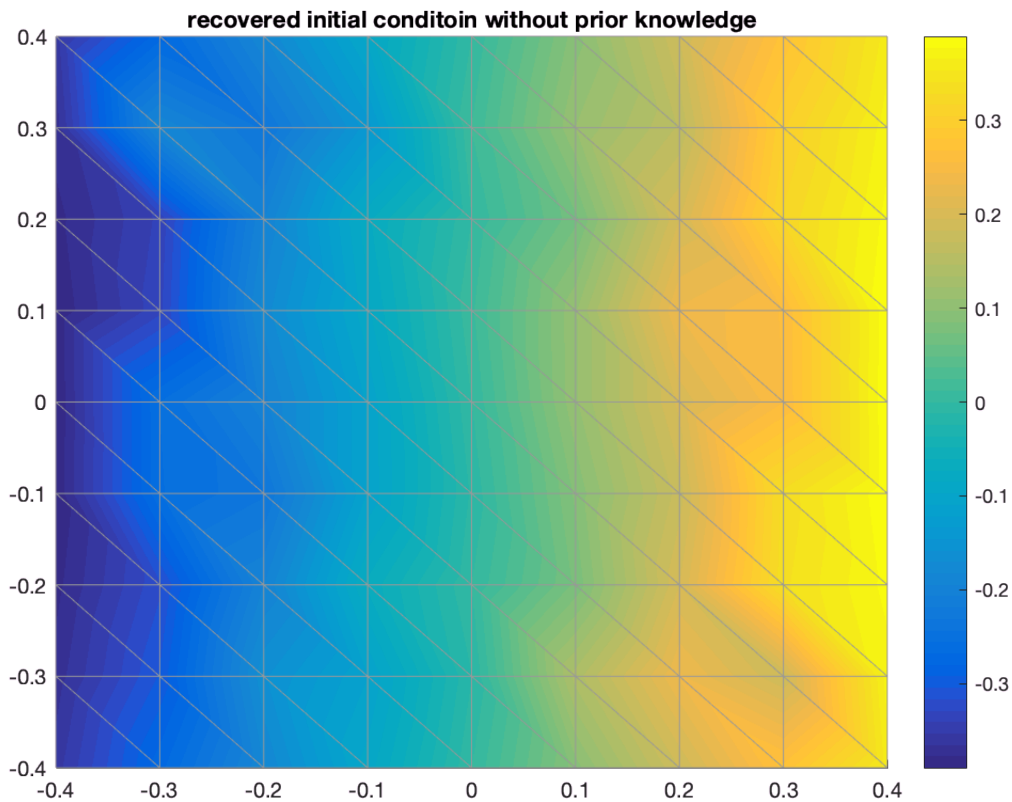
Solve for u_0

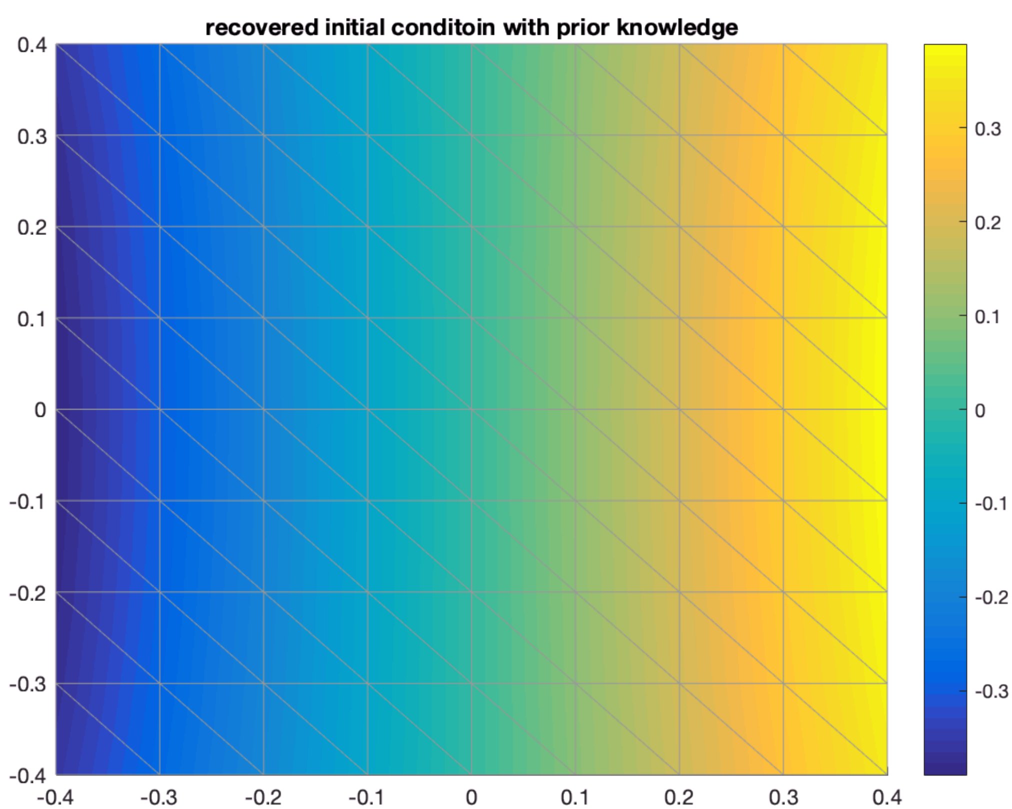
Now we are ready to solve for all α_i. We expect `numdata * alphas = homodata`. So

```
alphas = numdata \ homodata;
```

The guess of our initial condition is then

```
u0 = alphas2V(alphas) + u0star;
```






```
figure();
V.plotu(u0)
```

```
title('recovered initial conditoin without prior knowledge');
MatFem.errorNorm(V, u0, @(x,y)u_true(x,y,0), 'L2')
```

```
ans = 0.0145
```

Solve for u_0 with prior β

If we have some prior knowledge $\beta = u_0$, given weight $\kappa = 0.01$, then the least square solve becomes

```
beta  = V.project(@(x,y)u_true(x,y,0));
kappa = 0.01;

alphas = (numdata' * numdata + kappa * eye(nb)) \ (numdata' * homodata + kappa * V2alphas(beta

u0 = alphas2V(alphas) + u0star;
figure();
V.plotu(u0)
title('recovered initial conditoin with prior knowledge');
MatFem.errorNorm(V, u0, @(x,y)u_true(x,y,0), 'L2')
```

```
ans = 3.5642e-04
```

Functions

```
function u = F(u_old, t)
    % Solve the poisson system for 1 timestep.
    import MatFem.assemble
    global V bc Auv f g A dt

    b = dt * assemble(V, [0 0], @(x,y)f(x,y,t)); % Δt * integrate(f*v)
    b = b + Auv * u_old; % Auv * u_old gives integrate(u_old*v)
    b = bc.applyDir('d', @(x,y)g(x,y,t), b); % apply boundary condition
    u = A \ b;
end

function u = Fh(u_old)
    % Solve the homogenized poisson system for 1 timestep.
    import MatFem.assemble
    global bc Auv A

    b = Auv * u_old;
    b = bc.applyDir('d', 0, b);
    u = A \ b;
end

function v = alphas2V(alphas)
    % Map a vector of alphas to a function with
    %    v = \sum_i alpha_i \phi_i
    % where \phi_i is one basis of the functionspace V.
    %
    % This function is vectorized. `alphas` can be a 2D array
    % with each column represent a vector of alphas.
    global bases
    v = zeros(size(bases, 1), size(alphas, 2));
    v(bases,:) = alphas;
```

```
end

function alphas = V2alphas(v)
    % Map a vector in our function space to alphas.
    % I.e., remove all Dirichlet boundaries.
    global bases
    alphas = v(bases,:);
end

function data = observe(v)
    % Observe a state v in function space V.
    % Corresponds to sensors at locations:
    %     [-0.2 0 0.2] x [-0.2 0 0.2] (Cartesian product)
    % Vectorized so that alphas can be a row of vector of alphas.
    pos = [9 11 13 23 25 27 37 39 41]; % You can use the codes in section *Bases* to examine t
    alphas = V2alphas(v);
    data = alphas(pos, :);
end
```